Coalition and Relativised Group Announcement Logic*

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Abstract

There are several ways to quantify over public announcements. The most notable are reflected in arbitrary, group, and coalition announcement logics (APAL, GAL, and CAL correspondingly), with the latter being the least studied so far. In the present work, we consider coalition announcements through the lens of group announcements, and provide a complete axiomatisation of a logic with coalition announcements. To achieve this, we employ a generalisation of group announcements. Moreover, we study some logical properties of both coalition and group announcements that have not been studied before.

Keywords: Dynamic Epistemic Logic, Public Announcement Logic, Group Announcement Logic, Coalition Announcement Logic.

1 Introduction

The recent trend in *dynamic epistemic logic* (DEL) [van Ditmarsch et al., 2008] is quantification over various epistemic actions (see, for example, [Balbiani et al., 2008; Hales, 2013; Bozzelli et al., 2014; van Ditmarsch et al., 2017]). Arguably the simplest of such actions are public announcements [Plaza, 2007] that model the effects of agents simultaneously and publicly receiving the same piece of information. *Public announcement logic* (PAL) is an extension of epistemic logic (EL) [Meyer and van der Hoek, 1995] with constructs $[\psi]\varphi$ meaning that 'after a public announcement of ψ, φ is true in the resulting model'. Such a public announcement removes epistemic alternatives where ψ is not true.

Quantification over public announcements has been studied in [Balbiani et al., 2008], where the authors introduce *arbitrary public announcement logic*

*This paper is partially based on [Galimullin and Alechina, 2017] (and its corrected version available at `https://arxiv.org/abs/1810.02769`). Section 3, apart from Propositions 4 and 11, is entirely new. The axiomatisation and the completeness proof in Section 4 are also new, and many overlapping results (e.g. the Lindenbaum Lemma) are significantly improved.

(APAL) to deal with Fitch's knowability paradox [Brogaard and Salerno, 2013]. APAL is the extension of PAL with formulas $\Box\varphi$ meaning that 'after a public announcement of any epistemic formula, φ holds in the resulting model'. These additional formulas allow us to reason about the *existence* of an announcement leading to φ without providing such an announcement explicitly. For example, Ali Baba may know that there is a secret phrase to enter the thieves' cave, but he may not know which phrase it is.

Modalities of APAL do not take into account who makes an announcement and whether the announcement can be made by any set of agents modelled in a system. Hence, other possible quantified extensions of PAL were proposed. *Group announcement logic* (GAL) [Ågotnes and van Ditmarsch, 2008; Ågotnes et al., 2010] is an extension of PAL with group announcement modalities $[G]\varphi$ (and its dual $\langle G\rangle\varphi$). Alternatively, GAL can be considered as a restriction of APAL: instead of quantifying over *all* epistemic formulas, we quantify over formulas *known to agents in a group* G.

Formula $\langle G\rangle\varphi$ should be read as 'there is a truthful announcement by agents from group G such that φ holds after that announcement'. In this context a truthful announcement means that agents actually know formulas they announce. In other words, an announcement by a group is a conjunction of simultaneous announcements by each member of the group. Similarly, $[G]\varphi$ is read 'whatever agents from group G announce, φ holds afterwards'.

A logic of quantified announcements with a competitive flavour to it is *coalition announcement logic* (CAL), which was proposed in [Ågotnes and van Ditmarsch, 2008] as another generalisation of PAL. CAL modalities $[\langle G\rangle]\varphi$ and $\langle[G]\rangle\varphi$, as opposed to the GAL ones, are interpreted as double quantifications of the type $\forall\exists$ and $\exists\forall$ over epistemic formulas known to agents. Thus, $[\langle G\rangle]\varphi$ means 'whatever agents from G announce, there is an announcement by the agents outside of G such that φ holds afterwards'. And its dual, $\langle[G]\rangle\varphi$, is read as 'there is a truthful announcement made by the agents in G such that no matter what the agents not in G simultaneously announce, φ holds afterwards'. When dealing with coalition announcements, we refer to G as 'coalition', and to the set of agents not in G as 'anti-coalition'.

To illustrate the intuitive distinction between GAL and CAL consider the standard Prisoner's Dilemma scenario. If for some reason prosecutors interrogate only one of the prisoners (prisoner a, for example), then a can make an announcement to betray her accomplice (prisoner b) and avoid any punishment (let this outcome be denoted by φ). In GAL, we can express this situation with formula $\langle\{a\}\rangle\varphi$. However, if the prisoners are pitched against each other, none of them have the luxury of leaving the prison: if a decides to make an announcement to betray b, then b simultaneously makes a counter-announcement to betray agent a. Formally, $\neg\langle[\{a\}]\rangle\varphi$, or, equivalently, $[\langle\{a\}\rangle]\neg\varphi$.

CAL modalities were motivated by coalition logic [Pauly, 2002] (which is subsumed by CAL, see Appendix A) and van Benthem's playability operator [van Benthem, 2014, 2001]. Thus, CAL can be considered as one of the meeting points between DEL and game theory. Among other logics of quantified announcements — APAL and GAL — CAL is the least studied one.

In [Balbiani et al., 2008] the authors presented both finitary and infinitary versions of the proof system of APAL. However, it was later shown that the finitary variant is not sound[1]. Moreover, the completeness proof for APAL presented in [Balbiani et al., 2008] turned out to be incorrect. Subsequently, the proof was corrected in [Balbiani, 2015] and simplified in [Balbiani and van Ditmarsch, 2015].

As the proof system of GAL [Ågotnes et al., 2010] is quite similar to the one of APAL, it has suffered the same fate. The finitary version of the proof system is not sound [Fan, 2016, Footnote 4], and the correct completeness proof can be obtained via a relatively straightforward modification of the proof from [Balbiani and van Ditmarsch, 2015].

A complete axiomatisation of CAL is an open question. The satisfiability problem for all of APAL, GAL, and CAL is known to be undecidable [Ågotnes et al., 2016]. The relative expressivity of these logics is an open question. In [French et al., 2019] the authors solve one half of the problem by showing that APAL and CAL are not at least as expressive as GAL.

In this paper, we study coalition announcements through the lens of group announcements. In Section 3, we consider logical properties of CAL and compare them to the corresponding properties of GAL. In particular, we settle some open problems mentioned in the literature, and discuss the interaction between group and coalition announcements.

Continuing the theme of studying coalition announcements through the group ones, we present an axiomatiosation of a logic with coalition announcements in Section 4. In order to achieve completeness, we employ special group announcement operators, which we call relativised group announcements. The latter are like normal group announcements with some formula added as an additional conjunct to agents' announcements. This allows us to split a coalition's announcement and the anti-coalition's response. The completeness of the resulting formalism, *coalition and relativised group announcement logic* (CoRGAL), can be shown in a standard fashion for logics of quantified announcements (see [Galimullin, 2019] for the proof). In this work, however, we present an alternative completeness proof based on [Wang and Cao, 2013; Wang and Aucher, 2013]. This proof is interesting in its own right, since it allows us to treat the quantification over announcements similar to the box modality, i.e. as quantification over arrows in a model. In other words, we shift the perspective on public announcements, and treat them as static objects.

This, alternative, treatment of public announcements as classic modal operators also allows us to view models as tree-like structures, where nodes of the structure correspond to the initial model and various updates thereof, and edges are labeled with formulas being announced. This outlook may prove useful for dealing with problems that may require a tree-like representation of a model, e.g. showing that a logic has finite model property, or that some variant of a logic of quantified announcements is decidable. For example, in [van Ditmarsch

[1] A counterexample was given by Louwe Kuijer and can be found at `https://personal.us.es/hvd/APAL_counterexample.pdf`

and French, 2017] the authors, while proving the decidability of a restricted version of APAL, employ a ranked set of submodels of a given model, which is essentially a tree-like structure.

2 Background

2.1 Languages and Semantics

Let A be a finite set of agents, and P be a countable set of propositional variables.

Definition 1 (Languages)**.** *Languages of epistemic logic, public announcement logic, group announcement logic, and coalition announcement logic* are defined by the following BNFs:

$$\begin{array}{ll}
\mathcal{L}_{EL} & \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \land \varphi) \mid K_a\varphi \\
\mathcal{L}_{PAL} & \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \land \varphi) \mid K_a\varphi \mid [\varphi]\varphi \\
\mathcal{L}_{GAL} & \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \land \varphi) \mid K_a\varphi \mid [\varphi]\varphi \mid [G]\varphi \\
\mathcal{L}_{CAL} & \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \land \varphi) \mid K_a\varphi \mid [\varphi]\varphi \mid [\langle G\rangle]\varphi
\end{array}$$

where $p \in P$, $a \in A$, $G \subseteq A$, and all the usual abbreviations of propositional logic (such as $\vee$, $\rightarrow$, $\leftrightarrow$) and conventions for deleting parentheses hold. Diamond versions of modalities are defined as $\widehat{K}_a\varphi := \neg K_a\neg\varphi$, $\langle\psi\rangle\varphi := \neg[\psi]\neg\varphi$, $\langle G\rangle\varphi := \neg[G]\neg\varphi$, and $\langle[G]\rangle\varphi := \neg[\langle G\rangle]\neg\varphi$. Formula $K_a\varphi$ is read as 'agent a knows φ', $[\psi]\varphi$ is read as 'after public announcement of ψ, φ holds', $[G]\varphi$ is read as 'after any public announcement by agents from G, φ holds', and $[\langle G\rangle]\varphi$ is read as 'for any public announcement by agents from G, there is a *simultaneous* public announcement by agents from $A \setminus G$ such that φ holds after the joint announcement'.

Formulas of all logics we are dealing with in the paper are interpreted in epistemic models.

Definition 2 (Epistemic model)**.** An *epistemic model* is a triple $M = (S, \sim, V)$, where S is a non-empty set of states, $\sim: A \to 2^{S\times S}$ is an equivalence relation for each agent $a \in A$, $V : P \to 2^S$ is the valuation function for propositional variables $p \in P$. If necessary, we refer to the elements of the tuple as S^M, $\sim^M$, and V^M.

A model M with a designated state $s \in S$ is called a *pointed model* and denoted by M_s.

Model M is called *finite* if S is finite. Also, we write $M \subseteq N$ if $S^M \subseteq S^N$, $\sim^M$ and V^M are results of restricting $\sim^N$ and V^N to S^M, and call M a *submodel* of N.

An *updated model* M^φ is $(S^\varphi, \sim^\varphi, V^\varphi)$, where $S^\varphi = \{s \in S \mid M_s \models \varphi\}$ ($\models$ is defined in Definition 3), $\sim_a^\varphi = \sim_a \cap (S^\varphi \times S^\varphi)$ for all $a \in A$, and $V^\varphi(p) = V(p) \cap S^\varphi$. A model which results in subsequent updates of M_s with formulas $\varphi_1, \ldots, \varphi_n$ is denoted by $M_s^{\varphi_1,\ldots,\varphi_n}$. We will also sometimes write $M_s^X = (S^X, \sim^X, V^X)$, where $X \subseteq S$, $s \in X$, $S^X = X$, $\sim_a^X = \sim_a \cap (X \times X)$ for all $a \in A$, and $V^X(p) = V(p) \cap X$.

Let us denote by ψ_G formula $\bigwedge_{a \in G} K_a \psi_a$ such that $\psi_a \in \mathcal{L}_{PAL}$[2]. We refer to the set of all ψ_G's as $\mathcal{L}_{PAL}^G$. We will also write $\top_G$ to denote $\bigwedge_{a \in G} K_a(p \vee \neg p)$.

Definition 3 (Semantics)**.** Let M_s be a pointed epistemic model. The *semantics* is defined as follows:

$$
\begin{array}{lll}
M_s \models p & \text{iff} & s \in V(p) \\
M_s \models \neg\varphi & \text{iff} & M_s \not\models \varphi \\
M_s \models \varphi \wedge \psi & \text{iff} & M_s \models \varphi \text{ and } M_s \models \psi \\
M_s \models K_a\varphi & \text{iff} & \text{for all } t \in S : s \sim_a t \text{ implies } M_t \models \varphi \\
M_s \models [\psi]\varphi & \text{iff} & M_s \models \psi \text{ implies } M_s^\psi \models \varphi \\
M_s \models [G]\varphi & \text{iff} & \forall \psi_G : M_s \models [\psi_G]\varphi \\
M_s \models [\langle G \rangle]\varphi & \text{iff} & \forall \psi_G, \exists \chi_{A \setminus G} : M_s \models \psi_G \text{ implies } M_s \models \langle \psi_G \wedge \chi_{A \setminus G} \rangle \varphi
\end{array}
$$

Quantification in the definition of the semantics of group and coalition announcement operators $[G]$ and $[\langle G \rangle]$ is restricted to public announcement formulas known to agents. This allows us to avoid circularity in the definition. More on this issue and alternative semantics for quantified announcements is in [van Ditmarsch et al., 2016].

Note that $[\psi]\varphi$ is vacuously true if ψ is false, i.e. every φ is true after a false announcement. Also, it is easy to see that the diamond version of the public announcement operator implies the box one: $\langle \psi \rangle \varphi \to [\psi]\varphi$.

For convenience, let us also give the semantics of $\langle G \rangle \varphi$ and $\langle [G] \rangle \varphi$.

$$
\begin{array}{lll}
M_s \models \langle G \rangle \varphi & \text{iff} & \exists \psi_G : M_s \models \langle \psi_G \rangle \varphi \\
M_s \models \langle [G] \rangle \varphi & \text{iff} & \exists \psi_G, \forall \chi_{A \setminus G} : M_s \models \psi_G \text{ and } M_s \models [\psi_G \wedge \chi_{A \setminus G}]\varphi
\end{array}
$$

Definition 4 (Validity and satisfiability)**.** We call formula φ *valid* and write $\models \varphi$ if and only if for any pointed model M_s it holds that $M_s \models \varphi$. And φ is called *satisfiable* if and only if there is some M_s such that $M_s \models \varphi$.

2.2 Bisimulation

The basic notion of similarity in modal logic is bisimilation [Blackburn et al., 2001, Chapter 2].

Definition 5 (Bisimulation)**.** Let $M = (S^M, \sim^M, V^M)$ and $N = (S^N, \sim^N, V^N)$ be two models. A non-empty binary relation $Z \subseteq S^M \times S^N$ is called a *bisimulation* if and only if for all $s \in S^M$ and $u \in S^N$ with $(s, u) \in Z$:

- for all $p \in P$, $s \in V^M(p)$ if and only if $u \in V^N(p)$;

[2] As any formula of $\mathcal{L}_{PAL}$ can be translated into an equivalent formula of $\mathcal{L}_{EL}$ [Plaza, 2007], for succinctness' sake, we use the former rather than the latter in the scope of knowledge modalities in group announcements.

- for all $a \in A$ and all $t \in S^M$: if $s \sim_a^M t$, then there is a $v \in S^N$ such that $u \sim_a^N v$ and $(t, v) \in Z$;
- for all $a \in A$ and all $v \in S^N$: if $u \sim_a^N v$, then there is a $t \in S^M$ such that $s \sim_a^M t$ and $(t, v) \in Z$.

If there is a bisimulation between models M and N linking states s and u, we say that M_s and N_u are bisimilar, and write $M_s \rightleftarrows N_u$.

Definition 6 (Bisimulation Contraction)**.** Let $M = (S, \sim, V)$ be a model. The *bisimulation contraction* of M is the model $\|M\| = (\|S\|, \|\sim\|, \|V\|)$, where $\|S\| = \{[s] \mid s \in S\}$ and $[s] = \{t \in S \mid M_s \leftrightarrows M_t\}$, $[s]\|\sim\|_a[t]$ if and only if $\exists s' \in [s]$, $\exists t' \in [t]$ such that $s' \sim_a t'$, and $[s] \in \|V\|(p)$ if and only if $\exists s' \in [s]$ such that $s' \in V(p)$.

It is a standard result that $M_s \leftrightarrows \|M\|_{[s]}$ [Goranko and Otto, 2007]. Informally, bisimulation contraction $\|M\|$ is the minimal representation of M.

Theorem 1. Suppose M_s and N_t are bisimilar. Then for all $\varphi \in \mathcal{L}_{GAL \cup CAL}$, $M_s \models \varphi$ if and only if $N_t \models \varphi$.

Proof. Follows from the fact that PAL is invariant under bisimulation [van Ditmarsch et al., 2008]. □ □

This fact means that no GAL or CAL formula can distinguish two bisimilar states.

Corollary 2. $\|M\|_{[s]} \models \varphi$ if and only if $M_s \models \varphi$ for all $\varphi \in \mathcal{L}_{GAL \cup CAL}$.

3 Logical Properties of GAL and CAL

Validity and non-validity of certain logical formulas may shed light on some internal properties of the logic as well as build (or disprove) intuitions about how this logic may be (dis)similar to some other one. In Section 3.1 we study how uniting and decoupling groups and coalitions of agents affects their powers to achieve some configurations of a given model. Moreover, we investigate some relations between box and diamond versions of group and coalition announcement operators in Section 3.2. After that, in Section 3.3, we consider properties that capture some aspects of the interaction between CAL and GAL.

In what follows, we will frequently use axioms and validities of PAL. All of them can be found in [van Ditmarsch et al., 2008, Chapter 4].

3.1 Virtues of Cooperation

Intuition suggests that various groups and coalitions of agents, when united, can do no worse than if they were acting on their own. In this section we show that this intuition is indeed true.

We start with a somewhat obvious statement: if some configuration of a model can be achieved by a coalition, then the configuration can be achieved by a superset of the coalition.

Proposition 3. $\langle\!\langle G\rangle\!\rangle\varphi \rightarrow \langle\!\langle G\cup H\rangle\!\rangle\varphi$, where $G, H \subseteq A$, is valid.

