Tensor Induction As Left Kan Extension

Kaythi Aye

The Department of Mathematics
University of Bergen
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1 Introduction

A function between sets can be extended by many different ways! If A, B and C are sets and A is non-empty, B \hookrightarrow C, then a function \( f : B \rightarrow A \), can be extended as \( f' : C \rightarrow A \), by many different ways. But there is not a canonical or unique way. Besides, if A, B and C are even groups or Rings or Modules, \( f \) can be extended as many different functions. But it is not same in Category theory, if we have a functor \( T : M \rightarrow A \), and M is subcategory of C and all colimits and limits exist in A, there is ways to find two canonical extension functors from M to functors L, \( R : C \rightarrow A \). These extensions functors are called Left Kan Extension functor L and Right Kan Extension R. I am going to study here in my thesis the category which is all colimits exist and the Left Kan Extension of \( T \) along the inclusion functor will be found later in the chapter 2.

In the processes of constructing the left kan extension L, some tools are necessary to use. I have found the co-equalizer, co-product, bi-product diagrams in the category \( \text{Mod}_R \) as my tools. After I define our functor L as co-equalizer digram, the universal property of co-equalizer diagram gives beautifully the unique natural transformation between two functors T and L along the full and faithful functor M to C. which is necessary to prove L is the left kan extension.

Tensor product (\( \otimes \)) is though as another parallel functor with L in here. Tensor is bilinear as defining property but it is not a linear. As of Kan extension properties, another parallel functor is not an additive functor, tensor is not linear nor additive, we need to make a long proof to find the unique natural transformation between functors L.
and ⊗ by using universal property of co-equalizers. I could manage to prove that tensor product has a quality to use as a parallel functor of the left kan extension.

In the last part of chapter 2, the natural transformation γ between L and ⊗ is proved as a unique isomorphism. It becomes $L \cong \otimes$ and it shows that Tensor product is a kind of left kan extension.

In chapter (3), I introduce two category $C_G$ and $B^{OP}$, the category of the transitive G-set of finite group G and Category of finite G-sets. I construct these two categories with the maps between objects are composing three kinds of maps, the induction, restriction and transferring. I am going to use three kinds of functions when I need the finite g-sets to move between G’s subgroups. Then I prove that $C_G$ is full subcategory of $B^{OP}$. Being $C_G$ is full subcategory and the left Kan extension properties construct the left induction which is a functor category. This left induction functor category gives the connection between tensoring and the Grothendicks group representation.

End of chapter three I introduce the tensor induction with our categories $B^{OP}_H, B^{OP}_K, B_H$ and $B_K$. If we defining the $\text{Tens}^K_H$ to get well adjustment between the two Modules categories $\text{Mod}_R(B_H)$ and $\text{Mod}_R(B_K)$. It works and we get the commute diagram with $\text{Tens}^K_H$ as the left kan extension.

2 Tensor product

2.1 Tensor Product is not a additive functor

**Definition 2.1** (Tensor products of Rmodules, $\otimes$). : Tensor product is bilinear maps. For any two Rmodules $M$ and $N$, there exist a pair $(T, g)$, Rmodules $T$ and Rmodule morphism $g : M \times N \rightarrow T$, with the following property: Given any module $P$ and bilinear $f : M \times N \rightarrow P$, there exists a unique morphism $f' : T \rightarrow P$ such that $f = f' \circ g$. Every $R$-bilinear map on $M \times N$ factors through $T$. Moreover, $(T, g)$ and $(T', g')$ are two pairs with this property, then there exists unique isomorphism $j : T \rightarrow T'$ such that $j \circ g = g'$.

The modules $T$ constructed above is called the tensor product of $M$ an $N$, and is denoted by $M \otimes_R N$. It is generated as an Rmodule by the products $x \otimes_R y$. The elements $x_i \otimes_R y_j$ generate $M \otimes_R N$ if $x_i$ and $y_j$ are families of generators of $M$ and $N$.

The tensor product is not an additive functor.

**Definition 2.2** (Additive functor). A functor $T$ from additive categories $U$ to $V$ with properties $T(f+g) = Tf + Tg$ for any parallel pair of arrows $f, g : u \rightarrow u'$ in $U$ and $T$ send zero object to zero object of $V$ and binary bi-product diagram in $U$ to a bi-product diagram in $V$.

**Lemma 2.3.** : Tensor product is not additive functor.

**Proof.** Tensor product is though as a functor as follow: $\otimes : \text{Hom}(A,A') \times \text{Hom}(B,B') \rightarrow \text{Hom} (A \otimes B, A' \otimes B')$. If we consider our categories $A, A', B, B' = R$, then $\otimes : \text{Hom}(R,R) \times \text{Hom}(R,R) \rightarrow \text{Hom} (R \otimes R, R \otimes R)$ is $R \times R \rightarrow R$ since $\text{Hom}(R,R) = R$ and $R \otimes R = R$. 

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Let $\otimes (a,b) = a \cdot b$ and $f(1) = a \neq 0$ and $g(1) = b \neq 0$, $(a,b) \in \mathbb{R} \times \mathbb{R}$ and $(f,g) \mapsto f \otimes g$, $f,g$ are morphisms in $\text{Hom}(\mathbb{R}, \mathbb{R})$. We consider $(1, 1)$ in $\mathbb{R} \times \mathbb{R}$, $(1,1) = (1,0) + (0,1)$. $(f \otimes g)(1,1) = \otimes [f(1), g(1)] = \otimes (a,b) = ab \neq 0$. It is bilinear. But $(f \otimes g)[(1,0) + (0,1)] = a \cdot 0 + b \cdot 0 = 0$ and $(f \otimes g)[(1,1)] \neq (f \otimes g)[(0,1) + (1,0)]$.

Tensor product does not have the property as additive functor. So, Tensor product is not an additive functor.

2.2 Left Kan extension

In this chapter we are going to study about the left kan extension of the following diagram:

Let $L : \text{Mod}_\mathbb{R} \times \text{Mod}_\mathbb{R} \to \text{Mod}_\mathbb{R}$ be a functor together with a natural transformation $\eta : T \to L(F_\mathbb{R} \times F_\mathbb{R})$. I am going to prove that the functor $\otimes$ together with the natural transformation $\beta : T \to \otimes \circ F_\mathbb{R} \times F_\mathbb{R}$ is a Left Kan extension of $T$ along $F_\mathbb{R} \times F_\mathbb{R}$. Let $\gamma : L \to \otimes$ is natural transformation.

I am proving that the $\gamma$ such that $\beta = \gamma K \circ \eta$ is an unique natural transformation. That is as follow:

$$\text{Nat}(L, \otimes) \cong \text{Nat}(T, \otimes \circ F_\mathbb{R} \times F_\mathbb{R})$$

$$\gamma \mapsto (\gamma F_\mathbb{R} \times F_\mathbb{R} \circ \eta) = \beta.$$

**Definition 2.4 (Left Kan extension).** Let $T : M \to A$ and $K : M \to C$ be functors. In the diagram,

$$\begin{array}{ccc}
M & \xrightarrow{T} & A \\
\downarrow K & & \\
C & &
\end{array}$$

the left Kan extension of $T$ along $K$ is a functor $L_K T : C \to A$ together with a natural transformation $\eta : T \to L_K T K$ with the following properties: given any functor $S : C \to A$ together with the natural transformation $\beta : T \to SK$, there exist a unique natural transformation $\gamma : L_K T \to S$ such that $\beta = \gamma K \circ \eta$.

$$\text{Nat}(L, S) \cong \text{Nat}(T, S \circ K)$$

$$\gamma \mapsto (\gamma K \circ \eta) = \beta$$

is bijection.
We illustrate the concept at a left Kan extension in the following diagrams category and functors:

Given two functors $T : M \to \text{Mod}_R$ and $K : M \to C$, then the left Kan extension $L_KT = L$ of $T$ along $K$ exists and $L : C \to \text{Mod}_R$ is characterized by a universal property.

\[
\begin{array}{ccc}
M & \xrightarrow{T} & A = \text{Mod}_R \\
\downarrow{K} & & \downarrow{\mathcal{L}} \\
C & \xrightarrow{L} & S
\end{array}
\]

Natural transformations $\eta : T \to L_K$, $\beta : T \to S_K$, and $\gamma : L \to S$ with $\beta = \gamma_K \circ \eta$ give the diagram.

Now I want to explain a notation $\gamma_K$ which I am going to use.

**Definition 2.5.** $\gamma_K$: $\gamma$ is the natural transformation defined as above. $\gamma_K$ is the morphism $\gamma_c : L(c) \to S(c)$ for each object $c$ of $C$ such that $c=Km$. $\gamma_{Km} : L(c=Km) \to S(c=Km)$. Note that $L(Km)=(LK)(m)$ and $S(Km)=(SK)(m)$. The morphisms $\gamma_{Km}$ for $m$ in $M$ is a natural transformation from $L_K$ to $S_K$, which we call $\gamma_K$. Let $\alpha : m \to m'$ be a morphism in $M$ and the diagram

\[
\begin{array}{ccc}
L(c) & \xrightarrow{\gamma_{Km}} & S(c) \\
\downarrow{L_K(\alpha)} & & \downarrow{S_K(\alpha)} \\
L(c') & \xrightarrow{\gamma_{Km'}} & S(c')
\end{array}
\]

commutes because $\gamma$ is the unique natural transformation $\gamma : L(c) \to S(c)$ for all $c \in C$. So, $\gamma_K$ is natural too.

**Lemma 2.6.** If $[L, \eta : T \to LK]$ and $[L', \eta' : T \to L'K]$ are left Kan Extensions, then there exists a unique isomorphism $\gamma : L \to L'$ with $\eta' = \gamma_K \circ \eta$.

**Proof.** By the definition property of left Kan extension unique natural transformations $\gamma : L \to L'$ and $\gamma' : L' \to L$ with $\eta' = \gamma_K \circ \eta$ and $\eta = \gamma' K \circ \eta'$.

Now $\gamma' \circ \gamma : L \to L$ is a natural transformation, with $(\gamma' \circ \gamma)K \circ \eta = \gamma'K \circ \gamma K \circ \eta = \gamma'K \circ \eta' = \eta$ as a natural transformation $\eta : T \to LK$. Also $id_L : L \to L$ is a natural transformation with $\eta = (id_LK) \circ \eta$, so by uniqueness in the defining property of Left Kan extensions we have that $\gamma' \circ \gamma = id_L$. 

