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BALANCED JUDICIOUS BIPARTITION IS FIXED-PARAMETER TRACTABLE*

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Abstract. The family of judicious partitioning problems, introduced by Bollobás and Scott to the field of extremal combinatorics, has been extensively studied from a structural point of view for over two decades. This rich realm of problems aims to counterbalance the objectives of classical partitioning problems such as Min Cut, Min Bisection, and Max Cut. While these classical problems focus solely on the minimization/maximization of the number of edges crossing the cut, judicious (bi)partitioning problems ask the natural question of the minimization/maximization of the number of edges lying in the (two) sides of the cut. In particular, Judicious Bipartition (JB) seeks a bipartition that is "judicious" in the sense that neither side is burdened by too many edges, and Balanced JB (BJB) also requires that the sizes of the sides themselves are "balanced" in the sense that neither of them is too large. Both of these problems were defined in the work by Bollobás and Scott and have received notable scientific attention since then. In this paper, we shed light on the study of judicious partitioning problems from the viewpoint of algorithm design. Specifically, we prove that BJB is fixed parameter tractable (FPT) (which also proves that JB is FPT).

Key words. judicious partition, fixed-parameter tractable, minimum bisection, tree decomposition, randomized contractions

AMS subject classifications. 05C85, 68R99, 68Q25

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1. Introduction.

More than twenty years ago, Bollobás and Scott [3] defined the notion of *judicious partitioning problems*. Since then, the family of judicious partitioning problems has been extensively studied in the field of extremal combinatorics, as can be evidenced by the abundance of structural results described in surveys such as [7, 36]. This rich realm of problems aims to counterbalance the objectives of classical partitioning problems such as Min Cut, Min Bisection, Max Cut, and Max Bisection. While these classical problems focus solely on the minimization/maximization of the number of edges crossing the cut (or alternately, the total number of edges inside the parts of the partition), judicious (bi)partitioning problems ask the natural questions of the minimization/maximization of the number of edges lying inside each part of the partition *simultaneously*. Another significant feature of judicious partitioning problems that also distinguishes them from other classical partitioning problems is that they inherently and naturally encompass several objectives, aiming to minimize (or maximize) the number of edges in several sets *simultaneously*.

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In this paper, we shed light on properties of judicious partitioning problems from the viewpoint of the design of algorithms. Up until now, the study of such problems has essentially been overlooked at the algorithmic front, where one of the underlying reasons for this discrepancy might be that standard machinery does not seem to handle them effectively. Specifically, we focus on the Judicious Bipartition (JB) problem, where we seek a bipartition that is "judicious" in the sense that neither side has too many edges that lie entirely inside it, and on the Balanced Judicious Bipartition (BJB) problem, where we also require that the sizes of the sides themselves are "balanced" in the sense that the number of vertices in both the parts are almost same. Both of these problems were defined in the work by Bollobás and Scott and have received notable scientific attention since then. Formally, BJB is defined as follows.

Balanced Judicious Bipartition (BJB)	**Parameter:** $k_1 + k_2$
Input: A multigraph G, and integers μ, k_1, and k_2	
Question: Does there exist a partition (V_1, V_2) of $V(G)$ such that $\lvert V_1\rvert = \mu$ and for all $i \in \{1, 2\}$, it holds that $\lvert E(G[V_i])\rvert \le k_i$?	

We note that in the literature, the term BJB refers to the case where $\mu = \lceil \frac{\lvert V(G)\rvert}{2} \rceil$, and hence it is more restricted then the definition above. By dropping the requirement that $\lvert V_1\rvert = \mu$, we get the JB problem. By using new crucial insights into these problems on top of the most advanced machinery in parameterized complexity to handle partitioning problems,[1] we are able to resolve the question of the parameterized complexity of BJB (and hence also of JB). In particular, we prove the following theorem.

Theorem 1.1. BJB *can be solved in time* $2^{k^{\mathcal{O}(1)}} \cdot \lvert V(G)\rvert^{\mathcal{O}(1)}$.

Structural results. Denote $n = \lvert V(G)\rvert$ and $m = \lvert E(G)\rvert$. To survey several structural results about judicious partitioning problems, we first define the notions of *t-cut* and *max (min) t-judicious partitioning*. Given a partition of $V(G)$ into t parts, a t-cut is the number of edges going across the parts, while a max (min) judicious t-partitioning is the maximum (minimum) number of edges in any of the parts. When $t = 2$, we use the standard terms *bipartite-cut* and *judicious bipartitioning*, respectively. Furthermore, by t-judicious partitioning we mean max t-judicious partitioning. As stated earlier, Bollobás and Scott [3] defined the notion of *judicious partitioning problems* in 1993. In that paper, they showed that for any positive integer t and graph G, we can partition $V(G)$ into t sets, $V_1, \ldots, V_t$, so that $\lvert E(G[V_i])\rvert \le \frac{t}{t+1}m$ for all $i \in \{1, \ldots, t\}$. Bollobás and Scott also studied this problem on graphs of maximum degree Δ and showed that there exists a partition of $V(G)$ into t sets $V_1, \ldots, V_t$ so that it simultaneously satisfies an upper bound and a lower bound on the number of edges in each part as well as on edges between every pair of parts. Later, Bollobás and Scott [7] gave several new results concerning the extremal bounds of the k-judicious partitioning problem, leaving open other new questions concerning the tightness of their bounds in general and special cases. In [8] they showed an optimal bound for the number of edges inside a part for the judicious partitioning problem on bounded-degree graphs. These problems have also been studied on general hypergraphs [4], uniform hypergraphs [24], 3-uniform hypergraphs [6], and directed graphs [26].

[1] To the best of our knowledge, up until now, this machinery has actually only been proven useful to solve one natural problem which could not have been tackled using earlier tools.

The special cases of judicious partitioning problems called judicious bipartitioning and balanced judicious bipartitioning problems have also been studied intensively. Bollobás and Scott [5] proved an upper bound on judicious bipartitioning and proved that every graph that achieves the essentially best known lower bound on bipartite-cut, given by Edwards in [18] and [19], also achieves this upper bound for judicious bipartitioning. In fact, they showed that this is exact for complete graphs of odd order, which are the only extremal graphs without isolated vertices. Alon et al. [1] gave a nontrivial connection between the size of a bipartite-cut in a graph and judicious partitioning into two sets. In particular, they showed that if a graph has a bipartite-cut of size at least $\frac{m}{2}+\delta$, where $\delta \leq m/30$, then there exists a bipartition (V_1, V_2) of $V(G)$ such that $|E(G[V_i])| \leq \frac{m}{4} - \frac{\delta}{2} + \frac{10\delta^2}{m} + 3\sqrt{m}$ for $i \in \{1,2\}$. They complemented these results by showing an upper bound on the number of edges in each part when $\delta > m/30$. Bollobás and Scott [9] studied similar relations between t-cuts and t-judicious partitionings for $t \geq 3$. Recently, these results were further refined [39, 28]. Xu, Yan, and Yu [38] and Xu and Yu [40] studied balanced judicious bipartitioning where both parts are of almost equal size (that is, one of the sizes is $\lceil \frac{n}{2} \rceil$). Both of these papers concern the following conjecture of Bollobás and Scott [7]: if G is a graph with minimum degree of at least 2, then $V(G)$ admits a balanced bipartition (V_1, V_2) such that for each $i \in \{1,2\}$, $|E(G[V_i])| \leq \frac{m}{3}$. For further results on judicious partitioning, we refer to the surveys [7, 36].

Algorithmic results. While classical partitioning problems such as MIN CUT, MIN BISECTION, MAX CUT, and MAX BISECTION have been studied extensively algorithmically, the same is not true about judicious partitioning problems. Apart from MIN CUT, all the above-mentioned partitioning problems are NP-complete. These NP-complete partitioning problems were investigated by all algorithmic paradigms meant for coping with NP-completeness, including approximation algorithms and parameterized complexity. In what follows, we discuss known results related to these problems in the realm of parameterized complexity.

First, note that for every graph G, there always exists a bipartition of the vertex set into two parts (in fact equal parts [22, Corollary 1]) such that at least $m/2$ edges are going across. This immediately implies that MAX CUT and MAX BISECTION are fixed-parameter tractable (FPT) when parameterized by the cut size (the number of edges going across the partition). This led Mahajan and Raman [29] to introduce the notion of above-guarantee parameterization. In particular, they showed that one can decide whether a graph has a bipartite-cut of size $\frac{m}{2}+k$ in time $\mathcal{O}(m+n+k4^k)$. However, Edwards [18] showed that every connected graph G has a bipartite-cut of size $\frac{m}{2}+\frac{n-1}{4}$. Thus, a more interesting question asks whether finding a bipartite-cut of size at least $\frac{m}{2}+\frac{n-1}{4}+k$ is FPT. Crowston, Jones, and Mnich [16] showed that indeed this is the case as they design an algorithm with running time $\mathcal{O}(8^k n^4)$. Recently, Etscheid and Mnich [20] discovered a kernel with a linear number of vertices (improving upon a kernel by Crowston et al. [15]), and the aforementioned algorithm was sped up to run in time $\mathcal{O}(8^k m)$ [20]. Gutin and Yeo studied an above-guarantee version of MAX BISECTION [22], proving that finding a balanced bipartition such that it has at least $\frac{m}{2}+k$ edges is FPT (also see [33]).[2] In this context MAX BISECTION, it is also relevant to mention the $(k, n-k)$-MAX CUT, which asks for a bipartite-cut of size at least p where one of the sides is of size exactly k. Parameterized by k, this problem is W[1]-hard [11], but parameterized by p, this

[2]We refer to the surveys [30, 23] for details regarding above-guarantee parameterizations.

problem is solvable in time $\mathcal{O}^*(2^p)$ [35]. (This result improved upon algorithms given in [10, 37].)

Until recently, the parameterized complexity of Min Bisection was open. Approaches to tackle this problem materialized when the parameterized complexity of ℓ-Way Cut was resolved. Here, given a graph G and positive integers k and ℓ, the objective is to delete at most k edges from G such that it has at least ℓ components. Kawarabayashi and Thorup [25] showed that this problem is FPT (parameterized by k). Later, Chitnis et al. [13] developed a completely new tool based on this, called *randomized contractions*, to deal with a plethora of cut problems. Other cut problems that have been shown to be FPT include the generalization of Min Cut to Multiway Cut and Multicut [12, 31, 32]. Eventually, Cygan et al. [17], combining ideas underlying the algorithms developed for Multiway Cut, Multicut, ℓ-Way Cut, and randomized contractions together with a new kind of decomposition, showed Min Bisection to be FPT. Finally, let us also mention the min c-judicious partitioning (which is a maximization problem), called c-Load Coloring, where given a graph G and a positive integer k, the goal is to decide whether $V(G)$ can be partitioned into c parts so that each part has at least k edges. Barbero et al. [2] showed that this problem is FPT (also see [21]).

Despite the abundance of work described above, the parameterized complexity of JB and BJB has not yet been considered. We fill this gap in our studies by showing that both of these problems are FPT. It is noteworthy to remark that one can show that the generalization of Min Bisection to c-Min Bisection, where the objective is to find a partition into c-parts such that each of the parts are of almost the same size and there are at most k edges going across different parts, is FPT [17]. However, such a generalization is not possible for either JB or BJB. Indeed, even the existence of an algorithm with running time $n^{f(k)}$, for any arbitrary function f, would imply a polynomial-time algorithm for 3-Coloring, where k is set to 0.

Our approach. For the sake of readability, our strategy of presentation of our proof consists of the definition of a series of problems, each more “specialized” (in some sense) than the previous one, where each section shows that to eventually solve BJB, it is sufficient to focus on some such problem rather than the previous one. We start by showing that we can focus on the solution of the case of BJB where the input graph is bipartite at the cost of the addition of annotations. For this purpose, we present a (not complicated) Turing reduction that employs a known algorithm for the OCT problem (see section 3). The usefulness of the ability to assume that the input graph is bipartite is a key insight in our approach. In particular, the technical parts of our proof crucially rely on the observation that a connected bipartite graph has only two bipartitions. (Here, we consider bipartitions as ordered pairs.) Keeping this intuition in mind, our next step is to reduce the current annotated problem to one where the input graph is also assumed to be connected. (This specific argument relies on a simple application of dynamic programming.)

Having at hand an (annotated) problem where the input graph is assumed to be a connected bipartite graph, we proceed to the technical part of our proof, which employs the (heavy) machinery developed by Cygan et al. [17]. While this machinery primarily aims to tackle problems where one seeks small cuts in addition to some size constraint, our problem involves a priori seemingly different type of constraints. Nevertheless, we observe that once we handle a connected graph, the removal of any set of k edges (to deal with the size constraint and annotations) would not break the graph into more than $k+1$ connected components, and each of these components would clearly be a bipartite graph. Hence, we can view (in some sense) our problem as a cut problem. In practice, the relation between our problem and a cut problem is quite more intricate, and to realize our idea, we crucially rely on the fact that the connected components are bipartite graphs, which allows us to "guess" a binary vector specifying the biparition of their vertex sets in the final solution. This operation entails the employment of coloring functions (employing $k+1$ colors) and their translation into bipartitions (which at a certain point in our paper, we would start viewing as colorings employing two colors). Let us remark that the machinery introduced by [17] is the computation of a special type of tree decomposition. Accordingly, our approach would eventually involve the introduction of a specialization of BJB that aims to capture the work to perform when handling a bag of the tree decomposition. The definition of this specific problem is very technical, and hence we defer the description of related intuitive explanations to the appropriate locations in section 5, where we have already set up the required notation to discuss it.

2. Preliminaries.

General notation. For two sets A, B, $A \uplus B$ denotes the disjoint union of A and B. Let $f : A \to B$ be some function. Given $X \subseteq A$, the notation $f(X) = b$ indicates that for all $a \in X$, it holds that $f(a) = b$. The *restriction* $f|_X$ of f is a function from X to B such that for any $a \in X$, $f|_X(a) = f(a)$. An *extension* f' of the function f is a function whose domain, Y, is a superset of A and whose range is B, such that for all $a \in A$, it holds that $f'(a) = f(a)$. Bold face lowercase letters are used to denote tuples (vectors). For any tuple $\mathbf{v}$, we let $\mathbf{v}[i]$ denote the ith coordinate of $\mathbf{v}$. Given some condition ψ, we define $[\psi] = 1$ if ψ is true and $[\psi] = 0$ otherwise. For any positive integer x, we denote by $[x]$ the set $\{1, 2, \ldots, x\}$ and by $[x]_0$ the set $\{0, 1, \ldots, x\}$.

Graph theory. Given a graph G, we let $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of G, respectively. For any $u \in V(G)$, $N(u)$ denotes the set of neighbors of u in G, that is, $N(u) = \{v : \{u, v\} \in E(G)\}$. For a subset $A \subseteq V(G)$, $N(A) = \cup_{v \in A} N(v) \setminus A$. We denote by $\delta(A)$ the set of boundary vertices of A, that is, $\delta(A) = \{v \in A :$ there exists $u \in V(G) \setminus A$ such that $\{u, v\} \in E(G)\}$. We let $G \setminus A$ denote the subgraph of G induced by $V(G) \setminus A$. A *bipartite graph* is a graph G such that there exists a bipartition (X, Y) of $V(G)$ where X and Y are independent sets. In this paper, we treat such bipartitions as *ordered pairs*. That is, if (X, Y) is a bipartition of some bipartite graph G, then (Y, X) is assumed to be a *different* bipartition of the graph G. For *connected* bipartite graphs, we have the following simple yet powerful insight.

PROPOSITION 2.1 (folklore). *Any connected bipartite graph G has exactly two bipartitions, (X, Y) and (Y, X).*

The *treewidth* of a graph aims to measure how close the graph is to a tree. Formally, this notion is defined as follows.

DEFINITION 2.2. *A* tree decomposition *of a graph G is a pair (T, β) such that T is a rooted tree, $\beta : V(T) \to 2^{V(G)}$, and the following conditions are satisfied.*

1. *For all $\{u, v\} \in E(G)$, there exists $t \in V(T)$ such that $u, v \in \beta(t)$.*
2. *For all $v \in V(G)$, the subgraph of T induced by $X_v = \{t : v \in \beta(t)\}$ is a (connected) subtree of T on at least one node.*

Given $t, \widehat{t} \in V(G)$, the notation $\widehat{t} \preceq t$ indicates that $\widehat{t}$ is a descendant of t in T. Note that t is a descendant of itself. For any $t \in V(T)$, let t' denote the unique parent of t in T. We also need the standard notation $\sigma(t) = \beta(t) \cap \beta(t')$ and $\gamma(t) = \bigcup_{\widehat{t} \preceq t} \beta(\widehat{t})$.

PROPOSITION 2.3 (folklore). *Let (T, β) be a tree decomposition of a graph G. Given a node $t \in V(T)$, let $t_1, \ldots, t_s$ denote the children of t in T, and for all $i \in [s]$, define $V_{t_i} = \gamma(t_i) \setminus \beta(t)$. Let $V_{t'} = V(G) \setminus (\beta(t) \cup \bigcup_{i=1}^{s} V_{t_i})$. Then, the vertex-set of each connected component of $G \setminus \beta(t)$ is a subset of one of $V_{t_1}, \ldots, V_{t_s}, V_{t'}$.*

Let H be some hypergraph. A *spanning forest* of H is a subset $E' \subseteq E(H)$ of minimum size such that the hypergraph induced on E' has the same components as H.

Unbreakability. A *separation* of a graph G is a pair (X, Y) such that $X, Y \subseteq V(G)$, $X \cup Y = V(G)$ and there is no edge with one endpoint in $X \setminus Y$ and the other in $Y \setminus X$. The *order of a separation* (X, Y) is equal to $|X \cap Y|$.

DEFINITION 2.4. *Let G be a graph, $A \subseteq V(G)$, and $q, k \in \mathbb{N}$. The set A is said to be* (q, k)-unbreakable *in G if for* every *separation (X, Y) of G of order at most k, either $|(X \setminus Y) \cap A| \leq q$ or $|(Y \setminus X) \cap A| \leq q$.*

We also define a notion of unbreakability in the context of functions.

DEFINITION 2.5. *A function $g : U \to [k]_0$ is called* (q, k)-unbreakable *if there exists $i \in [k]_0$ such that $\sum_{j \in [k]_0 \setminus \{i\}} |g^{-1}(j)| \leq q$.*

Let us now claim that there do no exist "too many" (q, k)-unbreakable functions.