Proof. Let $M_s \models \langle\!\langle G\rangle\!\rangle\varphi$ for some arbitrary M_s. By the semantics of CAL this is equivalent to

$$\exists\psi_G, \forall\chi_{A\setminus G} : M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}]\varphi.$$

Let us consider formula $\chi_{A\setminus G}$. Since $A \setminus G = (A \setminus (G \cup H)) \cup (H \setminus G)$, we can split the formula into two parts $\chi_{A\setminus(G\cup H)}$ and $\chi_{H\setminus G}$. Hence we have

$$\exists\psi_G, \forall\chi_{H\setminus G}, \forall\chi_{A\setminus(G\cup H)} : M_s \models \psi_G \wedge [\psi_G \wedge \chi_{H\setminus G} \wedge \chi_{A\setminus(G\cup H)}]\varphi.$$

This implies

$$\exists\psi_G, \exists\top_{H\setminus G}, \forall\chi_{A\setminus(G\cup H)} : M_s \models \psi_G \wedge \top_{H\setminus G} \wedge [\psi_G \wedge \top_{H\setminus G} \wedge \chi_{A\setminus(G\cup H)}]\varphi,$$

where $\top_{H\setminus G} := \bigwedge_{a\in H\setminus G} K_a\top$. Combining ψ_G and $\top_{H\setminus G}$ into a single announcement $\psi_{G\cup H}$ by the united coalition $G \cup H$, we conclude that

$$\exists\psi_{G\cup H}, \forall\chi_{A\setminus(G\cup H)} : M_s \models \psi_{G\cup H} \wedge [\psi_{G\cup H} \wedge \chi_{A\setminus(G\cup H)}]\varphi.$$

This is equivalent to $M_s \models \langle\!\langle G\cup H\rangle\!\rangle\varphi$ by the semantics. □ □

It was shown in [Ågotnes et al., 2010] that $\langle G\rangle\varphi \leftrightarrow \langle G\rangle\langle G\rangle\varphi$. This property demonstrates that within the framework of GAL a multiple-step strategy of a group can be executed in a single step. Whether this is true for CAL is an open question. We show, however, that if the truth of some φ can be achieved by two consecutive coalition announcements by G, then whatever agents from $A \setminus G$ announce, they cannot preclude G from making φ true.

Proposition 4. $\langle\!\langle G\rangle\!\rangle\langle\!\langle G\rangle\!\rangle\varphi \rightarrow [\![A \setminus G]\!]\varphi$ is valid.

Proof. Suppose that for some M_s it holds that $M_s \models \langle\!\langle G\rangle\!\rangle\langle\!\langle G\rangle\!\rangle\varphi$. This is equivalent to

$$\exists\psi_G, \forall\chi_{A\setminus G}, \exists\psi'_G, \forall\chi'_{A\setminus G} : M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}](\psi'_G \wedge [\psi'_G \wedge \chi'_{A\setminus G}]\varphi).$$

Since $\chi'_{A\setminus G}$ quantifies over all epistemic formulas known to $A \setminus G$, it also quantifies over $\top_{A\setminus G} := \bigwedge_{a\in A\setminus G} K_a\top$. Hence it is implied that

$$\exists\psi_G, \forall\chi_{A\setminus G}, \exists\psi'_G : M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}](\psi'_G \wedge [\psi'_G \wedge \top_{A\setminus G}]\varphi),$$

which is equivalent to

$$\exists\psi_G, \forall\chi_{A\setminus G}, \exists\psi'_G : M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}]\psi'_G \wedge [\psi_G \wedge \chi_{A\setminus G}][\psi'_G]\varphi.$$

Using PAL validity $[\psi]\chi \wedge [\psi][\chi]\varphi \leftrightarrow [\psi]\chi \wedge [\psi]\langle\chi\rangle\varphi$, we get

$$\exists\psi_G, \forall\chi_{A\setminus G}, \exists\psi'_G : M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}]\psi'_G \wedge [\psi_G \wedge \chi_{A\setminus G}]\langle\psi'_G\rangle\varphi.$$

Next, we use PAL validity $[\psi]\varphi \leftrightarrow (\psi \to \langle\psi\rangle\varphi)$:

$$\exists\psi_G, \forall\chi_{A\setminus G}, \exists\psi'_G : M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}]\psi'_G \wedge (\psi_G \wedge \chi_{A\setminus G} \to \langle\psi_G \wedge \chi_{A\setminus G}\rangle\langle\psi'_G\rangle\varphi).$$

By propositional reasoning this implies

$$\exists\psi_G, \forall\chi_{A\setminus G}, \exists\psi'_G : M_s \models \psi_G \wedge (\psi_G \wedge \chi_{A\setminus G} \to \langle\psi_G \wedge \chi_{A\setminus G}\rangle\langle\psi'_G\rangle\varphi),$$

which is equivalent to

$$\exists\psi_G, \forall\chi_{A\setminus G}, \exists\psi'_G : M_s \models \psi_G \wedge (\psi_G \to (\chi_{A\setminus G} \to \langle\psi_G \wedge \chi_{A\setminus G}\rangle\langle\psi'_G\rangle\varphi)).$$

The latter implies

$$\exists\psi_G, \forall\chi_{A\setminus G}, \exists\psi'_G : M_s \models \chi_{A\setminus G} \to \langle\psi_G \wedge \chi_{A\setminus G}\rangle\langle\psi'_G\rangle\varphi.$$

By PAL axiom $\langle\psi\rangle\langle\chi\rangle\varphi \leftrightarrow \langle\psi \wedge [\psi]\chi\rangle\varphi$, we have that

$$\exists\psi_G, \forall\chi_{A\setminus G}, \exists\psi'_G : M_s \models \chi_{A\setminus G} \to \langle\psi_G \wedge \chi_{A\setminus G} \wedge [\psi_G \wedge \chi_{A\setminus G}]\psi'_G\rangle\varphi.$$

Let us consider the subformula $\psi_G \wedge \chi_{A\setminus G} \wedge [\psi_G \wedge \chi_{A\setminus G}]\psi'_G$. Here, ψ'_G is an abbreviation for $\bigwedge_{i\in G} K_i\psi'_i$. Using PAL axiom $[\psi](\varphi \wedge \chi) \leftrightarrow [\psi]\varphi \wedge [\psi]\chi$, we can push the public announcement within the scope of $\bigwedge_{i\in G}$: $\psi_G \wedge \chi_{A\setminus G} \wedge \bigwedge_{i\in G}[\psi_G \wedge \chi_{A\setminus G}]K_i\psi'_i$.

Next, by PAL axiom $[\psi]K_a\varphi \leftrightarrow (\psi \to K_a[\psi]\varphi)$, we obtain $\psi_G \wedge \chi_{A\setminus G} \wedge \bigwedge_{i\in G}(\psi_G \wedge \chi_{A\setminus G} \to K_i[\psi_G \wedge \chi_{A\setminus G}]\psi'_i)$. By propositional reasoning, the latter is equivalent to $\psi_G \wedge \chi_{A\setminus G} \wedge \bigwedge_{i\in G} K_i[\psi_G \wedge \chi_{A\setminus G}]\psi'_i$. Substituting this into the original formula, we get

$$\exists\psi_G, \forall\chi_{A\setminus G}, \exists\psi'_G : M_s \models \chi_{A\setminus G} \to \langle\psi_G \wedge \chi_{A\setminus G} \wedge \bigwedge_{i\in G} K_i[\psi_G \wedge \chi_{A\setminus G}]\psi'_i\rangle\varphi.$$

We can move $\exists\psi_G$ within the scope of $\forall\chi_{A\setminus G}$, and, using the fact that $K_a(\varphi \wedge \psi) \leftrightarrow K_a\varphi \wedge K_a\psi$, combine ψ_G and $\bigwedge_{i\in G} K_i[\psi_G \wedge \chi_{A\setminus G}]\psi'_i$ into a single announcement by G of the form $\bigwedge_{i\in G} K_i(\psi_i \wedge [\bigwedge_{i\in G} K_i\psi_i \wedge \chi_{A\setminus G}]\psi'_i)$. Since the latter belongs to $\mathcal{L}^G_{PAL}$, we abbreviate it as ψ^*_G

The resulting formula

$$\forall\chi_{A\setminus G}, \exists\psi^*_G : M_s \models \chi_{A\setminus G} \to \langle\chi_{A\setminus G} \wedge \psi^*_G\rangle\varphi$$

is equivalent to $M_s \models [\langle A \setminus G\rangle]\varphi$ by the semantics of CAL. $\square$ $\square$

Whether $\langle\!\langle G\rangle\!\rangle\langle\!\langle G\rangle\!\rangle\varphi \to \langle\!\langle G\rangle\!\rangle\varphi$ is valid is an open question. We conjecture that the property is not valid. Consider $\langle\!\langle G\rangle\!\rangle\langle\!\langle G\rangle\!\rangle\varphi$: after the initial announcement, coalition G has a consecutive announcement to make φ true. This announcement, however, depends on the choice of $A \setminus G$ in the first operator. In other words, a consecutive announcement by G may vary depending on the initial announcement by $A \setminus G$. Hence, it seems highly counterintuitive that G has a single announcement that can incorporate all possible simultaneous announcements by $A \setminus G$ in a general (infinite) case.

Formula $\langle G\rangle\langle H\rangle\varphi \to \langle G \cup H\rangle\varphi$ is a validity of GAL [Ågotnes et al., 2010]. Again, it is unknown whether the same property holds for coalition operators, and, for the same reasons as for $\langle\!\langle G\rangle\!\rangle\langle\!\langle G\rangle\!\rangle\varphi \to \langle\!\langle G\rangle\!\rangle\varphi$, we conjecture that the corresponding formula is not valid in CAL.

We show, however, a generalisation of Proposition 4.

Proposition 5. $\langle\!\langle G\rangle\!\rangle\langle\!\langle H\rangle\!\rangle\varphi \to [\![A \setminus (G \cup H)]\!]\varphi$ is valid.

Proof. Let $M_s \models \langle\!\langle G\rangle\!\rangle\langle\!\langle H\rangle\!\rangle\varphi$ for an arbitrary M_s. By Proposition 3, we have $M_s \models \langle\!\langle G \cup H\rangle\!\rangle\langle\!\langle H\rangle\!\rangle\varphi$. According to the semantics,

$$\exists \psi_{G\cup H}, \forall \chi_{A\setminus(G\cup H)} : M_s \models \psi_{G\cup H} \land [\psi_{G\cup H} \land \chi_{A\setminus(G\cup H)}]\langle\!\langle H\rangle\!\rangle\varphi.$$

This is equivalent to

$$\exists \psi_{G\cup H}, \forall \chi_{A\setminus(G\cup H)} : M_s \models \psi_{G\cup H} \text{ and}$$
$$(M_s \models \psi_{G\cup H} \land \chi_{A\setminus(G\cup H)} \text{ implies } M_s^{\psi_{G\cup H} \land \chi_{A\setminus(G\cup H)}} \models \langle\!\langle H\rangle\!\rangle\varphi).$$

By Proposition 3, from $M_s^{\psi_{G\cup H} \land \chi_{A\setminus(G\cup H)}} \models \langle\!\langle H\rangle\!\rangle\varphi$ follows $M_s^{\psi_{G\cup H} \land \chi_{A\setminus(G\cup H)}} \models \langle\!\langle G \cup H\rangle\!\rangle\varphi$. By propositional reasoning, we have

$$\exists \psi_{G\cup H}, \forall \chi_{A\setminus(G\cup H)} : M_s \models \psi_{G\cup H} \text{ and}$$
$$(M_s \models \psi_{G\cup H} \land \chi_{A\setminus(G\cup H)} \text{ implies } M_s^{\psi_{G\cup H} \land \chi_{A\setminus(G\cup H)}} \models \langle\!\langle G \cup H\rangle\!\rangle\varphi),$$

which is equivalent to $M_s \models \langle\!\langle G \cup H\rangle\!\rangle\langle\!\langle G \cup H\rangle\!\rangle\varphi$. Finally, By Proposition 4, the latter implies $M_s \models [\![A \setminus (G \cup H)]\!]\varphi$. □

□

In the following, we sometimes consider model updates by agents without presenting a formula that allows us to achieve the update, and instead we deal with sets of states that agents can force to be in the updated model. We refer to these sets as 'strategies' or 'choices'.

To have this interchangeability, we require models to be bisimulation contracted. In such models each state can be uniquely characterised by a formula of EL, which is called a *characteristic formula*. The construction of characteristic formulas is by induction and mimics the definition of bisimulation. Having constructed characteristic formulas for each state in a model, we can take a disjunction of them to uniquely characterise a subset of the set of states of the model. In models that are not bisimulation contracted, characteristic formulas cease to uniquely describe a state or a set. Hence there is no direct correspondence between a strategy of agents and characteristic formulas, as agents may 'choose' a subset which may split some bisimulation equivalence class. More on the construction of characteristic formulas and the correspondence between them and agents' announcements can be found in [Ågotnes and van Ditmarsch, 2011; van Ditmarsch et al., 2014; Galimullin et al., 2018].

Also note that, due to the truthfulness condition of public announcements, we require all strategies to include the current state, and therefore the corresponding equivalence class. Hence, if for a given agent a there are n a-equivalence classes, then a has 2^{n-1} strategies: all possible combinations to choose among remaining $n-1$ equivalence classes. Strategies of groups are intersections of strategies of individual agents.

For an example, consider model M in Figure 1. In state s agent a has two

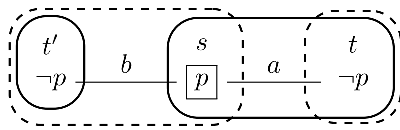

Figure 1: Model M, where a-equivalence classes are depicted by rounded rectangles, and b-equivalence classes are depicted by dashed rounded rectangles.

strategies: $\{s, t\}$ and $\{t', s, t\}$. Similarly, agent b's strategies are $\{t', s\}$ and $\{t', s, t\}$. Strategies available to group $\{a, b\}$ are intersections of strategies of the agents from the group, i.e. $\{s\}$, $\{s, t\}$, $\{t', s\}$, and $\{t', s, t\}$.

Returning back to the properties of GAL and CAL, we next show that splitting an announcement by a unified coalition into consecutive announcements by sub-coalitions may decrease their power to force certain outcomes. Whether $\langle\!\langle G \cup H \rangle\!\rangle \varphi \to \langle\!\langle G \rangle\!\rangle \langle\!\langle H \rangle\!\rangle \varphi$ is valid was mentioned as an open question in [Ågotnes et al., 2016]. We settle this problem by presenting a counterexample.

Proposition 6. $\langle\!\langle G \cup H \rangle\!\rangle \varphi \to \langle\!\langle G \rangle\!\rangle \langle\!\langle H \rangle\!\rangle \varphi$ is not valid.

Proof. Let $G = \{a\}, H = \{b\}$, and $\varphi := K_b(p \land q \land r) \land \neg K_a(p \land q \land r) \land \neg K_c(p \land q \land r)$. Formula φ says that agent b knows that the given propositional variables are true, and agents a and c do not. Consider model M_s in Figure 2.

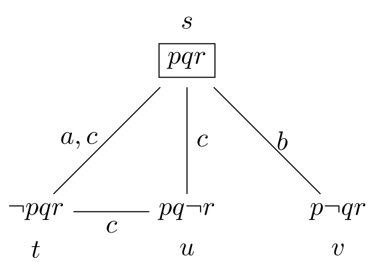

Figure 2: Model M

By the semantics $M_s \models \langle\!\langle \{a, b\} \rangle\!\rangle \varphi$ if and only if $\exists \psi_a, \exists \psi_b, \forall \chi_c : M_s \models \psi_a \land \psi_b \land [\psi_a \land \psi_b \land \chi_c]\varphi$. Let ψ_a be $K_a q$, and ψ_b be $K_b \top$. Observe that $M_s \models K_a q \land K_b \top$. This announcement leads to b learning that q. Moreover, c does not know any formula that she can announce to avoid φ. An informal argument is as follows. By announcing $K_a q$ agent a chooses a union of a-equivalence classes $\{s, t, u\}$ (and b chooses the whole model). Any simultaneous choice of c includes $\{s, t, u\}$ as a subset. Thus, intersection of $\{s, t, u\}$ and any of unions of c-equivalence classes is $\{s, t, u\}$, and φ is true in such a restriction of the model.

Let us show that $M_s \not\models \langle\!\langle\{a\}\rangle\!\rangle\langle\!\langle\{b\}\rangle\!\rangle\varphi$, or, equivalently, $M_s \models [\langle\{a\}\rangle][\langle\{b\}\rangle]\neg\varphi$. According to the semantics, $\forall\psi_a, \exists\chi_b, \exists\chi_c$: $M_s \models \psi_a \to \langle\psi_a \wedge \chi_b \wedge \chi_c\rangle[\langle\{b\}\rangle]\neg\varphi$. Assume that for an arbitrary ψ_a, announcements by b and c are $K_b p$ and $K_c\top$ correspondingly. Then $M_s \models \psi_a \wedge [\psi_a \wedge K_b p \wedge K_c\top][\langle\{b\}\rangle]\neg\varphi$. Note that no matter what a announces, $K_b p$ forces her to learn that $p \wedge q \wedge r$, and whatever is announced in the updated model $M_s^{\psi_a \wedge K_b p \wedge K_c \top}$, a's knowledge of $p \wedge q \wedge r$ and, hence, falsity of φ remains. Thus we reached a contradiction. $\square$ $\square$

The same counterexample can be used to demonstrate that $[\langle A \setminus (G \cup H)\rangle]\varphi \to \langle\!\langle G\rangle\!\rangle\langle\!\langle H\rangle\!\rangle\varphi$ is not valid, where $A \setminus (G \cup H) = \{c\}$. In the proof of Proposition 6 we show that $M_s \models \langle\!\langle\{a, b\}\rangle\!\rangle\varphi$. Using Proposition 9 we obtain $M_s \models [\langle c\rangle]\varphi$. The rest of the proof remains the same.

Corollary 7. $[\langle A \setminus (G \cup H)\rangle]\varphi \to \langle\!\langle G\rangle\!\rangle\langle\!\langle H\rangle\!\rangle\varphi$ is not valid.

To the best of our knowledge, the group announcement version of Proposition 6 has not been considered. We show that the property does not hold for GAL operators as well. To derive a contradiction, we use the intuition that separated groups, while being able to force a certain configuration of a model when united, may lack discerning power on their own. Contrast this to the proof of Proposition 6, where the contradiction was derived on the basis that former partners may spoil each other's strategies when pitched against one another.

Proposition 8. $\langle G \cup H\rangle\varphi \to \langle G\rangle\langle H\rangle\varphi$ is not valid.

Proof. Consider the model[3] in Figure 3.

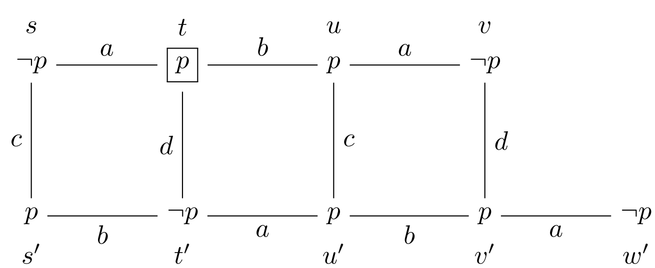

Figure 3: Model M^1

Note that M^1 is bisimulation contracted, and M^1_t can be distinguished from other pointed models in this proof by some distinguishing formula φ_1. Also let $G = \{a, d\}$ and $H = \{b, c\}$. Next consider model M^2 in Figure 4.

