Homotopy analysis of the Lippmann-Schwinger equation
for seismic wavefield modeling in strongly scattering media

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SUMMARY

We present an application of the homotopy analysis method for solving the integral equations of the
Lippmann-Schwinger type, which occurs frequently in acoustic and seismic scattering theory. In this method,
a series solution is created which is guaranteed to converge independent of the scattering potential. This
series solution differs from the conventional Born series because it contains two auxiliary parameters $\epsilon$
and $h$ and an operator $H$ that can be selected freely in order to control the convergence properties of the
scattering series. The $\epsilon$-parameter which controls the degree of dissipation in the reference medium (that
makes the wavefield updates localized in space) is known from the so-called convergent Born series theory;
but its use in conjunction with the homotopy analysis method represents a novel feature of this work. By
using $H = I$ (where $I$ is the identity operator) and varying the convergence control parameters $h$ and $\epsilon$, we
obtain a family of scattering series which reduces to the conventional Born series when $h = -1$ and $\epsilon = 0$.
By using $H = \gamma$ where $\gamma$ is a particular preconditioner and varying the convergence control parameters $h$
and $\epsilon$, we obtain another family of scattering series which reduces to the so-called convergent Born series
when $h = -1$ and $\epsilon \geq \epsilon_c$ where $\epsilon_c$ is a critical dissipation parameter depending on the largest value of the
scattering potential. This means that we have developed a kind of unified scattering series theory that in-
cludes the conventional and convergent Born series as special cases. By performing a series of 12 numerical
experiments with a strongly scattering medium, we illustrate the effects of varying the ($\epsilon, h, H$)-parameters
on the convergence properties of the new homotopy scattering series. By using ($\epsilon, h, H$) = (0.5, -0.8, $I$)
we obtain a new scattering series that converges significantly faster than the convergent Born series. The
use of a non-zero dissipation parameter $\epsilon$ seems to improve on the convergence properties of any scattering
series, but one can now relax on the requirement $\epsilon \geq \epsilon_c$ from the convergent Born series theory, provided
that a suitable value of the convergence control parameter $h$ and operator $H$ is used.
1 INTRODUCTION

There exist a range of different numerical methods for seismic wavefield modeling (Carcione, 2002), including differential equation methods (e.g., Robertsson et al., 2012) and integral equation methods (Pike and Sabatier, 2000; Oristaglio and Blok, 2012; Jakobsen, 2012; Jakobsen and Wu, 2016; Malovichko et al., 2018). The majority of researchers in the seismic community use differential equation methods (Carcione et al., 2002), but the integral equation approach has actually several advantages compared with the differential equation approach: (1) it is naturally target oriented (Haffinger et al., 2016; Huang et al., 2019), (2) it gives the sensitivity matrix directly in terms of Green’s functions (Jakobsen and Ursin, 2015) which is convenient for uncertainty estimation (Eikrem et al., 2018) and (3) it is compatible with the use of domain decomposition and renormalization methods from modern physics (Jakobsen and Wu, 2016, 2018). However, the integral equation approach can be less efficient than the differential equation approach, depending on how it is implemented (Malovichka et al., 2018; Jakobsen and Wu, 2018; Jakobsen et al. 2019). An integral equation solution based on matrix inversion can be very accurate, but very memory-depending and costly (Jakobsen and Wu, 2018). Efficient implementations of the integral equations approach are typically based on the use of iterative methods and/or scattering series solutions (Jakobsen and Wu, 2016, Malovichko et al., 2018; Jakobsen et al., 2019, Huang et al., 2019b,c).

Many geophysicists are familiar with the scattering series of Born that one can easily obtain from the Lippmann-Schwinger equation via simple iteration (Jakobsen and Wu, 2016). However, the Born series represents an example of a so-called naive perturbation expansion which is only guaranteed to converge in the special case of relatively small contrasts (Kirkinis, 2008; Jakobsen and Wu, 2016). To obtain a convergent scattering scattering series in the presence of strong contrasts, it may be required to use a non-perturbative method for strongly nonlinear systems. Previously, researchers have developed convergent scattering series solutions of the Lippmann-Schwinger equation by using renormalization procedures (Abubakar and Habashy, 2003; Wu et al., 2007; Osnabrugge et al., 2016; Jakobsen et al., 2016; Jakobsen et al., 2019). In this study, however, we have employed the so-called homotopy analysis method, which is based on concepts and ideas form topology (Liao, 2003; Hetmaniok et al., 2014).

