Ill Posedness Results for Generalized Water Wave Models

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Abstract

In the first part of the study, the weak asymptotic method is used to find singular solutions of the shallow water system in both one and two space dimensions. The singular solutions so constructed are allowed to contain Dirac-δ distributions (Espinosa & Omel'yanov, 2005). The idea is to construct complex-valued approximate solutions which become real-valued in the distributional limit. The approach, which extends the range f possible singular solutions, is used to construct solutions which contain combinations of hyperbolic shock waves and Dirac-δ distributions.
It is shown in the second part that the Cauchy problem for Korteweg-de Vries (KdV) type equations is locally ill-posed in a negative Sobolev space. The method is used to construct a solution which does not depend continuously on its initial data in H^{s_ε}, $s_\varepsilon = -1/2 - \varepsilon$.

Contents

Chapter 1

Introduction

1.1 Historical Overview of Solitary Wave Theory

The theory of solitary wave has caught the attention of many mathematicians beginning with Lagrange, Cauchy and Poison and continued by other French mathematicians [10, 17]. The theory later caught a great deal of attention from British mathematicians who made remarkable contributions. As knowledge of the solitary wave theory became prevalent in Britain, a Scottish engineer and naval architect, J. S. Russell (1808-1882), got himself involved in understanding such important theory. He observe a solitary wave on the Edinburgh-Glasgow canal in 1834. He discovered an interesting occurrence which he called "great wave of translation" and submitted his observations to the British Association in his "Report on Waves" paper [40]. In this paper, Russell wrote: "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".

This report constitutes one of his observations and has motivated him to carry out laboratory experiments in his own tank constructed for this purpose [7,20]. In the experiment, Russell sought to understand the generation and propagation of solitary waves by dropping a weight in the tank and observing the properties of the solitary wave produced. He discovered an interesting relation between the volume of water in the wave and the volume of water displaced. This relation is expressed between the solitary wave speed, c, and the wave amplitude, a, above the free surface in the form

$$c^2 = g(h + a), \tag{1.1}$$

where h is the undisturbed depth of the water and g is the acceleration due to gravity.

Many other researchers in the field made meaningful contributions to the subject but the work of G. B. Airy and G. G. Stokes caught much attention. They subjected Russell's ideas about solitary waves to careful examination and conjectured that solitary wave propagating in a liquid medium changes shape. In defence of his experimental findings, Russell wrote that "it so happens that their labours and mine do not in the least degree supersede or interfere with each other" rather, they should be seen "as supplementary the one to the other" (see [40], pg.332). Stokes [44] derived equations of motion for incompressible, inviscid fluid with constant vertical gravitational forces. Based on his equations, further assumptions were imposed to derive relevant approximate models to study shallow water waves. The shallow water wave is described with the assumption that the depth of the water is sufficiently small relative to the horizontal length scale of the wave. Interest in the study of the shallow water systems has grown in the past few years and many interesting results were obtained.

Russell's description of the solitary wave became of interest to Boussinesq [5] and Rayleigh [39] who independently confirmed the relation $c^2 = g(h+a)$ in their efforts to understand the subject. Using the equation for inviscid, incompressible fluid, they showed that the profile of a solitary wave takes the form

$$\eta(x,t) = asech^2(\beta(x - ct)) \tag{1.2}$$

where $\beta^2 = 3a/(4h^2(h + a))$ for any maximum wave height, $a > 0$ and water depth h. Further investigations by Boussinesq led to the discovery of new ideas among which is the non-linear evolution equation for long water waves

$$\partial_{tt}\eta = c^2 \left(\partial_{xx}\eta + \frac{2}{3}\partial_{xx}\left(\frac{\eta^2}{h}\right) + \frac{1}{3}\partial_{xxxx}\eta \right) \tag{1.3}$$

where $c = \sqrt{gh}$ is the wave speed. This equation is called the Boussinesq approximation.

Much later, D. Korteweg and his doctoral student Gustavo-de Vries (see [33]) made an important contribution to the theory of solitary wave by deriving a partial differential equation which models Russell's observation. The non-linear equation which is famously called the KdV equation is of the form

$$\partial_t \eta = \frac{c}{h}\left(\left(\varepsilon + \frac{3}{2}\eta\right)\partial_X \eta + \frac{1}{2}\sigma \partial_{XXX}\eta\right) \tag{1.4}$$

where $\sigma = h\left(\frac{h^2}{3} - \frac{T}{g\rho}\right)$ and $c = \sqrt{gh}$. ρ is the water density, T represents the surface tension and ε is assumed small. g denotes the gravitational constant, η is the surface elevation of the wave above equilibrium level h and X is a coordinate chosen to be moving almost with the wave.

The KdV equation is a non-linear partial differential equation which admits two types of solutions. The first is a periodic solution which is most often referred to as *cnoidal wave*. The reason for this is simply due to the fact that it can be expressed in terms of elliptic functions. The second possible solution of the KdV equation is called a *solitary wave* or *soliton* ([34], pg. 253).

We follow Debnath ([18], pp. 161-162) to establish a solution of the KdV equation. To find the solution of the KdV equation (4), it is suitable to rewrite it as

$$\partial_t \eta + c\left(1 + \frac{2}{3h}\eta\right)\partial_x \eta + \frac{c}{h^2}\partial_{xxx}\eta = 0, \tag{1.5}$$

where $c = \sqrt{gh}$ and $H = h + \eta$ is the total depth of the water. Note that the above equation is written in dimensional variables. The first two terms $(\partial_t \eta + c\partial_x \eta)$ are in conformity with the linear non-dispersive limit. They describe the propagation of waves (to the right) at constant shallow water speed c. The third term is a result of finite amplitude effect and the forth term is due to the weak dispersion resulting from the shallow water depth.

If the solution of (1.5) is stated in the frame X, then

$$\eta = \eta(X), \qquad X = x - Ut. \tag{1.6}$$

Substituting (1.6) into (1.5), we get

$$(c - U)\eta' + \frac{3c}{2h}\eta\eta' + \frac{ch^2}{6}\eta''' = 0, \tag{1.7}$$

where $\eta' = \partial_X \eta$. Equation (1.7) integrates with respect to X to

$$(c - U)\eta + \frac{3c}{4h}\eta^2 + \frac{ch^2}{6}\eta'' = A, \tag{1.8}$$

where A is the constant of integration. Multiply equation (1.8) by $2\eta'$ and integrate further with respect to X to get

$$(c-U)\eta^2 + \frac{c}{2h}\eta^3 + \frac{ch^2}{6}\left(\frac{\partial \eta}{\partial X}\right)^2 = 2A\eta + B, \tag{1.9}$$

where B is constant. We focus attention on the case where η and all its derivatives tend to zero at infinity so that $A = B = 0$. Then equation (1.9) simplifies to

$$\left(\frac{\partial \eta}{\partial X}\right)^2 = \frac{3}{h^3}\eta^2(a-\eta), \tag{1.10}$$

where

$$a = 2h\left(\frac{U}{c} - 1\right). \tag{1.11}$$

Note that the right hand side of equation (1.10) disappears at $\eta = 0$ and $\eta = 0$ and the solitary wave solution of (1.10) is given by

$$\eta = asech^2\left(\frac{3a}{4h^3}\right)^{\frac{1}{2}} X. \tag{1.12}$$

Transforming the solution back into the original variables gives

$$\eta(x,t) = asech^2\left[\left(\frac{3a}{4h^3}\right)^{\frac{1}{2}}(x-Ut)\right], \tag{1.13}$$

where

$$U = c\left(1 - \frac{a}{2h}\right). \tag{1.14}$$

This is the solution of the KdV equation for all $\frac{a}{h}$. However, it is important to note that the KdV equation is derived with the approximate assumption that $\frac{a}{h} \ll 1$. The solution (1.14) is called *solitary wave* or *soliton*. Hence, the solitary wave relation derived by Russell is a solution of the KdV equation (see [19]).

1.2 Motivation and Organisation

The motivation for this study comes from the realm of water wave theory and two main results are obtained. In the first part of the study, the method of weak asymptotic is used to find singular solutions of the shallow water system in both one and two dimensions. The second part is concerned with local ill-posedness of the generalized Korteweg-de Vries (KdV) type equations in negative Sobolev spaces.

1.2.1 The Weak Asymptotic Concept

The concept is used to establish singular solutions of the shallow water system which can contain Dirac-δ distributions. The study of such singular solution was initiated by Korchinsky [32] who used generalized delta functions to construct a unique solution to the Riemann problem for hyperbolic system of conservation laws. The use of the generalized delta functions (which are quiet more complicated than the commonly used Dirac-δ functions) was to overcome the non-uniqueness problem which naturally arises when a delta function is multiplied with a discontinuous function. Keyfitz and Kranzer [31] also showed that the singular solution for the Riemann problem for strictly hyperbolic system of conservation laws can be found only for states which are sufficiently close together. Interest in the subject has grown in the past few years and many meaningful results were found [26, 36, 37, 45, 47].

Over the past few years, efforts have been made at developing methods to study such low-regularity solutions. One convenient tool that has caught the attention of researchers is the method of weak asymptotics [11, 13, 15, 16, 22, 28, 38, 41]. This method has been applied to many problems involving existence and uniqueness of solutions of hyperbolic conservation laws [28, 29], formation of δ-shock wave in the case of triangular system of conservation laws [12] as well as propagation and interaction of δ-shock waves [14].

A problem similar to this work was recently considered by Kalisch and Mitrović [28]. The authors applied the method of weak asymptotics to find singular solutions of the classical one-dimensional shallow water system

$$\partial_t u + \partial_x \left(v + \frac{u^2}{2} \right) = 0, \tag{1.15}$$

$$\partial_t v + \partial_x (u + uv) = 0. \tag{1.16}$$

The definition given by the authors explicitly allows the approximating distributions to be complex-valued. Although the imaginary parts of the solution so constructed disappear in an appropriate limit, they noted that considering complex-valued weak asymptotic solutions which become real-valued in the distributional limit significantly extend the range of possible singular solutions.

To further demonstrate the power of the extension to complex-valued distributions, Kalisch and Mitrović [29] applied it to study the Brio system [6, 24] of the form

$$\partial_t u + \partial_x \left(\frac{u^2 + v^2}{2} \right) = 0,$$

$$\partial_t v + \partial_x (v(u-1)) = 0.$$

For the two cases mentioned above the authors convincingly demonstrated the significance of the complex-valued corrections in proving the existence of δ-shock wave solutions for the two systems of hyperbolic conservation laws.

In this work, the method of weak asymptotics is applied to find singular solutions of (1.15), (1.16) and also of the two-dimensional shallow water system of the form

$$\partial_t u + \partial_x \left(\frac{u^2}{2} + \eta \right) + v \partial_y u = 0 \tag{1.17}$$

$$\partial_t v + u \partial_x v + \partial_y \left(\frac{v^2}{2} + \eta \right) = 0 \tag{1.18}$$

$$\partial_t \eta + \partial_x [u(1+\eta)] + \partial_y [v(1+\eta)] = 0 \tag{1.19}$$

The unknown functions $u(x, y, t)$ and $v(x, y, t)$ denote the horizontal fluid velocity components which are assumed to be uniform throughout the vertical cross-section of the fluid and the unknown $\eta(x, y, t)$ represents the elevation of the free surface from its state of rest. It is also assumed that the $x - y$ plane is the quiescent free surface of the fluid with the z-axis positive upward. The motivation for studying this system comes form the realm of water wave theory.

The shallow water system is a widely known mathematical approximation that is used to simulate variety of problems arising in areas such as free-surface flows in rivers, reservoirs and oceans. In particular, it can be used to study river bores and surges in rivers and channels [43, 46] tidal fluctuations and tsunami waves [42] and energy balance [1, 2, 21, 27].

A different kind of approximation to the theory of water waves is the linearised formulation. For the full linear water wave problem or the linear Korteweg-de Vries (KdV) equation, Whitham [46] was convinced that the solution given by the Dirac-δ initial data defines the linear propagator.

1.2.2 Local Ill-posedness

The second part of this work is concerned with ill-posedness of the generalized KdV type equations

$$\partial_t u + u^p \partial_x u + \partial_x^q u = 0, \quad p, q \in \mathbb{N}, \quad x, t \in \mathbb{R}, \tag{1.20}$$

where $q = p + 1$ for even $p \geq 4$.

Efforts have been invested in investigating the solutions of non-linear hyperbolic system of equations. In particular, the KdV equation and other non-linear systems of its kind have been studied and ill-posed and well-posed results were found.

Many studies have focused on establishing ill-posedness of the KdV equation and other non-linear equations of its type. Birnir *et al.* [4] proved local ill-posedness of the modified KdV equation. The idea was to show that the solution of the modified KdV equation, if it exists, cannot depend continuously on its initial data in the negative Sobolev class H^s for $s < -1/2$. They convincingly demonstrated this by constructing a sequence of solitary wave solutions (of the modified KdV equation) which converges (strongly) to the data in H^s, $s < -1/2$ and then showed that the corresponding sequence of solutions does not converge (strongly) in H^s, $s < -1/2$. Birnir *et al.* [3] also showed that the initial value problem of the generalized KdV equation is ill-posed in the Sobolev space $H^{s_k}, s_k = 1/2 - 2/k, k \geq 4$.

Well-posed results are obtained by demonstrating existence, uniqueness and continuous dependence of the solution upon its initial data. In addition to these conditions, the persistence property is also shown as well. This property is based on the notion that the solution describes a continuous curve in the domain Ω whenever the initial data $u_0 \in \Omega$.

The concept of well-posedness of non-linear systems has made significant progress over the past few years and different techniques have been proposed to study it. Kenig *et al.* [30] used the iteration method to establish local well-posedness of the KdV equation in the Sobolev space $H^s, s > 3/4$ and of the modified KdV equation in $H^s, s \geq 1/4$.

The well-posedness concept has also been applied to study KdV equation with higher dispersion. Gorsky and Himonas [23] have shown that if the dispersion of the KdV equation is replaced by a higher order term ∂_x^m where $m \geq 3$ is an odd integer, then the critical Sobolev exponent for local well-posedness on the circle does not change. Thus, the resulting equation is locally well-posed in $H^s, s \geq -1/2$. By using bilinear estimates they noted that the critical Sobolev index $s = -1/2$ does not depend on the order of the dispersion term.

It is shown in this work that (1.20) is locally ill-posed in the Sobolev space H^{s_ε}, $s_\varepsilon = -1/2 - \varepsilon$ for arbitrary $\varepsilon > 0$. The equation is posed as a Cauchy problem where the initial data is taken to be a scalar multiple of the Dirac-δ function.

1.2.3 Structure of the work

The first chapter introduced the general concept of solitary waves. The discussion narrowed down on the KdV equation and the relation between the solution of this equation and Russell's solitary wave is established.

In Chapter 2, the wave problem is formulated and the linearised version is also specified. Elementary solution of the wave problem is presented as well. The shallow water theory is discussed and the governing equations are derived. The chapter ended with the derivation of the KdV equation which was discussed in the previous chapter.

Chapter 3 is dedicated to the weak asymptotic method applied to the shallow water equation in one space dimension. Basic concepts and terms are defined and interesting examples of singular solutions of the shallow water system are found.