[3]The original idea of an infinite-grid counterexample is by Tim French (personal communication). Here we present its finite and reworked version.

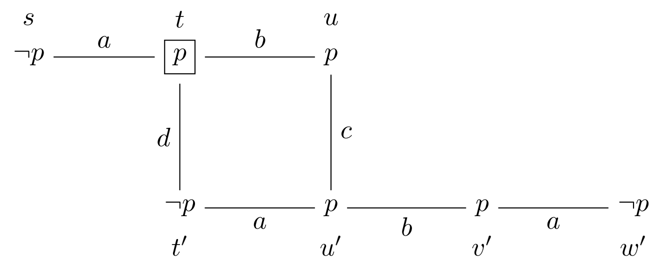

Figure 4: Model M^2

Again, M^2 is bisimulation contracted, and let some φ_2 be a distinguishing formula for M^2_t. The union of all agents in model M^1 can make φ_2 true, i.e. $M^1_t \models \langle\{a,d\} \cup \{b,c\}\rangle\varphi_2$. Indeed, a possible mutual choice for the agents is as follows: $X_a = \{s,t,u,v,t',u',v',w'\}$, $X_b = \{s,t,u,s',t',u',v',w'\}$, $X_c = \{s,t,u,s',t',u',v',w'\}$, and $X_d = \{s,t,u,v,t',u',v',w'\}$. Hence the corresponding group announcement reduce M^1 to $X_a \cap X_b \cap X_c \cap X_d = \{s,t,u,t',u',v',w'\}$ which is exactly model M^2.

Now we show that $M^1_t \models [\{a,d\}][\{b,c\}]\neg\varphi_2$, or, informally, any successive announcements by the corresponding groups do not result in M^2. Since we are interested only in group announcements that can lead to M^2, and due to the fact that M^2 is bisimulation contracted, we do not consider announcements that result in a model with fewer states than M^2.

There are only two such strategies for $\{a,d\}$. First strategy is the trivial one — a and d announce $K_a\top$ and $K_d\top$. Such an announcement leaves M^1 intact. It is easy to see that whatever $\{b,c\}$ announce afterwards, they cannot both retain only states of M^2. The closest they can get to M^2 is M^3, which is presented in Figure 5. Clearly, M^3 is not bisimilar to M^2, and hence $M^3_t \not\models \varphi_2$.

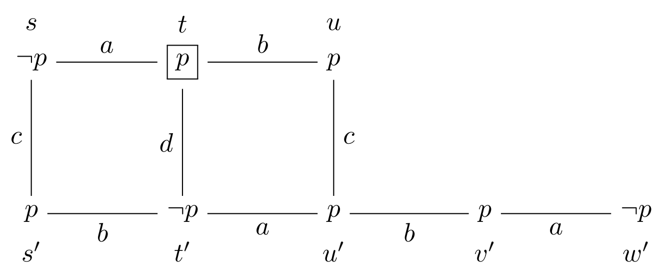

Figure 5: Model M^3

The second meaningful update of M^1 by $\{a,d\}$ is shown in Figure 6.

It might seem that the only difference between M^2 and M^4 is state v. Observe, however, that v is bisimilar to t', and any announcement by $\{b,c\}$ that removes v will also remove t' (see Figure 7).

Thus we showed that $M^1_t \models [\{a,d\}][\{b,c\}]\neg\varphi_2$, which is equivalent to $M^1_t \not\models$

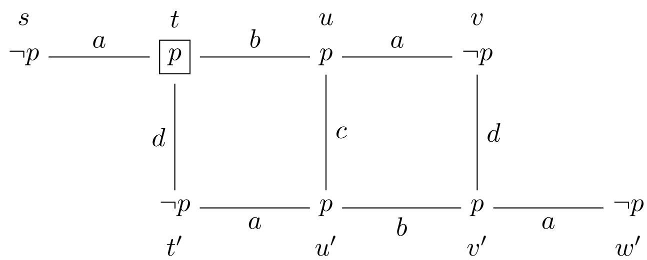

Figure 6: Model M^4

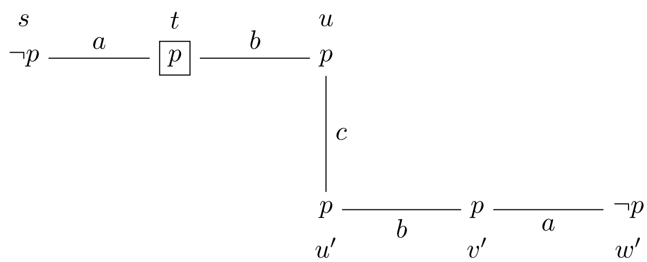

Figure 7: Model M^5

$\langle\{a,d\}\rangle\langle\{b,c\}\rangle\varphi_2$. $\square$ $\square$

3.2 Boxes, Diamonds, and Church-Rosser

In this section we consider two rather straightforward results for coalition announcement operators, and demonstrate that the Church-Rosser property does holds in neither GAL nor CAL (although it holds in APAL [Balbiani et al., 2008]).

We start with the fact that if coalition G has an announcement such that they can achieve φ no matter what agents outside of the coalition announce at the same time, then for every possible announcement by $A \setminus G$ there is a corresponding 'counter-announcement' such that φ holds afterwards.

Proposition 9. $\langle\![G]\!\rangle\varphi \to [\!\langle A \setminus G\rangle\!]\varphi$ is valid.

Proof. Assume that for some arbitrary M_s we have that $M_s \models \langle\![G]\!\rangle\varphi$. By the semantics this is equivalent to $\exists\psi_G, \forall\chi_{A\setminus G} : M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}]\varphi$, and the latter implies $\forall\chi_{A\setminus G}, \exists\psi_G : M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}]\varphi$. Using the validity of PAL $\models [\psi]\varphi \leftrightarrow (\psi \to \langle\psi\rangle\varphi)$, we have that $\forall\chi_{A\setminus G}, \exists\psi_G : M_s \models \psi_G \wedge (\psi_G \wedge \chi_{A\setminus G} \to \langle\psi_G \wedge \chi_{A\setminus G}\rangle\varphi)$, which implies, by propositional reasoning, $\forall\chi_{A\setminus G}, \exists\psi_G : M_s \models \chi_{A\setminus G} \to \langle\chi_{A\setminus G} \wedge \psi_G\rangle\varphi$. The latter is $M_s \models [\!\langle A \setminus G\rangle\!]\varphi$ by the semantics of CAL. $\square$ $\square$

The other direction of Proposition 9 is not valid. An intuitive explanation is that even though $A \setminus G$ may have a 'counter-announcement' to every announcement by G, they may, at the same time, lack the single 'universal' announcement for all possible G's announcements.

Proposition 10. $[\langle G \rangle]\varphi \to \langle [A \setminus G] \rangle\varphi$ is not valid.

Proof. We present a counterexample. Consider the model in Figure 8.

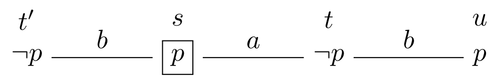

Figure 8: Model M^1

Pointed model M^1_s can be described by formula $\varphi_1 := p \wedge \widehat{K}_b(\neg p \wedge K_a \neg p) \wedge \widehat{K}_a(\neg p \wedge \widehat{K}_b K_a p)$.

Let us also consider some submodels of M^1 presented in Figure 9.

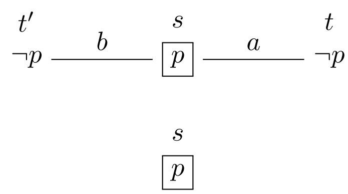

Figure 9: Models M^2 (top) and M^3 (bottom)

Corresponding distinguishing formulas for M^2_s and M^3_s are $\varphi_2 := p \wedge \widehat{K}_b K_a \neg p \wedge \widehat{K}_a(\neg p \wedge K_b \neg p)$ and $\varphi_3 := p \wedge K_a p \wedge K_b p$.

Let $G = \{a\}$, $A \setminus G = \{b\}$, and $\varphi := \varphi_1 \vee \varphi_2 \vee \varphi_3$.

First we show that $M^1_s \models [\langle a \rangle]\varphi$. By the semantics of CAL this is equivalent to $\forall \psi_a, \exists \chi_b : M_s \models \psi_a \to \langle \psi_a \wedge \chi_b \rangle \varphi$. Since M_1 is a finite bisimulation contracted model, there is only a finite number of non-equivalent model updates by either agent. Hence, we will abstract away from particular formulas and consider agents' strategies instead. In the model, agent a has four possible strategies: $\{s, t\}, \{t', s, t\}, \{s, t, u\}$, and $\{t', s, t, u\}$. Options $\{t', s, t\}$ and $\{t', s, t, u\}$ clearly satisfy φ as agent b announces $K_b \top$ at the same time. If agent a chooses to announce $\{s, t\}$ or $\{s, t, u\}$, agent b can announce $\{t', s\}$ at the same time, and such a joint announcement results in $\{s\}$ that satisfies φ_3.

Now, let us show that $M^1_s \not\models \langle [b] \rangle \varphi$, which is equivalent to $M^1_s \models [\langle b \rangle]\neg\varphi$. Agent b has only two options — $\{t', s, t, u\}$ and $\{t', s\}$. In the first case, agent a simultaneously 'chooses' $\{s, t\}$ that leads to φ being false. In the second case, agent a announces $K_a \top$, and φ is false in the resulting model. $\square$ $\square$

The Church-Rosser principle, $\Diamond\Box\varphi \to \Box\Diamond\varphi$, where $\Diamond$ and $\Box$ are some modal operators, corresponds to the confluence frame property $\forall x, y, z(xRy \wedge xRz \to \exists w(yRw \wedge zRw))$ (see [Blackburn et al., 2001, Chapter 3]). We are interested in how group boxes and diamonds commute together. In Proposition 11 we show that the generalised Church-Rosser principle does not hold for group announcements[4]. An intuitive explanation of this fact may be that knowledge of agents changes as a model is updated. Hence, they may lose their original strategies and discerning power as a result of an announcement by some other group. In other words, the order of announcements matters.

Proposition 11. $\langle G\rangle[H]\varphi \to [H]\langle G\rangle\varphi$ is not valid.

Proof. The counterexample model is presented in Figure 10.

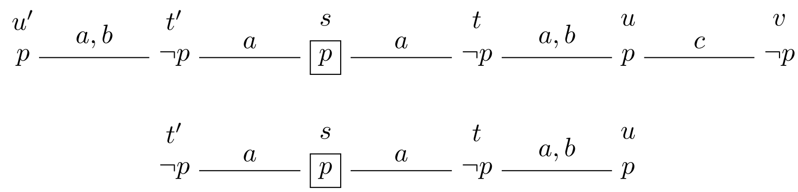

Figure 10: Models M^1 (top) and M^2 (bottom)

Formula φ is $\widehat{K}_a K_b \neg p \wedge \widehat{K}_a(\widehat{K}_b p \wedge \widehat{K}_b \neg p)$, and $M^2_s \models \varphi$ and $M^1_s \not\models \varphi$.

First we show that $M^1_s \models \langle\{a\}\rangle[\{b,c\}]\neg\varphi$. Let a's announcement be $K_a(\neg p \to K_c \neg p)$. Update of M^1_s with this announcement $(M^1_s)^{K_a(\neg p \to K_c \neg p)}$ is shown in Figure 11.

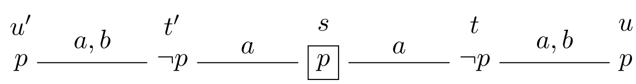

Figure 11: Model $(M^1_s)^{K_a(\neg p \to K_c \neg p)}$

Note that in this model states t and t', and u and u' became bisimilar. Hence, no matter what agents b and c announce, they cannot get rid of u' without removing u as well. In other words, agents b and c cannot make φ true. This establishes $M^1_s \models \langle\{a\}\rangle[\{b,c\}]\neg\varphi$.

The remaining part of the proof is to show that $M^1_s \not\models [\{b,c\}]\langle\{a\}\rangle\neg\varphi$, or, equivalently, $M^1_s \models \langle\{b,c\}\rangle[\{a\}]\varphi$. Let b and c's announcement be $K_b(\neg p \to \widehat{K}_b p)$ and $K_c(p \to (K_b p \vee \widehat{K}_c \neg p))$. Such a mutual announcement results in model M^2. Observe that in M^2_s, since the whole model is an a-equivalence class, agent a has no announcement to modify it. Moreover, $M^2_s \models \varphi$, and hence $M^1_s \models \langle\{b,c\}\rangle[\{a\}]\varphi$. □ □

The generalised Church-Rosser principle is not valid in CAL as well.

Proposition 12. $\langle\!\langle G\rangle\!\rangle[\![H]\!]\varphi \to [\![H]\!]\langle\!\langle G\rangle\!\rangle\varphi$ is not valid.

[4]Note that in [Ågotnes et al., 2010] (as well as in [van Ditmarsch, 2012]) it was claimed that the generalised Church-Rosser principle holds for GAL.

Proof. Consider models in Figures 8 and 9. Also let $G = \{a\}$, $H = \{b\}$, and $\varphi_2 := p \land \widehat{K}_b \neg p \land \widehat{K}_a \neg p$, $\varphi_3 := p \land K_a p \land K_b p$, and $\varphi := \varphi_2 \lor \varphi_3$. Note that $M_s^2 \models \varphi_2$ and $M_s^3 \models \varphi_3$.

First we show that $M_s^1 \models \langle\![\{a\}]\!\rangle[\!\langle\{b\}\rangle\!]\varphi$, which means that agent a has a strategy X_a such that whichever strategy X_b agent b simultaneously implements, $[\!\langle\{b\}\rangle\!]\varphi$ holds in the resulting model. Consider a's strategy $X_a = \{t', s, t\}$. Agent b has only two options in M_s^1: $X_b^1 = \{t', s, t, u\}$ and $X_b^2 = \{t', s\}$. Two possible resulting models are presented in Figure 12

$$\begin{tikzcd} \overset{t'}{\neg p} \arrow[r, dash, "b"] & \overset{s}{\boxed{p}} \arrow[r, dash, "a"] & \overset{t}{\neg p} \end{tikzcd}$$

$$\begin{tikzcd} \overset{t'}{\neg p} \arrow[r, dash, "b"] & \overset{s}{\boxed{p}} \end{tikzcd}$$

Figure 12: Resulting Models $(M_s^1)^{X_a \cap X_b^1}$ (top) and $(M_s^1)^{X_a \cap X_b^2}$ (bottom)

Next let us examine further model updates by coalition announcement $[\!\langle\{b\}\rangle\!]$. Again, there are only two options for agent b in $(M_s^1)^{X_a \cap X_b^1}$: $Y_b^1 = \{t', s, t\}$ and $Y_b^2 = \{t', s\}$. On Y_b^1 agent a responds with the same strategy, and on Y_b^2 she responds with $\{s, t\}$ that results in the model with single state s. In both cases φ holds. In $(M_s^1)^{X_a \cap X_b^2}$ agent b has only trivial strategy, and a responds with $\{s\}$ yielding the single-state model and making φ true. Hence, $M_s^1 \models \langle\![\{a\}]\!\rangle[\!\langle\{b\}\rangle\!]\varphi$.

Now we show that $M_s^1 \not\models [\!\langle\{b\}\rangle\!]\langle\![\{a\}]\!\rangle\varphi$, or, equivalently, that $M_s^1 \models \langle\![\{b\}]\!\rangle[\!\langle\{a\}\rangle\!]\neg\varphi$. Let b's strategy be the trivial one, i.e. $X_b = \{t', s, t, u\}$. Results of updates of M_s^1 with various a's strategies are presented in Figure 13.

$$\begin{tikzcd} \overset{s}{\boxed{p}} \arrow[r, dash, "a"] & \overset{t}{\neg p} \end{tikzcd}$$

$$\begin{tikzcd} \overset{s}{\boxed{p}} \arrow[r, dash, "a"] & \overset{t}{\neg p} \arrow[r, dash, "b"] & \overset{u}{p} \end{tikzcd}$$

$$\begin{tikzcd} \overset{t'}{\neg p} \arrow[r, dash, "b"] & \overset{s}{\boxed{p}} \arrow[r, dash, "a"] & \overset{t}{\neg p} \end{tikzcd}$$

$$\begin{tikzcd} \overset{t'}{\neg p} \arrow[r, dash, "b"] & \overset{s}{\boxed{p}} \arrow[r, dash, "a"] & \overset{t}{\neg p} \arrow[r, dash, "b"] & \overset{u}{p} \end{tikzcd}$$

Figure 13: From top to bottom: models $(M_s^1)^{X_b \cap X_a^1}$, $(M_s^1)^{X_b \cap X_a^2}$, $(M_s^1)^{X_b \cap X_a^3}$, and $(M_s^1)^{X_b \cap X_a^4}$

Finally, we consider further updates of the models in Figure 13 by $[\langle\{a\}\rangle]$. It is easy to see that any further announcements by a in models $(M_s^1)^{X_b \cap X_a^1}$ and $(M_s^1)^{X_b \cap X_a^2}$ can be countered by the trivial strategy of b so that $\neg\varphi$ is true in resulting models. In model $(M_s^1)^{X_b \cap X_a^3}$ agent b responds with $\{t', s, t\}$ on a's strategy $\{s, t\}$, and with $\{t', s\}$ on a's $\{t', s, t\}$; in both restrictions φ is false. In $(M_s^1)^{X_b \cap X_a^4}$ agent a has four strategies: $\{t', s, t, u\}$, $\{t', s, t\}$, $\{s, t, u\}$, and $\{s, t\}$. To make φ false, agent b responds with $\{t', s\}$ to the first two strategies, and with the trivial strategy to the last two. Thus, $M_s^1 \models \langle\!\langle\{b\}\rangle\!\rangle[\langle\{a\}\rangle]\neg\varphi$. $\square$ $\square$

3.3 Interaction Between Coalition and Group Announcements

We start this section with somewhat basic results concerning interaction between GAL and CAL operators.

In Proposition 13 formula 1 states that if a coalition can force some outcome, then they can achieve the outcome by a group announcement. Property 2 shows that coalition and group announcements are equivalent for the grand coalition A. That an anti-coalition cannot undo the result of a coalition announcement is presented in 3. Finally, property 4 states that if a coalition can force some outcome, then they can achieve the outcome by making one additional group announcement. The converse, however, is not valid (formula 5).

Proposition 13. 1–4 are valid, and 5 is not valid.