The homotopy analysis method (HAM) used in this study was developed by Liao (1998, 2003, 2004, 2009, 2012, 2014). However, the development of related globally convergent homotopy methods for solving nonlinear equations started around 1976 (see Watson, 1989). Historically, there have been several attempts to apply homotopy methods to model and invert geophysical data (see Watson, 1989), but the paper of Huang and Greenhalgh (2019a) appears to represent the first geophysical application of the modern homotopy analysis method developed by Liao (2003), which differs from the
one discussed by Watson (1989). In any case, the homotopy methods allows one to solve operator equations of any kind by using ideas and concepts of topology, which is a branch of pure and applied mathematics dealing with quantities that are preserved during continuous deformations. A homotopy describes a continuous transformation between two states and has been compared with the concept of scale-invariance in renormalization group theory (Palit and Datta, 2016; Jakobsen et al., 2019a; Pfeffer, 2019). The homotopy analysis method have been used to solve a range of different nonlinear problems, ranging from heat conduction problems (Abbasbandy, 2006) to problems within theoretical physics (Pfeffer, 2019). Most applications of the homotopy analysis method is based on a differential operator formulation, but there have also been successful attempts to solve integral equations of the Fredholm and Volterra types using the homotopy analysis method (see Hetmanio et al., 2014).

Although the homotopy analysis method may potentially be very useful for practical nonlinear inversion of seismic waveform data (see Han et al., 2005; Fu and Han, 2006), we shall focus on the forward problem. This is partially because there is still an important need for more work on the nonlinear direct scattering problem (Jakobsen et al., 2019a,b) and the corresponding nonlinear inverse scattering problem is much more difficult to solve due to its ill-posed nature. It will be demonstrated that the homotopy analysis method can be used to construct a scattering series solution of the Lippmann-Schwinger equation in the context of seismic wavefield modeling. Although such convergent scattering series have been developed on the basis of renormalization methods in the past (Abubakar and Habashy 1983; Osnabrugge et al., 2016; Jakobsen et al., 2019a), we think it is interesting to study convergence properties of the direct scattering series solution from different perspectives, since this may give us new ideas and insights that may be useful for future studies of nonlinear inverse scattering as well as direct scattering problems.

In section 2, we present fundamental equations and establish our notation. In section 3, we present a general method for obtaining convergent series solutions of nonlinear operator equations that does not depend on any parameter being small. In section 4, we derive a convergent scattering series solution of the Lippmann-Schwinger equation. In section 5, we show that the conventional Born series and the renormalized Born series of Osnabrugget al. (2016) and Huang et al. (2019) represents a special case of the HAM series. In section 6, we demonstrate that the HAM series converges for strongly scattering media where the conventional Born series diverges. In section 7, we also provide some ideas for further work.
2 THE LIPPMANN-SCHWINGER EQUATION AND CONVENTIONAL BORN SERIES

The scalar wave equation in the frequency domain (the inhomogeneous Helmholtz equation) can be written as (Morse and Feshbach, 1953; Osnabrugge et al., 2016; Huang et al., 2019b,c)

\[ \left( \nabla^2 + k^2(x) \right) \psi(x) = -S(x), \tag{1} \]

where \( k(x) \) is the wave number at position \( x \). Following Osnabrugge et al. (2016), we now decompose the actual medium with wavenumber \( k(x) \) into an arbitrary homogeneous dissipative reference medium with complex wave number \( k_d \) given by \( k_d^2 = k_0^2 + i\epsilon \) (where \( \epsilon \) is an arbitrary small positive number) and a corresponding complex scattering potential \( V(x) \) (with compensating gain, rather than dissipation). It follows that

\[ \left( \nabla^2 + k_d^2 \right) \psi(x) = -S(x) - V(x)\psi(x), \tag{2} \]

where

\[ V(x) = k^2(x) - k_d^2. \tag{3} \]

The second term on the right-hand side of equation (2) represents the so-called equivalent sources. By treating the contrast-sources just like real sources, one can derive the Lippmann-Schwinger equation (Jakobsen and Ursin, 2015)

\[ \psi(x) = \psi(0)(x) + \int_{\Omega} dx' G(0)(x - x')V(x')\psi(x'), \tag{4} \]

where \( G(0)(x - x') \) is the Green’s function for the homogeneous reference medium, that satisfies

\[ \left( \nabla^2 + k_d^2 \right) G(0)(x - x') = -\delta(x - x'). \tag{5} \]

Note that the introduction of a non-zero imaginary part \( \epsilon \) to the squared wave number \( k_0^2 \) in the homogeneous reference medium makes the energy associated with Green’s function finite and the wave fields more localized (Osnabrugge et al., 2016; Jakobsen et al., 2019a; Huang et al., 2019c). Although most workers tend to set \( \epsilon \) to zero, the use of a non-zero \( \epsilon \) parameter improves the convergence properties of any scattering series (Abubakar and Habashy, 2003; Osnabrugge et al., 2016; Huang et al., 2019c; Jakobsen et al., 2019a).