LEMMA 2.6. *For all $q, k \in \mathbb{N}$, the number of (q, k)-unbreakable functions from a universe U to $[k]_0$ is upper bounded by $\sum_{l=0}^{q} \binom{|U|}{l} \cdot q^k \cdot (k+1)$.*

Proof. Let $g : U \to [k]_0$ be some (q, k)-unbreakable function. By the definition of a (q, k)-unbreakable function, there exists $i \in [k]_0$ such that $\sum_{j \in [k]_0 \setminus i} |g^{-1}(j)| \leq q$. There are $(k+1)$ ways of choosing such an index i, $\sum_{l=0}^{q} \binom{|U|}{l}$ ways of choosing at most q elements that are not mapped to i, and at most q^k ways of partitioning this set of at most q elements into k parts. Thus, the total number of such functions g is upper bounded by $\sum_{l=0}^{q} \binom{|U|}{l} q^k (k+1)$. ☐

3. Solving Balanced Judicious Bipartition. In this section, we prove Theorem 1.1 under the assumption that we are given an algorithm for an annotated, yet restricted, variant of BJB. Throughout this section, an instance of BJB is denoted by BJB(G, μ, k_1, k_2), and we define $k = k_1 + k_2$. Given a partition (V_1, V_2) that witnesses that an instance BJB(G, μ, k_1, k_2) is a YES-instance, we think of the vertices in V_1 as colored 1 and the vertices in V_2 as colored 2; hence, we call such a partition a *witnessing coloring* of BJB(G, μ, k_1, k_2). To prove Theorem 1.1, we first define the OCT problem. Here, given a graph G, a set $S \subseteq V(G)$ is called an *odd cycle transversal* if $G \setminus S$ is a bipartite graph.

ODD CYCLE TRANSVERSAL (OCT)	**Parameter:** k
Input: An undirected multigraph G, and an integer k.	
Output: An odd cycle transversal of G of size at most k, if it exists; otherwise report NO.	

An instance of Odd Cycle Transversal is denoted by $\mathrm{OCT}(G,k)$. We say $\mathrm{OCT}(G,k)$ is a NO-instance if there is no odd cycle transversal of G of size at most k. The algorithm given by the result below shall be a central component in the design of our algorithm for BJB.

Proposition 3.1 (see [27]). *OCT can be solved in time* $2.3146^k n^{\mathcal{O}(1)}$.

Apart from OCT, we also need to define an auxiliary problem that we call Annotated Bipartite-BJB (AB-BJB). As we proceed with our proofs, we shall continue defining auxiliary problems, where each problem captures a task more specific and technically more challenging than the previous one. The choice of this structure aims to ease the readability of our paper. Intuitively, AB-BJB is basically the BJB problem on bipartite graphs, with an extra constraint that demands that certain vertices are assigned a particular color by the witnessing coloring. We remark that the necessity of the reduction to bipartite graphs stems from the fact that we would like to employ Proposition 2.1 later. The formal definition of AB-BJB is given below.

Annotated Bipartite-BJB (AB-BJB) **Parameter:** $k_1 + k_2$
Input: A bipartite multigraph G with bipartition (P,Q), $A,B \subseteq V(G)$ such that $A \cap B = \emptyset$, and integers μ, k_1, and k_2.
Question: Does there exist a partition (V_1,V_2) of $V(G)$ such that $A \subseteq V_1$, $B \subseteq V_2$, $|V_1| = \mu$ and for $i \in \{1,2\}$, $|E(G[V_i])| \leq k_i$?

An instance of AB-BJB is denoted by AB-BJB(G,A,B,μ,k_1,k_2). A partition (V_1,V_2) satisfying the above properties is called a *witnessing coloring* of AB-BJB (G,A,B,μ,k_1,k_2). Furthermore, we need the following theorem, proven later in this paper.

Theorem 3.2. AB-BJB *can be solved in time* $2^{k^{\mathcal{O}(1)}} \cdot n^{\mathcal{O}(1)}$.

Let us now turn to focus on the proof of Theorem 1.1.

Proof of Theorem 1.1. Given an instance $\mathrm{BJB}(G,\mu,k_1,k_2)$, call the algorithm given by Proposition 3.1 with the instance $\mathrm{OCT}(G,k)$ as input (recall that $k = k_1 + k_2$).

Claim 1. *If* $\mathrm{OCT}(G,k)$ *is a* NO-*instance, then* $\mathrm{BJB}(G,\mu,k_1,k_2)$ *is a* NO-*instance.*

Proof. Suppose $\mathrm{BJB}(G,\mu,k_1,k_2)$ is a YES-instance. Let (V_1,V_2) be a witnessing coloring for this instance. Let $E' = E(G[V_1]) \cup E(G[V_2])$. Then, observe that $G \setminus E'$ is a bipartite graph. Let V' be a set of vertices of minimum size such that every edge in E' has at least one endpoint in V'. Since $|E'| \leq k$, it holds that $|V'| \leq k$. Moreover, $G \setminus V'$ is bipartite. Therefore, V' is an odd cycle transversal of G of size at most k. Thus, $\mathrm{OCT}(G,k)$ is a YES-instance.

Henceforth, let S be an odd cycle transversal of G of size at most k. Then, $G \setminus S$ is a bipartite graph. Fix some bipartition (P,Q) of $G \setminus S$. Let $\mathcal{F}$ be the family of all subsets of S, that is, $\mathcal{F} = 2^S$. For any $F \in \mathcal{F}$, denote $l_1^F = |E(G[F])|$ and $l_2^F = |E(G[S \setminus F])|$, and let G_F be the graph constructed as follows (see Figure 1).

- $V(G_F) = V(G \setminus S) \cup \{w_F, x_F, y_F, z_F\}$, where w_F, x_F, y_F, z_F are new distinct vertices.
- $E(G_F) = E(G \setminus S) \cup E_{w_F} \cup E_{x_F} \cup E_{y_F} \cup E_{z_F}$, where the *multisets* $E_{w_F}, E_{x_F}, E_{y_F}$, and E_{z_F} are defined as follows.

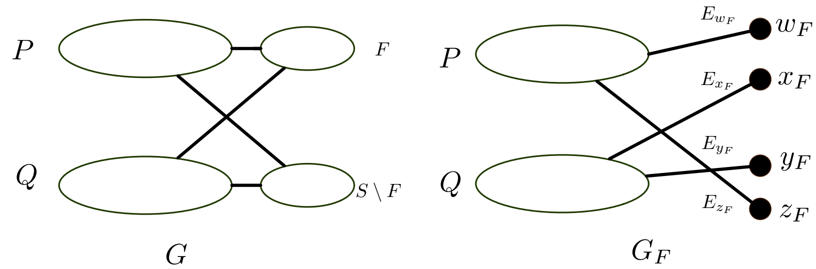

FIG. 1. *The construction in the proof of Theorem* 1.1.

- for each edge $(u, v) \in E(G)$, such that $u \in F$ and $v \in P$, there is an edge $(w_F, v) \in E_{w_F}$,
- for each edge $(u, v) \in E(G)$, such that $u \in F$ and $v \in Q$, there is an edge $(x_F, v) \in E_{x_F}$,
- for each edge $(u, v) \in E(G)$, such that $u \in S$ and $v \in Q$, there is an edge $(y_F, v) \in E_{y_F}$,
- for each edge $(u, v) \in E(G)$, such that $u \in S$ and $v \in P$, there is an edge $(z_F, v) \in E_{z_F}$.

Observe that G_F is a bipartite graph with $(P \cup \{x_F, y_F\}, Q \cup \{w_F, z_F\})$ as a bipartition.

CLAIM 2. BJB(G, μ, k_1, k_2) *is a* YES-*instance if and only if there exists* $F \in \mathcal{F}$ *such that* AB-BJB$(G_F, \{w_F, x_F\}, \{y_F, z_F\}, \mu - |F| + 2, k_1 - l_1^F, k_2 - l_2^F)$ *is a* YES-*instance.*

Proof. In the forward direction, suppose that BJB(G, μ, k_1, k_2) is a YES-instance, and let (V_1, V_2) be a witnessing coloring for BJB(G, μ, k_1, k_2). Moreover, let $F = V_1 \cap S$. Now, we define a partition (V_1', V_2') of $V(G_F)$ as follows: $V_1' = (V_1 \setminus S) \cup \{w_F, x_F\}$ and $V_2' = (V_2 \setminus S) \cup \{y_F, z_F\}$. Let us now argue that (V_1', V_2') is a witnessing coloring for AB-BJB$(G_F, \{w_F, x_F\}, \{y_F, z_F\}, \mu - |F| + 2, k_1 - l_1^F, k_2 - l_2^F)$. First, by the construction of (V_1', V_2'), we have that $\{w_F, x_F\} \subseteq V_1'$ and $\{y_F, z_F\} \subseteq V_2'$. Second, as $V_1' = (V_1 \setminus S) \cup \{w_F, x_F\}$, we also have that $|V_1'| = |V_1| - |F| + 2 = \mu + |F| + 2$. Third, observe that for any $|E(G[V_1'])| = |E(G[V_1])| - |E(G[F])|$ and $|E(G[V_2'])| = |E(G[V_2])| - |E(G[S \setminus F])|$. Thus, for $i \in [2]$, $|E(G[V_i])| \leq k_i - l_i^F$.

In the backward direction, suppose that there exists an $F \in \mathcal{F}$ such that AB-BJB $(G_F, \{w_F, x_F\}, \{y_F, z_F\}, \mu - |F| + 2, k_1 - l_1^F, k_2 - l_2^F)$ is a YES-instance, and let (V_1', V_2') be a witnessing coloring for this instance. We now define a partition (V_1, V_2) of $V(G)$ as follows: $V_1 = (V_1' \cap V(G)) \cup F$ and $V_2 = (V_2' \cap V(G)) \cup (S \setminus F)$. Let us now argue that (V_1, V_2) is a witnessing coloring for BJB(G, μ, k_1, k_2). From the definition of V_1, and since $V(G) = (V(G_F) \setminus \{w_F, x_F, y_F, z_F\}) \cup S$ and $S \cap V(G_F) = \emptyset$, we have that $|V_1| = |V_1'| - |\{x_F, y_F\}| + |F| = \mu - |F| + 2 - 2 + |F| = \mu$. Moreover, observe that $|E(G[V_1])| = |E(G[V_1'])| + |E(G[F])| \leq k_1 + l_1^F$ and $|E(G[V_2])| = |E(G[V_2'])| + |E(G[S \setminus F])| \leq k_2 + l_2^F$. This concludes the proof of the claim.

Thus, to solve an instance of BJB, it is enough to solve $2^{|S|} \leq 2^k$ instances of AB-BJB. Hence, by Theorem 3.2, BJB can be solved in time $2^{k^{\mathcal{O}(1)}} n^{\mathcal{O}(1)}$. ∎

4. Solving Annotated Bipartite-BJB. Recall the problem definition of AB-BJB from section 3. In this section, we prove Theorem 3.2. For this purpose, let us define another auxiliary problem, which we call Annotated Bipartite Connected-BJB (ABC-BJB). Intuitively, ABC-BJB is exactly the same problem as AB-BJB, where we are interested in an answer for every choice of $\mu \in [n]_0$, $l_1 \in [k_1]_0$ and $l_2 \in [k_2]_0$, and additionally we demand the input graph to be connected.

Annotated Bipartite Connected-BJB (ABC-BJB) **Parameter:** $k_1 + k_2$
Input: A connected bipartite multigraph $G = (P, Q)$, $A, B \subseteq V(G)$ such that $A \cap B = \emptyset$, and integers k_1 and k_2.
Output: For all $\mu \in [n]_0$, $l_1 \in [k_1]_0$ and $l_2 \in [k_2]_0$, output a binary value, $\mathtt{aJP}[\mu, l_1, l_2]$, which is 1 if and only if there exists a partition (V_1, V_2) of $V(G)$ such that

- $A \subseteq V_1$ and $B \subseteq V_2$,
- $|V_1| = \mu$, and
- for $i \in \{1, 2\}$, $|E(G[V_i])| \leq l_i$.

For any $\mu \in [n]_0$, $l_1 \in [k_1]_0$, $l_2 \in [k_2]_0$, a partition witnessing that $\mathtt{aJP}[\mu, l_1, l_2] = 1$ is called a *witnessing coloring* for $\mathtt{aJP}[\mu, l_1, l_2] = 1$. Moreover, an instance of ABC-BJB is denoted by ABC-BJB(G, A, B, k_1, k_2). In the rest of this paper, we prove the following theorem.

Theorem 4.1. ABC-BJB *can be solved in time* $2^{k^{\mathcal{O}(1)}} \cdot n^{\mathcal{O}(1)}$.

Having Theorem 4.1 at hand, a simple application of the method of dynamic programming results in the proof of Theorem 3.2.

Proof of Theorem 3.2. Let AB-BJB(G, A, B, μ, k_1, k_2) be an instance of AB-BJB. Let $C_1, \ldots, C_r$ be the connected components of G. For all $i \in [r]$, let $A_i = A \cap C_i$ and $B_i = B \cap C_i$. Let $I_i =$ ABC-BJB$(C_i, A_i, B_i, k_1, k_2)$. Let $\mathtt{aJP}_i$ be the output table for the instance I_i, returned by the algorithm of Theorem 4.1. For any $j \in [r]$, let $G_j = G[\bigcup_{i \in [j]} C_i]$. Note that $G = G_r$. Let us define a four-dimensional binary table $\mathtt{M}$ in the following way. For all $i \in [r]$, $\mu' \in [|V(G)|]_0$, $l_1 \in [k_1]_0$, and $l_2 \in [k_2]_0$, $\mathtt{M}[i, \mu', l_1, l_2] = 1$ if and only if there exists a partition (V_1, V_2) of $V(G_i)$ such that $(A \cap G_i) \subseteq V_1$, $(B \cap G_i) \subseteq V_2$, $|V_1| = \mu'$ and for $j \in \{1, 2\}$, $|E(G[V_j])| \leq l_j$. Observe that AB-BJB(G, A, B, μ, k_1, k_2) is a YES-instance if and only if $\mathtt{M}[r, \mu, k_1, k_2] = 1$. We now compute $\mathtt{M}[r, \mu, k_1, k_2]$ recursively using the following recurrences.

$$\mathtt{M}[1, \mu', l_1, l_2] = \mathtt{aJP}_1(\mu', l_1, l_2)$$

For all $i \in \{2, \ldots, r\}$, $\mu' \in [|V(G)|]_0$, $l_1 \in [k_1]_0$ and $l_2 \in [k_2]_0$,

$$\mathtt{M}[i, \mu', l_1, l_2] = \bigvee_{\substack{\mu' = \mu^1 + \mu^2 \\ l_1 = l_1^1 + l_1^2 \\ l_2 = l_2^1 + l_2^2}} (\mathtt{M}[i-1, \mu^1, l_1^1, l_2^1] \wedge \mathtt{aJP}_i[\mu^2, l_1^2, l_2^2]),$$

where for all $j \in \{1, 2\}$, μ^j, l_1^j, and l_2^j are nonnegative integers.

Note that the time taken to compute $\mathtt{M}[r, \mu, k_1, k_2]$ is at most $(r \cdot n^2 \cdot k_1^2 \cdot k_2^2 \cdot \tau)$, where τ is the time taken to solve an instance of ABC-BJB. Since from Theorem 4.1, an instance of ABC-BJB can be solved in time $2^{k^{\mathcal{O}(1)}} \cdot n^{\mathcal{O}(1)}$ and $r \leq n$, AB-BJB can be solved in time $2^{k^{\mathcal{O}(1)}} \cdot n^{\mathcal{O}(1)}$. ∎

5. Solving Annotated Bipartite Connected-BJB. Recall the problem definition of ABC-BJB from section 4. In this section, we prove Theorem 4.1. Let us start by stating a known result that is a crucial component of our proof. By this result, we would have an algorithm that efficiently computes a special type of tree decomposition, which we call a *highly connected tree decomposition*, where every bag is "highly connected" rather than "small" as in the case of standard tree decompositions. While this property is the main feature of this decomposition, it is also equipped with other beneficial properties, such as a (nontrivial) upper bound on the size of its adhesions, which are *all* exploited by our algorithm.

THEOREM 5.1 (see [17]). *There exists an $2^{\mathcal{O}(k^2)}n^2m$-time algorithm that, given a* connected *graph G together with an integer k, computes a tree decomposition (T,β) of G with at most n nodes such that the following conditions hold, where $\eta = 2^{\mathcal{O}(k)}$.*

1. *For each $t \in V(T)$, the graph $G[\gamma(t) \setminus \sigma(t)]$ is connected and $N(\gamma(t) \setminus \sigma(t)) = \sigma(t)$.*
2. *For each $t \in V(T)$, the set $\beta(t)$ is (η, k)-unbreakable in $G[\gamma(t)]$.*
3. *For each nonroot $t \in V(T)$, we have that $|\sigma(t)| \leq \eta$ and $\sigma(t)$ is $(2k,k)$-unbreakable in $G[\gamma(parent(t))]$.*

In order to process such a tree decomposition in a bottom-up fashion, relying on the method of dynamic programming, we need to address a specific problem associated with every bag, called HYPERGRAPH PAINTING (HP). We chose the name HP to be consistent with the choice of problem name in [17], yet we stress that our problem is more general than the one in [17] (since the handling of a bag in our case is more intricate than the one in [17]).

Roughly speaking, an input of HP would consist of the following components. First, we are given "budget" parameters k_1 and k_2 as in an instance of ABC-BJB. Second, we are given an argument b which would simply be n (to upper bound $|\gamma(t)|$) when we construct an instance of HP while processing some node t in the tree decomposition. Third, we are given a hypergraph H which would essentially be the graph $G[\beta(t)]$ to which we add hyperedges. Each hyperedge F of H is supposed to represent the sets $\sigma(\widehat{t})$ for each child $\widehat{t}$ of t. Fourth, we are given an integer q whose purpose is clarified in the discussion below the definition of HP (in Definition 5.4). Finally, for every hyperedge F, we are given a function $f_F : [k]_0^F \times [b]_0 \times [k_1]_0 \times [k_2]_0 \to \{0,1\}$. To roughly understand the meaning of this function, first recall that F is supposed to represent $\sigma(\widehat{t})$ for some child $\widehat{t}$ of t. Now, the function f_F aims to capture all information obtained while we processed the child $\widehat{t}$ of t that might be relevant to the node t. In particular, let us give an informal, intuitive interpretation of an element (Γ, μ, l_1, l_2) in the domain of f_F. For this purpose, note that when we remove at most k edges from the (connected) graph $G[\gamma(\widehat{t})]$, we obtain at most $k+1$ connected components. The function Γ can be thought of as a method to assign to each vertex in $\sigma(\widehat{t})$ the connected component in which it should lie. Such information is extremely useful since each such connected component is in particular a bipartite graph, and hence by relying on Proposition 2.1 and an exhaustive search, we would be able to use it to extract a witnessing coloring for an instance of ABC-BJB. The arguments μ, l_1, and l_2 can be thought of as those in the definition of an output of ABC-BJB. Now, the value $f_F(\Gamma, \mu, l_1, l_2)$ aims to indicate whether Γ, μ, l_1, and l_2 are "realizable" in the context of the child $\widehat{t}$. (The precise meaning of this value will become clearer later, once we establish additional necessary definitions.)