1. $\langle\!\langle G\rangle\!\rangle\varphi \to \langle G\rangle\varphi$,
2. $\langle\!\langle A\rangle\!\rangle\varphi \leftrightarrow \langle A\rangle\varphi$,
3. $\langle\!\langle G\rangle\!\rangle\varphi \leftrightarrow \langle\!\langle G\rangle\!\rangle[A \setminus G]\varphi$,
4. $\langle\!\langle G\rangle\!\rangle\varphi \to \langle\!\langle G\rangle\!\rangle\langle G\rangle\varphi$,
5. $\langle\!\langle G\rangle\!\rangle\langle G\rangle\varphi \to \langle\!\langle G\rangle\!\rangle\varphi$.

Proof. 1. If G can make φ true no matter what agents from $A \setminus G$ simultaneously announce, they can make φ true if all agents from coalition $A \setminus G$ announce $\top$.

2. Trivially by the semantics of the grand coalition.

3. $(\Rightarrow)$: We prove the contrapositive. Let $M_s \models [\langle G\rangle]\langle A \setminus G\rangle\varphi$ for some arbitrary M_s. By the semantics of CAL we have that $\forall\psi_G, \exists\chi_{A\setminus G}, \exists\chi'_{A\setminus G}$: $M_s \models \psi_G \to \langle\psi_G \wedge \chi_{A\setminus G}\rangle\langle\chi'_{A\setminus G}\rangle\varphi$. Due to PAL validity $\langle\psi\rangle\langle\chi\rangle\varphi \leftrightarrow \langle\psi \wedge [\psi]\chi\rangle\varphi$ the latter is equivalent to

$$M_s \models \psi_G \to \langle\psi_G \wedge \chi_{A\setminus G} \wedge [\psi_G \wedge \chi_{A\setminus G}]\chi'_{A\setminus G}\rangle\varphi.$$

Consider the subformula $\psi_G \wedge \chi_{A\setminus G} \wedge [\psi_G \wedge \chi_{A\setminus G}]\chi'_{A\setminus G}$, and recall that $\chi'_{A\setminus G}$ stands for $\bigwedge_{i \in A\setminus G} K_i\chi'_i$. Using PAL axiom $[\psi](\varphi \wedge \chi) \leftrightarrow [\psi]\varphi \wedge [\psi]\chi$, we obtain $\psi_G \wedge \chi_{A \setminus G} \wedge \bigwedge_{i \in A \setminus G} [\psi_G \wedge \chi_{A \setminus G}] K_i \chi'_i$. From the fact that $[\psi] K_a \varphi \leftrightarrow (\psi \rightarrow K_a [\psi] \varphi)$, we get

$$\psi_G \wedge \chi_{A \setminus G} \wedge \bigwedge_{i \in A \setminus G} (\psi_G \wedge \chi_{A \setminus G} \rightarrow K_i [\psi_G \wedge \chi_{A \setminus G}] \chi'_i),$$

which is equivalent to $\psi_G \wedge \chi_{A \setminus G} \wedge \bigwedge_{i \in A \setminus G} K_i [\psi_G \wedge \chi_{A \setminus G}] \chi'_i$ by propositional reasoning.

In order to form a single announcement by $A \setminus G$, we note that $\chi_{A \setminus G}$ is an abbreviation for $\bigwedge_{i \in A \setminus G} K_i \chi_i$, and using EL validity $K_a(\varphi \wedge \psi) \leftrightarrow K_a \varphi \wedge K_a \psi$, we have $\psi_G \wedge \bigwedge_{i \in A \setminus G} K_i(\chi_i \wedge [\psi_G \wedge \bigwedge_{i \in A \setminus G} K_i \chi_i] \chi'_i)$, the second conjunct of which is a knowledge formula of $A \setminus G$ and we denote it as $\chi^*_{A \setminus G}$. Substituting the result for $\psi_G \wedge \chi_{A \setminus G} \wedge [\psi_G \wedge \chi_{A \setminus G}] \chi'_{A \setminus G}$ in the original formula, we get $\forall \psi_G, \exists \chi^*_{A \setminus G} : M_s \models \psi_G \rightarrow \langle \psi_G \wedge \chi^*_{A \setminus G} \rangle \varphi$. The latter is equivalent to $M_s \models \langle\!\lbrack G \rbrack\!\rangle \varphi$ by the semantics of CAL.

($\Leftarrow$): Immediate by the fact that $A \setminus G$ can announce $\top_{A \setminus G}$.

4. Immediate by the fact that G can announce $\top_G$ after the coalition announcement.

5. The counterexample is the same as in Proposition 10 with $G = \{b\}$. Indeed, $(M_1, s) \models \langle\!\lbrack \{b\} \rbrack\!\rangle \langle \{b\} \rangle \varphi$, which is equivalent to $\exists \psi_b, \forall \chi_a, \exists \psi'_b$: $M_s \models \psi_b \wedge [\psi_b \wedge \chi_a] \langle \psi'_b \rangle \varphi$. Let $\psi_b := K_b \top$. Then we have that $\forall \chi_a, \exists \psi'_b$: $M_s \models K_b \top \wedge [K_b \top \wedge \chi_a] \langle \psi'_b \rangle \varphi$, or $\forall \chi_a, \exists \psi'_b$: $M_s \models [\chi_a] \langle \psi'_b \rangle \varphi$. The rest of the proof follows the one of Proposition 10 with substitution of b's simultaneous choice $\{t', s\}$ with the consecutive choice $\{s\}$. $\square$

$\square$

Whether CAL operators can be expressed in GAL is an open problem. The most probable definition of coalition announcements in terms of group announcements is $\langle\!\lbrack G \rbrack\!\rangle \varphi \leftrightarrow \langle G \rangle [A \setminus G] \varphi$. The validity of this formula was stated to be an open question in [van Ditmarsch, 2012; Ågotnes et al., 2016]. It was shown in [French et al., 2019] that the right-to-left direction of the formula is not valid. Here we prove the validity of the other direction.

Consider the left-to-right direction of the formula. In the antecedent all agents make a simultaneous announcement, whereas in the consequent agents from $A \setminus G$ know the announcement ψ_G made by G. Thus, in the updated model $M_s^{\psi_G}$ the agents in $A \setminus G$ may have learned some *new* epistemic formulas $\chi_{A \setminus G}$ that they did not know before the announcement. However, since ψ_G holds in the initial model, and $\chi_{A \setminus G}$ holds in the updated one, agents from $A \setminus G$ can always make an announcement in the initial model that they know that after the announcement of ψ_G, $\chi_{A \setminus G}$ is true. In other words, the set of announcements agents from $A \setminus G$ can make after G has announced ψ_G is a subset of the set of announcements that $A \setminus G$ can make in the initial model simultaneously with G.

Proposition 14. $\langle\!\lbrack G \rbrack\!\rangle \varphi \rightarrow \langle G \rangle [A \setminus G] \varphi$ is valid.

Proof. Assume that for some pointed model M_s it holds that $M_s \models \langle\!\langle G \rangle\!\rangle \varphi$. By the semantics of CAL this is equivalent to

$$\exists \psi_G, \forall \chi_{A \setminus G} : M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A \setminus G}]\varphi.$$

Since $\chi_{A\setminus G}$ quantifies over all possible announcements by $A \setminus G$, it also quantifies over a specific subset of these announcements — $\bigwedge_{i \in A \setminus G} K_i[\psi_G]\chi'_i$ for some ψ_G and for all $\chi'_i \in \mathcal{L}_{PAL}$.

Hence $\exists \psi_G \in, \forall \chi_{A\setminus G}$: $M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}]\varphi$ implies

$$\exists \psi_G \in \mathcal{L}^G_{PAL}, \forall \chi'_i \in \mathcal{L}_{PAL} : M_s \models \psi_G \wedge [\psi_G \wedge \bigwedge_{i \in A\setminus G} K_i[\psi_G]\chi'_i]\varphi.$$

Let us consider announcement $\psi_G \wedge \bigwedge_{i \in A \setminus G} K_i[\psi_G]\chi'_i$. By propositional reasoning it is equivalent to $\psi_G \wedge (\psi_G \rightarrow \bigwedge_{i \in A\setminus G} K_i[\psi_G]\chi'_i)$, which, in turn, is equivalent to $\psi_G \wedge \bigwedge_{i\in A\setminus G}(\psi_G \rightarrow K_i[\psi_G]\chi'_i)$.

Applying PAL axiom $[\psi]K_a\varphi \leftrightarrow (\psi \rightarrow K_a[\psi]\varphi)$, we get $\psi_G \wedge \bigwedge_{i \in A\setminus G}[\psi_G]K_i\chi'_i$. The latter is equivalent to $\psi_G \wedge [\psi_G] \bigwedge_{i\in A\setminus G} K_i\chi'_i$ due to the fact that $[\psi](\varphi \wedge \chi) \leftrightarrow [\psi]\varphi \wedge [\psi]\chi$ is also an axiom of PAL. Finally, denoting $\bigwedge_{i \in A\setminus G} K_i\chi'_i$ by $\chi'_{A\setminus G}$, we have that

$$\exists \psi_G, \forall \chi'_{A\setminus G} : M_s \models \psi_G \wedge [\psi_G \wedge [\psi_G]\chi'_{A\setminus G}]\varphi.$$

Using axiom $[\psi][\chi]\varphi \leftrightarrow [\psi \wedge [\psi]\chi]\varphi$, we get

$$\exists \psi_G, \forall \chi'_{A\setminus G} : M_s \models \psi_G \wedge [\psi_G][\chi'_{A\setminus G}]\varphi.$$

Due to validity $\psi \wedge [\psi]\varphi \leftrightarrow \langle\psi\rangle\varphi$, this is equivalent to $\exists \psi_G, \forall \chi'_{A\setminus G} : M_s \models \langle\psi_G\rangle[\chi'_{A\setminus G}]\varphi$. The latter is equivalent $M_s \models \langle G\rangle[A \setminus G]\varphi$ by the semantics of GAL. □ □

4 A Logic of Coalition and Relativised Group Announcements

A sound and complete axiomatisation of CAL is an open question. One of the reasons why finding one seems hard is the inherent alternation of quantifiers in the semantics of the coalition announcement operators. In order to mitigate this, we introduce relativised group announcements that allow us to separate a coalition's announcements from counter-announcements by their opponents. The resulting formalism, *coalition and relativised group announcement logic* (CoRGAL), is sound and complete. CoRGAL is reminiscent of alternating-time temporal dynamic epistemic logic (ATDEL) [de Lima, 2014]. The latter, however, is a more PDL-style logic [Harel et al., 2000] with postconditions and factual change. Moreover, in ATDEL agents are not required to know the formulas they announce.

The completeness proof given in this section deviates from a standard approach for logics of quantified announcements (see, for example, [Balbiani and van Ditmarsch, 2015] and [van Ditmarsch et al., 2017]). Instead of treating public announcements as *dynamic* operators, we treat them as *static* relations given in a model akin to any other standard box modality. Originally, such an approach was conceived in [Wang and Cao, 2013], where the authors presented a complete axiomatisation of PAL that does not rely on reduction axioms. We extend this technique to the case of quantified announcements, and as a result quantification over possible announcements becomes the quantification over arrows in a model. The standard completeness proof for CoRGAL can be found in [Galimullin, 2019, Chapter 6].

4.1 Relativised Group Announcements[5]

Let P denote a countable set of propositional variables, and A be a finite set of agents.

Definition 7 (Language of CoRGAL)**.** The *language of coalition and relativised group announcement logic* $\mathcal{L}_{CoRGAL}$ is as follows:

$$\mathcal{L}_{CoRGAL} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid [\varphi]\varphi \mid [G,\varphi]\varphi \mid [\langle G\rangle]\varphi$$

where $p \in P$, $a \in A$, $G \subseteq A$, and all the usual abbreviations of propositional logic and conventions for deleting parentheses hold. Diamond version of the operator $[G,\chi]\varphi$ is defined as $\langle G,\chi\rangle\varphi := \neg[G,\chi]\neg\varphi$.

Relativised group announcement $[G,\chi]\varphi$ says that 'given true announcement χ, *whatever* agents from G announce at the same time, they cannot avoid φ'. The diamond is read as 'given any announcement χ, *there is* a simultaneous announcement by G such that φ holds in the resulting model'.

Compare the intuitive reading of $[G,\chi]\varphi$ to the one of $[\langle G\rangle]\varphi$: '*whatever* agents from G announce, *there is* a simultaneous announcement by the agents from $A \setminus G$ such that φ holds in the resulting model'. Though they may look similar, the crucial difference is that there are two quantifiers in the case of $[\langle G\rangle]\varphi$ and only one in the case of $[G,\chi]\varphi$. This difference becomes clearer in the following formal definition of the semantics.

Definition 8 (Semantics of CoRGAL)**.** Let M_s be a pointed model. The *semantics of coalition and relativised group announcement logic* is as in Definition 3 plus the following.

$$\begin{array}{lll} M_s \models [G,\chi]\varphi & \text{iff} & M_s \models \chi \text{ and } \forall \psi_G : M_s \models [\psi_G \wedge \chi]\varphi \\ M_s \models \langle G,\chi\rangle\varphi & \text{iff} & M_s \models \chi \text{ implies } \exists \psi_G : M_s \models \langle \psi_G \wedge \chi\rangle\varphi \end{array}$$

Note that as in GAL and CAL we restrict the quantification in $[G,\chi]\varphi$ to formulas of $\mathcal{L}^G_{PAL}$.

[5]In [French et al., 2019] the authors use the same operators under the name of 'half-coalition' announcements. In this paper we use different syntax, which we think is more succinct.

Observe that the semantics of coalition announcement operators are given in a 'classic' way. An equivalent definition is possible using relativised group announcements.

$$\begin{array}{lll} M_s \models [\langle G \rangle]\varphi & \text{iff} & \forall \psi_G : M_s \models \langle A \setminus G, \psi_G \rangle \varphi \\ M_s \models \langle [G] \rangle \varphi & \text{iff} & \exists \psi_G : M_s \models [A \setminus G, \psi_G]\varphi \end{array}$$

Relativised group announcements help us to split coalition announcements, and treat the coalition's announcement and anti-coalition responses separately.

Next we show some intuitive properties of relativised group announcements.

Proposition 15. All of the following are valid:

1. $[G]\varphi \leftrightarrow [G, \top]\varphi$
2. $[\emptyset, \psi]\varphi \leftrightarrow \langle \psi \rangle \varphi$
3. $[A, \psi]\varphi \rightarrow \langle \psi \rangle \varphi$
4. $\neg \chi \rightarrow \langle G, \chi \rangle \varphi$

Proof. All proofs are trivial applications of the definition of semantics (Definition 8). □ □

The first property states that classic group announcements can be defined using relativised group announcements. Indeed, announcing a tautology in conjunction with an announcement by a group does not have any additional effect on the resulting model. Validities 2 and 3 demonstrate the relation between public announcements and relativised group announcements with empty and grand groups. Note that the property 3 holds only in one direction. A counterexample for the other direction would be a model with two states such that p holds only in one of them, agent's a relation is universal, and agent's b relation is the identity. If $\psi := p \vee \neg p$ and $\varphi := \neg K_a p$, then $\langle p \vee \neg p \rangle \neg K_a p$ is true, and $[\{a, b\}, p \vee \neg p]\neg K_a p$ is false in the p-state (since agent b can announce $K_b p$). Formula 4 says that if a false formula is being announced, we can always add a group announcement such that any φ holds vacuously afterwards.

4.2 Axiom System of CoRGAL

In this section, we present an axiomatisation of CoRGAL and show its soundness. The axiomatisation is based on the axiom system for PAL without reduction axioms [Wang and Cao, 2013], and have two additional axioms and two additional rules of inference.

There are multiple rules of inference we would like to use in an axiomatisation to deduce formulas with relativised group and coalition announcements. It is easy to see that rule

$$\text{If } \forall \psi_G : \vdash \chi \wedge [\psi_G \wedge \chi]\varphi, \text{then} \vdash [G, \chi]\varphi$$

is truth preserving. So is, for example,

$$\text{If } \forall \psi_G : \vdash K_a(\chi \land [\psi_G \land \chi]\varphi), \text{then} \vdash K_a([G, \chi]\varphi),$$

or

$$\text{If } \forall \psi_G : \vdash \tau \to [\theta](\chi \land [\psi_G \land \chi]\varphi), \text{then} \vdash \tau \to [\theta]([G, \chi]\varphi)$$

and so on. In order to have one general rule of inference that will encapsulate all these variations, necessity forms are introduced.

Definition 9. (Necessity forms) Let $\varphi \in \mathcal{L}_{CoRGAL}$, then *necessity forms* are inductively defined as follows:

$$\eta(\sharp) ::= \sharp \mid \varphi \to \eta(\sharp) \mid K_a\eta(\sharp) \mid [\varphi]\eta(\sharp).$$

The dual of a necessity form $\eta(\varphi)$ is a *possibility form* $\eta\{\varphi\}$ that is defined as $\eta\{\varphi\} := \neg\eta(\neg\varphi)$. The atom $\sharp$, which acts as a placeholder, has a unique innermost occurrence in each necessity form. Note that $\sharp$ is not a part of the language, i.e. $\sharp \notin \mathcal{L}_{CoRGAL}$, and hence a necessity (possibility) form is not a formula *per se*. However, it becomes one, once $\sharp$ is replaced by some $\varphi \in \mathcal{L}_{CoRGAL}$. The result of the replacement of $\sharp$ with φ in $\eta(\sharp)$ is denoted as $\eta(\varphi)$ and is inductively defined as follows:

- $\sharp(\varphi) = \varphi$,
- $(\psi \to \eta)(\varphi) = \psi \to \eta(\varphi)$,
- $(K_a\eta)(\varphi) = K_a\eta(\varphi)$,
- $([\psi]\eta)(\varphi) = [\psi]\eta(\varphi)$.

Definition 10 (Axiomatisation of CoRGAL)**.** The *axiom system for* CoRGAL is an extension of PAL with axioms and rules of inference for relativised group and coalition announcements.

