Differentiability of Products of Formal Power Series

by

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Master's Thesis in Mathematical Analysis

Fall 2020

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Acknowledgments

First and foremost, I would like to thank my excellent supervisor Alexander Schmeding for providing me with a very interesting project and for making the process of writing it so much fun. He has been a wonderful teacher and a great companion throughout our time working on this project. A special thanks to Steven Gray and Rafael Dahmen for all the excellent help and feedback I have received. It has been most valuable. I also want to thank the people in the administration at the University of Bergen for always being so helpful as well as all my previous teachers for inspiring me with their enthusiasm for mathematics. Last but not least, I would like to thank my family and friends for the support and companionship over all these years.

Contents

1 Introduction

Chen series are formal power series over a non-commutative alphabet of indeterminates whose coefficients are a type of iterated integrals. They are named after Kuo-Tsai Chen who in the papers [4] and [3] studied some of their algebraic properties. The first to realize that they had a connection with nonlinear system theory was Michel Fliess with his example in [8, (II.3) and (II.4)], where he considered a control problem whose output map could be represented by a type of functional series corresponding to the series studied by Chen. In the years following this work it has been shown that for certain control systems the input-output map $F_c\colon u \mapsto F_c[u]$ can be described in terms of these functional series, as first described by Fliess. They are now known as Chen-Fliess series and are parameterized by a non-commutative formal power series over an alphabet of indeterminates.

Specifically let X^* denote the monoid under concatenation of monomials over an alphabet of indeterminates $X = \{x_0, \dots x_m\}$ and let $c := \sum_{\eta \in X^*} (c, \eta)\eta$ be a formal power series in this alphabet, where (c, η) are real or complex coefficients. We denote by $L_p^m([0, T])$ the p'th Lebesgue space of m component mappings, all of whose components are elements of $L^p([0, T])$, where $p \in [1, \infty]$. Let $(u_i)_{i=1}^m$ be a sequence of $L^1([0, T])$ functions forming the input of the control system. Then the Chen-Fliess series (or Fliess operator) corresponding to the formal power series c is

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta) E_\eta[u](t), \tag{1}$$

where the real number $E_\eta[u](t)$ is for each $t \in [0, T]$ a type of iterated integral as studied by Chen. It was shown in [16] that with the assumptions

$$|(c, \eta)| \leq K M^{|\eta|} |\eta|! \tag{2}$$

for all $\eta \in X^*$ and $\max_{1 \leq i \leq m} \|u_i\|_{L_\infty} \leq R$, for some $R > 0$ depending on c, then the functional series (1) converges absolutely and uniformly for some $T > 0$. More generally, it was shown there that if each $u_i \in L^p([0, T]) \subset$

$L^1([0,T])$ then $F_c[u] \in B_q^m(R)$, where $B_q^m(R)$ denotes the norm ball of radius $R > 0$ in $L_q^m([0,T])$. Here p and q are conjugate exponents. Recently in [6] it was proven that the Fliess operators are jointly continuous with respect to their generating series as well as mappings from a norm ball in $L_p^m([0,T])$ to a norm ball in $L_q^m([0,T])$ again with p and q conjugate. The topology on the set of generating series for the Fliess operators was taken to be a so called Silva topology.

This thesis grew out of a research paper in preparation which in some respects continues the work in [6]. The papers main results are the continuity of certain products on the space of generating series. These products are induced ones in the sense that they originate from various ways of interconnecting Fliess operators. The interconnections arise in application, for example when a control system consists of several components whose input-output operators are interdependent in some fashion. Specifically, the interconnections relevant to this thesis are $F_c[u]F_d[u] = F_{c \sqcup\sqcup d}[u]$, $F_c[F_d[u]] = F_{c \circ d}[u]$ and $F_c[u + F_d[u]] = F_{c\,\tilde{\circ}\,d}[u]$ [17, Theorems 3.1 and 4.1], where $\sqcup\sqcup$, $\circ$ and $\tilde{\circ}$ are products on the space of generating series known respectively as the *shuffle*, *composition* and *modified composition* products. It is desirable that under the same Silva topology as constructed in [6], these induced products are continuous. Because this ensures for instance the continuity of the corresponding product of interconnecting Fliess operators.

This text will not consider Chen-Fliess series but instead focuses on Chen series and especially the aforementioned induced products. It is structured as follows:

In Chapter 2 we introduce the basic theory needed to understand the subsequent chapters. In particular we introduce the relevant theory of locally convex spaces, a calculus on these spaces as well as the basics of locally convex Lie groups and Lie algebras.

In Chapter 3 we shall introduce some notation, which will be used throughout the rest of the text, and consider the differentiability properties of Chen series.

In Chapter 4 we follow [6] in the construction of the Silva space of generating series corresponding to these Fliess operators. In addition, we prove a useful result regarding the complexification of the space of generating series with real coefficients.

In Chapter 5 we will discuss the induced products as well as a certain pre-Lie product. We will mainly be interested in showing that these are smooth (holomorphic) in the sense of the calculus introduced in Chapter 2. Some of these products, directly or indirectly, define inverses and hence unit groups which we will turn into locally convex Lie groups. Their associated Lie algebras and regularity properties are also then investigated.

In Chapter 6 we outline some possibilities for future work and include some results that are needed from the paper of which this thesis is affiliated.

2 Preliminaries

The purpose of this preliminary chapter is mainly to give a quick introduction to the relevant theory of locally convex vector spaces, which will be the underlying spaces we will work with, as well as a certain calculus on these spaces, which will allow us to talk about differentiability of maps between them. It has already been mentioned that some of the maps (products) we will consider defines groups which can be turned into so called locally convex Lie groups. Thus it is also necessary to introduce some infinite dimensional geometry. The prerequisite knowledge we assume of the reader is to be familiar with the basic theory of functional analysis and topology as can be found in for example the first three chapters of [22]. Some knowledge of finite dimensional differential geometry is useful for understanding the infinite dimensional one but is not needed.

The theory of locally convex spaces is vast and there are many good references on the subject. A relatively brief but thorough presentation is provided in [18, Chapters 22-25] and the more standard material in Chapter 2.1 will mostly follow this. The topic of infinite dimensional geometry is on the other hand fairly new. One reference is Hideki Omori's book [21] on infinite dimensional Lie groups in the setting of Banach spaces. For the more general setting of locally convex vector spaces, a good reference is Karl Herman Neebs lecture notes [19] which is what we will mainly follow in Chapter 2.3 and partly for Chapter 2.2. Throughout this text $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We will use the convention that $\mathbb{N} = \{1, 2, \dots\}$ and put $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

2.1 Locally Convex Spaces

Definition 3. *A topological vector space is a $\mathbb{K}$-vector space equipped with a Hausdorff topology turning addition and scalar multiplication into continuous operations.*

In particular the translations $\lambda_x \colon E \to E,\ y \mapsto x + y$ on a topological vector space E are continuous bijections with continuous inverses $\lambda_x^{-1}(y) =$

$y - x$, in other words homeomorphisms. This means that given any zero-neighbourhood $U \subset E$ and any point $x \in E$, $x + U$ is an x-neighbourhood in E. An x-neighbourhood basis in E is a collection of x-neighbourhoods $\{U_\alpha\}_{\alpha \in A}$ with the property that if $V \subset E$ is any x-neighbourhood then there is $\alpha \in A$ for which $U_\alpha \subset V$.

Definition 4. *A topological vector space is said to be a locally convex space if every zero-neighbourhood contains a convex zero-neighbourhood.*

By the remarks above any element of a locally convex space E has a basis consisting of convex neighbourhoods. Moreover any point in E has a neighbourhood basis consisting of absolutely convex neighbourhoods, where a subset $U \subset E$ is absolutely convex if $x, y \in U$ and $|\lambda| + |\mu| \leq 1$ with $\lambda, \mu \in \mathbb{K}$ implies $\lambda x + \mu y \in U$. This is in fact an if and only if [18, Lemma 22.2]. In the case that E is a normed linear space then any zero neighbourhood contains an ϵ-ball $B_\epsilon = \{x \in E : \|x\| \leq \epsilon\}$ which is an absolutely convex zero-neighbourhood. Thus locally convex spaces generalize normed linear spaces.

One can equivalently define locally convex spaces in terms of certain families of real-valued mappings known as seminorms.

Definition 5. *Let E be a $\mathbb{K}$-vector space. A mapping $p \colon E \to \mathbb{R}_{\geq 0}$ is called a seminorm if p satisfies the two relations*

1. $p(\lambda x) = |\lambda| p(x)$
2. $p(x + y) \leq p(x) + p(y)$

for all $\lambda \in \mathbb{K}$ and $x, y \in E$.

In a locally convex space any absolutely convex zero-neighbourhood U has a continuous seminorm associated to it called the Minkowski functional. It is defined as $\|\cdot\|_U \colon x \mapsto \|x\|_U := \inf_{t>0}\{t : x \in tU\}$ [18, Lemma 22.3]. Let $\{p_\alpha\}_{\alpha \in A}$ denote the collection of all Minkowski functionals corresponding to a basis of absolutely convex zero-neighbourhoods U_α and let $V_\alpha := \{x \in E : p_\alpha(x) < 1\}$. Then $\{V_\alpha\}_{\alpha\in A}$ forms what is known as a fundamental system of zero-neighbourhoods.

Definition 6. *A fundamental system of zero-neighbourhoods in a locally convex space E is a family of zero-neighbourhoods $\{V_\alpha\}_{\alpha\in A}$ such that given any zero-neighbourhood $V \subset E$ there is an $\epsilon > 0$ and $\alpha \in A$ such that $\epsilon V_\alpha \subset V$*

Thus any locally convex space has such a system. Moreover one can show that it satisfies the following two properties [18, Lemma 22.4]

1. For any non-zero $x \in E$, $p_\alpha(x) > 0$ for some $\alpha \in A$.

2. Given p_α, p_β there is $C > 0$ and p_γ such that $\max\{p_\alpha, p_\beta\} \leq C\, p_\gamma$.

Conversely any $\mathbb{K}$-vector space E which has a family of seminorms $\{p_\alpha\}_{\alpha\in A}$ that satisfies 1. and 2. above induces a unique locally convex topology on E, turning the family $\{p_\alpha\}_{\alpha\in A}$ into a fundamental system of seminorms [18, Lemma 22.5].

It is important to remark that as in the case for normed linear spaces, a linear map between locally convex spaces is continuous if and only if it is continuous at zero [18, Proposition 22.6]. Thus if $A\colon E \to F$ is a linear map, A is continuous if and only if for each zero neighbourhood $U \subset F$ the preimage $A^{-1}(U)$ is a zero neighbourhood in E. This will be useful when showing that certain topologies are identical.

Definition 7. *A subset $B \subset E$ of a locally convex space E is said to be bounded if for any zero-neighbourhood $V \subset E$ there is an $\epsilon > 0$ for which $\epsilon B \subset V$.*

Lemma 8. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a convergent sequence in a locally convex space E. Then $\{x_n\}_{n\in\mathbb{N}}$ forms a bounded set in E.*

Proof. Let U be any zero-neighbourhood in E and suppose the sequence converges to $x \in E$. The fact that E is a locally convex space allows us to find an absolutely convex zero-neighbourhood $V \subset U$. Since $\frac{1}{m}x \to 0$ as $m \to \infty$, there is $k \in \mathbb{N}$ such that $\frac{1}{k}x \in V$ or equivalently $x \in kV$. Moreover since kV is a neighbourhood of x, there is $N \in \mathbb{N}$ such that $x_n \in kV$ for all $n \geq N$ and consequently there is $\epsilon > 0$ such that $\epsilon x_n \in kV$ for all $n \in \mathbb{N}$. Then $\frac{\epsilon}{k}\{x_n\}_{n\in\mathbb{N}} \subset V \subset U$. □

Lemma 9. *Let E be a locally convex space with a fundamental system of seminorms $\{p_\alpha\}_{\alpha\in A}$. Then $B \subset E$ is bounded if and only if $\sup_{x\in B} p_\alpha(x) < \infty$ for every $\alpha \in A$.*

Proof. Suppose B is a bounded set in E. The continuity of each of the seminorms in question implies that $U_\alpha := p_\alpha^{-1}([0,\delta))$ is a zero-neighbourhood in E. Whence there is $\epsilon > 0$ such that $\epsilon B \subset U_\alpha$ and so $\sup_{x\in B} p_\alpha(x) \leq \frac{\delta}{\epsilon} < \infty$. Conversely if B is any subset of E such that $\sup_{x\in B} p_\alpha(x) < \infty$ for all p_α then B is bounded. For if V is any zero-neighbourhood in E, there is p_α such that $\delta U_\alpha \subset V$ for some $\delta > 0$. Since $\sup_{x\in B} p_\alpha(x) \leq K$ for some real $K > 0$ we get that $\frac{\delta}{K} B \subset \delta U_\alpha \subset V$. □

In the case that E is a normed linear space, a fundamental family of seminorms consist simply of the norm on E. Consequently the fundamental family of bounded sets has as sole element the closed norm ball $B_1 = \{x : \|x\| \leq 1\}$ and as per usual a set $A \subset E$ is bounded if and only if $\sup_{x\in A} \|x\| < \infty$.

Lemma 10. *Let $(E_i)_{i\in I}$ be a family of locally convex spaces. Then their direct product $E := \prod_{i\in I} E_i$, with componentwise vector space structure and the product topology, is again a locally convex space.*

Proof. It is a topological vector space because its topology is initial with respect to the projections $\pi_j \colon E \to E_j$, $(e_i)_{i\in I} \mapsto e_j$. Indeed it is then Hausdorff and moreover any mapping $f \colon F \to E$, from a locally convex space F, is continuous if and only if $\pi_j \circ f \colon F \to E_j$ is continuous for each $j \in I$. Let $j \in I$ be arbitrary and let A_E and A_{E_j} denote the addition on E

and E_j respectively. Then

$$\pi_j \circ A_E((c_i)_i, (d_i)_i) = c_j + d_j = A_{E_j} \circ (\pi_j, \pi_j)((c_i)_i, (d_i)_i)$$

Using that A_{E_j} and (π_j, π_j) are continuous we see that $\pi_j \circ A_E$ is continuous. As j is arbitrary we may conclude that A_E is continuous. The scalar multiplication $S_E \colon \mathbb{K} \times E \to E$ is also continuous because

$$\pi_j \circ S_E(\lambda, (c_i)_i) = \lambda\, c_j = S_{E_j} \circ (\mathrm{id}_{\mathbb{K}}, \pi_j)(\lambda, (c_i)_i),$$

and both S_{E_j} and $(\mathrm{id}_{\mathbb{K}}, \pi_j)$ are continuous. To see that it is a locally convex space we let U be an arbitrary open zero-neighbourhood in E. Again since the topology of E is initial to $(\pi_i)_{i \in I}$ there is a neighbourhood of zero $\bigcap_{1 \le j \le N_U} \pi_j^{-1}(U_j)$ contained in U. Since each E_j is a locally convex space we can find convex zero-neighbourhoods $B_j \subset U_j$ and since each $\pi_j^{-1}(B_j)$ is also convex and an intersection of convex sets is convex, $\bigcap_{1 \le j \le N_U} \pi_j^{-1}(B_j)$ is a convex zero-neighbourhood contained in U. □

Henceforth we always endow products of locally convex spaces with the structure of a locally convex space as described above. Now dually to the definition of a fundamental system of neighbourhoods we define fundamental systems of bounded sets as follows.

Definition 11. *A family $\{B_i\}_{i \in I}$ of bounded sets in a locally convex space is said to be a fundamental system of bounded sets if for any bounded $B \subset E$ there is $i \in I$ and $\epsilon > 0$ for which $B \subset \epsilon B_i$.*

Lemma 12. *Consider a finite product $E = \prod_{i=1}^{N} E_i$ of locally convex spaces. If the family $\mathcal{B}_i$ denotes a fundamental system of bounded sets in E_i then $\prod_{i=1}^{N} \mathcal{B}_i$ is a fundamental system bounded sets in E.*

Proof. If $B \subset E$ is bounded then also each $\pi_i(B) \subset E_i$ is bounded. This follows readily from the fact that the π_i are linear and continuous. Hence there is $B_i \in \mathcal{B}_i$ and $r > 0$ such that $\pi_i(B) \subset r\, B_i$ for $i = 1, 2, \ldots, N$ and consequently $B \subset r \prod_{i=1}^{N} B_i$. □

Definition 13. *Let X be a topological space and $U \subset X$ any subset. We say that U is sequentially open if whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in X converging to an element $x \in U$ then the sequence x_n is eventually in U. That is, there is $N \in \mathbb{N}$ such that $n > N$ implies $x_n \in U$. We say U is sequentially closed if whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in U converging to some $x \in X$, then $x \in U$.*

Definition 14. *A topological space X is said to be sequential if for every subset $U \subset X$ we have that U is open if and only if U is sequentially open.*

One could equivalently give the definition of a sequential space as a space where being sequentially closed is equivalent to being closed. Clearly every open (closed) subset of a topological space is sequentially open (closed). The converse however is in general not true.

Definition 15. *A locally convex space E is said to be sequentially complete if every Cauchy sequence in E converges in E.*

For the stronger notion of *completeness* one has to introduce so called nets and Cauchy nets. As these will never be of any use in the text it suffices for us to only introduce sequential completeness.

Lemma 16. *Let X be a sequential topological space and let $Y \subset X$ be an open subset. Then as a topological subspace, Y is sequential.*

Proof. Let $A \subset Y$ be a sequentially open subset of Y and suppose $\{x_n\}_{n\in\mathbb{N}}$ is any sequence in X converging to $x \in A$. Since Y is open in X, it is in particular sequentially open. Hence there is $N \in \mathbb{N}$ such that the sequence $\{x_n\}_{n\geq N}$ is in Y and of course still converging to $x \in A$. Since A is sequentially open in Y there is an $M \in \mathbb{N}$ such that $x_m \in A$ for all $m > M \geq N$, i.e. the sequence $\{x_n\} \subset X$ is eventually in A. This means that A is sequentially open in X and hence open in X. As Y is open in X, A is open in Y. □

Recall that a mapping of topological spaces $f\colon X \to Y$ is sequentially continuous if $x_n \to x$ implies $f(x_n) \to f(x)$.

Lemma 17. *Let X be a sequential topological space and consider $f\colon X \to Y$ where Y is a topological space. Then f is continuous if and only if f is sequentially continuous.*

Proof. Suppose f is continuous, let $x_n \to x$ in X and let $U \subset Y$ be any open set containing $f(x)$. Then $f^{-1}(U)$ is open in X and so there is $N \in \mathbb{N}$ such that $x_n \in f^{-1}(U)$ for all $n > N$. In particular $f(x_n) \in U$ for all $n > N$ and hence f is sequentially continuous. Conversely suppose f is sequentially continuous and let $U \subset Y$ be any open subset. Since X is sequential, $f^{-1}(U)$ is open if and only if it is sequentially open. So let $x_n \to x$ where $x \in f^{-1}(U)$. Since f is sequentially continuous, also $f(x_n) \to f(x)$ so that $\{f(x_n)\}_{n\in\mathbb{N}}$ is eventually in U which in turn means that x_n is eventually in $f^{-1}(U)$. In other words $f^{-1}(U)$ is open in X and so f is continuous. □

We will mainly be working with a special type of locally convex space called a *Silva space* (also known as *DFS*-spaces). Before stating its definition we need to first introduce inductive limits and imbedding spectres.

Definition 18. *A $\mathbb{K}$-vector space E together with a family of locally convex spaces $(E_i)_{i\in I}$ and linear maps $(j_i : E_i \to E)_{i\in I}$ is called an inductive system if*

$$\bigcup_{i\in I} j_i(E_i) = E$$

If there is a finest locally convex topology on E for which the maps $(j_i)_{i\in I}$ are all continuous, we call it the inductive topology of the system $(j_i : E_i \to E)_{i\in I}$.

We should remark that there always exist a finest topology on E for which all the maps $j_i\colon E_i \to E$ are continuous, namely the final topology with respect to this family. However, this topology may fail to be a Hausdorff and hence is not a locally convex topology.

The next lemma characterizes when the inductive topology of an inductive system exists. By E' and E^* we mean respectively the set of continuous linear functionals and the set of linear functionals on E. E' will be referred to as the dual of E and E^* its algebraic dual.

