



Well-Posedness for a Whitham–Boussinesq System with Surface Tension

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Abstract

We regard the Cauchy problem for a particular Whitham–Boussinesq system modelling surface waves of an inviscid incompressible fluid layer. The system can be seen as a weak nonlocal dispersive perturbation of the shallow water system. The proof of well-posedness relies on energy estimates. However, due to the symmetry lack of the nonlinear part, in order to close the a priori estimates one has to modify the traditional energy norm in use. Hamiltonian conservation provides with global well-posedness at least for small initial data in the one dimensional settings.

Keywords Water waves · Nonlinear dispersive equations · Well-posedness

Mathematics Subject Classification (2010) 35Q53 · 35Q55 · 35A01

1 Introduction

Consideration is given to the following one-dimensional Whitham-type system

$$\begin{cases} \partial_t \eta = -\partial_x v - i \tanh D(\eta v), \\ \partial_t v = -i \tanh D(1 + \varkappa D^2)\eta - i \tanh Dv^2/2, \end{cases} \quad (1.1)$$

where $D = -i\partial_x$ and $\tanh D$ are Fourier multiplier operators in the space of tempered distributions $S'(\mathbb{R})$. The positive parameter \varkappa stands for the surface tension here. The space variable is $x \in \mathbb{R}$ and the time variable is $t \in \mathbb{R}$. The unknowns η , v are real valued functions of these variables. We pick the initial values $\eta(0)$, $v(0)$ corresponding to the time moment $t = 0$ in Sobolev spaces as follows

$$\eta(0) = \eta_0 \in H^{s+1/2}(\mathbb{R}), \quad v(0) = v_0 \in H^s(\mathbb{R}), \quad (1.2)$$

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where $s \geq 1/2$. System (1.1) has the Hamiltonian structure

$$\partial_t(\eta, v)^T = \mathcal{J}\nabla\mathcal{H}(\eta, v)$$

with the skew-adjoint matrix

$$\mathcal{J} = \begin{pmatrix} 0 & -i \tanh D \\ -i \tanh D & 0 \end{pmatrix}$$

and the energy functional

$$\mathcal{H}(\eta, v) = \frac{1}{2} \int \left(\eta^2 + \varkappa(\partial_x \eta)^2 + v \frac{D}{\tanh D} v + \eta v^2 \right) dx \quad (1.3)$$

well defined on $H^1 \times H^{1/2}$. The latter conserves on solutions together with momentum $\mathcal{I}(\eta, v)$ that has the same view as in the pure gravity case

$$\mathcal{I}(\eta, v) = \int \eta \frac{D}{\tanh D} v dx.$$

In case of the trivial surface tension $\varkappa = 0$, System (1.1) was proposed in [6] as an approximate model for the study of water waves to provide a two-directional alternative to the well-known Whitham equation [23]. The latter was proved to be consistent with the KdV equation [18] in the long wave regime [19]. We also refer to [10] for another version of the fully-dispersive Boussinesq type. Importance of such models is supported by experiments [4]. The unknown η denotes the deflection of the free surface from its equilibrium position, corresponding to the vertical level $z = 0$. The bottom is assumed to be flat and located at the level $z = -1$. The variable v is associated with the free surface velocity as explained in [6].

The initial value problem for Model (1.1) was studied in [5, 9] in the case of vanishing surface tension $\varkappa = 0$. In the same framework existence of solitary waves was proved in [8]. A natural extension of the existing results is to consider the case of non-trivial capillarity $\varkappa > 0$. Note that the term $1 + \varkappa D^2$ could be applied to $-v_x$ in the first equation instead, as it is done in [12], for example, to regularise the system regarded in [21]. However, the case regarded here is physically more relevant [7]. Indeed, repeating the Hamiltonian perturbation analysis from [6] to the full Hamiltonian with the surface tension, that can be found in [7], one naturally arrives to (1.1), (1.3).

It turns out that surface tension changes the nature of the equations. Indeed, the multiplication operator $(\eta, v) \mapsto v\eta$ is not bounded in the natural Sobolev-based energy space arising from the linear equations, that is $H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})$. And moreover, the dispersive properties of the corresponding linear semigroup are insufficient to counterbalance the loss of $1/2$ derivatives. As a result the proof of well-posedness demands a technique different from the one used in [9].

As to additional initial conditions, apart from inclusions given in (1.2), one has to impose a restriction essentially similar to the one used in [9], namely, smallness of the $H^1 \times H^{1/2}$ -norm of (η_0, v_0) . This is important for the global-in-time existence. The meaning of this condition is that the total energy $\mathcal{H}(\eta_0, v_0)$ should be positive and not too big. We point out that this condition cannot be significantly weakened even for

the proof of the local result, which is also different from the non-capillarity situation. More precisely, for the local regular (s is large enough in (1.2)) well-posedness result it is enough to assume non-cavitation instead.

Definition 1 Let $d = 1, 2$. We say that elevation $\eta \in C([0, T]; L^\infty(\mathbb{R}^d))$ satisfies the non-cavitation condition if there exist $h, H > 0$ such that $H \geq \eta \geq h - 1$ on $\mathbb{R}^d \times [0, T]$. Analogously, one defines non-cavitation at a particular time moment.

The non-cavitation condition is a physical condition meaning that the elevation of the wave should not touch the bottom of the fluid for System (1.1) to be a relevant model. For convenience we have also included boundedness from above in this definition. We exploit the definition for providing with more general local existence formulation at high regularity level. However, in the low regularity case this condition cannot be controlled without imposing a stronger assumption, as we shall see below. We turn now to the formulation of the main results.

Theorem 1 Let $s > 3/2$. Suppose that the initial data (1.2) satisfies the noncavitation condition. Then there exist $T > 0$ and a unique solution

$$(\eta, v) \in C([0, T]; H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})) \cap C^1((0, T); H^{s-1}(\mathbb{R}) \times H^{s-3/2}(\mathbb{R}))$$

of System (1.1) with the initial data (η_0, v_0) . The time moment T depends on s, ν and the norm $\|\eta_0, v_0\|_{H^{s+1/2} \times H^s}$. With respect to the capillarity and the initial data norm, the time of existence T is a non-increasing function. Moreover, the solution depends continuously on the initial data with respect to $C(H^{s+1/2} \times H^s)$ -norm.

It is worth to emphasize here that the time of existence does not shrink as the surface tension parameter goes to zero. Making a bit stronger assumption on the initial data (1.2), one obtains a stronger result.

Theorem 2 Let $s > 1/2$. For any $\eta_0 \in H^{s+1/2}(\mathbb{R})$ and $v_0 \in H^s(\mathbb{R})$ having sufficiently small $H^1 \times H^{1/2}$ -norm there exists a unique global solution

$$(\eta, v) \in C([0, \infty); H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})) \cap C^1((0, \infty); H^{s-1}(\mathbb{R}) \times H^{s-3/2}(\mathbb{R}))$$

of System (1.1) with the initial data (η_0, v_0) . Moreover, the solution depends continuously on the initial data with respect to $C(H^{s+1/2} \times H^s)$ -norm on any finite time interval $[0, T]$.

As we shall see below, the smallness of $H^1 \times H^{1/2}$ -norm plays an essential role in proving the following two statements. The Cauchy problem (1.1) and (1.2) is locally well-posed for $1/2 < s \leq 3/2$. The solution can be extended to the global one for any $s > 1/2$. Whereas for the local result in the case $s > 3/2$, it is enough to impose a weaker assumption, namely, the noncavitation of η_0 .

The proof is essentially based on the energy method, that is natural to apply to quasilinear equations. The scaling $H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})$ is needed to rule out the linear terms, after the differentiation the corresponding energy norm. The main difficulty lies in the lack of symmetry of the nonlinearity. Indeed, a direct time differentiation of the norm $\|\eta, v\|_{H^{s+1/2} \times H^s}$ leads to the term $\int (J^{s-1/2} \partial_x \eta) \eta J^{s+1/2} v$, where J^σ stands for the Bessel potential of order $-\sigma$ (see the proof of Lemma 6 below). Note that this term cannot be handled by integration by parts or commutator estimates, and so cannot be estimated via the energy norm. To overcome this difficulty we modify the energy norm adding the cubic term $\int \eta (J^{s-1/2} v)^2$. The linear contribution of the derivative of this term will cancel out the mentioned inconvenient term. Meanwhile, the contribution coming from the nonlinear terms can easily be controlled. As we point out below a hint on the choice of the modifier comes from Hamiltonian (1.3). Note that after adding the cubic term the energy loses coercivity, and so one has to impose an additional condition. Either the noncavitation for big s or the smallness for small s of the initial data, both propagating through the flow of System (1.1), is enough to ensure that the modified energy is coercive.

Additionally, consideration is also given to a system posed on \mathbb{R}^{2+1} of the form

$$\begin{cases} \partial_t \eta + \nabla \cdot \mathbf{v} = -K^2 \nabla \cdot (\eta \mathbf{v}), \\ \partial_t \mathbf{v} + K^2 \nabla (1 + \varkappa |D|^2) \eta = -K^2 \nabla (|\mathbf{v}|^2/2), \end{cases} \tag{1.4}$$

that is a direct two dimensional extension of Model (1.1). Here $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ is a curl free vector field, that is $\nabla \times \mathbf{v} = 0$, and

$$K = \sqrt{\tanh |D|/|D|}$$

with $D = -i\nabla$. So the corresponding symbol $K(\xi) = \sqrt{\tanh(|\xi|)/|\xi|}$. We complement (1.4) with the initial data

$$\eta(0) = \eta_0 \in H^{s+1/2}(\mathbb{R}^2), \quad \mathbf{v}(0) = \mathbf{v}_0 \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2). \tag{1.5}$$

As above the variables η and \mathbf{v} stand for the surface elevation and the surface fluid velocity, respectively. The system enjoys the Hamiltonian structure

$$\partial_t (\eta, \mathbf{v})^T = \mathcal{J} \nabla \mathcal{H}(\eta, \mathbf{v})$$

with the skew-adjoint matrix

$$\mathcal{J} = \begin{pmatrix} 0 & -K^2 \partial_{x_1} & -K^2 \partial_{x_2} \\ -K^2 \partial_{x_1} & 0 & 0 \\ -K^2 \partial_{x_2} & 0 & 0 \end{pmatrix},$$

which in particular, guarantees conservation of the energy functional

$$\mathcal{H}(\eta, \mathbf{v}) = \frac{1}{2} \int \left(\eta^2 + \varkappa |\nabla \eta|^2 + |K^{-1} \mathbf{v}|^2 + \eta |\mathbf{v}|^2 \right) dx. \tag{1.6}$$

The noncavitation definition in the two dimensional problem has exactly the same view as in Definition 1 with the real line \mathbb{R} substituted by the plane \mathbb{R}^2 .