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Similarly, \((\gamma \circ \gamma')K \circ \eta' = \gamma K \circ \eta = \eta'\) as a natural transformation \(\eta' : T \to L'K\). \(id_{L'} : L' \to L'\) is natural transformation with \(\eta' = (id_{L'} K) \circ \eta'\). Again uniqueness of natural transformation gives \(\gamma \circ \gamma' = id_{L'}\).

\(\gamma' \circ \gamma = id_L\) and \(\gamma \circ \gamma' = id_{L'}\) give that \(\gamma\) and \(\gamma'\) are bijections and one of them is the inverse of the other.

\(\gamma\) is isomorphism. \(\square\)

2.3 All colimit exist in \(\text{Mod}_R\), then a left Kan extension of \(T\) along \(K\) exists.

\(\text{Mod}_R\) is a cocomplete category by the Theorem 3.13 of the research paper named "Limits, colimits and how to calculate them in the category of modules over a PID" by KAIRUI WANG. The theorem states that:

**Theorem 2.7 (Theorem 3.13).** Cocompleteness Theorem.: A category \(C\) is cocomplete if and only if the coproduct of any set of objects in \(C\) exists and the coequalizer between any two morphisms with the same source and target exists.

**Definition 2.8 (Cocomplete category).** A cocomplete category is a category where colimits over diagrams \(F\) with a small source category \(J\) exist. \(F\) is an object of the category of functors \(C^J\), \(J\) is a small category.

**Definition 2.9 (Coequalizer).**

\[
\begin{array}{ccc}
a & \xrightarrow{f} & b \\ \downarrow{g} & & \downarrow{u} \\ h & \swarrow{\exists h'} & e \\
& h & \downarrow{c}
\end{array}
\]

Given in a category a pair of maps \(f\) and \(g\) with the same domain \(a\) and codomain \(b\), a coequalizer of \([^f, g]\) is a pair \((u,e)\) of a morphism \(u : b \to e\) and codomain \(e\) such that (1) \(uf=ug\) (2) if \(h : b \to c\) has \(hf=hg\) then \(h = h'u\) for a unique \(h' : e \to c\).

**Definition 2.10 (A map of co-equalizer diagrams).** A map of co-equalizer diagrams is a diagram of the form:

\[
\begin{array}{ccc}
a & \xrightarrow{f} & b \\ \downarrow{g} & & \downarrow{u} \\ \alpha & \beta & \gamma \\
\end{array}
\]

\[
\begin{array}{ccc}
a' & \xrightarrow{f'} & b' \\ \downarrow{g'} & & \downarrow{u'} \\ \alpha' & \beta' & \gamma' \\
\end{array}
\]

So that the rows are co-equalizer diagrams and

\[\beta f = f'\alpha, \beta g = g'\alpha\quad \text{and} \quad \gamma u = u'\alpha.\]
Lemma 2.11. If in a map of co-equalizers diagrams (1), the maps $\alpha$ and $\beta$ are isomorphisms, then $\gamma$ is an isomorphism.

Proof.

Given diagram, maps $f$ and $g$ are such that: $uf = ug$ and if $h : b \to c$ has $hf = hg$ then $h = h'u$ for a unique $h' : e \to c$.

$h$ is the surjective map and the maps $f', g'$ and $u'$ are such that: $u'f' = u'g'$ and if $j : b' \to c'$ has $jf' = jg'$ then $j = j'u'$ for a unique $j' : e' \to c'$. We get the diagram below:

If $\alpha$ and $\beta$ are isomorphisms, $\gamma$ must be a isomorphism because in this diagram, we know that $\beta f = f'\alpha$, $\beta g = g'\alpha$ and $\gamma u = u'\alpha$.

$h = h'u$ and $j = j'u'$, then $c \cong c'$.

Definition 2.12 (co-product diagram).

is a coproduct diagram. $i_1$ and $i_2$ are injectives. If there exists $d, f : a \to d$ and $g : b \to d$, then there always exists unique $\mu$ such that $f = \mu \circ i_1$ and $g = \mu \circ i_2$. 8
2.3.1 Proposition

Given diagram of the form,

\[
M \xrightarrow{T} A = \text{Mod}_R \xrightarrow{K} C
\]

a left Kan extension of \( T \) along \( K \) exists. The functor \( L : C \to A \) and natural transformation \( \eta : T \to LK \) can be constructed as follows: For \( c \), an object of \( C \), the value \( L(c) \) of \( L \) of \( c \) is given by the coequalizer of the diagram

\[
\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) \xrightarrow{\begin{array}{cc} a \\ b \end{array}} \bigoplus_{Km \to c} Tm
\]

where the upper map \( a \) takes an element

\[
x = (\alpha : m_1 \to m_0, f : Km_0 \to c, t \in Tm_1)
\]

to the element \( (f \circ K\alpha, t) \) of \( \bigoplus_{Km \to c} Tm \), and the lower map \( b \) takes \( x \) to the element \( (f, T(\alpha)(t)) \) of \( \bigoplus_{Km \to c} Tm \). The natural transformation \( \eta : T \to LK \) takes and element \( t \) of \( Tm \) to the element in the co-equalizer \( LKm \) represented by the element

\[
[id : Km \to Km, t \in Tm] \text{ of } \bigoplus_{Km_0 \to c} (Tm_0).
\]

**Proof.** First we define \( L \). Given an object \( c \) of \( C \), let \( Lc \) be the co-equalizer described in the statement of the proposition. Given \( h : c \to c' \),

\[
\begin{array}{ccc}
\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) & \xrightarrow{\begin{array}{cc} a \\ b \end{array}} & \bigoplus_{Km \to c} Tm \\
& \xrightarrow{\begin{array}{cc} c \\ d \end{array}} & \bigoplus_{Km \to c'} Tm \\
\downarrow_{h_0} & & \downarrow_{h_0} \\
\bigoplus_{Km_0 \to c'} \bigoplus_{m_1 \to m_0} (Tm_1) & \xrightarrow{\begin{array}{cc} \mu \\ \nu \end{array}} & \bigoplus_{Km \to c'} Tm \\
& \xrightarrow{\begin{array}{cc} \exists Lh \\ \theta \end{array}} & Lc'
\end{array}
\]

we define \( Lh : Lc \to Lc' \) as follow;

an element \( x = (\alpha : m_1 \to m_0, f : Km_0 \to c, t \in Tm_1) \) of \( \bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) \)

will be sent to \( x' \) by composing with \( h \)

\[
x' = (\alpha : m_1 \to m_0, h \circ f : Km_0 \to c \to c', t \in Tm_1)
\]
And the map \( c \) send \( x' \) to \((h \circ f \circ K_\alpha, t)\) in \( \bigoplus_{Km \to c'} Tm \).

The map \( a \) sent \( x \) to 
\((f \circ K_\alpha, t)\) in \( (\bigoplus_{Km \to c} Tm) \) and it is sent to \((h \circ f \circ K_\alpha, t)\) in \( \bigoplus_{Km \to c'} Tm \).

So,
\[(h \circ a)(x) = (c \circ h)(x)\]

We get the commute diagram for the upper maps \( a \) and \( c \). For maps \( b \) and \( d \).

an element \( x = (\alpha : m_1 \to m_0, f : Km_0 \to c, t \in Tm_1) \) of \( \bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) \)

will be sent to \( x' \) by composing with \( h \)

\[x' = (\alpha : m_1 \to m_0, h \circ f : Km_0 \to c \to c', t \in Tm_1) \text{ in } \bigoplus_{Km_0 \to c'} \bigoplus_{m_1 \to m_0} (Tm_1)\]

and the map \( d \) send \( x' \) to \((h \circ f, T(\alpha)(t))\) in \( \bigoplus_{Km \to c'} Tm \).

the map \( b \) sent \( x \) to

\[(f, T(\alpha)(t)) \text{ in } (\bigoplus_{Km \to c} Tm) \text{ and it is sent to } (h \circ f, T(\alpha)(t)) \text{ in } \bigoplus_{Km \to c'} Tm.\]

So we get commute diagram for both of the pairs of maps \( a \) and \( c \) and \( b \) and \( d \). It gives the commuted diagram below and the defined properties of \( Lc \) gives the unique morphism \( Lh \) from \( Lc \) to \( Lc' \) which gives the commute diagram as \((Lh \circ \mu)(t) = (\theta \circ h)(t)\), for all follow \( t \in (\bigoplus_{Km \to c} Tm) \).

\( L \) is defined for all map \( h \) in \( C \).

We are going to show that \( L \) is a functor \( L : C \to \text{Mod}_R \) and \( \eta \) is a natural transformation. We have proved that \( Lh \) exist in \( \text{Mod}_R \) for all \( h \) in \( C \). In \( C \), there exists \( Id_c : c \to c \) in \( C \). Composing with \( Id_c \) to \( x \) and get the commute diagram below and get \( Id_{Lc} \).

\[
\begin{array}{ccc}
\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) & \xrightarrow{\alpha} & \bigoplus_{Km \to c} Tm \\
\downarrow{id} & & \downarrow{id} \\
\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) & \xrightarrow{\alpha} & \bigoplus_{Km \to c} Tm \\
\end{array}
\]

\[
\begin{array}{ccc}
\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) & \longrightarrow & Lc \\
\downarrow{\Xi} & & \downarrow{\Xi} \\
\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) & \longrightarrow & Lc \\
\end{array}
\]

\( \Xi \) is the unique morphism \( Lh \) from \( Lc \) to \( Lc' \) which gives the commute diagram as \((Lh \circ \mu)(t) = (\theta \circ h)(t)\), for all follow \( t \in (\bigoplus_{Km \to c} Tm) \).

\( \Xi \) is the unique morphism \( Lh \) from \( Lc \) to \( Lc' \) which gives the commute diagram as \((Lh \circ \mu)(t) = (\theta \circ h)(t)\), for all follow \( t \in (\bigoplus_{Km \to c} Tm) \).

\[
\begin{array}{ccc}
\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) & \xrightarrow{\alpha} & \bigoplus_{Km \to c} Tm \\
\downarrow{id} & & \downarrow{id} \\
\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) & \xrightarrow{\alpha} & \bigoplus_{Km \to c} Tm \\
\end{array}
\]

\[
\begin{array}{ccc}
\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) & \longrightarrow & Lc \\
\downarrow{\Xi} & & \downarrow{\Xi} \\
\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) & \longrightarrow & Lc \\
\end{array}
\]

\( \Xi \) is the unique morphism \( Lh \) from \( Lc \) to \( Lc' \) which gives the commute diagram as \((Lh \circ \mu)(t) = (\theta \circ h)(t)\), for all follow \( t \in (\bigoplus_{Km \to c} Tm) \).
$Id_{Lc} = L(Id_c)$ exists.

