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Relative Persistent Homology

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Abstract

The alpha complex efficiently computes persistent homology of a point cloud X in Euclidean space when the dimension d is low. Given a subset A of X, relative Čech persistent homology can be computed as the persistent homology of the relative Čech complex $\check{C}(X, A)$. However, this is not computationally feasible for larger point clouds X. The aim of this note is to present a method for efficient computation of relative Čech persistent homology in low dimensional Euclidean space. We introduce the relative Delaunay–Čech complex $\mathrm{Del}\check{C}(X, A)$ whose homology is the relative Čech persistent homology. It is constructed from the Delaunay complex of an embedding of X in $(d+1)$-dimensional Euclidean space.

Keywords Topological data analysis · Relative homology · Delaunay–Čech complex · Alpha complex

Mathematics Subject Classification 62R40 · 55N31 · 55U05

1 Introduction

Persistent homology is receiving growing attention in the machine learning community. In that light, the scalability of persistent homology computations is of increasing importance. To date, the alpha complex is the most widely used method to compute persistent homology for large low-dimensional data sets.

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Relative persistent homology has been considered several times in recent years. For example, Edelsbrunner and Harer [8] have presented an application of relative persistent homology to estimate the dimension of an embedded manifold. Relative persistent homology is also a way to introduce the concept of extended persistence [6]. De Silva and others have shown that the relative persistent homology of the union K_∞ of a filtered simplicial complex $\{K_t\}_{t\geq 0}$ with respect to the simplicial complexes K_t and its corresponding relative persistent cohomology have the same barcode [13]. They also show that absolute persistent homology of K_t can be computed from this relative persistent homology. More recently, Pokorny and others [10] have used relative persistent homology to cluster two-dimensional trajectories. Some software, such as PHAT [2], even allows for the direct computation of relative persistent homology. For an example see the PHAT github repository.

Despite the fact that relative persistent homology has been considered in many different situations, we are not aware of a relative version of the alpha- or Delaunay–Čech complexes being used before the conference version of this paper [4]. Since then, Reani and Bobrowski [11] described the relative alpha complex with similar methods.

Our contributions are as follows.

1. We give a new elementary proof that the Delaunay–Čech complex is level homotopy equivalent to the Čech complex. This has previously been shown using discrete Morse theory [1].
2. We extend this proof to the relative versions of the Delaunay–Čech complex and the Čech complex.
3. We explain how the relative Delaunay–Čech complex can be constructed through embedding in a higher dimension.

Given finite $A \subseteq X \subseteq \mathbb{R}^d$, these contributions lead to the construction of a filtered simplicial complex $\mathrm{Del\check{C}}(X, A)$ with persistent homology isomorphic to the relative persistent homology of Čech persistence modules $\check{\mathrm{C}}_*(X; k)/\check{\mathrm{C}}_*(A; k)$. The underlying simplicial complex of $\mathrm{Del\check{C}}(X, A)$ is the Delaunay complex of an embedding Z of X in $\mathbb{R}^{d+1}$ with the property that a projection $\mathrm{pr}\colon \mathbb{R}^{d+1} \to \mathbb{R}^d$ takes Z onto X. All simplices in the Delaunay complex of Z projecting to a subset of A are given filtration value zero. The filtration value of the remaining simplices in the Delaunay complex of Z is defined to be the Čech filtration value of their projection to $\mathbb{R}^d$. This is the content of Theorem 3.2.

This manuscript is structured as follows. In Sect. 2, we introduce relative Čech persistent homology, and in Sect. 3 we construct the relative Delaunay–Čech complex. The rest of the paper serves to prove that the relative Delaunay–Čech complex is level homotopy equivalent to the relative Čech complex. Section 4 introduces Dowker nerves, the theoretical foundation used in the proof. In Sect. 5, we introduce the alpha- and Delaunay–Čech complexes using the Dowker nerve terminology and show that they are level homotopy equivalent to the Čech complex. Section 6 introduces the relative alpha- and Delaunay–Čech dissimilarities, and proves that their nerves are level homotopy equivalent to the relative Čech complex. Finally, in Sect. 7 we show that the nerve of the relative Delaunay–Čech dissimilarity is level homotopy equivalent to the relative Delaunay–Čech complex.

2 Relative Čech Persistent Homology

Let X be a finite subset of Euclidean space $\mathbb{R}^d$. Given $t > 0$, the Čech complex $\check{C}_t(X)$ of X is the abstract simplicial complex with vertex set X and with $\sigma \subseteq X$ a simplex of $\check{C}_t(X)$ if and only if there exists a point $p \in \mathbb{R}^d$ with distance less than t to every point in σ. Varying t we obtain the filtered Čech complex $\check{C}(X)$.