Lemma 19. *[18, Lemma 24.6] Let $(j_i\colon E_i \to E)_{i\in I}$ be an inductive system. Then the following are equivalent.*

1. *The inductive topology of the system exists*
2. *There is a locally convex topology on E for which $j_i : E_i \to E$ is continuous for all $i \in I$*
3. *For each non-zero $x \in E$ there is $y \in E^*$ such that $y(x) \neq 0$ and for all $i \in I$ we have $y \circ j_i \in E_i'$*

Lemma 20. *Suppose that the inductive topology of the system $(j_i\colon E_i \to E)_{i\in I}$ exists and denote it by τ. Then an absolutely convex set $V \subset E$ is a τ-zero neighbourhood if and only if $j_i^{-1}(V)$ is a zero-neighbourhood in E_i for all $i \in I$.*

Proof. By definition of τ, each j_i is continuous and thus $j_i^{-1}(V)$ is a zero-neighbourhood in E_i for each $i \in I$. Conversely suppose that $j_i^{-1}(V)$ is a zero-neighbourhood in E_i for each $i \in I$. A-priori the map $\|\cdot\|_V\colon E \to \mathbb{R}$ is not necessarily continuous but at least it is a seminorm because V is absolutely convex and contains zero. However $\|\cdot\|_V \circ j_i$ is continuous for each $i \in I$ as follows from the easily verified fact that $\|\cdot\|_V \circ j_i = \|\cdot\|_{j_i^{-1}(V)}$ which is the Minkowski functional of the absolutely convex zero-neighbourhood $j_i^{-1}(V)$.

Claim. *$\|\cdot\|_V$ is continuous :*

If this is true then $V = \|\cdot\|_V^{-1}([0,1])$, implying that V is a τ-zero neighbourhood. Consider the collection

$$\mathcal{P} = \{p \text{ seminorm on } E \,:\, p \circ j_i \text{ is continuous for each } i \in I\}$$

This is in fact a fundamental system of seminorms because firstly, $x \neq 0$ in E implies that there is an absolutely convex zero-neighbourhood in E, say U, which does not contain x. Then $\|x\|_U > 0$ and since $\|\cdot\|_U$ and each j_i are continuous, $\|\cdot\|_U \in \mathcal{P}$. Secondly the linearity of each j_i ensures that $p_1 + p_2$ is also in $\mathcal{P}$ and dominates the maximum of the two. Thus $\mathcal{P}$ induces a locally convex topology on E which we denote by t. By Lemma

19 the mappings $j_i \colon E_i \to (E, \tau)$ are continuous and thus the collection of all τ-continuous seminorms are elements of $\mathcal{P}$, so that $t \geq \tau$. Conversely using that the family $\mathcal{P}$ is a fundamental system of seminorms for (E, t) one can easily show that the linear maps j_i are continuous in (E, t). Then the definition of τ ensures that $\tau \geq t$. Whence $t = \tau$ and so $\|\cdot\|_V$ is continuous in (E, τ) (as it is an element of $\mathcal{P}$), proving the claim. □

Definition 21. *A countable inductive system $(j_k : E_k \to E)_{k\in\mathbb{N}}$ is called an imbedding spectre if the following two conditions hold for all integers k:*

1. *E_k is a linear subspace of E and j_k is the inclusion map*
2. *$E_k \subset E_{k+1}$ and the inclusions $i_k : E_k \to E_{k+1}$ are continuous*

If the inductive topology exists for an imbedding spectre then we call E the inductive limit of the system and we write $\operatorname{ind}_{k\to\infty} E_k = E$. In the case when the E_k are normed linear spaces and $i_k \colon E_k \to E_{k+1}$ are compact operators, the inductive topology of the imbedding spectre $(j_k \colon E_k \to E)_{k\in\mathbb{N}}$ will always exist, as is shown in [18, Lemma 25.18]. Consequently the next definition makes sense.

Definition 22. *A Silva space is the inductive limit of an imbedding spectre of Banach spaces in which the inclusion mappings $i_k : E_k \to E_{k+1}$ are compact operators.*

Silva spaces have many useful properties. For our purposes the most important ones are the following.

Proposition 23. *For a Silva space $E = \operatorname{ind}_{n\to\infty} E_n$ the following holds*

1. *E is complete [18, Proposition 25.19 1.] and in particular sequentially complete*
2. *The unit balls $\{B_n\}_{n\in\mathbb{N}}$ form a fundamental system of bounded sets in E [18, Proposition 25.19 2.]*

3. *Finitely many direct products of Silva spaces are again Silva spaces [27, Proposition 3]*

4. *A subset $A \subset E$ is closed if and only if it is sequentially closed [27, Proposition 6]. In other words E is sequential.*

From this we may deduce

Proposition 24. *Let $E = \mathrm{ind}_{k\to\infty} E_k$ be a Silva space, F a locally convex space and consider a mapping $f \colon E \to F$. Then f is continuous if and only if $f\big|_{E_k} \colon E_k \to F$ is continuous for each $k \in \mathbb{N}$.*

Proof. Suppose f is continuous. Then f is sequentially continuous by Lemma 17. Since E_k is a Banach space it is in particular sequential. Thus since $x_n \to x$ in E_k implies $f\big|_{E_k}(x_n) = f(x_n) \to f(x) = f\big|_{E_k}(x)$ we see that $f\big|_{E_k}$ is sequentially continuous, and hence continuous. Conversely let $x_n \to x$ in E. As every convergent sequence is bounded by Lemma 8 and the unit balls $\{B_m\}_{m\in\mathbb{N}}$ form a fundamental sequence of bounded sets in E by Proposition 23, $(x_n)_{n\in\mathbb{N}} \subset r\, B_M \subset E_M$ for some $M \in \mathbb{N}$ and $r > 0$. Then $f(x_n) = f\big|_{E_M}(x_n) \to f\big|_{E_M}(x) = f(x)$. Hence f is sequentially continuous and thus continuous by Proposition 23.4 and Lemma 17. □

Corollary 25. *Let $E = \mathrm{ind}_{k\to\infty} E_k$ be a Silva space and F any locally convex space. Then $f \colon E \times E \to F$ is continuous if and only if $f\big|_{E_n \times E_m} \colon E_n \times E_m \to F$ is continuous for all $n, m \in \mathbb{N}$.*

Proof. Follows by applying Lemma 12 with the preceding proof. □

It will be important to remark the following: Suppose $f \colon E \times E \to E$ is a mapping of Silva spaces and that for each pair $n, m \in \mathbb{N}$ there is $k(n,m) \in \mathbb{N}$ for which $h := f\big|_{E_n \times E_m}^{E_k} \colon E_n \times E_m \to E_k$ is well defined. Then since $j_i \colon E_i \to E$ is continuous for all $i \in \mathbb{N}$, continuity of h implies continuity of $f\big|_{E_n \times E_m} = j_k \circ h$ and by the preceding corollary this implies in turn the continuity of f. In conclusion, f is continuous if it induces continuous mappings of Banach spaces $E_n \times E_m \to E_k$.

Next we define and discuss the process of complexification of a locally convex space. In particular the complexification of inductive limits.

Definition 26. *Let E be an arbitrary real locally convex space. Consider the product $E \times E$. We define for $x, y \in \mathbb{R}$ and $u, v \in E$ the scalar multiplication $(x + iy)(u, v) := (xu - yv, xv + yv)$ where i is the imaginary unit. The multiplication is continuous and hence the product $E \times E$ becomes a complex locally convex space which we denote by $E_{\mathbb{C}}$. E will be identified with the closed real subspace $E \times \{0\}$.*

In the case that E is a real normed linear space, $E_{\mathbb{C}} = E \times E$ is a complex normed linear space with norm $\|(x, y)\| := \max\{\|x\|, \|y\|\}$.

A natural and important question to consider for our purposes is whether the operation of complexification commutes with the operation of taking inductive limit. The next lemma shows that this is indeed the case for imbedding spectres of normed linear spaces.

Lemma 27. *Suppose $(i_n \colon X_n \to X)_{n \in \mathbb{N}}$ is an imbedding spectre of real normed linear spaces for which the locally convex inductive topology on X exists. Then $X_{\mathbb{C}} = (\mathrm{ind}_{n \to \infty} X_n)_{\mathbb{C}} = \mathrm{ind}_{n \to \infty} (X_n)_{\mathbb{C}}$*

Proof. In any case as sets, without any topological consideration

$$X_{\mathbb{C}} = X \times X = \bigcup_{n \in \mathbb{N}} X_n \times \bigcup_{n \in \mathbb{N}} X_n = \bigcup_{n \in \mathbb{N}} X_n \times X_n = \bigcup_{n \in \mathbb{N}} (X_n)_{\mathbb{C}}$$

Thus it only remains to show that the topologies are identical. Let t denote the inductive limit topology of $X_{\mathbb{C}}$ and τ its product topology. The continuity of the inclusion maps $i_{n \times n} \colon (X_n)_{\mathbb{C}} \to (X_{\mathbb{C}}, \tau)$, $(u, v) \mapsto (i_n(u), i_n(v))$ together with the definition of the inductive topology ensures that $t \geq \tau$. Conversely let $U = U_1 \times U_2$ be an absolutely convex zero-neighbourhood in $(X_{\mathbb{C}}, t)$. Then U_1 and U_2 are zero neighbourhoods in X. Indeed by Lemma 20, U is a t-zero neighbourhood if and only if $i_{n \times n}^{-1}(U_1 \times U_2) = (i_n^{-1}(U_1), i_n^{-1}(U_2))$ is a zero-neighbourhood in $(X_n)_{\mathbb{C}}$ for each $n \in \mathbb{N}$, and this is if and only if $i_n^{-1}(U_1)$ and $i_n^{-1}(U_2)$ are both zero-neighbourhoods in X_n for each $n \in \mathbb{N}$. Hence both U_1 and U_2 are zero neighbourhoods in X so that U is indeed a τ zero-neighbourhood. Then also $\tau \geq t$. □

Lemma 28. *Suppose $(i_n \colon X_n \to X)_{n\in\mathbb{N}}$ and $(j_n \colon Y_n \to Y)_{n\in\mathbb{N}}$ are imbedding spectres of locally convex spaces such that for each $n \in \mathbb{N}$ there is an isomorphism $\phi_n \colon X_n \to Y_n$ with the property that $\phi_{n+1}\big|_{X_n} = \phi_n$. If the inductive topology of both systems exists, then $X \cong Y$*

Proof. Let $\phi \colon X \to Y$ be defined by $x \mapsto \phi(x) := \phi_n(x)$ if $x \in X_n$. By our hypothesis about the collection $(\phi_n)_{n\in\mathbb{N}}$, ϕ is well defined. Linearity and the fact that it is bijective follows directly from the ϕ_n's. To show continuity we use Lemma 20. Let V be an absolutely convex zero neighbourhood in Y. By the linearity of ϕ, $\phi^{-1}(V)$ is absolutely convex in X and by Lemma 20 it is a zero neighbourhood in X if and only if $i_n^{-1}(\phi^{-1}(V))$ is a zero neighbourhood for each $n \in \mathbb{N}$. But note that $\phi_n = \phi \circ i_n$ and since ϕ_n is an isomorphism, $\phi_n^{-1}(V) = i_n^{-1}(\phi^{-1}(V))$ is a zero neighbourhood in X_n for each $n \in \mathbb{N}$. Thus $\phi^{-1}(V)$ is a zero neighbourhood in X which shows that ϕ is continuous. In a similar way we can show that its inverse is bijective, linear and continuous so that ϕ is indeed an isomorphism of locally convex spaces. □

2.2 Calculus in Locally Convex Spaces

In this chapter we introduce a calculus on locally convex spaces called Bastiani Calculus, named after Andrée Bastiani who first introduced it in [1]. We begin by defining derivatives of curves taking values in a locally convex space and then move on to consider differentiability of mappings between locally convex spaces. After this some useful examples of smooth (holomorphic) mappings will be considered as well as some properties of this calculus that will be useful in subsequent chapters.

Definition 29. *Let E be a locally convex space and consider a curve $\gamma \colon I \to$*

E defined on an open subset $I \subset \mathbb{R}$. γ is said to be differentiable at $t \in I$ if

$$\gamma'(t) = \lim_{z \to 0} \frac{\gamma(t+z) - \gamma(t)}{z}$$

exists and differentiable in I if it exist for all $t \in I$. In this case if the induced mapping γ' is continuous then γ is called a continuously differentiable curve. We say γ is of class C^k with $k \in \mathbb{N}_0$ if $\gamma^{(n)}$ defines a continuous mapping for all integers $n \leq k$, where we set $\gamma^{(0)} := \gamma$. We say that γ is smooth if γ is of class C^k for all $k \in \mathbb{N}_0$. Whenever $I \subset \mathbb{R}$ is a closed interval then $\gamma \colon I \to E$ is differentiable or C^k if it extends to a differentiable or C^k mapping $\tilde{\gamma} \colon \tilde{I} \to E$ on an open set $\tilde{I}$ containing I.

Definition 30. *Let E and F be locally convex $\mathbb{K}$-vector spaces and $U \subset E$ an open subset. Given a mapping $f \colon U \to F$ and $x \in U$, we say that f is Bastiani differentiable at x if the following limit exists for all $y \in E$*

$$df(x; y) := D_y f(x) := \lim_{t \to 0} \frac{f(x+ty) - f(x)}{t}$$

where $t \in \mathbb{K} \setminus \{0\}$. We will leave out the name Bastiani *and simply say that f is differentiable at x. The function f is called differentiable in U if the induced mapping $df \colon U \times E \to F$ is well defined and continuously differentiable if df is continuous. The map f is k times differentiable in U if the iterated directional derivatives*

$$d^{(k)} f(x; y_1, \ldots, y_{k-1}, y_k) := D_{y_k}(D_{y_{k-1}} \ldots D_{y_1} f)(x)$$

exist for all $(y_k, y_{k-1}, \ldots y_1) \in E^k$ and $x \in U$. As before f is called a C^k mapping if the induced map $d^{(n)} f \colon U \times E^n \to F$ is continuous for all integers $n \leq k$, where again we set $d^{(0)} f := f$. We say f is smooth if f is C^k for all $k \in \mathbb{N}_0$ in the case $\mathbb{K} = \mathbb{R}$ and holomorphic in the case that $\mathbb{K} = \mathbb{C}$.

We immediately see that

$$d^{(2)} f(x; y, z) = D_z(D_y f)(x)$$
$$= \lim_{t \to 0} \frac{D_y f(x+tz) - D_y f(x)}{t} = \lim_{t \to 0} \frac{df(x+tz; y) - df(x; y)}{t}$$

iterating this we obtain

$$d^{(k)}f(x;y_1,\dots,y_k) = \frac{d}{dt}\bigg|_{t=0} d^{(k-1)}f(x+ty_k;y_1,\dots,y_{k-1}) \tag{31}$$

It follows that f is C^k if and only if f is C^{k-1} and $d^{(k-1)}f$ is C^1. Moreover in the case that $f\colon \mathbb{R} \to E$ is a mapping between the Banach space $\mathbb{R}$ and any locally convex space E then the relationship between the derivatives in the sense of Definition 29 and 30 is $f'(t) = df(t;1)$.

Definition 32. *Let E and F be complex locally convex spaces and $U \subset E$ an open subset. A continuous mapping $f\colon U \to F$ is said to be complex analytic (or just analytic) if there is a sequence of continuous homogeneous polynomials β_n of degree $n \in N_0$ (β_n is said to be a homogeneous polynomial of degree n if $\beta_n(x) = f_n(x,\dots,x)$ for some n-linear map $f\colon E^n \to F$), for which at any $x \in U$ there is a zero neighbourhood V such that $x+V \subset U$ and for all $h \in V$ we have*

$$f(x+h) = \sum_{n\geq 0} \beta_n(h)$$

In the case of mappings between complex locally convex spaces, the property of being analytic and holomorphic are equivalent by [5, Proposition 1.1.16 (a)]. The two notions will be used interchangeably.

Next we provide some examples of smooth (holomorphic) mappings which will be extensively used in later chapters.

Proposition 33. *The following maps are smooth (holomorphic)*

1. *Continuous linear maps of locally convex spaces. In particular addition on any locally convex space.*
2. *Continuous bilinear maps of locally convex spaces*

Proof.

1 . Let $f\colon E \to F$ be a continuous linear map of locally convex spaces, both real or both complex. For any $x, y \in E$ we have

$$df(x;y) = \lim_{t\to 0} \frac{f(x+ty)-f(x)}{t} = \lim_{t\to 0} \frac{f(x)+tf(y)-f(x)}{t} = f(y)$$

which shows that df is continuous and that it only depends on its second argument. For any $x, y, z \in E$ we have

$$d^{(2)}f(x;y,z) = \lim_{t\to 0} \frac{df(x+tz;y)-df(x;y)}{t} = \lim_{t\to 0} \frac{f(y)-f(y)}{t} = 0$$

so that $d^{(2)}f$ is the zero map and hence continuous. Higher derivatives of the zero maps are also zero, whence $d^{(k)}f = 0$ is continuous for all $k > 2$ and so f is smooth (holomorphic).

2 . Let $\mu\colon E \times E \to E$, $(z_1, z_2) \mapsto \mu(z_1, z_2)$ be a continuous bilinear map on E. To simplify notation write $\mu(z_1, z_2) = z_1 z_2$. We compute for $z_1, z_2 \in E$

$$\begin{aligned} d\mu((z_1,z_2)\,;\,(y_1,y_2)) &= \lim_{t\to 0} t^{-1}\left(\mu((z_1+y_1t, z_2+y_2t)) - \mu((z_1,z_2))\right) \\ &= \lim_{t\to 0} t^{-1}\left(z_1z_2 + z_1y_2t + z_2y_1t + y_1y_2t^2 - z_1z_2\right) \\ &= \lim_{t\to 0} y_2z_1 + z_2y_1 + y_1y_2t = z_1y_2 + z_2y_1 \end{aligned}$$

which shows that $d\mu\colon E^4 \to E$ is continuous. Moreover for $x_1, x_2 \in E$

$$\begin{aligned} & d^{(2)}\mu((z_1,z_2);(y_1,y_2);(x_1,x_2)) \\ &= \lim_{t\to 0} t^{-1}\left(d\mu((z_1+x_1t, z_2+x_2t);(y_1,y_2)) - d\mu((z_1,z_2);(y_1,y_2))\right) \\ &= \lim_{t\to 0} t^{-1}\left(z_1y_2 + x_1y_2t + y_1z_2 + x_2y_1t - z_1y_2 - z_2y_1\right) \\ &= \lim_{t\to 0} t^{-1}\left(x_1y_2t + y_1x_2t\right) = x_1y_2 + y_1x_2 \end{aligned}$$

so $d^{(2)}\mu$ is continuous and moreover that it does not depend on its first entry. Then we may argue as in 1. that μ is C^k for all $k \in \mathbb{N}_0$.

□

There is the following version of the chain rule.

Proposition 34. *[26, Proposition A.1.11] Let $U \subset E$ and $V \subset F$ be open subsets of locally convex spaces and consider the mappings $f\colon U \to V$ and $g\colon V \to L$ where L is another locally convex space. If f and g are C^r then $g \circ f$ is C^r. In particular a composition of smooth (holomorphic) maps are again smooth (holomorphic). Moreover for any $x \in U$ and $y \in E$ we have*

$$d(g \circ f)(x; y) = dg(f(x); df(x; y))$$

Proposition 35. *Suppose F is a locally convex space and let $E = \prod_{i \in I} E_i$ be a product of locally convex spaces E_i. Consider a map $f\colon F \to E$, $x \mapsto (f_i(x))_{i \in I}$. Then f is C^r if and only if $f_i\colon F \to E_i$ is C^r for all $i \in I$. Moreover $df(x; y) = (df_i(x; y))_{i \in I}$ for any $x, y \in F$.*

Proof. Suppose first that f is C^r. As continuous linear maps, the projections π_i are smooth (holomorphic) and in particular C^r. Whence $f_i = \pi_i \circ f$ is C^r by Proposition 34. Conversely suppose f_i is C^r for all $i \in I$. Since the product topology on E is initial with respect to the projections π_i, $x_n \to x$ in E if and only if $\pi_i(x_n) \to \pi_i(x)$ in E_i for all $i \in E_i$. Thus

$$\begin{aligned} df(x; y) &= \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} = \lim_{t \to 0} \left(\frac{f_i(x + ty) - f_i(x)}{t}\right) \\ &= \left(\lim_{t \to 0} \frac{f_i(x + ty) - f_i(x)}{t}\right) = (df_i(x; y))_{i \in I} \end{aligned}$$

Computing higher derivatives using Equation 31 we see that

$$d^{(k)} f(x; y_1, \ldots, y_k) = (d^{(k)} f_i(x; y_1, \ldots, y_k))_{i \in I}$$

Again since the topology on E is initial and the $d^{(k)} f_i$ are continuous for all integers $k \leq r$, the same holds for $d^{(k)} f$. Thus f is C^r if every f_i is C^r. □

Definition 36. *Let E and F be real locally convex spaces and consider $f\colon U \to F$ where $U \subset E$ is an open set. We say that f is real analytic if it has an extension to an analytic map of the complexified locally convex spaces $f_{\mathbb{C}}\colon \tilde{U} \to F_{\mathbb{C}}$ on an open subset $\tilde{U} \subset E_{\mathbb{C}}$ containing U.*

Whenever we write that a mapping is $\mathbb{R}$-analytic (resp. $\mathbb{C}$-analytic) we mean that it is real analytic (resp. analytic). At later stages we will prove smoothness of various products on a real Silva space. It will then turn out that this reduces to proving analyticity for the corresponding products on the complex Silva space. This is because these products will be of such a form that the complexified versions are analytic extensions of the real versions. Then if we are able to show that the complexified versions are analytic, the real ones will be real analytic and we can apply the following proposition.