Theorem 3 *Let $s > 1$. Suppose that the initial data (1.5) has curl free velocity $\nabla \times \mathbf{v}_0 = 0$ and either has small enough $H^1 \times H^{1/2} \times H^{1/2}$ -norm if $s \leq 2$ or satisfies the noncavitation condition if $s > 2$. Then there exist $T > 0$ and a unique solution*

$$(\eta, \mathbf{v}) \in C\left([0, T]; H^{s+1/2}(\mathbb{R}^2) \times \left(H^s(\mathbb{R}^2)\right)^2\right) \cap C^1\left((0, T); H^{s-1}(\mathbb{R}^2) \times \left(H^{s-3/2}(\mathbb{R}^2)\right)^2\right)$$

of System (1.4) associated with this initial data. The time of existence T is a non-increasing function of the surface tension \varkappa and the initial data norm $\|\eta_0, \mathbf{v}_0\|_{H^{s+1/2} \times H^s \times H^s}$. Moreover, the solution depends continuously on the initial data with respect to $C(H^{s+1/2} \times H^s \times H^s)$ -norm.

Note that the theorem has the local character, in the opposite of the one dimensional case.

Remark 1 The same results hold in the periodic case as well. The proof is similar up to some small changes in the commutator estimates [15].

In the next section some important inequalities are recalled. In Section 3 we introduce the modified energy and obtain the corresponding energy estimate for System (1.1). In Section 4 we obtain the energy estimate for the difference of two solutions of System (1.1). Note that Sections 3 and 4 provide with the motivation for studying the parabolic regularisation later in Section 5, where the corresponding energy estimate is deduced for the regularised system. In Section 6 a priori estimates are obtained. Finally, in Section 7 we comment on the last steps in the proof of Theorem 2, omitting only the thorough discussion of the initial data regularisation. In Section 8 we discuss some peculiarities of the two dimensional problem. In the last section we study System (1.1) with $\varkappa \ll 1$.

2 Preliminary Estimates

We start this section by recalling all the necessary standard notations. For any positive numbers a and b we write $a \lesssim b$ if there exists a constant C independent of a, b such that $a \leq Cb$. The Fourier transform is defined by the formula

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) = \int f(x)e^{-i\xi x} dx$$

on Schwartz functions. By the Fourier multiplier operator $\varphi(D)$ with symbol φ we mean the line $\mathcal{F}(\varphi(D)f) = \varphi(\xi)\widehat{f}(\xi)$. In particular, $D = -i\partial_x$ is the Fourier multiplier associated with the symbol $\varphi(\xi) = \xi$. For any $\alpha \in \mathbb{R}$ the Riesz potential of order $-\alpha$ is the Fourier operator $|D|^\alpha$ and the Bessel potential of order $-\alpha$ is the Fourier operator $J^\alpha = \langle D \rangle^\alpha$, where we exploit the notation $\langle \xi \rangle = \sqrt{1 + \xi^2}$. The L^2 -based Sobolev space $H^\alpha(\mathbb{R})$ is defined by the norm $\|f\|_{H^\alpha} = \|J^\alpha f\|_{L^2}$, whereas

the homogeneous Sobolev space $\dot{H}^\alpha(\mathbb{R})$ is defined by $\|f\|_{\dot{H}^\alpha} = \| |D|^\alpha f \|_{L^2}$. We also exploit the notation $H^\infty(\mathbb{R}) = \bigcap_{\alpha \in \mathbb{R}} H^\alpha(\mathbb{R})$.

Introduce the operator

$$K_\varkappa = \sqrt{(1 + \varkappa|D|^2) \frac{\tanh |D|}{|D|}}, \tag{2.1}$$

where \varkappa is the surface tension. Note that $\varkappa > 0$ is a fixed constant. We implement the notation $K = K_0 = \sqrt{\tanh D/D}$ used in [9]. Its inverse K^{-1} and K_\varkappa both have the domain $H^{1/2}(\mathbb{R})$ and are equivalent to the Bessel potential $J^{1/2}$. Below we will need to compare J , $|D|$ and K^{-2} and so we prove the following simple estimates.

Lemma 1 *For any $f \in L^2(\mathbb{R})$ it holds that*

$$\left\| (J - K^{-2}) Df \right\|_{L^2} \leq \| (J - |D|) Df \|_{L^2} \leq \frac{1}{2} \| f \|_{L^2}.$$

Proof By the Plancherel identity it is enough to check the following inequalities

$$0 \leq \langle \xi \rangle - \frac{\xi}{\tanh \xi} \leq \langle \xi \rangle - |\xi| \leq \frac{1}{2|\xi|},$$

where the middle one is trivial. The rightmost inequality follows from

$$\langle \xi \rangle - |\xi| = \frac{1}{\langle \xi \rangle + |\xi|} \leq \frac{1}{2|\xi|}.$$

The leftmost one follows from the tanh-definition via exponents and the obvious

$$e^{2\xi} + e^{-2\xi} \geq 2 + 4\xi^2. \quad \square$$

Throughout the text we make an extensive use of the following bilinear estimates. Firstly, we state the Kato-Ponce commutator estimate [14].

Lemma 2 (Kato-Ponce commutator estimate) *Let $s \geq 1$, $p, p_2, p_3 \in (1, \infty)$ and $p_1, p_4 \in (1, \infty]$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. Then*

$$\| [J^s, f]g \|_{L^p} \lesssim \| \partial_x f \|_{L^{p_1}} \| J^{s-1} g \|_{L^{p_2}} + \| J^s f \|_{L^{p_3}} \| g \|_{L^{p_4}} \tag{2.2}$$

for any f, g defined on \mathbb{R} .

By the commutator $[A, B]$ between operators A and B we mean the operator $[A, B]f = ABf - BAf$. Secondly, we state the fractional Leibniz rule proved in the appendix of [16].

Lemma 3 *Let $\sigma = \sigma_1 + \sigma_2 \in (0, 1)$ with $\sigma_i \in (0, \sigma)$ and $p, p_1, p_2 \in (1, \infty)$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then*

$$\| |D|^\sigma (fg) - f |D|^\sigma g - g |D|^\sigma f \|_{L^p} \lesssim \| |D|^{\sigma_1} f \|_{L^{p_1}} \| |D|^{\sigma_2} g \|_{L^{p_2}} \tag{2.3}$$

for any f, g defined on \mathbb{R} . Moreover, the case $\sigma_2 = 0, p_2 = \infty$ is also allowed.

We also state an estimate, firstly appeared in [17] in a weaker form, and later sharpened in [22].

Lemma 4 *Suppose $a, b, c \in \mathbb{R}$. Then for any $f \in H^a(\mathbb{R})$, $g \in H^b(\mathbb{R})$ and $h \in H^c(\mathbb{R})$ the following inequality holds*

$$\|fgh\|_{L^1} \lesssim \|f\|_{H^a} \|g\|_{H^b} \|h\|_{H^c} \tag{2.4}$$

provided that

$$a + b + c > \frac{1}{2},$$

$$a + b \geq 0, \quad a + c \geq 0, \quad b + c \geq 0.$$

Proving a global-in-time a priori estimate we will use the following limiting case of the Sobolev embedding theorem, that in the one dimensional case $d = 1$ reads as follows.

Lemma 5 (Brezis-Gallouet inequality) *Suppose $f \in H^s(\mathbb{R})$ with $s > 1/2$. Then*

$$\|f\|_{L^\infty} \leq C_s \left(1 + \|f\|_{H^{1/2}} \sqrt{\log(1 + \|f\|_{H^s})} \right). \tag{2.5}$$

Inequality (2.5) was firstly put forward and proved in $H^2(\mathbb{R}^2)$ in the work by Brezis, Gallouet [2]. It was extended to more general Sobolev spaces and any dimension in [3], but in a slightly different form. For the sake of completeness, we provide here with the proof based on the idea introduced in [2].

Proof Let $f \in H^s(\mathbb{R})$ with $s > 1/2$. Then

$$\|f\|_{L^\infty} \leq \frac{1}{2\pi} \|\widehat{f}\|_{L^1} = \frac{1}{2\pi} \int_{|\xi| \leq R} |\widehat{f}(\xi)| d\xi + \frac{1}{2\pi} \int_{|\xi| > R} |\widehat{f}(\xi)| d\xi = I_1(R) + I_2(R),$$

where $R > 0$ is an arbitrary positive number. In the first integral I_1 we multiply and divide \widehat{f} by $(1 + \xi^2)^{1/4}$. Afterwards, we apply the Hölder inequality to get

$$I_1(R) \leq \frac{1}{2\pi} \left(\int_{-R}^R |\widehat{f}(\xi)|^2 (1 + \xi^2)^{1/2} d\xi \right)^{1/2} \left(\int_{-R}^R \frac{d\xi}{\sqrt{1 + \xi^2}} \right)^{1/2}$$

$$\leq \frac{1}{\pi} \|f\|_{H^{1/2}} \sqrt{\log(1 + R)},$$

and similarly,

$$I_2(R) \leq \frac{1}{2\pi} \|f\|_{H^s} \left(\int_{|\xi| > R} \frac{d\xi}{(1 + \xi^2)^s} \right)^{1/2}.$$

Now it is left to choose R depending on f, s . If $\|f\|_{H^s} \leq 1$ then taking $R = \|f\|_{H^s}$ we immediately obtain the desired inequality (2.5). In the case $\|f\|_{H^s} > 1$, we estimate the second integral as follows

$$I_2(R) \leq \frac{\|f\|_{H^s}}{2\pi \sqrt{s - 1/2} R^{s-1/2}},$$

and so if $s \geq 3/2$ one takes $R = \|f\|_{H^s}$ again to bound $I_2(R) \leq (2\pi\sqrt{s - 1/2})^{-1}$ and to come to (2.5). In the last case $\|f\|_{H^s} > 1$ and $\alpha_s = 1/(s - 1/2) > 1$, we can take $R = \|f\|_{H^s}^{\alpha_s}$ to bound $I_2(R)$ by the same constant. Note that $I_1(R)$ in this case is bounded as

$$I_1(R) \leq \frac{1}{\pi} \|f\|_{H^{1/2}} \sqrt{\log(1 + \|f\|_{H^s}^{\alpha_s})} \leq \frac{1}{\pi} \|f\|_{H^{1/2}} \sqrt{\alpha_s \log(1 + \|f\|_{H^s})},$$

and so we again obtain Inequality (2.5). □

3 Modified Energy

As we shall see in the proof of the next lemma, a direct use of $H^{s+1/2} \times H^s$ -norm as the energy does not allow us to close the estimates, and so we modify it as follows. Firstly, for each $\varkappa > 0$ and $s \geq 1/2$ we introduce the norm

$$\|\eta, v\|_{H_\varkappa^{s+1/2} \times H^s}^2 = \varkappa \|\partial_x \eta\|_{H^{s-1/2}}^2 + \|\eta\|_{H^{s-1/2}}^2 + \|K^{-1}v\|_{H^{s-1/2}}^2, \tag{3.1}$$

which is obviously equivalent to the standard norm in $H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})$. Such choice will be convenient later for analysis of dependence of solution on the capillarity \varkappa . Secondly, we define the modified energy

$$E^s(\eta, v) = \frac{1}{2} \|\eta, v\|_{H_\varkappa^{s+1/2} \times H^s}^2 + \frac{1}{2} \int \eta \left(J^{s-1/2} v \right)^2, \tag{3.2}$$

where the pair (η, v) represents a possible solution of System (1.1). Note that in the limit case $s = 1/2$ this quantity coincides with the Hamiltonian given in (1.3), $E^{1/2}(\eta, v) = \mathcal{H}(\eta, v)$. This gives us a small hint for the choice of the right cubic modifier that is basically a guess.