In Chapter 4, the concepts discussed in Chapter 3 are applied to the shallow water system in two dimensional space. This explicitly demonstrated the power of extending the weak asymptotic method to a higher space dimension.

Chapter 5 is concerned with the local ill-posedness of KdV type equations in a negative Sobolev space. It is demonstrated that if there exists a solution of the KdV type equations where the initial data is taken to be the Dirac-δ function in the Sobolev space $H^{s_\varepsilon}, s_\varepsilon = -1/2 - \varepsilon$, then it does not depend continuously on its data.

In Chapter 6, summary of the work and concluding remarks are given.

Chapter 2

Theory of Water Waves

The theory of water waves is a fascinating subject which embodies the general ideas about generation and propagation of water waves and other mathematically and physically related problems. Waves are generated due to the existence of restoring forces and are commonly classified into three different categories: interface gravity waves, internal gravity waves and compression and expansion waves. These waves occur in different mediums but our discussion shall be restricted in general to the case where water is the medium under consideration.

2.1 Formulation of the Wave Problem

In this section, the wave problem is formulated by considering the motion of waves in two dimensional space; the $x - z$ plane. We assume that the x-axis is in the horizontal direction and that the waves propagate in this direction only. The z-axis is assumed vertically upward from the undisturbed free surface. The fluid is assumed to be of uniform depth H, with the free surface at $z = 0$. The deflection of the free surface from its rest state is taken to be $z = \eta(x,t)$ so that the actual surface is located at $z = h = \eta + H$.

We start the formulation with the assumption that the fluid under consideration is inviscid, incompressible and homogeneous with constant density ρ. Consequently, the continuity equation ([34], pg. 86) is given by

$$\nabla \cdot \mathbf{u} = 0, \tag{2.1}$$

where $\mathbf{u} = (u, w)$ is the velocity vector. We further assume that the fluid has no vortical effect so that waves propagating at zero initial vorticity remain so for all times ([46], pg. 432). In addition, we restrict the discussion to irrotational flow so that the velocity potential φ, is defined by

$$u = \frac{\partial \varphi}{\partial x}, \qquad w = \frac{\partial \varphi}{\partial z}. \tag{2.2}$$

Substituting this equation into the continuity equation (2.1) gives the Laplace equation

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2}. \tag{2.3}$$

Equation (2.3) can be solved by specifying appropriate boundary conditions at the free surface and at the bottom. The bottom boundary condition is the zero normal velocity

$$w = \frac{\partial \varphi}{\partial z} = 0 \quad \text{at} \quad z = -H. \tag{2.4}$$

At the free surface, two boundary conditions are needed; a kinematic and a dynamic boundary conditions ([46], pg. 434). The reason for this is simply due to the fact that the free surface elevation, $\eta(x, t)$, and the velocity potential, φ, are unknown.

The kinematic boundary condition of the free surface states that fluid particles at the surface remain there at any given time so that

$$\frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \varphi}{\partial z} \quad \text{at} \quad z = \eta. \tag{2.5}$$

The dynamic condition states that the pressure just below the surface is equal to the atmospheric pressure. If we take the atmospheric pressure to be zero, the dynamic condition can be stated as

$$P = 0 \quad \text{at} \quad z = \eta. \tag{2.6}$$

The assumption that the fluid is irrotational allows us to calculate the pressure from the Bernoulli equation ([34], pg. 121)

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(u^2 + w^2) + \frac{p}{\rho} + gz = F(t), \tag{2.7}$$

where F(t) is independent of location and can be absorbed into $\frac{\partial \varphi}{\partial t}$ by redefining φ ([34], pg.221). Substituting this into equation (2.8) gives

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(u^2 + w^2) + g\eta = 0 \quad \text{at} \quad z = \eta. \tag{2.8}$$

The equations derived above are non-linear and finding solution requires more efforts. As noted above, the elevation of the free surface, $z = \eta(x, t)$, from its rest state is not known in advance. To add to this difficulty, the domain in which the velocity potential, $\varphi(x,z,t)$, is to be determined is also unknown.

Furthermore, if we make the assumption that $\varphi(x,z,t)$ is analytic and uniformly bounded throughout the fluid medium, then the solution would probably not exist for all times $t > 0$. This is simply due to the fact that any mathematical formulation of the problem which would fit observed wave phenomenon would necessarily require that we assume existence of singularities of unknown location in both space and time ([43], pg. 17).

These challenges have led to the simplification of the non-linear equation. The simplification process requires that we make special assumptions on the basis of general physical circumstances and mathematical interests.

2.2 The Linearised Formulation

We impose further assumption to simplify the wave equation formulated in the previous section. Assume that the amplitude of the surface wave is small so that the velocity potential, the free surface elevation and their derivatives are all small quantities. Consequently, the term $\partial_x\varphi\partial_x\eta$ will be significantly small and equation (2.5) simplifies to

$$\partial_t\eta = \partial_z\varphi \quad \text{at} \quad z = \eta. \tag{2.9}$$

Using Taylor series we can evaluate $\partial_z\varphi$ around $z = 0$ to get

$$\partial_z\varphi|_{z=\eta} = \partial_z\varphi|_{z=0} + \eta\partial_{zz}\varphi|_{z=0} + \eta^2\partial_{zz}\varphi|_{z=0} + \cdots .$$

If we approximate the above equation to first order term, we get

$$\partial_z\varphi|_{z=\eta} \approx \partial_z\varphi|_{z=0}$$

and equation (2.9) becomes

$$\partial_t\eta = \partial_z\varphi \quad \text{at} \quad z = 0. \tag{2.10}$$

Since we have assumed that the wave propagates with small amplitude, the non-linear term $u^2 + w^2$ in equation (2.7) can be omitted and the equation consequently simplifies to the linear form

$$\partial_z\varphi + \frac{p}{\rho} + gz = 0 \quad \text{at} \quad z = 0. \tag{2.11}$$

Note that the terms in the equation were evaluated at $z = 0$ instead of $z = \eta$ since the wave amplitude is assumed small. Although some error may occur in the above calculations, it is sufficiently small that the linearised formulation is considered a valid approximation.

2.3 Elementary Solution of the Wave Problem

We establish the solution of equation (2.3)

$$\partial_{xx} + \partial_{zz} = 0$$

subject to the boundary conditions

$$\begin{aligned} \partial_z\varphi &= 0 \quad \text{at} \quad z = -H, \\ \partial_z\varphi &= \partial_t\eta \quad \text{at} \quad z = 0, \\ \partial_t\varphi &= -g\eta \quad \text{at} \quad z = 0. \end{aligned}$$

This three conditions are equations (2.4),(2.10) and (2.12) respectively as established in the first section. For frequency ω and wavenumber k, assume the surface

$$\eta(x,t) = a\cos(kx - \omega t), \tag{2.12}$$

where a is the wave amplitude. The wavenumber can be expressed as $k = \frac{2\pi}{\lambda}$ where λ is the wavelength. The dependence of η on $(kx - \omega t)$ and conditions (2.10) and (2.12) require the velocity potential, φ, to be a sine function of $(kx - \omega t)$. Hence, we work with the assumption that the Laplace equation (2.3) has a separable solution

$$\varphi(z,x,t) = f(z)\sin(kx - \omega t), \tag{2.13}$$

where $f(z)$ and $\omega = \omega(k)$ are to be determined. Substitute this solution into equation (2.3) to get

$$\partial_{zz}f - k^2 f = 0 \tag{2.14}$$

which admits the general solution

$$f(z) = Ae^{kz} + Be^{-kz}, \tag{2.15}$$

where A and B are constants to be determined. Using this equation, the velocity potential can be written as

$$\varphi(z,x,t) = (Ae^{kz} + Be^{-kz})\sin(kx - \omega t). \tag{2.16}$$

Differentiate with respect to t and use equation (2.4) to get

$$B = Ae^{-2kH}. \tag{2.17}$$

Substitute (2.13) and (2.17) into equation (2.10) to obtain

$$k(A - B) = \omega a. \tag{2.18}$$

The constants A and B can now be determined by solving equations (2.18) and (2.19) to get

$$A = \frac{a\omega}{k(1-e^{-2kH})}, \qquad B = \frac{a\omega e^{-2kH}}{k(1-e^{-2kH})}. \tag{2.19}$$

Substitute these into equation (2.17) to yield

$$\varphi = \frac{a\omega}{k}\frac{\cosh(k(z+H))}{\sin(kH)}\sin(kx-\omega t) \tag{2.20}$$

from which the velocity components can be readily found ([34], pg.223).

As we have stated earlier, the surface elevation must also be calculated and this is done using the dynamic boundary condition (2.12). Substitution of (2.13) and (2.20) into (2.12) gives

$$\frac{-a\omega^2}{k}\frac{\cosh(kH)}{\sinh(kH)}\cosh(kx-\omega t) = -ag\cosh(kx-\omega t).$$

This equation can be simplified further to obtain

$$\omega = \sqrt{gk\tan(kH)}. \tag{2.21}$$

This equation expresses the relation between the frequency and the wave number and is commonly referred to as the *dispersion relation*. It expresses the nature of the dispersive process ([34], pg. 223). The phase speed of the surface wave can be calculated by using (2.21) to get

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tan(kH)} = \sqrt{\frac{g\lambda}{2\pi}\tan\left(\frac{2\pi H}{\lambda}\right)}. \tag{2.22}$$

Equation (2.22) shows that the wave speed is dependent on the wavelength $\lambda = \frac{2\pi}{k}$. When the water depth is greater compared with the wavelength, thus $kH \gg 1$, then $\tan(kH) \approx 1$ and the dispersion relation simplifies to $\omega = \sqrt{gk}$. Waves with this dispersive nature are called *deep water waves*. However, when the ratio of the water depth to the wavelength is small, thus $\frac{H}{\lambda} \ll 1$, the waves are called *shallow water waves*. In this case, $\tan(kH) \approx kH$ and the dispersion relation simplifies to $\omega = gk^2H$ ([34], pp. 230-231).

2.4 Shallow Water Theory

In Section 1.2 we worked with the assumption that the amplitude of the surface wave is small leading to the linearised approximation of the wave problem formulated in Section 1.1. A different kind of approximation is the result of the hypothesis that the depth of the water is significantly small compared with the wavelength. This assumption leads to a non-linear equation for initial valued problems which in its lowest order becomes useful in studying wave propagation in compressible gases [43].

We give the derivation of the theory in two dimensional space where the y-axis is taken vertically upward and h represents the depth of the quiescent water. Assume further that the quiescent free surface of the water is located at the x-axis and the bottom at $y = -h$. Let $u(x, y, t)$ and $v(x, y, t)$ denote the velocity components. Then the continuity equation takes the form

$$\partial_x u + \partial_y v = 0. \tag{2.23}$$

By following Stoker ([43], pp. 23-25), the bottom boundary condition is given by

$$u\partial_x h + v = 0 \quad \text{at} \quad y = -h. \tag{2.24}$$

In addition to the bottom boundary condition, free surface conditions must be specified. These are the kinematic boundary condition

$$\partial_t \eta + u\partial_x \eta - v = 0 \quad \text{at} \quad y = \eta, \tag{2.25}$$

and the dynamic boundary condition

$$P = 0 \quad \text{at} \quad y = \eta. \tag{2.26}$$

Integrate the continuity equation (2.23) with respect to y to get

$$\int_{-h}^{\eta} (\partial_x u) dy + v|_{-h}^{\eta} = 0. \tag{2.27}$$

From the bottom boundary condition (2.24) and the kinematic condition (2.25), we have

$$v|_{-h}^{\eta} = \partial_t \eta|_\eta + u\partial_x \eta|_\eta + u\partial_x h|_{-h}.$$

Substitution of this equation into (2.27) gives

$$\int_{-h}^{\eta} (\partial_x u) dy + \partial_t \eta|_\eta + u\partial_x \eta|_\eta + u\partial_x h|_{-h} = 0. \tag{2.28}$$

We introduce the relation

$$\frac{\partial}{\partial_x} \int_{-h(x)}^{\eta(x)} u dy = u\partial_x \eta|_{y=\eta} + u\partial_x h|_{y=-h} + \int_{-h}^{\eta} (\partial_x u) dy.$$

Using this relation, equation (2.28) simplifies to

$$\frac{\partial}{\partial_x}\int_{-h}^{\eta} u dy = -\partial_t \eta. \tag{2.29}$$

For water density ρ and acceleration due to gravity g, the hydrostatic pressure is given by

$$P = g\rho(\eta - y). \tag{2.30}$$

This equation agrees with the dynamic boundary condition at the surface. Differentiate (2.30) with respect to x to obtain

$$\partial_x P = g\rho\partial_x \eta. \tag{2.31}$$

This simply means that $\partial_x P$ is independent of the y-axis and that the acceleration component of the x-axis is also independent of y. Consequently, the velocity component of the x-axis is independent of y for all times t, thus $u = u(x, t)$. Using (2.31), the equation of motion in the x-direction takes the Eulerian form ([43], pg. 24)

$$\partial_t u + u\partial_x u = -g\partial_x \eta. \tag{2.32}$$

Note that $\partial_y u = 0$ since u is independent of y. In light of this observation, equation (2.29) can be written as

$$\partial_x(u(\eta + h)) = -\partial_t \eta. \tag{2.33}$$

Equations (2.32) and (2.33) form the non-linear shallow water equation for the functions $u(x, t)$ and $\eta(x, t)$.

The two hypotheses made in Sections 1.2 and 1.4 can be combined to derive the classical linear wave equation. Assume that both hypotheses are true and that $u(x, t)$ and $\eta(x, t)$ and all their derivatives are small. Then $u\partial_x u$ and $\partial_x(u\eta)$ will be small compared with the linear terms and equations (2.32) and (2.33) simply to

$$\partial_t u = -g\partial_x \eta \tag{2.34}$$

and

$$\partial_x(uh) = -\partial_t \eta \tag{2.35}$$

respectively. Differentiate (2.34) with respect to t and (2.35) with respect to x and eliminate η to get

$$\partial_{xx} u - \frac{1}{gh}\partial_{tt} u = 0, \tag{2.36}$$

where the water depth, h, is assumed constant.

2.5 Derivation of the Shallow Water Equations

Attention will be focused on wave motion in shallow water and the governing equations will be derived in both one and two dimensional spaces.

2.5.1 The One Dimensional Case

Consider a channel along the x-axis and assume that the horizontal velocity $v(x,t)$ is uniform throughout the fluid column. Suppose that the vertical velocity of the fluid is negligible. Assume further that the fluid is inviscid with constant density ρ and let $h = h(x,t)$ denote the non-uniform depth of the fluid. Then the fluid flow is governed by conservation of mass and conservation of momentum [25].