Let us now give the formal definition of HP. In this definition, we denote $k = k_1 + k_2$.

HYPERGRAPH PAINTING (HP)
Input: Integers k_1, k_2, b, d, and q, a multihypergraph H with hyperedges of size at most d, and for all $F \in E(H)$, a function $f_F : [k]_0^F \times [b]_0 \times [k_1]_0 \times [k_2]_0 \to \{0,1\}$.
Output: For all $0 \le \mu \le b$, $0 \le l_1 \le k_1$, $0 \le l_2 \le k_2$, output the binary value

$$\texttt{aHP}[\mu, l_1, l_2] = \bigvee_{\Upsilon : V(H) \to [k]_0} \bigvee_{\substack{\{\mu^F\}|_{F\in E(H)} \\ \{l_1^F\}|_{F\in E(H)} \\ \{l_2^F\}|_{F\in E(H)}}} \bigwedge_{F \in E(H)} f_F(\Upsilon|_F, \mu^F, l_1^F, l_2^F),$$

where $\sum_{F\in E(H)} \mu^F = \mu$, $\sum_{F\in E(H)} l_1^F \le l_1$, $\sum_{F\in E(H)} l_2^F \le l_2$, and each of μ^F, l_1^F, and l_2^F is a nonnegative integer.

For a particular choice of μ, l_1, and l_2, a function Υ witnessing that $\texttt{aHP}[\mu, l_1, l_2] = 1$ is called a witnessing coloring for $\texttt{aHP}[\mu, l_1, l_2]$. An instance of HP is denoted by $\mathrm{HP}(k_1, k_2, b, d, q, H, \{f_F\}|_{F\in E(H)})$. Observe that q is part of the input to an instance of HP, but does not appear in the problem definition. The reason for putting q in the input will become clear when we define *favorable* instances of HP. These are the instances that will be of interest to us throughout this article. Although we are not able to tackle HP efficiently at its full generality, we are still able to solve those instances that are constructed when we would like to "handle" a single bag in a highly connected tree decomposition. Such instances are formalized as favorable instances. For the sake of clarity, let us now address the beneficial properties that these instances satisfy individually, where each of them ultimately aims to ease our search for a witnessing coloring. The first property, called local unbreakability, unconditionally restricts the way a function $\Gamma : F \to [k]_0$, to be thought of as a restriction of the witnessing coloring we seek, can color a hyperedge F so that the value of f_F is 1.[3]

DEFINITION 5.2 (local unbreakability). *An instance*

$$\mathrm{HP}(k_1, k_2, b, d, q, H, \{f_F\}|_{F\in E(H)})$$

is locally unbreakable *if every $F \in E(H)$ satisfies the following property: for any $\Gamma : F \to [k]_0$ that is not $(3k^2, k)$-unbreakable, $f_F(\Gamma, \mu, l_1, l_2) = 0$ for all $0 \le \mu \le b$, $0 \le l_1 \le k_1$, and $0 \le l_2 \le k_2$.*

The second property, called connectivity, implies that if we would like to use a function $\Gamma : F \to [k]_0$ to color a hyperedge (as a restriction of a witnessing coloring) with more than one color, then we would have to "pay" at least 1 from our budget $l_1 + l_2$.

DEFINITION 5.3 (connectivity). *An instance* $\mathrm{HP}(k_1, k_2, b, d, q, H, \{f_F\}|_{F\in E(H)})$ *is* connected *if every $F \in E(H)$ satisfies the following property: for any $\Gamma : F \to [k]_0$ for which there exist distinct $i, j \in [k]_0$ such that $|\Gamma^{-1}(i)|, |\Gamma^{-1}(j)| > 0$, it holds that $f_F(\Gamma, \mu, l_1, l_2) = 1$ only if $l_1 + l_2 \ge 1$.*

The third property, called global unbreakability, directly restricts our "solution space" by implying that we only need to determine whether there exists a (q, k)-unbreakable witnessing coloring.

DEFINITION 5.4 (global unbreakability). *An instance*

$$\mathrm{HP}(k_1, k_2, b, d, q, H, \{f_F\}|_{F\in E(H)})$$

[3] In this context, it may be insightful to recall Lemma 2.6.

is globally unbreakable *if for all* $0 \leq \mu \leq b$, $0 \leq l_1 \leq k_1$, $0 \leq l_2 \leq k_2$*: if* $\mathtt{aHP}[\mu, l_1, l_2] = 1$*, there exists a witnessing coloring* $\Upsilon : V(H) \to [k]_0$ *that is* (q,k)*-unbreakable.*

An instance $HP(k_1, k_2, b, d, q, H, \{f_F\}|_{F \in E(H)})$ is called a *favorable instance of* HP if it is locally unbreakable, connected, and globally unbreakable. For such instances we have the following theorem.

THEOREM 5.5. HP *on favorable instances is solvable in time*

$$2^{\mathcal{O}(\min(k,q)\log(k+q))} d^{\mathcal{O}(k^3)} |E(H)|^{\mathcal{O}(1)}.$$

The proof of this theorem is very technical, involving nontrivial analysis of a very "messy" picture obtained by guessing part of a hypothetical witnessing coloring via the method of color coding. We defer the proof of Theorem 5.5 to section 6.

From now onward, to simplify the presentation of arguments ahead with respect to ABC-BJB, we would abuse notation and directly define a witnessing coloring as a function rather than a partition. More precisely, the term witnessing coloring for $\mathtt{aJP}[\mu, l_1, l_2] = 1$ would refer to a function $col : V(G) \to \{V_1, V_2\}$ such that $A \subseteq V_1$, $B \subseteq V_2$, $|V_1| = \mu$, and for $i \in \{1,2\}$, $|E(G[V_i])| \leq l_i$. To proceed to our proof of Theorem 4.1, we first need to introduce an additional notation. Roughly speaking, this notation translates a coloring Υ of the form that witnesses some $\mathtt{aHP}[\mu, l_1, l_2] = 1$ to a coloring of the form that witnesses $\mathtt{aJP}[\mu, l_1, l_2] = 1$ via some tuple $\mathbf{v} \in \{0,1\}^{k+1}$. Formally, we have the following.

DEFINITION 5.6. *For a tuple* $\mathbf{v} \in \{0,1\}^{k+1}$*, bipartite graph* G *with bipartition* (P,Q)*,* $X \subseteq V(G)$*, and* $\Upsilon : X \to [k]_0$*, define* $\widehat{\Upsilon}_{\mathbf{v}} : X \to \{V_1, V_2\}$ *as follows.*

- *For all* $v \in P \cap X$*,* $\widehat{\Upsilon}_{\mathbf{v}}(v) = V_1$ *if and only if* $\mathbf{v}[\Upsilon(v)] = 0$.
- *For all* $v \in Q \cap X$*,* $\widehat{\Upsilon}_{\mathbf{v}}(v) = V_1$ *if and only if* $\mathbf{v}[\Upsilon(v)] = 1$.

Suppose we are given an instance ABC-BJB(G, A, B, k_1, k_2). Fix some bipartition (P,Q) of G. Let (T, β) be the highly connected tree decomposition computed by the algorithm of Theorem 5.1, and let r be the root of T. In what follows, $\eta = 2^{\mathcal{O}(k)}$ as in Theorem 5.1, and $q = (\eta + k)k$. We now proceed to define a binary variable that is supposed to represent the answer we would like to compute when we process the bag of a specific node of the tree. Hence, one of the arguments is a node t, and three additional arguments are $\mu \in [n]_0, l_1 \in [k_1]_0$, and $l_2 \in [k_2]_0$. However, we cannot be satisfied with one answer, but need an answer for every possible "interaction" between the bag of t and the bag of its parent t'. Thus, the definition also includes a coloring of $\sigma(t)$. The tuple $\mathbf{v} \in \{0,1\}^{k+1}$ is necessary for the translation process described in Definition 5.6. (The way in which we shall obtain such a "right" tuple later in the proof would essentially rely on brute-force.)

DEFINITION 5.7. *Given* $t \in V(T)$*, a* $(3k^2, k)$*-unbreakable function* $\Upsilon^\sigma : \sigma(t) \to [k]_0$*, a tuple* $\mathbf{v} \in \{0,1\}^{k+1}$*, and integers* $\mu \in [n]_0, l_1 \in [k_1]_0$*, and* $l_2 \in [k_2]_0$*, the binary variable* $y[t, \Upsilon^\sigma, \mathbf{v}, \mu, l_1, l_2]$ *is* 1 *if and only if there exists* $\Upsilon : \gamma(t) \to [k]_0$ *extending* Υ^σ *such that the following hold.*

1. *The translation* $\widehat{\Upsilon}_{\mathbf{v}}$ *maps to* V_1 *exactly* μ *vertices, that is,* $|\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)| = \mu$.
2. *The translation* $\widehat{\Upsilon}_{\mathbf{v}}$ *maps* $A \cap \gamma(t)$ *to* V_1 *and* $B \cap \gamma(t)$ *to* V_2*, that is,* $A \cap \gamma(t) \subseteq \widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)$ *and* $B \cap \gamma(t) \subseteq \widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)$.
3. *For all* $i \in \{1,2\}$*, it holds that* $|E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_i)])| \leq l_i$.
4. *The set of edges between vertices receiving different colors by* Υ *is exactly the set of edges between vertices that are mapped to the same side by the*

translation $\widehat{\Upsilon}_{\mathbf{v}}$*, that is,*

$$\bigcup_{i,j\in[k]_0,i\neq j} E(\Upsilon^{-1}(i),\Upsilon^{-1}(j)) = E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)]) \cup E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)]).$$

A function Υ as above is called a *witnessing coloring* for $y[t,\Upsilon^\sigma,\mathbf{v},\mu,l_1,l_2]$. Recall that r refers to the root of the tree decomposition (T,β).

LEMMA 5.8. *For any* $\mu \in [n]_0$*,* $l_1 \in [k_1]_0$*, and* $l_2 \in [k_2]_0$*,* $\texttt{aJP}[\mu,l_1,l_2]=1$ *if and only if there exists* $\mathbf{v}\in\{0,1\}^{k+1}$ *such that* $y[r,\emptyset,\mathbf{v},\mu,l_1,l_2]=1$*.*

Proof. Let us prove the backward direction first. Let $\mathbf{v}\in\{0,1\}^{k+1}$ be such that $y[r,\emptyset,\mathbf{v},\mu,l_1,l_2]=1$ and let $\Upsilon : V(G)\to[k]_0$ be one of its witnessing colorings. Then, Definition 5.7 directly implies that $\widehat{\Upsilon}_{\mathbf{v}}$ is a witnessing coloring for $\texttt{aJP}[\mu,l_1,l_2]=1$.

For the forward direction, let $col : V(G)\to\{V_1,V_2\}$ be a witnessing coloring for $\texttt{aJP}[\mu,l_1,l_2]$. Let $X=E(G[col^{-1}(V_1)])\cup E(G[col^{-1}(V_2)])$. Let $C_0,\dots,C_s$ be the connected components of $G\setminus X$. Since $X\subseteq E(G)$ and $|X|\le l_1+l_2\le k_1+k_2=k$, $G\setminus X$ has at most $k+1$ connected components, and therefore $s\le k$. For any $i\in[s]_0$, let $(P_i=(P\cap C_i),Q_i=(Q\cap C_i))$ be a bipartition of C_i. (Recall that G is a connected bipartite graph with fixed bipartition (P,Q).)

CLAIM 3. *For any* $i\in[s]_0$*, either both* $P_i\subseteq col^{-1}(V_1)$ *and* $Q_i\subseteq col^{-1}(V_2)$ *or both* $P_i\subseteq col^{-1}(V_2)$ *and* $Q_i\subseteq col^{-1}(V_1)$*.*

Proof. Consider a bipartition $({P'}_i,{Q'}_i)$ of C_i, where ${P'}_i=col^{-1}(V_1)\cap C_i$ and ${Q'}_i=col^{-1}(V_2)\cap C_i$. Since C_i is connected, from Proposition 2.1, either $P_i={P'}_i$ and $Q_i={Q'}_i$, or $P_i={Q'}_i$ and $Q_i={P'}_i$. Hence the claim follows.

Let us now construct a k-length binary string, $\mathbf{v}$, as follows. For any $i\in[s]_0$, $\mathbf{v}[i]=0$ if and only if $P_i\subseteq col^{-1}(V_1)$ and $Q_i\subseteq col^{-1}(V_2)$. For $i\in\{s+1,\dots,k\}$, $\mathbf{v}[i]=0$.

Define $\Upsilon : V(G)\to[k]_0$ as follows. For any $v\in V(G)$, $\Upsilon(v)=i$ if and only if $v\in C_i$.

CLAIM 4. $\widehat{\Upsilon}_{\mathbf{v}}=col$*.*

Proof. Consider some vertex $v\in V(G)$. Denote $V_j=col(v)$, $i=\Upsilon(v)$ and $b=\mathbf{v}[i]$, and note that $j\in\{1,2\}$, $i\in[k]_0$, and $b\in\{0,1\}$. We divide the argument into two cases corresponding to whether $v\in P_i$ or $v\in Q_i$. Since $v\in col^{-1}(V_j)$, if $v\in P_i$, then by Claim 3, $P_i\subseteq col^{-1}(V_j)$ and $Q_i\subseteq col^{-1}(V_{3-j})$. Thus, by the construction of $\mathbf{v}$, $b=j-1$. Hence, by the definition of $\widehat{\Upsilon}_{\mathbf{v}}$, $\widehat{\Upsilon}_{\mathbf{v}}(v)=V_j$. Similarly, if $v\in Q_i$, then by Claim 3, $Q_i\subseteq col^{-1}(V_j)$ and $P_i\subseteq col^{-1}(V_{3-j})$. Thus, by the construction of $\mathbf{v}$, $b=2-j$. Hence, by the definition of $\widehat{\Upsilon}_{\mathbf{v}}$, $\widehat{\Upsilon}_{\mathbf{v}}(v)=V_j$.

Since the choice of v was arbitrary, by the definition of $\widehat{\Upsilon}_{\mathbf{v}}$, we have that $\widehat{\Upsilon}_{\mathbf{v}}(v)=V_j$.

CLAIM 5. *For the binary string* $\mathbf{v}$ *constructed as above, the function* Υ *constructed above is a witnessing coloring for* $y[r,\emptyset,\mathbf{v},\mu,l_1,l_2]=1$*.*

Proof. Since $\widehat{\Upsilon}_{\mathbf{v}}=col$, from the definition of col, we have that $|\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)|=\mu$, $A\subseteq\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)$, $B\subseteq\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)$, and for all $i\in\{1,2\}$, $|E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_i)])|\le l_i$. Observe that $\bigcup_{i,j\in[k]_0,i\neq j}E(\Upsilon^{-1}(i),\Upsilon^{-1}(j))=X$. Therefore, $\bigcup_{i,j\in[k]_0,i\neq j}E(\Upsilon^{-1}(i),\Upsilon^{-1}(j))=$

$E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)]) \cup E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)])$. Thus, Υ is a witnessing coloring for

$$y[r, \emptyset, \mathbf{v}, \mu, l_1, l_2] = 1.$$

This concludes the proof of the lemma. ∎

By Lemma 5.8, it is sufficient to compute $y[r, \emptyset, \mathbf{v}, \mu, l_1, l_2]$ for all $\mu \in [n]$, $l_1 \in [k_1]_0$ and $l_2 \in [k_2]_0$. To this end, we need to compute $y[t, \Upsilon^\sigma, \mathbf{v}, \mu, l_1, l_2]$ for every node $t \in V(T)$, function $\Upsilon^\sigma : \sigma(t) \to [k]_0$ that is $(3k^2, k)$-unbreakable, tuple $\mathbf{v} \in \{0,1\}^{k+1}$, and integers $\mu \in [n]_0, l_1 \in [k_1]_0$ and $l_2 \in [k_2]_0$. Here, we employ bottom-up dynamic programming over the tree decomposition (T, β). Let us now zoom into the computation of $y[t, \Upsilon^\sigma, \mathbf{v}, \mu, l_1, l_2]$ for all $\mu \in [n]$, $l_1 \in [k_1]_0$ and $l_2 \in [k_2]_0$, for some specific t, Υ^σ and $\mathbf{v}$. Note that we now assume that values corresponding to the children of t (if such children exist) have been already computed correctly. Moreover, note that $|\sigma(t)| \leq \eta$, the number of $(3k^2, k)$-unbreakable functions $\Upsilon^\sigma : \sigma(t) \to [k]_0$ is at most $|\eta|^{k^{\mathcal{O}(1)}} = 2^{k^{\mathcal{O}(1)}}$ (by Lemma 2.6), and the number of binary vectors of size $k+1$ is at most 2^{k+1}. Thus, the total running time would consist of the computation time of (T, β), and at most $2^{k^{\mathcal{O}(1)}} \cdot n^2$ times the computation time for a set of values as the one we examine now. Hence, it remains to show how to compute the current set of values in time $2^{k^{\mathcal{O}(1)}} \cdot n^{\mathcal{O}(1)}$.

To compute our current set of values, let us construct an instance

$$\mathrm{HP}(k_1, k_2, n, \eta, q, H, \{f_F\}|_{F \in E(H)})$$

of HP where $V(H) = \beta(t)$, and $E(H)$ and $\{f_F\}|_{F \in E(H)}$ are defined as follows.