In symbolic operator notation, the Lippmann-Schwinger equation (4) can be written as

\[ \psi = \psi(0) + G(0)V\psi, \tag{6} \]

where the scattering potential operator \( V \) is local (but see Jakobsen and Wu, 2017) and can be represented by a diagonal matrix in the real-space representation (Jakobsen and Ursin, 2015). The above equation has the following exact formal solution:

\[ \psi = (I - G(0)V)^{-1}\psi(0) \tag{7} \]
where \( I \) is the identity operator.

The solution (7) is valid independently of the contrast volume, but it involves the inversion of a huge operator or matrix (in the coordinate representation), which can be very costly in the case of a realistic model. In principle, one could try to solve the Lippmann-Schwinger equation by iteration. The well-known Born series can be regarded as the simplest possible iterative solution of the Lippmann-Schwinger equation and can be presented as

\[
\psi = \sum_{m=0}^{\infty} \psi_m
\]

where \( \psi_0 = \psi^{(0)} \) and

\[
\psi_m = G^{(0)} V \psi_{m-1}, \quad m = 1, 2, 3, \ldots
\]

The Born series is very popular due to its simplicity. However, the Born series represents an example of a naive perturbation expansion (Kirkinis, 2008) which is only guaranteed to converge if the contrast is relatively small, in the sense that the largest eigenvalue of the operator \( G^{(0)} V \) must be smaller than unity (Weinberg, 1983, Newton, 2002; Osnabrugge et al., 2016).

### 3 THE HOMOTOPY ANALYSIS METHOD

The homotopy analysis method can be used to solve operator equations of the form (Liao, 2003)

\[
N \left[ \psi \right] = 0,
\]

where \( N \) denotes a nonlinear operator and \( \psi \) is the unknown function (or state vector). The first step is to define the homotopy operator \( \mathcal{H} \) by (Liao, 2003)

\[
\mathcal{H} \left[ \Phi, \lambda \right] \equiv (1 - \lambda) L \left[ \Phi(\lambda) - \psi_0 \right] - \lambda h N \left[ \Phi(\lambda) \right],
\]

where \( \lambda \in [0, 1] \) is the so-called embedding parameter, \( h \neq 0 \) is the so-called convergence control parameter, \( H \) is a convergence control operator (see section 4), \( \psi_0 \) is our initial guess of the solution to equation (10) and \( L \) is an auxiliary linear operator that can be selected arbitrarily as long as \( L \left[ 0 \right] = 0 \).

By setting \( \mathcal{H} \left[ \Phi, \lambda \right] = 0 \) we get the so-called zero-order deformation equation (Liao, 2003)

\[
(1 - \lambda) L \left[ \Phi(\lambda) - \psi_0 \right] = \lambda h N \left[ \Phi(\lambda) \right].
\]

If \( \lambda = 0 \) then \( L \left[ \Phi(0) - \psi_0 \right] = 0 \), which implies that \( \Phi(0) = \psi_0 \). If \( \lambda = 1 \) then \( N \left[ \Phi(1) \right] = 0 \), which implies that \( \Phi(1) = \psi \), where \( \psi \) is the solution of equation (10) we are looking for. A gradual change in the embedding parameter \( \lambda \) from 0 to 1 therefore means a continuous transition of \( \Phi(\lambda) \) from the initial guess \( \psi_0 \) to the exact solution \( \psi \) of the original equation (10).

If we now expand the auxiliary field \( \Phi(\lambda) \) into a Maclaurin series with respect to the embedding parameter \( \lambda \) then we obtain (Liao, 2003)
\[ \Phi(\lambda) = \Phi(0) + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \Phi(\lambda)}{\partial \lambda^m} \bigg|_{\lambda=0} \lambda^m. \] 

(13)

By introducing the definition (Liao, 2003)

\[ \psi_m \equiv \left. \frac{1}{m!} \frac{\partial^m \Phi(\lambda)}{\partial \lambda^m} \right|_{\lambda=0}, \quad m = 1, 2, 3, \ldots, \]

(14)

the above equation (13) can be expressed as (Liao, 2003)

\[ \Phi(\lambda) = \Phi(0) + \sum_{m=1}^{\infty} \psi_m \lambda^m. \]

(15)

If the above series (15) is convergent for \( \lambda = 1 \) then the solution we are looking for is given by (Liao, 2003)

\[ \psi = \sum_{m=0}^{\infty} \psi_m. \]

(16)

It is of course not obvious that the series (15) is convergent for \( \lambda = 1 \), but by adjusting the auxiliary parameter \( h \) and the auxiliary operator \( H \) we can make sure that this series is indeed convergent (Liao, 2003).