Lemma 6 *Suppose $s \geq 1/2$. Then there exists $C_s > 0$ such that for any $\varkappa > 0$ and any functions $\eta, v \in C^1((0, T); H^\infty(\mathbb{R}))$ solving System (1.1) it holds that*

$$\frac{d}{dt} E^s(\eta, v) \leq C_s (1 + \varkappa) \left(\|\eta, v\|_{H_\varkappa^{s+1/2} \times H^s}^2 + \|\eta, v\|_{H_\varkappa^{s+1/2} \times H^s}^4 \right).$$

Proof We have already noticed that $E^{1/2}(\eta, v)$ is a conserved quantity, which proves the statement for the limit case $s = 1/2$.

Assuming $s > 1/2$ we calculate the derivatives

$$\varkappa \frac{d}{dt} \|\partial_x \eta\|_{H^{s-1/2}}^2 = -\varkappa \int \left(J^{s-1/2} \partial_x \eta \right) J^{s-1/2} \partial_x^2 v - i \varkappa \int \left(J^{s-1/2} \partial_x \eta \right) J^{s-1/2} \partial_x \tanh D(\eta v),$$

$$\frac{1}{2} \frac{d}{dt} \|\eta\|_{H^{s-1/2}}^2 = - \int \left(J^{s-1/2} \eta \right) J^{s-1/2} \partial_x v - i \int \left(J^{s-1/2} \eta \right) J^{s-1/2} \tanh D(\eta v)$$

and the derivative of velocity norm

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|K^{-1}v\|_{H^{s-1/2}}^2 &= -i \int (J^{s-1/2}K^{-1}v)J^{s-1/2}K^{-1} \tanh D \left(1 + \varkappa D^2\right) \eta \\ &\quad - \frac{i}{2} \int (J^{s-1/2}K^{-1}v)J^{s-1/2}K^{-1} \tanh Dv^2. \end{aligned}$$

Summing up these derivatives and simplifying the corresponding expression via integration by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\eta, v\|_{H^{\varkappa s+1/2} \times H^s}^2 = I_1 + I_2 + I_3,$$

where

$$I_1 = i\varkappa \int \left(J^{s-1/2}D^2 \tanh D\eta \right) J^{s-1/2}(\eta v),$$

$$I_2 = i \int \left(J^{s-1/2} \tanh D\eta \right) J^{s-1/2}(\eta v),$$

$$I_3 = \frac{i}{2} \int \left(J^{s-1/2}|D|^{1/2} \operatorname{sgn} Dv \right) J^{s-1/2}|D|^{1/2}(v^2).$$

The second integral I_2 can be estimated with the help of Lemma 4, by setting $f = J^{2s-1} \tanh D\eta$, $g = \eta$, $h = v$ and $a = 1/2 - s$, $b = s - 1/2$, $c = s$. Thus one obtains

$$I_2 \lesssim \|\eta\|_{H^{s-1/2}}^2 \|v\|_{H^s}.$$

Applying Hölder’s inequality to the third integral I_3 , we get

$$I_3 \lesssim \|v\|_{H^s} \|v^2\|_{H^s} \lesssim \|v\|_{H^s}^2 \|v\|_{L^\infty}.$$

We would like to point out here that the first integral I_1 cannot be estimated via the energy norm (3.1), using only integration by parts or commutator estimates. Turning our attention to the modifier of energy E^s , we calculate its time derivative as follows

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int \eta \left(J^{s-\frac{1}{2}}v \right)^2 &= -i \int \eta \left(J^{s-\frac{1}{2}}v \right) J^{s-\frac{1}{2}} \tanh D\eta - i\varkappa \int \eta \left(J^{s-\frac{1}{2}}v \right) J^{s-\frac{1}{2}}D^2 \tanh D\eta \\ &\quad - \frac{i}{2} \int \eta \left(J^{s-\frac{1}{2}}v \right) J^{s-\frac{1}{2}} \tanh Dv^2 - \frac{1}{2} \int \partial_x v \left(J^{s-\frac{1}{2}}v \right)^2 - \frac{i}{2} \int \tanh D(\eta v) \left(J^{s-\frac{1}{2}}v \right)^2. \end{aligned} \quad (3.3)$$

Let I_4, \dots, I_8 represent these integrals, respectively. The first summand, that we notate by I_4 , is estimated easily as

$$I_4 = -i \int \eta \left(J^{s-\frac{1}{2}}v \right) J^{s-\frac{1}{2}} \tanh D\eta \lesssim \|\eta\|_{H^{s-1/2}}^2 \|v\|_{H^s}$$

by applying Inequality (2.4).

The third integral in (3.3), notated by I_6 , is estimated in a similar way

$$I_6 \lesssim \|\eta\|_{L^2} \|v\|_{H^s} \|v^2\|_{H^s} \lesssim \|\eta\|_{L^2} \|v\|_{L^\infty} \|v\|_{H^s}^2 \lesssim \|\eta\|_{H^{s-1/2}} \|K^{-1}v\|_{H^{s-1/2}}^3.$$

The fourth integral in (3.3) equals

$$I_7 = -\frac{i}{2} \int \left(\operatorname{sgn} D|D|^{\frac{1}{2}} v \right) |D|^{\frac{1}{2}} \left(J^{s-\frac{1}{2}} v \right)^2 = -i \int \left(\operatorname{sgn} D|D|^{\frac{1}{2}} v \right) \left(J^{s-\frac{1}{2}} v \right) J^{s-\frac{1}{2}} |D|^{\frac{1}{2}} v - \frac{i}{2} \int \left(\operatorname{sgn} D|D|^{\frac{1}{2}} v \right) \left[|D|^{\frac{1}{2}} \left(J^{s-\frac{1}{2}} v \right)^2 - 2 \left(J^{s-\frac{1}{2}} v \right) J^{s-\frac{1}{2}} |D|^{\frac{1}{2}} v \right]$$

where the first integral can be treated with interpolation in Sobolev spaces and the second integral by the fractional Leibniz rule as follows

$$I_7 \lesssim \|\operatorname{sgn} D|D|^{\frac{1}{2}} v\|_{H^{s-1/2}} \|J^{s-\frac{1}{2}} v\|_{H^{1/2}} \|J^{s-\frac{1}{2}} |D|^{\frac{1}{2}} v\|_{L^2} + \|\operatorname{sgn} D|D|^{\frac{1}{2}} v\|_{L^2} \|J^{s-\frac{1}{2}} |D|^{\frac{1}{2}} v\|_{L^2}^2 \lesssim \|v\|_{H^s}^3.$$

The last integral in (3.3), that we notate by I_8 , is bounded by

$$I_8 \leq \frac{1}{2} \|\eta\|_{L^2} \|v\|_{L^\infty} \|J^{s-1/2} v\|_{L^4}^2 \lesssim \|\eta\|_{H^{s-1/2}} \|K^{-1} v\|_{H^{s-1/2}}^3.$$

It is left to regard the second integral in (3.3), denoted by I_5 , and the integral I_1 appeared after the differentiation of the energy norm (3.1). Firstly, let us note that

$$J^{s+1/2}(\eta v) = \left[J^{s+1/2}, \eta \right] v + \eta J^{s+1/2} v, \\ J \left(\eta J^{s-1/2} v \right) = [J, \eta] J^{s-1/2} v + \eta J^{s+1/2} v,$$

and so summing I_1, I_5 together one can easily obtain

$$I_1 + I_5 = i\kappa \int \left(J^{s-3/2} D^2 \tanh D\eta \right) \left(\left[J^{s+1/2}, \eta \right] v - [J, \eta] J^{s-1/2} v \right).$$

Applying the Kato-Ponce estimate to the first commutator one obtains

$$\left\| \left[J^{s+1/2}, \eta \right] v \right\|_{L^2} \lesssim \|\partial_x \eta\|_{L^{p_1}} \|J^{s-1/2} v\|_{L^{p_2}} + \|J^{s+1/2} \eta\|_{L^2} \|v\|_{L^\infty}.$$

Taking $p_1(s) = \frac{1}{1-s}, p_2(s) = \frac{2}{2s-1}$ for $s \in (\frac{1}{2}, 1)$ and $p_1 = p_2 = 4$ in case $s \geq 1$ one deduces

$$\left\| \left[J^{s+1/2}, \eta \right] v \right\|_{L^2} \lesssim \|\eta\|_{H^{s+1/2}} \begin{cases} \|v\|_{H^{1/2}} + \|v\|_{L^\infty} & \text{for } s \in (1/2, 1) \\ \|v\|_{H^{s-1/4}} & \text{for } s \geq 1 \end{cases}$$

after implementing the Sobolev embedding. Similarly,

$$\left\| [J, \eta] J^{s-1/2} v \right\|_{L^2} \lesssim \|\partial_x \eta\|_{L^{p_1}} \|J^{s-1/2} v\|_{L^{p_2}} + \|J\eta\|_{L^{p_3}} \|J^{s-1/2} v\|_{L^{p_4}}$$

follows from the Kato-Ponce inequality. Now taking $p_1 = p_3 = \frac{1}{1-s}, p_2 = p_4 = \frac{2}{2s-1}$ for $s \in (\frac{1}{2}, 1)$ and $p_1 = p_2 = p_3 = p_4 = 4$ for $s \geq 1$ one deduces

$$\left\| [J, \eta] J^{s-1/2} v \right\|_{L^2} \lesssim \|\eta\|_{H^{s+1/2}} \begin{cases} \|v\|_{H^{1/2}} & \text{for } s \in (1/2, 1) \\ \|v\|_{H^{s-1/4}} & \text{for } s \geq 1 \end{cases}$$

after implementing the Sobolev embedding.

Thus applying Hölder’s inequality to the sum $I_1 + I_5$ one obtains

$$I_1 + I_5 \lesssim \varkappa \|\partial_x \eta\|_{H^{s-1/2}} \|\eta\|_{H^{s+1/2}} \begin{cases} \|v\|_{H^{1/2}} + \|v\|_{L^\infty} & \text{for } s \in (1/2, 1) \\ \|v\|_{H^{s-1/4}} & \text{for } s \geq 1 \end{cases},$$

and so

$$I_1 + I_5 \lesssim \varkappa \|\partial_x \eta\|_{H^{s-1/2}} (\|\partial_x \eta\|_{H^{s-1/2}} + \|\eta\|_{H^{s-1/2}}) \|v\|_{H^s} \lesssim \|\eta, v\|_{H^{s+1/2} \times H^s}^2 + \varkappa \|\eta, v\|_{H^{s+1/2} \times H^s}^4.$$

Finally, summing Derivative (3.3) with the derivative of square of $H^{s+1/2} \times H^s$ -norm according to Definition (3.2) we obtain

$$\frac{d}{dt} E^s(\eta, v) = I_1 + \dots + I_8 \lesssim \|\eta, v\|_{H^{s+1/2} \times H^s}^2 + \varkappa \|\eta, v\|_{H^{s+1/2} \times H^s}^4,$$

which concludes the proof. □

In the following obvious statement the non-cavitation condition plays a crucial role.