If $g : c' \to c''$ in $C$, $g \circ h : c \to c''$ will induced a unique map $Lg \circ Lh : Lc \to Lc''$ as follow:

$$
\begin{align*}
\bigoplus_{K_{m_0 \to c}} \bigoplus_{m_1 \to m_0} (T_{m_1}) & \xrightarrow{a/b} (\bigoplus_{K_{m \to c}} T_m) \to Lc \\
\bigoplus_{K_{m_0 \to c'}} \bigoplus_{m_1 \to m_0} (T_{m_1}) & \xrightarrow{c/d} (\bigoplus_{K_{m \to c'}} T_m) \to Lc'
\end{align*}

\begin{align*}
\bigoplus_{K_{m_0 \to c''}} \bigoplus_{m_1 \to m_0} (T_{m_1}) & \xrightarrow{u/v} (\bigoplus_{K_{m \to c''}} T_m) \to Lc''
\end{align*}

$$

$x = (\alpha : m_1 \to m_0, f : K_{m_0} \to c, t \in T_{m_1})$ of $\bigoplus_{K_{m_0 \to c}} \bigoplus_{m_1 \to m_0} (T_{m_1})$

is sent same as above by map $a, b, c$ and $d$. Again, sent $x'$ to $x''$ by composing with $g$.

$$
x'' = (\alpha : m_1 \to m_0, g \circ h \circ f : K_{m_0} \to c'', t \in T_{m_1}) \quad \text{in} \quad \bigoplus_{K_{m_0 \to c''}} \bigoplus_{m_1 \to m_0} (T_{m_1}).
$$

We get

$$(g \circ h \circ a)(x) = (u \circ g \circ h)(x) = (g \circ h \circ f \circ K\alpha, t)$$

and

$$(g \circ h \circ b)(x) = (v \circ g \circ h)(x) = (g \circ h \circ f, T(\alpha)(t))$$

and the commute diagrams with the unique map

$Lh : Lc' \to Lc''$.

Again, we consider map $L(g \circ h)$, we get

$$
\begin{align*}
\bigoplus_{K_{m_0 \to c'}} \bigoplus_{m_1 \to m_0} (T_{m_1}) & \xrightarrow{a/b} (\bigoplus_{K_{m \to c'}} T_m) \to Lc \\
\bigoplus_{K_{m_0 \to c'''}} \bigoplus_{m_1 \to m_0} (T_{m_1}) & \xrightarrow{u/v} (\bigoplus_{K_{m \to c'''}} T_m) \to Lc''
\end{align*}

$$

This diagram works same way and get the same equations above,

$$(g \circ h \circ a)(x) = (u \circ g \circ h)(x) = (g \circ h \circ f \circ K\alpha, t)$$
and 
\[(g \circ h \circ b)(x) = (v \circ g \circ h)(x) = (g \circ h \circ f, T(\alpha)(t))\]

So,
\[L(g \circ h) = Lg \circ Lh.\]

L is a functor.

Then we are going to show that \(\eta\) is natural transformation. The morphisms 
\[\eta_m : Tm \rightarrow LKm\]
is such that:
\[t \mapsto L(id_{Km})(\eta_m t) = \eta_m t.\]

and for any \(t \in TM\) and morphism \(f\),
\[(f : Km \rightarrow c, t) \mapsto (L(f \circ K\alpha)(\eta_m t)).\]

Then the diagrams
\[
\begin{array}{ccc}
Tm_1 & \xrightarrow{T\alpha} & Tm_0 \\
\downarrow^{\eta_m_1} & & \downarrow^{\eta_m_0} \\
L(Km_1) & \xrightarrow{LK\alpha} & LKm_0 \\
& \xrightarrow{Lf} & Lc
\end{array}
\]

Any element \(t\) in \(Tm_1\) is sent by map \((LK\alpha \circ \eta_m_1)\)
\[(f : Km \rightarrow c, t) \mapsto (L(f \circ K\alpha)(\eta_m t))\]

\(t\) is sent by map \((T\alpha \circ \eta_m_0)\)
\[(f : Km \rightarrow c, t) \mapsto (L(f)\eta_m_0(T(\alpha)t))\]

\[(L(f \circ K\alpha)(\eta_m t)) = (L(f)\eta_m_0(T(\alpha)t)).\]

It makes the previous diagram commute. And \(\eta\) is natural.

Let S is the another functor \(C \rightarrow \text{Mod}_R\) together with \(\beta : T \rightarrow SK\). I am going to prove that there is a unique natural transformation \(\gamma : L \rightarrow S\) such that \(\beta = \gamma K \circ \eta\),

\[
\begin{array}{ccc}
M & \xrightarrow{T} & A = \text{Mod}_R \\
\downarrow^{K} & & \downarrow^{L} \\
C & \xrightarrow{S} & S
\end{array}
\]
The morphisms $\beta_m : T_m \to SK_m$ induces a morphism $\bigoplus_{K_m \to c} T_m \to Sc$

\[
[f : K_m \to c, t \in T_m] \mapsto S(f)(\beta_m t).
\]

And it gives a commute diagramas

\[
\begin{array}{ccc}
T_m_1 & \xrightarrow{T_\alpha} & T_m_0 \\
\downarrow{\beta_m} & & \downarrow{\beta_m} \\
S(K_m_1) & \xrightarrow{SK_\alpha} & SK_m_0 \\
& \xrightarrow{Sf} & Sc
\end{array}
\]

Any element $t$ in $T_m_1$ is sent by map $(SK_\alpha \circ \beta_m_1)$

\[
(f : K_m \to c, t) \mapsto (S(f \circ K_\alpha)(\beta_m_1 t))
\]

t is sent by map $(T_\alpha \circ \beta_m_0)$

\[
((f : K_m \to c, t)) \mapsto (S(f)\beta_m_0(T_\alpha t))
\]

\[
(S(f \circ K_\alpha)(\beta_m_1 t)) = (S(f)\beta_m_0(T_\alpha t)).
\]

It gives a commute diagram and the map $\phi$ as follow;

\[
\begin{array}{ccc}
\bigoplus_{K_m_0 \to c} \bigoplus_{m_1 \to m_0} (T_m_1) & \xrightarrow{a} & (\bigoplus_{K_m \to c} T_m) \\
\downarrow{b} & & \downarrow{\phi} \\
& Sc &
\end{array}
\]

By universal property of coequalizer , we get a uniquely determined morphism $\gamma_c : Lc \to Sc$

\[
\bigoplus_{K_m_0 \to c} \bigoplus_{m_1 \to m_0} (T_m_1) \xrightarrow{a} (\bigoplus_{K_m \to c} T_m) \xrightarrow{\psi} Lc
\]

for any modules $c$ in $C$ such that $\phi = \gamma_c \circ \psi$. Then we get the unique natural transformation $\gamma : L \to S$, $\forall c \in C$. It holds for any free module of finite set $m$ in $M$, so we get $\gamma K_m : LK_m \to SK_m$. We have defined $\beta_m : T_m \to SK_m$ which gives $\phi$ in above co-equalizerby composing with $f : K_m \to c$ and $\eta_m : T_m \to LK_m$ which gives $\psi$ in above co-equalizer by composing with $f : K_m \to c$. The composite of $\gamma K_m$ and $\eta_m$ is

\[
\beta_m = \gamma K_m \circ \eta_m : T_m \to SK_m, \forall m \in M.
\]
We can express it as
\[
\beta = \gamma K \circ \eta.
\]
and the diagram is,

\[
\begin{array}{ccc}
T & \xrightarrow{\eta} & LK \\
\downarrow{\beta} & & \downarrow{\gamma K} \\
& SK &
\end{array}
\]

So defining functor L as coequqlizer and unique natural transformation γ as above make the L is left Kan extension.

Definition 2.13. \( R[\text{fin}] \) is the category with finite sets as objects and the hom set in \( R[\text{fin}](X, Y) \) is \( R \)-modules generated by maps between two finite sets \( X \) and \( Y \).

\[
\sum_i a_i f_i, a_i \in R, f_i \in \text{hom}(X, Y).
\]

Definition 2.14 (Full subcategory). We say that \( S \) is a full subcategory of \( C \) when the inclusion functor \( T : S \to C \) is full. If every function \( T_{(c,c')} : \text{hom}(c,c') \to \text{hom}(Tc, Tc') \), for all pair \( (c, c') \) of \( C \), is surjective, \( T \) is full.