1. *Type*-1 *Hyperedges.* For all $v \in \beta(t)$, insert $F = \{v\}$ into $E(H)$. Define $f_F : [k]_0^F \times [n]_0 \times [k_1]_0 \times [k_2]_0 \to \{0,1\}$ as

$$f_F(\Gamma, \mu, l_1, l_2) = \begin{cases} 0 & \text{if } v \in \sigma(t) \text{ and } \Gamma(v) \neq \Upsilon^\sigma(v), \\ 1 & \text{if } v \in A,\ \widehat{\Gamma}_{\mathbf{v}}(F) = V_1,\ l_1 = l_2 = 0, \text{ and } \mu = 1, \\ 1 & \text{if } v \in B,\ \widehat{\Gamma}_{\mathbf{v}}(F) = V_2,\ l_1 = l_2 = 0, \text{ and } \mu = 0, \\ 1 & \text{if } v \notin A \cup B,\ l_1 = l_2 = 0, \text{ and } \mu = [\widehat{\Gamma}_{\mathbf{v}}(F) = V_1], \\ 0 & \text{otherwise.} \end{cases}$$

Informally speaking, we introduce this kind of hyperedges to account for the number of vertices in $\beta(t)$ that go to V_1 (and hence contribute to μ).

2. *Type*-2 *Hyperedges.* For all $(u, v) \in E(G[\beta(t)])$, add $F = \{u, v\}$ in $E(H)$. Define $f_F : [k]_0^F \times [n]_0 \times [k_1]_0 \times [k_2]_0 \to \{0,1\}$ as

$$f_F(\Gamma, \mu, l_1, l_2) = \begin{cases} 0 & \text{if } \mu \neq 0, \\ 1 & \text{if } \widehat{\Gamma}_{\mathbf{v}}(u) \neq \widehat{\Gamma}_{\mathbf{v}}(v) \text{ and } \Gamma(u) = \Gamma(v), \\ 1 & \text{if } \widehat{\Gamma}_{\mathbf{v}}(u) = \widehat{\Gamma}_{\mathbf{v}}(v) = V_1, l_1 \geq 1, \text{ and } \Gamma(u) \neq \Gamma(v), \\ 1 & \text{if } \widehat{\Gamma}_{\mathbf{v}}(u) = \widehat{\Gamma}_{\mathbf{v}}(v) = V_2, l_2 \geq 1, \text{ and } \Gamma(u) \neq \Gamma(v), \\ 0 & \text{otherwise.} \end{cases}$$

We introduce this kind of hyperedges to account for the number of edges in $G[\beta(t)]$ that contribute toward the budget k_1 and k_2.

3. *Type*-3 *Hyperedges.* For all $\widehat{t} \in V(T)$ that are a child of t in the tree T, insert $F = \sigma(\widehat{t})$ into $E(H)$. Define $f_F : [k]_0^F \times [n]_0 \times [k_1]_0 \times [k_2]_0 \to \{0,1\}$ as

$$f_F(\Gamma, \mu, l_1, l_2) = \begin{cases} 0 & \text{if } \Gamma \text{ is not } (3k^2, k)\text{-unbreakable or} \\ & y[\widehat{t}, \Gamma, \mathbf{v}, \mu + \mu', l_1 + l_1', l_2 + l_2'] = 0, \\ 1 & \text{otherwise,} \end{cases}$$

where $\mu' = |\widehat{\Gamma}_{\mathbf{v}}^{-1}(V_1)|$, and $l'_i = |\{\{u,v\} \in E(G[\sigma(\widehat{t})]) : \widehat{\Gamma}_{\mathbf{v}}(u) = \widehat{\Gamma}_{\mathbf{v}}(v) = V_i\}|$ for $i \in [2]$.

This kind of hyperedge encapsulates the partial partitions of the graphs induced by $\gamma(\widehat{t})$, where $\widehat{t}$ is a child of t.

Let us first claim that witnessing colorings related to HP$(k_1, k_2, n, \eta, q, H, \{f_F\}|_{F\in E(H)})$ are useful in the sense that they can be extended to witnessing colorings for the binary values in which we are interested.

LEMMA 5.9. *For all $\mu \in [n]$, $l_1 \in [k_1]_0$, and $l_2 \in [k_2]_0$, if* aHP$[\mu, l_1, l_2] = 1$*, then $y[t, \Upsilon^\sigma, \mathbf{v}, \mu, l_1, l_2] = 1$. In fact, for any witness $\Upsilon : \beta(t) \to [k]_0$ of* aHP$[\mu, l_1, l_2] = 1$*, there exists a function $\Upsilon' : \gamma(t) \to [k]_0$ that extends Υ and witnesses $y[t, \mathbf{v}, \Upsilon^\sigma, \mu, l_1, l_2] = 1$.*

Proof. If aHP$[\mu, l_1, l_2] = 1$, let $\Upsilon : \beta(t) \to [k]_0$ be a witnessing coloring for aHP$[\mu, l_1, l_2] = 1$. Then, there exist $\sum_{F\in E(H)} \mu^F = \mu$, $\sum_{F\in E(H)} l_1^F \leq l_1$ and $\sum_{F\in E(H)} l_2^F \leq l_2$, such that for all $F \in E(H)$, $f_F(\Upsilon|_F, \mu^F, l_1^F, l_2^F) = 1$. In particular, the following holds.

1. Since for any *type*-1 hyperedge F, it holds that $f_F(\Upsilon|_F, \mu^F, l_1^F, l_2^F) = 1$, we overall have that $\Upsilon^\sigma \subseteq \Upsilon$, $A \cap \beta(t) \subseteq \widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)$, $B \cap \beta(t) \subseteq \widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)$, and
$$\sum_{F \text{ is a type-1 hyperedge}} \mu^F = |\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1) \cap \beta(t)|.$$
2. Since for any *type*-2 hyperedge F and $i \in \{1,2\}$, it holds that $f_F(\Upsilon|_F, \mu^F, l_1^F, l_2^F) = 1$, we overall have that
$$|E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_i) \cap \beta(t)])| \leq \sum_{F \text{ is a type-2 hyperedge}} l_i^F.$$
3. For any *type*-3 hyperedge $F = \sigma(t_i)$, since $f_F(\Upsilon|_F, \mu^F, l_1^F, l_2^F) = 1$, we have that $\Upsilon|_F$ is $(3k^2, k)$-unbreakable and $y[t_i, \Upsilon|_F, \mathbf{v}, \mu^F + \mu', l_1^F + l'_1, l_2^F + l'_2] = 1$, where $\mu' = |\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1) \cap F|$, $l'_1 = |\{(u,v) \in E(G[\sigma(t_i)])|\widehat{\Upsilon}_{\mathbf{v}}(u) = \widehat{\Upsilon}_{\mathbf{v}}(v) = V_1\}|$, and $l'_2 = |\{(u,v) \in E(G[\sigma(t_i)])|\widehat{\Upsilon}_{\mathbf{v}}(u) = \widehat{\Upsilon}_{\mathbf{v}}(v) = V_2\}|$.
We thus derive that there exists a witnessing coloring $\Upsilon^i : \gamma(t_i) \to [k]_0$ for the condition $y[t_i, \Upsilon|_F, \mathbf{v}, \mu^F + \mu', l_1^F + l'_1, l_2^F + l'_2] = 1$. Specifically, the following conditions are satisfied.
   (a) Υ^i extends $\Upsilon|_F$.
   (b) $|\widehat{\Upsilon^i}_{\mathbf{v}}^{-1}(V_1)| = \mu^F + \mu'$.
   (c) $A \cap \gamma(t_i) \subseteq \widehat{\Upsilon^i}_{\mathbf{v}}^{-1}(V_1)$ and $B \cap \gamma(t_i) \subseteq \widehat{\Upsilon^i}_{\mathbf{v}}^{-1}(V_2)$.
   (d) $|E(G[\widehat{\Upsilon^i}_{\mathbf{v}}^{-1}(V_1)])| \leq l_1^F + l'_1$, and $|E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)])| \leq l_2^F + l'_2$.
   (e) $\bigcup_{\ell, j\in[k]_0, \ell\neq j} E({\Upsilon^i}^{-1}(\ell), {\Upsilon^i}^{-1}(j)) = E(G[\widehat{\Upsilon^i}_{\mathbf{v}}^{-1}(V_1)]) \cup E(G[\widehat{\Upsilon^i}_{\mathbf{v}}^{-1}(V_2)])$.

Keeping the above items in mind, we proceed to identify a witnessing coloring for $y[t, \Upsilon^\sigma, \mathbf{v}, \mu, l_1, l_2] = 1$. We construct such a coloring $\Upsilon' : \gamma(t) \to [k]_0$ as follows. For all $v \in \gamma(t)$, if $v \in \beta(t)$, then define $\Upsilon'(v) = \Upsilon(v)$, and otherwise there exists a unique child t_i of t such that $v \in \gamma(t_i)$, in which case we define $\Upsilon'(v) = \Upsilon^i(v)$. For the sake of clarity, let us extract the required argument to the proof of a separate claim.

CLAIM 6. *The aforementioned Υ' is a witnessing coloring for $y[t, \Upsilon^\sigma, \mathbf{v}, \mu, l_1, l_2] = 1$.*

Proof. First, note that by item 1 in the proof of Lemma 5.9, we have that $\Upsilon_\sigma \subseteq \Upsilon$ and therefore $\Upsilon_\sigma \subseteq \Upsilon'$. Let us now verify that all the conditions specified in Definition 5.7 are satisfied.

- Let us first prove condition 1. To this end, we observe that by items 1, 3(a) and 3(b), we have that the three following equalities hold.
  - $|\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1)| = |\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1) \cap \beta(t)| + \sum_{t_i \text{ is a child of } t \text{ in } T} |\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1) \cap (\gamma(t_i) \setminus \sigma(t_i))|$.
  - $|\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1) \cap \beta(t)| = |\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1) \cap \beta(t)| = \sum_{F \text{ is a type-1 hyperedge}} \mu^F$.
  - For every child t_i of t, $|\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1) \cap (\gamma(t_i) \setminus F)| = \mu^F$, where $F = \sigma(t_i)$.

  Thus, since $\sum_{F \text{ is a type-2 hyperedge}} \mu^F = 0$, we conclude that $|\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1)| = \sum_{F \in E(H)} \mu^F = \mu$.
- Next, we prove condition 2. However, by items 1 and 3(c), we directly deduce that both $A \cap \gamma(t) \subseteq \widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1)$ and $B \cap \gamma(t) \subseteq \widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_2)$ as required.
- We now turn to prove condition 3. First observe that there are no edges between a vertex of $\beta(t) \setminus \sigma(t_i)$ and a vertex of $\gamma(t_i) \setminus \sigma(t_i)$. In light of item 3(a), note that

$$\begin{aligned}|E(G[\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1)])| = {} & |E(G[\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1) \cap \beta(t)])| \\ & + \sum_{t_i \text{ is a child of } t \text{ in } T} |E(G[\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1) \cap \gamma(t_i)])| \\ & - \sum_{t_i \text{ is a child of } t \text{ in } T} |E(G[\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1) \cap \sigma(t_i)])|.\end{aligned}$$

  Now, observe that by items 2, 3(a), and 3(d), the two following equations hold.
  - $|E(G[\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1) \cap \beta(t)])| = |E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1) \cap \beta(t)])| \leq \sum_{F \text{ is a type-2 hyperedge}} l_1^F$.
  - For every child t_i of t, $|E(G[\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1) \cap \gamma(t_i)])| = l_1^F + |E(G[\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1) \cap \sigma(t_i)])|$, where $F = \sigma(t_i)$.

  Since $\sum_{F \text{ is a type-1 hyperedge}} l_1^F = 0$, we conclude that

$$|E(G[\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1)])| \leq \sum_{F \in E(H)} l_1^F \leq l_1.$$

  Similarly, we derive that $|E(G[\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_2)])| \leq \sum_{F \in E(H)} l_2^F \leq l_2$.
- Finally, we prove condition 4. In the first direction, consider some edge $e \in E(G[\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1)]) \cup E(G[\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_2)])$. Let us denote $e = \{u, v\}$, and observe that $\widehat{\Upsilon'}_{\mathbf{v}}(v) = \widehat{\Upsilon'}_{\mathbf{v}}(u)$. If $u, v \in \gamma(t_i)$ for some child t_i of t, then by item 3(e), we have that $e \in \bigcup_{\substack{i,j \in [k]_0, \\ i \neq j}} E(\Upsilon'^{-1}(i), \Upsilon'^{-1}(j))$. Otherwise, from point 1 of Theorem 5.1, $u, v \in \beta(t)$, and thus e is some type-2 hyperedge F. Since $f_F(\Upsilon|_F, \mu^F, l_1^F, l_2^F) = 1$, the definition of $f_F(\Upsilon|_F, \mu^F, l_1^F, l_2^F)$ directly implies that $\Upsilon(u) \neq \Upsilon(v)$, and therefore again $e \in \bigcup_{\substack{i,j \in [k]_0, \\ i \neq j}} E(\Upsilon'^{-1}(i), \Upsilon'^{-1}(j))$.

  In the other direction, consider some edge $e \in \bigcup_{\substack{i,j \in [k]_0, \\ i \neq j}} E(\Upsilon'^{-1}(i), \Upsilon'^{-1}(j))$. Let us denote $e = \{u, v\}$, and observe that $\Upsilon'(v) \neq \Upsilon'(u)$. If $u, v \in \gamma(t_i)$ for some child t_i of t, then by item 3(e), we have that $e \in E(G[\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1)]) \cup E(G[\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_2)])$. Otherwise, from point 1 of Theorem 5.1, $u, v \in \beta(t)$, and thus e is some type-2 hyperedge F. Since $f_F(\Upsilon|_F, \mu^F, l_1^F, l_2^F) = 1$, the definition of $f_F(\Upsilon|_F, \mu^F, l_1^F, l_2^F)$ directly implies that $\widehat{\Upsilon'}_{\mathbf{v}}(v) = \widehat{\Upsilon'}_{\mathbf{v}}(u)$, and therefore again $e \in E(G[\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_1)]) \cup E(G[\widehat{\Upsilon'}_{\mathbf{v}}^{-1}(V_2)])$.

Thus, we have proved that Υ' is a witnessing coloring for $y[t, \Upsilon^\sigma, \mathbf{v}, \mu, l_1, l_2]$. Moreover, Υ', which extends Υ, is the desired function for the second part of the lemma.

This concludes the proof of the lemma. ◻

In light of Lemma 5.9, we now turn to verify that $\mathrm{HP}(k_1, k_2, n, \eta, q, H, \{f_F\}|_{F\in E(H)})$ is of the form that we are actually able to solve.

LEMMA 5.10. $\mathrm{HP}(k_1, k_2, n, \eta, q, H, \{f_F\}|_{F\in E(H)})$ *is a favorable instance of* HP.

Proof. First note that $\mathrm{HP}(k_1, k_2, n, \eta, q, H, \{f_F\}|_{F\in E(H)})$ is indeed a favorable instance of HP. This is clear from the construction of $\{f_F\}|_{F\in E(H)}$ and the fact that each edge of $F \in E(H)$ has size at most η because of point 3 of Theorem 5.1. Let us now verify that each of the three properties of a favorable instance is satisfied.

- *Local unbreakability.* Let us choose an arbitrary $F \in E(H)$. If F is a type-1 or a type-2 hyperedge, then since $|F| \leq 2$, we have that local unbreakability is trivially satisfied. Otherwise, if F is a type-3 hyperedge, then the satisfaction of local unbreakability directly follows from the construction of f_F.
- *Connectivity.* Choose an arbitrary $F \in E(H)$ along with a tuple (Γ, μ, l_1, l_2) in the domain of f_F such that $f_F(\Gamma, \mu, l_1, l_2) = 1$. If F is a type-1 hyperedge, then connectivity trivially holds. If F is a type-2 hyperedge, then connectivity follows from the construction of f_F. Indeed, to see this, let us denote $F = \{u, v\}$. Then, if $\Gamma(u) \neq \Gamma(v)$, by the second through last case in the definition of f_F, we deduce that $\widehat{\Gamma}_{\mathbf{v}}(u) = \widehat{\Gamma}_{\mathbf{v}}(v)$, else we contradict the supposition that $f_F(\Gamma, \mu, l_1, l_2) = 1$. Then, connectivity directly follows from the third and fourth cases.
  Now, suppose that $F = \sigma(\widehat{t})$ is a type-3 hyperedge, and, say, $\Gamma : F \to [k]_0$ is such that there exist $i, j \in [k]_0$, $i \neq j$, satisfying $|\Gamma^{-1}(i)| > 0$ and $|\Gamma^{-1}(j)| > 0$. We need to show that $l_1 + l_2 \geq 1$. Since $f_F(\Gamma, \mu, l_1, l_2) = 1$, it holds that $y[\widehat{t}, \Gamma, \mathbf{v}, \mu + \mu', l_1 + l'_1, l_2 + l'_2] = 1$, where μ', l'_1 and l'_2 are as defined at the construction of f_F. Let $\Upsilon : \gamma(\widehat{t}) \to [k]_0$ denote some witnessing coloring for this condition. Since (T, β) is a *highly connected tree decomposition*, property 1 of such a decomposition implies that $G^* = G[\gamma(\widehat{t})] \setminus E(G[\sigma(\widehat{t})])$ is connected and that every vertex in $\sigma(\widehat{t})$ is adjacent in G to some vertex in $\gamma(\widehat{t}) \setminus \sigma(\widehat{t})$. Since only the edges internal to $\sigma(\widehat{t})$ were removed in forming G^*, it follows that every two vertices in $\sigma(\widehat{t})$ are connected by a path in G^*. Let $u \in \Gamma^{-1}(i)$ and $v \in \Gamma^{-1}(j)$. Note that $u \neq v$ and $i \neq j$. Since u and v are connected by a path in G^*, we derive that G^* has an edge e such that
$$e \in \left(\bigcup_{c,d\in[k]_0, c\neq d} E(\Upsilon^{-1}(c), \Upsilon^{-1}(d)) \right) \setminus E(G[\sigma(t')]).$$
  Recall that $\bigcup_{c,d\in[k]_0, c\neq d} E(\Upsilon^{-1}(c), \Upsilon^{-1}(d)) = E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)]) \cup E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)])$. Therefore, we have that $e \in (E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)]) \cup E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)])) \setminus E(G[\sigma(\widehat{t})])$. Thus, $l_1 + l_2 \geq 1$.
- *Global unbreakability.* Suppose that $\mathtt{aHP}[\mu, l_1, l_2] = 1$. Then, by Lemma 5.9, there exists $\Upsilon' : \gamma(t) \to [k]_0$ satisfying the properties listed in that lemma.

From here, we get that $\sum_{i,j\in[k]_0,i<j}|E(\Upsilon'^{-1}(i),\Upsilon'^{-1}(j))|\le l_1+l_2\le k_1+k_2\le k$. We argue that $\Upsilon'|_{\beta(t)}$ is a witnessing coloring for global unbreakability, that is, this function is (q,k)-unbreakable. In this context, we remind the reader that $q=(\eta+k)k$. To prove our argument, we first prove the following claim.