Lemma 7 (Coercivity) *Let $s \geq 1/2$ and $(\eta, v) \in C([0, T]; H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R}))$ be a solution of System (1.1) for some $T > 0$. If in addition η satisfies the non-cavitation condition then*

$$E^s(\eta, v) \sim \|\eta, v\|_{H^{s+1/2} \times H^s}^2.$$

Corollary 1 (Energy estimate) *If the conditions of the previous two lemmas are satisfied then it holds true that*

$$\frac{d}{dt} E^s(\eta, v) \lesssim (1 + \varkappa) \left(E^s(\eta, v) + E^s(\eta, v)^2 \right)$$

with the implicit constant independent of $\varkappa > 0$.

As we shall see below, the non-cavitation condition is convenient to work with only in the case of high regularity $s > 3/2$. Then the time interval on which the condition holds true can be easily estimated through the first equation in (1.1). Our goal is to study well-posedness in spaces of low regularity as well. So in case of $s \leq 3/2$ we will have to impose a stronger condition, instead of non-cavitation, namely smallness of the initial data norm, that we can control in time with the help of the Hamiltonian conservation, as the following lemma demonstrates.

Lemma 8 *There exists a constant $H > 0$ independent of the surface tension $\varkappa > 0$ such that for any $\epsilon \in (0, H]$ if a pair $u = (\eta, v) \in C([0, T]; H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R}))$, having initial condition $\|u(0)\|_{H^{s+1/2} \times H^s} \leq \epsilon/2$, solves System (1.1) then $\|u(t)\|_{H^{s+1/2} \times H^s} \leq \epsilon$ for any time $t \in [0, T]$.*

Proof We use a continuity argument. We simply write

$$\|u\|^2 = \frac{1}{2} \|u(t)\|_{H^{s+1/2} \times H^s}^2.$$

Then there exists $C > 0$ independent of $\varkappa > 0$ such that

$$\left| \int \eta v^2 \right| \leq \| \eta \|_{L^2} \| v \|_{L^4}^2 \leq C \| u \|^3,$$

and so

$$\| u \|^2 (1 - C \| u \|) \leq \mathcal{H}(u) \leq \| u \|^2 (1 + C \| u \|),$$

where $u = u(t)$ is a solution of (1.1) defined on some interval. Take $H = (2C)^{-1}$, any $0 < \epsilon \leq H$ and a solution with $u_0 = u(0)$ having $\| u_0 \| \leq \epsilon/2$. By continuity $\| u \| \leq \epsilon$ on some $[0, T_\epsilon]$ and so

$$\| u \| \leq \sqrt{2\mathcal{H}(u)} = \sqrt{2\mathcal{H}(u_0)} \leq \sqrt{\frac{1 + C\epsilon/2}{2}} \epsilon < \epsilon.$$

Hence the function u satisfies that $\| u(t) \|$ does not reach the level ϵ at any time t . \square

As a consequence of the lemma we can control $\| \eta \|_{L^\infty}$ for any $s \geq 1/2$ in time, admitting only small initial data, by the inequality

$$\| \eta \|_{L^\infty} \lesssim \left(1 + \frac{1}{\varkappa} \right) \| \eta, v \|_{H^s_\varkappa \times H^{s+1/2}},$$

which guarantees non-cavitation, in particular.

4 Uniqueness Type Estimate

Suppose that we have two solution pairs η_1, v_1 and η_2, v_2 of System (1.1) on some time interval. Define functions $\theta = \eta_1 - \eta_2, w = v_1 - v_2$. Then θ and w satisfy the following system

$$\begin{cases} \theta_t = -\partial_x w - i \tanh D(\theta v_2 + \eta_1 w), \\ w_t = -i \tanh D(1 + \varkappa D^2)\theta - i \tanh D((v_1 + v_2)w)/2. \end{cases} \tag{4.1}$$

We need an a priori estimate similar to one obtained in the previous section for the difference of solutions. For this purpose we introduce the difference energy

$$E^r(\eta_1, v_1, \eta_2, v_2) = \frac{\varkappa}{2} \| \theta \|_{H^{r+1/2}}^2 + \frac{1}{2} \| w \|_{H^r}^2 + \frac{1}{2} \int \eta_1 \left(J^{r-\frac{1}{2}} w \right)^2. \tag{4.2}$$

Lemma 9 *Let $\eta_1, v_1, \eta_2, v_2 \in C^1((0, T); H^\infty(\mathbb{R}))$ be solutions of System (1.1) for some $T > 0$ and $s > 1/2$. Their difference is denoted by (θ, w) . Let $0 < r \leq s - 1/2$. Then*

$$\frac{d}{dt} E^r(\eta_1, v_1, \eta_2, v_2) \lesssim \left(1 + \| \eta_1 \|_{H^{s+1/2}}^2 + \| v_1 \|_{H^s}^2 + \| v_2 \|_{H^s}^2 \right) \left(\| \theta \|_{H^{r+1/2}}^2 + \| w \|_{H^r}^2 \right),$$

where the implicit constant depends on \varkappa, r, s .

Proof We follow the same arguments as in the proof of Lemma 6. The derivative of squared norm

$$\begin{aligned} & \frac{\varkappa}{2} \frac{d}{dt} \|\theta\|_{H^{r+1/2}}^2 + \frac{1}{2} \frac{d}{dt} \|w\|_{H^r}^2 = -\varkappa \int (J^{r+\frac{1}{2}}\theta) J^{r+\frac{1}{2}} \partial_x w - i\varkappa \int (J^{r+\frac{1}{2}}\theta) J^{r+\frac{1}{2}} \tanh D(\theta v_2) \\ & \quad - i\varkappa \int (J^{r+\frac{1}{2}}\theta) J^{r+\frac{1}{2}} \tanh D(\eta_1 w) - i \int (J^r w) J^r \tanh D\theta \\ & \quad - i\varkappa \int (J^r w) J^r D^2 \tanh D\theta - \frac{i}{2} \int (J^r w) J^r \tanh D(v_1 + v_2) w \\ & = I_1 + \mathcal{O} \left(\|\theta\|_{H^r} \|w\|_{H^r} + \|v_2\|_{H^{r+1/2}} \|\theta\|_{H^{r+1/2}}^2 + \|\eta_1\|_{H^r} \|\theta\|_{H^r} \|w\|_{H^r} + \|v_1 + v_2\|_{H^s} \|w\|_{H^r}^2 \right), \end{aligned}$$

where

$$I_1 = i\varkappa \int \left(J^{r-\frac{1}{2}} D\theta \right) J^{r+\frac{1}{2}} (\eta_1 w).$$

In the case $r \geq 1/2$ we have the commutator estimate

$$\left\| \left[J^{r+\frac{1}{2}}, \eta_1 \right] w \right\|_{L^2} \lesssim \|\partial_x \eta_1\|_{L^4} \left\| J^{r-\frac{1}{2}} w \right\|_{L^4} + \left\| J^{r+\frac{1}{2}} \eta_1 \right\|_{L^4} \|w\|_{L^4} \lesssim \|\eta_1\|_{H^{s+1/2}} \|w\|_{H^r},$$

and so

$$I_1 = i\varkappa \int \left(J^{r-\frac{1}{2}} D\theta \right) \eta_1 J^{r+\frac{1}{2}} w + \mathcal{O} \left(\|\eta_1\|_{H^{s+1/2}} \|\theta\|_{H^{r+1/2}} \|w\|_{H^r} \right). \tag{4.3}$$

For $r \in (0, 1/2)$ we apply the Leibniz rule

$$\left\| |D|^{r+\frac{1}{2}} (\eta_1 w) - w |D|^{r+\frac{1}{2}} \eta_1 - \eta_1 |D|^{r+\frac{1}{2}} w \right\|_{L^2} \lesssim \left\| |D|^{\sigma_1} \eta_1 \right\|_{L^{p_1}} \left\| |D|^{\sigma_2} w \right\|_{L^{p_2}} \lesssim \|\eta_1\|_{H^1} \|w\|_{H^r},$$

where $p_2 > 2$ is such that $\sigma_2 = r - 1/2 + 1/p_2 > 0$. The last estimate is due to Sobolev’s embedding. Operator $J^{r+\frac{1}{2}} - |D|^{r+\frac{1}{2}}$ is bounded in L^2 . Thus

$$\begin{aligned} & I_1 = i\varkappa \int \left(J^{r-\frac{1}{2}} D\theta \right) |D|^{r+\frac{1}{2}} (\eta_1 w) + \mathcal{O} \left(\|\eta_1\|_{H^{s+1/2}} \|\theta\|_{H^{r+1/2}} \|w\|_{H^r} \right) \\ & = i\varkappa \int \left(J^{r-\frac{1}{2}} D\theta \right) w |D|^{r+\frac{1}{2}} \eta_1 + i\varkappa \int \left(J^{r-\frac{1}{2}} D\theta \right) \eta_1 |D|^{r+\frac{1}{2}} w + \mathcal{O} \left(\|\eta_1\|_{H^{s+1/2}} \|\theta\|_{H^{r+1/2}} \|w\|_{H^r} \right), \end{aligned}$$

where the first integral can be estimated by interpolation in Sobolev spaces. In the second integral the fractional derivative $|D|^{r+\frac{1}{2}}$ can be approximated by $J^{r+\frac{1}{2}}$ to come again to (4.3) now for $0 < r < 1/2$.

Differentiation of the energy modifier gives

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \eta_1 \left(J^{r-\frac{1}{2}} w \right)^2 = -i \int \eta_1 \left(J^{r-\frac{1}{2}} w \right) J^{r-\frac{1}{2}} \tanh D\theta - i\varkappa \int \eta_1 \left(J^{r-\frac{1}{2}} w \right) J^{r-\frac{1}{2}} D^2 \tanh D\theta \\ & - \frac{i}{2} \int \eta_1 \left(J^{r-\frac{1}{2}} w \right) J^{r-\frac{1}{2}} \tanh D(v_1 + v_2) w - \frac{1}{2} \int \partial_x v_1 \left(J^{r-\frac{1}{2}} w \right)^2 - \frac{i}{2} \int \tanh D(\eta_1 v_1) \left(J^{r-\frac{1}{2}} w \right)^2 \\ & = I_2 + \mathcal{O} \left(\|\eta_1\|_{H^s} \|\theta\|_{H^{r-1/2}} \|w\|_{H^{r-1/2}} + (1 + \|\eta_1\|_{H^s}) (\|v_1\|_{H^s} + \|v_2\|_{H^s}) \|w\|_{H^r}^2 \right), \end{aligned}$$

where

$$\begin{aligned}
 I_2 &= -i\kappa \int \left(J^{r-\frac{1}{2}} D\theta \right) J(\eta_1 J^{r-\frac{1}{2}} w) = -i\kappa \int \left(J^{r-\frac{1}{2}} D\theta \right) \eta_1 J^{r+\frac{1}{2}} w \\
 &\quad + \|\theta\|_{H^{r+1/2}} \mathcal{O} \left(\|\partial_x \eta_1\|_{L^{p_1}} \|J^{r-\frac{1}{2}} w\|_{L^{p_2}} + \|J\eta_1\|_{L^{p_3}} \|J^{r-\frac{1}{2}} w\|_{L^{p_4}} \right) \\
 &= -i\kappa \int \left(J^{r-\frac{1}{2}} D\theta \right) \eta_1 J^{r+\frac{1}{2}} w + \mathcal{O} \left(\|\eta_1\|_{H^{s+1/2}} \|\theta\|_{H^{r+1/2}} \|w\|_{H^r} \right)
 \end{aligned}$$

following from the Kato–Ponce inequality with $p_1 = p_3 = \frac{1}{1-s}$, $p_2 = p_4 = \frac{2}{2s-1}$ for $s \in \left(\frac{1}{2}, 1\right)$ and $p_1 = p_2 = p_3 = p_4 = 4$ for $s \geq 1$. Summing I_2 together with I_1 calculated in (4.3) we conclude the proof. \square