Given a subset A of X we obtain an inclusion $\check{C}(A) \subseteq \check{C}(X)$ of filtered simplicial complexes and an induced inclusion $\check{C}_*(A; k) \subseteq \check{C}_*(X; k)$ of associated chain complexes of persistence modules over the field k. The *relative Čech persistent homology of the pair* (X, A) is defined as the homology of the factor chain complex of persistence modules $\check{C}_*(X; k)/\check{C}_*(A; k)$.

For X of small cardinality, the relative Čech persistent homology can be calculated as the reduced persistent homology of the relative Čech complex $\check{C}(X, A)$, where $\sigma \subseteq X$ is a simplex of $\check{C}_t(X, A)$ if either $\sigma \subseteq A$ or $\sigma \in \check{C}_t(X)$. Note that, since $\check{C}_\infty(A)$ is contractible, this simplicial complex is of the same homotopy type as the mapping cone $\check{C}_t(A) \subseteq \check{C}_t(X)$. However, as the cardinality of X grows, this quickly becomes computationally infeasible.

Note that our construction of the relative Čech complex is a simplicial complex. It is possible to instead construct a relative Čech complex as a cell complex where A is collapsed. This would lead to smaller constructions, in particular if the cardinality of A is large. However, we chose to construct the relative Čech complex and the relative Delaunay–Čech complex as simplicial complexes.

3 The Relative Delaunay–Čech Complex

Before delving into theory we present a filtered simplicial complex that is level homotopy equivalent to the relative Čech complex $\check{C}(X, A)$ of a pair of finite subsets $A \subseteq X$ of Euclidean space $\mathbb{R}^d$. Two filtered simplicial complexes $K = (K_t)_{t \geq 0}$ and $L = (L_t)_{t \geq 0}$ are *level homotopy equivalent* if there exists a filtered simplicial map $f : K \to L$ so that the geometric realization of $f_t : K_t \to L_t$ is a homotopy equivalence for each t.

Recall that the Delaunay complex $\mathrm{Del}(X)$ is the simplicial complex consisting of subsets σ of X contained in an empty sphere. That is, there exists a center $p \in \mathbb{R}^d$ and a radius $r > 0$, so that all elements of σ have distance r to p and no element of X is closer to p.

For convenience, we let $B = X - A$ so that X is the disjoint union of A and B. Choose $s > 0$ bigger than the diameters of A and B. The set

$$Z = A \times \{s\} \cup B \times \{-s\}$$

is an embedding of X in $\mathbb{R}^{d+1}$. Let $\mathrm{Del}(Z)$ be the Delaunay complex of Z. Here, the reason for our choice of s is to ensure that we have an inclusion $\mathrm{Del}(A) = \mathrm{Del}(A \times \{s\}) \subseteq \mathrm{Del}(Z)$.

Definition 3.1 The *relative Delaunay–Čech complex* of the finite subsets $A \subseteq X$ of $\mathbb{R}^d$ is the filtered simplicial complex $\mathrm{Del\check{C}}(X, A)$ with $\mathrm{Del}(Z)$ as underlying simplicial complex and with filtration $R\colon \mathrm{Del}(Z) \to \mathbb{R}$ defined as follows: Given $\sigma \in \mathrm{Del}(Z)$, let $\mathrm{pr}(\sigma)$ be the projection of $\sigma \subseteq \mathbb{R}^{d+1}$ to $\mathbb{R}^d$ away from the last coordinate. If $\mathrm{pr}(\sigma)$ is contained in A we let $R(\sigma) = 0$. Otherwise we let $R(\sigma)$ be the radius of the smallest enclosing ball of $\mathrm{pr}(\sigma)$.

By construction, considered as a filtered simplicial complex, the relative Delaunay–Čech complex is a subcomplex of the relative Čech complex.

Theorem 3.2 *For each $t \geq 0$, the inclusion $\mathrm{Del\check{C}}_t(X, A) \subseteq \check{C}_t(X, A)$ is a homotopy equivalence. In particular, the persistent homology of $\mathrm{Del\check{C}}(X, A)$ is isomorphic to the relative Čech persistent homology of the pair (X, A). If $X \subseteq \mathbb{R}^d$ is of cardinality n, then $\mathrm{Del\check{C}}(X, A)$ contains $O(n^{\lceil (d+1)/2 \rceil})$ simplices.*

The statement about the size of the relative Delaunay–Čech complex is a direct consequence of the result that the Delaunay triangulation of n points in $d+1$ dimensions contains $O(n^{\lceil (d+1)/2 \rceil})$ simplices [9, 12].