The change of mass between any two points x_1 and x_2 along the channel at time t is given by ([25], pg. 167)

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2}\int_0^{h(x,t)} \rho dy dx = -\int_0^{h(x_2,t)} \rho v(x_2,t)dy + \int_0^{h(x_1,t)} \rho v(x_1,t)dy. \quad (2.37)$$

We introduce the relation

$$\int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\rho v(x,t)h(x,t)\right)dx = \int_0^{h(x_2,t)} \rho v(x_2,t)dy - \int_0^{h(x_1,t)} \rho v(x_1,t)dy. \quad (2.38)$$

Using this relation, equation (2.37) can be written as

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2}\int_0^{h(x,t)} \rho dy dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\rho v(x,t)h(x,t)\right)dx, \quad (2.39)$$

and this simplifies further to

$$\int_{x_1}^{x_2}\left[\frac{\partial}{\partial t}\left(\rho h(x,t)\right) + \frac{\partial}{\partial x}\left(\rho v(x,t)h(x,t)\right)\right]dx = 0. \quad (2.40)$$

Since this equation holds for any two arbitrary points $x_1 < x_2$ at any time t, we have

$$\partial_t h + \partial_x(vh) = 0. \quad (2.41)$$

The conservation of momentum equation is derived by the hypothesis that the fluid is in hydrostatic balance. Let the pressure be denoted by $P = P(x,y,t)$ and consider a small region $[x_1,x_2] \times [y, y+\Delta y]$ of the channel. Then the hydrostatic law requires that ([25], pg.167)

$$(P(\tilde{x}, y+\Delta y, t) - P(\tilde{x}, y, t)(x_2 - x_1) = -(x_2 - x_1)\rho g \Delta y,$$

where g is the acceleration due to gravity and $\tilde{x} \in [x_1, x_2]$. Divide by $(x_1 - x_2)\Delta y$ and let $x_1, x_2 \longrightarrow x$, $\Delta y \longrightarrow 0$ to get

$$\partial_y P(x, y, t) = -\rho g. \tag{2.42}$$

Assume pressure is zero at the surface of the fluid and integrate (41) to obtain

$$P(x, y, t) = -\rho g(h(x, y) - y). \tag{2.43}$$

The change of momentum in $[x_1, x_2]$ at time t is given by ([25], pg. 168)

$$\begin{aligned}\frac{\partial}{\partial t}\int_{x_1}^{x_2}\int_0^{h(x,t)} \rho v(x,t)dydx &= -\int_0^{h(x_2,t)} P(x_2, y, t)dy \; + \\ \int_0^{h(x_1,t)} P(x_1, y, t)dy &- \int_0^{h(x_2,t)} \rho v^2(x_2, t)dy + \int_0^{h(x_1,t)} \rho v^2(x_1, t)dy.\end{aligned}$$

Using (2.38) and (2.43), the above equation can be rewritten as

$$\begin{aligned}\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho v h dx = -\rho g(h^2(x_2, t) - \frac{1}{2}h^2(x_2, t) + \rho g(h^2(x_1, t) - \frac{1}{2}h^2(x_1, t) - \\ \int_{x_1}^{x_2} \frac{\partial}{\partial x}(\rho h v^2)dx\end{aligned} \tag{2.44}$$

Using the relations

$$\int_{x_1}^{x_2} \frac{\partial}{\partial x} h^2(x, t)dx = h^2(x-2, t) - h^2(x-1, t)$$

and

$$\int_{x_1}^{x_2} \frac{\partial}{\partial x}\frac{1}{2}h^2(x, t)dx = \frac{1}{2}h^2(x-2, t) - \frac{1}{2}h^2(x-1, t),$$

equation (2.44) can be simplified further to yield

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho v h dx = -\rho g \int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\frac{1}{2}h^2\right)dx - \int_{x_1}^{x_2} \frac{\partial}{\partial x}(\rho h v^2)dx. \tag{2.45}$$

This equation can be rewritten as

$$\int_{x_1}^{x_2} \left[\frac{\partial}{\partial t}(vh) + \frac{\partial}{\partial x}\left(v^2 h + \frac{1}{2}h^2\right)\right] dx = 0, \tag{2.46}$$

where $g = 1$. Since (2.46) holds for arbitrary interval $[x_1, x_2]$ and for any time t, the integrand must be identically zero. Consequently, (2.46) becomes

$$\partial_t(vh) + \partial_x\left(v^2 h + \frac{1}{2}h^2\right) = 0. \tag{2.47}$$

The system of conservation laws (2.41) and (2.47) constitute the shallow water equations in one space dimension.

2.5.2 The Two Dimensional Case

In this section we derive the non-linear shallow water equations (1.17), (1.18) and (1.19) by following Debnath ([18], pp. 141-145). We first assume an inviscid liquid with constant mean depth h and constant density ρ with no surface tension. The deflection of the free-surface from its rest position is taken to be $z = \eta(x, y, t)$ so that the actual surface is located at $z = H = h+\eta$ and the flat bottom at $z = 0$.

We introduce the following non-dimensional flow variables

$$(\tilde{x}, \tilde{y}) = \frac{1}{l}(x, y), \quad \tilde{z} = \frac{z}{h}, \quad \tilde{t} = \frac{ct}{l}, \quad \tilde{\eta} = \frac{\eta}{a}, \quad \tilde{\varphi} = \frac{h\varphi}{alc} \tag{2.48}$$

where l denotes horizontal length and h represents the vertical height. As usual we assume that a is the surface wave amplitude, k is the wave number and $c = \sqrt{gh}$ is the typical horizontal fluid velocity. In addition to the above variables, we introduce other fundamental parameters to characterize the non-linear shallow water waves

$$\varepsilon = \frac{a}{h} \quad \text{and} \quad \delta = \frac{h^2}{l^2}. \tag{2.49}$$

Furthermore, we will make use of the following basic equations for water waves

$$\nabla^2 = 0, \quad -h < z < \eta, \quad -\infty < x, y < \infty, \tag{2.50}$$

$$\varphi_t + \frac{1}{2}(\nabla\varphi)^2 + g\eta = 0 \quad \text{on} \quad z = \eta, \tag{2.51}$$

$$\eta_t + \eta_x\varphi_x + \eta_y\varphi_y - \varphi_z = 0 \quad \text{on} \quad z = \eta, \tag{2.52}$$

$$\varphi_x h_x + \varphi_y h_y + \varphi_z = 0 \quad \text{on} \quad z = -h. \tag{2.53}$$

The system (2.50)-(2.53) is a valid model for classical water waves. The unknown function $\varphi(x, y, t)$ represents the velocity potential of the fluid and h denotes the depth. It is assumed that the free surface of the fluid is located at $z = \eta(x, y, t)$ and the rigid bottom at $z = -h(x, y)$. For more on this system see Debnath ([18], pp. 10-12).

Using the non-dimensional variables and the fundamental parameters specified in equations (2.48) and (2.49), the basic equations for water waves (2.50)-(2.53) can be rewritten in the non-dimensional form (dropping the tilde)

$$\delta(\partial_{xx}\varphi + \partial_{yy}\varphi) + \partial_{zz}\varphi = 0 \tag{2.54}$$

$$\partial_t\varphi + \frac{\varepsilon}{2}(\partial_x\varphi^2 + \partial_y\varphi^2) + \frac{\varepsilon}{2\delta}\partial_z\varphi^2 + \eta = 0 \quad \text{on} \quad z = 1 + \varepsilon\eta \tag{2.55}$$

$$\delta[\partial_t\eta + \varepsilon(\partial_x\varphi\partial_x\eta + \partial_y\varphi\partial_y\eta)] - \partial_z\varphi = 0 \quad \text{on} \quad z = 1 + \varepsilon\eta \tag{2.56}$$

$$\partial_z\varphi = 0 \quad \text{on} \quad z = 0. \tag{2.57}$$

Note that the parameter $\kappa = ak$ was not used explicitly in the above equations instead, an equivalent parameter $\gamma = \frac{a}{l}$ is associated with ε and δ by the relation

$$\gamma = \left(\frac{a}{h}\right)\left(\frac{h}{l}\right) = \varepsilon\sqrt{\delta}.$$

If ε is sufficiently small then the linearised free surface conditions can be obtained by dropping the ε term in equations (2.55) and (2.56). However, if we assume that δ is negligible, then we will get an interpretation of the characteristics feature of the shallow water theory. Th second assumption will imply that we expand the variable φ in terms of δ alone without making any assumption about ε. The expansion can be written as

$$\varphi = \varphi_0 + \delta\varphi_1 + \delta^2\varphi_2 + \delta^3\varphi_3 + \cdots \tag{2.58}$$

Using equation (2.58) and the assumption that δ is significantly small, we can rewrite equation (2.54) as

$$\partial_{zz}\varphi_0 \equiv 0 \tag{2.59}$$

and equation (2.57) as

$$\partial_z\varphi_0 \equiv 0 \quad \text{for all} \quad z \;\; \text{or} \;\; \varphi_0 = \varphi_(x, y, t). \tag{2.60}$$

Note that only the lowest order terms were retained in equations (2.59) and (2.60). These equations lead to the realization that the horizontal velocity components are independent of the vertical coordinate z in lowest order. Hence, it is more appropriate to introduce the notations

$$\partial_x\varphi_0 = u(x, y, t), \tag{2.61}$$

$$\partial_y\varphi_0 = v(x, y, t). \tag{2.62}$$

substituting equation (2.61) into (2.54) and then using equation (2.58), the first and second order terms in equation (2.54) can be expressed respectively as

$$\delta\partial_{xx}(\varphi_0 + \delta\varphi_1 + \cdots) + \delta\partial_{yy}(\varphi_0 + \delta\varphi_1 + \cdots) + \partial_{zz}(\varphi_0 + \delta\varphi_1 + \delta^2\varphi_2 + \cdots) = 0$$

$$\Rightarrow \qquad \partial_{xx}\varphi_0 + \partial_{yy}\varphi_0 + \partial_{zz}\varphi_1 = 0, \tag{2.63}$$

and

$$\delta\partial_{xx}(\varphi_0 + \delta\varphi_1 + \cdots) + \delta\partial_{yy}(\varphi_0 + \delta\varphi_1 + \cdots) + \partial_{zz}(\varphi_0 + \delta\varphi_1 + \delta^2\varphi_2 + \cdots) = 0$$

$$\Rightarrow \qquad \partial_{xx}\varphi_1 + \partial_{yy}\varphi_1 + \partial_{zz}\varphi_2 = 0. \tag{2.64}$$

Integrate equation (2.63) with respect to z and use equations (2.61) and (2.62) to get

$$\begin{aligned} & z\partial_{xx}\varphi_0 + z\partial_{yy}\varphi_0 + \partial_z\varphi_1 = C(x,y,t) \\ \Rightarrow \quad & z\partial_x u + z\partial_y v + \partial_z\varphi_1 = C(x,y,t) \end{aligned} \tag{2.65}$$

where $C(x,y,t)$ is an arbitrary constant function. By the bottom boundary condition specified in equation (2.57), it is easy to see that $C(x,y,t) = 0$. Hence, equation (2.65) becomes

$$\partial_z\varphi_1 = -z(\partial_x u + \partial_y v). \tag{2.66}$$

Again, integrate equation (2.66) with respect to z to get

$$\varphi_1 = -\frac{z^2}{2}(\partial_x u + \partial_y v). \tag{2.67}$$

Note that the arbitrary constant function has been dropped by the analogy that led to equation (2.66). At the bottom, $z = 0$ and $\varphi_1 = 0$ in equation (2.67) hence, u and v are the horizontal velocity components at the bottom boundary.

Substitute equation (2.67) into equation (2.64) and integrate, with respect to z, the resulting expression and use equation (2.57) to obtain

$$\partial_{xx}\left[-\frac{z^2}{2}(\partial_x u + \partial_y v)\right] + \partial_{yy}\left[-\frac{z^2}{2}(\partial_x u + \partial_y v)\right] + \partial_{zz}\varphi_2 = 0$$

$$\begin{aligned} \Rightarrow \quad \partial_z\varphi_2 &= \frac{z^3}{6}\left[(\partial_{xx} + \partial_{yy})\partial_x u + (\partial_{xx} + \partial_{yy})\partial_y v\right] \\ &= \frac{z^3}{6}\left[\nabla^2(\partial_x u) + \nabla^2(\partial_y v)\right] \end{aligned} \tag{2.68}$$

where ∇^2 is the two dimensional Laplace operator. We further integrate equation (2.68) with respect to z to get

$$\varphi_2 = \frac{z^4}{24}\left[\nabla^2(\partial_x u) + \nabla^2(\partial_y v)\right]. \tag{2.69}$$

We now turn attention to the free surface boundary conditions specified in equations (2.55) and (2.56). Firstly, substitute equation (2.58) into (2.55) and use equations (2.60)-(2.62), (2.66), (2.67) and retain terms up to order δ and ε to get

$$\partial_t(\varphi_0+\delta\varphi_1)+\frac{\varepsilon}{2}\left[\partial_x(\varphi_0 + \delta\varphi_1)^2 + \partial_y(\varphi_0 + \delta\varphi_1)^2\right]+\frac{\varepsilon}{2\delta}\left[\partial_z(\varphi_0 + \delta\varphi_1)^2\right]+\eta = 0.$$

Simplify the above equation to obtain

$$\partial_t\varphi_0 - \frac{\delta}{2}\left(\partial_{tx}u + \partial_{ty}v\right) + \frac{\varepsilon}{2}(u^2+v^2) + \eta = 0 \tag{2.70}$$

Similarly, substitute equation (2.58) into(2.56) and manipulate using equations (2.60)-(2.62) and (2.66)-(2.68). In this case, retain terms up to order δ^2, ε^2 and $\delta\varepsilon$ to get

$$\delta[\partial_t\eta + \varepsilon\partial_x(\varphi_0+\delta\varphi_1)\partial_x\eta + \partial_y(\varphi_0+\delta\varphi_1)\partial_y\eta] - \partial_z(\varphi_0+\delta\varphi_1+\delta^2\varphi_2) = 0,$$

$$\delta[\partial_t\eta + \varepsilon(u\partial_x\eta + v\partial_y\eta)] + z\delta(\partial_x u + \partial_y v) - \frac{z^3\delta^2}{6}\left[\nabla^2(\partial_x u) + \nabla^2(\partial_y v)\right] = 0,$$

which simplifies to

$$\partial_t\eta + \partial_x[u(1+\varepsilon\eta)] + \partial_y[v(1+\varepsilon\eta)] = \frac{\delta}{6}\left[\nabla^2(\partial_x u) + \nabla^2(\partial_y v)\right] \tag{2.71}$$

Now differentiate equation (2.70) firstly, with respect to x to get

$$\begin{aligned} &\partial_x\partial_t\varphi_0 - \frac{\delta}{2}\partial_x\left(\partial_{tx}u + \partial_{ty}v\right) + \frac{\varepsilon}{2}\partial_x\left(u^2+v^2\right) + \partial_x\eta = 0 \\ \Rightarrow\quad &\partial_t u + \varepsilon(u\partial_x u + v\partial_x v) + \partial_x\eta - \frac{\varepsilon}{2}\left(\partial_{txx}u + \partial_{txy}v\right) = 0 \end{aligned} \tag{2.72}$$

and secondly, with respect to y to obtain

$$\begin{aligned} &\partial_y\partial_t\varphi_0 - \frac{\delta}{2}\partial_y\left(\partial_{tx}u + \partial_{ty}v\right) + \frac{\varepsilon}{2}\partial_y\left(u^2+v^2\right) + \partial_y\eta = 0 \\ \Rightarrow\quad &\partial_t v + \varepsilon(u\partial_y u + v\partial_y v) + \partial_y\eta - \frac{\varepsilon}{2}\left(\partial_{txy}u + \partial_{tyy}v\right) = 0 \end{aligned} \tag{2.73}$$

Taking $\varepsilon = 1$ and using the fact that φ_0 is irrotational so that $\partial_x u = \partial_y v$ and then deleting the δ terms, equations (2.71)-(2.73) can be rewritten respectively as

$$\partial_t u + \partial_x\left(\frac{u^2}{2} + \eta\right) + v\partial_y u = 0 \tag{2.74}$$

$$\partial_t v + u\partial_x v + \partial_y\left(\frac{v^2}{2} + \eta\right) = 0 \tag{2.75}$$

$$\partial_t\eta + \partial_x[u(1+\eta)] + \partial_y[v(1+\eta)] = 0 \tag{2.76}$$

Equations (2.74)-(2.76) form the fundamental shallow water system in two-dimensional space. This system of non-linear equations admits solutions for u, v and η.