Corollary 2 (Energy estimate for difference) *If in addition to the conditions of the previous lemma we assume non-cavitation for η_1 then*

$$\frac{d}{dt} E^r(\eta_1, v_1, \eta_2, v_2) \lesssim \left(1 + \|\eta_1\|_{H^{s+1/2}}^2 + \|v_1\|_{H^s}^2 + \|v_2\|_{H^s}^2 \right) E^r(\eta_1, v_1, \eta_2, v_2).$$

Proof Non-cavitation implies coercivity for E^r and the rest is obvious. \square

Remark 2 The restriction $s > 1/2$ appeared in the lemma and its corollary is inconvenient. It comes from the loss of Hamiltonian structure of System (4.1). This results in the fact that we can obtain only a weak solution in case $s = 1/2$ and probably not unique.

5 Parabolic Regularisation

For application of the energy method we need to do a parabolic regularisation of the view

$$\begin{cases} \eta_t + v_x + i \tanh D(\eta v) = -\kappa \mu |D|^p \eta, \\ v_t + i \tanh D(1 + \kappa D^2)\eta + i \tanh Dv^2/2 = -\kappa \mu |D|^p v \end{cases} \tag{5.1}$$

where $\mu \in (0, 1)$. We want to prove solution existence for (5.1) for any given μ , by the contraction mapping principal and so p should be big enough. However, we also do not want to spoil our energy estimates, and so p should be small enough. As we shall see below, this bounds us to $p \in (1/2, 1]$. Here the left number comes from the following lemma.

Lemma 10 *For any $s \geq 1/2$, $\mu > 0$ and $p > 1/2$ there exists a finite positive bound $C(T)$, tending to zero as $T \rightarrow 0$, such that*

$$\int_0^T \left\| e^{-\mu t |D|^p} (f(t)g(t)) \right\|_{H^r} dt \leq C(T) \|f\|_{C_T H^r} \|g\|_{C_T H^s}$$

for any functions f, g defined on $[0, T]$. Here either $r = s + 1/2$ or $r = s$.

Proof In the case $r = s > 1/2$ the statement is obvious due to boundedness of $\exp(-\mu t |D|^p)$ and the algebraic property $\|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{H^s}$. Hence $C(T) = c_s T$ with some constant c_s depending only on s .

Otherwise we use

$$\left\| e^{-\mu t |D|^p} (fg) \right\|_{H^r} \leq \left\| \xi \mapsto e^{-\mu t |\xi|^p} \langle \xi \rangle^{1/2} \right\|_{L^\infty} \|fg\|_{H^{r-1/2}}$$

where in the case $r = s = 1/2$ by the Hölder inequality we have

$$\|fg\|_{H^{r-1/2}} = \|fg\|_{L^2} \leq \|f\|_{L^4} \|g\|_{L^4} \lesssim \|f\|_{H^{1/4}} \|g\|_{H^{1/4}} \lesssim \|f\|_{H^s} \|g\|_{H^s}$$

and in the case $r = s + 1/2$ we obviously have

$$\|fg\|_{H^{r-1/2}} \lesssim \|f\|_{H^r} \|g\|_{H^s}.$$

It is left to check that the L^∞ -norm above is locally integrable. Indeed, we can estimate the function at $\xi \in [0, 1]$ and at $\xi \geq 1$ separately

$$e^{-\mu t |\xi|^p} \langle \xi \rangle^{1/2} \leq \max \left\{ 2^{1/4}, \sup_{\xi \geq 1} 2^{1/4} \xi^{\frac{1}{2p}} e^{-\mu t |\xi|} \right\} \leq 2^{1/4} \max \left\{ 1, (2pe\mu t)^{-\frac{1}{2p}} \right\}$$

that is an integrable function with respect to time over any bounded interval for $p > 1/2$. The integral of this function over $[0, T]$ defines the bound $C(T)$. \square

With Lemma 10 in hand we can prove the local well-posedness in $H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})$ with $s \geq 1/2$ for System (5.1) by the fixed-point argument. Diagonalization has the matrix form

$$S(t) = \exp(-\varkappa \mu t |D|^p) \mathcal{K} \begin{pmatrix} \exp(-it K_\varkappa D) & 0 \\ 0 & \exp(it K_\varkappa D) \end{pmatrix} \mathcal{K}^{-1}, \tag{5.2}$$

where

$$\mathcal{K} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ K_\varkappa & -K_\varkappa \end{pmatrix}, \quad \mathcal{K}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & K_\varkappa^{-1} \\ 1 & -K_\varkappa^{-1} \end{pmatrix}$$

with K_\varkappa defined by (2.1). For any fixed $u_0 = (\eta_0, v_0)^T \in X^s = H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})$ the function $S(t)u_0$ solves the linear initial-value problem associated with (5.1). Let $X_T^s = C([0, T]; X^s)$ and regard a mapping $\mathcal{A} : X_T^s \rightarrow X_T^s$ defined by

$$\mathcal{A}(\eta, v; u_0)(t) = S(t)u_0 + \int_0^t S(t-t')(-i \tanh D) \begin{pmatrix} \eta v \\ v^2/2 \end{pmatrix} (t') dt'. \tag{5.3}$$

Then the Cauchy problem for System (5.1) with the initial data u_0 may be rewritten equivalently as an equation in X_T^s of the form

$$u = \mathcal{A}(u; u_0) \tag{5.4}$$

where $u = (\eta, v)^T \in X_T^s$.

Lemma 11 *Let $s \geq 1/2$, $p > 1/2$, $\mu \in (0, 1)$ and $u_0 = (\eta_0, v_0)^T \in X^s$. Then there is $T = T(s, p, \varkappa, \mu, \|u_0\|_{X^s}) > 0$, decreasing to zero with increase of the norm of u_0 , such that there exists a unique solution $u = (\eta, v)^T \in X_T^s$ of Problem (5.4).*

Moreover, for any $R > 0$ there exists a $T = T(s, p, \varkappa, \mu, R) > 0$ such that the flow map associated with (5.4) is a real analytic mapping of the open ball $B_R(0) \subset X^s$ to X_T^s .

Proof We need to show that the restriction of \mathcal{A} on some closed ball B_M with the center at point $\mathcal{S}(t)u_0$ is a contraction mapping. Note that $\|\mathcal{S}(t)u\|_{X^s} \lesssim \|\exp(-\varkappa\mu t|D|^p)u\|_{X^s}$. Hence by Lemma 10 for any $T, M > 0$ and $u, u_1, u_2 \in B_M \subset X_T^s$ hold

$$\|\mathcal{A}(u) - \mathcal{S}(t)u_0\|_{X_T^s} \leq C(T)\|u\|_{X_T^s}^2 \leq C(T)(M + \|u_0\|_{X^s})^2,$$

$$\|\mathcal{A}(u_1) - \mathcal{A}(u_2)\|_{X_T^s} \leq C(T)\|u_1 - u_2\|_{X_T^s} (\|u_1\|_{X_T^s} + \|u_2\|_{X_T^s}) \leq 2C(T)(M + \|u_0\|_{X^s})\|u_1 - u_2\|_{X_T^s},$$

and so taking $M = \|u_0\|_{X^s}$ one can find a $T > 0$ such that \mathcal{A} will be a contraction in the closed ball B_M . The first statement of the lemma follows from the contraction mapping principle. Smoothness of the flow map can be proved in the same spirit applying the implicit function theorem instead, and so we omit it. Some details can be found in [9]. □

By a standard argumentation, see for example [11], one can show that if $u = (\eta, v)^T \in X_T^s$ is the solution of Problem (5.4) then $u \in C^1((0, T); H^{s-1}(\mathbb{R}) \times H^{s-3/2}(\mathbb{R}))$ and it solves the regularised system (5.1) as well with the initial data $u_0 \in X^s$. Clearly, in order to be able to use the following energy and a priori estimates, one has to pick up a smooth initial data. The justification is discussed briefly in Section 7.

Lemma 12 *Suppose $s \geq 1/2$. Then there exists $C_s > 0$ such that for any $\varkappa > 0$ and any functions $\eta, v \in C^1((0, T); H^\infty(\mathbb{R}))$ solving System (5.1) with $\mu \in (0, 1)$ and $p \in (1/2, 1]$ it holds that*

$$\frac{d}{dt} E^s(\eta, v) \leq C_s(1 + \varkappa) \left(\|\eta, v\|_{H_\varkappa^{s+1/2} \times H^s}^2 + \|\eta, v\|_{H_\varkappa^{s+1/2} \times H^s}^4 \right).$$

In other words, the parabolic regularisation (5.1) does not spoil the energy estimate. Note that $C_s > 0$ does not depend on \varkappa, μ, p .

Proof Following the proof of Lemma 6 one arrives at

$$\frac{d}{dt} E^s(\eta, v) = \tilde{I}_1 + \tilde{I}_2 + I_1 + \dots + I_8, \tag{5.5}$$

where

$$\tilde{I}_1 = -\varkappa^2\mu \left\| \partial_x |D|^{p/2} \eta \right\|_{H^{s-1/2}}^2 - \varkappa\mu \left\| |D|^{p/2} \eta \right\|_{H^{s-1/2}}^2 - \varkappa\mu \left\| K^{-1} |D|^{p/2} v \right\|_{H^{s-1/2}}^2 \leq 0,$$

$$\tilde{I}_2 = -\frac{\varkappa\mu}{2} \int (J^{s-1/2} v)^2 |D|^p \eta - \varkappa\mu \int \eta (J^{s-1/2} v) J^{s-1/2} |D|^p v \lesssim \varkappa \|\eta\|_{H^{s+1/2}} \|v\|_{H^s}^2$$

for $p \leq 1$ and the rest integrals I_1, \dots, I_8 are the same as in Lemma 6. □

As was noticed at the end of Section 3, one has to make sure that the modified energy is coercive. An effective way to do it at the low level of regularity is to control $\|\eta\|_{L^\infty}$ via the energy conservation. One can get the same controllability for the regularised problem via the energy dissipation due to the following result.