In order to determine the different \( \psi_m \) terms, we now differentiate the left and right side of the 0th-order deformation equation (12) \( m \) times with respect to the auxiliary parameter \( \lambda \), divide the result by \( m! \) and set \( \lambda = 0 \). In this way suggested by Liao (2003), we obtain the so-called \( m \)th-order deformation equation \( (m > 0) \):

\[ L \left[ \psi_m - \chi_m \psi_{m-1} \right] = hHR_m, \]

(17)

where

\[ \chi_m = \begin{cases} 
0 & \text{if } m \leq 1 \\
1 & \text{if } m \geq 2 
\end{cases} \]

(18)

and

\[ R_m = \frac{1}{(m-1)!} \left( \frac{\partial^{m-1} \Phi(0) - \partial \lambda^{m-1} N \left[ \sum_{i=0}^{\infty} \psi_i \lambda^i \right]}{\partial \lambda^{m-1}} \right) \bigg|_{\lambda=0}. \]

(19)

The different \( R_m \) parameters will depend on the nature of the non-linear operator \( N \). In the next section, we shall evaluate the \( R_m \)-parameters for the nonlinear operator corresponding with the Lippmann-Schwinger equation.

The selection of the convergence control parameter \( h \) is very important. In order to select a suitable value of \( h \), one can either use the \( h \)-parameter curve method or an optimization method (Liao, 2003; Hetmaniok et al., 2014). We shall discuss the selection of \( h \) in connection with the results we have obtained for the homotopy analysis of the Lippmann-Schwinger equation.
Hetmaniok et al. (2014) discuss the usage of the homotopy analysis method for solving nonlinear and linear integral equations of the second kind. However, their analysis is restricted to 1D media and slightly different from the analysis presented below. In order to derive a convergent scattering series solution of the Lippmann-Schwinger equation based on the homotopy analysis method, we define the linear and nonlinear integral operators $L$ and $N$ by

$$L[\psi] = \psi, \quad N[\psi] = \psi - \psi^{(0)} - G^{(0)}V\psi,$$

(20)

By using the above definitions of the linear and nonlinear operators $L$ and $N$ in conjunction with the $m$th-order deformation equation 17, we obtain

$$\psi_m = \chi_m \psi_{m-1} + hHR_m.$$

(21)

By using the definition of the nonlinear operator $N$ given in equation (20) in conjunction with the expression for the $R_m$ parameters in equation (18), we get

$$R_m = \frac{1}{(m - 1)!} \left( \frac{\partial^{m-1}}{\partial \lambda^{m-1}} \left[ \sum_{i=0}^{\infty} \psi_i \lambda^i - \psi^{(0)} - G^{(0)} V \sum_{i=0}^{\infty} \psi_i \lambda^i \right] \right)_{\lambda=0}.$$

(22)

The above equation implies that

$$R_m = \frac{1}{(m - 1)!} \left( (m - 1)! \psi_{m-1} - (1 - \chi_m) \psi^{(0)} - (1 - m)! G^{(0)} V \psi_{m-1} \right),$$

(23)

or

$$R_m = \psi_{m-1} - \frac{(1 - \chi_m)}{(1 - m)!} \psi^{(0)} - G^{(0)} V \psi_{m-1}.$$

(24)

By using the above expression for $R_m$ in conjunction with the recursive formula (21), we obtain

$$\psi_1 = hH \left( \psi_0 - \psi^{(0)} - G^{(0)} V \psi_0 \right),$$

(25)

and for $m \geq 2$:

$$\psi_m = \mathcal{M} \psi_{m-1}.$$  

(26)

where

$$\mathcal{M} \equiv I + hH - hHG^{(0)} V.$$

(27)

Equations (25) and (A.1) for the first and higher-order terms in the homotopy analysis scattering series represents the main results of this paper. This homotopy analysis method iterative solution of the Lippmann-Schwinger equation differs from the conventional Born series via the convergence control parameter $h$ and the operator $H$ that can be selected arbitrarily to ensure that the series is convergent. The HAM series converges if the spectral radius $\sigma$ of $M$ is smaller than unity; which can occur even
if the spectral radius of the operator $G_0V$ is larger than unity; that is, when the conventional scattering series of Born diverges.

5 COMPARISON WITH EXISTING ANALYTICAL RESULTS

If we use our freedom to set $\psi_0 = \psi(0)$, $h = -1$ and $H = I$ then the homotopy series in equations (25-26) reduces to the conventional Born series (9). As discussed earlier, the conventional Born series have a rather small range of convergence, since the largest eigenvalue of the operator $G^{(0)}V$ must be smaller than unity.

Osnabrugge et al. (2016) presented a modified Born series (CBS) which is guaranteed to converge independent of the scattering potential. If we set $\psi_0 = \gamma \psi(0)$ where $\gamma = iV/\epsilon$ is the preconditioner of Osnabrugge et al. (2016) then it follows from equation (23) that

$$\psi_1 = h H \left( I - \gamma^{-1} - G^{(0)}V \right) \psi_0.$$  \hspace{1cm} (28)

If we now set $h = -1$ and $H = \gamma$ then the above equation becomes

$$\psi_1 = M \psi_0,$$  \hspace{1cm} (29)

where

$$M \equiv I - \gamma + \gamma G^{(0)}V.$$  \hspace{1cm} (30)

Also, it follows from equation that

$$\psi_m = M \psi_{m-1},$$  \hspace{1cm} (31)

which implies that the convergent Born series of Osnabrugge et al. (2016) is a special case of our new HAM series.