$(A0)$	propositional tautologies,
$(A1)$	$K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$,
$(A2)$	$K_a\varphi \to \varphi$,
$(A3)$	$K_a\varphi \to K_aK_a\varphi$,
$(A4)$	$\neg K_a\varphi \to K_a\neg K_a\varphi$,
$(A5)$	$[\varphi](\psi \to \chi) \leftrightarrow ([\varphi]\psi \to [\varphi]\chi)$,
$(A6)$	$(p \to [\psi]p) \land (\neg p \to [\psi]\neg p)$,
$(A7)$	$\langle\psi\rangle\varphi \leftrightarrow (\psi \land [\psi]\varphi)$,
$(A8)$	$\langle\psi\rangle K_a\varphi \to K_a[\psi]\varphi$,
$(A9)$	$K_a[\psi]\varphi \to [\psi]K_a\varphi$,
$(A10)$	$[G, \chi]\varphi \to \chi \land [\psi_G \land \chi]\varphi$ for any ψ_G,
$(A11)$	$[\langle G\rangle]\varphi \to \langle A \setminus G, \psi_G\rangle\varphi$ for any ψ_G,
$(R0)$	If $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$,
$(R1)$	If $\vdash \varphi$, then $\vdash K_a\varphi$,
$(R2)$	If $\vdash \varphi$, then $\vdash [\psi]\varphi$,
$(R3)$	If $\forall \psi_G : \vdash \eta(\chi \land [\psi_G \land \chi]\varphi)$, then $\vdash \eta([G, \chi]\varphi)$,
$(R4)$	If $\forall \psi_G : \vdash \eta(\langle A \setminus G, \psi_G\rangle\varphi)$, then $\vdash \eta([\langle G\rangle]\varphi)$.

We call CoRGAL the smallest subset of $\mathcal{L}_{CoRGAL}$ that contains all the axioms $A0$ – $A11$ and is closed under rules of inference $R0$ – $R4$. Elements of CoRGAL are called *theorems*. Note that $R3$ and $R4$ are infinitary rules: they require an infinite number of premises.

Axiom $A5$ is a stronger version of a classic modal distributivity axiom for public announcements. That no public announcement can alter the value of a propositional variable is captured in $A6$. $A7$ expresses the fact that the execution of an announcement is deterministic, and that the precondition for an announcement to be executed is truthfulness of that announcement. Axiom $A8$ is usually called in literature *no miracles*, and means that if an agent learns φ after an announcement of ψ, then she knows that after any announcement of ψ, φ will hold. *Perfect recall* is expressed in $A9$: if an agent knows that φ will hold after an announcement, then she will know φ after that announcement. Axiom $A10$ says that if given some χ, agents from G cannot avoid φ no matter what they announce, they cannot avoid φ by making any particular announcement in this situation. The property expressed by $A11$ is as follows: if for every announcement by G there is a counter-announcement by $A \setminus G$, then for some particular announcement ψ_G by G there is a counter-announcement by $A \setminus G$. Rules $R3$ and $R4$ demonstrate how to infer formulas with relativised group and coalition announcements from an infinite number of premises.

Proposition 16. Axioms $A10$ and $A11$ are valid.

Proof. Follows directly from the definition of semantics (Definition 8). We just show the validity of ($A11$).

Assume that for some arbitrary pointed model M_s is holds that $M_s \models [\langle G \rangle]\varphi$. By the semantics this is equivalent to $\forall \psi_G, \exists \chi_{A\setminus G}$: $M_s \models \psi_G \rightarrow \langle \psi_G \land \chi_{A\setminus G} \rangle \varphi$. Since ψ_G quantifies over all epistemic formulas known to G, we can choose any particular ψ_G. Hence, we have that $\exists \chi_{A\setminus G}$: $M_s \models \psi_G \rightarrow \langle \psi_G \land \chi_{A\setminus G} \rangle \varphi$, which is equivalent to $M_s \models \langle A \setminus G, \psi_G \rangle \varphi$ by the semantics. □ □

Proposition 17. $R3$ and $R4$ are truth-preserving.

Proof. See Appendix B. □ □

Theorem 18 (Soundness)**.** For all $\varphi \in \mathcal{L}_{CoRGAL}$, if $\varphi \in$ CoRGAL, then φ is valid.

Proof. Soundness of $A0$–$A4$, $R0$, and $R1$ is due to soundness of EL. Axioms $A5$–$A9$ and rule of inference $R2$ are sound, since PAL is sound [Wang and Cao, 2013]. Soundness of $R3$ and $R4$ follows from Proposition 17, and validity of $A10$ and $A11$ is shown in Proposition 16. □

□

Note that in the axiomatisation of CoRGAL we do not include necessitation rules for relativised group and coalition announcements. In the next two lemmas we show that they are actually derivable in CoRGAL.

Lemma 19. If $\vdash \varphi$, then $\vdash \langle G, \chi \rangle \varphi$.

Proof.

1.	φ	Given
2.	$[\top_G \wedge \chi]\varphi$	$R2$
3.	$[\top_G \wedge \chi]\varphi \leftrightarrow (\chi \rightarrow \langle \top_G \wedge \chi \rangle \varphi)$	$A7$
4.	$\chi \rightarrow \langle \top_G \wedge \chi \rangle \varphi$	$R0$ (2) and (3)
5.	$(\chi \rightarrow \langle \top_G \wedge \chi \rangle \varphi) \rightarrow \langle G, \chi \rangle \varphi$	Contraposition of $A10$
6.	$\langle G, \chi \rangle \varphi$	$R0$ (4) and (5)

□ □

Lemma 20. If $\vdash \varphi$, then $\vdash [\langle G \rangle]\varphi$.

Proof.

1.	φ	Given
2.	$\{\langle A \setminus G, \psi_G \rangle \varphi \mid \psi_G \in \mathcal{L}^G_{PAL}\}$	From Lemma 19
3.	$[\langle G \rangle]\varphi$	$R4$

□ □

In the following lemma, which will be used in the sequel, we present an example of a derivation of a CoRGAL theorem.

Lemma 21. $\langle \psi \rangle (\varphi \wedge \chi) \leftrightarrow \langle \psi \rangle \varphi \wedge \langle \psi \rangle \chi$ is provable in CoRGAL.

Proof.

1.	$[\psi](\varphi \rightarrow \neg\chi) \leftrightarrow ([\psi]\varphi \rightarrow [\psi]\neg\chi)$	$A5$
2.	$[\psi]\neg(\varphi \wedge \chi) \leftrightarrow \neg([\psi]\varphi \wedge \neg[\psi]\neg\chi)$	Definition of $\rightarrow$
3.	$\langle \psi \rangle (\varphi \wedge \chi) \leftrightarrow ([\psi]\varphi \wedge \langle \psi \rangle \chi)$	Def. of $\langle \psi \rangle$ and prop. reasoning
4.	$\langle \psi \rangle (\varphi \wedge \chi) \leftrightarrow ([\psi]\varphi \wedge \psi \wedge [\psi]\chi)$	$A7$
5.	$\langle \psi \rangle (\varphi \wedge \chi) \leftrightarrow (\langle \psi \rangle \varphi \wedge \langle \psi \rangle \chi)$	$A7$ twice

□ □

4.3 Extended Semantics

In order to prove completeness, we make a detour through extended epistemic models. First, we show the completeness of CoRGAL relative to extended models, and then argue that these models are equivalent to the underlying classic ones.

Before we start with the main part of the proof, we define the size of a CoRGAL formula, which will be used in arguments based on induction.

Definition 11 (Size)**.** The *size* of some formula $\varphi \in \mathcal{L}_{CoRGAL}$ is defined as follows:

1. $Size(p) = 1$,
2. $Size(\neg\varphi) = Size(K_a\varphi) = Size([G,\chi]\varphi) = Size([\langle G\rangle]\varphi) = Size(\varphi) + 1$,
3. $Size(\varphi \wedge \psi) = Size(\varphi) + Size(\psi) + 1$,
4. $Size([\psi]\varphi) = Size(\psi) + 3 \cdot Size(\varphi)$.

The $[,]$-*depth* is defined as follows:

1. $d_{[,]}(p) = 0$,
2. $d_{[,]}(\neg\varphi) = d_{[,]}(K_a\varphi) = d_{[,]}([\langle G\rangle]\varphi) = d_{[,]}(\varphi)$,
3. $d_{[,]}(\varphi \wedge \psi) = \max\{d_{[,]}(\varphi), d_{[,]}(\psi)\}$,
4. $d_{[,]}([\psi]\varphi) = d_{[,]}(\psi) + d_{[,]}(\varphi)$,
5. $d_{[,]}([G,\chi]\varphi) = d_{[,]}(\varphi) + d_{[,]}(\chi) + 1$.

The $[\langle\rangle]$-*depth* is the same as $[,]$, with the following exceptions.

1. $d_{[\langle\rangle]}([G,\chi]\varphi) = d_{[\langle\rangle]}(\varphi) + d_{[\langle\rangle]}(\chi)$,
2. $d_{[\langle\rangle]}([\langle G\rangle]\varphi) = d_{[\langle\rangle]}(\varphi) + 1$.

Definition 12 (Size Relation)**.** The binary relation $<^{Size}_{[,],[\langle\rangle]}$ between $\varphi, \psi \in \mathcal{L}_{CoRGAL}$ is defined as follows:

$\varphi <^{Size}_{[,],[\langle\rangle]} \psi$ iff $d_{[\langle\rangle]}(\varphi) < d_{[\langle\rangle]}(\psi)$, or, otherwise, $d_{[\langle\rangle]}(\varphi) = d_{[\langle\rangle]}(\psi)$, and either $d_{[,]}(\varphi) < d_{[,]}(\psi)$, or $d_{[,]}(\varphi) = d_{[,]}(\psi)$ and $Size(\varphi) < Size(\psi)$. The relation is a well-founded strict partial order between formulas. Note that for all epistemic formulas ψ we have that $d_{[,]}(\psi) = d_{[\langle\rangle]}(\psi) = 0$.

Lemma 22. Let $\varphi, \psi, \chi \in \mathcal{L}_{CoRGAL}$. The following inequalities hold.

1. $\varphi <^{Size}_{[,],[\langle\rangle]} \neg\varphi$,
2. $\varphi <^{Size}_{[,],[\langle\rangle]} \varphi \wedge \chi$,
3. $\varphi <^{Size}_{[,],[\langle\rangle]} K_a\varphi$,
4. $\varphi <^{Size}_{[,],[\langle\rangle]} [\psi]\varphi$ and $\psi <^{Size}_{[,],[\langle\rangle]} [\psi]\varphi$,
5. $\chi \wedge [\psi_G \wedge \chi]\varphi <^{Size}_{[,],[\langle\rangle]} [G,\chi]\varphi$,
6. $\langle A \setminus G, \psi_G\rangle\varphi <^{Size}_{[,],[\langle\rangle]} [\langle G\rangle]\varphi$.

Proof. See Appendix B. $\square$ $\square$

Now, we are ready to define extended epistemic models and tie them together with the classic ones.

Definition 13 (Extended Epistemic Model)**.** An *extended epistemic model* $\mathcal{M}$ is a tuple $(S, \sim, V, \{\xrightarrow{\psi} \mid \psi \in \mathcal{L}_{CoRGAL}\})$, where $\xrightarrow{\psi}$ is a binary relation (possibly empty) over S for every ψ. Unless otherwise stated, in this section we deal with extended epistemic models, and use M to denote the underlying epistemic model $(S, \sim, V)$.

Next, we define the semantics of CoRGAL on extended epistemic models.

Definition 14 (Semantics of CoRGAL)**.** Let $\mathcal{M} = (S, \sim, V, \{\xrightarrow{\psi} \mid \psi \in \mathcal{L}_{CoRGAL}\})$ be an extended epistemic model. The *semantics of coalition and relativised group announcement logic* is the same as in Definition 8 with the only exception for public announcement.

$$\mathcal{M}_s \models [\psi]\varphi \quad \text{iff} \quad \text{for all } t \in S : s \xrightarrow{\psi} t \text{ implies } \mathcal{M}_t \models \varphi$$

As it is clear from a font used for a model — $\mathcal{M}$ or M — which semantics is being employed, we use the same symbol $\models$ for both extended and usual semantics. The usual semantics and the semantics with extended epistemic models do not generally coincide [Wang and Cao, 2013]. In order to make them equivalent, we restrict the class of extended epistemic models.

Definition 15 (Normal Extended Epistemic Model)**.** An extended model $\mathcal{M}$ is called *normal* if the following holds for all s and t:

U-Functionality For any $\psi \in \mathcal{L}_{CoRGAL}$, if $\mathcal{M}_s \models \psi$, then there is a *unique* t such that $s \xrightarrow{\psi} t$. If $\mathcal{M}_s \not\models \psi$, then there are no outgoing ψ-arrows at s.

U-Invariance If $s \xrightarrow{\psi} t$, then for all $p \in P$: $\mathcal{M}_s \models p$ if and only if $\mathcal{M}_t \models p$.

U-Zig If for some $a \in A$ it holds that $s \sim_a s'$, $s \xrightarrow{\psi} t$, and $s' \xrightarrow{\psi} t'$, then $t \sim_a t'$.

U-Zag If $s \xrightarrow{\psi} t$ and $t \sim_a t'$, then there exists an s' such that $s \sim_a s'$ and $s' \xrightarrow{\psi} t'$.

An extended model is called *ψ-normal* if U-functionality holds for a particular ψ.

Next lemma shows the equivalence of ψ-updates and ψ-transitions in normal extended models.

Lemma 23 (Lemma 35 of [Wang and Cao, 2013])**.** Let some $\psi \in \mathcal{L}_{CoRGAL}$ and a ψ-normal extended model $\mathcal{M}$ be given. We have that $M_s^\psi \rightleftarrows M_t$, if two conditions hold:

1. $s \xrightarrow{\psi} t$,
2. for all $u \in S$: $M_u \models \psi$ if and only if $\mathcal{M}_u \models \psi$.

Now we show that classic and extended semantics for normal models coincide.

Theorem 24. For all $\varphi \in \mathcal{L}_{CoRGAL}$, all normal extended epistemic models $\mathcal{M}$, and all s in $\mathcal{M}$, $\mathcal{M}_s \models \varphi$ if and only if $M_s \models \varphi$.

Proof. The proof is by induction on $<^{Size}_{[,],[\langle]}$-size of φ. Classic and extended semantics are the same for boolean and knowledge formulas.

Case $[\psi]\varphi$. $(\Rightarrow)$. Assume that $\mathcal{M}_s \models [\psi]\varphi$. According to the definition of semantics, this is equivalent to the fact that for all t: $s \xrightarrow{\psi} t$ implies $\mathcal{M}_t \models \varphi$.

If $\mathcal{M}_s \not\models \psi$, then, by the induction hypothesis, $M_s \not\models \psi$ and hence $M_s \models [\psi]\varphi$ for any φ.

Suppose that $\mathcal{M}_s \models \psi$. By U-Functionality, there is a unique t such that $s \xrightarrow{\psi} t$ and $\mathcal{M}_t \models \varphi$. Using Lemma 22 and the induction hypothesis twice, we yield $M_s \models \psi$, and by Lemma 23 we have that $M_s^\psi \models \varphi$. Finally, by the semantics we obtain $M_s \models \langle\psi\rangle\varphi$ which implies $M_s \models [\psi]\varphi$.

($\Leftarrow$). Let $M_s \models [\psi]\varphi$, and let $M_s \not\models \psi$. Then, by the induction hypothesis we have that $\mathcal{M}_s \not\models \psi$. Since $\mathcal{M}$ is a normal extended model, by U-Functionality there are no outgoing ψ-arrows at s, which is equivalent to $\mathcal{M}_s \models [\psi]\varphi$ for all φ by the semantics.

Let $M_s \models [\psi]\varphi$ and let $M_s \models \psi$. By Lemma 22 and the induction hypothesis we have that $\mathcal{M}_s \models \psi$, and by U-Functionality it holds that there is a unique $t \in S$ such that $s \xrightarrow{\psi} t$. By the induction hypothesis and Lemma 23, this is equivalent to $M_s^\psi \rightleftarrows M_t$. The latter means that for all $\varphi \in \mathcal{L}_{CoRGAL}$, $M_s^\psi \models \varphi$ if and only if $M_t \models \varphi$. By the induction hypothesis, $M_s^\psi \models \varphi$ if and only if $\mathcal{M}_t \models \varphi$. From $\mathcal{M}_s \models \psi$, $s \xrightarrow{\psi} t$, and $\mathcal{M}_t \models \varphi$ by the semantics and U-Functionality, we have that $\mathcal{M}_s \models [\psi]\varphi$.

Case $[G, \psi]\varphi$. ($\Rightarrow$). Assume that $\mathcal{M}_s \models [G, \psi]\varphi$. According to the definition of semantics, this is equivalent to $\forall\psi_G$: $\mathcal{M}_s \models \psi \wedge [\psi_G \wedge \psi]\varphi$. By Lemma 22 and the induction hypothesis, the latter is equivalent to $\forall\psi_G$: $M_s \models \psi \wedge [\psi_G \wedge \psi]\varphi$, which is $M_s \models [G, \psi]\varphi$ by the semantics.

($\Leftarrow$). Let $M_s \models [G, \psi]\varphi$. By the semantics this is equivalent to $M_s \models \psi$ and $M_s \models [\psi \wedge \psi_G]\varphi$ for all ψ_G. By Lemma 22 and the induction hypothesis, we have $\mathcal{M}_s \models \psi$ and $\mathcal{M}_s \models [\psi \wedge \psi_G]\varphi$, which is equivalent to $\mathcal{M}_s \models [G, \psi]\varphi$ by the semantics.

Case $[\langle G\rangle]\varphi$. ($\Rightarrow$). Assume that $\mathcal{M}_s \models [\langle G\rangle]\varphi$. According to the definition of semantics, this is equivalent to $\forall\psi_G$: $\mathcal{M}_s \models \langle A \setminus G, \psi_G\rangle\varphi$. By Lemma 22 and the induction hypothesis, the latter is equivalent to $\forall\psi_G$: $M_s \models \langle A \setminus G, \psi_G\rangle\varphi$, which is $M_s \models [\langle G\rangle]\varphi$ by the semantics.

($\Leftarrow$). Let $M_s \models [\langle G\rangle]\varphi$. By the semantics this is equivalent to $M_s \models \langle A \setminus G, \psi_G\rangle\varphi$ for all ψ_G. By Lemma 22 and the induction hypothesis, we have $\mathcal{M}_s \models \langle A \setminus G, \psi_G\rangle\varphi$, which is equivalent to $\mathcal{M}_s \models [\langle G\rangle]\varphi$ by the semantics. □ □

4.4 Completeness I: Theories and Lindenbaum Lemma

In order to prove the Lindenbaum Lemma, we expand and modify proofs from [Balbiani et al., 2008] and [Goldblatt, 1982, Chapter 2].

First, we prove a useful auxiliary lemma.

Lemma 25. Let $\varphi, \psi \in \mathcal{L}_{CoRGAL}$. If $\varphi \to \psi$ is a theorem, then $\eta(\varphi) \to \eta(\psi)$ is a theorem as well.

Proof. See Appendix B. □ □

Next, we introduce theories that will be used in the construction of the canonical model.