Proposition 37. *[10, Proposition 2.4] Let E and F be real locally convex spaces and $U \subset E$ an open subset. If $f\colon U \to F$ is real analytic, then f is smooth.*

The next lemma will make for a powerful tool in establishing analyticity of mappings. Whenever we say a family of continuous linear functionals $\{\lambda_\alpha\}_{\alpha \in A}$ on a locally convex space E seperates points we mean that if $0 \neq x \in E$ then there is an $\alpha \in A$ for which $\lambda_\alpha(x) \neq 0$.

Lemma 38. *[2, Lemma A.3] Let E and F be complex locally convex spaces and let $U \subset E$ be open. Suppose there is a family of continuous linear functionals $\{\lambda_\alpha\}_{\alpha \in A}$, separating points on F. Then a continuous mapping $f\colon U \to F$ is holomorphic if and only if $\lambda_\alpha \circ f\colon U \to \mathbb{C}$ is holomorphic for each $\alpha \in A$.*

For normed linear spaces there is of course the classical calculus built from the notion of a Fréchet derivative. Before ending this chapter, we will briefly compare the two calculi: If the domain of a mapping of normed linear spaces is a subset of a finite dimensional normed linear space, then the two calculi are actually equivalent [26, Proposition A.3.5]. That is, a mapping is C^k in the Bastiani sense if and only if it is C^k in the Fréchet sense. However in the infinite dimensional case Bastiani differentiability is weaker than the Fréchet differentiability. Specifically let X, Y be two normed linear spaces and let $f\colon U \to Y$ be a map on an open subset $U \subset X$. Then [26, Lemma A.3.1] tells us that if f is C^k in the sense of Fréchet then f is also C^k in the sense of Bastiani and moreover $D^k f(x) = d^k f(x;)$, where D^k denotes the k'th derivative operator in the Fréchet calculus [26, Definition A.2.13]. However if we only know that f is Bastiani C^k, with $k \geq 1$, then f is Fréchet C^{k-1} [26, Lemma A.3.3].

2.3 Locally Convex Lie Groups

Similarly to finite dimensional Lie groups, locally convex Lie groups are manifolds as well as groups with smooth (holomorphic) group operations. However, whereas finite dimensional Lie groups are modelled on finite dimensional Euclidean space, locally convex Lie groups are, as the name suggests, modelled on locally convex spaces. Thus the notion of differentiability is different because in the locally convex case whenever we talk about a map being C^r or smooth (holomorphic) we mean always in the sense of the calculus introduced in Chapter 2.2. In what follows we will very briefly discuss infinite dimensional manifolds, locally convex Lie groups and their associated Lie algebras. We start with the definition of infinite dimensional manifolds.

Definition 39. *A manifold M modelled on a locally convex space E is a topological Hausdorff space for which at each point $x \in M$ there is a homeomorphism $\phi\colon U_\phi \to E$ of an open subset U_ϕ containing x onto an open subset $V_\phi := \phi(U_\phi)$ of the locally convex space E. Such a homeomorphism is called a chart. Moreover given $r \in \mathbb{N} \cup \{\infty\}$ a collection of charts $\mathcal{A}$ is called a C^r-atlas if the following holds*

1. *Whenever $\phi, \psi \in \mathcal{A}$ are such that $U_\phi \cap U_\psi \neq \emptyset$, then the transition maps*

$$\psi \circ \phi^{-1}\colon \phi(U_\phi \cap U_\psi) \to \psi(U_\phi \cap U_\psi)$$

   and

$$\phi \circ \psi^{-1}\colon \psi(U_\psi \cap U_\phi) \to \phi(U_\psi \cap U_\phi)$$

   are of class C^r as maps between open subsets of the locally convex space E

2. $\bigcup_{\phi\in\mathcal{A}} U_\phi = M$

Given a manifold M modelled on a locally convex space, we say that two C^r-atlases $\mathcal{A}_1, \mathcal{A}_2$ are equivalent if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a C^r-atlas for M. This is clearly an equivalence relation on the set of C^r atlases for M, allowing for the next definition.

Definition 40. *A C^r-manifold modelled on a locally convex space E is a pair $(M, \mathcal{A})$, with M a topological Hausdorff space and $\mathcal{A}$ an equivalence class of C^r-atlases, for which the charts of M are homeomorphisms onto open subsets of E.*

Definition 41. *A continuous function $f\colon M \to N$ of C^r manifolds M and N modelled on locally convex spaces E and F is said to be C^r, $r \in \mathbb{N} \cup \{\infty\}$, if for any charts ϕ of M and ψ of N the map*

$$\psi \circ f \circ \phi^{-1} \colon \phi(U_\phi \cap f^{-1}(U_\psi)) \to \psi(U_\psi)$$

is C^r.

In order to even talk about smooth (holomorphic) group operations we need the following standard result.

Proposition 42. *Suppose M and N are C^r manifolds modelled on locally convex spaces. Then $M \times N$, with the product topology, is a C^r manifold modelled on the locally convex space $E \times F$.*

Proof. The direct product of Hausdorff spaces is again Hausdorff. Let $\mathcal{A}$ and $\mathcal{B}$ be C^r atlases for M and N respectively.

Claim. *$\mathcal{A} \times \mathcal{B}$ is a C^r atlas for $M \times N$*

Indeed we clearly have that

$$\bigcup_{(\phi,\psi)\in\mathcal{A}\times\mathcal{B}} U_\phi \times U_\psi = M \times N$$

Since any charts $\phi \in \mathcal{A}$ and $\psi \in \mathcal{B}$ are homeomorphisms between open subsets M and E, and N and F respectively, the mapping $\phi \times \psi$ also becomes a homeomorphism of open subsets of $M \times N$ and $E \times F$. Thus it only remains to check that the transition charts, when defined, are C^r. Indeed $(\phi_2, \psi_2) \circ (\phi_1, \psi_1)^{-1} = (\phi_2, \psi_2) \circ (\phi_1^{-1}, \psi_1^{-1}) = (\phi_2 \circ \phi_1^{-1}, \psi_2 \circ \psi_1^{-1})$ is C^r by Proposition 35 because both $\phi_2 \circ \phi_1^{-1}$ and $\psi_2 \circ \psi_1^{-1}$ are C^r. $\square$

There is also the notion of a real analytic manifold. This is a manifold modelled on a real locally convex space with real analytic transition maps. The preceding proposition is still valid when exchanging C^r with real analytic. With the notions of infinite-dimensional manifolds, differentiable maps between them and the preceding proposition, we can give the definition of Lie groups modelled on locally convex spaces.

Definition 43. *A (analytic, real analytic, smooth) locally convex Lie group G is a group as well as a manifold modelled on a locally convex space E, for which the group operations multiplication $G \times G \to G$ and inversion $G \to G$ are (analytic, real analytic, smooth) with respect to the manifold structure.*

The easiest example of a smooth (analytic) locally convex Lie group is the additive group $(E, +)$, where E is any locally convex space. The identity mapping serves as a global chart and Proposition 33 shows that addition is smooth (holomorphic). Moreover it is easily verified that the inversion $x \mapsto -x$ is smooth (holomorphic). Thus $(E, +)$ is a smooth (analytic) locally convex Lie group. In contrast a nontrivial and important example of a locally convex Lie group is a so called continuous inverse algebra. Its definition is as follows.

Definition 44. *Let A be a locally convex space equipped with a continuous bilinear product that is associative and which has a multiplicative identity. Let $A^\times$ denote the unit group of A under the bilinear product. If $A^\times$ is open and the inversion mapping $a \mapsto a^{-1}$ is continuous, then A is said to be a continuous inverse algebra (or CIA for short).*

Proposition 45. *The unit group of a CIA forms a smooth (analytic) locally convex Lie group.*

Proof. As an open subset of A, the unit group $A^\times$ is a manifold modelled on A with the identity map as a global chart. Proposition 33 ensures the smoothness (analyticity) of the continuous bilinear product on $A^\times$ and thus it only remains to show that the inverse is also smooth (holomorphic). Denote by I the inverse mapping on A. To simplify notation write $\beta_A(x, y) = xy$, where β_A is the bilinear product on A. We begin by showing that $dI(x; y) = -x^{-1}yx^{-1}$. Indeed, using the relation

$$b^{-1} - a^{-1} = b^{-1}(a - b)a^{-1}$$

and recalling that inversion and multiplication are continuous, we compute

$$dI(x;y) = \lim_{t\to 0} t^{-1}(I(x+yt) - I(x)) = \lim_{t\to 0} t^{-1}((x+yt)^{-1} - x^{-1})$$
$$= \lim_{t\to 0} t^{-1}((x+yt)^{-1}\,(x-(x+yt))\,x^{-1}) = \lim_{t\to 0}(x+yt)^{-1}\,y\,x^{-1} = -x^{-1}yx^{-1}$$

Thus dI is also continuous so that I is C^1. We continue by induction to prove that I is smooth (holomorphic) by showing it is C^k for any integer k. The case $k = 1$ has just been handled. Let $k > 1$ and assume that I is C^{k-1}. Then since the algebra product is smooth (holomorphic), inversion is C^{k-1} and $dI(x; y) = -x^{-1}yx^{-1}$, dI is C^{k-1} by Proposition 34 as a composition of C^{k-1} maps. Whence $d^{(k-1)}I$ is C^1 and so the inversion mapping I is C^k. □

We next consider Lie algebras and in particular Lie algebras associated to locally convex Lie groups. In addition we also define the notion of regularity of Lie groups. The results in the thesis relying on what comes next are of a secondary nature. In any case a rigorous treatment of the topic would be very lengthy. For these reasons we omit proofs and refer instead to [19, Chapters II.3 and III.1] for a detailed exposition.

In order to introduce the Lie algebra associated to a Lie group we need to first define tangent bundles of manifolds and vector fields. Since we are only interested in these related to Lie groups, we state the definitions and result in the case that the manifolds are smooth (holomorphic).

Definition 46. *Let M be a smooth (holomorphic) manifold and let $p \in M$. Two smooth curves γ, ξ passing through p (i.e. $\xi(0) = \gamma(0) = p$) are said to be equivalent if $(\phi \circ \gamma)'(0) = (\phi \circ \xi)'(0)$ for some (in particular any) chart ϕ with chart domain containing p. A tangent vector v at p is then defined to be an equivalence class of smooth curves passing through p and the tangent space at p is defined to be the set of all tangent vectors at p. The tangent space will be denoted T_pM.*

As for finite dimensional smooth manifolds also locally convex manifolds have tangent spaces isomorphic to the modelling space. For locally convex manifolds the locally convex structure is inherited via the bijection

$$h_\phi \colon E \to T_pM, y \mapsto [t \mapsto \phi^{-1}(\phi(p) + ty)]$$

whose inverse is the map

$$h_\phi^{-1} \colon T_pM \to E, [\gamma] \mapsto (\phi \circ \gamma)'(0)$$

Here ϕ is any chart whose chart-domain contains p and E is the modelling space for M. One then defines the topology and linear structure on T_pM such that h_ϕ becomes an isomorphism of locally convex spaces. Moreover the locally convex structure can be shown to be independent of choice of chart in the above construction. With the tangent spaces introduced we can define the tangent bundle.

Definition 47. *Let M be a smooth (holomorphic) manifold modelled on a locally convex space with tangent spaces T_pM. The tangent bundle of M is the disjoint union $TM := \bigcup_{p \in M} T_pM$.*

Again similarly to the finite dimensional case the tangent bundle of a smooth manifold M is again a smooth manifold. If M is modelled on E

via the homeomorphisms (U_ϕ, ϕ) then TM is modelled on $E \times E$ via the homeomorphisms

$$T\phi^{-1} \colon V_\phi \times E \to TM\,,\ (x,y) \mapsto [t \mapsto \phi^{-1}(x+ty)])$$

For any $p \in M$ its tangent map at p is defined as

$$T_p f \colon T_pM \to T_{f(p)}N\,,\ [\gamma] \mapsto [f \circ \gamma]$$

and is a linear map. The tangent map Tf is defined as

$$Tf \colon TM \to TN\,,\ (p,v) \mapsto (f(p), T_pf(v))$$

The tangent map Tf can be shown to be smooth (holomorphic) if the map f is smooth. One can moreover show for smooth (holomorphic) maps $f \colon M \to N$ and $g \colon N \to P$ that $T(g \circ f) = Tg \circ Tf$.

Definition 48. *A vector field X on a smooth (holomorphic) manifold M is a smooth (holomorphic) map $X \colon M \to TM$ for which $\pi_M \circ X = \mathrm{id}_M$. Here π_M is the projection $\pi_M(x,v) = x$. In other words a tangent vector associates to each point $p \in M$ a tangent vector in T_pM. By abuse of notation one often denotes this tangent vector as $X(p)$.*

We define the sum and scalar multiplication of vector fields as

$$X + Y \colon M \to TM\,,\ p \mapsto (p, X(p) + Y(p))$$
$$r\,X \colon M \to TM\,,\ p \mapsto (p, rX(p))$$

Using that each tangent space is a locally convex space and that by Proposition 33, addition and scalar multiplication (which is a continuous bilinear map) is smooth (holomorphic), the sum and scalar multiples of smooth (holomorphic) vector fields is again a smooth (holomorphic) vector field. Hence the set of smooth (holomorphic) vector fields form a vector space, which is denoted by $\mathcal{V}(M)$. For a smooth (holomorphic) map $f \colon M \to N$ we say that $X \in \mathcal{V}(M)$ is f-related to $Y \in \mathcal{V}(N)$ if

$$Tf \circ X = Y \circ f$$

For $M = G$ a locally convex Lie group the left multiplication

$$\lambda_g \colon G \mapsto G\,,\; h \mapsto gh$$

is a diffeomorphism. Indeed it is smooth (holomorphic) as a composition of the smooth (holomorphic) maps $\lambda_g = m_G \circ \rho \colon G \to G \times G \to G$ where $\rho(h) := (g, h)$ is smooth (holomorphic) and m_G denotes the multiplication in G. A smooth (holomorphic) inverse is then $\lambda_g^{-1} = \lambda_{g^{-1}}$. In particular the tangent map $T\lambda_g$ is a diffeomorphism with inverse $T\lambda_{g^{-1}}$ since

$$\mathrm{id}_{TM} = T(\mathrm{id}_M) = T(\lambda_g \circ \lambda_{g^{-1}}) = T\lambda_g \circ T\lambda_{g^{-1}} = T\lambda_{g^{-1}} \circ T\lambda_g$$

A vector field $X \colon G \to TG$ is said to be left invariant if X is λ_g related to itself for all $g \in G$. The set of all left invariant vector fields form a vector subspace of $\mathcal{V}(M)$ and is denoted $\mathcal{V}^l(M)$. In fact one can show that $\mathcal{V}^l(M)$ has the structure of a locally convex space.

Definition 49. *A Lie algebra $\mathfrak{g}$ is a vector space equipped with a bilinear mapping called the Lie bracket $[\,,\,] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying the following two properties*

1. $[x, x] = 0$*, for all* $x \in \mathfrak{g}$.
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$*, for all* $x, y, z \in \mathfrak{g}$

A Lie algebra that is also a locally convex space with a continuous Lie bracket is called a locally convex Lie algebra.

The Lie algebra associated to a Lie group G will be the tangent space at the identity and is denoted $L(G) := T_eG$, where $e \in G$ is the identity element. For now we only know that it is a locally convex space so we need to define a bracket on $L(G)$ satisfying the conditions 1. and 2. above. To introduce this bracket consider the following isomorphism of vector spaces

$$\theta \colon L(G) \to \mathcal{V}^l(G)\,,\; v \mapsto (g \mapsto T_e\lambda_g(v))$$

Its inverse is given by

$$\theta^{-1}\colon \mathcal{V}^l(G) \to L(G)\,,\; X \mapsto X(e)$$

One can show that θ is in fact an isomorphism of locally convex spaces. Assuming for the moment that the space $\mathcal{V}^l(G)$ carries a Lie bracket, we can define the bracket on $L(G)$ as

$$[v, w] = \theta^{-1}([\theta(v), \theta(w)]) = [\theta(v), \theta(w)](e) \tag{50}$$

Since θ is an isomorphism it is easily shown that this bracket satisfies the conditions in the definition of a Lie algebra. The bracket can also be shown to be continuous so that the Lie algebra associated to a locally convex Lie group will always be a locally convex Lie algebra. In what comes next we will discuss how, for M a smooth manifold, the space $\mathcal{V}(M)$ is a Lie algebra of which $\mathcal{V}^l(M)$ is a Lie sub-algebra. For holomorphic manifolds the construction and result is exactly the same.

The Lie Bracket of Smooth Vector Fields

The local case is handled first. Consider an open subset $U \subset E$. Then $TU = U \times E$ and any vector field on U is given by $X = (\mathrm{id}_U, X_E)$ where $X_E\colon U \to E$ is smooth. For any smooth map $f\colon U \to E$ and vector field $X \in \mathcal{V}(U)$, we define $X.f := df \circ X$ where $df = pr_2 \circ Tf$. In particular $df \circ (X(u)) = df(u; X_E(u))$. Then $X.f$ defines a smooth function $U \to E$. We define the bracket of two vector fields $X, Y \in \mathcal{V}(U)$ as the vector field determined by $[X, Y]_E := X.Y_E - Y.X_E$. One can show that with this bracket, $\mathcal{V}(U)$ satisfies the Lie algebra axioms. Moreover one can show that relatedness is inherited by the bracket. That is if X_1, X_2 are correspondingly f-related to Y_1, Y_2, then $[X_1, X_2]$ is f-related to $[Y_1, Y_2]$.

Now consider the general case of a smooth manifold M with atlas A. Given any vector field $X \in \mathcal{V}(M)$ its local representative in any chart is the smooth vector field on $\phi(U_\phi) = V_\phi \subset E$ given by $X_\phi = T\phi \circ X \circ \phi^{-1}$. For two overlapping chart domains U_ϕ and U_ψ the local representatives X_ϕ and X_ψ are $\psi \circ \phi^{-1}$-related. Conversely if given a collection $(X_\phi)_{\phi\in A}$ of vector fields on $(V_\phi)_{\phi\in A}$ that are all $\psi\circ\phi^{-1}$-related to each other whenever we have overlapping domains, then it is easily shown that the collection glues together to form a vector field $Y \in \mathcal{V}(M)$, with local representative $Y_\phi = X_\phi$ for each $\phi \in A$. Thus the local representatives uniquely determine any vector field on M. Now given $X, Y \in \mathcal{V}(M)$ we define their bracket $[X, Y]$ to be the vector field on M with local representatives $[X, Y]_\phi = [X_\phi, Y_\phi]$. Since relatedness is inherited by the bracket, $[X_\phi, Y_\phi]$ is $\psi \circ \phi^{-1}$-related to $[X_\psi, Y_\psi]$, whence the vector field $[X, Y]$ is well defined. Since $[X, Y]$ is uniquely determined by its local representatives and the bracket on $\mathcal{V}(V_\phi)$ satisfies the Lie algebra axioms, so too does $\mathcal{V}(M)$. In the case that $M = G$ is a Lie group, $\mathcal{V}^l(G)$ becomes a Lie sub-algebra of $\mathcal{V}(G)$ since the property of being left-invariant is inherited by the bracket. In particular this means that $L(G) = T_eG$ becomes a Lie algebra, in fact a locally convex Lie algebra, with bracket (50).

The preliminary chapter is concluded with the definition of regularity of Lie groups.

Definition 51. *Let G a be Lie group modelled on a locally convex space with identity element e, $L(G)$ its Lie algebra and let $k \in \{\infty\} \cup \mathbb{N}_0$. Denote by $\lambda_g\colon G \to G$ the diffeomorphism given by left multiplication with $g \in G$ and by $T_e\lambda_g\colon L(G) \to T_gG$ its corresponding tangent map. We say that G is C^kregular if for each C^k curve $f\colon [0,1] \to L(G)$ the initial value problem*

$$\begin{cases} \gamma'(t) = T_e\lambda_{\gamma(t)}(f(t)) \\ \gamma(0) = e \end{cases}$$

has a unique C^{k+1} solution $\mathrm{Evol}(f) := \gamma\colon [0,1] \to G$ *and the map*

$$\mathrm{evol}\colon C^k([0,1], L(G)) \to G\,,\ f \mapsto \mathrm{Evol}(f)(1)$$

is smooth.

Lie groups modelled on finite dimensional Euclidean (hence Banach) spaces are automatically C^0-regular [20]. However this is not necessarily the case when the Lie group under question is modelled on an infinite-dimensional locally convex space.

3 Differentiability of Chen Series

This chapter considers the differentiability of the formal power series known as *Chen series*. Before actually defining Chen series we introduce some notation which will be used throughout the rest of this text.