CLAIM 7. *Suppose that there exists $i\in[k]_0$ such that $|\Upsilon'^{-1}(i)\cap\beta(t)|>\eta+k$. Then, $\sum_{j\in[k]_0,i\neq j}|\Upsilon'^{-1}(j)\cap\beta(t)|\le\eta+k$.*

Proof. Suppose that the claim is false. Then, both $|\Upsilon'^{-1}(i)\cap\beta(t)|>\eta+k$ and $\sum_{j\in[k]_0,i\neq j}|\Upsilon'^{-1}(j)\cap\beta(t)|>\eta+k$. Thus,

$$\left(X=\Upsilon'^{-1}(i)\cap\beta(t),Y=\left(\bigcup_{j\in[k]_0,i\neq j}\Upsilon'^{-1}(j)\cap\beta(t)\right)\cup\delta(\Upsilon'^{-1}(i)\cap\beta(t))\right)$$

is a separation of order at most k of $G[\gamma(t)]$ as we have already shown that

$$\sum_{i,j\in[k]_0,i\le j}|E(\Upsilon'^{-1}(i),\Upsilon'^{-1}(j))|\le l_1+l_2\le k_1+k_2\le k.$$

Moreover, $|(X\setminus Y)\cap\beta(t)|>\eta$ and $|(Y\setminus X)\cap\beta(t)|>\eta$, which contradicts point 2 of Theorem 5.1, that $\beta(t)$ is (η,k)-unbreakable in $G[\gamma(t)]$.

Thus, if there exist $i\in[k]_0$ as defined in Claim 7, then we are done. That is, we conclude that $\Upsilon'|_{\beta(t)}$ is (q,k)-unbreakable. Otherwise, for all $i\in[k]_0$, it holds that $|\Upsilon'^{-1}(i)|\le\eta+k$. In particular, for any $i\in[k]_0$, $\sum_{j\in[k]_0,i\neq j}|\Upsilon'^{-1}(j)|\le(\eta+k)k=q$. Thus, we again conclude that $\Upsilon'|_{\beta(t)}$ is (q,k)-unbreakable.

Finally, we turn to address the statement complementary to the one of Lemma 5.9.

LEMMA 5.11. *For all $\mu\in[n]$, $l_1\in[k_1]_0$, and $l_2\in[k_2]_0$, if $y[t,\Upsilon^\sigma,\mathbf{v},\mu,l_1,l_2]=1$, then $\mathtt{aHP}[\mu,l_1,l_2]=1$.*

Proof. Fix some $\mu\in[n]$, $l_1\in[k_1]_0$, and $l_2\in[k_2]_0$ such that $y[t,\Upsilon^\sigma,\mathbf{v},\mu,l_1,l_2]=1$. Our objective is to show that $\mathtt{aHP}[\mu,l_1,l_2]=1$. To this end, let Υ be a witnessing coloring for $y[t,\Upsilon^\sigma,\mathbf{v},\mu,l_1,l_2]=1$. We would like to prove that $\Upsilon|_{\beta(t)}$ is a witnessing coloring for $\mathtt{aHP}[\mu,l_1,l_2]=1$, which would complete the proof of the lemma. To do so, we proceed as follows.

First, for any hyperedge $F\in E(H)$, let us define μ^F, ${l_1}^F$, and ${l_2}^F$ as follows.

- *If F is a type-1 hyperedge.* Set $\mu^F=1$ if $\widehat{\Upsilon}_{\mathbf{v}}(F)=V_1$, and $\mu^F=0$ otherwise. Set $l_1^F=0$ and $l_2^F=0$.
- *If $F=\{u,v\}$ is a type-2 hyperedge.* Set $\mu^F=0$. If $\widehat{\Upsilon}_{\mathbf{v}}(u)\neq\widehat{\Upsilon}_{\mathbf{v}}(v)$ and $\Upsilon(u)=\Upsilon(v)$, set ${l_1}^F={l_2}^F=0$. Otherwise, if $\widehat{\Upsilon}_{\mathbf{v}}(u)=\widehat{\Upsilon}_{\mathbf{v}}(v)=V_1$, set ${l_1}^F=1$ and ${l_2}^F=0$, and if $\widehat{\Upsilon}_{\mathbf{v}}(u)=\widehat{\Upsilon}_{\mathbf{v}}(v)=V_2$, set ${l_1}^F=0$ and ${l_2}^F=1$. The other cases cannot arise. Indeed, since Υ is a witnessing coloring for $y[t,\Upsilon^\sigma,\mathbf{v},\mu,l_1,l_2]=1$, we have that

$$\bigcup_{i,j\in[k]_0,i\neq j}E(\Upsilon^{-1}(i),\Upsilon^{-1}(j))=E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)])\cup E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)]).$$

- *If F is a type-3 hyperedge.* Denote $F=\sigma(\widehat{t})$, where $\widehat{t}$ is a child of t in T. Set $\mu^F=|\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)\cap(\gamma(\widehat{t})\setminus\sigma(\widehat{t}))|$, $l_1^F=|E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)\cap\gamma(\widehat{t})])|-|E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)\cap\sigma(\widehat{t})])|$, and $l_2^F=|E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)\cap\gamma(\widehat{t})])|-|E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)\cap\sigma(\widehat{t})])|$.

Let us proceed by proving three claims that would together imply that $\Upsilon|_{\beta(t)}$ is a witnessing coloring for $\mathtt{aHP}[\mu, l_1, l_2] = 1$.

CLAIM 8. *Let $\widehat{t}$ be a child of t in T, and let $i \in [k]_0$ be such that $|\Upsilon^{-1}(i) \cap \sigma(\widehat{t})| > 3k$. Then, $\sum_{j\in[k]_0, i\neq j} |\Upsilon^{-1}(j) \cap \sigma(\widehat{t})| \leq 3k$.*

Proof. Suppose, by way of contradiction, that the claim is false. That is, we have that both $|\Upsilon^{-1}(i) \cap \sigma(\widehat{t})| > 3k$ and $\sum_{j\in[k]_0, i\neq j} |\Upsilon^{-1}(j) \cap \sigma(\widehat{t})| > 3k$. Consider the separation (X, Y) of $G[\gamma(t)]$, where $X = \Upsilon^{-1}(i)$ and $Y = (\gamma(t) \setminus \Upsilon^{-1}(i)) \cup \delta(\Upsilon^{-1}(i))$. Observe that $X \cap Y = \delta(\Upsilon^{-1}(i))$. Since Υ is a witnessing coloring for $y[t, \Upsilon^\sigma, \mathbf{v}, \mu, l_1, l_2]$, we have that

$$\bigcup_{i,j\in[k]_0, i\neq j} E(\Upsilon^{-1}(i), \Upsilon^{-1}(j)) = E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)]) \cup E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)])$$

and $|E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)]) \cup E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)])| \leq l_1 + l_2 \leq k_1 + k_2 \leq k$. Therefore, $|\delta(\Upsilon^{-1}(i))| \leq k$, and thus the order of the separation (X, Y) is at most k. Moreover, since $|\Upsilon^{-1}(i) \cap \sigma(\widehat{t})| > 3k$, we have that $|(X \setminus Y) \cap \sigma(\widehat{t})| > 3k - k = 2k$, and since $\sum_{j\in[k]_0, i\neq j} |\Upsilon^{-1}(j) \cap \sigma(\widehat{t})| > 3k$, we also have that $|(Y \setminus X) \cap \sigma(\widehat{t})| > 3k$. This implies that $\sigma(\widehat{t})$ is not $(2k, k)$-unbreakable in $G[\gamma(t)]$, which means that $\sigma(\widehat{t})$ is not $(2k, k)$-unbreakable in $G[\gamma(\text{parent}(\widehat{t}))]$. This is a contradiction to the fact that (T, β) is a highly connected tree decomposition—specifically, it should satisfy Property 3 in Theorem 5.1.

Having Claim 8 at hand, we now verify that each function f_F assigns 1 to the required tuple.

CLAIM 9. *For any $F \in E(H)$, $f_F(\Upsilon|_F, \mu^F, l_1^F, l_2^F) = 1$.*

Proof. First, noting that since Υ is a witnessing coloring for $y[t, \Upsilon^\sigma, \mathbf{v}, \mu, l_1, l_2] = 1$, we have that $\Upsilon \subseteq \Upsilon_\sigma$, $A \cap \gamma(t) \subseteq \widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)$, and $B \cap \gamma(t) \subseteq \widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2)$. Thus, from the construction of a type-1 hyperedge F and the corresponding function f_F with respect to $HP(k_1, k_2, n, \eta, q, H, \{f_F\}_{F\in E(H)})$, it is clear that $f_F(\Upsilon|_F, \mu^F, l_1^F, l_2^F) = 1$. Second, suppose F is a type-2 hyperedge. The specifications of f_F, together with our definition of μ^F, l_1^F, and l_2^F, directly imply that $f_F(\Upsilon|_F, \mu^F, l_1^F, l_2^F) = 1$.

Third, suppose that F is a type-3 hyperedge, and denote $F = \sigma(t_i)$ for some t_i that is a child of t in T. Note that $y[t_i, \Upsilon|_F, \mathbf{v}, \mu^F + \mu', l_1^F + l_1', l_2^F + l_2'] = 1$ because $\Upsilon|_{\gamma(t_i)}$ is a witnessing coloring for this equality, where $\mu' = |\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1) \cap \sigma(\widehat{t})|$, $l_1' = |E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1) \cap \sigma(\widehat{t})])|$, and $l_2' = |E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_2) \cap \sigma(\widehat{t})])|$. We now need to show that $\Upsilon|_F$ is $(3k^2, k)$-unbreakable, as then we would be able to conclude that $f_F(\Upsilon|_F, \mu^F, l_1^F, l_2^F) = 1$. By Claim 8, if there exists $i \in [k]_0$ such that $|\Upsilon^{-1}(i) \cap \sigma(\widehat{t})| > 3k$, then we deduce that $\Upsilon|_{\sigma(\widehat{t})}$ is $(3k^2, k)$-unbreakable. Otherwise, for all $i \in [k]_0$, $|\Upsilon^{-1}(i) \cap \sigma(\widehat{t})| \leq 3k$. Hence, for any $i \in [k]_0$, $\sum_{j\in[k]_0, i\neq j} |\Upsilon^{-1}(j) \cap \sigma(\widehat{t})| \leq 3k^2$. Thus, we have proved that $\Upsilon_{|F}$ is $(3k^2, k)$-unbreakable.

Finally, we present our third claim.

CLAIM 10. $\mu = \sum_{F\in E(H)} \mu^F$, $\sum_{F\in E(H)} l_1^F \leq l_1$, *and* $\sum_{F\in E(H)} l_2^F \leq l_2$.

By the property of (T, β) being a tree decomposition, for any two children t_i and t_j of t in T, $\gamma(t_i) \cap \gamma(t_j) \subseteq \beta(t)$, and also by the definition, $\sigma(t_i) \subseteq \beta(t)$ for any child t_i of t. Now, note that $\mu = |\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)|$. Thus, to show that $\mu = \sum_{F\in E(H)} \mu^F$, it is sufficient to show that $|\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)| = \sum_{F\in E(H)} \mu^F$. However, keeping the above argument in mind, the claim that $|\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)| = \sum_{F\in E(H)} \mu^F$ directly follows from the satisfaction of the three following conditions. We remark that the satisfaction of these conditions is a direct consequence of the supposition that Υ is a witnessing coloring for $y[t, \Upsilon^\sigma, \mathbf{v}, \mu, l_1, l_2] = 1$, together with our definition of the values μ^F, l_1^F, and l_2^F.

1. For any type-1 hyperedge F, we have that $\mu^F = 1$ only if $\widehat{\Upsilon}_{\mathbf{v}}(F) = V_1$. In particular, $\sum_{F \in E(H) \text{ of type-1}} \mu^F = |\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1) \cap \beta(t)|$.
2. For any type-2 hyperedge F, $\mu^F = 0$. Thus, $\sum_{F \in E(H) \text{ of type-2}} \mu^F = 0$.
3. For any type-3 hyperedge $F = \sigma(t_i)$, $\mu^F = |\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1) \cap (\gamma(t_i) \setminus \sigma(t_i))|$.

Similarly, let us observe that $|E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)])| \le l_1$. Thus, to show that $\sum_{F \in E(H)} l_1^F \le l_1$, it is sufficient to show that $\sum_{F \in E(H)} l_1^F \le |E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)])|$. However, the latter inequality directly follows from the satisfaction of all of the following conditions.

1. For any type-1 hyperedge F, $l_1^F = 0$. Thus, $\sum_{F \in E(H) \text{ of type-1}} l_1^F = 0$.
2. For any type-2 hyperedge $F = \{u, v\}$, ${l_1}^F = 1$ only if $\widehat{\Upsilon}_{\mathbf{v}}(u) = \widehat{\Upsilon}_{\mathbf{v}}(v) = V_1$. In particular, $\sum_{F \in E(H) \text{ of type-2}} l_1^F = |E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1)]) \cap E(G[\beta(t)])|$.
3. For any type-3 hyperedge $F = \sigma(t_i)$, $|E(G[\widehat{\Upsilon}_{\mathbf{v}}^{-1}(V_1) \cap (\gamma(t_i) \setminus \sigma(t_i))])| \le l_1^F$.

Symmetrically, $\sum_{F \in E(H)} l_2^F \le l_2$. This concludes the proof of the claim.

As we have proved Claims 9 and 10, we derive that $\Upsilon|_{\beta(t)}$ is a witnessing coloring for $\mathtt{aHP}[\mu, l_1, l_2] = 1$. This concludes the proof of the lemma. ∎

Recall that we have argued that to prove Theorem 4.1, it is sufficient to show that the current set of values $y[t, \Upsilon^\sigma, \mathbf{v}, \mu, l_1, l_2]$ can be computed in time $2^{k^{\mathcal{O}(1)}} n^{\mathcal{O}(1)}$. Here, n refers to $|V(G)|$. By Lemmas 5.9 and 5.11, this set of values can be derived from the solution of $\mathrm{HP}(k_1, k_2, n, \eta, q, H, \{f_F\}|_{F \in E(H)})$. Since $\mathrm{HP}(k_1, k_2, n, \eta, q, H, \{f_F\}|_{F \in E(H)})$ is a favorable instance of HP (by Lemma 5.10), it can be solved in time $2^{\mathcal{O}(\min(k,q)\log(k+q))} \eta^{\mathcal{O}(k^3)} |E(H)|^{\mathcal{O}(1)} = 2^{k^{\mathcal{O}(1)}} n^{\mathcal{O}(1)}$, using Theorem 5.5.

6. Solving favorable instances of HP.

Recall the problem statement of HP and the definition of a *favorable* instance of HP from section 5. In this section, we prove Theorem 5.5. We prove this theorem in two steps. In the first step we prove Lemma 6.1. In the second step, we perform a dynamic programming procedure exploiting the structure given in Lemma 6.1.

6.1. Color coding the instance.

Again, recall that our goal is to solve the HP problem on a favorable instance. In this section, given a hypergraph, our goal is to somehow partition the vertex set of the hypergraph such that, if the given instance of HP is a YES instance, then the witnessing coloring for it does not color the parts of this partition in a very "unpredictable" way. This is formally captured in the conditions of Lemma 6.1. Before stating the lemma, we first define what we mean by a sets-colorings tuple.

A *sets-colorings* tuple of a hypergraph H is a tuple consisting of a partition of $V(H)$, $V(H) = C_0 \uplus C_{11} \uplus \ldots \uplus C_{1a} \uplus C_{21} \uplus \ldots \uplus C_{2b}$ ($C_0, C_{11}, \ldots, C_{1a}, C_{21}, \ldots, C_{2b}$ are called the sets of this tuple), and coloring functions $\Phi_i : C_{1i} \to [k]_0$ for all $i \in [a]$, such that for each $F \in E(H)$, either F is contained in some set of this tuple or intersects at most two sets of this tuple, one of which necessarily being C_0 and the other being one of $\{C_{11}, \ldots, C_{1a}\}$. A sets-colorings tuple looks like $(C_0 \uplus C_{11} \uplus \ldots \uplus C_{1a} \uplus C_{21} \uplus \ldots \uplus C_{2b}, \Phi_1, \ldots, \Phi_a)$.

LEMMA 6.1. *Let $H = (V(H), E(H))$ be a hypergraph and k, d, x, y, q be positive integers. For each $F \in E(H)$, let $|F| \le d$. Let $\Upsilon : V(H) \to [k]_0$ be a coloring of $V(H)$ satisfying the following conditions.*

1. *The number of hyperedges $F \in E(H)$, such that F is not monochromatic under Υ, is at most x.*
2. *For each $F \in E(H)$, $\Upsilon|_F$ is (y,k)-unbreakable. This condition is called the* local unbreakabilty *condition of* Υ.
3. Υ *is (q,k)-unbreakable. This condition is called the* global unbreakability *condition of* Υ. *Let 0 be the globally dominant color of Υ with respect to this global unbreakability.*

Then, given H,k,d,x,y,q, one can, in time $\mathcal{O}(2^{\mathcal{O}(\min(x,q)\log(x+q))} \max\{d,y\}^{\mathcal{O}(\max\{xy,xk\})} \cdot |E(H)|^{\mathcal{O}(1)})$, find a family of size $\mathcal{O}(2^{\mathcal{O}(\min(x,q)\log(x+q))} \max\{d,y\}^{\mathcal{O}(\max\{xy,xk\})} \cdot \log^{\mathcal{O}(1)}|E(H)|)$, consisting of sets-colorings tuples of H, such that there exists a tuple $\boldsymbol{t} = (C_0, C_{11} \uplus \ldots \uplus C_{1a} \uplus C_{21} \uplus \ldots \uplus C_{2b}, \Phi_1, \ldots, \Phi_a)$ in the family where

1. $\Upsilon|_{C_0} = 0$,
2. *for each $i \in [b]$, $\Upsilon|_{C_{2i}}$ is monochromatic in Υ,*
3. *for each $i \in [a]$, either $\Upsilon|_{C_{1i}} = 0$, or $\Upsilon|_{C_{1i}} = \Phi_i$,*

A sets-colorings tuple satisfying the properties mentioned in Lemma 6.1 is called a *good* sets-colorings tuple for Υ. The rest of the section is devoted to the proof of Lemma 6.1.