Definition 16 (Theory)**.** A set of formulas x is called a *theory* if and only if it contains CoRGAL, and is closed under $R0, R3$, and $R4$. A theory x is *consistent* if and only if there is no $\varphi \in \mathcal{L}_{CoRGAL}$ such that $\varphi \land \neg\varphi \in x$, and is *maximal* if and only if for all $\varphi \in \mathcal{L}_{CoRGAL}$ it holds that either $\varphi \in x$ or $\neg\varphi \in x$. The smallest theory is CoRGAL itself.

We will also use the following equivalent definition of the consistency of x. Theory x is consistent if and only if there is no φ such that $\varphi, \neg\varphi \in x$. To see that this definition is equivalent to the one in Definition 16 it is enough notice that $\varphi \to (\neg\varphi \to \varphi \land \neg\varphi) \in x$, $\varphi \land \neg\varphi \to \varphi \in x$, $\varphi \land \neg\varphi \to \neg\varphi \in x$, and x is closed under $R0$.

Note that theories are not closed under necessitation rules. The reason for this is that while these rules preserve validity, they do not preserve truth, whereas $R0, R3$, and $R4$ preserve both validity and truth.

Lemma 26. Let x be a theory, $\varphi, \psi \in \mathcal{L}_{CoRGAL}$, and $a \in A$. The following are theories: $x + \varphi = \{\psi \mid \varphi \to \psi \in x\}, K_a x = \{\varphi \mid K_a\varphi \in x\}$, and $[\varphi]x = \{\psi \mid [\varphi]\psi \in x\}$.

Proof. See Appendix B. □ □

Lemma 27. For any $\varphi \in \mathcal{L}_{CoRGAL}$ and any theory x, we have $x \subseteq x + \varphi$.

Proof. Let $\psi \in x$ for some $\psi \in \mathcal{L}_{CoRGAL}$. Since x is a theory, $\psi \to (\varphi \to \psi) \in x$. Moreover, $\varphi \to \psi \in x$ as x is closed under $R0$. By Lemma 26, $\psi \in x+\varphi$. □ □

Proposition 28. Let $\varphi \in \mathcal{L}_{CoRGAL}$ and x be a theory. Then $x+\varphi$ is consistent if and only if $\neg\varphi \notin x$.

Proof. ($\Rightarrow$). Suppose to the contrary that $x + \varphi$ is consistent and $\neg\varphi \in x$. First, note that since $\varphi \to \varphi \in x$, we have $\varphi \in x + \varphi$. Then observe that $\neg\varphi \in x + \varphi$ by Lemma 27. Since $x + \varphi$ contains all propositional tautologies, $\varphi \to (\neg\varphi \to \varphi \land \neg\varphi) \in x + \varphi$. Applying $R0$ twice, we yield $\varphi \land \neg\varphi \in x + \varphi$, which contradicts $x + \varphi$ being consistent.

($\Leftarrow$). Let us consider the contrapositive: if $x+\varphi$ is inconsistent, then $\neg\varphi \in x$. Since $x + \varphi$ is inconsistent, there is some ψ such that $\psi \land \neg\psi \in x + \varphi$. Since $\psi \land \neg\psi \to \neg\varphi$ is a propositional tautology and $x + \varphi$ is closed under $R0$, it holds that $\neg\varphi \in x + \varphi$. From the latter it follows that $\varphi \to \neg\varphi \in x$. Finally, $(\varphi \to \neg\varphi) \to \neg\varphi \in x$, and hence $\neg\varphi \in x$. □ □

The following proposition is a variation of the Lindenbaum Lemma. In order to prove it, we rely heavily on rules of inference $R3$ and $R4$.

Lemma 29 (Lindenbaum)**.** Every consistent theory x can be extended to a maximal consistent theory y.

Proof. Let $\psi_0, \psi_1, \ldots$ be an enumeration of formulas of the language, and let $y_0 = x$. Suppose that for some $n \geq 0$, y_n is a consistent theory, and $x \subseteq y_n$. If $y_n + \psi_n$ is consistent (i.e. if $\neg\psi_n \notin y_n$ by Proposition 28), then $y_{n+1} = y_n + \psi_n$. Otherwise, if ψ_n is not a conclusion of either $R3$ or $R4$, $y_{n+1} = y_n$.

If ψ_n is a conclusion of $R3$, that is if ψ_n is of the form $\eta([G,\chi]\varphi)$, then $y_{n+1} = y_n + \neg\eta(\chi \wedge [\psi_G \wedge \chi]\varphi)$, where $\neg\eta(\chi \wedge [\psi_G \wedge \chi]\varphi)$ is the first formula in the enumeration such that $\eta(\chi \wedge [\psi_G \wedge \chi]\varphi) \notin y_n$. Theory y_{n+1} is consistent due to the fact that if $\neg\eta([G,\chi]\varphi) \in y_n$, then there must exist some ψ_G such that $\eta(\chi \wedge [\psi_G \wedge \chi]\varphi) \notin y_n$, for otherwise $R3$ would lead to $\eta([G,\chi]\varphi) \in y_n$, which contradicts the assumption that $\neg\eta([G,\chi]\varphi) \in y_n$ and consistency of y_n. We pick the first formula $\neg\eta(\chi \wedge [\psi_G \wedge \chi]\varphi)$ in the enumeration, and have that $\neg\eta(\chi \wedge [\psi_G \wedge \chi]\varphi) \in y_{n+1}$ and $\eta(\chi \wedge [\psi_G \wedge \chi]\varphi) \notin y_{n+1}$. Note that adding such a witness ψ_G corresponds to the semantics of relativised group announcements, i.e. for formula $\eta\{\langle G,\chi\rangle\neg\varphi\}$ we have ψ_G such that $\eta\{\chi \to \langle\psi_G \wedge \chi\rangle\neg\varphi\}$.

If ψ_n is a conclusion of $R4$, that is if ψ_n is of the form $\eta([\langle G\rangle]\varphi)$, then $y_{n+1} = y_n + \neg\eta(\langle A \setminus G, \psi_G\rangle\varphi)$, where $\neg\eta(\langle A \setminus G, \psi_G\rangle\varphi)$ is the first formula in the enumeration such that $\eta(\langle A \setminus G, \psi_G\rangle\varphi) \notin y_n$. Theory y_{n+1} is consistent due to the fact that if $\neg\eta([\langle G\rangle]\varphi) \in y_n$, then there must exist some ψ_G such that $\eta(\langle A \setminus G, \psi_G\rangle\varphi) \notin y_n$, for otherwise $R4$ would lead to $\eta([\langle G\rangle]\varphi) \in y_n$, which contradicts the assumption that $\neg\eta([\langle G\rangle]\varphi) \in y_n$ and consistency of y_n. We pick the first formula $\neg\eta(\langle A \setminus G, \psi_G\rangle\varphi)$ in the enumeration, and have that $\neg\eta(\langle A \setminus G, \psi_G\rangle\varphi) \in y_{n+1}$ and $\eta(\langle A \setminus G, \psi_G\rangle\varphi) \notin y_{n+1}$. Note that since for all $\chi_{A\setminus G}$: $\eta([A \setminus G, \psi_G]\varphi) \to \eta(\psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}]\varphi)$ are theorems, they and their contrapositions (due to Lemma 25) are already in y_n (because CoRGAL $\subseteq x \subseteq y_n$). Thus, adding $\neg\eta(\langle A \setminus G, \psi_G\rangle\varphi)$ to y_n adds all the $\neg\eta(\psi_G \to \langle\psi_G \wedge \chi_{A\setminus G}\rangle\varphi)$ for $\chi_{A\setminus G}$ as well. This satisfies the semantics of coalition announcements, i.e. for formula $\eta\{\langle\!\langle G\rangle\!\rangle\neg\varphi\}$ we have some ψ_G such that for all $\chi_{A\setminus G}$: $\eta\{\psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}]\neg\varphi\}$.

Next we need to show that $y = \bigcup_{n=0}^{\infty} y_n$ is a maximal consistent theory. First, observe that each y_n is a consistent theory. Theory y_0 is consistent because x is consistent. And y_{n+1} is consistent by its construction and the consistency of y_n: either $y_{n+1} = y_n + \psi_n$ is consistent, or, otherwise, if ψ_n is not a conclusion of the infinitary rules, then $y_{n+1} = y_n$ and hence y_{n+1} is consistent, or if ψ_n is a conclusion of the infinitary rules, then y_{n+1} is consistent by the argument presented in the above two paragraphs. Next, we argue that y is consistent, i.e. that there is no ψ such that $\psi \wedge \neg\psi \in y$. Suppose towards a contradiction that $\psi \wedge \neg\psi \in y$. This means that there is n, such that $\psi \wedge \neg\psi \in y_n$, which contradicts y_n being a consistent theory.

Now we argue that y is a theory, i.e. CoRGAL $\subset y$ (1), and y is closed under $R0$ (2), $R3$ (3), and $R4$ (4).

1. Since $x \subseteq y$, we have that CoRGAL $\subseteq x \subset y$.

2. Assume that $\psi \to \varphi \in y$, and $\psi \in y$. This means that there is n such that $\psi \to \varphi, \psi \in y_n$. The latter implies that $\varphi \in y_n$, and thus $\varphi \in y$. Therefore, y is closed under $R0$.

3. Let some ψ_n be $\eta([G,\chi]\varphi)$. By the construction of y_{n+1}, if $\neg\eta([G,\chi]\varphi) \notin y_n$ (or, equivalently, if $y_n + \eta([G,\chi]\varphi)$ is consistent), then $\eta([G,\chi]\varphi) \in y_{n+1}$, where $y_{n+1} = y_n + \eta([G,\chi]\varphi)$, and hence $\eta([G,\chi]\varphi) \in y$. Since CoRGAL $\subset y$ and y is closed under $R0$, this means that $\eta(\chi \wedge [\psi_G \wedge$

$\chi]\varphi) \in y$ for all ψ_G. If $\neg\eta([G,\chi]\varphi) \in y_n$, then there is ψ_G such that $\neg\eta(\chi \wedge [\psi_G \wedge \chi]\varphi) \in y_{n+1}$, and hence $\neg\eta(\chi \wedge [\psi_G \wedge \chi]\varphi) \in y$. By the consistency of y and Proposition 28, we have that $\eta(\chi \wedge [\psi_G \wedge \chi]\varphi) \notin y$, and therefore y is closed under $R3$.

4. Let some ψ_n be $\eta([\![G]\!]\varphi)$. By the construction of y_{n+1}, if $\neg\eta([\![G]\!]\varphi) \notin y_n$ (or, equivalently, if $y_n + \eta([\![G]\!]\varphi)$ is consistent), then $\eta([\![G]\!]\varphi) \in y_{n+1}$, where $y_{n+1} = y_n + \eta([\![G]\!]\varphi)$, and hence $\eta([\![G]\!]\varphi) \in y$. Since CoRGAL $\subset y$ and y is closed under $R0$, this means that $\eta(\langle A \setminus G, \psi_G\rangle\varphi) \in y$ for all ψ_G. If $\neg\eta([\![G]\!]\varphi) \in y_n$, then there is ψ_G such that $\neg\eta(\langle A \setminus G, \psi_G\rangle\varphi) \in y_{n+1}$, and hence $\neg\eta(\langle A \setminus G, \psi_G\rangle\varphi) \in y$. By the consistency of y and Proposition 28, we have that $\eta(\langle A \setminus G, \psi_G\rangle\varphi) \notin y$, and therefore y is closed under $R4$.

Finally, we show that y is maximal. For any formula ψ_n in the enumeration, we have that either $\neg\psi_n \notin y_n$, and thus $\psi_n \in y_{n+1} \subseteq y$, or we have that $\neg\psi_n \in y_n$ and hence $\neg\psi_n \in y$. Therefore, for any ψ_n we have that either $\psi_n \in y$ or $\neg\psi_n \in y$. □ □

4.5 Completeness II: Extended Truth Lemma

In the rest of the section we prove Truth Lemma for the extended canonical model, and argue that the model is normal. This allows us to show the completeness of CoRGAL with respect to the classic semantics using Theorem 24.

Definition 17 (Extended Canonical Model)**.** The *extended canonical model* $\mathcal{M}^C$ is a tuple $(S^C, \sim^C, V^C, \{\xrightarrow{\psi} \mid \psi \in \mathcal{L}_{CoRGAL}\})$, where set $S^C = \{x \mid x \text{ is a maximal consistent theory}\}$, $x \sim_a^C y$ if and only if $K_a x \subseteq y$, $V^C(p) = \{x \mid p \in x\}$, and $x \xrightarrow{\psi} y$ if and only if for all φ: $\varphi \in y$ implies $\langle\psi\rangle\varphi \in x$. We denote the standard canonical model $(S^C, \sim^C, V^C)$ by M^C.

Lemma 30. Let x and y be maximal consistent theories, $x \xrightarrow{\psi} y$ if and only if for all φ: $[\psi]\varphi \in x$ implies $\varphi \in y$.

Proof. ($\Rightarrow$). Let $x \xrightarrow{\psi} y$ and $[\psi]\varphi \in x$ for an arbitrary φ. Since x is a maximal consistent theory, we have that $\neg[\psi]\varphi \notin x$, which is equivalent to $\langle\psi\rangle\neg\varphi \notin x$. According to the definition of the extended canonical model, this implies that $\neg\varphi \notin y$. Since y is a maximal consistent theory, $\varphi \in y$.

($\Leftarrow$). Let $\forall\varphi$: $[\psi]\varphi \in x$ implies $\varphi \in y$. If $\varphi \in y$, then $\neg\varphi \notin y$. This implies that $[\psi]\neg\varphi \notin x$. By the maximality of x, we have that $\neg[\psi]\neg\varphi \in x$, which is equivalent to $\langle\psi\rangle\varphi \in x$. We yield $x \xrightarrow{\psi} y$ by the definition of the extended canonical model. □ □

Proposition 31 (Extended Truth Lemma)**.** For all $\varphi \in \mathcal{L}_{CoRGAL}$ and all maximal consistent theories x, $\mathcal{M}^C_x \models \varphi$ if and only if $\varphi \in x$.

Proof. The proof is by induction on the size of φ.

Base case. $\mathcal{M}^C_x \models p$ is equivalent to $p \in x$ by the definition of the extended canonical model.

Induction hypothesis. For all $\varphi <^{Size}_{[\cdot],[\langle\rangle]} \psi$ and all maximal consistent theories x, $\mathcal{M}^C_x \models \varphi$ if and only if $\varphi \in x$.

Case $\neg\varphi$. Let $\mathcal{M}^C_x \models \neg\varphi$. By the semantics this is equivalent to $\mathcal{M}^C_x \not\models \varphi$. By Lemma 22 and the induction hypothesis, $\mathcal{M}^C_x \not\models \varphi$ if and only if $\varphi \notin x$. Since x is a maximal consistent theory, $\varphi \notin x$ if and only if $\neg\varphi \in x$.

Case $\varphi \wedge \chi$. Let $\mathcal{M}^C_x \models \varphi \wedge \chi$. By the semantics this is equivalent to $\mathcal{M}^C_x \models \varphi$ and $\mathcal{M}^C_x \models \chi$. By Lemma 22 and the induction hypothesis, $\mathcal{M}^C_x \models \varphi$ and $\mathcal{M}^C_x \models \chi$ if and only if $\varphi \in x$ and $\chi \in x$. Since x is a maximal consistent theory, it contains $\varphi \wedge \chi \to \varphi$, $\varphi \wedge \chi \to \chi$, and $\varphi \to (\chi \to \varphi \wedge \chi)$, and is closed under $R0$. Thus $\varphi \in x$ and $\chi \in x$ if and only if $\varphi \wedge \chi \in x$.

Case $K_a\varphi$. ($\Rightarrow$) Let $\mathcal{M}^C_x \models K_a\varphi$. By the semantics this is equivalent to the fact that for all y: $x \sim^C_a y$ implies $\mathcal{M}_y \models \varphi$. According to the definition of the extended canonical model, and by the induction hypothesis and Lemma 22, this is equivalent to $\forall y \in S^C : K_a x \subseteq y$ implies $\varphi \in y$. We need to show that $K_a\varphi \in x$. Suppose towards a contradiction that $K_a\varphi \notin x$. This means that $K_a x + \neg\varphi$ is a consistent theory. By Lindenbaum Lemma it can be expanded to a maximal consistent theory z such that $z \in S^C$, $K_a x \subseteq z$ and $\neg\varphi \in z$. A contradiction.

($\Leftarrow$). Let $K_a\varphi \in x$, and let $x \sim^C_a y$ for some y. This means that $\varphi \in y$ by the definition of the extended canonical model. By Lemma 22 and the induction hypothesis, this is equivalent to $\mathcal{M}^C_y \models \varphi$. Since y was arbitrary, the latter holds for all y such that $x \sim^C_a y$, which is equivalent to $\mathcal{M}^C_x \models K_a\varphi$ by the semantics.

Case $[\psi]\varphi$. ($\Rightarrow$). Assume that $\mathcal{M}^C_x \models [\psi]\varphi$. By the semantics this means that for all $y \in S^C$: $x \xrightarrow{\psi} y$ implies $\mathcal{M}^C_y \models \varphi$. Let for some arbitrary y it be the case that $x \xrightarrow{\psi} y$. Hence, $\mathcal{M}^C_y \models \varphi$, which is equivalent to $\varphi \in y$ by the induction hypothesis and Lemma 22. By the definition of the extended canonical model, $x \xrightarrow{\psi} y$ is equivalent to the fact that for all φ: if $\varphi \in y$, then $\langle\psi\rangle\varphi \in x$. Thus, we have that $\langle\psi\rangle\varphi \in x$, which implies $[\psi]\varphi \in x$ due to the fact that x is a theory.

($\Leftarrow$). Let $[\psi]\varphi \in x$, and let $x \xrightarrow{\psi} y$ for some y. This means that $[\psi]\varphi \in x$ implies $\varphi \in y$ by Lemma 30. Thus, we have that $\varphi \in y$. By the induction hypothesis and Lemma 22, this is equivalent to $\mathcal{M}^C_y \models \varphi$. Since y was arbitrary, the latter holds for all y such that $x \xrightarrow{\psi} y$, which is equivalent to $\mathcal{M}^C_x \models [\psi]\varphi$ by the semantics.

Case $[G,\psi]\varphi$. Let $[G,\psi]\varphi \in x$. Since x is a theory and thus closed under $R3$, we have that $[G,\psi]\varphi \in x$ if and only if for all ψ_G: $\psi \wedge [\psi_G \wedge \psi]\varphi \in x$. By the induction hypothesis and Lemma 22, the latter is equivalent to $\mathcal{M}^C_x \models \psi \wedge [\psi_G \wedge \psi]\varphi$ for all ψ_G, and this is equivalent to $\mathcal{M}^C_x \models [G,\psi]\varphi$ by the semantics.