2.6 Derivation of the KdV Equation

We expand our discussion about the shallow water theory to the KdV equation which was originally derived to modelled shallow water waves which have long wavelength and small amplitude. The motivation for including the KdV equation is the fact that it incorporates dispersive effect into the shallow water theory. The derivation is based on Whitham ([46], pp. 463-466).

In addition to the hypotheses made in Sections 2.2 and 2.4, let the distance measured from the horizontal bottom be denoted by $Z = z + H$. Then the velocity potential, φ, satisfies the Laplace equation

$$\varphi_{xx} + \varphi_{ZZ} = 0. \tag{2.77}$$

Assume that the solution of equation (2.77) can be expressed by an expansion in Z by

$$\varphi = \sum_{n=0}^{\infty} Z^n f_n(x,t). \tag{2.78}$$

Substitute equation (2.78) into the Laplace equation (2.77) and use the boundary condition in (2.4) to obtain

$$\varphi = \sum_{m=0}^{\infty} (-1)^m \frac{Z^{2m}}{(2m)!} \frac{\partial^{2m} f}{\partial_x^{2m}}, \tag{2.79}$$

where $f = f_0$. The change in the Laplace equation from (2.3) to (2.77) requires a corresponding change in the boundary conditions (2.4), (2.5) and (2.6). We introduce the following normalisations for the purpose of convenience

$$x = \lambda x', \quad Z = HZ', \quad t = \frac{\lambda t'}{c}, \quad \eta = a\eta', \quad \varphi = \frac{g\lambda a \varphi'}{c}. \tag{2.80}$$

The original variables are non-primed and the normalised variables are primed. The amplitude is denoted by a and c represents the phase speed. In the normalised variables the Laplace equation (2.77) takes the form (dropping the prime)

$$\beta\varphi_{xx} + \varphi_{ZZ} = 0 \quad \text{for} \quad 0 < Z < 1 + \alpha\eta, \tag{2.81}$$

where $\alpha = \frac{a}{H}$ and $\beta = \frac{H^2}{\lambda^2}$. The boundary conditions (2.4)-(2.6) in normalised variables respectively take the form

$$\varphi_Z = 0 \quad \text{on} \quad Z = 0, \tag{2.82}$$

$$\eta_t + \alpha\varphi_x - \frac{1}{\beta}\varphi_Z = 0 \quad \text{on} \quad Z = 1 + \alpha\eta, \tag{2.83}$$

$$\eta + \varphi_t + \frac{1}{2}\alpha\varphi_x^2 + \frac{1}{2}\frac{\alpha}{\beta}\varphi_Z^2 = 0 \quad \text{on} \quad Z = 1 + \alpha\eta. \tag{2.84}$$

From the Laplace equation (2.77) and the bottom boundary condition (2.82), the expression (2.79) becomes

$$\varphi = \sum_{m=0}^{\infty} (-1)^m \frac{Z^{2m}}{(2m)!} \frac{\partial^{2m} f}{\partial_x^{2m}} \beta^m. \tag{2.85}$$

Substituting this equation into the free surface conditions (2.83) and (2.84) respectively lead to

$$\eta_t + (1+\alpha\eta) f_{xx} - \frac{1}{6}(1+\alpha\eta)^3 f_{xxxx} + \frac{1}{2}\alpha(1+\alpha\eta)^2 \eta_x f_{xxx} \beta + O(\beta^2) = 0 \tag{2.86}$$

and

$$\eta + f_t + \frac{1}{2}\alpha f_x^2 - \frac{1}{2}(1+\alpha\eta)^2 f_{xxt} + \alpha f_x f_{xxx} - \alpha f_{xx}^2 \beta + O(\beta^2) = 0. \tag{2.87}$$

Simplify these equations by dropping terms containing β and then differentiate equation (2.87) with respect to x to get the non-linear shallow water equations

$$\eta_t + ((1+\alpha\eta)w)_x = 0,$$
$$w_t + \alpha w w_x + \eta_x = 0,$$

where $w = f_x$. However, if terms of order $\alpha\beta$ are dropped in (2.86) and (2.87) and equation (2.87) is differentiated with respect to x, a variant of Boussinesq's equation are obtained in the form

$$\eta_t + ((1+\alpha\eta)v)_x - \frac{1}{6}\beta v_{xxx} + O(\alpha\beta, \beta^2) = 0, \tag{2.88}$$
$$v_t + \alpha v v_x + \eta_x - \frac{1}{2}\beta v_{xxt} + O(\alpha\beta, \beta^2) = 0, \tag{2.89}$$

where $v = f_x$ and the horizontal velocity correspondingly takes the form

$$\varphi_x = v - \beta \frac{Z^2}{2} v_{xx} + O(\beta^2).$$

Integrating this equation over the depth gives the averaged depth value

$$\tilde{u} = v - \frac{1}{6}\beta v_{xx} + O(\alpha\beta, \beta^2) = 0$$

and the inverse

$$v = \tilde{u} + \frac{1}{6}\beta \tilde{u}_{xx} + O(\alpha\beta, \beta^2) = 0.$$

The KdV equation is derived from equations (2.88) and (2.89) by focusing on waves moving to the right. Neglecting terms of order α and β, (2.88) and (2.89) simplify respectively to

$$\eta_t + v_x = 0 \tag{2.90}$$

and

$$v_t + \eta_x = 0. \tag{2.91}$$

We get a plain linear transport equation $\eta_t + \eta_x = 0$ if $\eta = v$. However, we seek a solution containing first order terms in α and β in the form

$$v = \eta + \alpha A + \beta B + O(\alpha^2 + \beta^2), \tag{2.92}$$

where A and B are functions of η and its derivatives. Substituting equation (2.92) into (2.88) and (2.89) yield

$$\eta_t + \eta_x + \alpha(A_x + 2\eta\eta_x) + \beta(B_t - \frac{1}{2}\eta xxx) + O(\alpha\beta, \beta^2) = 0 \tag{2.93}$$

and

$$\eta_t + \eta_x + \alpha(A_t + \eta\eta_x) + \beta(B_t - \frac{1}{2}\eta xxt) + O(\alpha\beta, \beta^2) = 0 \tag{2.94}$$

respectively. These equations are coherent if

$$A = -\frac{1}{4}\eta^2, \qquad B = \frac{1}{3}\eta_{xx}.$$

Substituting these into (2.93) and (2.94) yield the normalised KdV equation

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx} + O(\alpha\beta, \beta^2) = 0. \tag{2.95}$$

The expression for v takes the form

$$v = \eta + \frac{1}{4}\alpha\eta^2 + \frac{1}{3}\beta\eta_{xx} + O(\alpha\beta, \beta^2) = 0.$$

Changing (2.95) back to the original variables using (2.80) gives the KdV equation

$$\eta_t + c\eta_x + \frac{3}{2}\frac{c}{H}\eta\eta_x + \frac{1}{6}cH^2\eta_{xxx} + O(\alpha\beta, \beta^2) = 0. \tag{2.96}$$

Chapter 3

The Weak Asymptotic Method - 1D Case

The method of weak asymptotic is used to find singular solution of the classical shallow water system of equations (1.15), (1.16). The singular solution to be constructed is allowed to contain Dirac-δ distributions. The idea is to construct complex-valued approximate solution which becomes real-valued in the distributional limit. This extends the range of possible singular solutions of (1.15),(1.16). Let $\mathcal{D}$ represent the space of text functions and $\mathcal{D}'$ be the space of distributions.

3.1 Definition of Terms

Definition 3.1 Let $f_\varepsilon(x) \in \mathcal{D}'(\mathbb{R})$ be a family of distributions which depend on $\varepsilon \in (0, 1)$. If the estimate $\langle f_\varepsilon, \varphi\rangle \ = o(1)$ as $\varepsilon \longrightarrow 0$ holds for any test function $\varphi(x) \in \mathcal{D}(\mathbb{R})$, then the function $f_\varepsilon = o_{\mathcal{D}}(1)$.

Definition 3.2 Let (u_ε) and (v_ε) and be a family of smooth complex-valued distributions and u, v and be smooth real-valued distributions. We say that u_ε and v_ε represent a weak asymptotic solution to (1.15), (1.16) if for every fixed $t \in \mathbb{R}^+$

$$u_\epsilon \rightharpoonup u, \quad v_\epsilon \rightharpoonup v, \qquad as \quad \varepsilon \to 0$$

in the sense of distributions in $\mathcal{D}'(\mathbb{R})$ and

$$\partial_t u_\varepsilon + \partial_x \left(v_\varepsilon + \frac{u_\varepsilon^2}{2} \right) = o_{\mathcal{D}'}(1), \tag{3.1}$$

$$\partial_t v_\varepsilon + \partial_x (u_\varepsilon + u_\varepsilon v_\varepsilon) = o_{\mathcal{D}'}(1). \tag{3.2}$$

In addition, initial data are given by

$$u_\varepsilon(x,0) \rightharpoonup u(x,0) \quad \text{and} \quad v_\varepsilon(x,0) \rightharpoonup v(x,0). \tag{3.3}$$

The weak convergence used above is in the sense of distributions as $\varepsilon \longrightarrow 0$. Before applying the definition we will make an intelligent guess for a possible weak solution by first considering a smooth function $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the following conditions:

(i) $\rho(x) \geq 0$

(ii) $\int_{\mathbb{R}} \rho(x)\ dx = 1$

(iii) $\operatorname{supp}(\rho) \subset (0,1)$

Denote

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon}\rho\left(\frac{x+3\varepsilon}{\varepsilon}\right), \qquad R_\varepsilon(x) = \frac{1}{\sqrt{\varepsilon}}\rho\left(\frac{x-3\varepsilon}{\varepsilon}\right) \tag{3.4}$$

It is easy to verify that

$$R_\varepsilon(x)\delta_\varepsilon(x) = 0 \ \text{ for every } \ x \in \mathbb{R}. \tag{3.5}$$

In addition, we define $\rho_0 = \int_{\mathbb{R}} \rho^2(x)\ dx$ and obtain the following distributional limits

$$R_\varepsilon(x) \rightharpoonup 0, \quad R_\varepsilon^2(x) \rightharpoonup \rho_0\delta(x) \ \text{ and } \ \delta_\varepsilon(x) \rightharpoonup \delta(x) \tag{3.6}$$

3.2 Non-uniqueness

A special example of a weak asymptotic solution for the shallow water equations is a stationary delta distribution in v centred at the origin. The example specifies the important role of complex-valued extension of the weak asymptotic method.

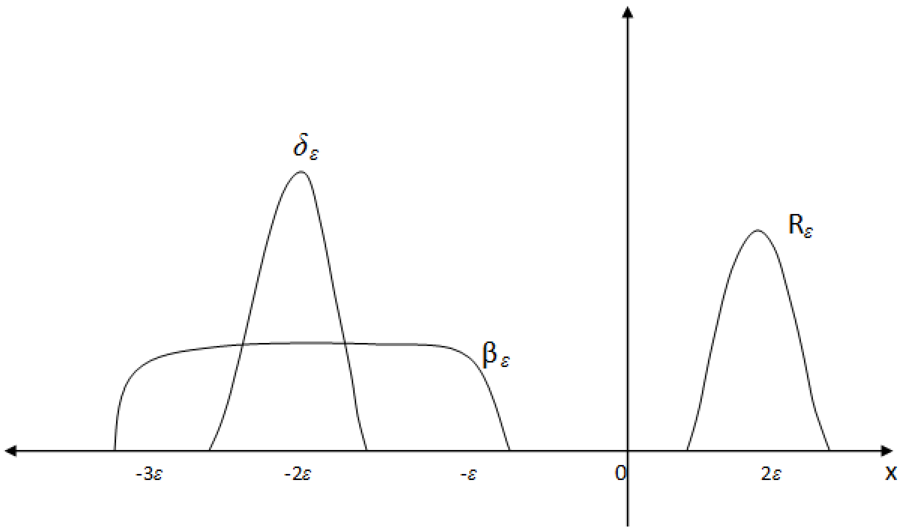

Figure 3.1: Combination of hyperbolic shock wave and Dirac delta distribution

Proposition 3.1. *Define the constant*

$$m = \begin{cases} \frac{1}{2}\rho_0, & if \;\; a{=}0,\; b{=}1 \\ -\frac{1}{2}\rho_0, & if \;\; a{=}1,\; b{=}0 \end{cases}$$

The pair of families of smooth functions defined by

$$u_\varepsilon(x,t) = (a+ib)R_\varepsilon(x-ct) + \beta_\varepsilon(x), \tag{3.7}$$

$$v_\varepsilon(x,t) = m\delta_\varepsilon(x-ct), \tag{3.8}$$

where c is the shock speed and

$$\beta_\varepsilon(x) = \begin{cases} 0, & if \quad x < -3\varepsilon \\ c, & if \quad -3\varepsilon < x < -\varepsilon \\ 0, & if \quad x > -\varepsilon \end{cases}$$

represents a weak asymptotic solution of the shallow water system (1.15), (1.16).