Outline of the proof of Lemma 6.1. We begin by classifying the hyperedges of H based on Υ. The algorithm highlights a set of hyperedges and the colorings of them as given by Υ using color coding. In the next phase, based on this highlighting, an auxiliary graph is constructed and later tweaked to clean the unwanted highlighting—the side effect of color coding. Eventually another auxiliary graph is constructed which is finally exploited to give the desired output.

6.1.1. Classifying hyperedges. By the global unbreakability of $\Upsilon : V(H) \to [k]_0$, $\sum_{j\in[k]_0, j\neq i} |\Upsilon^{-1}(j)| \leq q$ for some index $i \in [k]_0$. Without loss of generality, suppose that $i = 0$ is such an index, that is, $\sum_{j\in[k]} |\Upsilon^{-1}(j)| \leq q$. We first categorize the hyperedges of H into the following types, based on the coloring Υ. In this context, we recall that the notation $f(A') = b$ indicates that for all $a \in A'$, it holds that $f(a) = b$ (see section 2).

- Let $E_b = \{F \in E(H) : \Upsilon(F) = 0\}$. Here, "b" stands for *big*.
- For each $i \in [k]$, let $E_{s_i} = \{F \in E(H) : \Upsilon(F) = i\}$. Here, "s" stands for *small*.
- Let $E_m = \{F \in E(H) :$ there exist $u, v \in F$ such that $\Upsilon(u) \neq \Upsilon(v)\}$. Here, "m" stands for *multichromatic*.

Observe that each hyperedge $F \in E(H)$ belongs to exactly one of the sets $E_b, E_m, E_{s_1}, \ldots, E_{s_k}$. Furthermore, let E'_{s_i} denote the edge set of some arbitrary spanning forest of the hypergraph on the vertex set $V(H)$ and the edge set E_{s_i}. Let $E_s = \bigcup_{i\in[k]} E'_{s_i}$ denote the union of these edge sets. From the properties of Υ, $|E_m| \leq x$. Also, as we will see in Lemma 6.2, $|E_s| \leq q$. We exploit these bounds to highlight the hyperedges in E_m and E_s (Lemma 6.8) efficiently. In addition to this, as we shall see in Lemma 6.3, the total number of possible restrictions of Υ on any hyperedge can also be bounded effectively. Thus, we cannot only highlight the hyperedges in E_m and E_s, but we can also guess the restrictions of Υ to these hyperedges. The proof of Lemma 6.8 would capture the idea of the performance of highlighting and guessing. As one would expect, this highlighting does not conclude our arguments, as it does not just highlight the hyperedges in E_m and E_s, but also some hyperedges from E_b. We deal with the inherent challenges of handling such a "messy picture" in our proof.

LEMMA 6.2. $|E_s| \leq q$.

Proof. Recall that for each $i \in [k]$, we defined E'_{s_i} as the edge set of a spanning forest of the hypergraph with the vertex set $V(H)$ and the edge set E_{s_i}. Hence, by this definition, $|E'_{s_i}| \leq |\Upsilon^{-1}(i)|$. Now, recall that since Υ is (q,k)-unbreakable, we assumed without loss of generality that $\sum_{i\in[k]} |\Upsilon^{-1}(i)| \leq q$. We thus have that $\sum_{i\in[k]} |E'_{s_i}| \leq q$. Therefore, $|E_s| \leq q$. ∎

6.1.2. Introducing good assignments. Let us first note that by Lemma 2.6, for any hyperedge $F \in E(H)$, the number of (y,k)-unbreakable functions (that we call (y,k)-unbreakable colorings) from F (recall $|F| \leq d$) to $[k]_0$ is at most $\alpha = \sum_{l=1}^{y} \binom{d}{l} \cdot y^k \cdot (k+1) = \max\{d,y\}^{\mathcal{O}(\max\{y,k\})}$. For each hyperedge F, let us arbitrarily order all possible (y,k)-unbreakable colorings. For each $i \in [\alpha]$, let $\lambda_{F,i}$ denote the ith such coloring. If for an hyperedge F, the number of such colorings is strictly smaller than α, then we extend its list of possible colorings to be of size α by letting some colorings be present multiple times. Thus, for each $F \in E(H)$ and $i \in [\alpha]$, we ensure $\lambda_{F,i}$ is well-defined.

LEMMA 6.3. *For any $F \in E(H)$, there exists $i \in [\alpha]$ such that $\Upsilon|_F = \lambda_{F,i}$.*

Proof. This follows from the fact that $\Upsilon|_F$ is (y,k)-unbreakable. ∎

Here, we are interested in assignments that are functions associating each hyperedge $F \in E(H)$ with a coloring $\lambda_{F,i}$. Let us proceed by defining which assignments would be useful for us to have at hand.

DEFINITION 6.4. *An assignment $p : E(H) \to [\alpha]_0$ is said to be a* good *assignment if the following conditions hold.*

1. *For all $F \in E_s$, $p(F) = 0$.*
2. *For all $F \in E_m$, $p(F) = i > 0$ and $\Upsilon|_F = \lambda_{F,i}$.*

To employ color coding, we first mention the required derandomization tools.

PROPOSITION 6.5 (see [14, Lemma 1.1]). *Given a set U of size n and $c, d \in [n]_0$, we can construct in time $\mathcal{O}(2^{\mathcal{O}(\min(c,d)\log(c+d))} n \log n)$ a family $\mathcal{F}$ of at most $\mathcal{O}(2^{\mathcal{O}(\min(c,d)\log(c+d))} \log n)$ subsets of U, such that the following holds: for all sets $C, D \subseteq U$ such that $C \cap D = \emptyset$, $|C| \leq c$, and $|D| \leq d$, there exists a set $S \in \mathcal{F}$ with $C \subseteq S$ and $D \cap S = \emptyset$.*

DEFINITION 6.6 ((N,r)-perfect family). *For any universe N, an (N,r)-perfect family is a family of functions from N to $[r]$, such that for any subset $X \subseteq N$ of size r, there exists a function in the family that is injective on X.*

PROPOSITION 6.7 (see [34]). *An (N,r)-perfect family of size $\mathcal{O}(e^r r^{\mathcal{O}(\log r)} \log |N|)$ can be computed in time $\mathcal{O}(e^r r^{\mathcal{O}(\log r)} |N| \log |N|)$.*

We are now ready to present our color coding phases.

LEMMA 6.8. *There exists a set $\mathcal{A}$ of assignments from $E(H)$ to $[\alpha]_0$, such that $|\mathcal{A}| \leq 2^{\mathcal{O}(\min(x,q)\log(x+q))} \cdot \max\{d,y\}^{\mathcal{O}(\max\{xy,xk\})} \cdot \log^2 |E(H)|$ and there exists a good assignment in $\mathcal{A}$. Moreover, such a set $\mathcal{A}$ is computable in time $\mathcal{O}(2^{\mathcal{O}(\min(x,q)\log(x+q))}$ $\max\{d,y\}^{\mathcal{O}(\max\{xy,xk\})} \cdot |E(H)|^{\mathcal{O}(1)})$.*

Proof. We start by defining three families, which would guide us through the construction of $\mathcal{A}$. For $U = E(H)$, $c = x$, and $d = q$, let $\mathcal{F} = \{S_1, \ldots, S_\nu\}$ be the family of size $\nu = 2^{\mathcal{O}(\min(x,q)\log(x+q))} \log |E(H)|$ obtained by calling the algorithm of

Proposition 6.5. For each $j \in [\nu]$, let $\mathcal{P}_j$ be a $(E(H) \setminus S_j, x)$-perfect family of size at most $\zeta \leq e^x x^{\mathcal{O}(\log x)} \log |E(H)|$ computed by the algorithm of Proposition 6.7. Let $\mathcal{Q}$ be the family of all possible functions from $[x]$ to $[\alpha]$. Observe that $|\mathcal{Q}| = \alpha^x$.

For each set $S_j \in \mathcal{F}$, function $\kappa \in \mathcal{P}_j$, and function $\kappa_0 \in \mathcal{Q}$, let $p[S_j, \kappa, \kappa_0] : E(H) \to [\alpha]_0$ be defined as follows.

$$p[S_j, \kappa, \kappa_0](F) = \begin{cases} 0 & \text{if } F \in S_j, \\ \kappa_0(\kappa(F)) & \text{otherwise.} \end{cases}$$

Let $\mathcal{A} = \{p[S_j, \kappa, \kappa_0] : S_j \in \mathcal{F}, \kappa \in \mathcal{P}_j, \kappa_0 \in \mathcal{Q}\}$. We claim that there exists a good assignment in $\mathcal{A}$. Since $|E_m| \leq x$ (from the preconditions of Lemma 6.1) and $|E_s| \leq q$ (from Lemma 6.2), from Proposition 6.5 there exists $S_j \in \mathcal{F}$ such that $E_s \subseteq S_j$ and $E_m \cap S_j = \emptyset$. By Proposition 6.7, there exists a function $\kappa \in \mathcal{P}_j$ which is injective on E_m. Let $E_m = \{F_1, \ldots, F_c\}$, where $c \leq x$. Without loss of generality, $\kappa(F_y) = y$ for all $y \in [c]$. Since $\mathcal{Q}$ contains all possible functions from $[x]$ to $[\alpha]$, and for each $F \in E_m$ there exists $i \in [\alpha]$ such that $\Upsilon|_F = \lambda_{F,i}$ (from Lemma 6.3), there exists $\kappa_0 \in \mathcal{Q}$ such that for each $F \in E_m$, $\Upsilon|_F = \lambda_{F,\kappa_0(\kappa(F))}$. Moreover, since $E_s \subseteq S_j$, we have that $p[S_j, \kappa, \kappa_0](E_s) = 0$. Thus, $p[S_j, \kappa, \kappa_0] \in \mathcal{A}$ is a good assignment.

Recall that $\alpha = \max\{d, y\}^{\mathcal{O}(\max\{y,k\})}$. Now, as we have upper bounded ν and ζ, we observe that $|\mathcal{A}| \leq \nu\zeta\alpha^x = 2^{\mathcal{O}(\min(x,q)\log(x+q))} e^x x^{\mathcal{O}(\log x)} \max\{d, y\}^{\mathcal{O}(\max\{xy,xk\})} \log^2 |E(H)|$. This proves the desired bound on the size of $\mathcal{A}$.

The time taken to compute $\mathcal{A}$ is proportional to the time taken to compute $\mathcal{F}, \mathcal{P}_j$ for each $j \in \{\nu\}$ and $\mathcal{Q}$. By Propositions 6.5 and 6.7, we thus derive that the running time is upper bounded by $\mathcal{O}(2^{\mathcal{O}(\min(x,q)\log(x+q))} \cdot \max\{d, y\}^{\mathcal{O}(\max\{xy,xk\})} \cdot |E(H)|^{\mathcal{O}(1)})$. ∎

In the next section, we work with a fixed assignment $p \in \mathcal{A}$. For each such assignment, we eventually compute a sets-colorings tuple of H. The family as described in Lemma 6.1 is then the union of these tuples for each $p \in \mathcal{A}$. We also prove that if p is a good assignment, then the sets-colorings tuple corresponding to it is a good sets-colorings tuple for Υ. Since, from Lemma 6.8, there exists a $p \in \mathcal{A}$ such that p is good, the family of sets-colorings tuples obtained in the end contains a good sets-colorings tuple for Υ.

6.1.3. Associating the graph L_p with an assignment p. For our assignment $p : E(H) \to [\alpha]_0$, let us now construct an undirected simple graph L_p with $V(L_p) = V(H)$. For each $F \in E(H)$ such that $p(F) = 0$, make F a clique in L_p. We say that the edges of this clique are the edges that *correspond* to the hyperedge F. For any $F \in E(H)$ such that $p(F) = i > 0$, for each $j \in [k]_0$, make the set ${\lambda_{F,i}}^{-1}(j)$ a clique in L_p. We say that the edges of all such cliques are the edges that *correspond* to the hyperedge F. Since we want L_p to be a simple graph, between any two vertices of L_p we retain at most one copy of the edge between them (if one exists). If a deleted copy of some edge e in L_p corresponds to some hyperedge F, then in the simple graph the retained copy of that edge e is the one that is said to correspond to that hyperedge F (even if we originally added the retained copy of e due to a different hyperedge). Note that it may thus be the case that one edge in L_p corresponds to several hyperedges in $E(H)$.

We proceed by analyzing the connected components of L_p. Informally, we first argue that if p is a good assignment, then every connected component of L_p behaves as a single unit with respect to Υ.

LEMMA 6.9. *Let p be a good assignment and let D be any connected component of L_p. Then, $\Upsilon(D) = i$ for some $i \in [k]_0$, that is, all the vertices in D are assigned the same color by Υ.*

Proof. For any $\mathcal{F} \subseteq E(H)$, let $L_p[\mathcal{F}]$ be the simple graph on the same vertex set as L_p, whose edge set contains only those edges of L_p that correspond to some hyperedge in $\mathcal{F}$. Observe that $L_p[E(H)] = L_p$. Moreover, observe that if a set of vertices is connected in $L_p[\mathcal{F}]$, then it is also connected in $L_p[\mathcal{F}']$ for any $\mathcal{F}' \supseteq \mathcal{F}$.

Let $E(H) = \{F_1, \ldots, F_r\}$. Moreover, for any $j \in [r]$, denote $\mathcal{F}_j = \bigcup_{c=1}^{j} F_c$. Let us prove by induction on j that for each component D of $L_p[\mathcal{F}_j]$, we have that $\Upsilon(D) = i$ for some $i \in [k]_0$. The proof of this claim would conclude the proof of the lemma, as by setting $j = r$, we thus derive that for each component D of $L_p[\mathcal{F}_r] = L_p$, we have that $\Upsilon(D) = i$ for some $i \in [k]_0$. Hence, we next focus only on the proof of the claim.

To prove the base case, where $j = 1$, consider the graph $L_p[\mathcal{F}_1]$. If $F_1 \notin E_m$, then $\Upsilon(F_1) = i$ for some $i \in [k]_0$ (by the definition of E_m). Hence, for each connected component D of $L_p[\mathcal{F}_1]$, $\Upsilon(D) = i$ for some $i \in [k]_0$. Otherwise, $F_1 \in E_m$. In this case, let $p(F_1) = s > 0$. Since p is a good assignment, $\lambda_{F_1,s} = \Upsilon|_{F_1}$. Since each component D of $L_p[\mathcal{F}_1]$ is either an isolated vertex or $\lambda^{-1}_{F_1,s}(i)$ for some $i \in [k]_0$, we conclude that $\Upsilon(D) = i$ for some $i \in [k]_0$.

We now suppose that $j \geq 2$. By induction hypothesis, for each connected component D of $L_p[\mathcal{F}_{j-1}]$, we have that $\Upsilon(D) = i$ for some $i \in [k]_0$. Let us now examine the graph $L_p[\mathcal{F}_j]$ and the hyperedge F_j. Note that $F_j = \mathcal{F}_j \setminus \mathcal{F}_{j-1}$. If $F_j \notin E_m$, then $\Upsilon(F_j) = i$ for some $i \in [k]_0$ (from the definition of E_m). Let $\mathcal{D}$ be the collection of every connected components of $L_p[\mathcal{F}_{j-1}]$ which intersects F_j. Then, the definition of L_p and the inductive hypothesis directly imply that $\Upsilon(\bigcup \mathcal{D}) = i$ for some $i \in [k]_0$. Thus, by the inductive hypothesis, for each connected component D of $L_p[\mathcal{F}_j]$, we have that $\Upsilon(D) = i$ for some $i \in [k]_0$. Otherwise, $F_j \in E_m$. Then, denote $p(F_1) = s > 0$. Since p is a good assignment, $\lambda_{F_1,s} = \Upsilon|_{F_1}$. For each $i \in [k]_0$, let $\mathcal{D}_i$ be the collection of all connected components of $L_p[\mathcal{F}_{j-1}]$ that intersect $\lambda^{-1}_{F_j,s}(i)$. Then, the definition of L_p and the inductive hypothesis directly imply $\Upsilon(\mathcal{D}_i) = i$. Hence, by the inductive hypothesis, for each connected component D of $L_p[\mathcal{F}_j]$, we have that $\Upsilon(D) = i$ for some $i \in [k]_0$. ☐

Roughly speaking, we now show that given a good assignment p, if a hyperedge F of H intersects multiple components of L_p and Υ assigns a color $i > 0$ to at least one of the components, then $F \in E_m$.

LEMMA 6.10. *Let p be a good assignment and let D be any connected component of L_p such that $\Upsilon(D) = i > 0$ for some $i \in [k]$. For any $F \in E(H)$ such that $F \cap D \neq \emptyset$ and $F \setminus D \neq \emptyset$, $F \in E_m$.*

Proof. Suppose that the statement is false, that is, there exists $F \in E(H) \setminus E_m$ such that $F \cap D \neq \emptyset$ and $F \setminus D \neq \emptyset$. Since $F \notin E_m$, $F \cap D \neq \emptyset$ and $\Upsilon(D) > 0$, there exists $j \in [k]$ such that $F \in E_{s_j}$. Since $F \cap D \neq \emptyset$ and $\Upsilon(D) = i$, we have that $j = i$, that is, $F \in E_{s_i}$. Recall that E'_{s_i} is a spanning forest of the hypergraph with vertex set $V(H)$ and edge set E_{s_i}. Observe that, since p is a good assignment, by the definition of L_p, for any spanning forest E'_{s_i}, all vertices of F lie in the same component of L_p, which contradicts that $F \setminus D \neq \emptyset$. ☐

6.1.4. Rules to modify a good assignment. We now modify the assignment p by applying the following rule exhaustively. Note that whenever we change p, we update L_p accordingly.

Rule 1. If there exist a connected component D of L_p and a hyperedge $F \in E(H)$ such that $F \subseteq D$ and $p(F) > 0$, then update $p(F) = 0$.

LEMMA 6.11. *If p was a good assignment, then after any application of Rule 1, it remains a good assignment.*

Proof. From Lemma 6.9, $\Upsilon(D) = i$ for some $i \in [k]_0$. Thus, if $F \subseteq D$, then $F \notin E_m$. Hence, when we redefine $p(F) = 0$, p remains a good assignment. for the sake of contradiction, that $F_1 \in E_m$. Since p is a good assignment, $\lambda_{F_1,i} = \Upsilon|_F$. Denote $\lambda_{F_1,i}(v_1) = c$, where $c \in [k]$. Since $v_1 \in D$ and $\lambda_{F_1,i}(v_1) = c > 0$, from Lemma 6.9, $\Upsilon(D) = c > 0$. From Lemma 6.10, $F_2 \in E_m$. Again, since p is a good assignment, $\lambda_{F_2,j} = \Upsilon|_F$. Since $\lambda_{F_2,j}(v_2) = 0$ and $v_2 \in D$, this implies that $\Upsilon(D) = 0$, which is a contradiction. ∎

For each connected component D of L_p, let us now define a *label set* $L(D) \subseteq [k]_0$ as follows. For any $i \in [k]_0$, we insert i into $L(D)$ if and only if there exists $F \in E(H)$ such that $F \cap D \neq \emptyset$, $p(F) = j > 0$ and $\lambda_{F,j}(F \cap D) = i$. Observe that $L(D)$ could be empty.