A collection of indeterminates $X = \{x_0\,, x_1\,, \dots, x_m\}$ is said to be an alphabet, the indeterminates called letters. Any finite sequence of letters $\eta = x_{i_1}\, x_{i_2} \dots x_{i_n}$ is a word with associated length $|\eta| := n$. By $|\eta|_{x_i}$ we mean the number of letters in η which are equal to $x_i \in X$. The empty word is written $\emptyset$ and is the only word of length zero. For any $x \in X$ we put $x^0 := \emptyset$. By X^n we mean the set of all words of length n and by $X^* = \bigcup_{n\in\mathbb{N}_0} X^n$ the set of all words over the alphabet X. With the associative catenation product of words defined for $\eta = x_{i_1} \dots x_{i_n}$ and $\xi = x_{j_1} \dots x_{j_k}$ as

$$\eta \cdot \xi := \eta\xi = x_{i_1} \dots x_{i_n} x_{j_1} \dots x_{j_1},$$

X^* becomes a monoid with identity element $\emptyset$. Any function $c\colon X^* \to \mathbb{K}^m$ is called a formal power series and its value at a word η is denoted $(c, \eta) \in \mathbb{K}^m$. The i'th component function $c[i]\colon X^* \to \mathbb{K}$ is the formal power series such that $(c[i], \eta) = (c, \eta)[i]$ for all $\eta \in X^*$. It is customary to write the mapping c as a sum $\sum_{\eta\in X^*}(c, \eta)\,\eta$. Equipped with the componentwise vector space structure, the collection of all such formal power series becomes a $\mathbb{K}$-vector space and will be denoted by $\mathbb{K}^m\langle\langle X\rangle\rangle$. We extend the catenation product of words first to $\mathbb{K}\langle\langle X\rangle\rangle$ and then coordinate-wise to $\mathbb{K}^m\langle\langle X\rangle\rangle$ by

$$c \cdot d = \sum_{u,v\in X^*} (c, u)(d, v)\, u \cdot v$$

for $c, d \in \mathbb{K}\langle\langle X\rangle\rangle$. In this way $\mathbb{K}^m\langle\langle X\rangle\rangle$ becomes a non-commutative, associative and unital $\mathbb{K}$-algebra. The concatenation product $\mathbb{K}^m\langle\langle X\rangle\rangle$ will not be explored any further but will frequently used later on in the following way: If $x_i \in X$ and $d \in \mathbb{K}\langle\langle X\rangle\rangle$ then $x_i d = \sum_{\eta\in X^*}(d, \eta)\, x_i\eta$ and we notice immediately that $(x_i d, x_j\xi)$ equals zero if $j \neq i$ and equals (d, ξ) if $j = i$.

For $c \in \mathbb{K}\langle\langle X\rangle\rangle$ nonzero we define

$$\text{ord}(c) := \min\{|\eta| \,:\, (c, \eta) \neq 0\}$$

and $\mathrm{ord}(0) := \infty$. When $c \in \mathbb{K}^m\langle\langle X\rangle\rangle$ with $m > 1$ we put

$$\mathrm{ord}(c) := \min_{1\leq i\leq m} \mathrm{ord}(c[i])$$

The definition of Chen series is as follows.

Definition 52. *Suppose $T > 0$ and that $u\colon [0,T] \to \mathbb{R}^m$ is an element of $L_1^m([0,T])$ (i.e. all its components $\{u_i\}_{i=1}^m$ are of class $L^1([0,T])$). Given such a function u and a word η we define the map $E_\eta[u](t)$ inductively by setting $E_\emptyset[u](t) = 1$ and*

$$E_{x_i\bar{\eta}}[u](t) = \int_0^t E_{\bar{\eta}}[u](s)\, u_i(s)\, ds$$

where $x_i \in X$, $\bar{\eta} \in X^*$ and $u_0 := 1$. The Chen series associated to u is then for every $0 \leq t \leq T$ a formal power series of the form

$$P[u](t) = \sum_{\eta\in X^*} E_\eta[u](t)\, \eta$$

Thus a Chen series is a map $P[u] : [0,T] \to \mathbb{R}\langle\langle X\rangle\rangle$. In [12, Page 4] such a series was formally differentiated in a way that the derivative of the Chen series was equal to the derivative of the components in its series. Applying the calculus introduced in Chapter 2.2 we will show that in this sense the derivative of a Chen series has indeed such a form.

The first step is to turn $\mathbb{K}^m\langle\langle X\rangle\rangle$ into a locally convex space. To this end consider the vector space $\prod_{\eta\in X^*}\mathbb{K}^m$. This is a locally convex space as a direct product of the Banach spaces $\mathbb{K}^m$. We use $\prod_{\eta\in X^*}\mathbb{K}^m$ to define a locally convex topology on $\mathbb{K}^m\langle\langle X\rangle\rangle$. Specifically the mapping

$$f\colon \mathbb{K}^m\langle\langle X\rangle\rangle \to \prod_{\eta\in X^*}\mathbb{K}^m\,,\ c \mapsto ((c,\eta))_{\eta\in X^*}$$

is a linear bijection of $\mathbb{K}$-vector spaces. Equip $\mathbb{K}^m\langle\langle X\rangle\rangle$ with the topology for which f becomes a homeomorphism. Thus $U \subset \mathbb{K}^m\langle\langle X\rangle\rangle$ is open if and only if $f(U) \subset \prod_{\eta\in X^*}\mathbb{K}^m$ is open. In this way, f becomes a linear topological isomorphism of locally convex spaces.

As $\mathbb{K}^m\langle\langle X\rangle\rangle$, and in particular $\mathbb{R}\langle\langle X\rangle\rangle$, is a locally convex space, we may apply Chapter 2.2 to the context of Chen series. Indeed now $P[u](t)\colon [0,T] \to \mathbb{R}\langle\langle X\rangle\rangle$ is a curve into a locally convex space and $f\colon \mathbb{R}\langle\langle X\rangle\rangle \to \prod_{\eta\in X^*}\mathbb{R}$ is an isomorphism of locally convex spaces.

Proposition 53. *Suppose that a Chen Series*

$$P[u](t)\colon [0,T] \to \mathbb{R}\langle\langle X\rangle\rangle\,,\; t \mapsto \sum_{\eta\in X^*} E_\eta[u](t)\,\eta$$

is differentiable at $t \in [0,T]$. Then its derivative is of the form

$$\frac{d}{dt}\,P[u](t) = \frac{d}{dt}\sum_{\eta\in X^*} E_\eta[u](t)\,\eta = \sum_{\eta\in X^*}\frac{d}{dt}E_\eta[u](t)\,\eta$$

Proof. Since the topology on $\prod_{\eta\in X^*}\mathbb{R}$ is initial with respect to the projections and f is a linear continuous map, we have

$$f(\frac{d}{dt}P[u](t)) = f(dP[u](t;1)) = d(f\circ P[u])(t;1)$$
$$= \lim_{r\to 0}\frac{f(P[u](t+r)) - f(P[u](t))}{r} = \lim_{r\to 0}(\frac{E_\eta[u](t+r) - E_\eta[u](t)}{r})_{\eta\in X^*}$$
$$= (\lim_{r\to 0}\frac{E_\eta[u](t+r) - E_\eta[u](t)}{r})_{\eta\in X^*} = (\frac{d}{dt}E_\eta[u](t))_{\eta\in X^*}$$

Thus $f(\frac{d}{dt}P[u](t)) = (\frac{d}{dt}E_\eta(t))_{\eta\in X^*}$ and consequently

$$\frac{d}{dt}\,P[u](t) = \sum_{\eta\in X^*}\frac{d}{dt}E_\eta[u](t)\,\eta$$

□

The above shows that whenever the derivative of a Chen series exists it necessarily has to equal the series corresponding to the derivative of the components $E_\eta[u](t)$. In fact iterating the computations in Proposition 53 we see that

$$\frac{d^k}{dt^k}P[u](t) = \sum_{\eta\in X^*} \frac{d^k}{dt^k}E_\eta[u](t)\,\eta$$

Consequently the question of differentiability is completely determined by the differentiability of the component functions $E_\eta[u](t)$. Moreover from the locally convex topology on $\mathbb{R}\langle\langle X\rangle\rangle$ inherited by the product $\prod_{\eta\in X^*}\mathbb{R}$, we have that that any curve with values in $\mathbb{R}\langle\langle X\rangle\rangle$ is continuous if and only if it is continuous in each of its components. From this we deduce that $P[u](t)$ is C^k if and only if each $E_\eta[u](t)$ is C^k. Since

$$E_{x_i\bar{\eta}}[u](t) = \int_0^t u_i(s)\,E_{\bar{\eta}}[u](s)\,ds$$

it is clear that the differentiability of $E_\eta[u](t)$ is in turn determined by the functions u_i. In particular if the u_i are all C^k then $E_\eta[u](t)$ is C^{k+1} for all words η and hence $P[u](t)$ is C^{k+1}. In the case that the u_i are nothing more than L^1 functions then each $E_\eta[u](t)$ is an absolutely continuous function and hence differentiable almost everywhere ([25] Corollary 11 page 105). This means that the derivative of $E_\eta[u](t)$ will exist outside a set of Lebesgue measure zero. As the countable union of sets of measure zero is also a set of measure zero, there is a common set outside of which $\frac{d}{dt}E_\eta[u](t)$ exists for each $\eta\in X^*$. Hence outside of this set the derivative $\frac{d}{dt}P[u](t)$ will exist. Summarizing the above we obtain the following corollary

Corollary 54. *Given a Chen series $P[u](t) = \sum_{\eta\in X^*} E_\eta[u](t)\,\eta$ corresponding to an $L_1^m([0,T])$ mapping $u\colon [0,T]\to\mathbb{R}^m$. Then*

1. *$P[u](t)$ is differentiable almost everywhere.*
2. *If each component function $u_i\colon [0,T]\to\mathbb{R}$ is of class C^k then $P[u](t)$ is of class C^{k+1}.*

4 The Silva Space of Locally Convergent Series

We now follow the construction in [6] of the so called Silva space of locally convergent series, on which we will later consider various products. Its elements turn out to be formal power series (as introduced in the preceding chapter) which satisfies a growth condition on its coefficients. Specifically for any formal power series $c \in \mathbb{K}^m\langle\langle X\rangle\rangle$ we define $|(c,\eta)| := \max_{1\leq i\leq m} |(c[i],\eta)|$, where in the case that $\mathbb{K} = \mathbb{C}$ we set

$$|(c[i],\eta)| := \max\{|\mathrm{re}(c[i],\eta)|, |\mathrm{im}(c[i],\eta)|\}$$

If c is such that there are $K, M \geq 0$ for which

$$|(c,\eta)| \leq K M^{|\eta|}\,|\eta|! \quad \forall \eta \in X^* \tag{55}$$

then c is said to be locally convergent. The vector subspace

$$\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle := \{c \in \mathbb{K}^m\langle\langle X\rangle\rangle \;:\; (55) \text{ holds for some } K, M \geq 0\}$$

will be referred to as the space of locally convergent series. It is a locally convex space with subspace topology inherited from $\mathbb{K}^m\langle\langle X\rangle\rangle$. However we will show that one can turn $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ into a Silva space. For this we need to produce an imbedding spectre of Banach spaces whose union is $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ such that the inclusion mappings are compact operators. To this end consider for $M \in \mathbb{N}_0$ the set

$$\ell_{\infty,M}(X^*,\mathbb{K}^m) = \{c \in \mathbb{K}^m\langle\langle X\rangle\rangle \;:\; \sup_{\eta\in X^*} \frac{|(c,\eta)|}{M^{|\eta|}|\eta|!} < \infty\}$$

This is a vector subspace of $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ and the assignment

$$\|c\|_M = \sup_{\eta\in X^*} \frac{|(c,\eta)|}{M^{|\eta|}|\eta|!}$$

is easily seen to be a norm on $\ell_{\infty,M}(X^*, \mathbb{K}^m)$. Moreover the mapping

$$\ell_{\infty,M}(X^*, \mathbb{K}^m) \to \ell_\infty(X^*, \mathbb{K}^m)$$
$$c \mapsto \frac{c}{M^{|\eta|}|\eta|!} \colon X^* \to \mathbb{K}^m \, , \, \eta \mapsto \frac{(c,\eta)}{M^{|\eta|}|\eta|!}$$

is an isometry of normed linear spaces and hence as $\ell_\infty(X^*, \mathbb{K}^m)$ is a Banach space so too is $\ell_{\infty,M}(X^*, \mathbb{K}^m)$. Clearly if $M \leq N$ are integers then for any $c \in \mathbb{K}^m\langle\langle X\rangle\rangle$ and all $\eta \in X^*$

$$\frac{|(c,\eta)|}{N^{|\eta|}|\eta|!} \leq \frac{|(c,\eta)|}{M^{|\eta|}|\eta|!}$$

so that $\|\cdot\|_N \leq \|\cdot\|_M$ and consequently we have continuous inclusions

$$\ell_{\infty,M}(X^*, \mathbb{K}^m) \subset \ell_{\infty,N}(X^*, \mathbb{K}^m)$$

In fact these inclusion operators are not only continuous but compact [7, Lemma B.6]. Thus the inclusion mappings are compact operators between Banach spaces.

Lemma 56. *As sets*

$$\bigcup_{k\in\mathbb{N}} \ell_{\infty,k}(X^*, \mathbb{K}^m) = \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$$

Proof. Suppose $c \in \ell_{\infty,k}(X^*, \mathbb{K}^m)$ for some $k \in \mathbb{N}$. Then $\|c\|_k = R$ for some $R > 0$ implies that $\frac{|(c,\eta)|}{k^{|\eta|}|\eta|!} \leq R$ for all $\eta \in X^*$ or equivalently $|(c,\eta)| \leq R\, k^{|\eta|}\, |\eta|!$ for all $\eta \in X^*$. Thus by definition $c \in \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$. Conversely if c is such that for some $R, M > 0$ we have $|(c,\eta)| \leq R\, M^{|\eta|}|\eta|! \leq R\bar{M}^{|\eta|}|\eta|!$ for all $\eta \in X^*$, where $\bar{M}$ is any fixed integer larger than M. Then $\|c\|_{\bar{M}} \leq R$ and so $c \in \ell_{\infty,\bar{M}}(X^*, \mathbb{K}^m)$. □

From the above lemma we may conclude that the system

$$(j_k : \ell_{\infty,k}(X^*, \mathbb{K}^m) \to \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle)_{k\in\mathbb{N}}$$

is a countable imbedding spectre of Banach Spaces for which the inclusions maps

$$i_k : \ell_{\infty,k}(X^*, \mathbb{K}^m) \to \ell_{\infty,k+1}(X^*, \mathbb{K}^m)$$

are compact operators. It follows that $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ is a Silva space.

Proposition 57. *The Silva topology on $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ is finer than the subspace topology inherited from $\mathbb{K}^m\langle\langle X\rangle\rangle$.*

Proof. Suppose $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ carries the subspace topology. It suffices to prove that each inclusion $j_k\colon \ell_{\infty,k}(X^*, \mathbb{K}^m) \to \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ is continuous by the definition of the inductive limit topology of the inductive system

$$(j_k\colon \ell_{\infty,k}(X^*, \mathbb{K}^m) \to \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle)_{k\in\mathbb{N}}$$

Since the topology of $\mathbb{K}^m\langle\langle X\rangle\rangle$ is initial with respect to the linear projections

$$\pi_\eta\colon \mathbb{K}^m\langle\langle X\rangle\rangle \to \mathbb{K}^m\,,\ c \mapsto (c, \eta),$$

the linear map j_k will be continuous if and only if the composition

$$\pi_\eta \circ j_k\colon \ell_{\infty,k}(X^*, \mathbb{K}^m) \to \mathbb{K}^m$$

is a bounded linear operator of Banach spaces for each $\eta \in X^*$. But this is indeed the case because

$$|\pi_\eta \circ j_k(c)| = |(c, \eta)| \leq \|c\|_k k^{|\eta|}|\eta|!,$$

and so

$$\|\pi_\eta \circ j_k\| = \sup_{\|c\|_k \leq 1} |\pi_\eta \circ j_k(c)| \leq k^{|\eta|}|\eta|!$$

□

Proposition 58. *As locally convex spaces*

$$(\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle)_\mathbb{C} \cong \mathbb{C}^m_{LC}\langle\langle X\rangle\rangle$$

Proof. There is a canonical isometric isomorphism

$$(l_{\infty,n}(X^*, \mathbb{R}^m))_{\mathbb{C}} \cong l_{\infty,n}(X^*, \mathbb{C}^m)$$

via the map

$$\phi_n \colon \ell_{\infty,n}(X^*, \mathbb{C}^m) \to (\ell_{\infty,n}(X^*, \mathbb{R}^m))_{\mathbb{C}}\,,\ c \mapsto (\mathrm{re}(c), \mathrm{im}(c))$$

where we are viewing $(\ell_{\infty,n}(X^*, \mathbb{R}^m))_{\mathbb{C}}$ as a Banach space with the norm $\|(u, v)\| = \max\{\, \|u\|_n, \|v\|_n\}$. We are justified in doing this since the topology induced by this norm is identical to the product topology on the space. Now linearity of ϕ_n is clear. Moreover if we are given an element $(u, v) \in (\ell_{\infty,n}(X^*, \mathbb{R}^m))_{\mathbb{C}}$ then $u + iv \in \ell_{\infty,n}(X^*, \mathbb{C}^m)$. This follows from the fact that

$$\begin{aligned} \sup_{\eta\in X^*} \frac{|u(\eta) + iv(\eta)|}{n^{|\eta|}|\eta|!} &= \sup_{\eta\in X^*} \frac{\max\{\, |u(\eta)|\,,\, |v(\eta)|\}}{n^{|\eta|}|\eta|!} \\ &\leq \sup_{\eta\in X^*} \frac{|u(\eta)|}{n^{|\eta|}|\eta|!} + \sup_{\eta\in X^*} \frac{|v(\eta)|}{n^{|\eta|}|\eta|!} < \infty \end{aligned}$$

so that ϕ_n is onto. It is moreover an isometry. Indeed given $c \in \ell_{\infty,n}(X^*, \mathbb{C}^m)$

$$\begin{aligned} \|\phi_n(c)\| &= \|(\mathrm{re}(c), \mathrm{im}(c))\| = \max\{\, \|\mathrm{re}(c)\|_n, \|\mathrm{im}(c)\|_n\} \\ = \max\{\sup_{\eta\in X^*} \frac{|\mathrm{re}(c)(\eta)|}{n^{|\eta|}|\eta|!}, \sup_{\eta\in X^*} \frac{|\mathrm{im}(c)(\eta)|}{n^{|\eta|}|\eta|!}\} &= \sup_{\eta\in X^*} \max\{\frac{|\mathrm{re}(c)(\eta)|}{n^{|\eta|}|\eta|!}, \frac{|\mathrm{im}(c)(\eta)|}{n^{|\eta|}|\eta|!}\} \\ = \sup_{\eta\in X^*} \frac{\max\{|\mathrm{re}(c)(\eta)|, |\mathrm{im}(c)(\eta)|\}}{n^{|\eta|}|\eta|!} &= \sup_{\eta\in X^*} \frac{|c(\eta)|}{n^{|\eta|}|\eta|!} = \|c\|_n \end{aligned}$$

This establishes the isometric isomorphism of Banach spaces

$$(\ell_{\infty,n}(X^*, \mathbb{R}^m))_{\mathbb{C}} \cong \ell_{\infty,n}(X^*, \mathbb{C}^m)$$

From the definition of the collection $(\phi_n)_{n\in\mathbb{N}}$ it is clear that also

$$\phi_{n+1}\big|_{\ell_{\infty,n}(X^*,\mathbb{C}^m)} = \phi_n$$

Thus we may apply the Lemma 27 and 28 to conclude that

$$\begin{aligned} (\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle)_{\mathbb{C}} &= (\mathrm{ind}_{n\to\infty}\, \ell_{\infty,n}(X^*, \mathbb{R}^m))_{\mathbb{C}} = \mathrm{ind}_{n\to\infty}\, (\ell_{\infty,n}(X^*, \mathbb{R}^m))_{\mathbb{C}} \\ &\cong \mathrm{ind}_{n\to\infty}\, \ell_{\infty,n}(X^*, \mathbb{C}^m) = \mathbb{C}^m_{LC}\langle\langle X\rangle\rangle \end{aligned}$$

□

5 Products on the Silva Space of Locally Convergent Series

Having shown that $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ carries the structure of a Silva space we now proceed to consider some products on $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$. The main goal is to prove that these are smooth (holomorphic). However we will also consider the locally convex Lie groups some of them define, to which regularity class they belong to and the form of their associated locally convex Lie algebra. The first product considered is the most fundamental one of them all in the sense that every other product is defined in terms of this one.

5.1 Shuffle Product

The *shuffle product* between words is defined recursively as

$$(x\eta) \sqcup\!\sqcup (y\psi) = x(\eta \sqcup\!\sqcup (y\psi)) + y(x\eta \sqcup\!\sqcup \psi)$$

for $x, y \in X$ and $\eta, \psi \in X^*$ and $\eta \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \eta = \eta$ for all $\eta \in X^*$. The resulting shuffle product of two words, η and ξ of length n and m respectively, is a finite sum of words of length $n + m$ in which the summands are words obtained from interlacing η with ξ in such a way that the original ordering of the letters in the two are preserved.