A comparison of equations (26-27) and (30-31) clearly suggests that we have generalized the convergent Born series of Osnabrugge et al. (2016). We can construct a family of convergent Born series similar to the convergent Born series of Osnabrugge et al. (2016) if we set $H = \gamma$ but use different values of the convergence control parameter $h$. The CBS is based on the use of a dissipative reference medium, which makes the Green’s function finite and localized (Osnabrugge et al., 2016; Huang et al., 2019c; Jakobsen et al., 2019a). Since our generalized convergent Born series based on HAM contains the additional convergence control parameter $h$, we can decrease the value of the dissipation parameter $\epsilon$ if we compensate by using a suitable $h$-value. The convergence control parameter $h$ is a global convergence parameter, in the sense that it acts globally on the whole model, whereas the dissipation parameter $\epsilon$ can be regarded as a local convergence parameter, since a higher value for $\epsilon$ implies a higher degree of wavefield localization. By a suitable choice of the local and global convergence parameters, we can ensure that the CBS is guaranteed to converge independent of the scattering potential.
Convergent scattering series via homotopy analysis

control parameters $\epsilon$ and $h$ we can accelerate the convergence of the HAM series. This point will be illustrated in the next section dealing with numerical experiments based on a strongly scattering seismic model. By introducing an imaginary part to the wavevector of the background medium, we make the total energy represented by the background Greens function finite and localized. The imaginary term in the background medium is compensated exactly by an imaginary term in the scattering potential. Therefore, the final solution remains the same as the solution without any dissipation.

6 NUMERICAL RESULTS AND DISCUSSION

We performed a series of 12 different numerical experiments to study the effects of the auxiliary parameters $(\epsilon, h, H)$ on the convergence properties of the homotopy analysis method for solving the Lippmann-Schwinger equation. The numerical experiments are based on a resampled version of the SEG/EAGE salt model (Figure 1, left). We used a homogeneous reference medium with wavespeed $c_0 = 2870$ m/s (Figure 1, right). We employed a single delta function source with frequency 10 Hz located in the middle of the top of the model and we used a grid size equal to 5 m in each direction. In each experiment, we used one of the combinations of the $(\epsilon, h, H)$ -parameters given in Table 1 and generated a scattering series solution of the Lippmann-Schwinger equation by using the recursive formula (25-27).

We quantified the convergence properties of the different scattering series by calculating the normalized overall error $\delta_k$ as a function of the number of iterations $k$, where

$$
\delta_k = \frac{\| \sum_{i=1}^{k} \psi_i - \psi^{(r)} \|}{\| \psi^{(r)} \|},
$$

and $\psi^{(r)}$ is a reference solution obtained by solving equation (7) via matrix inversion (see Figure 2), which is exact apart from very small numerical discretization errors (Jakobsen, 2012). In each numerical experiment, we iterated until the normalized overall error became smaller than $10^{-3}$ (in the case of convergence) or larger than 10 (in the case of divergence). However, this stopping criteria is of course flexible and dependent on the desired accuracy.

If the scattering series diverges then the resulting wavefield (Figure 3) will of course look very different from the reference wavefield (Figure 2). If the scattering series converges in the sense that the overall normalized error becomes smaller than $10^{-3}$ than the resulting wavefield (Figure 4) will necessarily be very similar to the true wavefield (Figure 2). Since the resulting wavefield is independent of the auxiliary parameters in the case of convergence, we focus on the behaviour of $\delta_k$ rather than the wavefield itself.

In numerical experiments 1-6 (Figure 5) we assumed $H = I$ and varied the dissipation parameter $\epsilon$ and the convergence control parameter $h$. As discussed in the previous section, when $\epsilon = 0$ and $h =
The numerical results correspond with the conventional Born series, whereas the use of different \( \epsilon \) - and \( h \)-values represents different modifications of the conventional Born series. Clearly, one can see from the blue curve in Figure 5 that the conventional Born series corresponding with \( \epsilon = 0 \) and \( h = -1 \) diverges for this strongly scattering medium. When \( \epsilon = 0 \) and \( h = -0.95 \) corresponding with the green curve in Figure 5, the scattering series still diverges. When \( \epsilon = 0 \) and \( h = -0.9 \) corresponding with the red curve in Figure 5, the scattering series is starting to converge, but extremely slowly. When \( \epsilon = 0 \) and \( h = -0.8 \) corresponding with the cyan curve in Figure 5, the scattering series converges faster. When \( \epsilon = 0 \) and \( h = -0.1 \) corresponding to the black curve in Figure 5, the scattering series is still convergent, but the convergence rate is much smaller than when using \( h = -0.8 \). When \( \epsilon = 0.5 \) and \( h = -0.8 \) corresponding to the black curve in Figure 5 than the scattering series converges faster than for all the other experiments 1-5. Therefore, it appears that the use of a non-zero \( \epsilon \)-value in conjunction with an optimal \( h \)-value helps to accelerate an already convergent scattering series.