Proof. It is not difficult to check that $\partial_t u_\varepsilon \longrightarrow 0$ from equation (3.7). Using equations (3.6), (3.7) and (3.8), we have

$$\langle \partial_x v_\varepsilon, \varphi\rangle = m\langle \partial_x \delta_\varepsilon, \varphi\rangle \longrightarrow -m\langle \delta, \varphi\prime\rangle = -m\varphi\prime(0).$$

and

$$\begin{aligned}
\left\langle \partial_x \frac{u_\varepsilon^2}{2}, \varphi \right\rangle &= \frac{1}{2}\langle \partial_x u_\varepsilon^2, \varphi\rangle \\
&= \frac{1}{2}\langle \partial_x[(a+ib)R_\varepsilon + \beta_\varepsilon(x)]^2, \varphi\rangle \\
&= \frac{1}{2}\langle \partial_x[(a+ib)^2 R_\varepsilon^2 + 2(a+ib)R_\varepsilon\beta_\varepsilon(x) + \beta_\varepsilon^2(x)], \varphi\rangle \\
&\longrightarrow \frac{1}{2}\rho_0(a+ib)^2\langle \partial_x\delta, \varphi\rangle \\
&= -\frac{1}{2}\rho_0(a+ib)^2\langle \delta, \varphi'\rangle \\
&= -\frac{1}{2}\rho_0(a+ib)^2\varphi'(0).
\end{aligned}$$

The estimates $\partial_x\beta_\varepsilon^2(x) = 0$ and $R_\varepsilon\beta_\varepsilon(x) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ were used in the above calculation. See Figure 3.1. Firstly, assume that $m > 0$ so that $a = 0$ and $b = 1$. Using the definitions of m and ρ_0, we have

$$\begin{aligned}
\left\langle \partial_x \frac{u_\varepsilon^2}{2}, \varphi \right\rangle &\longrightarrow \frac{1}{2}\rho_0\varphi'(0) = m\varphi'(0), \\
\langle \partial_x v_\varepsilon, \varphi\rangle &\longrightarrow -m\varphi'(0)
\end{aligned}$$

and these terms cancel in the limit as $\varepsilon \longrightarrow 0$. Likewise, the case when $m < 0$ such that $a = 1$ and $b = 0$ gives

$$\begin{aligned}
\left\langle \partial_x \frac{u_\varepsilon^2}{2}, \varphi \right\rangle &\longrightarrow -\frac{1}{2}\rho_0\varphi'(0) = -m\varphi'(0), \\
\langle \partial_x v_\varepsilon, \varphi\rangle &\longrightarrow m\varphi'(0)
\end{aligned}$$

and the terms cancel in the limit as $\varepsilon \longrightarrow 0$. These conclusively satisfy equation (3.1) in the sense of Definition (3.1). Focusing on equation (3.2) and using equation (3.7) and the definition of m, it is not difficult to verify that $\partial_x u_\varepsilon \rightharpoonup 0$ as $\varepsilon \longrightarrow 0$. Moreover,

$$\begin{aligned}
\partial_t v_\varepsilon &= m\partial_t\delta_\varepsilon(x-ct) \\
&= -cm\delta_\varepsilon'(x-ct) \rightharpoonup -cm\delta'(x-ct) \quad \text{as} \quad \varepsilon \longrightarrow 0,
\end{aligned}$$

and

$$\begin{aligned}
\partial_x(u_\varepsilon v_\varepsilon) &= \partial_x([(a+ib)R_\varepsilon + \beta_\varepsilon(x)]m\delta_\varepsilon) \\
&= m(a+ib)\partial_x(R_\varepsilon\delta_\varepsilon) + m\partial_x(\beta_\varepsilon(x)\delta_\varepsilon) \\
&= cm\partial_x\delta_\varepsilon \\
&\longrightarrow cm\delta'(x-ct) \quad \text{as} \quad \varepsilon \longrightarrow 0.
\end{aligned}$$

and these estimates cancel in the limit as $\varepsilon \longrightarrow 0$. Finally, it is straightforward to check that the initial data are satisfied in the sense of Definition 3.2. $\square$

It is noted from this example that if only real-valued distributions were permitted as weak asymptotic solutions, then the constant m defined in Proposition 3.1 would only be negative.

3.3 Riemann Problem with a Delta Singularity

It is shown next that a more general weak asymptotic solution than the one given in Proposition 3.1 could be found. This is achieved by imposing a jump discontinuity in u. In this case non-uniqueness is not shown. Let the initial data be given by

$$u(x,0) = \begin{cases} u_1, x < 0, \\ u_2, x > 0, \end{cases} \tag{3.9}$$

$$v(x,0) = m\delta(x), \tag{3.10}$$

where $u_1 > u_2$ is assumed. Let R_ε and δ_ε be given as in equation (3.4) and suppose that

$$c = \frac{u_1 + u_2}{2} \tag{3.11}$$

is the shock speed [35]. In addition, let F be a regularized step function defined by

$$F_\varepsilon(z) = \begin{cases} u_1, & z \leqslant -5\varepsilon, \\ c, & -3\varepsilon \leqslant z \leqslant 3\varepsilon, \\ u_2, & z \geqslant 5\varepsilon, \end{cases} \tag{3.12}$$

where F_ε continuous smoothly in the intervals $(-5\varepsilon, -3\varepsilon)$ and $(3\varepsilon, 5\varepsilon)$ such that $u_2 \leqslant F_\varepsilon \leqslant u_1$.

We begin by proving a result which would be used to establish the global solution.

Lemma 3.2. *The function $u(x,t)$ defined by*

$$u(x,t) = \begin{cases} u_1, & x < ct, \\ u_2, & x > ct, \end{cases}$$

where $c = (u_1 + u_2)/2$ *represents weak solution of the inviscid Burgers equation*

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0,$$

if

$$\int_0^\infty \int_{-\infty}^\infty (\varphi_t u + \varphi_x f(u)) dx dt = - \int_{-\infty}^\infty \varphi(x,0) u(x,0) dx \tag{3.13}$$

is satisfied for all functions $\varphi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$.

Proof. For $t = 0$, we have the initial data

$$u(x,0) = \begin{cases} u_1, & x < 0, \\ u_2, & x > 0. \end{cases}$$

The right-hand side of (3.13) can be expressed as

$$- \int_{-\infty}^\infty \varphi(x,0) u(x,0) dx = -u_1 \int_{-\infty}^0 \varphi(x,0) dx - u_2 \int_0^\infty \varphi(x,0) dx.$$

The left-hand side of (3.13) could also be expressed as

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty \left(\varphi_t u + \varphi_x \frac{u^2}{2} \right) dx dt \quad &= \\ \int_0^\infty \int_{-\infty}^{ct} \left(\varphi_t u_1 + \varphi_x \frac{u_1^2}{2} \right) \ dx dt \ &+ \int_0^\infty \int_{ct}^\infty \left(\varphi_t u_2 + \varphi_x \frac{u_2^2}{2} \right) dx dt \end{aligned}$$

$$\begin{aligned} = u_1 \int_0^\infty \int_{-\infty}^{ct} \varphi_t dx dt \ &+ \ \frac{u_1^2}{2} \int_0^\infty \int_{-\infty}^{ct} \varphi_x dx dt \\ &+ \ u_2 \int_0^\infty \int_{ct}^\infty \varphi_t dx dt + \frac{u_2^2}{2} \int_0^\infty \int_{ct}^\infty \varphi_x dx dt. \end{aligned}$$

Assume that the shock speed $c > 0$ (noting that the same procedure works for the case $c < 0$). Using the appropriate limits for the first and the third integrands gives

$$\begin{aligned} &= u_1 \left(\int_{-\infty}^0 \int_0^\infty \varphi_t + \int_0^\infty \int_{\frac{x}{c}}^\infty \varphi_t \right) dt dx \ + \\ &\frac{u_1^2}{2} \int_0^\infty \int_{-\infty}^{ct} \varphi_x dx dt + u_2 \int_0^\infty \int_0^{\frac{x}{c}} \varphi_t dt dx \ + \ \frac{u_2^2}{2} \int_0^\infty \int_{ct}^\infty \varphi_x dx dt. \end{aligned}$$

This simplifies to

$$= -u_1 \int_{-\infty}^{0} \varphi(x,0)dx - u_1 \int_{0}^{\infty} \varphi\left(x, \frac{x}{c}\right) dx \quad +$$

$$u_2 \int_0^\infty \Big(\varphi\left(x,\frac{x}{c}\right) - \varphi(x,0)\Big)\, dx + \frac{u_1^2}{2}\int_0^\infty \varphi(ct,t)dt \quad - \quad \frac{u_2^2}{2}\int_0^\infty \varphi(ct,t)dt.$$

Letting $y = ct$ in the above equation gives

$$= -u_1 \int_{-\infty}^{0} \varphi(x,0)dx - u_2 \int_0^\infty \varphi(x,0)dx \quad -$$

$$(u_1 - u_2)\int_0^\infty \varphi\left(x,\frac{x}{c}\right) dx \quad + \quad \frac{(u_1^2 - u_2^2)}{2c}\int_0^\infty \varphi\left(y,\frac{y}{c}\right) dy.$$

Using the expression for the shock speed, c, defined in the Lemma reduces the equation to

$$\int_0^\infty \int_{-\infty}^\infty \left(\varphi_t u + \varphi_x \frac{u^2}{2}\right) dxdt = -u_1 \int_{-\infty}^0 \varphi(x,0)dx - u_2 \int_0^\infty \varphi(x,0)dx.$$

□

Proposition 3.3. *Let* (u_ε) *and* (v_ε) *be defined by*

$$u_\varepsilon(x,t) \;=\; F_\varepsilon(x-ct) + (p(t)+iq(t))R_\varepsilon(x-ct) + \beta_\varepsilon(x), \qquad (3.14)$$

$$v_\varepsilon(x,t) \;=\; \lambda(t)\delta_\varepsilon(x-ct), \qquad (3.15)$$

where $\lambda(t) = (u_1 - u_2)t + m$ *and* p *and* q *are chosen such that* $\rho_0(p+iq)^2 = -2\lambda$*. Then* (u_ε) *and* (v_ε) *represent weak asymptotic solution to the shallow water system (1.15), (1.16) with initial data (3.9) and (3.10).*

Proof. Begin by substituting (u_ε) and (v_ε) into equations (3.1) and (3.2) and use the distributional limits $R_\varepsilon \rightharpoonup 0$, $R_\varepsilon F_\varepsilon \equiv cR_\varepsilon \rightharpoonup 0$ and $R_\varepsilon \delta_\varepsilon \rightharpoonup 0$ to obtain

$$\partial_t F_\varepsilon(x-ct) + \partial_x\left(\lambda(t)\delta_\varepsilon + \frac{F_\varepsilon^2 + (p+iq)^2 R_\varepsilon^2}{2}\right) = o_{\mathcal{D}'}(1), \qquad (3.16)$$

$$\partial_t(\lambda(t)\delta_\varepsilon(x-ct)) + \partial_x(F_\varepsilon + F_\varepsilon\lambda(t)\delta_\varepsilon + \beta_\varepsilon(x)\lambda(t)\delta_\varepsilon) = o_{\mathcal{D}'}(1). \qquad (3.17)$$

Consider equation (3.16) and note that as $\varepsilon \to 0$,

$$\partial_x(\lambda(t)\delta_\varepsilon(x-ct)) \rightharpoonup \lambda(t)\delta'(x-ct)$$

and

$$\frac{(p+iq)^2}{2}R_\varepsilon^2(x-ct) \rightharpoonup \frac{(p+iq)^2}{2}\rho_0\delta'(x-ct).$$

These terms cancel in the limit if p and q are chosen in a way such that

$$\rho_0(p+iq)^2 = -2\lambda.$$

This then reduces equation (3.16) to

$$\partial_t F_\varepsilon(x-ct) + \frac{1}{2}\partial_x F_\varepsilon^2(x-ct) = o_{\mathcal{D}'}(1) \tag{3.18}$$

Observe that $F_\varepsilon \to u$ in L'_{loc} where

$$u(x,t) = \begin{cases} u_1, & x < ct, \\ u_2, & x > ct, \end{cases}$$

is the weak solution of the inviscid Burgers' equation $\partial_t u + \frac{1}{2}\partial_x u^2 = 0$ and the shock speed is given by $c = (u_1+u_2)/2$. This argument implies that equation (3.18) is satisfied and consequently, equation (3.16) is also satisfied. To show that equation (3.17) is also satisfied, differentiate each term in the limit as $\varepsilon \to 0$ to get

$$\begin{aligned} &\lambda'\delta_\varepsilon - c\lambda\delta'_\varepsilon + \partial_x F_\varepsilon + \lambda F_\varepsilon \delta'_\varepsilon + \lambda\delta_\varepsilon \partial_x F_\varepsilon = 0, \\ \Rightarrow \quad & \lambda'\delta - c\lambda\delta' + (u_2-u_1)\delta + c\lambda\delta' = 0, \\ \Rightarrow \quad & (\lambda' + (u_2-u_1))\delta = 0. \end{aligned}$$

This gives the formula for $\lambda(t)$. □

Chapter 4

The Weak Asymptotic Method - 2D Case

The method of weak asymptotic is used to find singular solution of the two-dimensional shallow water system of equations (1.17), (1.18) and (1.19). The singular solution to be constructed is allowed to contain Dirac-δ distributions. The idea is to construct complex-valued approximate solutions which become real-valued in the distributional limit. This extends the range of possible singular solutions of (1.17)-(1.19).

4.1 Definitions and Basic Concepts

Let $\mathcal{D}$ be the space of text functions and $\mathcal{D}'$ be the space of distributions as defined in [9].

Definition 4.1 Let $f_\varepsilon(x,y) \in \mathcal{D}'(\mathbb{R}^2)$ be a family of distributions which depend on $\varepsilon \in B(0,1)$. If the estimate $\langle f_\varepsilon, \varphi\rangle \ = o(1)$ as $\varepsilon \longrightarrow 0$ holds for any test function $\varphi(x,y) \in \mathcal{D}(\mathbb{R}^2)$, then the function $f_\varepsilon = o_{\mathcal{D}'}(1)$.

This means that for any test function φ, the family of distributions f_ε, converges to zero in the sense of Definition 4.1 if the pairing $\langle f_\varepsilon, \varphi\rangle$ converges to zero as $\varepsilon \to 0$. If the estimate above holds uniformly in t for families of distributions $f_\varepsilon(x,y,t)$, then we write $f_\varepsilon = o_{\mathcal{D}'}(1)$.

Definition 4.2 Let (u_ε), (v_ε) and (η_ε) be a family of smooth complex-valued distributions and u, v and η be smooth real-valued distributions. We say that u_ε, v_ε and η_ϵ represent a weak asymptotic solution to (1.17), (1.18), (1.19) if for every fixed $t \in \mathbb{R}^+$

$$u_\epsilon \rightharpoonup u, \quad v_\epsilon \rightharpoonup v, \quad \eta_\epsilon \rightharpoonup \eta \quad as \quad \varepsilon \to 0$$

in the distributional sense in $\mathcal{D}'(\mathbb{R}^2)$ and

$$\partial_t u_\varepsilon + \partial_x \left(\frac{u_\varepsilon^2}{2} + \eta_\varepsilon \right) + v_\varepsilon \partial_y u_\varepsilon = o_{\mathcal{D}'}(1), \tag{4.1}$$

$$\partial_t v_\varepsilon + u_\varepsilon \partial_x v_\varepsilon + \partial_y \left(\frac{v_\varepsilon^2}{2} + \eta_\varepsilon \right) = o_{\mathcal{D}'}(1), \tag{4.2}$$

$$\partial_t \eta_\varepsilon + \partial_x [u_\varepsilon(1+\eta_\varepsilon)] + \partial_y [v_\varepsilon(1+\eta_\varepsilon)] = o_{\mathcal{D}'}(1). \tag{4.3}$$

Furthermore, initial data are given by

$$\begin{aligned} u_\varepsilon(x,y,0) &\rightharpoonup u(x,y,0) \\ v_\varepsilon(x,y,0) &\rightharpoonup v(x,y,0) \\ \eta_\varepsilon(x,y,0) &\rightharpoonup \eta(x,y,0) \end{aligned}$$

as $\varepsilon \longrightarrow 0$.