Example 59. *If $\eta = x_0x_1$ and $\xi = x_1$ then*

$$\eta \sqcup\!\sqcup \xi = x_0x_1 \sqcup\!\sqcup x_1 = x_0x_1x_1 + x_0x_1x_1 + x_1x_0x_1 = 2x_0x_1^2 + x_1x_0x_1$$

From the shuffle product of words we define the shuffle product on $\mathbb{K}^m\langle\langle X\rangle\rangle$

Definition 60. *Let $c, d \in \mathbb{K}^m\langle\langle X\rangle\rangle$. Their shuffle product is denoted $c \sqcup\!\sqcup d$ and is the series whose components are given by*

$$(c \sqcup\!\sqcup d)[i] := (c[i] \sqcup\!\sqcup d[i]) := \sum_{v_1, v_2 \in X^*} (c[i], v_1)\,(d[i], v_2)\, v_1 \sqcup\!\sqcup v_2 \quad (i = 1, \ldots, m)$$

Given any $\eta, u, v \in X^*$, $(u \sqcup\!\sqcup\, v, \eta) \neq 0$ only if $|u| + |v| = |\eta|$. This means that we can write the value at η of the i'th component series in the shuffle product as

$$((c \sqcup\!\sqcup\, d)[i], \eta) = \sum_{k=0}^{|\eta|} \sum_{\substack{u \in X^k \\ v \in X^{|\eta|-k}}} (c[i], u)(d[i], v)\,(u \sqcup\!\sqcup\, v, \eta) \tag{61}$$

and consequently $((c \sqcup\!\sqcup\, d)[i], \eta) < \infty$ for all $\eta \in X^*$. Thus the shuffle product of formal power series is well defined.

Example 62. *Suppose $c, d \in \mathbb{R}^2\langle\langle X\rangle\rangle$ are such that $c[1] = c[2] = 4\emptyset + 2x_0$, $d[1] = x_1x_1 + x_2$ and $d[2] = 1\emptyset$ then*

$$\begin{aligned}
(c \sqcup\!\sqcup\, d)[1] &= (4\emptyset + 2x_0) \sqcup\!\sqcup\, (x_1x_1 + x_2) \\
&= 4\emptyset \sqcup\!\sqcup\, (x_1x_1 + x_2) + 2x_0 \sqcup\!\sqcup\, (x_1x_1 + x_2) \\
&= 4(\emptyset \sqcup\!\sqcup\, x_1x_1) + 4(\emptyset \sqcup\!\sqcup\, x_2) + 2(x_0 \sqcup\!\sqcup\, x_1x_1) + 2(x_0 \sqcup\!\sqcup\, x_2) \\
&= 4x_1x_1 + 4x_2 + 2(x_0x_1x_1 + x_1x_0x_1 + x_1x_1x_0) + 2(x_0x_2 + x_2x_0) \\
&= 4x_2 + 4x_1^2 + 2x_0x_2 + 2x_2x_0 + 2x_0x_1x_1 + 2x_1x_0x_1 + 2x_1x_1x_0 \\
(c \sqcup\!\sqcup\, d)[2] &= (4\emptyset + 2x_0) \sqcup\!\sqcup\, 1\emptyset = 4\emptyset + 2x_0 = c[2]
\end{aligned}$$

The shuffles product turns the vector space $\mathbb{K}\langle\langle X\rangle\rangle$ into a commutative and associative $\mathbb{K}$-algebra [23, Page 24]. By our definition of the shuffle product in the case when $m > 1$, this is also true for $\mathbb{K}^m\langle\langle X\rangle\rangle$. In fact it is even an integral domain [8, Page 8]. From its definition we immediately see that the product is bilinear on $\mathbb{K}^m\langle\langle X\rangle\rangle$.

Lemma 63. *For any $c, d \in \mathbb{K}\langle\langle X\rangle\rangle$, $\operatorname{ord}(c \sqcup\!\sqcup\, d) = \operatorname{ord}(c) + \operatorname{ord}(d)$. If $c, d \in \mathbb{K}^m\langle\langle X\rangle\rangle$ with $m > 1$, then $\operatorname{ord}(c \sqcup\!\sqcup\, d) \geq \operatorname{ord}(c) + \operatorname{ord}(d)$.*

Proof. The first statement holds in the case that $c = 0$ or $d = 0$ since then $c \sqcup\!\sqcup\, d = 0$ and $\operatorname{ord}(0) = \infty$. Suppose that neither are the zero element. Clearly $\operatorname{ord}(c \sqcup\!\sqcup\, d) \geq \operatorname{ord}(c) + \operatorname{ord}(d)$. To see the converse inequality we put $\operatorname{ord}(c) =: n$ and $\operatorname{ord}(d) =: k$ and define

$$p := \sum_{\eta \in X^n} (c, \eta)\, \eta$$

and

$$q := \sum_{\xi \in X^k} (d, \xi)\, \xi$$

The fact that $\mathbb{K}\langle\langle X \rangle\rangle$ is an integral domain and $p \neq 0$ and $q \neq 0$ implies together that $p \sqcup\!\sqcup\, q \neq 0$. So there is $\gamma \in X^{n+m}$ for which $(p \sqcup\!\sqcup\, q, \gamma \neq 0)$. Using equation 61 we see that

$$(c \sqcup\!\sqcup\, d, \gamma) = \sum_{\substack{u \in X^n \\ v \in X^k}} (c, u)(d, v)(u \sqcup\!\sqcup\, v, \gamma) = (p \sqcup\!\sqcup\, q, \gamma) \neq 0.$$

Thus $\mathrm{ord}(c \sqcup\!\sqcup\, d) \leq n+m = \mathrm{ord}(c) + \mathrm{ord}(d)$, which proves the first statement. The second statement now follows from the first since

$$\begin{aligned} \mathrm{ord}(c \sqcup\!\sqcup\, d) &= \min_{1 \leq i \leq m} \mathrm{ord}(c[i] \sqcup\!\sqcup\, d[i]) = \min_{1 \leq i \leq m} (\mathrm{ord}(d[i]) + \mathrm{ord}(c[i])) \\ &\geq \min_{1 \leq i \leq m} (\mathrm{ord}(d[i])) + \min_{1 \leq i \leq m} (\mathrm{ord}(c[i])) = \mathrm{ord}(d) + \mathrm{ord}(c) \end{aligned}$$

□

In Lemma 99 and 100 it is shown that the shuffle product on $\mathbb{K}^m_{LC}\langle\langle X \rangle\rangle$ induces a continuous mapping of Banach spaces

$$\sqcup\!\sqcup\colon \ell_{\infty,n}(X^*, \mathbb{K}^m) \times \ell_{\infty,m}(X^*, \mathbb{K}^m) \to \ell_{\infty,k(n,m)}(X^*, \mathbb{K}^m)$$

By the remarks following Corollary 25, $\sqcup\!\sqcup$ is a continuous product on the Silva space $\mathbb{K}^m_{LC}\langle\langle X \rangle\rangle$.

Theorem 64. *The shuffle product on* $\mathbb{K}^m_{LC}\langle\langle X \rangle\rangle$ *is smooth and* $\mathbb{K}$*-analytic.*

Proof. It is analytic and smooth by Proposition 33 because the shuffle product is a continuous bilinear map on the Silva space $\mathbb{K}^m_{LC}\langle\langle X \rangle\rangle$. To see that it is $\mathbb{R}$-analytic, note first that since any isomorphism of locally convex spaces is a diffeomorphism and by Proposition 58

$$(\mathbb{R}^m_{LC}\langle\langle X \rangle\rangle)_{\mathbb{C}} \cong \mathbb{C}^m_{LC}\langle\langle X \rangle\rangle$$

we see that the composition

$$(\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle \times \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle)_{\mathbb{C}} \cong \mathbb{C}^m_{LC}\langle\langle X\rangle\rangle \times \mathbb{C}^m_{LC}\langle\langle X\rangle\rangle$$
$$\to \mathbb{C}^m_{LC}\langle\langle X\rangle\rangle \cong (\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle)_{\mathbb{C}}$$

is holomorphic. Here the middle map is the shuffle product on $\mathbb{C}^m_{LC}\langle\langle X\rangle\rangle$. Since the shuffle product of locally convergent series whose coefficients are all real has again only real coefficients, this mapping is an analytic extension of the shuffle product on $\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$. □

Next we show that the set of invertible elements under the shuffle product on $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ forms a locally convex Lie group. These invertible elements turn out to be the so called non-proper series. A series in $c \in \mathbb{K}_{LC}\langle\langle X\rangle\rangle$ is said to be *proper* if $(c, \emptyset) = 0$ and is otherwise said to be *non-proper*. For the case $m > 1$ we extend the definition by saying that a series $c \in \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ is *non-proper* if

$$(c[i], \emptyset) \neq 0 \text{ for every } i = 1, \ldots, m$$

and is *proper* if

$$(c[i], \emptyset) = 0 \text{ for every } i = 1, \ldots, m$$

Note however that it is possible for series in $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ with $m > 1$ to be neither proper nor non-proper.

In what follows, whenever we write $c^{\sqcup\!\sqcup k}$ we mean the series obtained from taking the shuffle product of c with itself $k-1$ times. We define $c^{\sqcup\!\sqcup 1} := c$ and

$$c^{\sqcup\!\sqcup 0} := \mathbb{1}\colon X^* \to \mathbb{K}^m\,,\, \eta \mapsto \begin{cases} 0 \in \mathbb{K}^m & \text{if } \eta \neq \emptyset \\ \vec{1} \in \mathbb{K}^m & \text{else} \end{cases}$$

G will denote the set of non-proper series in $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$.

Proposition 65. *$(G, \sqcup\!\sqcup)$ forms an abelian group. The inverse of any element $c \in G$ is $c^{-1} \in \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ whose component series are given by*

$$c[i]^{-1} = (c[i], \emptyset)^{-1}\, (c[i]')^{\sqcup\!\sqcup *}$$

where $c[i]' = \mathbb{1} - (c[i], \emptyset)^{-1} c[i]$ *and*

$$(c[i]')^{\sqcup\!\sqcup *} = \sum_{n \geq 0} (c[i]')^{\sqcup\!\sqcup n}$$

Proof. The identity element is the series $\mathbb{1}$ because for each $i = 1, \dots, m$ we have

$$c[i] \sqcup\!\sqcup \mathbb{1}[i] = \sum_{u,v \in X^*} (c[i], u)(\mathbb{1}[i], v)\, u \sqcup\!\sqcup v = \sum_{u \in X^*} (c[i], u)(\mathbb{1}[i], \emptyset)\, u \sqcup\!\sqcup \emptyset$$
$$= \sum_{u \in X^*} (c[i], u)\, u = c[i] = \mathbb{1}[i] \sqcup\!\sqcup c[i]$$

We have already remarked that the shuffle product is both associative and commutative. It has also been remarked that for c, d locally convergent, their shuffle product is a locally convergent series. Moreover since $\mathbb{K}$ is a field, $(c[i] \sqcup\!\sqcup d[i], \emptyset) = (c[i], \emptyset)(d[i], \emptyset) \neq 0$ for each $i = 1, \dots, m$, so that $c \sqcup\!\sqcup d$ is also non-proper. To show that the inverse is as stated we compute for each $i \in \{1, \dots, m\}$

$$c[i]^{-1} \sqcup\!\sqcup c[i] = c[i] \sqcup\!\sqcup c[i]^{-1} = c[i] \sqcup\!\sqcup \big((c[i], \emptyset)^{-1} \sum_{n \geq 0} (c[i]')^{\sqcup\!\sqcup n}\big)$$
$$= (c[i], \emptyset)^{-1} (\sum_{n \geq 0} c[i] \sqcup\!\sqcup (c[i]')^{\sqcup\!\sqcup n})$$
$$= (c[i], \emptyset)^{-1} (c[i] \sqcup\!\sqcup \mathbb{1}[i] + c[i] \sqcup\!\sqcup c[i]' + c[i] \sqcup\!\sqcup c[i]' \sqcup\!\sqcup c[i]' + \dots)$$

Using the relation $c[i] = (c[i], \emptyset)(\mathbb{1}[i] - c[i]')$ we obtain that

$$c[i] \sqcup\!\sqcup c[i]^{-1} = (c[i], \emptyset)^{-1} ((c[i], \emptyset)(\mathbb{1}[i] - c[i]') \sqcup\!\sqcup \mathbb{1}[i]$$
$$+ ((c[i], \emptyset)(\mathbb{1}[i] - c[i]')) \sqcup\!\sqcup c[i]' + ((c[i], \emptyset)(\mathbb{1}[i] - c[i]')) \sqcup\!\sqcup c[i]' \sqcup\!\sqcup c[i]' + \dots)$$
$$= (\mathbb{1}[i] - c[i]' + c[i]' - c[i]' \sqcup\!\sqcup c[i]' + c[i]' \sqcup\!\sqcup c[i]' + \dots) = \mathbb{1}[i]$$

In the above we have used that

$$c[i] \sqcup\!\sqcup \big((c[i], \emptyset)^{-1} \sum_{n \geq 0} (c[i]')^{\sqcup\!\sqcup n}\big) = (c[i], \emptyset)^{-1} (\sum_{n \geq 0} c[i] \sqcup\!\sqcup (c[i]')^{\sqcup\!\sqcup n})$$

This is valid because c' proper implies $\operatorname{ord}(c') \geq 1$ and by Lemma 63, $\operatorname{ord}((c')^{\sqcup\!\sqcup n}) \geq n$. Thus

$$(\sum_{n\geq 0}(c[i]')^{\sqcup\!\sqcup\, n}, \eta) = (\sum_{n=0}^{|\eta|}(c[i]')^{\sqcup\!\sqcup\, n}, \eta)$$

Using this when looking at $(c[i] \sqcup\!\sqcup \big((c[i], \emptyset)^{-1} \sum_{n\geq 0}(c[i]')^{\sqcup\!\sqcup\, n}\big), \eta)$ for any $\eta \in X^*$, one can show that

$$(c[i] \sqcup\!\sqcup \big((c[i], \emptyset)^{-1} \sum_{n\geq 0}(c[i]')^{\sqcup\!\sqcup\, n}\big), \eta) = ((c[i], \emptyset)^{-1} \sum_{n\geq 0} c_i \sqcup\!\sqcup (c[i]')^{\sqcup\!\sqcup\, n}, \eta)$$

Finally the inverse is an element of G. Indeed it is clearly non-proper and moreover in [14, Theorem 5] it is shown that each component series $c[i]^{-1} \in \mathbb{K}_{LC}\langle\langle X\rangle\rangle$ and hence $c^{-1} \in \mathbb{K}_{LC}^m\langle\langle X\rangle\rangle$. □

An important remark for our purposes of turning G into a locally convex Lie group is that it forms an open subset of the Silva space $\mathbb{K}_{LC}^m\langle\langle X\rangle\rangle$.

Lemma 66. *G is open in $\mathbb{K}_{LC}^m\langle\langle X\rangle\rangle$*

Proof. The evaluation maps $\pi_\eta(c) = (c, \eta)$ are bounded linear operators of Banach spaces when restricted to each Banach step. Indeed linearity is clear and given $c \in \ell_{\infty,M}(X^*, \mathbb{K}^m)$ we have that

$$|\pi_\eta(c)| = |(c, \eta)| \leq \|c\|_M \, M^{|\eta|}|\eta|!$$

and so

$$\|\pi_\eta\| = \sup_{\|c\|\leq 1} |\pi_\eta(c)| \leq M^{|\eta|}\, |\eta|!$$

In particular Proposition 24 shows that they are continuous on $\mathbb{K}_{LC}^m\langle\langle X\rangle\rangle$. Since

$$G = \pi_\emptyset^{-1}(\prod_{i=1}^{m} \mathbb{K} \setminus \{0\}),$$

the group G is open, as the preimage of an open set under a continuous map. □

Next the shuffle inverse is shown to be a continuous map on G. The argument will rely on uniformly bounding the Neumann type series which defines the inverse. To do this we will bound the infinite sum by a geometric one using the worst case estimates for the element defining the series. The subsequent four lemmas will provide the necessary tools for this.

Lemma 67. *For K positive and M a natural number, let d be the proper series in $\ell_{\infty,M}(X^*,\mathbb{K}^m)$ whose components are defined as $(d[i],\eta) := K\, M^{|\eta|}|\eta|!$ for $\eta \neq \emptyset$. Then for all $\eta \in X^*$ and each i,*

$$(d[i]^{\sqcup\!\sqcup\, n},\eta) = |(d^{\sqcup\!\sqcup},\eta)| \leq K^n\, M^{|\eta|} \binom{(n-1)+|\eta|}{n-1} |\eta|!$$

Proof. The proof is by induction on $n \in \mathbb{N}$. The case $n=1$ is valid by definition. Let $n > 1$ and $i \in \{1,\dots,m\}$. We have then for any $\eta \in X^*$

$$(d[i]^{\sqcup\!\sqcup n},\eta) = (d[i] \sqcup\!\sqcup\, d[i]^{\sqcup\!\sqcup\,(n-1)},\eta)$$

$$= \sum_{k=0}^{|\eta|} \sum_{\substack{u\in X^k\\ v\in X^{|\eta|-k}}} (d[i],v)(d[i]^{\sqcup\!\sqcup\,(n-1)},u)\,(u \sqcup\!\sqcup\, v,\eta)$$

$$\leq \sum_{k=0}^{|\eta|} \sum_{\substack{u\in X^k\\ v\in X^{|\eta|-k}}} K\,M^{|\eta|-k}\,(|\eta|-k)!\,M^k\,K^{n-1}\,k!\binom{(n-2)+k}{n-2}(u \sqcup\!\sqcup\, v,\eta)$$

$$= K^n\,M^{|\eta|}\sum_{k=0}^{|\eta|} k!(|\eta|-k)!\binom{(n-2)+k}{n-2}\sum_{\substack{u\in X^k\\ v\in X^{|\eta|-k}}}(u \sqcup\!\sqcup\, v,\eta)$$

$$= K^n M^{|\eta|}\sum_{k=0}^{|\eta|} k!\,(|\eta|-k)!\binom{|\eta|}{k}\binom{(n-2)+k}{n-2}$$

$$= K^n\,M^{|\eta|}\,|\eta|!\sum_{k=0}^{|\eta|}\binom{(n-2)+k}{n-2} = K^n\,M^{|\eta|}\,|\eta|!\binom{(n-1)+|\eta|}{n-1}$$

where the identities

$$\sum_{k=0}^{|\eta|}\binom{(n-2)+k}{n-2} = \binom{(n-1)+|\eta|}{n-1}$$

and

$$\sum_{\substack{u\in X^k\\ v\in X^{|\eta|-k}}} (u \sqcup\!\sqcup v, \eta) = \binom{|\eta|}{k} \qquad (0 \leq k \leq |\eta|)$$

were used. The first is well known and can be shown using a standard induction argument. To justify the second recall that the shuffle product of two words $u, v \in X^*$ is the linear combination of all the words with length $|u| + |v|$ one can obtain by interlacing u and v in such a way that the order in which the letters originally appeared in the two are preserved. In our case where $u \in X^k$ and $v \in X^{|\eta|-k}$, $(u \sqcup\!\sqcup v, \eta) = 0$ if u or v cannot be obtained by choosing respectively k or $|\eta| - k$ letters from the word η. So if u is obtained from η by choosing k of its letters then necessarily v must consist of the remaining letters. Moreover whenever $(u \sqcup\!\sqcup v, \eta) \geq 1$ it must equal the number of ways in which k letters can be chosen from η to equal u and thus the remaining to equal v. Summing over all k-letter words u, the second identity follows. □

Lemma 68. *For each $n \in \mathbb{N}$ let f_n be the positive function defined on integers $m \geq 1$ as*

$$f_n(m) = \binom{(n-1)+m}{n-1} \frac{1}{4^m}, \qquad m \geq n$$

and $f_n(m) = 0$ for $m < n$. Then there is a positive constant K for which $f_n(m) \leq K$ for all $n, m \in \mathbb{N}$.

Proof. Firstly the sequence of positive numbers $\{f_n(n)\}_{n\in\mathbb{N}}$ converges to zero. Indeed

$$f_n(n) = \frac{(2n-1)\dots(1+n)\,n!}{n!\,(n-1)!\,4^n} = \frac{(2n)!}{2n\,(n-1)!\,n!\,4^n} = \frac{(2n)!}{2\,n!\,n!\,4^n}.$$

Using the following estimate for $n!$ in [24, Page 28]

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}\, e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}\, e^{\frac{1}{12n}},$$

valid for positive integers n, we obtain

$$f_n(n) \le \frac{1}{2}\,\frac{\sqrt{2\pi}}{2\pi}\, n^{-(2n+1)}\,(2n)^{2n+\frac{1}{2}}\,e^{2n}\,e^{-2n}\,e^{\frac{1}{24n}}\,e^{-\frac{2}{12n+1}}\,\frac{1}{4^n}$$
$$= \frac{1}{2\sqrt{\pi}}\,(\frac{4}{4})^n\,\frac{1}{\sqrt{n}}\,e^{\frac{1}{24n}-\frac{2}{12n+1}}.$$

The latter sequence tends to zero as n tends to infinity, implying the same for $f_n(n)$. Thus there is $K > 0$ for which $f_n(n) \le K$ for all $n \ge 1$.