Case $[\langle G\rangle]\varphi$. Let $[\langle G\rangle]\varphi \in x$. Since x is a theory and thus closed under $R4$, we have that $[\langle G\rangle]\varphi \in x$ if and only if for all ψ_G: $\langle A \setminus G, \psi_G\rangle\varphi \in x$. By the induction hypothesis and Lemma 22, the latter is equivalent to $\mathcal{M}^C_x \models \langle A \setminus G, \psi_G\rangle\varphi$ for all ψ_G, and this is equivalent to $\mathcal{M}^C_x \models [\langle G\rangle]\varphi$ by the semantics. $\square$ $\square$

The next step in the proof is to show that the extended canonical model is normal, i.e. that it possesses properties of U-Functionality, U-Invariance, U-Zig, and U-Zag. But before that, we need one additional lemma.

Lemma 32. $\neg\psi \to [\psi]\varphi$ is a theorem of CoRGAL.

Proof. Consider the left-to-right direction of $A7$: $\langle\psi\rangle\neg\varphi \to \psi \wedge [\psi]\neg\varphi$. By propositional reasoning, the latter implies $\langle\psi\rangle\neg\varphi \to \psi$, and this is equivalent to $\neg\psi \to [\psi]\varphi$. □ □

In the following, Lemmas 33–38 are similar to the corresponding lemmas of Wang and Cao [2013]. The difference is that we use maximal consistent theories, in contrast to maximal consistent sets therein.

Lemma 33 (U-Invariance)**.** For all $x \in S^C$: if $x \xrightarrow{\psi} y$, then for all $p \in P$: $p \in x$ if and only if $p \in y$.

Proof. Assume that $x \xrightarrow{\psi} y$. If $p \in x$, then by $A6$ and $R0$ we have $[\psi]p \in x$. From Lemma 30 we have that $p \in y$. For the other direction, assume that $p \in y$. By the definition of the extended canonical model, we have that $\langle\psi\rangle p \in x$. The latter implies that $p \in x$ by the contraposition of $A6$. □ □

Lemma 34. Every state in $\mathcal{M}^C$ has at most one ψ-successor.

Proof. Suppose that some x has two ψ-successors y and z. Since y and z are two different maximal consistent theories, then there is some φ such that $\varphi \in y$ and $\neg\varphi \in z$. According to the definition of the extended canonical model, we have that $\langle\psi\rangle\varphi \in x$ and $\langle\psi\rangle\neg\varphi \in x$, which implies that $[\psi]\varphi \in x$ and $[\psi]\neg\varphi \in x$. Since $x \xrightarrow{\psi} z$ and by Lemma 30, we have that $\varphi \in z$ and $\neg\varphi \in z$, which contradicts z being a maximal consistent theory. □ □

Lemma 35. For all $\psi \in \mathcal{L}_{CoRGAL}$ and $x \in S^C$, if $\psi \in x$, then there is a unique y such that $x \xrightarrow{\psi} y$ and $y = \{\varphi \mid \langle\psi\rangle\varphi \in x\} = \{\varphi \mid [\psi]\varphi \in x\}$. If $\psi \notin x$, then x does not have ψ-successors.

Proof. Let $\psi \in x$. We need to show that y is a maximal consistent theory. First, note that as $y = \{\varphi \mid [\psi]\varphi \in x\} = [\psi]x$, it is a theory by Lemma 26. Assume towards a contradiction that y is inconsistent. This means that $p, \neg p \in y$. By the construction of y, this means that $[\psi]p, [\psi]\neg p \in x$. Taking into account that $\psi \in x$, we have that $\langle\psi\rangle p$ and $\langle\psi\rangle\neg p$ are both in x by $A7$. By the contraposition $A6$ we have that $p \in x$ and $\neg p \in x$, which contradicts x being a maximal consistent theory.

For maximality, suppose towards a contradiction that $\varphi, \neg\varphi \notin y$. By the construction of y, this means that $[\psi]\varphi, [\psi]\neg\varphi \notin x$. Since x is maximal, $\neg[\psi]\varphi \in x$. Also, from $[\psi]\neg\varphi \notin x$ we have that $\neg[\psi]\neg\varphi \in x$, which is equivalent to $\langle\psi\rangle\varphi \in x$. By $A7$ the latter implies $[\psi]\varphi \in x$, and together with $\neg[\psi]\varphi \in x$, this contradicts to x being a maximal consistent theory.

From the fact that $\psi \in x$, it follows that $\{\varphi \mid \langle\psi\rangle\varphi \in x\} = \{\varphi \mid [\psi]\varphi \in x\}$. By the construction of y and the definition of the extended canonical model, it follows that $x \xrightarrow{\psi} y$. Moreover, by Lemma 34 y is the unique ψ-successor of x.

Let $\psi \notin x$. Assume towards a contradiction that there is a maximal consistent theory y such that $x \xrightarrow{\psi} y$. Since x is a maximal consistent theory, we have that $\neg\psi \in x$. By Lemma 32, $\neg\psi \to [\psi](p \wedge \neg p) \in x$. From $[\psi](p \wedge \neg p) \in x$, $x \xrightarrow{\psi} y$, and Lemma 30, we conclude that $p \wedge \neg p \in y$, which contradicts to y being a maximal consistent theory. □ □

Lemma 36 (U-Functionality)**.** For any $\psi \in \mathcal{L}_{CoRGAL}$, if $\mathcal{M}^C_x \models \psi$, then there is a unique $y \in S^C$ such that $x \xrightarrow{\psi} y$. If $\mathcal{M}^C_x \not\models \psi$, then there are no outgoing ψ-arrows at x.

Proof. Suppose that for an arbitrary ψ we have that $\mathcal{M}^C_x \models \psi$. By Extended Truth Lemma this is equivalent to $\varphi \in x$, and by Lemma 35 x has a unique ψ-successor y.

If $\mathcal{M}^C_x \not\models \psi$, then, by Extended Truth Lemma, $\psi \notin x$, and by Lemma 35 x does not have ψ-successors. □ □

Lemma 37 (U-Zig)**.** For all $x, x', y, y' \in S^C$, if $x \xrightarrow{\psi} y$, $x \sim^C_a x'$, and $x' \xrightarrow{\psi} y'$, then $y \sim^C_a y'$.

Proof. Let $x \sim^C_a x'$, $x \xrightarrow{\psi} y$, and $x' \xrightarrow{\psi} y'$. Assume that for an arbitrary φ it holds that $K_a\varphi \in y$. We need to show that $\varphi \in y'$. From $x \xrightarrow{\psi} y$ and the definition of the extended canonical model, we conclude that $\langle\psi\rangle K_a\varphi \in x$. By $A8$ we also have that $K_a[\psi]\varphi \in x$. From $x \sim^C_a x'$ and the definition of the extended canonical model, it follows that $[\psi]\varphi \in x'$. Finally, by Lemma 30 and $x' \xrightarrow{\psi} y'$, we have that $\varphi \in y'$. □ □

Lemma 38 (U-Zag)**.** Let $x, y, y' \in S^C$. If $x \xrightarrow{\psi} y$ and $y \sim^C_a y'$, then there exists an x' such that $x \sim^C_a x'$ and $x' \xrightarrow{\psi} y'$.

Proof. Let $x \xrightarrow{\psi} y$ and $y \sim^C_a y'$. We need to show that there exists an x' such that $x \sim^C_a x'$ and $x' \xrightarrow{\psi} y'$. Let z be the closure of the set $\{\langle\psi\rangle\varphi \mid \varphi \in y'\} \cup \{\varphi \mid K_a\varphi \in x\}$ under $R0$, $R3$, and $R4$.

Obviously, z is a theory. We need to show that z is consistent.

Assume towards a contradiction that z is inconsistent. As $R0$, $R3$, and $R4$ are truth-preserving, this means that there are $\langle\psi\rangle\varphi_1, \ldots, \langle\psi\rangle\varphi_n \in \{\langle\psi\rangle\varphi \mid \varphi \in y'\}$ and $\phi_1, \ldots, \phi_m \in \{\varphi \mid K_a\varphi \in x\}$ such that $\langle\psi\rangle\varphi_1 \wedge \ldots \wedge \langle\psi\rangle\varphi_n \wedge \phi_1 \wedge \ldots \wedge \phi_m$ is inconsistent, i.e. $\vdash \langle\psi\rangle\varphi_1 \wedge \ldots \wedge \langle\psi\rangle\varphi_n \wedge \phi_1 \wedge \ldots \wedge \phi_m \to \chi \wedge \neg\chi$. By the propositional reasoning we have that $\phi_1 \wedge \ldots \wedge \phi_m \to \neg(\langle\psi\rangle\varphi_1 \wedge \ldots \wedge \langle\psi\rangle\varphi_n) \in$ CoRGAL, and hence $K_a(\phi_1 \wedge \ldots \wedge \phi_m) \to K_a(\neg(\langle\psi\rangle\varphi_1 \wedge \ldots \wedge \langle\psi\rangle\varphi_n)) \in$ CoRGAL by $R1$, $A1$, and $R0$. Since x is a theory, $K_a(\phi_1 \wedge \ldots \wedge \phi_m) \to K_a(\neg(\langle\psi\rangle\varphi_1 \wedge \ldots \wedge \langle\psi\rangle\varphi_n)) \in x$. From the fact that $K_a\varphi \wedge K_a\psi \leftrightarrow K_a(\varphi \wedge \psi)$ is a theorem of EL, it follows that $K_a(\phi_1 \wedge \ldots \wedge \phi_m) \in x$. Hence, $K_a(\neg(\langle\psi\rangle\varphi_1 \wedge \ldots \wedge \langle\psi\rangle\varphi_n)) \in x$. By

Lemma 21, this is equivalent to $K_a(\neg\langle\psi\rangle(\varphi_1 \wedge \ldots \wedge \varphi_n)) \in x$ and $K_a[\psi]\neg(\varphi_1 \wedge \ldots \wedge \varphi_n) \in x$. By $A9$ we have that $[\psi]K_a\neg(\varphi_1 \wedge \ldots \wedge \varphi_n) \in x$. From $x \xrightarrow{\psi} y$ and Lemma 30, we can infer that $K_a\neg(\varphi_1 \wedge \ldots \wedge \varphi_n) \in y$. Since $y \sim_a^C y'$, we also have that $\neg(\varphi_1 \wedge \ldots \wedge \varphi_n) \in y'$, which contradicts $\varphi_1, \ldots, \varphi_n \in y'$.

Since z is a consistent theory, it can be extended to a maximal consistent theory x' by the Lindenbaum Lemma, and $x \sim_a^C x'$ and $x' \xrightarrow{\psi} y'$ by the construction of z. □ □

From Lemmas 33, 36, 37, and 38 it follows that the extended canonical model is normal.

Proposition 39. $\mathcal{M}^C$ is normal.

Finally, we prove the completeness of CoRGAL.

Theorem 40 (Completeness w.r.t. classic semantics)**.** For all $\varphi \in \mathcal{L}_{CoRGAL}$, if φ is valid , then $\varphi \in$ CoRGAL.

Proof. Suppose towards a contradiction that φ is valid, and $\varphi \notin$ CoRGAL. Since CoRGAL is a consistent theory, from Lemma 26 and Proposition 28 it follows that CoRGAL$+\neg\varphi$ is a consistent theory. By the Lindenbaum Lemma, there is a maximal consistent theory x such that CoRGAL $+ \neg\varphi \subseteq x$. By Lemma 26, $\neg\varphi \in$ CoRGAL $+ \neg\varphi$, and hence $\varphi \notin x$. From Extended Truth Lemma we can infer that $\mathcal{M}_x^C \not\models \varphi$, and from Proposition 39 and Theorem 24 it follows that $M_x^C \not\models \varphi$, which contradicts φ being a validity w.r.t. to classic semantics. □ □

5 Conclusion

We presented CoRGAL, the logic of coalition announcements with added relativised group announcements. The new operators allowed us to tackle the double quantification in coalition announcement operators. The completeness of CoRGAL was shown using an alternative technique where public announcements are static transitions rather than dynamic updates. Let us note that CoRGAL and its completeness proof can be straightforwardly modified to yield a complete axiomatisation of *relativised group announcement logic* without coalition announcements. We also considered various (in)validities of GAL and CAL some of which were mentioned as open problems in the literature. We hope that this sheds more light on coalition announcements and how they are related to group announcements.

Arguably the most intriguing open problem in the area is a complete and sound axiomatisation of CAL. Another interesting question is whether there exist finitary axiomatisations of APAL, GAL, CAL, and CoRGAL. The negative answer has been given to a somewhat related arbitrary arrow update logic with common knowledge (AAULC) [Kuijer, 2017]. Another related result has been presented in [van Ditmarsch and French, 2017], where the authors considered a restriction of APAL to arbitrary announcements of boolean formulas only. The resulting logic, boolean arbitrary public announcement logic, has a complete finitary axiomatisation. The positive answer was also obtained for APAL with Memory (APALM) [Baltag et al., 2018], where epistemic models are extended with the memory of the initial configuration, and special operators of APALM can access it.

A host of open research questions deal with extending the logics of quantified announcements with group knowledge modalities. These modalities include common knowledge [Vanderschraaf and Sillari, 2014], distributed knowledge [Wáng and Ågotnes, 2013], relativised common knowledge [van Benthem et al., 2006], and resolution [Ågotnes and Wáng, 2017]. GAL with distributed knowledge was studied in [Galimullin et al., 2019]. To the best of our knowledge, the closest work that touches upon the problem of expanding a logic of quantified epistemic actions with common knowledge is [Kuijer, 2017]. However, an axiomatisation of AAULC is an open problem.

Finally, relativised group announcements are interesting in their own right. They are used to show an expressivity result in [French et al., 2019], and further exploration of their possible use is an exciting avenue of future research.

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Appendix A: Coalition Annoucement Logic Subsumes Coalition Logic

It is known [Ågotnes and van Ditmarsch, 2008] that CAL subsumes CL, i.e. all axioms of CL are valid in CAL, and rules of inference of CL are validity preserving in CAL. However, to the best of our knowledge, a formal proof has not yet been presented.

Proposition 41. All of the following are valid and validity preserving in CAL.

$$
\begin{array}{ll}
(C0) & \text{all instantiations of propositional tautologies,}\\
(C1) & \neg\langle\!\langle G\rangle\!\rangle\bot,\\
(C2) & \langle\!\langle G\rangle\!\rangle\top,\\
(C3) & \neg\langle\!\langle\emptyset\rangle\!\rangle\neg\varphi \to \langle\!\langle A\rangle\!\rangle\varphi,\\
(C4) & \langle\!\langle G\rangle\!\rangle(\varphi_1 \wedge \varphi_2) \to \langle\!\langle G\rangle\!\rangle\varphi_1,\\
(C5) & \langle\!\langle G\rangle\!\rangle\varphi_1 \wedge \langle\!\langle H\rangle\!\rangle\varphi_2 \to \langle\!\langle G \cup H\rangle\!\rangle(\varphi_1 \wedge \varphi_2),\ \text{if } G \cap H = \emptyset,\\
(R0) & \vdash \varphi, \varphi \to \psi \Rightarrow \vdash \psi,\\
(R1) & \vdash \varphi \leftrightarrow \psi \Rightarrow \vdash \langle\!\langle G\rangle\!\rangle\varphi \leftrightarrow \langle\!\langle G\rangle\!\rangle\psi.
\end{array}
$$

Proof. $C0$ and $R0$ are obvious.

$C1$: It holds that $\models \top$, and $\top$ is true in every restriction of a model, i.e. $\models [\psi]\top$. In particular, for some model M_s and all true formulas ψ_G and $\chi_{A\setminus G}$: $M_s \models \langle\psi_G \wedge \chi_{A\setminus G}\rangle\top$. We can relax the requirement of ψ_G being true by adding the formula as an antecedent. Formally, for all (true and false) ψ_G and some (true) $\chi_{A\setminus G}$: $M_s \models \psi_G \to \langle\psi_G \wedge \chi_{A\setminus G}\rangle\top$. The latter is $M_s \models [\langle G\rangle]\top$ by the semantics, and this is equivalent to $M_s \models \neg\langle\!\langle G\rangle\!\rangle\bot$ by the duality of the coalition announcement operators.

$C2$: For any pointed model M_s and any announcement $\psi_G \wedge \chi_{A\setminus G}$ it holds that $M_s \models [\psi_G \wedge \chi_{A\setminus G}]\top$. The latter implies that for some true ψ_G and for all $\chi_{A\setminus G}$: $M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}]\top$, which is $M_s \models \langle\!\langle G\rangle\!\rangle\top$ by the semantics.

$C3$: Let $\neg\langle\!\langle\emptyset\rangle\!\rangle\neg\varphi$ be true in some arbitrary pointed model M_s. This is equivalent to $\exists\psi_A$: $M_s \models \neg[\psi_A]\neg\varphi$, which is $M_s \models \langle\!\langle A\rangle\!\rangle\varphi$ by the semantics.

$C4$: Suppose that for some M_s, $M_s \models \langle\!\langle G\rangle\!\rangle(\varphi_1 \wedge \varphi_2)$ holds. By the semantics, $\exists\psi_G, \forall\chi_{A\setminus G}$: $M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}](\varphi_1 \wedge \varphi_2)$. Then, by the axiom of PAL $[\psi](\varphi \wedge \chi) \leftrightarrow [\psi]\varphi \wedge [\psi]\chi$, we have $\exists\psi_G, \forall\chi_{A\setminus G}$: $M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}]\varphi_1 \wedge [\psi_G \wedge \chi_{A\setminus G}]\varphi_2$. The latter implies $\exists\psi_G, \forall\chi_{A\setminus G}$: $M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}]\varphi_1$, which is $M_s \models \langle\!\langle G\rangle\!\rangle\varphi_1$ by the semantics.