In numerical experiments 7-12 (Figure 6) we assumed \( H = \gamma \) and varied the dissipation parameter \( \epsilon \) as well as the convergence control parameter \( h \). When \( \epsilon \leq \epsilon_c \) where \( \epsilon_c \) is a critical value depending on the velocity model and \( h = -1 \) then the numerical results corresponds with the convergent Born series of Osnabrugge et al. (2016), whereas the use of different \( \epsilon \) and \( h \)-parameters correspond with different modifications of the convergent Born series of Osnabrugge et al. (2016). Clearly, one can see from the blue curve in Figure 6 that the convergent Born series of Osnabrugge et al. (2016) corresponding with \( \epsilon = \epsilon_c \) and \( h = -1 \) is indeed convergent. When \( h = -1 \) but \( \epsilon = 0.5 \epsilon_c \) corresponding to the green curve in Figure 6, the scattering series is as expected divergent. When \( \epsilon = 0.5 \epsilon_c \) and \( h = -0.5 \) corresponding with the red curve in Figure 6, the scattering series is still divergent. However, when \( \epsilon = 0.5 \epsilon_c \) and \( h = -0.25 \) corresponding with the cyan curve in Figure 6, the scattering series become convergent again. When \( \epsilon = 0.5 \epsilon_c \) and \( h = -0.125 \) corresponding with the curve in Figure 6, the scattering series is still convergent, but the convergence rate is smaller than when using \( h = -0.25 \). When using \( \epsilon = 0.25 \epsilon_c \) corresponding with the black curve in Figure 6 the scattering series converges extremely slowly. Therefore, it appears that the \( \epsilon \) and \( h \)-parameters corresponding with the original convergent Born series of Osnabrugge et al. (2016) are optimal when \( H = \gamma \).

Figure 7 represents a comparison of the optimal HAM series (the black curve in Figure 6 corresponding to \( \epsilon = 0.5, h = -0.8 \) and \( H = I \)) and the original convergent Born series of Osnabrugge et al. (2016) (the blue curve in Figure 6 corresponding with \( \epsilon = 1, h = -1 \) and \( H = -\gamma \)). Clearly, one can see that the optimal HAM series requires much less iterations than the original convergent Born series. Therefore, one can say that we have generalized and improved on the convergent Born series of Osnabrugge et al. (2016) by using the homotopy analysis method.

The auxiliary parameters \( \epsilon \) and \( h \) may be referred to as local and global convergence control pa-
rameters, respectively. This is because a non-zero $\epsilon$ value leads to dissipation in the reference medium (and gain in the scattering potential), which makes the wavefield update more localized in space; and different $h$-values are associated with different degrees of global wavefield scaling. Having both local and global convergence control parameters in addition to the auxiliary convergence control operator $H$ makes this homotopy analysis method very general and flexible.
7 CONCLUDING REMARKS

We have used the homotopy analysis method (HAM) to derive a general scattering series solution of the Lippmann-Schwinger equation which is guaranteed to converge independent of the scattering potential, provided that one select the dissipation parameter $\epsilon$ as well as the convergence control parameter $h$ and operator $H$ in a suitable manner. We have found that the conventional Born series and the convergent Born series of Osnabrugge et al. (2016) are special cases of the new scattering series based on HAM. We have performed a series of 12 numerical experiments and found that a scattering series with $\epsilon = 0.5$, $h = -0.8$ and $H = I$ requires much less iterations to converge than the original convergent Born series of Osnabrugge et al. (2016). Other choices of the $(\epsilon, h, H)$ may lead to even higher convergence rate, but existing guidelines for selecting $h$ and $H$ (see Liao, 2003) needs to be modified in the presence of the new parameter $\epsilon$.