Secondly $f_n(n)$ is the maximum of f_n for any $n \in \mathbb{N}$. To see this fix $n \ge 1$ and suppose $m = n + i$ with $i \ge 0$ an integer. Then

$$\begin{aligned} & f_n(m) \ge f_n(m+1) \\ \iff & \frac{(n-1+m)(n-2+m)\dots(1+m)}{(n-1)!\,4^m} \\ \ge & \frac{(n-1+m+1)(n-2+1+m)\dots(m+1+1)}{(n-1)!4^{m+1}} \\ \iff & 4(1+m) \ge (n-1+m+1) \iff 4(1+n+i) \ge (2n+i), \end{aligned}$$

and the last statement is certainly true for any integer $i \ge 0$. Then $f_n(m) \le K$ for all $n, m \ge 1$. □

Lemma 69. *Let c and c_j be proper series such that $\|c - c_j\|_M \to 0$ for some $M \in \mathbb{N}$. Then there is $N > M$ for which*

$$\sup_{j\in\mathbb{N}} \|c_j\|_N \le \frac{1}{2} \sup_{j\in\mathbb{N}} \|c_j\|_M.$$

In other words, we may make $\sup_{j\in\mathbb{N}} \|c_j\|_N$ as small as we like by choosing a large enough $N \in \mathbb{N}$.

Proof. If $\|c_j - c\|_M \to 0$ then $\sup_j \|c_j\|_M =: K < \infty$. We have that for any $N \ge M$, $\|c_j\|_N \le \|c\|_N + \|c - c_j\|_N \le \|c\|_N + \|c - c_j\|_M$. Choose $J \in \mathbb{N}$ so that $j > J$ implies $\|c - c_j\|_M \le \frac{K}{4}$. Whenever $N > M$ we have for nonzero proper elements $d \in \ell_{\infty,M}(X^*\mathbb{K}^m)$ that $\|d\|_N < \|d\|_M$. Thus for any $1 \le j \le J$ there is N_j so that $\|c - c_j\|_{N_j} \le \frac{K}{4}$. As $J < \infty$ we may choose $N \geq M$, for which $\|c - c_j\|_N \leq \frac{K}{4}$ for all $1 \leq j \leq J$ and $\|c\|_N \leq \frac{K}{4}$. Then $\|c_j\|_N \leq \|c\|_N + \|c - c_j\|_N \leq \frac{K}{4} + \frac{K}{4} = \frac{K}{2}$ for each $j \in \mathbb{N}$. That is, $\sup_j \|c_j\|_N \leq \frac{1}{2} \sup_j \|c_j\|_M$. □

Lemma 70. *If $\|c - c_j\|_M \to 0$ for some $M \in \mathbb{N}$ then*

$$\|c^{\sqcup\!\sqcup\, n} - c_j^{\sqcup\!\sqcup\, n}\|_{M_n} \to 0 \textit{ for all } n \geq 1$$

where $2M > M_{n+1} > M_n \geq M_1 = M$, and $M_n = M(1 + (1 - \frac{1}{n}))$

Proof. The proof is by induction on $n \geq 1$. When $n = 1$ the statement is valid by definition. Let $n > 1$. Then using the bilinearity of the shuffle product we get

$$\|c^{\sqcup\!\sqcup\, n} - c_j^{\sqcup\!\sqcup\, n}\|_{M_n} = \|(c - c_j) \sqcup\!\sqcup\, c^{\sqcup\!\sqcup\,(n-1)} + c_j \sqcup\!\sqcup\, (c^{\sqcup\!\sqcup\,(n-1)} - c_j^{\sqcup\!\sqcup\,(n-1)})\|_{M_n}$$
$$\leq K_n \|c - c_j\|_{M_{n-1}} \|c^{\sqcup\!\sqcup\,(n-1)}\|_{M_{(n-1)}} + K_n \|c_j\|_{M_{(n-1)}} \|c^{\sqcup\!\sqcup\,(n-1)} - c_j^{\sqcup\!\sqcup\,(n-1)}\|_{M_{(n-1)}}$$

where $K_n > 0$ is the constant $K_n = \sup_{\eta \in X^*} \frac{|\eta|+1}{(1+\epsilon_n)^{|\eta|}}$ from Lemma 99 and ϵ_n is the positive constant such that $M_n = (1+\epsilon_n)M_{n-1}$. The term $\|c^{\sqcup\!\sqcup\,(n-1)}\|_{M_{n-1}}$ is also a positive constant since it is bounded by $\|c\|_M K_2 \ldots K_{n-1}$. All of this implies as desired that for any $n \geq 1$

$$\lim_{j \to \infty} \|c^{\sqcup\!\sqcup\, n} - c_j^{\sqcup\!\sqcup\, n}\|_{M_n} = 0$$

□

Proposition 71. *If for proper series c_j, c we have $\|c_j - c\|_M \to 0$ for some $M \in \mathbb{N}$ then*

$$\|\sum_{n \geq 0} c^{\sqcup\!\sqcup\, n} - c_j^{\sqcup\!\sqcup\, n}\|_{N_0} \to 0$$

for a sufficiently large $N_0 \in \mathbb{N}$.

Proof. We will show that the sum

$$\sum_{n \geq 0} \|c^{\sqcup\!\sqcup\, n} - c_j^{\sqcup\!\sqcup\, n}\|_{N_0}$$

is uniformly bounded for some $N_0 \geq 2M$. With this and Lemma 70 we see that

$$\lim_{j\to\infty} \|\sum_{n\geq 0} c^{\sqcup\!\sqcup\, n} - c_j^{\sqcup\!\sqcup\, n}\|_{N_0} \leq \lim_{j\to\infty} \sum_{n\geq 0} \|c^{\sqcup\!\sqcup\, n} - c_j^{\sqcup\!\sqcup\, n}\|_{N_0}$$
$$= \sum_{n\geq 0} \lim_{j\to\infty} \|c^{\sqcup\!\sqcup\, n} - c_j^{\sqcup\!\sqcup\, n}\|_{N_0} \leq \sum_{n\geq 0} \lim_{j\to\infty} \|c^{\sqcup\!\sqcup n} - c_j^{\sqcup\!\sqcup n}\|_{M_n} = 0$$

We start by letting $K_0 := \sup_n f_n(n)$ where f_n is as defined in Lemma 68. Using Lemma 69 we can find an $N \in \mathbb{N}$ so that $\|c\|_N + \sup_j \|c_j\|_N \leq \frac{1}{2}$. Define $d \in l_{\infty,N}(X^*, \mathbb{K}^m)$ as the proper series all of whose components $d[i]$ satisfy

$$(d[i], \eta) := (\|c\|_N + \sup_{j\in\mathbb{N}} \|c_j\|_N)\, N^{|\eta|}\, |\eta|!,$$

for $\eta \neq \emptyset$. Then from its definition we have for every i that $|(d, \eta)| = (d[i], \eta) \geq |(c, \eta)| + |(c_j, \eta)|$ for all $j \in \mathbb{N}$ and all $\eta \in X^*$. Moreover for any fixed $r = 1, \ldots, m$

$$(d[i] \sqcup\!\sqcup\, d[i], \eta) = \sum_{k=0}^{|\eta|} \sum_{\substack{u\in X^k \\ v\in X^{|\eta|-k}}} (d[i], u)(d[i], v)(u \sqcup\!\sqcup\, v, \eta)$$
$$\geq \sum_{k=0}^{|\eta|} \sum_{\substack{u\in X^k \\ v\in X^{|\eta|-k}}} (|(c[r], u)| + |(c_j[r], u)|)(|(c[r], v)| + |(c_j[r], v)|)(u \sqcup\!\sqcup\, v, \eta)$$
$$\geq \sum_{k=0}^{|\eta|} \sum_{\substack{u\in X^k \\ v\in X^{|\eta|-k}}} |(c[r], u)|\, |(c[r], v)|(u \sqcup\!\sqcup\, v, \eta)$$
$$+ \sum_{k=0}^{|\eta|} \sum_{\substack{u\in X^k \\ v\in X^{|\eta|-k}}} |(c_j[r], u)|\, |(c_j[r], v)|(u \sqcup\!\sqcup\, v, \eta)\big)$$
$$\geq |(c[r] \sqcup\!\sqcup\, c[r], \eta)| + |(c_j[r] \sqcup\!\sqcup\, c_j[r], \eta)|$$

for any $\eta \in X^*$ and so in particular we have that

$$|(d \sqcup\!\sqcup\, d, \eta)| = (d[i] \sqcup\!\sqcup\, d[i], \eta) \geq |(c \sqcup\!\sqcup\, c, \eta)| + |(c_j \sqcup\!\sqcup\, c_j, \eta)|$$

In fact $|(d^{\sqcup\!\sqcup\, n},\eta)| \geq |(c^{\sqcup\!\sqcup\, n},\eta)| + |(c_j^{\sqcup\!\sqcup\, n},\eta)|$ for all $\eta \in X^*$, $j \in \mathbb{N}$ and $n \geq 1$ by a standard induction argument. Thus for all $N' \geq N$, $n \geq 1$ and $j \in \mathbb{N}$ we have

$$\|d^{\sqcup\!\sqcup\, n}\|_{N'} \geq \|c^{\sqcup\!\sqcup\, n} - c_j^{\sqcup\!\sqcup\, n}\|_{N'}$$

Using Lemma 67 and 68 we see that for any $n \geq 1$, $i \in \{1, \ldots, m\}$ and all $\eta \in X^*$ we have

$$\begin{aligned} |(d^{\sqcup\!\sqcup n},\eta)| &= (d[i]^{\sqcup\!\sqcup\, n},\eta) \leq (\|c\|_N + \sup_{j\in\mathbb{N}} \|c_j\|_N)^n \, N^{|\eta|} \binom{(n-1)+|\eta|}{n-1} |\eta|! \\ &\leq \frac{1}{2^n}\, N^{|\eta|} \binom{(n-1)+|\eta|}{n-1} |\eta|! = \frac{1}{2^n}\,(4N)^{|\eta|}\,|\eta|! \binom{(n-1)+|\eta|}{n-1} \frac{1}{4^{|\eta|}} \\ &= \frac{1}{2^n}\,(4N)^{|\eta|}\,|\eta|!\, f_n(|\eta|) \leq \frac{1}{2^n}\,(4N)^{|\eta|}\,|\eta|!\, K_0 \end{aligned}$$

So that for any $n \geq 1$, $\|d^{\sqcup\!\sqcup\, n}\|_{4N} \leq \frac{1}{2^n} K_0$. This implies that

$$\begin{aligned} \sum_{n\geq 0} \|c^{\sqcup\!\sqcup\, n} - c_j^{\sqcup\!\sqcup\, n}\|_{4N} &= \sum_{n\geq 1} \|c^{\sqcup\!\sqcup\, n} - c_j^{\sqcup\!\sqcup\, n}\|_{4N} \\ &\leq \sum_{n\geq 1} \|d^{\sqcup\!\sqcup\, n}\|_{4N} \leq \sum_{n\geq 1} \frac{K_0}{2^n} = K_0 < \infty \end{aligned}$$

□

Theorem 72. *The shuffle inverse*

$$\sqcup\!\sqcup^{-1} \colon G \to G\,,\; c \mapsto \sqcup\!\sqcup^{-1}(c)$$

with component series

$$\sqcup\!\sqcup^{-1}(c)[i] = (c[i],\emptyset)^{-1} \sum_{n\geq 0} (c[i]')^{\sqcup\!\sqcup\, n},$$

is continuous.

Proof. Lemma 66 shows that G is an open subset of the sequential space $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$, whence Lemma 16 shows that G is also sequential. Thus $\sqcup\!\sqcup^{-1}$ is continuous if it is sequentially continuous by Lemma 17. Let $\{c_j\}_{j\in\mathbb{N}}$ be a sequence in G converging to $c \in G$. Using Lemma 8 together with Proposition 23 2. we see that any convergent sequence $\{c_j\}_{j\in\mathbb{N}} \subset G \subset \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ is contained entirely in $\ell_{\infty,M}(X^*, \mathbb{K}^m)$ for some $M \in \mathbb{N}$. In particular $\|c - c_j\|_M \to 0$. Then also $\|c'_j - c'\|_M \to 0$, where as before

$$c'[i] = \mathbb{1}[i] - \frac{c[i]}{(c[i], \emptyset)} \text{ and } c'_j[i] = \mathbb{1}[i] - \frac{c_j[i]}{(c_j[i], \emptyset)}$$

Since c', c'_j are proper series with $\|c'_j - c'\|_M \to 0$ we can use Proposition 71 to conclude that

$$\| \sum_{n\geq 0} (c')^{\sqcup\! \sqcup n} - (c'_j)^{\sqcup\! \sqcup n} \|_N \to 0$$

for some fixed N large enough. Hence also

$$\| \sqcup\! \sqcup^{-1}(c) - \sqcup\! \sqcup^{-1}(c_j)\|_N = \|(c, \emptyset)^{-1} \sum_{n\geq 0} (c')^{\sqcup\! \sqcup n} - (c_j, \emptyset)^{-1} \sum_{n\geq 0} (c'_j)^{\sqcup\! \sqcup n} \|_N \to 0$$

or in other words, $\sqcup\! \sqcup^{-1}(c_j) \to \sqcup\! \sqcup^{-1}(c)$ in G. □

Lemma 73. $G = (\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle, \sqcup\! \sqcup)^{\times}$

Proof. Suppose that $c \notin G$ say $(c[i], \emptyset) = 0$ for some $i \in \{1, \dots, m\}$. Let $d[i] \in \mathbb{K}_{LC}\langle\langle X\rangle\rangle$. Then $(c[i] \sqcup\! \sqcup d[i], \emptyset) = (c[i], \emptyset)(d[i], \emptyset) = 0$ and so $c[i] \sqcup\! \sqcup d[i] \neq \mathbb{1}[i]$. In particular there is no $d \in \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ for which $c \sqcup\! \sqcup d = \mathbb{1}$ so that $c \notin (\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle, \sqcup\! \sqcup)^{\times}$. □

Summarizing this chapter up till now we know that $(\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle, \sqcup\! \sqcup)$ is a commutative $\mathbb{K}$-algebra with smooth (holomorphic) bilinear product and continuous inverse. Moreover by Lemma 66 and 73 we know that its unit group G is an open subset. Thus $(\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle, \sqcup\! \sqcup)$ is a *continuous inverse algebra* and we may apply Proposition 45 to obtain the following result.

Theorem 74. *G is a smooth and $\mathbb{K}$-analytic Lie group.*

Proof. From Proposition 45 we know that G is an analytic Lie group in the case $\mathbb{K} = \mathbb{C}$ and a smooth Lie group when $\mathbb{K} = \mathbb{R}$. Theorem 64 shows that the shuffle product is real analytic and in a similar way to the proof there, we can find an analytic extension of the shuffle inverse. Consequently G is a real analytic Lie group. □

Proposition 75. *The Lie algebra of G is $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ with constant zero bracket.*

Proof. As G is an open subset of $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ we see that

$$L(G) = \mathrm{T}_{\mathbb{1}}G = \mathrm{T}_{\mathbb{1}}\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle = \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$$

Let λ_c denote the left multiplication by c, i.e. $\lambda_c(d) = c \sqcup\!\sqcup d$. Its tangent map at the identity is given by

$$\mathrm{T}_{\mathbb{1}}\lambda_c(d) = \left.\frac{d}{dt}\right|_{t=0} c \sqcup\!\sqcup (\mathbb{1} + t\, d) = c \sqcup\!\sqcup d$$

The left invariant vector field corresponding to $d \in L(G)$ has the form $X^d(c) = \mathrm{T}_{\mathbb{1}}\lambda_c(d) = c \sqcup\!\sqcup d$. Consequently the bracket of $L(G)$ is

$$\begin{aligned} [c, d] &= X^c.X^d(\mathbb{1}) - X^d.X^c(\mathbb{1}) = \mathrm{d}X^d \circ X^c(\mathbb{1}) - \mathrm{d}X^c \circ X^d(\mathbb{1}) \\ &= \mathrm{d}X^d(\mathbb{1}; c) - \mathrm{d}X^c(\mathbb{1}; d) = d \sqcup\!\sqcup c - c \sqcup\!\sqcup d = 0 \end{aligned}$$

since the shuffle product is commutative. □

Proposition 76. *The Lie group G is C^0-regular*

Proof. G is the unit group of a sequentially complete (in particular Mackey-complete) commutative CIA. Then the proof in [11, Corollary page 3] together with [11, Proposition 3.4 (a)] shows that G is C^0-regular. □

5.2 Composition Products

Using the shuffle product from the preceding chapter we define various new products on $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ whose smoothness (analyticity) we prove. Before defining the so called composition product of formal power series we first define it as a product between words and series.

Definition 77. *For any $\eta \in X^*$ and $d \in \mathbb{K}^m\langle\langle X\rangle\rangle$*

$$\eta \circ d := \begin{cases} \eta & \text{if } \eta = x_0^n,\ \ n \geq 0, \\ x_0^{n+1}(d[i] \sqcup\!\sqcup (\eta' \circ d)) & \text{if } \eta = x_0^n x_i \eta',\ \ i \neq 0,\ \ n \geq 0 \end{cases}$$

Note in particular that $x_0\eta \circ d = x_0(\eta \circ d)$. Also since we may write any $\eta \in X^*$ as

$$\eta = x_0^{n_k} x_{i_k} x_0^{n_{k-1}} x_{i_{k-1}} \ldots x_0^{n_1} x_{i_1} x_0^{n_0}$$

with $k = |\eta| - |\eta|_{x_0}$, $i_j \neq 0$ and $n_i \geq 0$, we see that

$$\eta \circ d = x_0^{n_k+1}(d[i_k] \sqcup\!\sqcup (x_0^{n_{k-1}+1}(d[i_{k-1}] \sqcup\!\sqcup \ldots x_0^{n_1+1}(d[i_1] \sqcup\!\sqcup x_0^{n_0})) \ldots))$$

Definition 78. *Given $c, d \in \mathbb{K}^m\langle\langle X\rangle\rangle$ their composition product is defined as*

$$c \circ d := \sum_{\eta \in X^*} (c, \eta)\, \eta \circ d$$

Its components are of the form

$$(c \circ d)[i] = \sum_{\eta \in X^*} (c[i], \eta)\, \eta \circ d$$

Example 79. *Let $c, d \in \mathbb{K}\langle\langle X\rangle\rangle$ with $d = 0$. Then $x_0^n \circ d = x_0^n$ and $\eta \circ d = 0$ for all $\eta \in X^*$ that is not a power of x_0. Consequently*

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\eta \circ d = \sum_{n \geq 0} (c, x_0^n) x_0^n$$

If we instead set $d = 1\emptyset$ then for $\eta = x_0^n x_i \eta'$ with $i \neq 0$ we have $\eta \circ d = x_0^{n+1}(\emptyset \sqcup\!\sqcup \eta' \circ \emptyset) = x_0^{n+1}(\eta' \circ \emptyset)$. Iterating we see that for any $\eta \in X^*$, $\eta \circ \emptyset = x_0^{|\eta|}$ and so

$$c \circ d = \sum_{\eta \in X^*} (c, \eta) x_0^{|\eta|}$$

For a more typical example put $d = x_1$. Then the first few terms of the series defining $c \circ d$ is given by

$$
\begin{aligned}
&(c,\emptyset)\,\emptyset \circ x_1 + (c,x_0)\,x_0 \circ x_1 + (c,x_1)\,x_1 \circ x_1 + (c,x_0^2)\,x_0^2 \circ x_1 \\
&+(c,x_1^2)\,x_1^2 \circ x_1 + (c,x_0x_1)\,x_0x_1 \circ x_1 + (c,x_1x_0)\,x_1x_0 \circ x_1 + \ldots \\
&= (c,\emptyset)\,\emptyset + (c,x_0)\,x_0 + (c,x_1)\,x_0x_1 + (c,x_0^2)\,x_0^2 + (c,x_1^2)\,x_0x_1x_0x_1 \\
&+2(c,x_1^2)\,x_0^2x_1^2 + (c,x_0x_1)\,x_0^2x_1 + (c,x_1x_0)\,x_0x_1x_0 + (c,x_1x_0)\,x_0^2x_1 + \ldots
\end{aligned}
$$

Proposition 80. *The composition product is well defined on $\mathbb{K}^m\langle\langle X\rangle\rangle$.*

Proof. Consider the value of the composition product of two series at an arbitrary word $\eta \in X^*$, $(c \circ d, \eta)$. We claim that

$$(c \circ d, \eta) = \sum_{i=0}^{|\eta|} \sum_{u \in X^i} (c,u)\,(u \circ d, \eta) \tag{81}$$

If the above holds then $|(c \circ d, \eta)| < \infty$, as a finite sum of real (complex) numbers and well definedness is proven. To prove the claim let $\xi \in X^*$ be any word with $|\xi| > |\eta|$. We want show that $(\xi \circ d, \eta) = 0$. Write

$$\xi = x_0^{n_k} x_{i_k} x_0^{n_{k-1}} x_{i_{k-1}} \ldots x_0^{n_1} x_{i_1} x_0^{n_0},$$

with $k = |\xi| - |\xi|_{x_0}$. Then since

$$\xi \circ d = x_0^{n_k+1}(d[i_k] \sqcup\!\sqcup (x_0^{n_{k-1}+1}(d[i_{k-1}] \sqcup\!\sqcup \ldots x_0^{n_1+1}(d[i_1] \sqcup\!\sqcup x_0^{n_0}))\ldots)),$$

and by Lemma 63, $\mathrm{ord}(c \sqcup\!\sqcup d) = \mathrm{ord}(c) + \mathrm{ord}(d)$ whenever $c, d \in \mathbb{K}\langle\langle X\rangle\rangle$, we get

$$
\begin{aligned}
\mathrm{ord}(\xi \circ d) &= k + n_0 + \sum_{i=1}^{k}(n_i + \mathrm{ord}(d[i])) \\
&\geq |\xi| - |\xi|_{x_0} + |\xi|_{x_0} + k\,\mathrm{ord}(d) = |\xi| + k\,\mathrm{ord}(d) > |\eta|,
\end{aligned}
$$

and thus $(\xi \circ d, \eta) = 0$. □

The composition product of series is linear in its first argument since for any $r \in \mathbb{K}$ and $c_1, c_2 \in \mathbb{K}^m\langle\langle X\rangle\rangle$,

$$(c_1 + r\, c_2) \circ d = \sum_{\eta \in X^*} (c_1 + r\, c_2, \eta)\, (\eta \circ d)$$
$$= \sum_{\eta \in X^*} (c_1, \eta)\, (\eta \circ d) + \sum_{\eta \in X^*} r\, (c_2, \eta)\, (\eta \circ d) = c_1 \circ d + r\, (c_2 \circ d)$$

It is in general non-linear in its second argument, because taking for instance $c = x_0$ and $d = 1\emptyset$ we have that $c \circ 3d = x_0$ while $3(c \circ d) = 3x_0$. Moreover it is not commutative as is seen by taking c and d as above since $c \circ d = x_0 \neq 1\emptyset = d \circ c$. However it is associative and distributes to the left over the shuffle product [14, Page 2]. In Lemma 101 and Theorem 102 it is shown that the composition product induces a well defined and continuous product on the Silva space $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$. We will next show that the composition product is both holomorphic and smooth which, contrary to the case for the shuffle product, we cannot immediately conclude from Proposition 33 since it is not bilinear. The argument will instead rely on Lemma 38 and the next two lemmas.