Let us now turn to analyze the label sets we have just defined.

LEMMA 6.12. *For any assignment p, let D be a connected component of L_p such that $L(D) = \emptyset$. Then, for any $F \in E(H)$ such that $F \cap D \neq \emptyset$, $F \setminus D = \emptyset$.*

Proof. Observe that if there exists $F \in E(H)$ such that $p(F) > 0$ and $F \cap D \neq \emptyset$, then $|L(D)| \geq 1$. Therefore, if $L(D) = \emptyset$, then for all $F \in E(H)$ such that $F \cap D \neq \emptyset$, we have that $p(F) = 0$. Thus, from the construction of L_p, we have that $F \setminus D = \emptyset$. ∎

LEMMA 6.13. *Let p be a good assignment such that Rule 1 is no longer applicable to L_p. Then, for any connected component D of L_p, if $\Upsilon(D) = i > 0$, then either $L(D) = \emptyset$ or $L(D) = \{i\}$.*

Proof. Suppose that $L(D) \neq \emptyset$. Then, there exists $F \in E(H)$ such that $F \cap D \neq \emptyset$ and $p(F) = j > 0$. Let $\lambda_{F,j}(F \cap D) = s$. We will now show that $s = i$. First, let us argue that $F \setminus D \neq \emptyset$. Indeed, if $F \setminus D = \emptyset$, then $F \subseteq D$. In this case, since p is a good assignment, where Rule 1 has been exhaustively applied, $p(F)$ should be equal to 0, which is a contradiction. Thus, since $\Upsilon(D) = i > 0$, $F \cap D \neq \emptyset$, and $F \setminus D \neq \emptyset$, from Lemma 6.10, we have that $F \in E_m$. Then, since p is a good assignment, $\lambda_{F,j}(F \cap D) = \Upsilon|_{F \cap D}$. Since $\Upsilon(D) = i$, we derive that indeed $\lambda_{F,j}(F \cap D) = i$. Thus, $L(D) = \{i\}$. ∎

By Lemma 6.13, we have that if p is a good assignment and D is a connected component of L_p such that either $L(D) = \{0\}$ or $|L(D)| \geq 2$, then $\Upsilon(D) = 0$.

LEMMA 6.14. *If p is a good assignment such that Rule 1 is no longer applicable to L_p, and D is a connected component of L_p such that $L(D) = \{l_d\}$, then either $\Upsilon(D) = l_d$ or $\Upsilon(D) = 0$.*

Proof. Since $L(D) = \{l_d\}$, there exists $F \in E(H)$ such that $p(F) = i > 0$, $F \cap D \neq \emptyset$, and $\lambda_{F,i}(F \cap D) = l_d$. Since Rule 1 has been applied exhaustively, $F \setminus D \neq \emptyset$. Denote $\Upsilon(D) = j$, and suppose that $j \neq 0$, else we are done. Since $j \neq 0$, from Lemma 6.10 we have that $F \in E_m$. Then, since p is a good assignment, $\lambda_{F,i} = \Upsilon|_F$. Finally, since all the vertices of D are assigned the same color by Υ (by Lemma 6.9), we have that $\Upsilon(D) = l_d$. ∎

For a connected component D of L_p such that $|L(D)| \geq 2$, let us redefine the label set of D to be $L(D) = \{0\}$. Now, for any connected component D of L_p, $|L(D)| \leq 1$. Moreover, if p is a good assignment and $L(D) = \{0\}$, then $\Upsilon(D) = 0$ (by Lemma 6.13). We call a connected component D of L_p such that $L(D) = \{0\}$ is a 0-*component*. Thus, from Lemma 6.13, if p is a good assignment and D is a 0-component of L_p, then $\Upsilon(D) = 0$.

Let us continue modifying the assignment p, now with the following rule. Again, whenever we modify p, we update L_p accordingly.

Rule 2. If there exist $F \in E(H)$ and two distinct 0-components of L_p, D_1 and D_2, such that $F \cap D_1 \neq \emptyset$ and $F \cap D_2 \neq \emptyset$, then update $p(F) = 0$.

LEMMA 6.15. *If p is a good assignment, then after the application of Rule 2, it remains good.*

Proof. To prove the lemma, it is sufficient to show that $F \notin E_m$. Suppose that this claim is false, that is, $F \in E_m$ and hence after the update, we obtain an assignment that is not good. Since (the original) p is a good assignment, we have that $p(F) = i > 0$ such that $\lambda_{F,i} = \Upsilon|_F$. Since D_1 and D_2 are different connected components of L_p, $(F \cap D_1) \subseteq \lambda^{-1}_{F,i}(j_1)$, $(F \cap D_2) \subseteq \lambda^{-1}_{F,i}(j_2)$, and $j_1 \neq j_2$. However, since D_1 and D_2 are 0-components of L_p, $\Upsilon(D_1) = 0$ and $\Upsilon(D_2) = 0$. Hence, $\lambda_{F,i}(F \cap (D_1 \cup D_2)) = 0$, and so, $F \cap (D_1 \cup D_2)$ is a clique in L_p. This contradicts that D_1 and D_2 are two different components of L_p. ∎

To further analyze 0-components, define B as the set containing every vertex $v \in V(H)$ such that $\Upsilon(v) = 0$ and there exists $F \in E_m$ that is incident to v.

LEMMA 6.16. *Let p be a good assignment and let D be a connected component of L_p containing a vertex $v \in B$. Then, D is a 0-component.*

Proof. From the definition of the set B, there exists $F \in E_m$ such that $v \in F$. Since p is a good assignment, $p(F) = i > 0$ such that $\lambda_{F,i} = \Upsilon|_F$. Since $\Upsilon(v) = 0$, $v \in F$, and $v \in D$, we have that $\lambda_{F,i}(F \cap D) = 0$. Hence, $0 \in L(D)$. Therefore, by Lemma 6.13, we conclude that D is a 0-component of L_p. ∎

6.1.5. Constructing a supergraph L_p^* of L_p. Let us now construct another simple undirected graph L_p^*, which is a supergraph of L_p with the same vertex set as of L_p and the following additional edges. If there exists $F \in E(H)$ and two distinct connected components of L_p, D_1, and D_2, such that $F \cap D_1 \neq \emptyset$, $F \cap D_2 \neq \emptyset$, $L(D_1) \neq \{0\}$ and $L(D_2) \neq \{0\}$, then insert an edge between some vertex of D_1 and some vertex of D_2 into L_p^*. Clearly, any connected component D of L_p is contained in some connected component of L_p^*. This leads us to the following definition.

DEFINITION 6.17. *Given a connected component D^* of L_p^*, we say that a connected component D of L_p is a* constituent *of D^* if $D \subseteq D^*$.*

A component D^* of L_p^* is called a 0-component of L_p^* if it has only one constituent component and that constituent component is a 0-component in L_p. We now proceed to analyze the new graph L_p^*.

LEMMA 6.18. *Let D^* be some connected component of L_p^* that has a constituent component D such that $L(D) = \emptyset$ or $L(D) = \{0\}$. Then, D is the only constituent component of D^*, that is, $D^* = D$.*

Proof. When $L(D) = \{0\}$, the lemma follows from the construction of L_p^*. When $L(D) = \emptyset$, by Lemma 6.12, for any $F \in E(H)$ such that $F \cap D \neq \emptyset$, we have that $F \setminus D = \emptyset$. Thus, by the construction of L_p^*, it holds that $D^* = D$. ∎

From Lemma 6.18, we have that a component of L_p^* is a 0-component of L_p^* if and only if it is a 0-component of L_p.

LEMMA 6.19. *Let D^* be a connected component of L_p^*. Let D be some constituent component of D^*. If $\Upsilon(D) = 0$, then $\Upsilon(D^*) = 0$.*

Proof. If $D^* = D$, then we are done. Otherwise, for the sake of contradiction, suppose that $\Upsilon(D^*) \neq 0$. Then there exists a constituent component $\tilde{D}$ of D^* such that $\Upsilon(\tilde{D}) \neq 0$. Since $\Upsilon(D) = 0$ and D^* is connected, there exist constituent components D', D'' of D^*, $D' \neq D''$, such that there is an edge between D' and D'' in D^* and $\Upsilon(D') = 0$ and $\Upsilon(D'') \neq 0$. is an edge between D' and D'' in D^*, from the construction of L_p^*, there exists $F \in E(H)$ such that $F \cap D' \neq \emptyset$ and $F \cap D'' \neq \emptyset$. Since $\Upsilon(D') = 0$, $\Upsilon(D'') \neq 0$, $F \cap D' \neq \emptyset$, and $F \cap D'' \neq \emptyset$, $F \in E_m$. Since p is a good assignment, from the construction of L_p and the assigning of label sets, $L(D') = \{0\}$. From Lemma 6.18, this implies that D' is the only constituent component of D^*, which is a contradiction. ∎

LEMMA 6.20. *For any $F \in E(H)$, F intersects exactly one non 0-component of L_p^*.*

Proof. If there exists $F \in E(H)$ which intersects two non 0-components of L_p^*, then from the construction of L_p^*, those two components are joined by an edge in L_p^* and hence are the same component in L_p^*. If there exists $F \in E(H)$ which intersects two 0-components of L_p^*, then this violates that Rule 2 has been applied. ∎

We are now ready to output the sets-colorings tuple $\mathbf{t}_p = (C_0 \uplus C_{11} \uplus \ldots \uplus C_{1a} \uplus C_{21} \uplus \ldots \uplus C_{2b}, \Phi_1, \ldots, \Phi_a)$ corresponding to the assignment p. The sets of $\mathbf{t}_p$ correspond to the connected components of L_p^*. C_0 is the collection of the 0-components of L_p^*. $\{C_{11}, \ldots, C_{1a}\}$ are the components of L_p^* whose constituents have a nonempty label set. $\{C_{21}, \ldots, C_{2b}\}$ are the components of L_p^* whose unique constituent has an empty label set. For any $i \in [a]$, $\Phi_i : C_{1i} \to [k]_0$ is defined as follows. Since C_{1i} is a connected component of L_p^*, let $C_{1i}^1, C_{1i}^2, \ldots, C_{1i}^j$ be its constituent components. Then for any $r \in [j]$, $\Phi_i(C_{1i}^r) = L(C_{1i}^r)$ (recall $L(C_{1i}^r)$ has a unique label for the constituent component C_{1i}^r). From Lemma 6.20, each hyperedge intersects at most one of $\{C_{11}, \ldots C_{1a}, C_{21}, \ldots, C_{2b}\}$. Also, from Lemmas 6.13 and 6.12, if $F \cap C_{2i} \neq \emptyset$, then $F \setminus C_{2i} = \emptyset$. This proves that the tuple $\mathbf{t}_p$ is indeed a sets-colorings tuple for H.

We will now prove that if p is a good assignment, then $\mathbf{t}_p$ is the tuple with the properties desired in Lemma 6.1.

LEMMA 6.21. *If p is a good assignment, then the sets-colorings tuple $\mathbf{t}_p$ corresponding to it is a good tuple for Υ.*

Proof. We show that $\mathbf{t}_p$ satisfies all the properties described in Lemma 6.1.

1. By the definition of C_0, 0-component of L_p^*, 0-component of L_p, and Lemma 6.13, $\Upsilon|_{C_0} = 0$.
2. From Lemma 6.18 and the definition of C_{2i}, C_{2i} is some connected component of L_p. Thus, from Lemma 6.9, $\Upsilon|_{C_{2i}}$ is monochromatic in Υ.
3. Consider any C_{1i}. From the construction of C_{1i}, C_{1i} is a connected component of L_p^* each of whose constituent components have a nonempty label set. Thus, from Lemmas 6.14 and 6.19, either $\Upsilon|_{C_{1i}} = 0$ or $\Upsilon|_{C_{1i}} = \Phi_i$. ∎

Thus, the algorithm for Lemma 6.1, for each $p \in \mathcal{A}$, constructs L_p^* and computes a corresponding sets-colorings tuple as discussed before. Here, $\mathcal{A}$ is the family in Lemma 6.8. It then outputs the family containing these sets-colorings tuples for each $p \in \mathcal{A}$. From Lemma 6.8, there exists a $p \in \mathcal{A}$, such that p is a good assignment. Also, from Lemma 6.21, if p is a good assignment, then $\mathbf{t}_p$ is a good tuple for Υ. Thus, the output family of sets-colorings tuples contains a good tuple for $\mathbf{t}_p$. We are now left to analyze the running time of the algorithm.

Running time analysis. The algorithm begins by computing a family $\mathcal{A}$ of assignments from $E(H)$ to $[\alpha]_0$ using Lemma 6.8. Then for each assignment $p \in \mathcal{A}$, the algorithm constructs the graph L_p (in polynomial time), modifies it using Rules 1 and 2 (in polynomial time), assigns it labels (in polynomial time), constructs L_p^* (in polynomial time) and finally constructs the sets-colorings tuple for it (in polynomial time). Since, from Lemma 6.8, $|\mathcal{A}| = \mathcal{O}(2^{\mathcal{O}(\min(x,q)\log(x+q))} \max\{d,y\}^{\mathcal{O}(\max\{xy,xk\})} \cdot \log^2 |E(H)|)$ and the time taken to compute $|\mathcal{A}|$ is

$$\mathcal{O}(2^{\mathcal{O}(\min(x,q)\log(x+q))} \max\{d,y\}^{\mathcal{O}(\max\{xy,xk\})} \cdot |E(H)|^{\mathcal{O}(1)}),$$

the total time taken by the algorithm is $\mathcal{O}(2^{\mathcal{O}(\min(x,q)\log(x+q))} \cdot \max\{d,y\}^{\mathcal{O}(\max\{xy,xk\})} \cdot |E(H)|^{\mathcal{O}(1)})$.

6.2. Dynamic programming. Recall that our aim is to prove Theorem 5.5, that is, we need to design an algorithm to solve favorable instances of HP. To do so, we will use Lemma 6.1 followed by a dynamic programming procedure for each sets-colorings tuple in the family returned by the algorithm of Lemma 6.1. Recall a favorable instance of HP, $I = (k_1, k_2, b, d, q, H, \{f_F\}|_{F \in E(H)})$. If $\mathtt{aHP}[\mu, l_1, l_2] = 1$, then there exists a witnessing coloring $\Upsilon : V(H) \to [k]_0$ for $\mathtt{aHP}[\mu, l_1, l_2]$. We will show that since I is a favorable instance of HP, Υ satisfies the prerequisites of Lemma 6.1. Then, for each sets-colorings tuple in the family returned by Lemma 6.1, we define $k+1$ coloring functions for each hyperedge of H. These coloring functions are defined in such a way that when we later compute $\mathtt{aHP}[\mu, l_1, l_2]$ using dynamic programming, these coloring functions together give a coloring for $V(H)$. Moreover, if $\mathtt{aHP}[\mu, l_1, l_2] = 1$, then since there exists a witnessing coloring Υ for $\mathtt{aHP}[\mu, l_1, l_2]$ that satisfies the preconditions of Lemma 6.1, the dynamic programming procedure corresponding to the sets-colorings tuple that satisfies the conditions of Lemma 6.1 will return 1 (or Yes).

Proof of Theorem 5.5. Given a favorable instance

$$I = (k_1, k_2, b, d, q, H, \{f_F\}|_{F \in E(H)})$$

of HP, our algorithm proceeds by calling the algorithm of Lemma 6.1 on the instance (H, k, d, x, y, q), where $k = k_1 + k_2, x = k$, and $y = 3k^2$. The output is a family, say, $\mathcal{T}$, of sets-colorings tuples of H.

For each sets-colorings tuple $\mathbf{t} \in \mathcal{T}$, for each $F \in E(H)$, we define $k+1$ coloring functions from F to $[k]_0$, $\Psi_F^1, \ldots, \Psi_F^{k+1}$ (defined later). Let $\mathbf{t} = (C_0 \uplus C_{11} \uplus \cdots \uplus C_{1a} \uplus C_{21} \uplus \cdots \uplus C_{2b}, \Phi_1, \ldots \Phi_a)$. Rename the sets in the tuple $\mathbf{t}$ as $\{S_0, S_1, \ldots, S_z\}$, where $z = a + b$, such that $S_0 = C_0$, for all $i \in [a]$ $S_i = C_{1i}$, and for all $i \in [b]$ $S_{a+i} = C_{2i}$. For each $i \in [z]$ and $j \in [k]_0$, define a function $\Psi_i^j : S_i \to [k]_0$ as follows. $\Psi_0^j(S_0) = 0$ for all $j \in [k]_0$. For each $i \in [a]$, $\Psi_i^0(S_i) = 0$ and $\Psi_i^j(S_i) = \Phi_i$ for all $j \in [k]$. For each $i \in \{a+1, \ldots, z\}$ and $j \in [k]_0$, $\Psi_i^j(S_i) = j$. Based on these coloring functions for the sets in the tuple, we now define coloring functions for the hyperedges of H.

For that, let us first classify the hyperedges of H based on the sets in the tuple $\mathbf{t}$. For each $i \in \{0, a+1, a+2, \ldots, z\}$, let $E_{S_i} = \{F \in E(H) : F \subseteq S_i\}$. For each $i \in [a]$, let $E_{S_i} = \{F \in E(H) : F \cap S_i \neq \emptyset\}$. Since $\mathbf{t}$ is a sets-colorings tuple and $S_0, S_1, \ldots, S_z$ are the sets of this tuple, renamed as described above, $E(H) = \uplus_{i \in [z]_0} E_{S_i}$. We now define the coloring functions for the hyperedges of H. For each $i \in \{0, a+1, \ldots, z\}$, $F \in E_{S_i}$ and $j \in [k]_0$, $\Psi^j_F = \Psi^j_i|_F$. For each $i \in [a]$, for each $F \in E_{S_i}$ and $j \in [k]_0$, $\Psi^j_F(v) = \Psi^j_i(v)$ if $v \in S_i$; $\Psi^j_F(v) = 0$ if $v \in S_0$.