Historically, this paper represents a rare example of the application of HAM for solving integral equations and the first example in the context of seismic wavefield modeling. The introduction of the dissipation parameter $\epsilon$ into the HAM formalism also represents a novel feature of this work. Theoretically, the embedding parameter $\lambda$ reminds us about the running coupling constant in the renormalization group theory of Jakobsen et al. (2019a) as well as the homotopy parameter $\lambda$ of Watson (1989), but these relations requires further investigation. Computationally, it is interesting to note that the computational cost of the reference solution (7) we have obtained via matrix inversion and the scattering series solution (25-27) scales like $N^3$ and $N^2$, respectively, where $N$ is the number of grid blocks in a discretized seismic model. Since our formulation is based on a homogeneous reference medium, it allows for the use of efficient and memory-saving Fast Fourier Transform methods that scales like $N$ and $N \log N$, respectively (see Osnabrugge et al., 2016; Jakobsen et al., 2019a). The present work can also be combined with convergence acceleration techniques (Eftekhar et al., 2018). Practically, it is important that the theory and method developed in this study can be generalized to anisotropic elastic media, since the corresponding wave equation can also be transformed into an integral equation of the Lippmann-Schwinger type (Jakobsen et al., 2019b). Having developed a convergent forward scattering series, the next step could be to apply this series in the context of inverse scattering theory. Weglein et al (2003) have pioneered inverse acoustic scattering methods that do not require an assumed propagation velocity model within the medium (Zou and Weglein, 2018). Their approach (Zhang and Weglein, 2009; Zou et al. 2019) is based on the Born series solution of the Lippmann-Schwinger equation and a concomitant expansion of the interaction in orders of the data. In principle, the method is completely general and requires no prior information about the target or the propagation details of the probe signal within the target. The only fundamental limitation of the approach appears to the convergence of the conventional Born series (Kouri, 2003). Weglein et al., (2003) have made significant progress using
this approach by introducing subseries. We hope to develop convergent inverse scattering series using the HAM scattering series. However, it is not obvious that the inverse scattering series will converge even though the forward scattering series is convergent. Finally, we note that the work reported here may be useful in future applications of the homotopy analysis method within the context of nonlinear inverse scattering to solve the so-called regularized normal equations (Jegen et al., 2001).

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Jakobsen, M., Psenik, I., Iversen, E. and Ursin, B., 2019b. Transition operator approach to seismic full waveform


Zou, Y., Ma, C. and Weglein, A. B. 2019. A new multidimensional method that eliminates internal multiples that interfere with primaries, without damaging the primary, without knowledge of subsurface properties, for off-shore and on-shore conventional and unconventional plays, *SEG Technical Program Expanded Abstracts 2019*, 4525-4529.
Table 1. Convergence control parameters used in 4 different numerical experiments focusing on the convergence properties of the homotopy analysis method. The parameter $\epsilon_c$ is the critical value which is required for the convergent Born series of Osnabrugge et al. (2016) to converge. The color refers to the different colors used in Figure 5 (experiments 1-6) and Figure 6 (experiments 7-12).

<table>
<thead>
<tr>
<th>Experiment</th>
<th>$\epsilon/\epsilon_c$</th>
<th>$h$</th>
<th>$H$</th>
<th>color</th>
<th>$|M|$</th>
<th>$\sigma(M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00</td>
<td>-1.00</td>
<td>$I$</td>
<td>1.46</td>
<td>1.06</td>
<td>b</td>
</tr>
<tr>
<td>2</td>
<td>0.00</td>
<td>-0.950</td>
<td>$I$</td>
<td>1.40</td>
<td>1.03</td>
<td>g</td>
</tr>
<tr>
<td>3</td>
<td>0.00</td>
<td>-0.900</td>
<td>$I$</td>
<td>1.35</td>
<td>0.99</td>
<td>r</td>
</tr>
<tr>
<td>4</td>
<td>0.00</td>
<td>-0.800</td>
<td>$I$</td>
<td>1.27</td>
<td>0.94</td>
<td>c</td>
</tr>
<tr>
<td>5</td>
<td>0.00</td>
<td>-0.100</td>
<td>$I$</td>
<td>1.01</td>
<td>0.97</td>
<td>m</td>
</tr>
<tr>
<td>6</td>
<td>0.50</td>
<td>-0.800</td>
<td>$I$</td>
<td>0.96</td>
<td>0.82</td>
<td>k</td>
</tr>
<tr>
<td>7</td>
<td>1.00</td>
<td>-1.000</td>
<td>$\gamma$</td>
<td>0.94</td>
<td>1.98</td>
<td>b</td>
</tr>
<tr>
<td>8</td>
<td>0.50</td>
<td>-1.000</td>
<td>$\gamma$</td>
<td>0.85</td>
<td>1.93</td>
<td>g</td>
</tr>
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<td>9</td>
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<td>$\gamma$</td>
<td>0.92</td>
<td>1.09</td>
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<td>$\gamma$</td>
<td>0.96</td>
<td>0.93</td>
<td>c</td>
</tr>
<tr>
<td>11</td>
<td>0.50</td>
<td>-0.125</td>
<td>$\gamma$</td>
<td>0.98</td>
<td>0.96</td>
<td>m</td>
</tr>
<tr>
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<td>-0.125</td>
<td>$\gamma$</td>
<td>0.96</td>
<td>1.00</td>
<td>k</td>
</tr>
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Figure 1. The true velocity model and the homogeneous reference model.
**Figure 2.** Real and imaginary parts of the frequency domain wavefield at 10 Hz computed by solving the Lippmann-Schwinger equation exactly using a real space matrix representation.
Figure 3. The real and imaginary parts of the frequency domain wavefield at 10 Hz computed using the HAM series with auxiliary parameters corresponding to experiment 1 in Table 1.
Figure 4. The real and imaginary parts of the frequency domain wavefield at 10 Hz computed using the HAM series with auxiliary parameters corresponding with experiment 2 in Table 1.
Figure 5. Comparison of overall errors vs the number of iterations for numerical experiments 1-6 described in Table 1.
Figure 6. Comparison of overall errors vs the number of iterations for numerical experiments 7-12 described in Table 1.
Figure 7. Comparison of overall errors vs the number of iterations for numerical experiment 6 (optimal HAM series with some dissipation in the reference medium) and 7 (original convergent Born series of Osnabrugge et al. 2016). Note that the optimal HAM series (black curve) requires much less iterations than the convergent Born series (blue curve) of Osnabrugge et al. (2016).
APPENDIX A: DESCRIPTION AND IMPLEMENTATION OF THE HAM SERIES