An important part of this definition is that it allows the approximating distributions to be complex-valued and hence, broadens the range of possible singular solutions as noted by Kalisch and Mitrovic [28]. The imaginary parts of (u_ε), v_ε and η_ε finally vanishes in the limit as ε approaches zero.

In addition to the definitions above, we will make an intelligent guess for a possible weak solution by first considering a smooth function $\rho : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ satisfying the following conditions:

(i) $\rho(x,y) \geq 0$

(ii) $\int_{\mathbb{R}} \int_{\mathbb{R}} \rho(x,y)\, dxdy = 1$

(iii) $\text{supp}(\rho) \subset B(0,1)$

Denote

$$R_\varepsilon(x,y) = \frac{1}{\varepsilon}\rho\left(\frac{x-3\varepsilon}{\varepsilon}, \frac{y}{\varepsilon}\right) \tag{4.4}$$

$$S_\varepsilon(x,y) = \frac{1}{\varepsilon}\rho\left(\frac{x-5\varepsilon}{\varepsilon}, \frac{y}{\varepsilon}\right) \tag{4.5}$$

$$\delta_\varepsilon(x,y) = \frac{1}{\varepsilon^2}\rho\left(\frac{x+3\varepsilon}{\varepsilon}, \frac{y}{\varepsilon}\right) \tag{4.6}$$

By using the expressions above, it is easy to verify that

$$R_\varepsilon(x,y)\delta_\varepsilon(x,y)=0, \quad S_\varepsilon(x,y)\delta_\varepsilon(x,y)=0 \ \text{ for every } \ x,y\in\mathbb{R} \tag{4.7}$$

In addition to equations (3.4), (3.5) and (3.6), we define

$$\rho_0=\int_{\mathbb{R}}\int_{\mathbb{R}}\rho^2(x,y)\ dxdy$$

and obtain the following distributional limits

$$\begin{aligned}\langle R_\varepsilon,\varphi\rangle &= \frac{1}{\varepsilon}\int_{\mathbb{R}}\int_{\mathbb{R}}\rho\left(\frac{x-3\varepsilon}{\varepsilon},\frac{y}{\varepsilon}\right)\varphi(x,y)\ dxdy\\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}}\int_{\mathbb{R}}\rho\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\varphi(x+3,y)\ dxdy\end{aligned}$$

Let $\tilde{x}=\frac{x}{\varepsilon}$ and $\tilde{y}=\frac{y}{\varepsilon}$, where $\tilde{x},\tilde{y}\in\mathbb{R}$. This gives

$$\begin{aligned}&= \varepsilon\int_{\mathbb{R}}\int_{\mathbb{R}}\rho(\tilde{x},\tilde{y})\varphi(\varepsilon\tilde{x}+3,\varepsilon\tilde{y})\ d\tilde{x}d\tilde{y}\\ &\le \varepsilon\sup_{\tilde{x},\tilde{y}\in\mathbb{R}}|\varphi(\varepsilon\tilde{x}+3,\varepsilon\tilde{y})|\int_{\mathbb{R}}\int_{\mathbb{R}}\rho(\tilde{x},\tilde{y})\ d\tilde{x}d\tilde{y}\\ &= \varepsilon\sup_{\tilde{x},\tilde{y}\in\mathbb{R}}|\varphi(\varepsilon\tilde{x},\varepsilon\tilde{y})|\rightharpoonup 0 \quad \text{as} \quad \varepsilon\longrightarrow 0\end{aligned} \tag{4.8}$$

Similarly as above, we estimate the limit of $S_\varepsilon(x,y)$ as $\varepsilon\longrightarrow 0$ to get

$$\langle S_\varepsilon,\varphi\rangle \ \rightharpoonup 0 \ \text{ as } \ \varepsilon\longrightarrow 0. \tag{4.9}$$

We follow a similar process to estimate the limit of $\delta_\varepsilon(x,y)$

$$\begin{aligned}\langle \delta_\varepsilon,\varphi\rangle &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\rho\left(\frac{x+3\varepsilon}{\varepsilon},\frac{y}{\varepsilon}\right)\varphi(x,y)\ dxdy\\ &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\rho(\frac{x}{\varepsilon},\frac{y}{\varepsilon})\varphi(x-3,y)\ dxdy\end{aligned}$$

Let $\tilde{x}=\frac{x}{\varepsilon}$, $\tilde{y}=\frac{y}{\varepsilon}$ where x, y $\in\mathbb{R}$. By substitution the expression becomes

$$\begin{aligned}&= \int_{\mathbb{R}}\int_{\mathbb{R}}\rho(\tilde{x},\tilde{y})\varphi(\varepsilon\tilde{x}-3,\varepsilon\tilde{y})\ d\tilde{x}d\tilde{y}\\ &\le \sup_{\tilde{x},\tilde{y}\in\mathbb{R}}|\varphi(\varepsilon\tilde{x}-3,\varepsilon\tilde{y})|\int_{\mathbb{R}}\int_{\mathbb{R}}\rho(\tilde{x},\tilde{y})\ d\tilde{x}d\tilde{y}\\ &= \sup_{\tilde{x},\tilde{y}\in\mathbb{R}}|\varphi(\varepsilon\tilde{x}-3,\varepsilon\tilde{y})|\rightharpoonup \delta(x,y) \ \text{ as } \ \varepsilon\longrightarrow 0\end{aligned} \tag{4.10}$$

The limit for $R^2_\varepsilon(x,y)$ can be estimated in like-manner as was done for $R_\varepsilon(x,y)$ by using ρ_0 as defined above.

$$\begin{aligned}\langle R^2_\varepsilon, \varphi\rangle &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \rho^2(\frac{x-3\varepsilon}{\varepsilon}, \frac{y}{\varepsilon})\varphi(x,y)\ dxdy \\ &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \rho^2(\frac{x}{\varepsilon}, \frac{y}{\varepsilon})\varphi(x+3,y)\ dxdy\end{aligned}$$

Here again we let $\tilde{x} = \frac{x}{\varepsilon}, \tilde{y} = \frac{y}{\varepsilon}$ where $x, y \in \mathbb{R}$. Substituting these into the above expression gives

$$\begin{aligned}&= \int_{\mathbb{R}}\int_{\mathbb{R}} \rho^2(\tilde{x},\tilde{y})\phi(\varepsilon\tilde{x}+3,\varepsilon\tilde{y})\ d\tilde{x}d\tilde{y} \\ &\leq \sup_{\tilde{x},\tilde{y}\in\mathbb{R}} |\varphi(\varepsilon\tilde{x}+3,\varepsilon\tilde{y})| \int_{\mathbb{R}}\int_{\mathbb{R}} \rho^2(\tilde{x},\tilde{y})\ d\tilde{x}d\tilde{y} \\ &= \sup_{\tilde{x},\tilde{y}\in\mathbb{R}} |\phi(\varepsilon\tilde{x}+3,\varepsilon\tilde{y})| \int_{\mathbb{R}}\int_{\mathbb{R}} \rho^2(\tilde{x},\tilde{y})\ d\tilde{x}d\tilde{y} \rightharpoonup \rho_0\delta \ \text{ as } \ \varepsilon \longrightarrow 0\end{aligned} \quad (4.11)$$

Note that for every $x, y \in \mathbb{R}$,

$$\langle S^2_\varepsilon(x,y), \varphi(x,y)\rangle \ \rightharpoonup \rho_0\delta(x,y) \ \text{ as } \ \varepsilon \longrightarrow 0 \quad (4.12)$$

4.2 Non-uniqueness

In this section we shall give a special example which clearly demonstrates the power of the weak asymptotic method. This special case is a stationary delta distribution in η, centred at the origin.

Proposition 4.1. *Define the constant*

$$m = \begin{cases} \frac{1}{2}\rho_0, & if\ \ a=0,\ b=1, \\ -\frac{1}{2}\rho_0, & if\ \ a=1,\ b=0. \end{cases}$$

The pair of families of smooth functions defined by

$$u_\varepsilon(x,y,t) = (a+ib)R_\varepsilon(x,y) \quad (4.13)$$

$$v_\varepsilon(x,y,t) = (a-ib)S_\varepsilon(x,y) \quad (4.14)$$

$$\eta_\varepsilon(x,y,t) = m\delta_\varepsilon(x,y) \quad (4.15)$$

represents a weak asymptotic solution of the shallow water system (1.17), (1.18), (1.19).

Proof. Starting with equation (4.1), it is easy to see that $\partial_t u_\varepsilon \longrightarrow 0$. By equations (4.11) and (4.13) we have

$$\begin{aligned}\left\langle \partial_x \frac{u_\varepsilon^2}{2}, \varphi \right\rangle &= \frac{1}{2}\langle \partial_x u_\varepsilon^2, \varphi\rangle \\ &= \frac{1}{2}(a+ib)^2\langle \partial_x R_\varepsilon^2, \varphi\rangle \\ &\longrightarrow \frac{1}{2}\rho_0(a+ib)^2\langle \partial_x \delta, \varphi\rangle \\ &= -\frac{1}{2}\rho_0(a+ib)^2\langle \delta, \varphi'\rangle \\ &= -\frac{1}{2}\rho_0(a+ib)^2\varphi'(0).\end{aligned}$$

and

$$\langle \partial_x \eta_\varepsilon, \varphi\rangle \ = m\langle \partial_x \delta_\varepsilon, \varphi\rangle \ \longrightarrow -m\langle \delta, \varphi'\rangle \ = -m\varphi'(0).$$

Similarly,

$$\langle v_\varepsilon \partial_y u_\varepsilon, \varphi\rangle \ = (a-ib)(a+ib)\langle S_\varepsilon \partial_y R_\varepsilon, \varphi\rangle \ \longrightarrow (a^2+b^2)\langle 0, \varphi\rangle \ = 0$$

Firstly, assume that $m > 0$ so that $a = 0$ and $b = 1$. Using the definitions of m and ρ_0, we have

$$\begin{aligned}\left\langle \partial_x \frac{u_\varepsilon^2}{2}, \varphi \right\rangle &\longrightarrow \frac{1}{2}\rho_0\varphi'(0) = m\varphi'(0), \\ \langle \partial_x \eta_\varepsilon, \varphi\rangle &\longrightarrow -m\varphi'(0)\end{aligned}$$

and these terms cancel in the limit as $\varepsilon \longrightarrow 0$. Likewise, the case when $m < 0$ such that $a = 1$ and $b = 0$ gives

$$\begin{aligned}\left\langle \partial_x \frac{u_\varepsilon^2}{2}, \varphi \right\rangle &\longrightarrow -\frac{1}{2}\rho_0\varphi'(0) = -m\varphi'(0), \\ \langle \partial_x \eta_\varepsilon, \varphi\rangle &\longrightarrow m\varphi'(0)\end{aligned}$$

and the terms cancel in the limit as $\varepsilon \longrightarrow 0$. These conclusively satisfy equation (4.1) in the sense of Definition (4.1). Focusing on equation (4.2), and using equation (4.12) and the definitions of ρ_0 and m, it is not difficult to check that $\partial_t v_\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Furthermore, note that

$$\begin{aligned}\langle u_\varepsilon \partial_x v_\varepsilon, \varphi\rangle &= (a+ib)(a-ib)\langle R_\varepsilon \partial_x S_\varepsilon, \varphi\rangle \\ &\longrightarrow (a^2+b^2)\langle 0, \varphi\rangle \\ &= 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\left\langle \partial_y \frac{v_\varepsilon^2}{2}, \varphi \right\rangle &= \frac{1}{2}\langle \partial_y v_\varepsilon^2, \varphi\rangle \\ &= \frac{1}{2}(a-ib)^2\langle \partial_y S_\varepsilon^2, \varphi\rangle \\ &\longrightarrow -\frac{1}{2}\rho_0(a-ib^2)\langle \delta, \varphi'\rangle \\ &= -\frac{1}{2}\rho_0(a-ib)^2\varphi'(0)\end{aligned}$$

and

$$\langle \partial_y \eta_\varepsilon, \varphi\rangle \ = m\langle \partial_y \delta_\varepsilon, \varphi\rangle \ \longrightarrow -m\langle \delta, \varphi'\rangle \ = -m\varphi'(0).$$

Just as we did earlier we assume firstly that $m > 0$ so that $a = 0$ and $b = 1$. The following limits are obtained

$$\begin{aligned}\left\langle \partial_y \frac{v_\varepsilon^2}{2}, \varphi \right\rangle &\longrightarrow \frac{1}{2}\rho_0\varphi'(0) = m\varphi'(0), \\ \langle \partial_y \eta_\varepsilon, \varphi\rangle &\longrightarrow -m\varphi'(0)\end{aligned}$$

and the limits cancel as $\varepsilon \longrightarrow 0$. Secondly, assume that $m < 0$ such that $a = 1$ and $b = 0$. Then

$$\begin{aligned}\left\langle \partial_y \frac{v_\varepsilon^2}{2}, \varphi \right\rangle &\longrightarrow -\frac{1}{2}\rho_0\varphi'(0) = -m\varphi'(0) \\ \langle \partial_y \eta_\varepsilon, \varphi\rangle &\longrightarrow m\varphi'(0)\end{aligned}$$

and these cancel in the limit as $\varepsilon \longrightarrow 0$. Hence, equation (4.2) is satisfied in the sense of Definition (4.1).
The proof of equation (4.3) follows immediately from the previous calculations. Note that $\partial_t \eta_\varepsilon \equiv 0$ and equations (4.8) and (4.9) respectively lead to the limits $u_\varepsilon \rightharpoonup 0$ and $v_\varepsilon \rightharpoonup 0$ as $\varepsilon \longrightarrow 0$. It is immediate, by equation (4.7), that $u_\varepsilon \eta_\varepsilon \equiv 0$ and $v_\varepsilon \eta_\varepsilon \equiv 0$. In addition, the initial data are satisfied in the sense of Definition (4.1). □

As demonstrated exclusively by this example, if only real-valued distributions were permitted as weak asymptotic solutions, then the constant m defined in Proposition 4.1 would only be negative.

4.3 Riemann Problem with a Delta Singularity

This section presents a more generalised solution in the framework of weak asymptotic defined above. This is achieved by imposing a jump discontinuity in both u and v in addition to the Dirac-δ initial data in η. Assume that

$$u(x,y,0) = \begin{cases} u_1, & x < 0, \\ u_2, & x > 0, \end{cases} \tag{4.16}$$

where $u_1 > u_2 > 0$,

$$v(x,y,0) = \begin{cases} v_1, & x < 0, \\ v_2, & x > 0, \end{cases} \tag{4.17}$$

where $v_2 < v_1 < 0$ and

$$\eta(x,y,0) = m\delta(x,y). \tag{4.18}$$

Let the functions R_ε, S_ε and δ_ε be given as in equations (4.4), (4.5) and (4.6) respectively and suppose that

$$c = \frac{u_1+u_2}{2}, \qquad \bar{c} = \frac{v_1+v_2}{2},$$

are the shock speeds [35]. In addition, Let F_ε and G_ε be regularized step functions defined by

$$F_\varepsilon(z) = \begin{cases} u_1, & z \leqslant -5\varepsilon, \\ c, & -3\varepsilon \leqslant z \leqslant 3\varepsilon, \\ u_2, & z \geqslant 5\varepsilon, \end{cases} \tag{4.19}$$

where F_ε continuous smoothly in the intervals $(-5\varepsilon, -3\varepsilon)$ and $(3\varepsilon, 5\varepsilon)$ such that $u_2 \leqslant F_\varepsilon \leqslant u_1$ and

$$G_\varepsilon(w) = \begin{cases} v_1, & w \leqslant -2\varepsilon, \\ \bar{c}, & -\varepsilon \leqslant w \leqslant \varepsilon, \\ v_2, & w \geqslant 2\varepsilon, \end{cases} \tag{4.20}$$

where G_ε continuous smoothly in the intervals $(-2\varepsilon, -\varepsilon)$ and $(\varepsilon, 2\varepsilon)$ so that $v_2 \leqslant G_\varepsilon \leqslant v_1$.