Lemma 13 Suppose $\eta, v \in C^1((0, T); H^\infty(\mathbb{R}))$ solve System (5.1) with $\varkappa > 0$, $\mu \in (0, 1)$ and $p \in (1/2, 1]$. Then there exists $\delta(p) > 0$ independent of the viscosity μ and the capillarity \varkappa such that $\mathcal{H}(\eta, v)$ is a non-increasing function of time t provided $\|\eta(t)\|_{L^2} + \|v(t)\|_{H^{1/2}} \leq \delta$ holds for any moment t .

Proof Hamiltonian (1.3) has the derivative

$$\frac{1}{\varkappa\mu} \frac{d}{dt} \mathcal{H}(\eta, v) = -\|\eta\|_{\dot{H}^{p/2}}^2 - \varkappa \|\eta\|_{\dot{H}^{p/2+1}}^2 - \|K^{-1}v\|_{\dot{H}^{p/2}}^2 - I_1 - I_2,$$

where the rest integrals

$$I_1 = \int \eta v |D|^p v, \quad I_2 = \frac{1}{2} \int v^2 |D|^p \eta$$

are of no definite sign. One has to check that I_1, I_2 are absorbed by the first and third norms.

Firstly, we rewrite I_1 in the form

$$I_1 = \int |D|^{p/2}(\eta v) |D|^{p/2} v = \int \left(|D|^{p/2}(\eta v) - v |D|^{p/2} \eta - \eta |D|^{p/2} v \right) |D|^{p/2} v + \int \left(|D|^{p/2} \eta \right) v |D|^{p/2} v + \int \eta \left(|D|^{p/2} v \right)^2.$$

Applying the Hölder inequality and the fractional Leibniz rule (2.3) for $|D|^{p/2}$ with L^2 -norm to the first integral, Lemma 4 to the second integral and the Hölder inequality to the third integral, one obtains

$$|I_1| \lesssim \left\| |D|^{p/2-1/4} \eta \right\|_{L^4} \left\| |D|^{1/4} v \right\|_{L^4} \left\| |D|^{p/2} v \right\|_{L^2} + \left\| |D|^{p/2} \eta \right\|_{L^2} \|v\|_{H^{1/2}} \left\| |D|^{p/2} v \right\|_{H^{1/2}} + \|\eta\|_{L^2} \left\| |D|^{p/2} v \right\|_{L^4}^2.$$

Using the Sobolev embedding $\dot{H}^{1/4} \hookrightarrow L^4$, one finally obtains

$$|I_1| \lesssim \left\| |D|^{p/2} \eta \right\|_{L^2} \|v\|_{H^{1/2}} \left\| |D|^{p/2} v \right\|_{H^{1/2}} + \|\eta\|_{L^2} \left\| |D|^{p/2} v \right\|_{H^{1/2}}^2.$$

The second integral I_2 can be treated by the Hölder inequality as follows

$$|I_2| = \frac{1}{2} \int \left(|D|^{p/2} v^2 \right) |D|^{p/2} \eta \leq \frac{1}{2} \left\| |D|^{p/2} v^2 \right\|_{L^2} \left\| |D|^{p/2} \eta \right\|_{L^2}.$$

Here the first norm is estimated with the help of the Leibniz rule in the way

$$\left\| |D|^{p/2} v^2 \right\|_{L^2} \lesssim \left\| v |D|^{p/2} v \right\|_{L^2} + \left\| |D|^{p/2} v \right\|_{L^4}^2 \lesssim \|v\|_{H^{1/2}} \left\| |D|^{p/2} v \right\|_{H^{1/2}} + \left\| |D|^{p/2+1/4} v \right\|_{L^2}^2,$$

where we have used Lemma 4 and the embedding $\dot{H}^{1/4} \hookrightarrow L^4$. Thus

$$|I_2| \lesssim \|v\|_{H^{1/2}} \left\| |D|^{p/2} v \right\|_{H^{1/2}} \left\| |D|^{p/2} \eta \right\|_{L^2}.$$

Eventually we obtain

$$|I_1| + |I_2| \lesssim \left(\|\eta\|_{\dot{H}^{p/2}}^2 + \|K^{-1}v\|_{\dot{H}^{p/2}}^2 \right) \max \{ \|\eta\|_{L^2}, \|v\|_{H^{1/2}} \},$$

that concludes the proof. Note that the implicit constant here does not depend on \varkappa . □

As a simple corollary with the proof similar to that of Lemma 8 one obtains the following.

Corollary 3 *There exists a constant $\delta > 0$, depending only on the parabolic regularization power p , such that if a pair $u = (\eta, v) \in C^1((0, T); (H^\infty(\mathbb{R}))^2)$, having initial condition $\|u_0\|_{H^s_\varkappa \times H^{1/2}} \leq \delta/2$, solves System (5.1) then $\|u(t)\|_{H^s_\varkappa \times H^{1/2}} \leq \delta$ for any time t .*

The dependence of δ on the parabolic regularisation power p is unimportant since below we stick only to the case $p = 1$.

6 A Priori Estimate

We have an a priori global bound for solutions of both systems (1.1) and (5.1) in $H^1(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ due to Lemma 8 and Corollary 3, respectively. Our aim is it to obtain estimates in $H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > 1/2$.

Lemma 14 (A priori estimate) *Suppose $s > 1/2$ and $\varkappa > 0$. Let*

$$(\eta, v) \in C\left([0, T^*]; H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})\right) \cap C^1\left((0, T^*); (H^\infty(\mathbb{R}))^2\right)$$

be a solution of System (1.1) (or of the regularised system (5.1) with $\mu \in (0, 1)$ and $p = 1$) defined on its maximal time interval of existence and satisfying the blow-up alternative

$$T^* < +\infty \text{ implies } \lim_{t \rightarrow T^*} \|\eta(t), v(t)\|_{H^{s+1/2}_\varkappa \times H^s} = +\infty. \tag{6.1}$$

Suppose that its initial data (1.2) either satisfies the non-cavitation condition for $s > 3/2$ or has small enough $H^1_\varkappa \times H^{1/2}$ -norm for $s \leq 3/2$. Then there exists $T_0 < T^$ such that*

$$\sup_{t \in [0, T_0]} \|\eta(t), v(t)\|_{H^{s+1/2}_\varkappa \times H^s} \leq C \|\eta_0, v_0\|_{H^{s+1/2}_\varkappa \times H^s} \tag{6.2}$$

for some $C > 0$ independent of \varkappa, μ . The time of existence T_0 is a non-increasing function of the surface tension \varkappa and of the initial data norm $\|\eta_0, v_0\|_{H^{s+1/2}_\varkappa \times H^s}$.

Proof We closely follow the arguments in [12] since we have essentially the same energy estimates. The main difference lies in the control of coercivity of the modified energy (3.2) for small s . Let h_0, H_0 define non-cavitation of η_0 according to

Definition 1. Regard $h = h_0/2$ and $H = H_0 + h_0/2$. If the wave η satisfies the non-cavitation condition associated with h, H then there exist positive constants $c_0(h), C_0(H)$ such that

$$c_0 \|\eta, v\|_{H^\varkappa^{s+1/2} \times H^s}^2 \leq E^s(\eta, v) \leq C_0 \|\eta, v\|_{H^\varkappa^{s+1/2} \times H^s}^2$$

by coercivity of the energy. These constants depend only on h_0, H_0 . They are used to define the time set

$$\mathcal{T} = \left\{ T \in (0, T^*) : \sup_{t \in [0, T]} \|\eta(t), v(t)\|_{H^\varkappa^{s+1/2} \times H^s} \leq 3\sqrt{C_0/c_0} \|\eta_0, v_0\|_{H^\varkappa^{s+1/2} \times H^s} \right\}$$

that is non-empty and closed in $(0, T^*)$ by the solution continuity. Moreover, for $\tilde{T} = \sup \mathcal{T}$ we have either $\tilde{T} < T^*$ and so $\tilde{T} \in \mathcal{T}$ or $\tilde{T} = T^* = +\infty$ by the blow-up alternative (6.1). Introduce $T_0 = \min\{T_1, T_2\}$ with

$$T_1 = \frac{1}{C_1(1 + \varkappa)} \log \left(1 + \frac{1}{1 + C_1(1 + \varkappa)C_0 \|\eta_0, v_0\|_{H^\varkappa^{s+1/2} \times H^s}^2} \right),$$

$$T_2 = \begin{cases} \frac{h_0}{C_2 \left(\|\eta_0, v_0\|_{H^\varkappa^{s+1/2} \times H^s} + \|\eta_0, v_0\|_{H^\varkappa^{s+1/2} \times H^s}^2 \right)} & \text{for } s > 3/2 \\ 1 & \text{otherwise} \end{cases},$$

where C_1, C_2 are two big positive constants to be fixed below in the proof. The idea is to show that these constants can be chosen, independently on the initial data, in such a way that $T_0 \in \mathcal{T}$ or equivalently $T_0 \leq \tilde{T}$.

Assume the opposite $\tilde{T} < T_0$. Firstly, we will check that the non-cavitation condition holds on $[0, \tilde{T}]$. Indeed, in the low regularity case $s \in (1/2, 3/2)$ it is assumed smallness of the initial data and so $H^\varkappa^1 \times H^{1/2}$ -norm of the solution stays small with time evolution by Lemma 8 and Corollary 3. In particular, the wave satisfies the non-cavitation condition. For $s > 3/2$ one can estimate η using the first equation in System (1.1) (or in System (5.1)) as follows

$$\eta(x, t) = \eta_0(x) + \int_0^t \partial_t \eta(x, t') dt',$$

where

$$\begin{aligned} \|\partial_t \eta\|_{L^\infty} &\leq \|\partial_x v\|_{L^\infty} + \|\tanh D(\eta v)\|_{L^\infty} + \varkappa \mu \| |D|\eta\|_{L^\infty} \\ &\lesssim \|\partial_x v\|_{H^{s-1}} + \|\eta\|_{H^{s-1}} \|v\|_{H^{s-1}} + \varkappa \|\partial_x \eta\|_{H^{s-1}} \end{aligned}$$

with the implicit constant independent on $\mu \in (0, 1)$, obviously. Hence

$$\|\partial_t \eta\|_{L^\infty} \lesssim \|\eta_0, v_0\|_{H^\varkappa^{s+1/2} \times H^s} + \|\eta_0, v_0\|_{H^\varkappa^{s+1/2} \times H^s}^2$$

uniformly on $(0, \tilde{T}] \subset \mathcal{T}$. Thus we have

$$\left\| \int_0^t \partial_t \eta(x, t') dt' \right\|_{L^\infty} \leq \tilde{T} \sup_{t \in (0, \tilde{T}]} \|\partial_t \eta(t)\|_{L^\infty} \leq \frac{h_0}{2}$$

for big enough C_2 since $\tilde{T} < T_2$. As a result the non-cavitation

$$h - 1 = h_0/2 - 1 \leq \eta \leq H_0 + h_0/2 = H$$

holds on $\mathbb{R} \times [0, \tilde{T}]$. Without loss of generality one can assume that for $s \leq 3/2$ the non-cavitation of η is governed by the same constants h, H .