This finishes the description of the coloring functions for the sets of the tuples and the hyperedges of H. Observe that the colorings Ψ defined for hyperedges are consistent with $V(H)$, that is, for each $F \in E(H)$, no matter which coloring out of Ψ^j_F, $j \in [z]_0$ is picked, it together colors $V(H)$, where each vertex in $V(H)$ gets a unique color (assuming $V(H) = \cup_{F \in E(H)} V(F)$, where $V(F)$ denotes the vertices in the hyperedge F). This is true because $\mathbf{t}$ is a sets-colorings tuple and $\Psi_{S_0} = 0$.

For each $i \in [z]$, let $E_{S_i} = \{F_{i,1}, \ldots, F_{i,z_i}\}$. Fix a set S_i and $j \in [k]_0$, and define

$$h^j_i[\mu', l'_1, l'_2] = \bigvee_{\substack{(\mu^r)_{r \in [z_i]} \\ (l^r_1)_{r \in [z_i]} \\ (l^r_2)_{r \in [z_i]}}} \bigwedge_{r \in [z_i]} f_{F_{i,r}}(\Psi^j_{F_{i,r}}, \mu^r, {l_1}^r, {l_2}^r),$$

where $\sum_{r \in [z_i]} \mu^r = \mu'$, $\sum_{r \in [z_i]} l^r_1 \leq l'_1$, $\sum_{r \in [z_i]} {l_2}^r \leq l'_2$, and each μ^r, l^r_1, l^r_2 is a nonnegative integer.

Now define $\mathcal{H}[i, \mu', l'_1, l'_2] = \bigvee_{j \in [k]_0} h^j_i[\mu', l'_1, l'_2]$.

Let

$$\texttt{computeHP}[\mu, l_1, l_2] = \bigvee_{\substack{(\mu^i)_{i \in [z]_0} \\ (l^i_1)_{i \in [z]_0} \\ (l^i_2)_{i \in [z]_0}}} \bigwedge_{i \in [z]_0} \mathcal{H}[i, \mu^i, l^i_1, l^i_2],$$

where $\sum_{i \in [z]_0} \mu^i = \mu$, $\sum_{i \in [z]_0} l^i_1 \leq l_1$, $\sum_{i \in [z]_0} l^i_2 \leq l_2$, and each μ^i, l^i_1, l^i_2 is a nonnegative integer. Note that each of the functions $\Psi^j_i, \Psi^j_F, h^j_i, \mathcal{H}$ and computeHP are defined with respect to a sets-colorings tuple $\mathbf{t}$.

LEMMA 6.22. *Suppose* $\texttt{aHP}[\mu, l_1, l_2] = 1$. *Then there exists a sets-colorings tuple* $\mathbf{t} \in \mathcal{T}$, *such that, for this tuple* $\mathbf{t}$, $\texttt{computeHP}[\mu, l_1, l_2] = 1$.

Proof. Recall that

$$\texttt{aHP}[\mu, l_1, l_2] = \bigvee_{\Upsilon : V(H) \to [k]_0} \bigvee_{\substack{(\mu^F)|_{F \in E(H)} \\ (l^F_1)|_{F \in E(H)} \\ (l^F_2)|_{F \in E(H)}}} \bigwedge_{F \in E(H)} f_F(\Upsilon|_F, \mu^F, l^F_1, l^F_2),$$

where $\sum_{F \in E(H)} \mu^F = \mu$, $\sum_{F \in E(H)} l^F_1 \leq l_1$, $\sum_{F \in E(H)} l^F_2 \leq l_2$, and each of μ^F, l^F_1, and l^F_2 is a nonnegative integer.

Since $E(H) = \uplus_{i \in [z]_0} E_{S_i}$, we have the following.

$$\texttt{aHP}[\mu, l_1, l_2] = \bigvee_{\substack{\Upsilon : V(H) \to [k]_0 \\ (\mu^F)_{F \in E(H)} \\ (l^F_1)_{F \in E(H)} \\ (l^F_2)_{F \in E(H)}}} \bigwedge_{i \in [z]_0} \bigwedge_{F \in E_{S_i}} f_F(\Upsilon|_F, \mu^F, l^F_1, l^F_2),$$

where $\sum_{F\in E(H)} \mu^F = \mu$, $\sum_{F\in E(H)} l_1^F \le l_1$, $\sum_{F\in E(H)} l_2^F \le l_2$, and for all $F \in E(H)$, μ^F, l_1^F, and l_2^F are nonnegative integers.

Since $\texttt{aHP}[\mu, l_1, l_2] = 1$, there exists a witnessing assignment $\Upsilon : V(H) \to [k]_0$. Since I is a favorable instance of HP, Υ clearly satisfies the local unbreakability and global unbreakability conditions of Lemma 6.1 (when the input to the algorithm of Lemma 6.1 is $(H, k, d, k, 3k^2, q)$). We first show that Υ also satisfies the first precondition of Lemma 6.1 with $x = k$. That is, the number of hyperedges of H that are nonmonochromatic under Υ is at most k. Since $\texttt{aHP}[\mu, l_1, l_2] = 1$, for all $F \in E(H)$ there exist μ^F, l_1^F, and l_2^F such that $f_F(\Upsilon_{|F}, \mu^F, l_1^F, l_2^F) = 1$. Hence, the connectivity property of Υ (which exists because I is a favorable instance) implies that for each F that is not monochromatic in Υ, we have that $l_1^F + l_2^F \ge 1$. However, $\sum_{F\in E(H)} l_1^F + l_2^F \le l_1 + l_2 \le k_1 + k_2 = k$. Thus, the number of nonmonochromatic hyperedges under Υ is at most k.

Thus, from Lemma 6.1, there exists a good tuple $\mathbf{t} \in \mathcal{T}$ for Υ. Consider $\texttt{computeHP}[\mu, l_1, l_2]$ (and thus corresponding $h_i^j, \Psi_F^j, \Psi_i^j$) defined for this good tuple. From the definition of a good tuple and Ψ_i^j, for each $i \in [z]_0$, $\Upsilon|_{S_i} = \Psi_i^j$ for some $j \in [k]_0$. Also, since $\mathbf{t}$ is a sets-colorings tuple, for each $F \in E(H)$, $\Upsilon|_F = \Psi_F^j$ for some $j \in [k]_0$.

Thus, if $\texttt{aHP}[\mu, l_1, l_2] = 1$, we get the following.

$$\texttt{aHP}[\mu, l_1, l_2] = \bigvee_{\substack{(\mu^i)_{i\in[z]_0}\\(l_1^i)_{i\in[z]_0}\\(l_2^i)_{i\in[z]_0}}} \bigwedge_{i\in[z]_0} \mathcal{H}[i, \mu^i, l_1^i, l_2^i] = \texttt{computeHP}[\mu, l_1, l_2],$$

where $\sum_{i\in[z]_0} \mu^i = \mu$, $\sum_{i\in[z]_0} l_1^i \le l_1$, $\sum_{i\in[z]_0} l_2^i \le l_2$, and each μ^i, l_1^i, l_2^i is a nonnegative integer. ∎

LEMMA 6.23. *If there exists a tuple $\mathbf{t} \in \mathcal{T}$, such that, for this tuple $\mathbf{t}$, $\texttt{computeHP}[\mu, l_1, l_2] = 1$, then $\texttt{aHP}[\mu, l_1, l_2] = 1$.*

Proof. From the definition of $\texttt{computeHP}$, we have the following.

$$\texttt{computeHP}[\mu, l_1, l_2] = \bigvee_{\substack{(\mu^i)_{i\in[z]_0}\\(l_1^i)_{i\in[z]_0}\\(l_2^i)_{i\in[z]_0}}} \bigwedge_{i\in[z]_0} \mathcal{H}[i, \mu^i, l_1^i, l_2^i] = 1,$$

where $\sum_{i\in[z]_0} \mu^i = \mu$, $\sum_{i\in[z]_0} l_1^i \le l_1$, $\sum_{i\in[z]_0} l_2^i \le l_2$, and each μ^i, l_1^i, l_2^i is a nonnegative integer.

Since $\mathcal{H}[i, \mu', l_1', l_2'] = \bigvee_{j\in[k]_0} h_i^j[\mu', l_1', l_2']$, we get the following for some $j_0, \ldots, j_z \in [k]_0$.

$$\bigvee_{\substack{(\mu^i)_{i\in[z]_0}\\(l_1^i)_{i\in[z]_0}.\\(l_2^i)_{i\in[z]_0}}} \bigwedge_{i\in[z]_0} h_i^{j_i}[\mu^i, l_1^i, l_2^i] = 1.$$

From the definition of $h_i^{j_i}$, we get the following.

$$\bigvee_{\substack{(\mu^{F_{i,r}})|_{i\in[z]_0, r\in[z_i]}\\(l_1^{F_{i,r}})|_{i\in[z]_0, r\in[z_i]}\\(l_2^{F_{i,r}})|_{i\in[z]_0, r\in[z_i]}}} \bigwedge_{i\in[z]_0} \bigwedge_{r\in[z_i]} f_{F_{i,r}}(\Psi_{F_{i,r}}^{j_i}, \mu^{F_{i,r}}, l_1^{F_{i,r}}, l_2^{F_{i,r}}) = 1,$$

where $\sum_{i\in[z]_0,r\in[z_i]}\mu^{F_{i,r}}=\mu$, $\sum_{i\in[z]_0,r\in[z_i]} l_1^{F_{i,r}}\le l_1$, $\sum_{i\in[z]_0,r\in[z_i]} l_2^{F_{i,r}}\le l_2$, and each of μ^F, l_1^F, and l_2^F is a nonnegative integer.

Since Ψ for hyperedges gives a consistent coloring for $V(H)$, let this coloring be $\Upsilon : V(H)\to[k]_0$. This coloring Υ then witnesses the following.

$$\bigvee_{\substack{(\mu^{F_{i,r}})|_{i\in[z]_0,r\in[z_i]}\\(l_1^{F_{i,r}})|_{i\in[z]_0,r\in[z_i]}\\(l_2^{F_{i,r}})|_{i\in[z]_0,r\in[z_i]}}}\ \bigwedge_{i\in[z]_0}\ \bigwedge_{r\in[z_i]} f_{F_{i,r}}(\Upsilon|_{F_{i,r}},\mu^{F_{i,r}},l_1^{F_{i,r}},l_2^{F_{i,r}})=1,$$

where $\sum_{i\in[z]_0,r\in[z_i]}\mu^{F_{i,r}}=\mu$, $\sum_{i\in[z]_0,r\in[z_i]} l_1^{F_{i,r}}\le l_1$, $\sum_{i\in[z]_0,r\in[z_i]} l_2^{F_{i,r}}\le l_2$, and each of μ^F, l_1^F, and l_2^F is a nonnegative integer.

Since $E(H)=\uplus_{i\in[z]_0}E_{S_i}$, we get the following.

$$\bigvee_{\substack{(\mu^F)|_{F\in E(H)}\\(l_1^F)|_{F\in E(H)}\\(l_2^F)|_{F\in E(H)}}}\ \bigwedge_{F\in E(H)} f_F(\Upsilon|_F,\mu^F,l_1^F,l_2^F)=1,$$

where $\sum_{F\in E(H)}\mu^F=\mu$, $\sum_{F\in E(H)} l_1^F\le l_1$, $\sum_{F\in E(H)} l_2^F\le l_2$, and for all $F\in E(H)$, μ^F, l_1^F, and l_2^F are nonnegative integers.

Thus, we conclude that $\texttt{aHP}[\mu,l_1,l_2]=1$. ◻

From Lemmas 6.22 and 6.23, we conclude that, to compute $\texttt{aHP}[\mu,l_1,l_2]$, it is enough to compute $\texttt{computeHP}[\mu,l_1,l_2]$ for each tuple $\mathbf{t}\in\mathcal{T}$. In the upcoming lemmas we analyze the time taken to compute $\texttt{computeHP}[\mu,l_1,l_2]$ for any tuple $\mathbf{t}\in\mathcal{T}$.

LEMMA 6.24. *For any $i\in[z]_0, j\in[k]_0, \mu'\le\mu, l_1'\le l_1\le k_1, l_2'\le l_2\le k_2$, $h_i^j[\mu',l_1',l_2']$ can be computed in time $\mathcal{O}(z_i\cdot(\mu\cdot k_1\cdot k_2)^2)$.*

Proof. Recall

$$h_i^j[\mu',l_1',l_2']=\bigvee_{\substack{(\mu^r)_{r\in[z_i]}\\(l_1^r)_{r\in[z_i]}\\(l_2^r)_{r\in[z_i]}}}\ \bigwedge_{r\in[z_i]} f_{F_{i,r}}(\Psi^j_{F_{i,r}},\mu^r,{l_1}^r,{l_2}^r),$$

where $\sum_{r\in[z_i]}\mu^r=\mu'$, $\sum_{r\in[z_i]}l_1^r\le l_1'$, $\sum_{r\in[z_i]}{l_2}^r\le l_2'$, and each μ^r, l_1^r, l_2^r is a nonnegative integer.

For any $c\in[z_i],\mu',l_1'$, and l_2', let

$$h_i^j[c,\mu',l_1',l_2']=\bigvee_{\substack{(\mu^r)_{r\in[c]}\\(l_1^r)_{r\in[c]}\\(l_2^r)_{r\in[c]}}}\ \bigwedge_{r\in[c]} f_{F_{i,r}}(\Psi^j_{F_{i,r}},\mu^r,{l_1}^r,{l_2}^r),$$

where $\sum_{r\in[c]}\mu^r=\mu'$, $\sum_{r\in[c]}l_1^r\le l_1'$, $\sum_{r\in[c]}{l_2}^r\le l_2'$, and each μ^r, l_1^r, l_2^r is a nonnegative integer.

Then $h_i^j[z_i,\mu',l_1',l_2']$ can be computed using the following recurrences in time $\mathcal{O}(z_i\cdot(\mu\cdot k_1\cdot k_2)^2)$

$$h_i^j[1,\mu',l_1',l_2']=f_{F_{i,1}}(\Psi^j_{F_{i,1}},\mu',l_1',l_2').$$

For all $c \in \{2 \ldots, z_i\}$,

$$h_i^j[c, \mu', l_1', l_2'] = \bigvee_{\substack{\mu'=\mu^1+\mu^2 \\ l_1' \geq l_1^1 + l_1^2 \\ l_2' \geq l_2^1 + l_2^2}} h_i^j[c-1, \mu^1, l_1^1, l_2^1] \wedge f_{F_{i,c}}(\Psi^j_{F_{i,c}}, \mu^2, l_1^2, l_2^2).$$

Observe that $h_i^j[\mu', l_1', l_2'] = h_I^j[z_i, \mu', l_1', l_2']$. This concludes the proof. ∎

Since $\mathcal{H}[i, \mu', l_1', l_2'] = \bigvee_{j \in [k]_0} h_i^j[\mu', l_1', l_2']$, from Lemma 6.24, for any i, μ', l_1', l_2', $\mathcal{H}[i, \mu', l_1', l_2']$ can be computed in time $\mathcal{O}(z_i \cdot (\mu)^2 \cdot (k_1 + k_2)^3)$.

LEMMA 6.25. *For any tuple $\boldsymbol{t} \in \mathcal{T}$, for any $\mu' \leq \mu, l_1' \leq l_1 \leq k_1$, $l_2' \leq l_2 \leq k_2$, **computeHP**$[\mu', l_1', l_2']$ can be computed in time $z \cdot z_i \cdot (\mu \cdot (k_1 + k_2)^{\mathcal{O}(1)})$.*

Proof. Recall

$$\texttt{computeHP}[\mu, l_1, l_2] = \bigvee_{\substack{(\mu^i)_{i \in [z]_0} \\ (l_1^i)_{i \in [z]_0} \\ (l_2^i)_{i \in [z]_0}}} \bigwedge_{i \in [z]_0} \mathcal{H}[i, \mu^i, l_1^i, l_2^i],$$

where $\sum_{i \in [z]_0} \mu^i = \mu$, $\sum_{i \in [z]_0} l_1^i \leq l_1$, $\sum_{i \in [z]_0} l_2^i \leq l_2$, and each μ^i, l_1^i, l_2^i is a nonnegative integer.

For any $c \in [z]_0, \mu', l_1'$, and l_2', let

$$\texttt{computeHP}[c, \mu, l_1, l_2] = \bigvee_{\substack{(\mu^i)_{i \in [c]_0} \\ (l_1^i)_{i \in [c]_0} \\ (l_2^i)_{i \in [c]_0}}} \bigwedge_{i \in [c]_0} \mathcal{H}[i, \mu^i, l_1^i, l_2^i],$$

where $\sum_{i \in [c]} \mu^i = \mu$, $\sum_{i \in [c]} l_1^i \leq l_1$, $\sum_{i \in [c]} l_2^i \leq l_2$, and each μ^i, l_1^i, l_2^i is a nonnegative integer.

Then $\texttt{computeHP}[z, \mu, l_1, l_2]$ can be computed using the following recurrences.

$$\texttt{computeHP}[0, \mu, l_1, l_2] = \mathcal{H}[0, \mu, l_1, l_2].$$

For all $c \in [z]$,

$$\texttt{computeHP}[c, \mu, l_1, l_2] = \bigvee_{\substack{\mu'=\mu^1+\mu^2 \\ l_1' \geq l_1^1 + l_1^2 \\ l_2' \geq l_2^1 + l_2^2}} \texttt{computeHP}[c-1, \mu^1, l_1^1, l_2^1] \wedge \mathcal{H}[c, \mu^2, l_1^2, l_2^2].$$

Observe that $\texttt{computeHP}[\mu, l_1, l_2] = \texttt{computeHP}[z, \mu, l_1, l_2]$. From Lemma 6.24, for any i, μ, l_1, l_2, $\mathcal{H}[i, \mu, l_1, l_2]$ can be solved in time $\mathcal{O}(z_i \cdot (\mu)^2) \cdot (k_1 + k_2)^3$. Thus, $\texttt{computeHP}[\mu, l_1, l_2]$ can be solved in time $z \cdot z_i \cdot (\mu \cdot (k_1 + k_2)^{\mathcal{O}(1)})$. This concludes the proof. ∎

From Lemma 6.1, the number of tuples in $\mathcal{T}$ is bounded and for each tuple, the time taken to compute $\texttt{computeHP}[\mu, l_1, l_2]$ is given by Lemma 6.25. Thus, Lemmas 6.1 and 6.25 together give the desired running time bound of Theorem 5.5.

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