Proposition 4.2. *Let (u_ε), (v_ε) and (η_ε) be defined by*

$$u_\varepsilon(x,y,t) = F_\varepsilon(x-ct,y) + (p(t)+iq(t))R_\varepsilon(x-ct,y), \tag{4.21}$$
$$v_\varepsilon(x,y,t) = G_\varepsilon(x-\bar{c}t,y) + (p(t)-iq(t))S_\varepsilon(x-\bar{c}t,y), \tag{4.22}$$
$$\eta_\varepsilon(x,y,t) = \lambda(t)\delta_\varepsilon(x-\kappa t,y), \tag{4.23}$$

where $\lambda(t) = (u_1-u_2)t+m$, $\kappa = (c+\bar{c})$ and p and q are chosen such that $\rho_0(p+iq)^2 = -2\lambda$ and $\rho_0(p-iq)^2 = -2\lambda$. Then (u_ε), (v_ε) and (η_ε) represent weak asymptotic solution to the shallow water system (1.17), (1.18), (1.19) with initial data (4.16), (4.17) and (4.18).

Proof. In addition to equations (4.8) and (4.9), it is not difficult to verify that

$$R_\varepsilon F_\varepsilon = cR_\varepsilon \rightharpoonup 0, \qquad S_\varepsilon G_\varepsilon = \bar{c}S_\varepsilon \rightharpoonup 0. \tag{4.24}$$

Substituting the expressions for (u_ε), (v_ε) and (η_ε) into (4.1), (4.2), and (4.3) and using the limits in (4.8), (4.9) and (4.24), we can simply rewrite equations (4.1), (4.2), and (4.3) respectively to

$$\partial_t F_\varepsilon + \partial_x \left(\frac{F_\varepsilon^2 + (p+iq)^2 R_\varepsilon^2(x-ct,y)}{2} + \lambda(t)\delta_\varepsilon \right) + G_\varepsilon \partial_y F_\varepsilon = 0 \tag{4.25}$$

$$\partial_t G_\varepsilon + F_\varepsilon \partial_x G_\varepsilon + \partial_y \left(\frac{G_\varepsilon^2 + (p-iq)^2 S_\varepsilon^2(x-ct,y)}{2} + \lambda(t)\delta_\varepsilon \right) = 0 \tag{4.26}$$

$$\partial_t(\lambda(t)\delta_\varepsilon) + \partial_x[F_\varepsilon(1+\lambda(t)\delta_\varepsilon)] + \partial_y[G_\varepsilon(1+\lambda(t)\delta_\varepsilon)] = 0 \tag{4.27}$$

We simplify the above equations further by focusing on one at a time. Beginning firstly with equation (4.25), it is easy to note that

$$\frac{(p+iq)^2}{2}\partial_x R_\varepsilon^2(x-ct,y) \rightharpoonup \frac{(p+iq)^2}{2}\rho_0\delta'(x-\kappa t,y),$$

and

$$\partial_x(\lambda(t)\delta_\varepsilon(x-\kappa t,y)) \rightharpoonup \lambda(t)\delta'(x-\kappa t,y).$$

Observe that these two terms will cancel out in the limit if we choose p and q such that

$$\rho_0(p+iq)^2 = -2\lambda. \tag{4.28}$$

This allows us to simplify equation (4.25) to

$$\partial_t F_\varepsilon(x-ct,y) + \frac{1}{2}\partial_x F_\varepsilon^2(x-ct,y) + G_\varepsilon(x-\bar{c}t)\partial_y F_\varepsilon(x-ct,y) = 0. \tag{4.29}$$

From the definitions of the regularized step functions F_ε and G_ε, it is explicit that these functions have disjoint support. Consequently, $G_\varepsilon \partial_y F_\varepsilon \equiv 0$ and equation (4.29) simplifies further to

$$\partial_t F_\varepsilon(x-ct,y) + \frac{1}{2}\partial_x F_\varepsilon^2(x-ct,y) = 0. \tag{4.30}$$

In like manner, equation (4.26) can be reduced by following the same easy and straight-forward calculation done above. In this case, $F_\varepsilon \partial_x G_\varepsilon \equiv 0$ yielding

$$\partial_t G_\varepsilon(x-\bar{c}t,y) + \frac{1}{2}\partial_x G_\varepsilon^2(x-\bar{c}t,y) = 0. \tag{4.31}$$

Observe that equations (4.30) and (4.31) are in the form of the inviscid Burgers equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0. \tag{4.32}$$

Observe further that $F_\varepsilon \to u$ in L^1_{loc} and $G_\varepsilon \to v$ in L^1_{loc} where

$$u(x,t) = \begin{cases} u_1, & x < ct, \\ u_2, & x > ct, \end{cases} \tag{4.33}$$

and

$$v(x,t) = \begin{cases} v_1, & x < \bar{c}t, \\ v_2, & x > \bar{c}t. \end{cases} \tag{4.34}$$

By Lemma (3.2), $u(x,t)$, $v(x,t)$ represents weak solution to the Burgers equation if the shock speed is given by $c = (u_1+u_2)/2$, $\bar{c} = (v_1+v_2)/2$ respectively. This implies that equations (4.30) and (4.31) are satisfied. Consequently, we conclude that equations (4.25) and (4.26) are satisfied. The proof of equation (4.27) is done quiet differently. Differentiating term by term gives

$$\lambda'\delta_\varepsilon - \kappa\lambda\delta'_\varepsilon + \partial_x F_\varepsilon + \lambda\delta_\varepsilon\partial_x F_\varepsilon + \lambda F_\varepsilon \partial_x \delta_\varepsilon + \partial_y G_\varepsilon + \lambda\delta_\varepsilon \partial_y G_\varepsilon + \lambda G_\varepsilon \partial_y \delta_\varepsilon = 0.$$

Note that $F_\varepsilon \partial_x \delta_\varepsilon \equiv 0$ and $G_\varepsilon \partial_y \delta_\varepsilon \equiv 0$ since F_ε, G_ε and δ_ε have disjoint supports. Letting $\varepsilon \to 0$ simplifies the equation to

$$\begin{aligned} \lambda'\delta - (c+\bar{c})\lambda\delta' + (u_2 - u_1)\delta + c\lambda\delta' + \bar{c}\lambda\delta' = 0, \\ \Rightarrow (\lambda' + (u_2 - u_1))\delta = 0, \end{aligned} \tag{4.35}$$

where $\kappa = (c + \bar{c})$ was used. This represents the required formula for $\lambda(t)$.

□

Chapter 5

Local Ill-posedness

This work is concerned with ill-posed results for the Cauchy problem for Korteweg-de Vries (KdV) type equations

$$\partial_t u + u^p \partial_x u + \partial_x^q u = 0, \quad p, q \in \mathbb{N}, \quad x, t \in \mathbb{R}, \tag{5.1}$$
$$u(x, 0) = u_0(x). \tag{5.2}$$

where $p \geqslant 4$ is even, in the Sobolev space H^{s_ε}, $s_\varepsilon = -1/2 - \varepsilon$ for arbitrary $\varepsilon > 0$. The positive integers p and q are related in a way which makes the system above interesting to study. The relation would be used to establish a generalised scaling argument which renders the system (5.1) invariant.

It is not difficult to check that (5.1) is invariant under the scaling

$$v(x, t) = \lambda u(\lambda x, \lambda^q t), \quad \lambda > 0, \tag{5.3}$$

where

$$q = p + 1, \quad \text{for} \quad p \in \mathbb{Z}^+.$$

In other words we say that if $u(x, t)$ is a solution of (5.1) then $v(x, t)$ defined in (5.3) also solves (5.1) with initial data given by

$$v(x, 0) = \lambda u_0(\lambda x).$$

Note that if $p = 2$, then $q = 3$ and equation (5.1) reduces to the modified KdV (MKdV) equation

$$\partial_t u + u^2 \partial_x u + \partial_x^3 = 0, \quad x, t \in \mathbb{R}. \tag{5.4}$$

Note that the scaling relation for this case is

$$v(x, t) = \lambda u(\lambda x, \lambda^3 t), \quad \lambda > 0.$$

The MKdV equation has received much attention in the past few years and many interesting results were obtained. Birnir *et al.* [4] established ill-posed result for the MKdV equation in a negative Sobolev space H^s, $s < -1/2$. They showed that if there exists a solution of the MKdV equation, then it does not depend continuously on its initial data in H^s, $s < -1/2$ by constructing a sequence which converges strongly to the data in H^s and then showed that the corresponding sequence of solutions does not converge strongly in H^s, $s < -1/2$.

5.1 Generalized Solution

Lemma 5.1. *Supposed that ϕ is a solution of*

$$-\phi + \frac{1}{q}\phi^q + \phi^{(p)} = 0, \tag{5.5}$$

where $q = p+1$, $p \geq 4$ and $\phi^{(p)}$ denotes the p^{th} derivative of ϕ. Define

$$\phi_c(x) = \lambda\phi(\lambda x),$$

where $c = \lambda^p$ is the propagation speed. Then ϕ_c represents a solution of

$$-c\phi_c + \frac{1}{q}\phi_c^q + \phi_c^{(p)} = 0. \tag{5.6}$$

Proof. To show that ϕ_c solves (5.6), substitute the expression for ϕ_c into (5.6) to get

$$\begin{aligned}
&= -c\lambda\phi(\lambda x) + \frac{1}{q}\lambda^q\phi^q(\lambda x) + \lambda\lambda^p\phi^{(p)}(\lambda x) \\
&= -\lambda^q\phi + \frac{1}{q}\lambda^q\phi^q + \lambda^q\phi^{(p)} \\
&= \lambda^q(-\phi + \frac{1}{q}\phi^q + \phi^{(p)}) \\
&= 0.
\end{aligned}$$

□

Note that equation (5.6) and $q = p+1$ were used in the proof. Solutions of KdV type equations have been established in recent works. Chen and Bona [8] established existence of solitary wave solutions of the Benjamin's model and used a recently developed theory to determine the spatial asymptotics of these solutions.

Lemma 5.2. *Define u by*

$$u(x,t) = \phi_c(x - ct), \quad x \in \mathbb{R},\ t \in \mathbb{R}^+. \tag{5.7}$$

The u represents a solution of (5.1).

Proof. By the definition of ϕ_c in Lemma 5.1, we have

$$u(x,t) = \phi_c(x - ct) = \lambda\phi(\lambda x - \lambda ct).$$

Using this equation we have the following derivatives

$$u_t = -\lambda^2 c\phi', \quad u^p u_x = \lambda^{p+2}\phi^p\phi', \quad \partial_x^q u = \lambda^{q+1}\phi^{(q)}.$$

Substitute these expressions into equation (5.1) and use Lemma 5.1 and $c = \lambda^p$ to obtain

$$\begin{aligned} -\lambda^2 c\phi' + \lambda^{p+2}\phi^p\phi' + \lambda^{q+1}\phi^{(q)} &= \lambda^2(-c\phi' + \lambda^p\phi^p\phi' + \lambda^p\phi^{(q)}) \\ &= \lambda^2\left(-c\phi + \frac{1}{q}\lambda^p\phi^q + \lambda^p\phi^{(p)}\right) \\ &= \lambda^{p+2}\left(-\phi + \frac{1}{q}\phi^q + \phi^{(p)}\right) \\ &= 0. \end{aligned}$$

□

5.2 Convergence Concepts

It is shown next that the initial data converges to a scalar multiple of the Dirac-δ function in the negative Sobolev space H^{s_ε}, $s_\varepsilon = -1/2 - \varepsilon$ for any $\varepsilon > 0$. Thus,

$$\lim_{c\to\infty} u_c(x,0) = \kappa\delta(x)$$

where κ is a constant to be determined.

Proposition 5.3. *The initial data $u_c(x,0) = \phi_c(x)$ has a finite H^{s_ε} norm for $s = -1/2 - \varepsilon$ and*

$$\lim_{c\to\infty} ||u_c(x,0)||_{s_\varepsilon} = ||\kappa\delta(x)||_{s_\varepsilon}$$

where the constant κ is given by

$$\kappa = \sqrt{2\pi}\hat{\phi}(0).$$

Proof. Let's first show that $u_c(x,0)$ is in H^{s_ε}. It is not difficult to check that the Fourier transform of ϕ_c is

$$\hat{\phi}_c(\xi) = \lambda \frac{1}{\lambda} \hat{\phi}\left(\frac{1}{\lambda}\xi\right) = \hat{\phi}\left(\frac{1}{\lambda}\xi\right).$$

Then

$$\begin{aligned} ||u_c(x,0)||_{s_\varepsilon}^2 = ||\phi_c(x)||_{s_\varepsilon}^2 &= ||\langle\xi\rangle^{s_\varepsilon}\hat{\phi}_c(\xi)||_{L^2(\mathbb{R})}^2 \\ &= \int_{-\infty}^{\infty} \frac{\hat{\phi_c}^2(\xi)}{(1+\xi^2)^{s_\varepsilon}} d\xi \\ &= \int_{-\infty}^{\infty} \frac{\hat{\phi}^2(\frac{1}{\lambda}\xi)}{(1+\xi^2)^{s_\varepsilon}} d\xi \\ &\leq \max\left|\hat{\phi}^2\left(\frac{1}{\lambda}\xi\right)\right| \int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)^{s_\varepsilon}} d\xi < \infty, \end{aligned}$$

for $s_\varepsilon = -1/2 - \varepsilon$. The integral converges uniformly in the limit as c goes to infinity. Note also that for $\varphi \in C_0^\infty(\mathbb{R})$, the Fourier transform of δ is given by

$$\begin{aligned} \langle\hat{\delta}, \varphi\rangle \equiv \langle\delta, \hat{\varphi}\rangle = \varphi(0) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \varphi(x) dx \\ &= \langle(2\pi)^{-1/2}, \varphi\rangle. \end{aligned}$$

Thus

$$\hat{\delta} = (2\pi)^{-1/2}.$$

So the H^{s_ε} norm of δ can be calculated by

$$||\delta||_{s_\varepsilon}^2 = \int_{-\infty}^{\infty} \frac{\hat{\delta}^2(\xi)}{(1+\xi^2)^{s_\varepsilon}} d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)^{s_\varepsilon}} d\xi$$

$$\Rightarrow \qquad ||\sqrt{2\pi}\delta||_{s_\varepsilon}^2 = \int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)^{s_\varepsilon}} d\xi.$$