Similar to the conventional Born series, every iteration is associated with multiple scattering processes of different orders. However, we have reorganized the different terms in the conventional Born series so that the spectral radius of $M$ is smaller than unity. This implies that each new term is smaller than the previous one, so that the scattering series does not diverge when the number of iterations becomes large. Mathematically, this is done by introducing an integral operator with spectral radius smaller than unity via the use of control parameter $h$ and the convergence control operator $H$. The series converges if the spectral radius of the operator $M$ is less than unity. We have also introduced an element of dissipation in the reference medium, which ensures that the energy associated with Greens function is finite and localized. It should be emphasized that the dissipation parameter $\epsilon$ can be selected arbitrary. This is because the dissipation is compensated exactly by a corresponding gain term in the scattering potential, suggesting that the final results are independent of this dissipation in the reference medium. The dissipation aspect of our HAM algorithm is similar to the convergent Born series of Osnabrugge et al. (2016). However, our convergent scattering series is more general than the convergent Born series, since the convergent control parameters can be selected rather arbitrary, as long as the spectral radius of the $M$-operator is smaller than unity. Some details for implementation of the new convergent scattering series using based on the HAM is provided in Algorithm 1. In addition, Table 1 shows the norm $\|M\|$ and spectral radius $\sigma(M)$ of the operator $M$ with numerical experiments.

The HAM algorithm is represented by equations (25)-(27). However, the formulation in the main text is based on the real-space coordinate representation of the relevant integral operators. As discussed by Osnabrugge et al., (2016), the operation of Green’s function with contrast-source terms has a convolution structure that can be implemented more efficiently by using the wave vector representation; that is, by using the Fast Fourier Transform (FFT) algorithm in this context. This is because convolution in real-space is equivalent to multiplication in the Fourier space, and the computational cost of the FFT-operation is much smaller than that of matrix multiplication and inversion. The memory requirements scales like $N^2$ and $N$ when using the position and wave vector representations, respectively. The computational cost should theoretically scale like $N^3$ and $N\log N$ when using the position and wave vector representations, respectively. The iterative FFT algorithm is implemented as

$$
\psi_m(r) = hH(\psi^{(0)}(r) - \psi_0(r) - \text{ifft}\left[ C^{(0)}(k) \text{ fft} \left[ V\psi_m(r) \right] \right]),
$$

(A.1)

where fft and ifft are the forward and inverse fast Fourier transform operators, $k$ is the Fourier transformed coordinates. Figures A1 and A2 show the frequency domain wavefield using exact solutions, the efficient and memory-saving FFT implementation with 100 iterations, respectively.
Algorithm 1 Pseudo code for the scattering series

Initialisation: frequency, maximum iteration number $N_{max}$, the parameter $\epsilon$

true model and background model;

$V = k^2 - k_0^2 - i\epsilon$

$k_b = \sqrt{k_0^2 + i\epsilon}$

$\psi^{(0)} = G^{(0)}S$

$\psi = \psi^{(0)}$

$M = I + hH - hG^{(0)}V$

$n = 1$

while $n < N_{max}$ do

$n = n + 1$

if $n == 1$ then

$\psi_m = hH(\psi^{(0)} - \psi_0 - G^{(0)}V\psi^{(0)})$

else

$\psi_m = M\psi_m$

end if

$\psi = \psi + \psi_m$

end while
Figure A1. The real and imaginary parts of the frequency domain wavefield at 10 Hz computed by solving the Lippmann-Schwinger equation exactly using a real space matrix representation.
Figure A2. The real and imaginary parts of the frequency domain wavefield computed using the HAM series with auxiliary parameters corresponding with experiment 11 in Table 1.