Now we have shown how to construct the relative Delaunay–Čech complex $\mathrm{Del\check{C}}(X, A)$. In the following sections we prove Theorem 3.2, i.e., that the relative Delaunay–Čech complex $\mathrm{Del\check{C}}(X, A)$ is level homotopy equivalent to the relative Čech complex $\check{C}(X, A)$.

4 Dowker Nerves

The main theoretical ingredients of our proof are dissimilarities, Dowker nerves, and partitions of unity. In this section, we introduce these concepts together with properties we need in the proof of Theorem 3.2.

A *dissimilarity* is a continuous function of the form $\Lambda\colon X \times Y \to [0, \infty]$, for topological spaces X and Y, where $[0, \infty]$ is given the order topology. A *morphism* $f\colon \Lambda \to \Lambda'$ of dissimilarities $\Lambda\colon X \times Y \to [0, \infty]$ and $\Lambda'\colon X' \times Y' \to [0, \infty]$ consists of a pair (f_1, f_2) of continuous functions $f_1\colon X \to X'$ and $f_2\colon Y \to Y'$ so that for all $(x, y) \in X \times Y$ the following inequality holds:

$$\Lambda'(f_1(x), f_2(y)) \leq \Lambda(x, y).$$

This notion of morphism is less general than for example [3, Defn. 2.10], but it is simpler and suffices for our purposes. The *Dowker nerve* $N\Lambda$ of Λ is the filtered simplicial complex described as follows: For $t > 0$, the simplicial complex $N\Lambda_t$ consists of the finite subsets σ of X for which there exists $y \in Y$ so that $\Lambda(x, y) < t$ for every $x \in \sigma$. Let $f\colon \Lambda \to \Lambda'$ be a morphism of dissimilarities as above and let $\sigma \in N\Lambda_t$. Given $y \in Y$ with $\Lambda(x, y) < t$ for every $x \in \sigma$ we see that

$$\Lambda'(f_1(x), f_2(y)) \leq \Lambda(x, y) < t,$$

so $f_1(\sigma) \in N\Lambda'_t$. Thus we have an induced simplicial map $f\colon N\Lambda \to N\Lambda'$. Given $x \in X$ and $t > 0$, the *Λ-ball of radius t centered at x* is the subset of Y defined as

$$B_\Lambda(x,t) = \{y \in Y \mid \Lambda(x,y) < t\}.$$

The *t-thickening of Λ* is the subset of Y defined as

$$\Lambda^t = \bigcup_{x \in X} B_\Lambda(x,t).$$

Note that by construction the set of Λ-balls of radius t is an open cover of the t-thickening of Λ.

The *geometric realization* $|K|$ of a simplicial complex K on the vertex set V is the subspace of the space $[0,1]^V$ of functions $\alpha\colon V \to [0,1]$ described as follows:

1. The subset $\alpha^{-1}((0,1])$ of V consisting of elements where α is strictly positive is a simplex in K. In particular it is finite.
2. The sum of the values of α is one, that is $\sum_{v \in V} \alpha(v) = 1$.

With respect to the product topology, the subspace topology on $|K|$ is called the *strong topology* on the geometric realization. It is convenient for construction of functions into $|K|$. The *weak topology* on $|K|$, which we are not going to use here, is convenient for construction of functions out of $|K|$. The homotopy type of $|K|$ is the same for these two topologies [7, p. 355, Cor. A.2.9]. Given a simplex $\sigma \in K$, the simplex $|\sigma|$ of $|K|$ is the closure of

$$\{\alpha\colon V \to [0,1] \mid \alpha(v) > 0 \text{ for all } v \in \sigma\} \cap |K|.$$

The simplices of $|K|$ are the sets of this form.

A *partition of unity* $\varphi = \{\varphi^t\colon \Lambda^t \to |N\Lambda_t|\}$ *subordinate to the dissimilarity* $\Lambda\colon X \times Y \to [0,\infty]$ consists of continuous maps $\varphi^t\colon \Lambda^t \to |N\Lambda_t|$ for $t \geq 0$, such that given $x \in X$ and $t \geq 0$, the closure of the set

$$\{y \in Y \mid \varphi^t(y)(x) > 0\}$$

is contained in $B_\Lambda(x,t)$. Given $t \geq 0$ and $y \in \Lambda^t$, the function $\varphi^t(y)\colon X \to [0,1]$ is a partition of unity because $\varphi^t(y) \in |N\Lambda_t|$ implies that $\sum_{x \in X} \varphi^t(y)(x) = 1$. We say that Λ is *numerable* if a partition of unity subordinate to Λ exists. If Y is paracompact, then every dissimilarity of the form $\Lambda\colon X \times Y \to [0,\infty]$ is numerable [7, p. 355, paragraph after Defn. A.2.10].