$C5$: Assume that for some M_s we have that $M_s \models \langle\!\langle G\rangle\!\rangle\varphi_1 \wedge \langle\!\langle H\rangle\!\rangle\varphi_2$. Let us consider the first conjunct $M_s \models \langle\!\langle G\rangle\!\rangle\varphi_1$. By the semantics it is equivalent to $\exists\psi_G, \forall\chi_{A\setminus G}$: $M_s \models \psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}]\varphi_1$. Since $G \cap H = \emptyset$, we can split $\chi_{A\setminus G}$ into χ_H and $\chi_{A\setminus G\cup H}$. Thus we have that $\exists\psi_G, \forall\chi_H, \forall\chi_{A\setminus(G\cup H)}$: $M_s \models \psi_G \wedge [\psi_G \wedge \chi_H \wedge \chi_{A\setminus(G\cup H)}]\varphi_1$. The same holds for the second conjunct: $\exists\psi_H, \forall\chi_G, \forall\chi_{A\setminus(G\cup H)}$: $M_s \models \psi_H \wedge [\psi_H \wedge \chi_G \wedge \chi_{A\setminus(G\cup H)}]\varphi_2$. Since χ_H (χ_G) quantifies over all formulas known to H (G), we can substitute χ_H (χ_G) with ψ_H (ψ_G). Hence we have

$$\exists\psi_G, \exists\psi_H, \forall\chi_{A\setminus(G\cup H)}:$$

$$M_s \models \psi_G \wedge \psi_H \wedge [\psi_G \wedge \psi_H \wedge \chi_{A\setminus(G\cup H)}]\varphi_1 \wedge [\psi_G \wedge \psi_H \wedge \chi_{A\setminus G\cup H}]\varphi_2.$$

By the axiom of PAL $[\psi](\varphi \wedge \chi) \leftrightarrow [\psi]\varphi \wedge [\psi]\chi$, we have that

$$\exists\psi_G, \exists\psi_H, \forall\chi_{A\setminus(G\cup H)} : M_s \models \psi_G \wedge \psi_H \wedge [\psi_G \wedge \psi_H \wedge \chi_{A\setminus(G\cup H)}](\varphi_1 \wedge \varphi_2),$$

and the latter is equivalent to $M_s \models \langle\!\langle G \cup H\rangle\!\rangle(\varphi_1 \wedge \varphi_2)$ by the semantics.

$R1$: Assume that $\models \varphi \leftrightarrow \psi$. This means that for any pointed model M_s the following holds: $M_s \models \varphi$ if and only if $M_s \models \psi$ (1). Now suppose that for some pointed model N_t it holds that $N_t \models \langle\!\langle G\rangle\!\rangle\varphi$. By the semantics, $\exists\psi_G, \forall\chi_{A\setminus G}$: $N_t \models \psi_G \wedge [\psi_G \wedge \chi_{A\setminus G}]\varphi$, which is equivalent to the following: $N_t \models \psi_G$ and ($N_t \models \psi_G \wedge \chi_{A\setminus G}$ implies $N_t^{\psi_G \wedge \chi_{A\setminus G}} \models \varphi$). By (1) we have that $\exists\psi_G, \forall\chi_{A\setminus G}$: $N_t \models \psi_G$ and ($N_t \models \psi_G \wedge \chi_{A\setminus G}$ implies $N_t^{\psi_G \wedge \chi_{A\setminus G}} \models \psi$), which is $N_t \models \langle\!\langle G\rangle\!\rangle\psi$ by the semantics. Since N_t was arbitrary, we have that $\models \langle\!\langle G\rangle\!\rangle\varphi \to \langle\!\langle G\rangle\!\rangle\psi$. The same argument holds in the other direction. □ □

Appendix B: Proofs from Section 4

Proposition 17. $R3$ and $R4$ are truth-preserving.

Proof. ($R3$) *Base case.* If for all ψ_G we have that $M_s \models \chi \wedge [\psi_G \wedge \chi]\varphi$, then this is equivalent to $M_s \models [G, \chi]\varphi$ by the semantics.

Induction Hypothesis. If for some M_s it holds that $M_s \models \eta(\chi \wedge [\psi_G \wedge \chi]\varphi)$ for all ψ_G, then $M_s \models \eta([G, \chi]\varphi)$.

Case $\forall\psi_G$: $\tau \to \eta(\chi \wedge [\psi_G \wedge \chi]\varphi)$ for some $\tau \in \mathcal{L}_{CoRGAL}$. This means that $M_s \models \neg\tau$ or $M_s \models \eta(\chi \wedge [\psi_G \wedge \chi]\varphi)$. By the induction hypothesis we have that $M_s \models \neg\tau$ or $M_s \models \eta([G, \chi]\varphi)$, which is equivalent to $M_s \models \tau \to \eta([G, \chi]\varphi)$.

Case $\forall\psi_G$: $K_a\eta(\chi \wedge [\psi_G \wedge \chi]\varphi)$ for some $a \in A$. By semantics we have that for every $t \in S$: $s \sim_a t$ implies $M_t \models \eta(\chi \wedge [\psi_G \wedge \chi]\varphi)$. By the induction hypothesis we conclude that for every $t \in S$: $s \sim_a t$ implies $M_t \models \eta([G,\chi]\varphi)$, which is equivalent to $M_s \models K_a\eta([G,\chi]\varphi)$.

Case $\forall\psi_G$: $[\tau]\eta(\chi\wedge[\psi_G\wedge\chi]\varphi)$ for some $\tau \in \mathcal{L}_{CoRGAL}$. This means that $M_s \models \tau$ implies $M_s^\tau \models \eta(\chi \wedge [\psi_G \wedge \chi]\varphi)$. By the induction hypothesis we have that $M_s \models \tau$ implies $M_s^\tau \models \eta([G,\chi]\varphi)$, which is equivalent to $M_s \models [\tau]\eta([G,\chi]\varphi)$.

(*R4*) *Base case.* If for all ψ_G we have that $M_s \models \langle A \setminus G, \psi_G\rangle\varphi$, then this is equivalent to $M_s \models [\langle G\rangle]\varphi$ by the semantics.

Induction Hypothesis. If for some M_s it holds that $M_s \models \eta(\langle A \setminus G, \psi_G\rangle\varphi)$ for all ψ_G, then $M_s \models \eta([\langle G\rangle]\varphi)$.

Case $\forall\psi_G$: $\tau \to \eta(\langle A \setminus G, \psi_G\rangle\varphi)$ for some $\tau \in \mathcal{L}_{CoRGAL}$. This means that $M_s \models \neg\tau$ or $M_s \models \eta(\langle A \setminus G, \psi_G\rangle\varphi)$. By the induction hypothesis we have that $M_s \models \neg\tau$ or $M_s \models \eta([\langle G\rangle]\varphi)$, which is equivalent to $M_s \models \tau \to \eta([\langle G\rangle]\varphi)$.

Case $\forall\psi_G$: $K_a\eta(\langle A \setminus G, \psi_G\rangle\varphi)$ for some $a \in A$. By semantics we have that for every $t \in S$: $s \sim_a t$ implies $M_t \models \eta(\langle A \setminus G, \psi_G\rangle\varphi)$. By the induction hypothesis we conclude that for every $t \in S$: $s \sim_a t$ implies $M_t \models \eta([\langle G\rangle]\varphi)$, which is equivalent to $M_s \models K_a\eta([\langle G\rangle]\varphi)$.

Case $\forall\psi_G$: $[\tau]\eta(\langle A \setminus G, \psi_G\rangle\varphi)$ for some $\tau \in \mathcal{L}_{CoRGAL}$. This means that $M_s \models \tau$ implies $M_s^\tau \models \eta(\langle A \setminus G, \psi_G\rangle\varphi)$. By the induction hypothesis we have that $M_s \models \tau$ implies $M_s^\tau \models \eta([\langle G\rangle]\varphi)$, which is equivalent to $M_s \models [\tau]\eta([\langle G\rangle]\varphi)$.
□ □

Lemma 22. Let $\varphi, \psi, \chi \in \mathcal{L}_{CoRGAL}$. The following inequalities hold.

1. $\varphi <^{Size}_{[,],[\langle\rangle]} \neg\varphi$,
2. $\varphi <^{Size}_{[,],[\langle\rangle]} \varphi \wedge \chi$,
3. $\varphi <^{Size}_{[,],[\langle\rangle]} K_a\varphi$,
4. $\varphi <^{Size}_{[,],[\langle\rangle]} [\psi]\varphi$ and $\psi <^{Size}_{[,],[\langle\rangle]} [\psi]\varphi$,
5. $\chi \wedge [\psi_G \wedge \chi]\varphi <^{Size}_{[,],[\langle\rangle]} [G,\chi]\varphi$,
6. $\langle A \setminus G, \psi_G\rangle\varphi <^{Size}_{[,],[\langle\rangle]} [\langle G\rangle]\varphi$.

Proof. The proof is straightforward. We just show cases 5 and 6.

5. Note that $[\langle\rangle]$-depth for both sides of the inequality is the same and equals $d_{[\langle\rangle]}(\chi) + d_{[\langle\rangle]}(\varphi)$. In particular, we have the following for the left-hand side: $d_{[\langle\rangle]}(\chi \wedge [\psi_G \wedge \chi]\varphi) = \max\{d_{[\langle\rangle]}(\chi), d_{[\langle\rangle]}([\psi_G \wedge \chi]\varphi)\} = d_{[\langle\rangle]}([\psi_G \wedge \chi]\varphi) = d_{[\langle\rangle]}(\psi_G \wedge \chi) + d_{[\langle\rangle]}(\varphi) = \max\{d_{[\langle\rangle]}(\psi_G), d_{[\langle\rangle]}(\chi)\} + d_{[\langle\rangle]}(\varphi) = d_{[\langle\rangle]}(\chi) + d_{[\langle\rangle]}(\varphi)$. For the right-hand side we have that $d_{[\langle\rangle]}([G,\chi]\varphi) = d_{[\langle\rangle]}(\chi) + d_{[\langle\rangle]}(\varphi)$.

Since $[\langle\rangle]$-depths are the same, we calculate $[,]$-depths. For the left-hand side we have that $d_{[,]}(\chi \wedge [\psi_G \wedge \chi]\varphi) = d_{[,]}(\chi) + d_{[,]}(\varphi)$. In particular, $d_{[,]}(\chi \wedge [\psi_G \wedge \chi]\varphi) = \max\{d_{[,]}(\chi), d_{[,]}([\psi_G \wedge \chi]\varphi)\} = d_{[,]}([\psi_G \wedge \chi]\varphi) = d_{[,]}(\psi_G \wedge \chi) + d_{[,]}(\varphi) = \max\{d_{[,]}(\psi_G), d_{[,]}(\chi)\} + d_{[,]}(\varphi) = d_{[,]}(\chi) + d_{[,]}(\varphi)$. Depth of the right-hand side formula is $d_{[,]}([G,\chi]\varphi) = 1 + d_{[,]}(\varphi) + d_{[,]}(\chi)$. Hence, $\chi \wedge [\psi_G \wedge \chi]\varphi <^{Size}_{[,],[\langle\rangle]} [G,\chi]\varphi$.

6. On the left-hand side we have that $d_{[\langle\rangle]}(\langle A \setminus G, \psi_G\rangle\varphi) = d_{[\langle\rangle]}(\varphi)$, and on the right-hand side the depth is $d_{[\langle\rangle]}[\langle G\rangle]\varphi = d_{[\langle\rangle]}(\varphi) + 1$. Hence, $\langle A \setminus G, \psi_G\rangle\varphi <^{Size}_{[,],[\langle\rangle]} [\langle G\rangle]\varphi$. □ □

Lemma 25. Let $\varphi, \psi \in \mathcal{L}_{CoRGAL}$. If $\varphi \to \psi$ is a theorem, then $\eta(\varphi) \to \eta(\psi)$ is a theorem as well.

Proof. Assume that $\varphi \to \psi$ is a theorem. We prove the lemma by induction on η.

Base case $\eta := \sharp$. Formula $\varphi \to \psi$ is a theorem by assumption.

Induction Hypothesis. Assume that for some η, $\eta(\varphi) \to \eta(\psi)$ is a theorem.

Case $(\tau \to \eta(\varphi)) \to (\tau \to \eta(\psi))$ for some $\tau \in \mathcal{L}_{CoRGAL}$. Formula $(\eta(\varphi) \to \eta(\psi)) \to ((\tau \to \eta(\varphi)) \to (\tau \to \eta(\psi)))$ is a propositional tautology, and, hence, a theorem of CoRGAL. Using the induction hypothesis and $R0$, we have that $(\tau \to \eta(\varphi)) \to (\tau \to \eta(\psi))$ is a theorem.

Case $K_a\eta(\varphi) \to K_a\eta(\psi)$ for some $a \in A$. Since $\eta(\varphi) \to \eta(\psi)$ is a theorem by the induction hypothesis, $K_a(\eta(\varphi) \to \eta(\psi))$ is also a theorem by $R1$. Next, $K_a(\eta(\varphi) \to \eta(\psi)) \to (K_a\eta(\varphi) \to K_a\eta(\psi))$ is an instance of $A1$, and, hence, a theorem. Finally, using $R0$ we have that $K_a\eta(\varphi) \to K_a\eta(\psi)$ is a theorem.

Case $[\tau]\eta(\varphi) \to [\tau]\eta(\psi)$ for some $\tau \in \mathcal{L}_{CoRGAL}$. Since $\eta(\varphi) \to \eta(\psi)$ is a theorem by the induction hypothesis, $[\tau](\eta(\varphi) \to \eta(\psi))$ is also a theorem by $R2$. Formula $[\tau](\eta(\varphi) \to \eta(\psi)) \to ([\tau]\eta(\varphi) \to [\tau]\eta(\psi))$ is implied by axiom schema $A5$. Using $R0$ we can conclude that $[\tau]\eta(\varphi) \to [\tau]\eta(\psi)$ is a theorem of CoRGAL. □ □

Lemma 26. Let x be a theory, $\varphi, \psi \in \mathcal{L}_{CoRGAL}$, and $a \in A$. The following are theories: $x + \varphi = \{\psi \mid \varphi \to \psi \in x\}, K_a x = \{\varphi \mid K_a\varphi \in x\}$, and $[\varphi]x = \{\psi \mid [\varphi]\psi \in x\}$.

Proof. Let ψ be a theorem, i.e. $\psi \in$ CoRGAL. Then $\varphi \to \psi$ is also a theorem, since $\psi \to (\varphi \to \psi) \in$ CoRGAL and CoRGAL is closed under $R0$. Moreover, $K_a\psi$ and $[\varphi]\psi$ are theorems as well due to the fact that CoRGAL is closed under $R1$ and $R2$. Therefore, $\psi \in x + \varphi$, $\psi \in K_a x$, and $\psi \in [\varphi]x$, and hence CoRGAL $\subseteq x + \varphi, K_a x, [\varphi]x$.

The rest of the proof is an extension of the one from [Balbiani et al., 2008], where it was shown that $x + \varphi$, $K_a x$, and $[\varphi]x$ are closed under $R0$. We argue that corresponding sets are closed under $R3$ and $R4$.

Case $x + \varphi$. Suppose that $\eta(\chi \wedge [\psi_G \wedge \chi]\tau) \in x + \varphi$ for some given χ, for all ψ_G, and for some $\tau \in \mathcal{L}_{CoRGAL}$. This means that $\varphi \to \eta(\chi \wedge [\psi_G \wedge \chi]\tau) \in x$ for all ψ_G. Since $\varphi \to \eta(\chi \wedge [\psi_G \wedge \chi]\tau)$ is a necessity form, and x is closed under $R3$ (by Definition 16), we infer that $\varphi \to \eta([G, \chi]\tau) \in x$, and, consequently, $\eta([G, \chi]\tau) \in x + \varphi$. So, $x + \varphi$ is closed under $R3$.

Now, let $\forall \psi_G$: $\eta(\langle A \setminus G, \psi_G\rangle\tau) \in x + \varphi$. By the definition of $x + \varphi$ this means that $\varphi \to \eta(\langle A \setminus G, \psi_G\rangle\tau) \in x$ for all ψ_G. Since $\varphi \to \eta(\langle A \setminus G, \psi_G\rangle\tau$ is a necessity form and x is closed under $R4$, we infer that $\varphi \to \eta([\![G]\!]\tau) \in x$, and, consequently, $\eta([\![G]\!]\tau) \in x + \varphi$. So, $x + \varphi$ is closed under $R4$.

Case $K_a x$. Suppose that $\eta(\chi \wedge [\psi_G \wedge \chi]\tau) \in K_a x$ for some given χ, for all ψ_G, and for some $\tau \in \mathcal{L}_{CoRGAL}$. This means that $K_a\eta(\chi \wedge [\psi_G \wedge \chi]\tau) \in x$ for all ψ_G. Since $K_a\eta(\chi \wedge [\psi_G \wedge \chi]\tau)$ is a necessity form, and x is closed under $R3$ (by Definition 16), we infer that $K_a\eta([G, \chi]\tau) \in x$, and, consequently, $\eta([G, \chi]\tau) \in K_a x$. So, $K_a x$ is closed under $R3$.

Now, let $\forall \psi_G$: $\eta(\langle A \setminus G, \psi_G\rangle\tau) \in K_a x$. By the definition of $K_a x$ this means that $K_a\eta(\langle A \setminus G, \psi_G\rangle\tau) \in x$ for all ψ_G. Since $K_a\eta(\langle A \setminus G, \psi_G\rangle\tau$ is a necessity form and x is closed under $R4$, we infer that $K_a\eta([\![G\rangle\!]\tau) \in x$, and, consequently, $\eta([\![G\rangle\!]\tau) \in K_a x$. So, $K_a x$ is closed under $R4$.

Case $[\varphi]x$. Finally, suppose that $\eta(\chi \wedge [\psi_G \wedge \chi]\tau) \in [\varphi]x$ for some given χ, for all ψ_G, and for some $\tau \in \mathcal{L}_{CoRGAL}$. This means that $[\varphi]\eta(\chi \wedge [\psi_G \wedge \chi]\tau) \in x$ for all ψ_G. Since $[\varphi]\eta(\chi \wedge [\psi_G \wedge \chi]\tau)$ is a necessity form, and x is closed under $R3$ (by Definition 16), we infer that $[\varphi]\eta([G, \chi]\tau) \in x$, and, consequently, $\eta([G, \chi]\tau) \in [\varphi]x$. So, $[\varphi]x$ is closed under $R3$.

Now, let $\forall\psi_G$: $\eta(\langle A \setminus G, \psi_G\rangle\tau) \in [\varphi]x$. By the definition of $[\varphi]x$ this means that $[\varphi]\eta(\langle A \setminus G, \psi_G\rangle\tau) \in x$ for all ψ_G. Since $[\varphi]\eta(\langle A \setminus G, \psi_G\rangle\tau$ is a necessity form and x is closed under $R4$, we infer that $[\varphi]\eta([\![G\rangle\!]\tau) \in x$, and, consequently, $\eta([\![G\rangle\!]\tau) \in [\varphi]x$. So, $[\varphi]x$ is closed under $R4$. $\square$ $\square$