Definition 2.15. Let \( X \) and \( Y \) are finite sets. \( F_R \) is a full embedding functor which makes a finite set to a free \( R \) modules.

\[
F_R X = \bigoplus_{x \in X} R.
\]

Every map of \( R[\text{fin}](X, Y) \) is sent the map in \( \text{map}(X, F_R Y) \) as follow:

\[
R[\text{fin}](X, Y) \to \text{map}(X, F_R Y)
\]

\[
(\sum_i a_i f_i) \mapsto (x \mapsto \sum_i a_i f_i(x)),
\]

and every map \( \alpha \) in \( (X_0, F_R Y) \) will send to a map in \( \text{Mod}_R(F_R X, F_R Y) \) as follow:

\[
\text{map}(X, F_R Y) \to \text{Mod}_R(F_R X, F_R Y).
\]

\[
(\alpha : X \to F_R Y) \mapsto [(\sum_i \lambda_i x_i) \mapsto \sum_i \lambda_i \alpha(x_i)].
\]

In this chapter I am going to prove that \( R[\text{Fin}] \) is full subcategory of \( \text{Mod}_R \) by using the left kan extension as co-equalizer.
Definition 2.16 (-X-). \( R[\text{fin}] \times R[\text{fin}] \to R[\text{fin}] \), -X- is a functor which makes pair of two finite sets to a Cartesian product of two finite sets.

\[ (X, Y) \mapsto X \times Y \]

and morphisms

\[ (\sum a_i f_i, \sum b_j g_j) \mapsto \sum_{i,j} a_i b_j (f_i, g_j). \]

Definition 2.17 (T and \( \eta \)). \( T \) is a functor of composition of two functors \( F_R \circ - \times - \),

\[ T(X, Y) = \bigoplus_{X \times Y} R, \]

with a natural transformation

\[ \eta : T \to \bigodot_R F_R \times F_R. \]

\[ \bigoplus_{X \times Y} R \to F_R X \bigodot F_R Y \]

\[ \sum_{(x, y)} c_{(x, y)}(x, y) \mapsto \sum_{(x, y)} c_{(x, y)}(x \otimes y). \]

Theorem 2.18. In the diagram (1) if there is a functor \( \bigotimes : \text{Mod}_R \times \text{Mod}_R \to \text{Mod}_R \) and an natural transformations \( \beta : T \to \bigotimes \circ F_R \times F_R \), then there exist a unique natural isomorphism \( \gamma : L \to \bigotimes \) such that \( \beta = (\gamma F_R \times F_R \circ \eta) : T \to S. \)

Lemma 2.19. There is a natural isomorphism

\[ LM \cong M \]
Proof. Let \( \alpha : LM \to M \)

such that: we have co-equalizer diagram

\[
\bigoplus_{FX_0 \to M} \bigoplus_{X_1 \to X_0} (FX_1) \xrightarrow{a} \bigoplus_{FX \to M} FX \xrightarrow{\psi} LM
\]

an element \( x = (g : X_1 \to X_0, f : FX_0 \to M, t \in FX_1) \)

will be sent to \( x' \) by map \( a \)

\[
x' = (f, F(g)(t)) \quad \text{of} \quad \bigoplus_{FX \to M} FX
\]

an element \( x = (g : X_1 \to X_0, f : FX_0 \to M, t \in FX_1) \)

will be sent to \( x'' \) by map \( b \)

\[
x'' = (f \circ Fg, t) \quad \text{of} \quad \bigoplus_{FX \to M} FX.
\]

An element

\[
y = (f : Km_0 \to M, t \in FX) \quad \text{of} \quad \bigoplus_{FX \to M} FX
\]

will be sent to \( (f(t)) \) in \( M \) by map \( \phi \) in diagram below.

In above diagram \( \psi \) is surjective and we get the unique \( \alpha \) according to the universal properties of Co-equalizer. It is factor out the map \( \phi \) such that \( \phi = \alpha \circ \psi \), \( (\alpha \circ \psi)(y) = f(t) \).

Case 1. If \( M \) is a free \( \mathbb{R} \)modules : \( M = FY \), \( Y \) is a finite set.

Let

\[
\alpha : LM \to M
\]

\[
[f : FX \to M = FY, t \in FX] \mapsto f(t),
\]
and
\[ \beta : M \to LM \]
\[ m \mapsto (id : FY \to M, m \in FY) , \]
we consider
\[ \alpha(\beta(m)) = \alpha(id : FY \to M = FY, m \in FY) = id(m) = m \]
\[ \bigoplus_{X \in X_0} (\sum_{X_1 \to X_0} (FX_1)) \xrightarrow{\psi} \bigoplus_{FX \to M} \bigoplus_{X \in X_0} (FX_1) \xrightarrow{f=\phi} (\bigoplus_{FX \to M} \bigoplus_{X \in X_0} (FX_1)) \]
\[ \xrightarrow{\beta} \bigoplus_{FX \to M} (M = FY) \xrightarrow{(\alpha \circ \beta) = id} (M = FY) \]

and
\[ \beta(\alpha(f : FX \to M = FY, t \in FX)) = \beta(f(t)) = (id : FY \to M, f(t) \in FY) \]

we know that, in LM,
\[ (f : FX \to M = FY, t \in FX)) = (id : FY \to M, f(t) \in FY) \]
because t \in FX will be sent to f(t) \in FY by f and f(t) \in M=FY will be to itself by id_{FY}. Both elements are in the same equivalence class of LM. So, we have prove LM \cong M for M, any finitely generated FREE module.

Case 2. If
\[ M = \bigoplus_{x \in X} R, \]
for X is an infinite set. Let
\[ \alpha_M : LM \to M \]
\[ [f : FX \to M, t \in FX] \mapsto f(t) . \]
Let any m \in M, m = \sum m_x[x], only finitely many m_x are not zero. Let Y = [ x \in X/m_x \neq 0 ]. So we can express m as m = \sum \lambda_y[y].

L is left adjoint functor. Then we get the diagram below,
\[ \begin{array}{ccc}
LM & \xrightarrow{=} & L(\bigoplus_{x \in X} R) \\
\downarrow{\alpha_M} & & \downarrow{\alpha_M} \\
M & \xrightarrow{=} & \bigoplus_{x \in X} R \\
\end{array} \xrightarrow{=} \begin{array}{ccc}
L(\bigoplus_{x \in X} R) & \xrightarrow{=} & \bigoplus_{x \in X} L(R) \\
\downarrow{\alpha_M} & & \downarrow{(\oplus \alpha_R) = \Xi} \\
\bigoplus_{x \in X} R & \xrightarrow{=} & \bigoplus_{x \in X} R.
\end{array} \]
We have $\bigoplus_{x \in X} L(R)$ is isomorphic to $\bigoplus_{x \in X} R$. Then we get

$$L(\bigoplus_{x \in X} R) \cong \bigoplus_{x \in X} L(R) \cong \bigoplus_{x \in X} R$$

and an isomorphism

$$\alpha_M : L(\bigoplus_{x \in X} R) \rightarrow \bigoplus_{x \in X} R.$$ 

$LM \cong M$.

Case 3. If $M$ is any $R$module: we can write $M$ as a co-equalizer of free $R$modules

$$K = \ker \left( \bigoplus_{m \in M} Rm \rightarrow M \right)$$

$$\sum_{m \in M} a_m [m] \mapsto \sum a_m m.$$

Get a surjective map

$$\bigoplus_{k \in K} Rk \rightarrow K$$

We have exact sequence

$$\bigoplus_{k \in K} Rk \xrightarrow{\beta} \bigoplus_{m \in M} Rm \rightarrow M \rightarrow 0$$

Thus,

$$\bigoplus_{k \in K} Rk \xrightarrow{0} \bigoplus_{m \in M} Rm \xrightarrow{\beta} M$$

(3)

is an coequalizer sequence. We get coequalizer sequence with $L$ too as $L$ is left adjont.

$$L(\bigoplus_{k \in K} Rk) \xrightarrow{L(0)} L(\bigoplus_{m \in M} Rm) \xrightarrow{L(\beta)} LM$$

(4)

As the lemma 2.11, these two co-equalizers diagram 3 and 4 have the same universal property of co-equalizer. We get commute diagram as follow:

$$\begin{align*}
L(\bigoplus_{k \in K} Rk) & \xrightarrow{L0} L(\bigoplus_{m \in M} Rm) \xrightarrow{LM} \\
\downarrow \alpha \circ Rk = \cong & \downarrow \alpha \circ Rm = \cong \downarrow \alpha M \\
\bigoplus_{k \in K} Rk & \xrightarrow{0} \bigoplus_{m \in M} Rm \xrightarrow{M}
\end{align*}$$
\( \alpha_M \) works same as above in case 2. It is an isomorphism. Therefore

\[ \alpha : LM \to M \]

is isomorphism for all modules \( M \in \text{Mod}_R \) and

\[ LM \cong M. \]

\[ \square \]

### 2.4 Defining \( L \), the left Kan extension and a co-equalizer

Let functors \(- \times -\), product of sets. \( \times (X,Y) = X \times Y \), \( F_R \) is a functor which makes free modules of finite sets, \( F_R(X \times Y) = \oplus_{X \times Y} R \) and \( L : \text{mod}_R \times \text{mod}_R \to \text{mod}_R \) be a co-equalizer functor of modules. \( T \) is a functor of composition of two functors \( F_R \circ - \times -\), \( T(X,Y) = \oplus_{X \times Y} R \).

In the diagram 5, Let \( L \) is a coequalizer such that:

\[
\begin{align*}
\bigoplus_{F_{X_1 \to M, F_{Y_1 \to N}} F(X_0 \times Y_0)} & F(X_0 \times Y_0) \\
\begin{array}{c}
\downarrow u \\
\bigoplus_{F_{X \to M, F_{Y \to N}} F(X \times Y)} F(X \times Y) \\
\downarrow \zeta \\
L(M, N)
\end{array}
\end{align*}
\]
Lemma 2.20. There is a coequalizer diagram in \( \Delta \) as follow:

\[
\begin{array}{ccc}
\bigoplus_{FX_1 \to M, FY_1 \to N} FX_0 \times FY_0 & \xrightarrow{u'} & \bigoplus_{FX \to M, FY \to N} FX \times FY \\
\downarrow & & \downarrow \\
M \times N & \xrightarrow{\zeta'} & M' \\
\end{array}
\]  \hspace{1cm} (7)

Proof. We have shown in 2.19 that LM is isomorphic to M and we have the co-equalizer diagram:

\[
\begin{array}{ccc}
\bigoplus_{FX_1 \to M, X_0 \to X_1} (FX_0) & \xrightarrow{a} & (\bigoplus_{FX \to M} FX) \\
\downarrow \quad \quad & & \downarrow \psi \\
M & \xrightarrow{h} & M' \\
\end{array}
\]  \hspace{1cm} (8)

\( h \) works

\[(h \circ a)(x) = (h \circ b)(x), \forall x \in \bigoplus_{FX_1 \to M, X_0 \to X_1} (FX_0), h = \xi \circ \psi. \]

The two maps work such that: en element \( x \) in \( FX_0 \) is send to different element in \( FX \) as follow:

\[x = (\alpha : FX_1 \to M, f : FX_0 \to FX_1, t \in FX_0) \mapsto (\alpha, f(t))\]

by map \( a \) and

\[x = (\alpha : FX_1 \to M, f : FX_0 \to FX_1, t \in FX_0) \mapsto (\alpha \circ f, t)\]

by map \( b \). But these two different elements in \( FX \) are sent to same elements \( f(t) \) of \( M \) by \( \psi \).