Let $E(t) = E^s(\eta, v)(t)$ be the energy defined by (3.2) and $E_0 = E(0)$. For System (1.1) (or for System (5.1)) we have the a priori energy estimate given in its differential form by Corollary 1. It can be rewritten in the form

$$\left(\frac{E}{1 + E} \right)' \leq c(1 + \varkappa) \frac{E}{1 + E}.$$

A straightforward use of Grönwall’s inequality gives

$$E(t) \left(1 - \frac{E_0}{1 + E_0} e^{c(1+\varkappa)t} \right) \leq \frac{E_0}{1 + E_0} e^{c(1+\varkappa)t}$$

for any $t \in [0, \tilde{T}]$ with c depending only on h, s . Note that

$$e^{c(1+\varkappa)t} \leq 1 + \frac{1}{1 + C_1(1 + \varkappa)E_0}$$

for any $C_1 \geq c$ and $0 \leq t \leq \tilde{T} < T_1$. In particular,

$$\frac{E_0}{1 + E_0} e^{c(1+\varkappa)t} \leq \frac{(C_1(1 + \varkappa))^{-1} + E_0}{1 + E_0} < 1$$

if in addition $C_1(1 + \varkappa) > 1$. Thus

$$E(t) \leq \frac{1}{\left(\frac{E_0}{1 + E_0} e^{c(1+\varkappa)t} \right)^{-1} - 1} \leq E_0 \frac{2 + C_1(1 + \varkappa)E_0}{1 + (C_1(1 + \varkappa) - 1)E_0} \leq 2E_0$$

if in addition $C_1(1 + \varkappa) \geq 2$. As a result setting $C_1 = \max\{2, c\}$ we have

$$\|\eta(t), v(t)\|_{H_\varkappa^{s+1/2} \times H^s} \leq \sqrt{2C_0/c_0} \|\eta_0, v_0\|_{H_\varkappa^{s+1/2} \times H^s}$$

for all $t \in [0, \tilde{T}]$. Taking into account $\tilde{T} < T^*$ and continuity of the solution one can find $\tilde{T} < T' < T^*, T_0$ such that on $[0, T']$ holds

$$\|\eta(t), v(t)\|_{H_\varkappa^{s+1/2} \times H^s} \leq 2\sqrt{C_0/c_0} \|\eta_0, v_0\|_{H_\varkappa^{s+1/2} \times H^s}$$

which contradicts the definition of \tilde{T} . Therefore, we showed that $T_0 \leq \tilde{T}$ concluding the main part of the proof.

It is left to specify the dependence of T_0 on the initial data and the surface tension. From its definition one can see that T_0 is non-increasing as a function of the initial data norm for each $\varkappa > 0$ fixed. One can also see straightaway that T_0 is non-increasing as a function of \varkappa for each (η_0, v_0) and $s > 3/2$ fixed. To the same conclusion one can easily come in the case $s \leq 3/2$, taking into account that the smallness assumption imposed on the initial data norm is \varkappa -independent according to Corollary 3. □

Studying the low surface tension regime in the last section, we will appeal to the following remark.

Remark 3 Suppose that $\varkappa \in (0, K]$. Then $\|u_0\|_{H_{\varkappa}^{s+1/2} \times H^s} \leq \|u_0\|_{H_K^{s+1/2} \times H^s}$ and $T_0(K) \leq T_0(\varkappa)$. Thus $T_0(K)$ and the smallness parameter of $H_K^1 \times H^{1/2}$ -norm of the initial data $u_0 = (\eta_0, v_0)$ can serve as bounds independent of \varkappa instead of the corresponding bounds given in the statement of the lemma.

Lemma 15 *Suppose $s > 1/2$, $\varkappa > 0$ and functions $\eta, v \in C^1((0, T); H^\infty(\mathbb{R}))$ solve System (1.1) (or the regularised system (5.1) with $\mu \in (0, 1)$ and $p = 1$). Then if $s < 1$ the following holds true*

$$\frac{d}{dt} E^s(\eta, v) \leq C_s(1 + \varkappa) \left(1 + \|v\|_{L^\infty} + \|\eta, v\|_{H_{\varkappa}^s \times H^{1/2}}^2 \right) \|\eta, v\|_{H_{\varkappa}^{s+1/2} \times H^s}^2,$$

and if $s \geq 1$ then

$$\frac{d}{dt} E^s(\eta, v) \leq C_s(1 + \varkappa) \left(1 + \|\eta, v\|_{H_{\varkappa}^{s+1/4} \times H^{s-1/4}}^2 \right) \|\eta, v\|_{H_{\varkappa}^{s+1/2} \times H^s}^2.$$

Moreover, the constant C_s does not depend on \varkappa, μ .

Proof The estimates obtained while proving Lemmas 6, 12 need to be refined for $s > 1/2$ as follows. We stick to the notations used in the corresponding proofs. Recall Identity (5.5) and note that $\tilde{I}_1, I_1 + I_5, I_3, I_6, I_8$ need not to be refined. So it is left to reconsider only the integrals I_2, I_2, I_4, I_7 .

Note that by Lemma 4 we have

$$\begin{aligned} \tilde{I}_2 &\lesssim \varkappa \left\| J^{s-1/2} v \right\|_{L^2} \left\| J^{s-1/2} v \right\|_{H^{1/2}} \left\| |D|\eta \right\|_{H^{s-1/2}} + \varkappa \|\eta\|_{H^s} \left\| J^{s-1/2} v \right\|_{H^{1/2}} \left\| J^{s-1/2} |D|v \right\|_{H^{-1/2}} \\ &\lesssim (1 + \varkappa) \|\eta, v\|_{H_{\varkappa}^s \times H^{s-1/2}} \|\eta, v\|_{H_{\varkappa}^{s+1/2} \times H^s}. \end{aligned}$$

In order to refine I_7 we need to estimate

$$\left\| \left(\operatorname{sgn} D |D|^{1/2} v \right) J^{s-1/2} v \right\|_{L^2} \lesssim \left\| \left(\operatorname{sgn} D |D|^{1/2} v \right) \right\|_{L^{p_1}} \left\| J^{s-1/2} v \right\|_{L^{p_2}}$$

following from Hölder’s inequality with $p_1(s) = \frac{1}{1-s}, p_2(s) = \frac{2}{2s-1}$ for $s \in (\frac{1}{2}, 1)$ and $p_1 = p_2 = 4$ in case $s \geq 1$. Implementing the Sobolev embedding and gathering the rest of I_7 one obtains

$$I_7 \lesssim \|v\|_{H^s}^2 \begin{cases} \|v\|_{H^{1/2}} & \text{for } s \in (1/2, 1) \\ \|v\|_{H^{s-1/4}} & \text{for } s \geq 1 \end{cases}.$$

It turns out that I_2 and I_4 should be estimated together in order to make sure that the constant C_s in the statement does not depend on \varkappa . Summing I_2 and I_4 one obtains

$$I_2 + I_4 = i \int \left(J^{s-1/2} \tanh D\eta \right) \left(J^{s-1/2}(\eta v) - \eta J^{s-1/2} v \right).$$

Firstly, we regard the case $s > 3/2$ and appeal to the Kato-Ponce inequality (2.2) to estimate the commutator above as

$$\left\| \left[J^{s-1/2}, \eta \right] v \right\|_{L^2} \lesssim \|\partial_x \eta\|_{L^{p_1}} \|J^{s-3/2} v\|_{L^{p_2}} + \|J^{s-1/2} \eta\|_{L^2} \|v\|_{L^\infty}.$$

Taking $p_1(s) = \frac{1}{2-s}, p_2(s) = \frac{2}{2s-3}$ for $s \in \left(\frac{3}{2}, 2\right)$ and $p_1 = p_2 = 4$ in case $s \geq 2$ one deduces

$$I_2 + I_4 \lesssim \|\eta\|_{H^{s-1/2}}^2 \begin{cases} \|v\|_{H^{1/2}} + \|v\|_{L^\infty} & \text{for } s \in (3/2, 2) \\ \|v\|_{H^{s-5/4}} + \|v\|_{L^\infty} & \text{for } s \geq 2 \end{cases}.$$

Secondly, in the case $s = 3/2$ the commutator is estimated straightforwardly as

$$\left\| \left[J^{s-1/2}, \eta \right] v \right\|_{L^2} \lesssim \|\eta\|_{H^1} \|v\|_{H^1} + \|\eta\|_{L^\infty} \|v\|_{H^1},$$

and so

$$I_2 + I_4 \lesssim \|\eta\|_{H^{s-1/2}}^2 \|v\|_{H^{s-1/2}}.$$

Finally, regarding the left case $s \in (1/2, 3/2)$ we firstly approximate the Bessel potential $J^{s-1/2}$ by the Riesz potential $|D|^{s-1/2}$ in the commutator as

$$\left\| \left(J^{s-1/2} - |D|^{s-1/2} \right) (\eta v) - \eta \left(J^{s-1/2} - |D|^{s-1/2} \right) v \right\|_{L^2} \lesssim \|\eta\|_{L^2} \|v\|_{L^\infty} + \|\eta\|_{L^2} \|v\|_{L^2},$$

and then appealing to the Leibniz rule (2.3) we obtain

$$\left\| |D|^{s-1/2} (\eta v) - \eta |D|^{s-1/2} v \right\|_{L^2} \lesssim \left\| |D|^{s-1/2} \eta \right\|_{L^2} \|v\|_{L^\infty}.$$

Hence for $s \in (1/2, 3/2)$ the sum of I_2 and I_4 is estimated as

$$I_2 + I_4 \lesssim \|\eta\|_{H^{s-1/2}}^2 (\|v\|_{L^\infty} + \|v\|_{L^2}).$$

Thus gathering all the parts one obtains

$$\tilde{I}_1 + \tilde{I}_2 + I_1 + \dots + I_8 \lesssim (1 + \varkappa) \|\eta, v\|_{H_\varkappa^{s+1/2} \times H^s}^2 \begin{cases} 1 + \|v\|_{L^\infty} + \|\eta, v\|_{H_\varkappa^1 \times H^{1/2}}^2 & \text{for } s \in (1/2, 1) \\ 1 + \|\eta, v\|_{H_\varkappa^{s+1/4} \times H^{s-1/4}}^2 & \text{for } s \geq 1 \end{cases}$$

which are the desired estimates. □

Knowing coercivity of the energy E^s , controlled either by the smallness or by the non-cavitation of the initial data, one can deduce from the lemma that the time of existence depends only on $\|\eta_0, v_0\|_{H_\varkappa^{s'+1/2} \times H^{s'}}$, where $1/2 < s' < s$. Taking into account the boundedness of $\|\eta, v\|_{H_\varkappa^1 \times H^{1/2}}$, holding true at least for small initial data, one can get a stronger result thanks to the Brezis-Gallouet limiting embedding (2.5). In order to exploit it we need the following Grönwall inequality.