Lemma 82. *Fix $v_1, \dots, v_N \in X^*$ and define the composition of maps*

$$\pi \colon \mathbb{C}^m_{LC}\langle\langle X\rangle\rangle \times \mathbb{C}^m_{LC}\langle\langle X\rangle\rangle \to \prod_{j=1}^{N} \mathbb{C} \to \mathbb{C}$$
$$(c, d) \mapsto (\pi^{i_j}_{v_j}(e_j))_{j=1}^N \mapsto \prod_{j=1}^{N} \pi^{i_j}_{v_j}(e_j)$$

where $e_j = c$ or $e_j = d$, $i_j \in \{1, \dots, m\}$ and the $\pi^{i_j}_{v_j}$ are the evaluation mappings

$$\pi^{i_j}_{v_j} \colon \mathbb{C}^m_{LC}\langle\langle X\rangle\rangle \to \mathbb{C}\,,\; c \mapsto (c[i_j], v_j)$$

Then π is holomorphic.

Proof. The map $(c, d) \mapsto (\pi^{i_j}_{v_j}(e_j))_{j=1}^N$, where $e_j = c$ or $e_j = d$, is continuous and complex linear implying that it is holomorphic by Proposition 33. Indeed it can be shown in an analogous way as in Lemma 66 that for any fixed $v \in X^*$ and $i \in \{1, \ldots, m\}$, the evaluation map π_v^i is continuous and complex linear on $\mathbb{C}_{LC}^m\langle\langle X \rangle\rangle$. Thus the map $(c, d) \mapsto (\pi_{v_j}^{i_j}(e_j))_{j=1}^N$ is a product of continuous complex linear mappings, whence continuous and complex linear itself. Now complex multiplication is also holomorphic by another application of Proposition 33. Then by Proposition 34, as a composition of holomorphic maps, π is holomorphic. □

Lemma 83. *Fix $v \in X^*$ and $d \in \mathbb{C}_{LC}^m\langle\langle X \rangle\rangle$. Then for any $u \in X^*$, $(u \circ d, v)$ is a polynomial in evaluation mappings*

$$\pi_\eta^i \colon \mathbb{C}_{LC}^m\langle\langle X \rangle\rangle \to \mathbb{C}\,,\; c \mapsto (c[i], \eta)$$

where $\eta \in X^*$ and $i \in \{1, \ldots, m\}$

Proof. The proof is by induction on the word length of u. If $u = \emptyset, u = x_0$ or $u = x_i$ with $i \neq 0$ then $u \circ d = \emptyset\,,\; x_0$ or $x_0\, d[i]$. In the first two cases $(u \circ d, v) = 0$ or 1, depending on v. In the last case $(u \circ d, v) = (x_0 d[i], v) = 0$ if $v \neq x_0\, v'$ with $v' \in X^*$ and $(u \circ d, v) = (d[i], v') = \pi_{v'}^i(d)$ else. Now let $u \in X^*$ with $|u| > 1$. Write $u = x_i\, u'$ with $u' \in X^*$.

If $i \neq 0$ then $u \circ d = x_0\,(d[i] \sqcup\!\sqcup (u' \circ d))$. Again if $v \neq x_0\, v'$ with $v' \in X^*$ then $(u \circ d, v) = 0$. If $v = x_0\, v'$ then

$$\begin{aligned}(u \circ d, v) &= (x_0\,(d[i] \sqcup\!\sqcup (u' \circ d)), x_0 v') = ((d[i] \sqcup\!\sqcup (u' \circ d), v') \\ &= \sum_{k=0}^{|v'|} \sum_{\substack{\xi \in X^k \\ \eta \in X^{|v'|-k}}} (d[i], \xi)\,(u' \circ d, \eta)\,(\xi \sqcup\!\sqcup \eta, v') \\ &= \sum_{k=0}^{|v'|} \sum_{\substack{\xi \in X^k \\ \eta \in X^{|v'|-k}}} \pi_\xi^i(d)\,(u' \circ d, \eta)\,(\xi \sqcup\!\sqcup \eta, v')\end{aligned}$$

By induction hypothesis, since u' is of length $|u|-1$, $(u' \circ d, \eta)$ is a polynomial in evaluation maps, implying that $(u \circ d, v)$ is also of this form.

If $i = 0$ then $(x_0 u' \circ d, v) = (x_0(u' \circ d), v)$ and this is again zero if $v \neq x_0 v'$ and $(u' \circ d, v')$ else. Thus again by induction hypothesis, $(u \circ d, v)$ is a polynomial in the evaluation maps. $\square$

Theorem 84. *The composition product is holomorphic on* $\mathbb{C}^m_{LC}\langle\langle X\rangle\rangle$.

Proof. Denote by P the composition product on $\mathbb{C}^m_{LC}\langle\langle X\rangle\rangle$. We will apply Lemma 38 which we recall states that the composition product is holomorphic if there is a family of complex valued continuous linear functionals separating points such that each of their compositions with P is holomorphic. For each $i = 1, \ldots, m$ and $v \in X^*$ let

$$\pi^i_v \colon \mathbb{C}^m_{LC}\langle\langle X\rangle\rangle \to \mathbb{C}\,,\; c \mapsto (c[i], v)$$

The collection of all such maps $\{\pi^i_v\}$ seperates points on $\mathbb{C}^m_{LC}\langle\langle X\rangle\rangle$ because $\pi^i_v(c) = (c[i], v) = 0$ for each $v \in X^*$ and $i = 1, \ldots, m$ implies of course that $c = 0$. Moreover $\pi^i_v \circ P$ is holomorphic for any $v \in X^*$. Indeed in Proposition 80 it was shown that for any word $v \in X^*$ we could write

$$(P(c,d), v) = (c \circ d, v) = \sum_{i=0}^{|v|} \sum_{v \in X^*} (c, u)\,(u \circ d, v)$$

and in particular

$$(P(c,d)[i], v) = (c[i] \circ d, v) = \sum_{i=0}^{|v|} \sum_{v \in X^*} (c[i], u)\,(u \circ d, v)$$

That is

$$\pi^i_v(P(c,d)) = \sum_{i=0}^{|v|} \sum_{v \in X^*} \pi^i_u(c)\,(u \circ d, v)$$

In Lemma 83 we showed that $(u \circ d, v)$ is a polynomial in evaluation maps $\pi^j_\xi \colon \mathbb{C}^m_{LC}\langle\langle X\rangle\rangle \to \mathbb{C}$, which in itself is just a finite linear combination of the holomorphic maps introduced in Lemma 82. Thus $\pi^i_v(P(c,d))$ is also of this form and hence holomorphic. $\square$

Theorem 85. *The composition product is real analytic on $\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$. In particular it is smooth.*

Proof. The proof of real analyticity in Theorem 64 carries over after exchanging the shuffle product with the composition product. This is because the composition product of locally convergent series with only real coefficients has again only real coefficients and because the composition product on $\mathbb{C}^m_{LC}\langle\langle X\rangle\rangle$ is holomorphic by Theorem 84. $\square$

It is easy to see that the composition product does not have a unit element. Because in the case $m = 1$, if we assume that $e \in \mathbb{K}\langle\langle X\rangle\rangle$ is a unit, we would need to have that $x_1 = x_1 \circ e = x_0 e$ which is impossible. However we shall later see that we can embed $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ as a closed subspace of an associative and unital locally convex algebra. The product this algebra carries is defined in terms of the modified composition product.

Definition 86. *For any $\eta \in X^*$ and $d \in \mathbb{K}^m\langle\langle X\rangle\rangle$*

$$\eta \,\tilde{\circ}\, d = \begin{cases} \eta & \text{if } \eta = x_0^n,\ \ n \geq 0, \\ x_0^n x_i(\eta' \,\tilde{\circ}\, d) + x_0^{n+1}(d[i] \sqcup\!\sqcup (\eta' \,\tilde{\circ}\, d)) & \text{if } \eta = x_0^n x_i \eta',\ \ i \neq 0\ \ n \geq 0 \end{cases}$$

From its definition it is clear that $\operatorname{ord}(\eta \,\tilde{\circ}\, d) \geq |\eta|$. Whence we may define the modified composition product of series in an analogous way to the composition product case and argue similarly as in Proposition 80 for the proof of well definedness.

Definition 87. *For any $c, d \in \mathbb{K}^m\langle\langle X\rangle\rangle$*

$$c \,\tilde{\circ}\, d := \sum_{\eta \in X^*} c(\eta)\, \eta \,\tilde{\circ}\, d$$

Its component series is given by

$$(c \,\tilde{\circ}\, d)[i] := \sum_{\eta \in X^*} (c[i], \eta)\, \eta \,\tilde{\circ}\, d$$

One immediately sees that the product is linear in its left argument. From [13, Page 4, Equation (7)] we know that the product is not associative and instead satisfies the below relation

$$(c \,\tilde{\circ}\, d) \,\tilde{\circ}\, e = c \,\tilde{\circ}\, (d \,\tilde{\circ}\, e + e) \tag{88}$$

Notice also that $0 \,\tilde{\circ}\, d = 0$, while $d \,\tilde{\circ}\, 0 = d$.

From [17, Page 662] the modified composition product of locally convergent series is also locally convergent and hence induces a well defined mapping

$$\tilde{\circ} : \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle \times \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle \to \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$$

The proof of continuity in Theorem 102 can be slightly altered to show continuity of the modified composition product. Specifically, the only change is in the definitions of $\bar{d}$ and $\bar{d}_j$, namely $\bar{d} = M\sum_{i=0}^m (x_i + x_0 d[i])$ and similarly $\bar{d}_j = M\sum_{i=0}^m (x_i + x_0 d_j[i])$, and these are still proper series. Moreover the proofs of analyticity, real analyticity and smoothness for the composition product can be easily modified to show that the mixed composition product is analytic, real analytic and smooth.

Proposition 89. *$\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ may be identified as a closed subspace of a locally convex algebra which is both associative and unital. The algebra multiplication is moreover holomorphic (smooth)*

Proof. The candidate locally convex space is $\mathbb{K} \times \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ endowed with componentwise vector space operations and the product topology. Then clearly

$$\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle \cong \{0\} \times \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$$

which is a closed subspace of $\mathbb{K} \times \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$. Define the product of two elements in $\mathbb{K} \times \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ as

$$(r, c) \bullet (s, d) := (rs, d + c \,\tilde{\circ}\, d)$$

Since the modified composition product on $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ is well defined, so is the product $\bullet$. It is moreover continuous because each of its components is continuous. The unit is $\delta := (1, 0)$ because

$$(1,0) \bullet (s,d) = (s, d + 0 \,\tilde{\circ}\, d) = (s,d) = (s, 0 + d \,\tilde{\circ}\, 0) = (s,d) \bullet (1,0)$$

To see that it is associative, we remember equation (88) in addition to the fact that the mixed composition product is linear in its left argument.

$$\begin{aligned}((r,c) \bullet (s,d)) \bullet (t,e) &= (rs, d + c \,\tilde{\circ}\, d) \bullet (t,e) = (rst, e + (d + c \,\tilde{\circ}\, d) \,\tilde{\circ}\, e) \\ &= (rst, e + d \,\tilde{\circ}\, e + (c \,\tilde{\circ}\, d) \,\tilde{\circ}\, e) = (rst, e + d \,\tilde{\circ}\, e + c \,\tilde{\circ}\, (d \,\tilde{\circ}\, e + e)) \\ &= (r,c) \bullet (st, e + d \,\tilde{\circ}\, e) = (r,c) \bullet ((s,d) \bullet (t,e))\end{aligned}$$

To see that $\bullet$ is holomorphic (smooth) we use Proposition 35 together with the fact that addition on $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$, scalar multiplication on $\mathbb{K}$ and the modified composition product on $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ are all holomorphic (smooth) operations. □

5.3 A pre-Lie Product

The final product considered on $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ is a type of pre-Lie product which will turn $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ into a so called pre-Lie algebra.

Definition 90. *A pre-Lie algebra is a pair $(V, \lhd)$ consisting of vector space V and a bilinear mapping $\lhd\colon V \times V \to V$, which satisfies the relation*

$$(x \lhd y) \lhd z - x \lhd (y \lhd z) = (x \lhd z) \lhd y - x \lhd (z \lhd y)$$

One can show that the commutator bracket $[v, u] = v \lhd u - u \lhd v$ is a Lie bracket, whence a pre-Lie algebra is always also a Lie algebra. The following pre-Lie algebra was first introduced in [9] where it was shown that the vector space of polynomials in two indeterminates, $\mathbb{K}\langle x_0, x_1\rangle$, carries a pre-Lie structure. In [15, Theorem 2] this was shown to extend to a pre-Lie structure on the space of formal power series $\mathbb{R}\langle\langle x_0, x_1\rangle\rangle$ and the proof generalizes immediately to a pre-Lie structure on $\mathbb{C}\langle\langle x_0, x_1\rangle\rangle$.

Definition 91. *[9] [15] Given $d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$, define a bilinear product $\triangleleft$ inductively by*

$$(x_0\eta) \triangleleft d := x_0(\eta \triangleleft d)$$
$$(x_1\eta) \triangleleft d := x_1(\eta \triangleleft d) + x_0(\eta \sqcup\!\sqcup d)$$

with $\eta \in X^*$ and $\emptyset \triangleleft d = 0$. For $c, d \in \mathbb{K}\langle\langle x_0, x_1\rangle\rangle$ let

$$c \triangleleft d := \sum_{\eta \in X^*} (c, \eta)\, \eta \triangleleft d$$

Then the pair $(\mathbb{K}\langle\langle x_0, x_1\rangle\rangle, \triangleleft)$ forms a pre-Lie algebra.

The proof of the pre-Lie structure on $\mathbb{K}\langle x_0, x_1\rangle$ in [9] generalizes to the case $\mathbb{K}^n\langle X\rangle$ where $X = \{x_0, x_1, \ldots x_n\}$. The main changes are as follows: The coordinate maps forming the generators for V now take the form

$$\pi_u^j \colon \mathbb{K}^n\langle x_0, x_1, \ldots x_n\rangle \to \mathbb{K}\,,\; p \mapsto (p[j], u)$$

where $u \in X^*$ and $j \in \{1, \ldots, n\}$. The statement in [9, Lemma 2.3] becomes $(x_i c) \,\tilde{\circ}\, d = x_i(c \,\tilde{\circ}\, d) + x_0(d[i] \sqcup\!\sqcup c \,\tilde{\circ}\, d)$, [9, Proposition 3.2 and 3.3] becomes respectively

$$\tilde{\Delta} \circ \theta_0 = (\theta_0 \otimes \mathrm{id}) \circ \tilde{\Delta} + \sum_{i=1}^{n} (\theta_i \otimes m) \circ (\tilde{\Delta} \otimes \mathrm{Id}) \circ \Delta_{\sqcup\!\sqcup}$$

and

$$\tilde{\Delta} \circ \theta_i = (\theta_i \otimes \mathrm{Id}) \circ \tilde{\Delta}$$

with the obvious additional definitions of the endomorphisms θ_i. The maps Δ and $\tilde{\Delta}$ are defined as in [9]. The definition of the degree is the same so that [9, Definition 5 and Lemma 6] are still valid. [9, Proposition 9.1 and 9.2] undergoes analogous changes as the ones for [9, Proposition 3.2 3.3] and the proofs for all of these carry over with the added internal sums. [9, Proposition 11.3] becomes

$$(x_i c) \triangleleft d = x_i(c \triangleleft d) + x_0(c \sqcup\!\sqcup d)$$

for words c, d identified as $c = \vec{1}c$ and $d = \vec{1}d$. An analogous procedure as in [9, Proposition 11] applies to show that the product is indeed of this form.

With the pre-Lie structure on $\mathbb{K}^m\langle X\rangle$ the proofs in [15, Lemma 1 and Theorem 2] applies componentwise with some minor changes to extend the pre-Lie structure to $\mathbb{K}^m\langle\langle X\rangle\rangle$. Thus we obtain the following definition.

Definition 92. *For $d \in \mathbb{K}^m\langle\langle X\rangle\rangle$ let*

$$(x_0\eta) \lhd d := x_0(\eta \lhd d)$$
$$(x_i\eta) \lhd d := x_i(\eta \lhd d) + x_0(\eta \sqcup\!\sqcup d), \quad i \neq 0$$

with $\eta \in X^*$ and $\emptyset \lhd d = 0$. For $c, d \in \mathbb{K}^m\langle\langle X\rangle\rangle$ let

$$c \lhd d := \sum_{\eta \in X^*} (c, \eta)\, \eta \lhd d$$

where the k'th component series is given by

$$(c \lhd d)[k] := \sum_{\eta \in X^*} (c[k], \eta)\, \eta \lhd d[k]$$

Then the pair $(\mathbb{K}^m\langle\langle X\rangle\rangle, \lhd)$ is a pre-Lie algebra.