There is a co-equalizer in \( FY \) too such that:

\[
\begin{array}{ccc}
\bigoplus_{FY_1 \to N} \bigoplus_{FY_0 \to FY_1} (FY_0) & \xrightarrow{\delta'} & (\bigoplus_{FY \to N} FY) \\
\downarrow \quad \quad \quad & & \downarrow \psi' \\
N & \xrightarrow{h'} & N' \\
\end{array}
\]  \hspace{1cm} (9)
h’ works

\[(h' \circ a')(y) = (h' \circ b')(y), \forall y \in (\bigoplus_{FY_1 \to N} \bigoplus_{FY_0 \to Y_1} (FY_0)), h' = \xi' \circ \psi'.\]

\[y = (\beta : FY_1 \to N, g : FY_0 \to FY_1, s \in FY_0) \mapsto (\beta, g(s))\]

by map \(a'\) and

\[y = (\beta : FY_1 \to N, g : FY_0 \to FY_1, s \in FY_0) \mapsto (\beta \circ g, s)\]

by map \(b'\). But these two different elements in FY are sent to same elements \(g(s)\) of \(N\) by \(\psi'\).

The co-product of co-equalizer diagrams is a co-equalizer diagram.

In our category \(R[\text{fin}]\) there are objects which co-product of its object, finite set. So these coproduct objects will be the

coproduct of 8 and 9 gives the following equation

\[\bigoplus_{FX_1 \to M} \bigoplus_{FX_0 \to FX_1} (FX_0) \bigoplus_{FY_1 \to N} \bigoplus_{FY_0 \to FY_1} (FY_0)\]

\[\xymatrix{ & \bigoplus_{FX \to N} FX \oplus \bigoplus_{FY \to N} FY \ar[d]^{\psi \oplus \psi'} \ar[r]_{M \oplus N} & M \oplus N \ar[d]^{\psi \oplus \psi'} \ar[r]_{M' \oplus N'} & M' \oplus N'}\]

This is equal to

\[\bigoplus_{FX_1 \to M, FY_1 \to N} \bigoplus_{FX_0 \to FX_1, FY_0 \to FY_1} (FX_0 \times FY_0) \xrightarrow{u \times u'} \bigoplus_{FX \to M, FY \to N} (FX \times FY) \xrightarrow{\psi \times \psi'} (M \times N)\]

Since we have co-product diagram:

\[\xymatrix{ (\bigoplus_{FX \to M} FX) \ar[r]^{i_1} \ar[dr]^{\psi} & (\bigoplus_{FX \to M} FX) \bigoplus (\bigoplus_{FY \to N} FY) \ar[d]^{\psi} \ar[r]^{i_2} \ar[dl]^{h'} & (\bigoplus_{FY \to N} FY) \ar[dr]^{h} \ar[dl]^{\Xi \times \xi'} \ar[r]^{\psi} & M \oplus N \ar[r]^{\psi} & M' \oplus N'}\]
and get unique map $\xi \oplus \xi'$ such that $h = (\xi \oplus \xi') \circ \psi$ and $h' = (\xi \oplus \xi') \circ \psi'$, we get co-equalizer diagram:

$$
\bigoplus_{FX \to M, FY \to N} \bigoplus_{FX_0 \to FX, FY_0 \to FY}(FX_0 \times FY_0) \xrightarrow{u \times u'} \bigoplus_{FX \to M, FY \to N}(FX \times FY) \xrightarrow{\psi \times \psi'} (M \times N) \xrightarrow{h \circ h'} M' \times N'
$$

(10)

Lemma 2.21. The bilinear map

$$
\hat{\phi} : FX \times FY \to F(X \times Y)
$$

$$
\hat{\phi}(\sum_i \lambda_i x_i, \sum_j \mu_j x_j) = \sum_{i,j} \lambda_i \mu_j (x_i, y_j)
$$

induces a map of coequalizer diagrams and the map

$$
\phi : M \times N \to L(M, N).
$$

Proof. We have defined the co-equalizer diagram 6 as follow

$$
\bigoplus_{FX \to M, FY \to N} \bigoplus_{FX_0 \to FX, FY_0 \to FY}(FX_0 \times FY_0) \xrightarrow{u \times u'} \bigoplus_{FX \to M, FY \to N}(FX \times FY) \xrightarrow{\psi \times \psi'} (M \times N)
$$

and I have got a co-equalizer in the lemma 2.20

$$
\bigoplus_{FX \to M, FY \to N} \bigoplus_{FX_0 \to FX, FY_0 \to FY}(FX_0 \times FY_0) \xrightarrow{u \times u'} \bigoplus_{FX \to M, FY \to N}(FX \times FY) \xrightarrow{\psi \times \psi'} (M \times N)
$$

From the commute diagram of the two co-equalizer diagrams, get a map $\phi$ as follow:

$$
\bigoplus_{FX \to M, FY \to N} \bigoplus_{FX_0 \to FX, FY_0 \to FY}(FX_0 \times FY_0) \xrightarrow{u \times u'} \bigoplus_{FX \to M, FY \to N}(FX \times FY) \xrightarrow{\psi \times \psi'} (M \times N)
$$

$$
\bigoplus_{FX \to M, FY \to N} \bigoplus_{FX_0 \to FX, FY_0 \to FY} F(X_0 \times Y_0) \xrightarrow{a \times b} \bigoplus_{FX \to M, FY \to N} F(X \times Y) \xrightarrow{\zeta} L(M, N)
$$
In the diagram, ∀ finite set X and Y,

\[ \hat{\phi} : FX \times FY \to F(X \times Y) \]

\[ (\sum_i \lambda_i x_i, \sum_j \mu_j x_j) \mapsto \sum_{i,j} \lambda_i \mu_j (x_i, y_j). \]

**Proposition 2.22.** \( \phi \) is bilinear.

**Proof.** \( \phi \) inherit bilinearlity from the bilinear \( \hat{\phi} \) such that: Given \( x_0, x'_0 \) and \( y_0 \), choose \( x, x', y \) such that

\[ \psi(x) = x_0, \psi(x') = x'_0, \psi(y) = y_0. \]

We define \( \hat{\phi} \) is bilinear map, then

\[ \hat{\phi}(x + x', y) = \hat{\phi}(x, y) + \hat{\phi}(x', y) \]

\[ (\xi)((\hat{\phi})(x, y) + \hat{\phi}(x', y)) = (\xi)(\hat{\phi})(x, y) + (\xi)(\hat{\phi})(x', y) \ldots \ldots (\ast), \]

since \( \xi \) is bilinear too. In the above commute diagram

\[ (\xi)(\hat{\phi})(x, y) = (\phi)(\psi)(x), (\phi)(\psi')(y) = \phi(x_0, y_0). \]

In the \((\ast)\)

\[ (\xi)((\hat{\phi})(x, y) + \hat{\phi}(x', y)) = (\xi)(\hat{\phi})(x, y) + (\xi)(\hat{\phi})(x', y) = \phi(x_0, y_0) + \phi(x'_0, y_0). \]

We have

\[ (\xi)((\hat{\phi})(x, y) + \hat{\phi}(x', y)) = \phi((\psi(x) + \psi(x'), \psi'(y)) = \phi(x_0 + x_0, y_0) \]

\[ \phi(x_0 + x'_0, y_0) = \phi(x_0, y_0) + \phi(x'_0, y_0). \]

\( \phi \) is a bilinear map. \( \square \)

**Proposition 2.23.** There is a homomorphism \( \bar{\phi} : M \otimes_R N \to L(M, N) \)

**Proof:** Universal properties for defining tensor product(this is the unique natural morphism \( \gamma \)).

**Lemma 2.24.** The maps \( FX \to M \) and \( FY \to N \) induces a homomorphism \( \bar{\psi} : L(M, N) \to M \otimes_R N \)

23
proof:

\[
FX \otimes_R FY \xrightarrow{\theta=(\psi \otimes \psi')} M \otimes_R N
\]
\[
F(X \times Y) \xrightarrow{\cong} (FX \otimes_R FY)
\]
\[
(FX \otimes_R FY) \xrightarrow{\cong} (FX \otimes_R FY)
\]
\[
L(M \times N) \xrightarrow{\bar{\psi}} M \otimes_R N
\]

\[
\begin{array}{ccc}
M \otimes_R N & \xrightarrow{\theta} & L(M, N) \\
\downarrow{\phi} & \downarrow{\xi} & \downarrow{\bar{\psi}} \\
M \otimes_R N & & M \otimes_R N
\end{array}
\]

Lemma 2.25. \( \bar{\psi} \circ \bar{\phi} = id[M \otimes_R N] \)

Proof. Tensor is bi-linearity, so \( \theta \) and \( \xi \) are modules homomorphisms and conjugacy of upper horizontal maps give the identity map \( \bar{\psi} \circ \bar{\phi} \).

\[
\oplus_{FX \to M, FY \to N} F(X \times Y) \xrightarrow{\cong} \oplus_{FX \to M, FY \to N} FX \otimes_R FY \xrightarrow{\cong} \oplus_{FX \to M, FY \to N} F(X \times Y)
\]
\[
\begin{array}{ccc}
M \otimes_R N & \xrightarrow{\theta} & L(M, N) \\
\downarrow{\phi} & \downarrow{\xi} & \downarrow{\bar{\psi}} \\
M \otimes_R N & & M \otimes_R N
\end{array}
\]

Lemma 2.26. \( \bar{\phi} \circ \bar{\psi} = id[L(M, N)] \)

Proof.

\[
\oplus_{FX \to M, FY \to N} F(X \times Y) \xrightarrow{\cong} \oplus_{FX \to M, FY \to N} FX \otimes_R FY \xrightarrow{\cong} \oplus_{FX \to M, FY \to N} F(X \times Y)
\]
\[
\begin{array}{ccc}
L(M, N) & \xrightarrow{\bar{\psi}} & M \otimes_R N \\
\downarrow{\bar{\psi}} & \downarrow{\theta} & \downarrow{\bar{\phi}} \\
L(M, N) & & L(M, N)
\end{array}
\]

Tensor is bi-linearity, \( \theta \) and \( \xi \) are modules homomorphisms and conjugacy of upper horizontal maps give the identity composing \( \bar{\phi} \circ \bar{\psi} \).
Lemma 2.27. 
\[ \psi : L(M,N) \to M \otimes_R N \]
is a natural isomorphism.