Lemma 16 (Grönwall inequality) *Let y be an absolutely continuous positive function defined on some interval $[0, T]$. Suppose that almost everywhere*

$$y' \leq Ay \log y$$

where $A > 0$ is constant. Then there exists $C > 0$ independent of T such that

$$y(t) \leq \exp \left(Ce^{At} \right).$$

Proof Denote the right hand side by $z(t) = \exp(Ce^{At})$, where we take $C > 0$ such that $z(0) > y(0)$. Regard the derivative

$$\left(\frac{y}{z}\right)' = \frac{y'z - yz'}{z^2} \leq A \frac{y}{z} \log \frac{y}{z},$$

where the latter is less than zero at least for $t = 0$. So the fraction y/z decreases and stays always below the unity. □

Corollary 4 (Persistence of regularity) *In the conditions of the a priori estimate lemma 14 the following holds true*

$$\|\eta(t), v(t)\|_{H_x^{s+1/2} \times H^s} \leq \exp\left(Ce^{C(1+\varkappa)t}\right)$$

provided $s < 1$, and if $s \geq 1$ then

$$\|\eta(t), v(t)\|_{H_x^{s+1/2} \times H^s} \leq \|\eta_0, v_0\|_{H_x^{s+1/2} \times H^s} \exp\left(C(1+\varkappa)t + C(1+\varkappa) \int_0^t \|\eta, v\|_{H_x^{s+1/4} \times H^{s-1/4}}^2\right)$$

where the constant $C > 0$ does not depend on \varkappa, μ . In particular, the maximal time of existence $T^* = +\infty$ provided $\|\eta_0, v_0\|_{H_x^1 \times H^{1/2}}$ is small enough.

Proof The statement is obvious for $s \geq 1$. Suppose $s \in (1/2, 1)$. By Lemma 8 and Corollary 3 the norm $\|\eta(t), v(t)\|_{H_x^1 \times H^{1/2}}$ stays bounded with time. Hence from the Brezis-Gallouet inequality (2.5) one deduces

$$\|v(t)\|_{L^\infty} \lesssim 1 + \log(3 + \|v(t)\|_{H^s}).$$

Thus applying Lemma 15 and taking into account that E^s is coercive one obtains

$$\frac{d}{dt} E^s \lesssim (1 + \varkappa) (1 + \log(3 + E^s)) E^s.$$

As a result, after the application of the previous lemma with $y = 3 + E^s$, we have the estimate

$$E^s \leq \exp\left(Ce^{C(1+\varkappa)t}\right),$$

which again due to coercivity of E^s leads to the first inequality of the corollary after renaming the constant. □

7 Proof of Theorems 1 and 2

With the a priori estimate (6.2) in hand we can reapply the local existence Lemma 11 for the regularised problem (5.1) with $\mu \in (0, 1)$ and $p = 1$ in order to obtain solutions $u^\mu = (\eta^\mu, v^\mu)$ on the time interval $[0, T_0]$ defined by Lemma 14. Convergence of u^μ as $\mu \rightarrow 0$ follows from an adaptation of Lemma 9 to the difference energy (4.2) with $\eta_j = \eta^{\mu_j}, v_j = v^{\mu_j}$ ($j = 1, 2$) and $0 < \mu_2 < \mu_1 < 1$. The proof repeats the arguments of Lemma 9 and Lemma 13. Moreover, using the Gagliardo–Nirenberg interpolation one can obtain that u^μ converges to some u in $C([0, T_0]; H^{r+1/2}(\mathbb{R}) \times H^r(\mathbb{R}))$ as $\mu \rightarrow 0$ for any $0 < r < s$. This u is a

solution of (1.1) in the distributional sense. Furthermore, to prove persistence $u \in C([0, T_0]; H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R}))$, justify all the previous steps and obtain continuity of the flow map one has to regularise the initial data (1.2) as $u_0^\epsilon = (\eta_0 * \rho_\epsilon, v_0 * \rho_\epsilon)$, where ρ_ϵ is an approximation of the identity parametrised by $0 < \epsilon < 1$ [1, 13]. An application of the Bona–Smith argument in a straightforward standard way [1, 15, 20] results in the persistence and continuous dependence. We omit further details.

8 The Two-Dimensional Problem

In this section we comment briefly on adaptation of the proof for the two dimensional case. Firstly, we define the energy norm

$$\|\eta, \mathbf{v}\|_{H_\varkappa^{s+1/2} \times H^s \times H^s}^2 = \varkappa \|\nabla \eta\|_{H^{s-1/2}}^2 + \|\eta\|_{H^{s-1/2}}^2 + \|K^{-1} \mathbf{v}\|_{H^{s-1/2} \times H^{s-1/2}}^2 \tag{8.1}$$

and the modified energy

$$E^s(\eta, \mathbf{v}) = \frac{1}{2} \|\eta, \mathbf{v}\|_{H_\varkappa^{s+1/2} \times H^s \times H^s}^2 + \frac{1}{2} \int \eta \left| J^{s-1/2} \mathbf{v} \right|^2, \tag{8.2}$$

and then notice that it is coercive provided the wave η either satisfies the noncavitation condition or has small H^1 -norm. Note that the latter does not imply the first one, since now we do not have embedding of H^1 to L^∞ . The smallness of $H_\varkappa^1 \times H^{1/2} \times H^{1/2}$ -norm can be controlled by the energy conservation. Indeed, by Hölder’s inequality and the Sobolev embedding the cubic part of Hamiltonian (1.6) is estimated as

$$\int \eta |\mathbf{v}|^2 dx \lesssim \|\eta\|_{L^2} \|\mathbf{v}\|_{H^{1/2} \times H^{1/2}}^2,$$

and so repeating the arguments given in the proof of Lemma 8 we arrive at the conclusion that the small enough initial data stays small through the flow. For $s > 2$ the noncavitation preserves locally-in-time due to the first equation in (1.4).

The assumption $\nabla \times \mathbf{v}_0 = 0$ is needed to correctly define the semigroup associated with the regularised linear problem. Indeed, instead of Semigroup (5.2), for the two dimensional problem we have

$$\mathcal{S}(t) = \exp(-\varkappa \mu t |D|^p) \mathcal{K} \begin{pmatrix} \exp(-it K_\varkappa |D|) & 0 \\ 0 & \exp(it K_\varkappa |D|) \end{pmatrix} \mathcal{K}^{-1},$$

where

$$\mathcal{K} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ K_\varkappa \frac{D_1}{|D|} & -K_\varkappa \frac{D_1}{|D|} \\ K_\varkappa \frac{D_2}{|D|} & -K_\varkappa \frac{D_2}{|D|} \end{pmatrix}, \quad \mathcal{K}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \frac{|D|}{2K_\varkappa D_1} & \frac{|D|}{2K_\varkappa D_2} \\ 1 & -\frac{|D|}{2K_\varkappa D_1} & -\frac{|D|}{2K_\varkappa D_2} \end{pmatrix}$$

with K_\varkappa defined by (2.1). Note that \mathcal{K}^{-1} is well defined on the subspace of $H^{s+1/2}(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2$ with the curl free second coordinate. Moreover, it is easy to show that the condition $\nabla \times \mathbf{v} = 0$ preserves through the flow. The energy estimates and the rest of the proof of Theorem 3 can be done in exactly the same manner as in the one dimensional case, and so we omit further details.

9 The Low Capillarity Regime

This section is devoted to analysis of the solution dependence on the surface tension $\varkappa \in (0, 1]$. It allows, for instance, to validate that solutions of Systems (1.1), (1.4) with $\varkappa = 0$, that are known to exist [9], do indeed approximate solutions of the same systems when $\varkappa \ll 1$. We restrict ourselves to the one dimensional case. The extension to the two dimensional situation is straightforward.

Theorem 4 *Let $s \geq 2$ and*

$$u^\varkappa = (\eta^\varkappa, v^\varkappa) \in C\left([0, T]; H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})\right) \cap C^1\left((0, T); H^{s-1}(\mathbb{R}) \times H^{s-3/2}(\mathbb{R})\right)$$

be the solution of Problem (1.1), (1.2) for each $\varkappa \in (0, 1]$. Then u^\varkappa converges to the solution $u = (\eta, v)$ of Problem (1.1), (1.2) with $\varkappa = 0$ in $C\left([0, T]; H^{s-1/2}(\mathbb{R}) \times H^s(\mathbb{R})\right)$ as $\varkappa \rightarrow 0$.

Proof By the Bona-Smith argument it is enough to prove the statement for the smooth initial data $u_0 = (\eta_0, v_0)$ with $\eta_0, v_0 \in H^\infty(\mathbb{R})$. Moreover, it is enough to prove convergence in $C\left([0, T]; L^2(\mathbb{R}) \times H^{1/2}(\mathbb{R})\right)$. Without loss of generality we can assume that T coincides with T_0 defined in Lemma 14. Note that it can be regarded as independent of $\varkappa \in (0, 1]$ according to Remark 3. Moreover, we can assume that on the same time interval $[0, T]$ the solution u , corresponding to the zero surface tension, also satisfies (6.2) with the same constant C and $\varkappa = 0$.

Define functions $\theta = \eta^\varkappa - \eta, w = v^\varkappa - v$. Then θ and w satisfy the following system

$$\begin{cases} \theta_t = -\partial_x w - i \tanh D(\theta v + \eta^\varkappa w), \\ w_t = -i \tanh D\theta - i\varkappa D^2 \tanh D\eta^\varkappa - i \tanh D((v^\varkappa + v)w)/2. \end{cases}$$

Introduce the norm

$$\|\theta, w\|^2 = \|\theta, w\|_{H_0^1 \times H^{1/2}}^2 = \|\theta\|_{L^2}^2 + \|K^{-1}w\|_{L^2}^2$$

and calculate its derivative

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta, w\|^2 &= -i \int \theta \tanh D(\theta v + \eta^\varkappa w) - i\varkappa \int (K^{-1}w) K^{-1} D^2 \tanh D\eta^\varkappa \\ &- \frac{i}{2} \int (K^{-1}w) K^{-1} \tanh D((v^\varkappa + v)w) \lesssim \|\theta\|_{L^2}^2 \|v\|_{H^1} + \|\theta\|_{L^2} \|K^{-1}w\|_{L^2} \|\eta^\varkappa\|_{H^{1/2}} \\ &+ \varkappa \|K^{-1}w\|_{L^2} \|\partial_x \eta^\varkappa\|_{H^{3/2}} + \|K^{-1}w\|_{L^2}^2 (\|v^\varkappa\|_{H^1} + \|v\|_{H^1}). \end{aligned}$$

Thus we have

$$\frac{d}{dt} \|\theta, w\| \lesssim \|\theta, w\| \left(\|\eta^\varkappa, v^\varkappa\|_{H_\varkappa^{3/2} \times H^1} + \|v\|_{H^1} \right) + \sqrt{\varkappa} \|\eta^\varkappa, v^\varkappa\|_{H_\varkappa^{5/2} \times H^2},$$

and so applying the a priori estimate (6.2) one deduces

$$\frac{d}{dt} \|\theta, w\| \lesssim \|u_0\|_{H_x^{5/2} \times H^2} (\|\theta, w\| + \sqrt{\varkappa}).$$

Taking into account that at the initial time moment $\theta(0) = w(0) = 0$, one easily obtains

$$\|\theta(t), w(t)\| \leq \sqrt{\varkappa} C \|u_0\|_{H_x^{5/2} \times H^2} \left(\exp\left(C \|u_0\|_{H_x^{5/2} \times H^2} t\right) - 1 \right),$$

that tends to zero as $\varkappa \rightarrow 0$ uniformly with respect to $t \in [0, T]$. This concludes the proof. \square

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