The last part of the proposition is proven by showing that given any arbitrary $\zeta > 0$, there exists $M(\xi,\zeta)$ such that

$$|\ ||u_c(x,0)||_{H^{s_\varepsilon}}^2 - ||\kappa\delta(x)||_{H^{s_\varepsilon}}^2| < \zeta$$

if $c > M(\xi,\zeta)$. Thus,

$$\lim_{c\to\infty} |\ ||u_c(x,0)||_{H^{s_\varepsilon}}^2 - ||\kappa\delta(x)||_{H^{s_\varepsilon}}^2| = 0.$$

Since the integrals above converge uniformly in c, use the dominated convergence theorem to get

$$\begin{aligned} |\; ||u_c(x,0)||^2_{H^{s_\varepsilon}} - ||\kappa\delta(x)||^2_{H^{s_\varepsilon}}| &= \left| \int_{-\infty}^{\infty} \frac{\hat{\phi_c}^2(\xi)}{(1+\xi^2)^{s_\varepsilon}} d\xi - \frac{\kappa^2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)^{s_\varepsilon}} d\xi \right| \\ &= \left| \int_{-\infty}^{\infty} \left(\hat{\phi_c}^2(\xi) - \frac{\kappa^2}{2\pi} \right) \frac{1}{(1+\xi^2)^{s_\varepsilon}} d\xi \right| \\ &= \left| \int_{-\infty}^{\infty} \left(\hat{\phi}^2(\frac{1}{\lambda}\xi) - \frac{\kappa^2}{2\pi} \right) \frac{1}{(1+\xi^2)^{s_\varepsilon}} d\xi \right| \end{aligned}$$

Observe that in the limit as $\lambda \to \infty$,

$$\left(\hat{\phi}^2 \left(\frac{1}{\lambda}\xi \right) - \frac{\kappa^2}{2\pi} \right) \to \left(\hat{\phi}^2(0) - \frac{\kappa^2}{2\pi} \right) = 0,$$

since $\kappa = \sqrt{2\pi}\hat{\phi}(0)$. Therefore,

$$\lim_{c\to\infty} |\; ||u_c(x,0)||^2_{H^{s_\varepsilon}} - ||\kappa\delta(x)||^2_{H^{s_\varepsilon}}| = 0, \tag{5.8}$$

as required. □

Observe that Proposition 5.3 holds for $u_c(x,t) = \varphi_c(x-ct)$ since the integrals are invariant under translation. Thus, for $s_\varepsilon = -1/2 - \varepsilon$ and $t > 0$,

$$||u_c(x,t)||_{s_\varepsilon} < \infty$$

and

$$\lim_{c\to\infty} ||u_c(x,t)||_{s_\varepsilon} = ||\kappa\delta(x)||_{s_\varepsilon}. \tag{5.9}$$

Proposition 5.3 explicitly showed that the initial data has a finite H^{s_ε} norm for $s = -1/2 = \varepsilon$ and also has a limit which converges to a scalar multiple of the Dirac-δ function. It is shown next that the convergence of $u_c(x,0)$ to $\kappa\delta(x)$ is in the strong sense.

Proposition 5.4. *The initial data $u_c(x,0)$ converges strongly to the limit $\kappa\delta(x)$ as $c \to \infty$, in H^{s_ε}, $s_\varepsilon = -1/2 - \varepsilon$.*

Proof. By definition, we have

$$\begin{aligned} ||u_c(x,0) - \kappa\delta(x)||^2_{H^{s_\varepsilon}} &= ||\langle\xi\rangle^{s_\varepsilon}(\hat{\phi_c} - \kappa\hat{\delta})||_{H^{s_\varepsilon}} \\ &= \int_{-\infty}^{\infty} (1+|\xi|^2)^{s_\varepsilon} \left(\hat{\phi}\left(\frac{\xi}{\lambda}\right) - \frac{\kappa}{\sqrt{2\pi}} \right) d\xi. \end{aligned}$$

Note that as $\lambda \to \infty$, the dominated convergence theorem (see appendix) can be applied to get the limit

$$\left(\hat{\phi}\left(\frac{\xi}{\lambda}\right) - \frac{\kappa}{\sqrt{2\pi}}\right) \to \left(\hat{\phi}(0) - \frac{\kappa}{\sqrt{2\pi}}\right) = 0,$$

where κ is given as in Proposition 5.3. Therefore,

$$\lim_{c\to\infty} ||u_c(x,0) - \kappa\delta(x)||^2_{H^{s_\varepsilon}} = 0. \tag{5.10}$$

□

Proposition 5.5. *The solution $u_c(x,t)$, $t > 0$ converges weakly to zero in the limit as $c \to \infty$, in $L^2(\mathbb{R})$.*

Proof. Note that by Proposition 3 and the discussions that followed, the H^{s_ε} norms of $u_c(x,0)$ and $u_c(x,t)$ are finite. Thus,

$$||u_c(x,0)|| < \infty, \qquad ||u_c(x,t)|| < \infty \quad \text{for} \quad s_\varepsilon = -1/2 - \varepsilon.$$

Hence, we can calculate the limit

$$\lim_{c\to\infty} \langle u_c, \psi \rangle, \quad \psi \in C_0^\infty(\mathbb{R}),$$

where $C_0^\infty(\mathbb{R})$ is a dense subset of L^2 and $\langle \cdot\ , \cdot \rangle$ denotes the inner product in $L^2(\mathbb{R})$.

$$\begin{aligned}
\lim_{c\to\infty} \langle u_c(x,t), \psi(x) \rangle &= \lim_{c\to\infty} \int_{-\infty}^{\infty} u_c(x,t)\psi(x)dx \\
&= \lim_{c\to\infty} \int_{-\infty}^{\infty} \phi_c(x - ct)\psi(x)dx \\
&= \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \lambda\phi(\lambda x - \lambda^q t)\psi(x)dx \\
&= \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \phi(y)\psi\left(\frac{y}{\lambda} + \lambda^p t\right) dy
\end{aligned}$$

where $y = \lambda x - \lambda^q t$. Note that in the limit as $\lambda \to \infty$, the support of ψ expands. To deal with this situation, assume that $\frac{y}{\lambda} + \lambda^p t > 0$ and that the $\text{supp}(\psi) \in (-1,1)$. Then

$$\int_{-\infty}^{\infty} \phi(y)\psi\left(\frac{y}{\lambda} + \lambda^p t\right) dy =$$

$$\int\limits_{|y|>N} \phi(y)\psi\left(\frac{y}{\lambda}+\lambda^p t\right)dy + \int\limits_{|y|<N} \phi(y)\psi\left(\frac{y}{\lambda}+\lambda^p t\right)dy.$$

Observe that for any arbitrary $\zeta > 0$, $N = N(\zeta)$ can be chosen large enough such that

$$\int\limits_{|y|>N} |\phi(y)|dy < \frac{\zeta}{|\psi|_\infty}.$$

Then

$$\int\limits_{|y|>N} \phi(y)\psi\left(\frac{y}{\lambda}+\lambda^p t\right)dy < |\psi|_\infty \frac{\zeta}{|\zeta|_\infty} = \zeta.$$

Consequently,

$$\int_{-\infty}^{\infty} \phi(y)\psi\left(\frac{y}{\lambda}+\lambda^p t\right)dy = \int\limits_{|y|<N} \phi(y)\psi\left(\frac{y}{\lambda}+\lambda^p t\right)dy.$$

To demonstrate that the right hand side can be made arbitrarily small in the limit as $\lambda \to 0$, we use the following estimate

$$\begin{aligned} -1 &< \frac{y}{\lambda} + \lambda^p t < 1 \\ -1 - \lambda^p t &< \frac{y}{\lambda} < 1 - \lambda^p t \\ -\lambda - \lambda^q t &< y < \lambda - \lambda^q t. \end{aligned}$$

Note that the support of $\psi\left(\frac{\cdot}{\lambda}+\lambda^p t\right)$ should lie outside the interval $[-N, N]$. Thus we need

$$-\lambda^p t + \lambda < -N.$$

Note further that for any $t > 0$, $t\lambda^q - \lambda$ is an increasing polynomial of λ. Hence, the above inequality is achievable by choosing λ large enough. Consequently,

$$\lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \phi(y)\psi\left(\frac{y}{\lambda}+\lambda^p t\right)dy = 0.$$

Therefore,

$$u_\varepsilon(x,t) \rightharpoonup 0 \quad \text{in} \quad L^2(\mathbb{R}). \tag{5.11}$$

□

We now prove the main result that if the Cauchy problem of the KdV type equation (1) with initial data $u_0(x) = \kappa\delta(x)$ has a solution in the Sobolev space H^{s_ε}, $s_\varepsilon = -1/2 - \varepsilon$, then it does not depend continuously on its initial data.

Theorem 5.6. *The Cauchy problem for the KdV type equations (5.1), (5.2) is ill-posed in the Sobolev space* H^{s_ε}, $s_\varepsilon = -1/2 - \varepsilon$.

Proof. By Proposition 5.4, $u_c(x, 0)$ converges strongly to the data $\kappa\delta(x)$, where κ is given as in Proposition 5.3. By equation (5.9), the H^{s_ε} norm of $u_c(x, t)$ converges to the H^{s_ε} norm of $\kappa\delta(x)$. However, Proposition 5.5 explicitly showed that $u_c(x, t)$ converges weakly to zero. Hence, $u_c(x, t)$ cannot converge strongly to the solution of the system (5.1) in the Sobolev space H^{s_ε}, $s_\varepsilon = -1/2 - \varepsilon$. □

Chapter 6

Summary and Conclusion

The main contributions of this thesis are the establishment of non-unique solutions for the shallow water system and also on the local ill posedness of KdV type equations in negative Sobolev spaces.

The weak asymptotic method is used to find singular solutions of the shallow water system which can contain Dirac-δ distributions. The solutions so constructed are allowed to be complex-valued. The complex part then vanishes in the distributional limit, thus, the complex-valued approximations become real-valued in the distributional limit and this extends the range of possible singular solutions. The method is applied to the shallow water system both in one and two space dimensions. It is shown how this method can be used to construct solutions which contain combinations of classical hyperbolic shock waves and Dirac delta distributions.

It is also shown that the Cauchy problem for KdV type equations is locally ill posed in negative Sobolev spaces H^{s_ε}, $s_\varepsilon = -1/2 - \varepsilon$ where $\varepsilon > 0$ is arbitrary. The goal is to show that if the KdV type equations have a solution, then it does not depend continuously on its data in H^{s_ε}. The initial data is taken to be the Dirac delta function. This is achieved by constructing a sequence which converges strongly to the data in H^{s_ε} and then proved that the corresponding sequence of solutions does not converge strongly in H^{s_ε}.

Appendix

Definition Let $\Omega \neq \emptyset$. The couple $(\Omega, \mathcal{M})$ is said to be a *measurable space* if $\mathcal{M}$ is a σ-algebra in Ω.

Remark The members of $\mathcal{M}$ are called *measurable sets*.

Definition A *measure* μ on a measurable space $(\Omega, \mathcal{M})$ is a function $\mu :\longrightarrow [0, \infty]$ such that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

for any countable collection of disjoint measurable sets $\{E_i\}_{i=1}^{\infty}$ and $\mu(E) < \infty$ for at least one $E \in \mathcal{M}$.

Definition A *measure space* $(\Omega, \mathcal{M}, \mu)$ is a measurable space which has a measure defined on its σ-algebra.

Remark $\mathcal{M}$ is a σ-algebra in Ω if the following properties hold

- $\Omega \in \mathcal{M}$,
- if $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$, where E^c is the complement of E,
- if $E_i \in \mathcal{M}$ for $i = 1, 2, \cdots$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

Theorem .1 (Lebesgue Dominated Convergence Theorem)**.** *Let $(\Omega, \mathcal{M}, \mu)$ be a measure space and suppose that $\{f_n\}$ is a sequence of measurable functions on Ω such that the pointwise limit*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

holds a.e. on Ω for a measurable function f. If there exists a non-negative function $g \in L^1(\mu)$ that dominates f in the sense that

$$|f_n(x)| \leq g(x) \quad a.e. \quad on \quad \Omega \quad \forall n,$$

then $f \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int_{\Omega} f_n d\mu = \int_{\Omega} f d\mu.$$

The existence of dominating function $g \in L^1(\mu)$ such that $|f_n(x)| \leq g(x)$ is an integral part of this theorem. This theorem would be weakened without this condition and the limit expression in the theorem would fail. To demonstrate the power of this condition, suppose that $g \in L^1(\mu)$ is omitted from the theorem and let $\Omega = \mathbb{R}$. Define the sequence f_n by

$$f_n(x) = \begin{cases} \infty, & x \in \left[0, \frac{1}{n}\right], \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_n \to 0$ as $n \to \infty$ and

$$\int_\Omega f_n d\mu = \infty \quad \text{for all} \quad n = 1, 2, \cdots$$

but

$$\int_\Omega 0 \; d\mu = 0$$

which contradicts the limit formula in the theorem.

The prove of the Dominated Convergence Theorem shall be based on the monotone convergence theorem and the Fatou's lemma. For the purpose of our discussion they shall only be stated.

Theorem .2 (Monotone Convergence Theorem). *Let $\{f_n\}$ be a sequence of measurable functions on the measure space $(\Omega, \mathcal{M}, \mu)$. Suppose that $0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq \infty$ for every $x \in \Omega$ and define*

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \forall x \in \Omega.$$

Then f is measurable on Ω and

$$\int_\Omega f d\mu = \lim_{n\to\infty} \int_\Omega f_n d\mu.$$

Lemma .3 (Fatou's Lemma). *Let $(\Omega, \mathcal{M}, \mu)$ be a measure space and suppose that $f_n : \Omega \longrightarrow [0, \infty]$ is measurable for each positive integer n, then*

$$\int_\Omega \left(\liminf_{n\to\infty} f_n\right) d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n d\mu.$$

Proof of the Lebesgue Dominated Convergence Theorem

Let $h_n = 2g - |f_n - f|, n = 1, 2, \cdots$ and note that $h_n \geq 0$ and $\liminf h_n = 2g$. Since $|f| \leq g$ and f is measurable, $f \in L^1(\mu)$.

It follows from the Fatou's lemma and the fact that $(g - f_n) \geq 0, \forall n$ that

$$\begin{aligned} \int_\Omega g d\mu - \int_\Omega f d\mu = \int_\Omega (g - f) d\mu &\leq \liminf \int_\Omega (g - f_n) d\mu \\ &= \int_\Omega g d\mu - \limsup \int_\Omega f_n d\mu \end{aligned}$$

and

$$\begin{aligned}\int_\Omega g d\mu + \int_\Omega f d\mu = \int_\Omega (g+f) d\mu &\leq \liminf \int_\Omega (g+f_n) d\mu \\ &= \int_\Omega g d\mu + \liminf \int_\Omega f_n d\mu.\end{aligned}$$

Note that the identity $\liminf(-g) = -\limsup(g)$ and the linearity of integration were used. Therefore, the two expressions above lead to

$$\limsup_{n\to\infty} \int_\Omega f_n d\mu \leq \int_\Omega f_n d\mu \leq \int_\Omega f d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n d\mu.$$

□

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