Proof: Lemma 2.26 and 2.27 give that both \( \phi \) and \( \psi \) are natural isomorphism. And the unique natural morphism \( \gamma \) of diagram 1 is 
\[ \gamma = \psi : L(M,N) \to M \otimes_R N \]

Conclusion is our two functor are isomorphic.

\( L \cong \otimes \)

Theorem 2.28. Let \( R[\text{fin}] \) be the category of finitely generated free \( R \)-modules (2.13). Let \( F_R : R[\text{fin}] \to \text{Mod}_R \) be the full embedding from (??) and \( T : R[\text{fin}] \times R[\text{fin}] \to \text{Mod}_R \) is a composing of \( F_R \) and \( - \times - \), \( T(X,Y) = F_R(X \times Y) \), as (2.17). Let \( L \) be the left Kan Extension of \( T \) along \( F_R \times F_R \).

\[ L : \text{Mod}_R \times \text{Mod}_R \to \text{Mod}_R \]

with the natural transformation \( \eta : T \to L \circ F_R \times F_R \), and the another functor \( \otimes \) with the natural transformation \( \beta : T \to \otimes \circ F_R \times F_R \), then there exist a unique natural isomorphism 
\[ \gamma : L \to \otimes \]
such that 
\[ \beta = (\gamma F_R \times F_R \circ \eta) \]

3 Tensor induction

3.1 Constructing the category \( B^{op} \)

I am going to start with the category of G-sets. The category \( B^+ \) will be constructed with objects of the category of G-sets but maps are only some kinds of G-maps we need. Then I will get the \( B \) from \( B^+ \) by Grothendieck construction. It is an additive category. Then a contra-variant functor will give the category of \( B^{op} \) which I am going to study.

There are two different categories of “Mackey functors” but I use the original one defined by Dress.
Definition 3.1 (A Mackey functor). Mackey functor is an additive functor from an additive category $B^{op}$ to Ab category $\text{Mod}_R$. We work with Mackey functors over a commutative ring $R$. A Mackey functor over $R$ is a functor

$$M : B^{op} \to \text{Mod}_R$$

Definition 3.2 (Additive Category). Additive category is an Ab Category which has a zero object and a bi-product for each pair of its objects.

Definition 3.3 (Bi-product diagram). Bi-product diagram for the objects $a, b \in A$ is a diagram

$$
\begin{array}{c}
a \\ \downarrow p_1 \\
\circlearrowright c \\
\downarrow p_2 \\
b
\end{array}
\begin{array}{c}
a \\ i_1 \\
\downarrow p_1 \\
\bigcirc c \\
\downarrow p_2 \\
b
\end{array}
\begin{array}{c}
c \\ i_2 \\
\downarrow p_2 \\
\bigcirc b
\end{array}
$$

So that,

$$
\begin{array}{c}
a \leftarrow p_1 \\
\bigcirc c \\
b
\end{array}
$$

is a product diagram and

$$
\begin{array}{c}
a \\ i_1 \\
i_2 \leftarrow b
\end{array}
$$

is a co-product diagram since

$$p_1 i_1 = 1_a, p_2 i_2 = 1_b \quad \text{and} \quad i_1 p_1 + i_2 p_2 = 1_c$$

Definition 3.4 (G-set). Let $G$ be a finite group. A left $G$-set is a set and a group homomorphism

$$f : G \times X \to X,$$

$$(g, x) \mapsto gx \in X,$$

such that the following conditions hold:

1) if $g, h \in G$ and $x \in X$, then $g(h.x) = (gh)x$,

2) if $1_G$ is identity element of $G$ then $1_G.x = x$.

Definition 3.5 (G-equivariant map or G-map). If $G$ is a group and $X$ and $Y$ are left $G$-set, a morphism of $G$-sets from $X$ to $Y$ is a map $f : X \to Y$ such that $f(gx) = g f(x)$, for any $g \in G$ and $x \in X$. Such a map is called a $G$-equivariant map from $X$ to $Y$ and the set of such a map is denoted by $\text{Hom}_G(X, Y)$. 
$B^+$ is constructed from category of $G$-sets by taking all objects, $G$-sets, $a, b, c, d, \text{etc}$ and $G$-maps which are able to be written by the composition of induction, transfer and restriction maps in representation ring. For example: a map $f$ from $a$ to $b$ of $B^+$ we can describe such that

$$a \leftarrow f_1 \cdots c \overset{f_2}{\rightarrow} b$$

$f_1$ get from $f_1'$, a $G$ map from $c$ to $a$. $f_1$ is from $a$ to $c$ rather than $c$ to $a$. $f_1$ and $f_2$ are $G$ equavariance maps. $f_1$ the map with dotted arrow in $B^+$, correspond to induction maps with indentity or a transfer maps in the familier makey functors like representation ring and so are called transfer. $f_2$ induces the restriction maps and are called restrictions. The hom set of $B^+$ are commutative monoids (semi group with identity).

If two maps are determin the same map in $B^+$, then there is an inner isomorphism of $c$ and $d$ as shown in diagram:

$$
\begin{array}{c}
c \overset{f_2}{\rightarrow} b \\
\downarrow \sim \\
\downarrow \\
a \overset{g_2}{\leftarrow} d
\end{array}
$$

Composition of two maps $f$ and $g$ is

$$a \leftarrow f_1 \cdots c \overset{f_2}{\rightarrow} b \overset{g_1}{\leftarrow} e \overset{g_2}{\rightarrow} d$$

and this compositions of two maps are given by the following pullback diagram:

$$
\begin{array}{c}
h \overset{P_2}{\rightarrow} e \\
\downarrow \\
P_1 \\
\downarrow \\
c \overset{f_2}{\rightarrow} b
\end{array}
$$

We get the composition map $a$ to $d$ in $B^+$ is as follow:

$$a \leftarrow (gf)_1 \cdots h \overset{(gf)_2}{\rightarrow} d$$

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$g \circ f : a \rightarrow d$.

In $B^+$ zero set is the initial and terminal object. Disjoin union of sets, in the $B^+$, get from the direct sum and direct produce of each map as follow:

This is a pair of maps out of $a \sqcup b$ and it is coproduct diagram in $B^+$ and

the above is a product diagram in $B^+$

Category $B$ is obtained from with $B^+$. They have same objects, finite G-sets but hom set are free abelian group. An abelian monoid, set of homomorphism of $B^+$ is quotient by the subgroup generated by the elements of the form

$$[f \sqcup g] - [f] - [g],$$

f and g are g-maps in of $B^+$, $[f]$ denotes the isomorphism class of f and $f \sqcup g$ is disjoint union of f and g. So, objects of $B$ are finite G-sets same as $B^+$ objects and the morphisms of $B$ are formal differences of maps in $B^+$. That’s why hom sets in $B$ become abelian groups. There is an obvious functor from $B$ to it’s opposite category $B^{OP}$.

3.2 The category $C_G$

G is a finite group. I am going to construct the $C_G$ from category C(G) and C(G) is constructed from category $C$. $C$ is the same category $C$ in the book Biset Functors for Finite Groups of Serge Bouc. It is the biset category of finite groups. Objects of the $C$ are finite groups and morphism from finite groups G to H are

$$Hom_C(G, H) = B(H, G).$$
Definition 3.6. \(B(H, G)\), the Grothendieck group of the category \((H, G)\) bisets, is defined as the quotient of the free abelian group on the set of isomorphism classes of finite \((H,G)\)-bisets by the subgroup generated by the element of the form, 

\[ [X \sqcup Y] - [X] - [Y], \]

where \(X\) and \(Y\) are finite \((H, G)\)-bisets, \([X]\) is an isomorphism class of \(X\) and \(X \sqcup Y\) is disjoint union of \(X\) and \(Y\).

Definition 3.7. \((H,G)\)-biset If \(H\) and \(G\) are finite groups and \(X\) is \((H, G)\)-biset. \((H,G)\)-biset is a left \(H\)-set and right \(G\)-set, such that

\[ \forall h \in H, \forall x \in X, \forall g \in G, (h.x).g = h.(x.g)inX. \]

In \(C\), the every morphism between finite groups \(G\) to \(H\) can be factored as the composition of \(\text{Ind}^H_B \circ \text{Ind}^D_C \circ \text{Iso}(f) \circ \text{Def}^B_H \circ \text{Res}^C_B\). \(f\) is isomorphism from \(B/A\) to \(D/C\). \(B\) and \(D\) are sub groups of \(G\) and \(H\), \(A\) and \(C\) are normal subgroups of \(B\) and \(D\).

\[ \text{Hom}_C(G, H) = B(H, G) \]

We have fundamental bisets in \(C(G)\) which connected with the three types of maps we are having in the category \(C(G)\). Let \(H\) is a subgroup of \(G\).

1. \(G\) is an \((H,G)\)-biset for the actions given by left and right multiplication in \(G\) and it is denoted by \(\text{Res}^G_H\).
2. \(G\) is an \((G,H)\)-biset for the actions given by left and right multiplication in \(G\) and it is denoted by \(\text{Ind}^G_H\).
3. If \(f : B \to D\) is a group isomorphism, then the set \(D\) is an \((D,B)\)-biset, for the left action of \(D\) by multiplication, the right action of \(B\) given by taking image by \(f\), and then multiplying on the right in \(D\). It is denoted by \(\text{Iso}(f)\).

Category \(C(G)\) can be constructed from \(C\) by taking a fixed finite group \(G\). \(C(G)\) has objects the group \(G\) and its subgroups \(H, K, A, B, C, D\) ect. The morphisms in \(C(G)\) can be shown as composition of only three types of maps, induction map (\(\text{Ind}\)), inner isomorphism (\(\text{Iso}\)) and restriction map (\(\text{Res}\)). Any Map from \(H\) to \(K\) can be factored as \(\text{Ind}^K_H \circ \text{Iso}(f) \circ \text{Res}^H_K\), induction maps from subgroup \(D\) to \(K\) (\(\text{Ind}^K_D\)), inner isomorphisms from \(B\) to \(D\) (\(\text{Iso}(f)\)) and restriction maps from \(H\) to \(B\) (\(\text{Res}^H_B\)). For any objects of \(C(G)\) \(H\) and \(K\),

\[ \text{Hom}_{C(G)}(H, K) \subset \text{Hom}_C(H, K) \]

\(\text{Mod}_{C(G)}\) is not equivalent to the category of \(\text{Mod}_{BOP}\) due to the Theorem A of Mackey Functors and Bisets, Hambleton, Taylor and Williams.