Lemma 93. *The pre-Lie product on $\mathbb{K}^m\langle\langle X\rangle\rangle$ induces a well defined mapping on $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ turning $(\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle, \lhd)$ into a pre-Lie algebra.*

Proof. We need to show that for any $c, d \in \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$, $c \lhd d \in \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$. This amounts to finding an integer N for which

$$\|c \lhd d\|_N = \sup_{\eta \in X^*} \frac{|(c \lhd d, \eta)|}{N^{|\eta|}\, |\eta|!} < \infty$$

To this end let $c, d \in \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$. Assume $c, d \in \ell_{\infty,M}(X^*, \mathbb{K}^m)$ for $M > 0$ and consider

$$c \lhd d = \sum_{\eta \in X^*} (c, \eta)\, \eta \lhd d$$

[15, Lemma 1] shows in particular, with our definition of ord(d) when $d \in \mathbb{K}^m\langle\langle X\rangle\rangle$, that ord$(\eta \lhd d) \geq |\eta|$. Moreover since $\emptyset \lhd d = 0$ we obtain

$$(c \lhd d, \eta) = \sum_{1 \leq |\xi| \leq |\eta|} (c, \xi)(\xi \lhd d, \eta)$$

A case analysis shows that for $\xi = x_j \xi'$ and $\eta = x_i \eta'$ with $i, j \in \{0, 1, \ldots, m\}$, $\xi', \eta' \in X^*$ we have

$$(\xi \lhd d, \eta) = \begin{cases} (\xi' \lhd d, \eta') & \text{or} \\ (\xi' \sqcup\!\sqcup d, \eta') & \text{or} \\ 0 \end{cases}$$

depending on the values of j and i. In any case we see that

$$(\xi \lhd d, \eta) = (\bar{\xi} \sqcup\!\sqcup d, \bar{\eta})$$

for some $\bar{\xi}$ and $\bar{\eta}$ with $\xi = \xi'\bar{\xi}$ and $\eta = \eta'\bar{\eta}$ or

$$(\xi \lhd d, \eta) = 0$$

Thus for any component series $(c \lhd d)[i]$, we have

$$|((c \lhd d)[i], \eta)| \leq \sum_{1 \leq |\xi| \leq |\eta|} |(c[i], \xi)| \, |(\xi \lhd d[i], \eta)|$$

$$\leq \sum_{1 \leq |\xi| \leq |\eta|} M^{|\xi|} \, |\xi|! \, \|c\|_M |(\xi \lhd d[i], \eta)| = \|c\|_M \sum_{1 \leq |\xi| \leq |\eta|} M^{|\xi|} \, |\xi|! \, |(\bar{\xi} \sqcup\!\sqcup d[i], \bar{\eta})|$$

where $\bar{\xi}$ and $\bar{\eta}$ are such that $\xi = \xi'\bar{\xi}$ and $\eta = \eta'\bar{\eta}$. Continuing

$$|((c \lhd d)[i], \eta)| \leq \|c\|_M \sum_{1 \leq |\xi| \leq |\eta|} M^{|\xi|} \, |\xi|! \, \|\bar{\xi} \sqcup\!\sqcup d[i]\|_{M_\epsilon} \, M_\epsilon^{|\bar{\eta}|} \, |\bar{\eta}|!$$

$$\leq K_\epsilon \, \|c\|_M \, \|d\|_M \sum_{1 \leq |\xi| \leq |\eta|} \frac{M^{|\xi|} \, M_\epsilon^{|\bar{\eta}|}}{M^{|\bar{\xi}|}} \, \frac{|\xi|! \, |\bar{\eta}|!}{|\bar{\xi}|!}$$

where K_ϵ and M_ϵ are the positive number coming from Lemma 99 and $\epsilon > 0$ is fixed. Note that $|\xi'| = |\eta'| =: n$, hence $|\xi| = |\bar{\xi}| + n$ and $|\eta| = |\bar{\eta}| + n$. Using this one can show the inequality $\frac{|\xi|!\,|\bar{\eta}|!}{|\bar{\xi}|!} \leq |\eta|!$. We then obtain that

$$\begin{aligned}
|((c \lhd d)[i], \eta)| &\leq \|c\|_M \, \|d\|_M \, K_\epsilon \sum_{1 \leq |\xi| \leq |\eta|} (M_\epsilon^2)^{|\eta|} \, |\eta|! \\
&= \|c\|_M \, \|d\|_M \, K_\epsilon \, (M_\epsilon^2)^{|\eta|} \, |\eta|! \sum_{1 \leq |\xi| \leq |\eta|} 1 \\
&= \|c\|_M \, \|d\|_M \, K_\epsilon \, (M_\epsilon^2)^{|\eta|} \, |\eta|! \sum_{1 \leq k \leq |\eta|} (m+1)^k \\
&\leq \|c\|_M \, \|d\|_M \, K_\epsilon \, (M_\epsilon^2)^{|\eta|} \, |\eta|! \, |\eta| (m+1)^{|\eta|} \\
&= \|c\|_M \, \|d\|_M \, K_\epsilon \, (M_\epsilon^2 \, (m+1) \, (1+\alpha))^{|\eta|} \, |\eta|! \, \frac{|\eta|}{(1+\alpha)^{|\eta|}}
\end{aligned}$$

where α is any fixed positive number. Put $K := \sup_{\eta \in X^*} \frac{|\eta|}{(1+\alpha)^{|\eta|}} \, K_\epsilon$. Then defining N to be the smallest integer larger than $(M_\epsilon^2 \, (m+1) \, (1+\alpha))$ and noting that i was arbitrary, we see that

$$|(c \lhd d, \eta)| = \max_{1 \leq i \leq m} |((c \lhd d)[i], \eta)| \leq K \, \|c\|_M \, \|d\|_M \, N^{|\eta|} \, |\eta|!$$

and so

$$\|c \lhd d\|_N = \sup_{\eta \in X^*} \frac{|(c \lhd d, \eta)|}{N^{|\eta|} |\eta|!} \leq \|c\|_M \, \|d\|_M \, K < \infty \tag{94}$$

□

Using the above estimates the following is readily proven.

Theorem 95. *The pre-Lie product on $\mathbb{K}^m_{LC}\langle\langle X \rangle\rangle$ is continuous*

Proof. Recall that $\mathbb{K}^m_{LC}\langle\langle X \rangle\rangle \times \mathbb{K}^m_{LC}\langle\langle X \rangle\rangle$ is a Silva space, as a direct product of Silva spaces, and is hence a sequential space. Thus it suffices to show continuity of the pre-Lie product $\lhd$ via sequences. Also recall that any convergent sequence in $\mathbb{K}^m_{LC}\langle\langle X \rangle\rangle$ converges as a sequence in one of the Banach steps $\ell_{\infty,M}(X^*, \mathbb{K}^m)$ from which the Silva space is constructed.

With this in mind let $c_j \to c$ and $d_j \to d$ in say $\ell_{\infty,M}(X^*, \mathbb{K}^m)$ where $M \in \mathbb{N}$. By the preceding lemma we know there is an $N \in \mathbb{N}$, $N > M$, such that $\|c_j \lhd d_j\|_N < \infty$ and $\|c \lhd d\|_N < \infty$. Using the bilinearity of the pre-Lie product we compute

$$\begin{aligned}\|c_j \lhd d_j - c \lhd d\|_N &\leq \|(c_j - c) \lhd d_j\|_N + \|c \lhd (d_j - d)\|_N \\ &\leq \|c_j - c\|_M\, K\, \|d_j\|_M + \|d_j - d\|_M\, K\, \|c\|_M,\end{aligned}$$

where $K > 0$ is the constant coming from inequality (94). Then

$$\|c_j \lhd d_j - c \lhd d\|_N \to 0$$

as $j \to \infty$ implying that $c_j \lhd d_j \to c \lhd d$ in $\ell_{\infty,N}(X^*, \mathbb{K}^m)$. In other words, $\lhd$ is continuous. □

Corollary 96. *Given the bracket $[c, d] = c \lhd d - d \lhd c$, $\mathbb{K}^m_{LC}\langle\langle X \rangle\rangle$ becomes a locally convex Lie algebra.*

Theorem 97. *The pre-Lie product is smooth and $\mathbb{K}$-analytic.*

Proof. We already know that the pre-Lie product is bilinear and by Theorem 95 it is continuous, whence we may apply Proposition 33 to conclude that it is smooth and analytic. Since the pre-Lie product of locally convergent series with real coefficients has again only real coefficients, the proof of real analyticity from Theorem 64 carries over. □

6 Appendix

Future work

There was not enough time to finalize the details of the next conjecture. This is unfortunate as it nicely relates the pre-Lie structure on $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ with the unit group of the locally convex algebra $(\mathbb{K} \times \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle, \bullet)$.

Conjecture 98. *The unit group of* $(\mathbb{K} \times \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle, \bullet)$ *is a smooth (analytic) locally convex Lie group. Its Lie algebra is* $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ *with the bracket*

$$[c, d] = c \lhd d - d \lhd c$$

The unit group of $(\mathbb{K} \times \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle, \bullet)$ turns out to be $(\{1\} \times \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle, \bullet)$ and the manifold structure of the group is given by the global chart

$$\psi\colon \{1\} \times \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle \to \{0\} \times \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle\,,\ (1, c) \mapsto (0, c)$$

The inverse of any element $(1, c)$ is $(1, c^{-1})$, where the value of c^{-1} at any word $\eta \in X^*$ and component $i \in \{1 \ldots, m\}$ is

$$(c^{-1}[i], \eta) = \pi^i_\eta(c^{-1}) = S(\pi^i_\eta)(c)$$

where S is the antipode of a certain Hopf algebra described in [13]. Moreover with [13, Theorem 6] we can compute the value of the inverse at any word. To prove continuity of the inverse one must first assume that $c_j \to c$ for $c_j, c \in \ell_{\infty,M}(X^*, \mathbb{K}^m)$. Then one must look at the difference of the inverses at any word, which is given recursively by the antipode formula in [13, Theorem 6], and somehow show that this satisfies the growth bound from (55) multiplied with some norm difference of c and c_j. If continuity of the inverse is proven, smoothness and analyticity can be proved in a similar manner as for the composition product. The Lie algebra of this group will then be $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ and moreover the bracket can be shown to be given by the commutator bracket of the pre-Lie product on $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ as described in Lemma 93.

External results

Some results used in the text along with their proofs are presented next. These are from a paper in preparation called "Continuity of formal power series products", authored by Rafael Dahmen, Steven Gray, Mathias Palmstrøm and Alexander Schmeding.

Lemma 99. *Fix $M > 0$. If $c, d \in \ell_{\infty,M}(X^*, \mathbb{K}^\ell)$, then $c \sqcup\!\sqcup d \in \ell_{\infty,M_\epsilon}(X^*, \mathbb{K}^\ell)$ for any $M_\epsilon = M(1+\epsilon)$, $\epsilon > 0$ and*

$$\|c \sqcup\!\sqcup d\|_{\ell_\infty,M_\epsilon} \leq K_\epsilon \, \|c\|_{\ell_\infty,M} \|d\|_{\ell_\infty,M},$$

where $K_\epsilon = \sup_{\eta \in X^*} (|\eta|+1)/(1+\epsilon)^{|\eta|}$.

Proof. For any $\eta \in X^*$

$$|(c \sqcup\!\sqcup d, \eta)| = |\sum_{k=0}^{|\eta|} \sum_{\substack{\nu \in X^k \\ \xi \in X^{|\eta|-k}}} (c,\nu)(d,\xi)(\nu \sqcup\!\sqcup \xi, \eta)|$$

$$\leq \sum_{k=0}^{|\eta|} \sum_{\substack{\nu \in X^k \\ \xi \in X^{|\eta|-k}}} \|c\|_M M^k k! \, \|d\|_M M^{|\eta|-k} (|\eta|-k)! \, (\nu \sqcup\!\sqcup \xi, \eta)$$

$$= \|c\|_M \|d\|_M M^{|\eta|} \sum_{k=0}^{|\eta|} k! \, (|\eta|-k)! \binom{|\eta|}{k}$$

$$= \|c\|_M \|d\|_M M^{|\eta|} \sum_{k=0}^{\eta} |\eta|! = \|c\|_M \|d\|_M M^{|\eta|} (|\eta|+1)!.$$

Note that this bound is achievable when $c = \sum_{\eta \in X^*} K_c M^{|\eta|} |\eta|! \, \eta$ and $d = \sum_{\eta \in X^*} K_d M^{|\eta|} |\eta|! \, \eta$ for any $K_c, K_d \geq 0$. Now define $M_\epsilon = M(1+\epsilon)$ with $\epsilon > 0$ and rewrite the final inequality above as

$$\frac{|(c \sqcup\!\sqcup d, \eta)|}{M_\epsilon^{|\eta|} |\eta|!} \leq \|c\|_M \|d\|_M \frac{|\eta|+1}{(1+\epsilon)^{|\eta|}} \;\; \forall \eta \in X^*.$$

Taking the supremum over X^* gives

$$\|c \sqcup\!\sqcup d\|_{M_\epsilon} \leq K_\epsilon \, \|c\|_M \|d\|_M \;\; \forall \eta \in X^*,$$

where $K_\epsilon = \sup_{\eta \in X^*}(|\eta| + 1)/(1 + \epsilon)^{|\eta|}$. □

Lemma 100. *The shuffle product is continuous as a mapping*

$$\ell_{\infty,M}(X^*, \mathbb{K}^m) \times \ell_{\infty,N}(X^*, \mathbb{K}^m) \to \ell_{\infty,K}(X^*, \mathbb{K}^m)$$

where $K > \max\{M, N\}$ *is arbitrarily fixed.*

Proof. We know that this induced map is well defined by Lemma 99. Suppose without loss of generality that $M > N$ and that $c_j \to c$ and $d_j \to d$ in respectively $\ell_{\infty,M}(X^*, \mathbb{K}^m)$ and $\ell_{\infty,N}(X^*, \mathbb{K}^m)$. Then in particular $d_j \to d$ in $\ell_{\infty,M}(X^*, \mathbb{K}^m)$ and

$$\begin{aligned}
\|(c \sqcup\!\sqcup d) - (c_j \sqcup\!\sqcup d_j)\|_K &= \|(c - c_j) \sqcup\!\sqcup d + c_j \sqcup\!\sqcup (d - d_j)\|_K \\
&\leq \|(c - c_j) \sqcup\!\sqcup d\|_K + \|c_j \sqcup\!\sqcup (d - d_j)\|_K \\
&\leq K_\epsilon \|(c - c_j)\|_M \|d\|_M + K_\epsilon \|c_j\|_M \|(d - d_j)\|_M,
\end{aligned}$$

where $K_\epsilon = \sup_{\eta \in X^*}(|\eta| + 1)/(1 + \epsilon)^{|\eta|}$ and $\epsilon > 0$ is such that $K = M(1 + \epsilon)$. Thus $\lim_{j\to\infty} \|(c \sqcup\!\sqcup d) - (c_j \sqcup\!\sqcup d_j)\|_K = 0$. □

Lemma 101. *Fix* $M > 0$. *If* $c \in \ell_{\infty,M}(X^*, \mathbb{K}^m)$ *and* $d \in \ell_{\infty,M}(X^*, \mathbb{K}^\ell)$, *then* $c \circ d \in \ell_{\infty,M_\epsilon}(X^*, \mathbb{K}^\ell)$ *for any* $M_\epsilon = M(1 + \epsilon)$, $\epsilon > \phi(m\|d\|_M)$ *with* $\phi(x) = x/2 + \sqrt{x^2/4 + x}$ *and*

$$\|c \circ d\|_{M_\epsilon} \leq \|c\|_M (K_\epsilon \circ \phi)(m\|d\|_M),$$

where $K_\epsilon(a) = \sup_{\eta \in X^*}(|\eta| + 1)(1 + a)^{|\eta|}/(1 + \epsilon)^{|\eta|}$.

Proof. It was shown in [17] that under the stated conditions

$$|(c \circ d, \eta)| \leq \|c\|_M ((1 + \phi(m\|d\|_M))M)^{|\eta|} (|\eta| + 1)!, \;\; \forall \eta \in X^*.$$

Therefore,

$$\frac{|(c \circ d, \eta)|}{M_\epsilon |\eta|!} \leq \|c\|_M (1 + \phi(m\|d\|_M))^{|\eta|} \frac{(|\eta| + 1)}{(1+\epsilon)^{|\eta|}}, \ \forall \eta \in X^*.$$

Taking the supremum over X^* proves the lemma. □

Theorem 102. *The composition product is continuous on* $\mathbb{K}^m_{LC}\langle\langle X \rangle\rangle$.

Proof. Left and right continuity of the composition product is first proved, beginning with left continuity. Let $M > 0$ be fixed. Let $c, d \in \ell_{\infty,M}(X^*, \mathbb{K}^m)$, and assume c_j, $j \geq 1$ is a sequence in $\ell_{\infty,M}(X^*, \mathbb{K}^m)$ converging to c. Applying Lemma 101 gives

$$\begin{aligned} \|(c \circ d) - (c_j \circ d)\|_{M_\epsilon} &= \|(c - c_j) \circ d\|_{M_\epsilon} \\ &\leq \|c - c_j\|_M (K_\epsilon \circ \phi)(m\|d\|_M). \end{aligned}$$

Thus, $\lim_{j \to \infty} \|(c \circ d) - (c_j \circ d)\|_{M_\epsilon} = 0$. Right continuity is addressed next. It is more complicated given the non-linearity in the right argument of the product. Let $c, d \in \ell_{\infty,M}(X^*, \mathbb{K}^m)$ and assume d_j, $j \geq 1$ is a sequence in $\ell_{\infty,M}(X^*, \mathbb{K}^m)$ converging to d. For a fixed $\xi \in X^*$, observe that

$$\begin{aligned} |((c \circ d) - (c \circ d_j), \xi)| &= \left| \sum_{\eta \in X^*} (c, \eta)(\eta \circ d - \eta \circ d_j, \xi) \right| \\ &\leq \sum_{n=0}^{\infty} \|c\|_M M^n \, n! \left| \sum_{\eta \in X^n} (\eta \circ d - \eta \circ d_j, \xi) \right| \\ &= \|c\|_M \sum_{n=0}^{\infty} M^n \, n! \left| \sum_{\substack{r_0 \geq 0, \ldots, r_m \geq 0 \\ r_0 + \cdots + r_m = n}} ((x_0^{r_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_m^{r_m}) \circ d \right. \\ &\qquad \left. - (x_0^{r_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_m^{r_m}) \circ d_j, \xi) \right|. \end{aligned}$$

Applying the identities $x_i^{\sqcup\!\sqcup n} = n!\, x_i^n$, $n \geq 0$ and $(c \sqcup\!\sqcup d) \circ e = (c \circ e) \sqcup\!\sqcup (d \circ e)$ gives

$$
\begin{aligned}
&|((c \circ d) - (c \circ d_j), \xi)| \\
&\leq \|c\|_M \sum_{n=0}^{\infty} M^n \left| \sum_{\substack{r_0 \geq 0, \ldots, r_m \geq 0 \\ r_0 + \cdots + r_m = n}} \binom{n}{r_0 \cdots r_m} ((x_0^{\sqcup\!\sqcup r_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_m^{\sqcup\!\sqcup r_m}) \circ d \right. \\
&\qquad \left. - (x_0^{\sqcup\!\sqcup r_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_m^{\sqcup\!\sqcup r_m}) \circ d_j, \xi) \right| . \\
&= \|c\|_M \sum_{n=0}^{\infty} M^n \left| \left(\left(\sum_{k=0}^{m} x_k \circ d \right)^{\sqcup\!\sqcup n} - \left(\sum_{k=0}^{m} x_k \circ d_j \right)^{\sqcup\!\sqcup n}, \xi \right) \right| \\
&= \|c\|_M \sum_{n=0}^{\infty} \left| \left(\bar{d}^{\sqcup\!\sqcup n} - \bar{d}_j^{\sqcup\!\sqcup n}, \xi \right) \right| ,
\end{aligned}
$$

where $\bar{d} := M x_0 \sum_{k=0}^{m} d[k]$ and $\bar{d}_j := M x_0 \sum_{k=0}^{m} d_j[k]$ are proper series in $\mathbb{K}_{LC}\langle\langle X \rangle\rangle$. Here $d[k]$ denotes the k-th component series of d. It is clear that $\bar{d} \in \ell_{\infty,M}(X^*, \mathbb{K})$, and $\bar{d}_j$ is a sequence in $\ell_{\infty,M}(X^*, \mathbb{K})$. Furthermore, $\bar{d}_j \to \bar{d}$ as $j \to \infty$ since

$$
\begin{aligned}
\|\bar{d} - \bar{d}_j\|_M &= \sup_{\eta \in X^*} \frac{|(\bar{d} - \bar{d}_j, \eta)|}{M^{|\eta|} |\eta|!} \\
\leq \sum_{k=1}^{m} \sup_{\eta \in X^*} \frac{M |(x_0(d[k] - d_j[k]), \eta)|}{M^{|\eta|} |\eta|!} &= \sum_{k=1}^{m} \sup_{x_0 \eta \in X^*} \frac{|(d[k] - d_j[k], \eta)|}{M^{|\eta|} (|\eta| + 1)!} \\
&= \sum_{k=1}^{m} \sup_{\eta \in X^*} \frac{|(d[k] - d_j[k], \eta)|}{M^{|\eta|} |\eta|! (|\eta| + 1)} \leq m \|d - d_j\|_M .
\end{aligned}
$$

It follows that for any $N \geq M$,

$$
\|(c \circ d) - (c \circ d_j)\|_N \leq \|c\|_M \sum_{n=0}^{\infty} \|\bar{d}^{\sqcup\!\sqcup n} - \bar{d}_j^{\sqcup\!\sqcup n}\|_N . \tag{103}
$$

The right continuity follows now by Proposition 71 with some large enough $N \in \mathbb{N}$.

Note that the estimates for left and right continuity imply joint continuity of the composition product due to the following simple observation that

$$\|(c \circ d) - (c_j \circ d_j)\|_N \leq \|(c \circ d) - (c_j \circ d)\|_N + \|(c_j \circ d) - (c_j \circ d_j)\|_N$$
$$\leq \|(c - c_j) \circ d\|_N + \|c_j\|_M \sum_{n=0}^{\infty} \|\bar{d}^{\sqcup\!\sqcup n} - \bar{d}_j^{\sqcup\!\sqcup n}\|_N,$$

where the last inequality is a direct consequence of (103). Again we may apply Proposition 71 to conclude that this tends to zero for some large enough $N \in \mathbb{N}$. □

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