Then \(C_G\) the category I aim, will be constructed from \(C(G)\) by a functor.

\[ F : C(G) \to C_G. \]

The paper of Hambleton, Taylor and Williams, I mention above, gives such a functor, where \(C_G\) is full subcategory of \(B^{OP}\) with objects \(G/H\) where \(H\) is a subgroup of \(G\).
3.3 $B^{OP}$ and $C_G$

Category $C_G$ is full subcategory of $B^{OP}$. All maps of $C_G$ is in $B^{OP}$ since maps are composition of Induction, inner Isomorphism and restriction maps. There is a functor

$$i : C_G \hookrightarrow B^{OP}.$$

**Definition 3.8** (Ab). $Ab$ is the category which whose objects are all small (additive) abelian groups and morphisms are all homomorphisms of abelian groups.

**Definition 3.9** (Left Induction $L\text{Ind}_{C_G}^{B^{OP}}$). The Left Induction functor is from the book named Biset Functors for Finite Groups by Serge Bouc.

Let the functor $i : C_G \hookrightarrow B^{OP}$,

$$G/H \mapsto iH = G/H$$

In the diagram

$$\begin{align*}
C_G & \xrightarrow{F} Ab \\
& \downarrow \\
B^{OP} & \xrightarrow{L}
\end{align*}$$

$L\text{Ind}_{C_G}^{B^{OP}}$ is a functor of Ab categories. It sends from $\text{Mod}_{C_G}$ to $\text{Mod}_{B^{OP}}$. Let $A$ is the Ab category.

$$\text{Mod}_{B^{OP}} = [B^{OP} \to A] \quad \text{and} \quad \text{Mod}_{C_G} = [C_G \to A]$$

$$L\text{Ind}_{C_G}^{B^{OP}} : \text{Mod}_{C_G} \to \text{Mod}_{B^{OP}}$$

$F$ is a functor in The functor category $\text{Mod}_{C_G}, F : C_G \to Ab$.

$$L\text{Ind}_{C_G}^{B^{OP}}(F)(iG/H) = L\text{Ind}_{C_G}^{B^{OP}}(F)(G/H)$$

$L\text{Ind}_{C_G}^{B^{OP}}$ works as follow,

$$L\text{Ind}_{C_G}^{B^{OP}}(F)(G/H) = \bigoplus_{K \in S} \text{Hom}_{B^{OP}}(iG/K, iG/H) \otimes F(K)/I.$$

$S$ is set of representative of objects of $C(G)$, set of subgroups of $G$. $I$ is the submodule generated by the elements

$$(u \circ \alpha) \otimes f - u \otimes F(\alpha)(f),$$

For any elements $J$ and $K$ of $S$, any morphism $\alpha \in \text{Hom}_{C_G}(G/J, G/K)$, any $f \in F(G/J)$, and any $u \in \text{Hom}_{B^{OP}}(iG/K, iG/H)$. $J$, $K$ and $H$ are subgroups of $G$.

$$i^* \circ L\text{Ind}_{C_G}^{B^{OP}} \text{ sends } F(G/H) \mapsto \bigoplus_{K \in S} \text{Hom}_{B^{OP}}(iG/K, iG/H) \otimes F(G/K)/I.$$
Let \( f \in F(G/H) \), then \( H \in S \) and \( \text{Hom}_{B^{\text{op}}}(G/H, G/H) \otimes F(G/H)/I \). So, \( f \mapsto [\text{id}_{G/H} \otimes f] \).

Any map \( v \in \text{Hom}_{B^{\text{op}}}, v : iG/K \to iG/J \), the map

\[
L\text{Ind}_{C_G}^{B^{\text{op}}} (F)(v) : L\text{Ind}_{C_G}^{B^{\text{op}}} (F)(iG/K) \to L\text{Ind}_{C_G}^{B^{\text{op}}} (F)(iG/J)
\]

is induced by composition on the left in \( B^{\text{op}} \).

**Theorem 3.10.** There is an equivalence of categories \( \text{Mod}_{B^{\text{op}}} \) to \( \text{Mod}_{C_G} \).

**Proof.** Let every objects of \( B^{\text{op}} \) is finite sum of objects of \( C_G \).

Claim 1. The functor \( i^* \) : \( \text{Mod}_{B^{\text{op}}} \to \text{Mod}_{C_G} \) is full and faithful.

Proof for claim 1,

\[
\text{Mod}_{B^{\text{op}}} = [B^{\text{op}} \to A],
\]

\( A \) is the \( \text{Ab} \) category, and

\[
\text{Mod}_{C_G} = [C_G \to A]
\]

Let functor \( i : C_G \hookrightarrow B^{\text{op}} \). Every object \( H \) in \( C_G \),

\[
i(G/H) = G/H \in \text{ob}(B^{\text{op}}).
\]

Let isomorphism \( f : G/B \to G/D \) in \( C_G \) \((D = gBg^{-1})\). \( i \) sent \( \text{Iso}(f) \) to

\[
G/D \xleftarrow{f} G/B \xrightarrow{id} G/B
\]

For \( \text{Ind}_{B}^{C_G} (D \subset K) \), \( \text{Ind}_{C_G} : G/K \to G/D \) will be sent

\[
G/K \xleftarrow{f_1} G/D \xrightarrow{id} G/D
\]

For \( \text{Res}_{B}^{H} (B \subset H) \), \( \text{Res}_{C_G} : G/B \to G/H \) will be sent to

\[
G/B \xleftarrow{id} G/B \xrightarrow{f_2} G/H
\]

\[
\text{Mod}_{B^{\text{op}}} = [B^{\text{op}} \to A],
\]

if we pre-compose \( i \) to any functor \( F' \) of \( \text{Mod}_{B^{\text{op}}} \), we will get the

\[
i \circ F' : C_G \to A
\]
$i^* : \text{Mod}_{B^{\text{op}}} \to \text{Mod}_{C_G}$.

If any pair of objects in $\text{Mod}_{B^{\text{op}}}$ are exist in $\text{Mod}_{C_G}$, every morphism between these objects will exist in $\text{Mod}_{C_G}$ too. So, $i^*$ is full and faithful.

Claim 2.

$i : C_G \hookrightarrow B^{\text{op}}$.

$i$ is a full and faithful functor from $C_G$ to $B^{\text{op}}$.

proof for claim 2.

Every object of $C_G$ are exist in $B^{\text{op}}$ since every object in $B^{\text{op}}$ is the finite sum objects of $C_G$. Any pair of objects of $C_G$ are exist in $B^{\text{op}}$ as I showed above. So, $i$ is a full and faithful functor from $C_G$ to $B^{\text{op}}$.

Let the functor $L\text{Ind}_{C_G}^{B^{\text{op}}} : \text{Mod}_{C_G} \to \text{Mod}_{B^{\text{op}}}$.

Both $B^{\text{op}}$ and $A$ are addictive categories. There exists Left Kan extension $L$ of $T$ along $i$. $L$ is an addictive functor and pair with the natural transformation $\epsilon : T \to Li$. It is a functor of the functor category $L\text{Ind}_{C_G}^{B^{\text{op}}}$. I give a short name

$i' = L\text{Ind}_{C_G}^{B^{\text{op}}}$.

According to the Corollary 3, Section X.3 of Categories for the Working Mathematician by Mac Lane, if the functor $i$ is full and faithful, then the universal arrow $\eta : T \to Li$ for Functor $L$ along $i$ is a Natural Isomorphism $\eta : T \cong Li$. But I know

$Li = i'T$ and $i^* i' = i'T$.

By the adjunction, there is a natural bijection map

$$(T \xrightarrow{\eta T} i^* i'T) \leftrightarrow (i'T \xrightarrow{i^*} i'T)$$

There exists $id \in [i'T \to i'T] \iff id \in [T \to i^* i'T]$

Then, get $i^* \circ i' \cong id_{\text{Mod}_{C_G}}$

On the other hand, the Theorem 1 of adjunction, chapter IV.1 of Saunders Mac Lane, gives a natural map

$$(i' \circ i^* L \xrightarrow{\epsilon L} L) \leftrightarrow (i^* L \xrightarrow{i^*} i^* L)$$

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There exists \( id \in [i^* L \to i^* L] \iff id \in [i' \circ i^* L \to L] \)

\[\begin{array}{ccc}
i^* \circ i' \circ i^* L & \xrightarrow{id} & i^* L \\
i^* \circ \eta & \downarrow & \\
i^* L & & \\
\end{array}\]

The two isomorphisms \( i^* \circ \eta \) and \( id \) are given that \( i^* \circ \epsilon \) is isomorphism in the naturally commute diagram. And the following proposition 3.11 gives that \( \epsilon \) is isomorphism for all \( L \) of \( \text{Mod}_{B^{op}} \).

\( i' \circ i^* L \xrightarrow{\epsilon} L \)

\( i' \circ i^* = \text{Id}_{\text{Mod}_{B^{op}}} \)

So, if every objects of \( B^{op} \) is finite sum of objectives of \( C_G \), then The functor \( i^*: \text{Mod}_{B^{op}} \to \text{Mod}_{C_G} \) is equivalence of categories.

**Proposition 3.11.** For every additive functor \( M: B^{op} \to A \), the natural map \( M(a \oplus b) \to M(a) \times M(b) \) is an isomorphism.

**Proof.** Image of disjoint union of \( \text{Gsets} \), \( a \bigcup b \) in \( B^{op} \) is \( M(a \bigcup b) \) in \( A \).

Claim. \( M(a \bigcup b) \) is isomorphic to \( M(a) \times M(b) \).

Due to definition of Additive functor 2.2, \( M \) send the bi-product diagram to a bi-product diagram in \( A \).

According to the Theorem 2 of the section VIII.2, Categories for working Mathematician of Mac Lane, for any two objects \( a \) and \( b \) in an \( \text{Ab category} \) \( A \), \( A \) has bi-product of them if and only if \( A \) has product of them.

According to the definitions of bi-product 3.3 and co-product 2.12,

\[
\begin{array}{ccc}
M(a) & \xrightarrow{i_1} & M(a \sqcup b) & \xleftarrow{i_2} & M(b) \\
\downarrow & \downarrow & \exists \alpha & \downarrow & \downarrow \\
M(a) \times M(b) & & & & \\
\end{array}
\]

there is the unique map between \( M(a \oplus b) \) and \( M(a) \times M(b) \) and the unique map \( \alpha \) should be an isomorphism since

\[\alpha \circ i_1 = i_1 \quad \text{and} \quad \alpha \circ i_2 = i_2 \]

\( M(a \oplus b) \cong M(a) \times M(b) \)

\( \square \)
3.4 Tensor induction of representations

Let $R$ be a commutative ring, then the tensor product $M \otimes_R N$ of two $R$-modules is itself an $R$-module (by functoriality). This allows us to iterate the tensor product construction. In particular, we can consider

$$\bigotimes_{x \in X} M = M \otimes_R M \otimes_R M \otimes_R \ldots \otimes_R M$$

This construction can also be considered as a left Kan extension: $F$ is functor for making free modules and $L(M) = \bigotimes_{x \in X} M \in \text{Mod}_R$, $L$ is a functor of left Kan extension. Let a finite set $X$ is fixed. In the following diagram

$$\begin{array}{ccc}
R[\text{fin}] & \xrightarrow{\text{map}(X,-)} & R[\text{fin}] \\
\downarrow_{F} & & \downarrow_{F} \\
\text{Mod}_R & & \text{Mod}_R
\end{array}$$

$$\text{map}(X, -) : R[\text{fin}] \to R[\text{fin}]$$

$Y \mapsto \text{map}(X, Y)$. The functor $\text{map}(X, -)$ sends the maps $f_i : Y \to Y' \in R[\text{fin}]$ and $a_i \in R$,

$$\sum_{i=1, \ldots, n} a_i f_i \mapsto \phi = \left[ \sum_{i=1, \ldots, n} \left( \prod_{x \in X} a_i(x) f_i(x) \right) \right] \in R[\text{fin}].$$

Map

$$f_i : \text{map}(X, Y) \to \text{map}(X, Y')$$

is given by the formula $f_i(k)(x) = f_i(k)(x)$ for $k \in \text{map}(X, Y)$ and $x \in X$. We can show the previous diagram as a commute diagram as follow too,

$$\begin{array}{ccc}
Y \in R[\text{fin}] & \xrightarrow{\text{map}(X,-)} & R[\text{fin}] \ni \text{map}(X, Y) \\
\downarrow_{F} & & \downarrow_{F} \\
FY \in \text{Mod}_R & \xrightarrow{M \to \bigotimes_{x \in X} M} & \text{Mod}_R \ni F(\text{map}(X, Y))
\end{array}$$

The total number of maps in $\text{map}(X, Y)$ is $|Y|^{|X|}$ maps and when we make the free module

$$F(\text{map}(X, Y)) \cong \bigoplus_{f \in \text{map}(X, Y)} R = R^{|Y|^{|X|}}.$$

and $FY = \bigoplus_{y \in Y} R$. So,

$$\bigotimes_{x \in X} FY = \bigotimes_{x \in X} (\bigoplus_{y \in Y} R) = R^{|Y|^{|X|}}.$$
\[ F(map(X, Y)) \cong \otimes_{x \in X} FY. \]

We define the functor L, left induction functor,
\[ LM = \bigotimes_{x \in X} M = M \otimes_R M \otimes_R M \otimes_R \ldots \otimes_R M \]
as a co-equalizer
\[ \bigoplus_{FY_1 \to M} \bigoplus_{FY_0 \to FY_1} F(map(X, Y_0)) \xrightarrow{a} \bigoplus_{FY \to M} F(map(X, Y)) \xrightarrow{u} LM \]

In equation 11, the element
\[ x = (\alpha : FY_1 \to M, f : FY_0 \to FY_1, t \in map(X, Y_0)) \]
of \[ \bigoplus_{FY_1 \to M} \bigoplus_{FY_0 \to FY_1} F(map(X, Y_0)) \]
will be sent by map a to \((\alpha \circ f, t) \in \bigoplus_{FY \to M} F(map(X, Y))\) and it will be sent to an element \(Lm \in LM\) by \(u\). The element \(x\) will be sent by map b to \((\alpha, f(t)) \in \bigoplus_{FY \to M} F(map(X, Y))\) and it will be sent to the same element \(Lm \in LM\) by \(u\).

**Lemma 3.12.** There is a coequalizer diagram as follow:
\[ \bigoplus_{FY_1 \to M} \bigoplus_{FY_0 \to FY_1} map(X, FY_0) \xrightarrow{a'} \bigoplus_{FY \to M} map(X, FY) \xrightarrow{u'} map(X, M) \]

In equation 12
Two parallel morphisms \(a'\) and \(b'\) send en map to two different maps of \(\bigoplus_{FY \to M} map(X, FY)\) but coequalizer \(u'\) make both of them send to same maps in \(map(X, LM)\) in In equation 11.

Let the element
\[ x' = (\alpha : FY_1 \to M, f : FY_0 \to FY_1, a \in map(X, FY_0)) \]
of \[ \bigoplus_{FY_1 \to M} \bigoplus_{FY_0 \to FY_1} map(X, FY_0) \]
send by map \(a'\) to \((\alpha \circ f, a) \in \bigoplus_{FY \to M} map(X, FY)\) and then we get the map \((\alpha \circ f \circ a) \in map(X, M)\) by map \(u'\).

Let \(x'\) send by map \(b'\) to the \((\alpha, f(a) \in map(X, FY))\) and get \((\alpha \circ f \circ a) \in map(X, M)\) by \(u'\).

**Proof.** The proof for this lemma is the same with case of Lemma 2.20 if the the fix set \(X\) has the only two elements. If \(X\) has more than two elements we can use the induction method to prove it is right for all finite set \(X\). I will omit this detail proof here in my thesis. \(\square\)
Definition 3.13 (The tensor induction in the diagram). The formula for the tensor induction of representations. Let G and H be finite groups and let X be a left H, right G -set which is free as an H -set. F is functor of free modules. We define a functor map_H(X,−) from HR[fin] to GR[fin] taking an object Y to map_H(X,Y).

\[
\begin{array}{c}
HR[\text{fin}] \\ ^{\text{map}_H(X,-)} \longrightarrow \ \ G \ - \ Set \\
\downarrow F \\
R[\text{H}] \ - \ Mod \\
\end{array}
\xrightarrow{\text{Tens}_H^G}
\begin{array}{c}
G \ - \ Set \\
\downarrow F \\
R[\text{G}] \ - \ Mod \\
\end{array}
\]

It takes a H-morphism \( f = \sum_{i=1,...,n} a_i f_i, f_i : Y \rightarrow Y' \) and \( a_i \in R \), to

\[
\phi = [\sum_{\mathfrak{I}: H \backslash X \rightarrow [1,...,n]} \prod_{u \in H \backslash X} a_{\mathfrak{I}(u)} f_{\mathfrak{I}op}] \in GR[fin],
\]

where \( p : X \rightarrow H \backslash X \) is the projection and given \( J : X \rightarrow [1,...,n] \), the map

\[
f_J : \text{map}(X,Y) \rightarrow \text{map}(X,Y')
\]

is given by the formula \( f_J(k)(x) = f_J(x)(k(x)) \) for \( k \in \text{map}(X,Y) \) and \( x \in X \). It is straight forward to check that

\[
\phi(gk) = g\phi(k)
\]

for every \( g \in G \). Using that H acts freely on X, f is an H-morphism we can verify that if k is a H-map, then \( \phi(k) \in F(\text{map}_H(X,Y')) \). This means that we have a G-morphism \( \phi : \text{map}(X,Y') \rightarrow \text{map}(X,Y') \). We define

\[
\text{map}(X,-)(f) := \phi.
\]

Here in the diagram, \( \text{Tens}_H^G M \) is the tensor induction functor, \( R[G] \)-Mod is R module with an action of G. That is \( \bigotimes(M) = \text{Tens}_H^G M \in R[G] \)-Mod. There is an isomorphism of R-modules

\[
\text{Tens}_H^G M \cong \bigotimes_{G/H} M.
\]

3.5 Tensor induction with the category of \( B_G^{OP} \)

Let H and K are subgroups of G and \( H \subset K \). We construct two categories \( B_H^{OP} \) and \( B_K^{OP} \) from H and K. Then the Mackey functors give \( \text{Mod}_{R(B_H)} \) and \( \text{Mod}_{R(B_K)} \) and we can have \( \text{Tens}_H^K \) as a functor between two categories of modules.

\[
\begin{array}{c}
R(B_H^{OP}) \\
F_H \\
Mod_{R(B_H)}
\end{array}
\xrightarrow{\text{Tens}_H^K} 
\begin{array}{c}
R(B_K^{OP}) \\
F_K \\
\text{Mod}_{R(B_K)}
\end{array}
\]
in the diagram $\text{Mod}_{R(B^H)}$ is the category of the functors from $RB^H$ to $\text{Mod}_R$, and the functor $P^K_H = \text{map}(K, -)$. Let $X, X'$ and $Y$ are objects of $RB^P_H$, $P^K_H$ takes a map in $RB^P_H$

$$X \xleftarrow{f_1} c \xrightarrow{f_2} Y$$

where $X$, $Y$ and $c$ are H-set, to

$$\text{map}(K, X) \xleftarrow{f_1'} \text{map}(K, c) \xrightarrow{f_2'} \text{map}(K, Y)$$

$F_H$ takes the map $X$ to $X'$ of $R(B^P_H)$

$$X \xleftarrow{f_1''} c' \xrightarrow{f_2''} X'$$

to $RB_H(X, -)$. For any $Y$ in $ob(B^P_H)$, there is $RB_H(X, Y)$. The map $(X, X')$ in $RB^P_H$ induces $RB_H(X', Y)$ by Yoneda embedding lemma as follow:

$$X' \xleftarrow{g_1} c' \xrightarrow{g_2} X$$

in $RB_H$, and

$$X \xleftarrow{f_1} c \xrightarrow{f_2} Y$$

give by composing and having pull back

$$X' \xleftarrow{h_1} c \xrightarrow{h_2} Y$$

Yoneda embedding : $RB^P_H \rightarrow [\text{Mod}_{RB_H} = (RB_H, \text{Mod}_R)]$

Another functor $F_K$ is working same as $F_H$. If we define Tensor induction $\text{Tens}^K_H$ similar as previous section, we get the functor which makes commute the diagram 13.
References