Let φ be a partition of unity subordinate to Λ, let $t \geq 0$, and let $y \in \Lambda^t$. If $x \in X$ with $\varphi^t(y)(x) > 0$, then $\Lambda(x,y) < t$. Therefore $\varphi^t(y)$ is contained in a simplex $|\sigma|$ in $|N\Lambda_t|$ with σ contained in $\{x \in X \mid \Lambda(x,y) < t\}$. Every finite subset of this set is an element of $N\Lambda_t$. This implies that for $s \leq t$ there is a simplex of $|N\Lambda_t|$ containing both $\varphi^s(y)$ and $\varphi^t(y)$. It also implies that given another partition of unity ψ subordinate to Λ there is a simplex of $|N\Lambda_t|$ containing both $\varphi^t(y)$ and $\psi^t(y)$. This is exactly the definition of contiguous maps, so φ^t and ψ^t are contiguous, and thus homotopic maps [7, Rem. 2.22, p. 350]. Similarly, the diagram

$$\begin{array}{ccc} \Lambda^s & \xrightarrow{\varphi^s} & |N\Lambda_s| \\ \downarrow & & \downarrow \\ \Lambda^t & \xrightarrow{\varphi^t} & |N\Lambda_t| \end{array}$$

commutes up to homotopy [7, paragraph on the nerve starting on p. 355 and ending on p. 356].

Recall that a cover $\mathcal{U}$ of Y is *good* if all non-empty finite intersections of members of $\mathcal{U}$ are contractible. We now state the Nerve Lemma in the context of dissimilarities.

Theorem 4.1 *If Y is paracompact and $\Lambda\colon X \times Y \to [0, \infty]$ is a dissimilarity, then there exists a partition of unity φ subordinate to Λ. Moreover, given $t \geq 0$, if the cover of Λ^t by Λ-balls of radius t is a good cover, then φ^t is a homotopy equivalence.*

Proof By the above discussion, we only need to note that the last statement about good covers is [14, Thm. 4.3]. □

A functorial version of the Nerve Lemma can be stated as follows:

Proposition 4.2 *Let $\Lambda\colon X\times Y \to [0, \infty]$ and $\Lambda'\colon X'\times Y' \to [0, \infty]$ be dissimilarities and let $f = f_1\times f_2\colon X\times Y \to X'\times Y'$ be a morphism $f\colon \Lambda \to \Lambda'$ of dissimilarities. If $\{\varphi^t\colon \Lambda^t \to |N\Lambda_t|\}$ is a partition of unity subordinate to Λ and $\{\psi^t\colon (\Lambda')^t \to |N\Lambda'_t|\}$ is a partition of unity subordinate to Λ', then for every $t \geq 0$ the diagram*

$$\begin{array}{ccc} \Lambda^t & \xrightarrow{\varphi^t} & |N\Lambda_t| \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle |f_1|} \\ (\Lambda')^t & \xrightarrow{\psi^t} & |N\Lambda'_t|, \end{array}$$

commutes up to homotopy, where $|f_1|$ is induced by the simplicial map $N\Lambda_t \to N\Lambda'_t$ extending the map $f_1\colon X \to X'$ of vertices.

Proof We show that the two compositions are contiguous. Recall that $|f_1|$ takes a point $\alpha\colon X \to [0, 1]$ of $|N\Lambda_t|$ to the point $|f_1|(\alpha)$ of $|N\Lambda'_t|$ with $|f_1|(\alpha)(x') = \sum_{f_1(x)=x'} \alpha(x)$. Recall further that $\varphi^t(y)$ is contained in a simplex $|\sigma|$ in $|N\Lambda_t|$, where σ is contained in $\{x \in X \mid \Lambda(x, y) < t\}$. Then we have that for $y \in \Lambda^t$, the elements $|f_1|(\varphi^t(y))$ and $\psi^t(f_2(y))$ of $|N\Lambda'_t|$ are contained in simplices $|\sigma'|$ and $|\tau'|$ respectively. Both σ' and τ' are subsets of the set $\{x' \in X' \mid \Lambda'(x', f_2(y)) < t\}$. However, every finite subset of this set is a simplex in $N\Lambda'_t$. In particular, so is the union $\sigma' \cup \tau'$. □

5 The Alpha- and Delaunay–Čech Complexes

Here we introduce the standard Čech, Delaunay, Delaunay–Čech, and alpha complexes using the dissimilarity notation introduced above. We then show that the Delaunay–Čech complex is homotopy equivalent with the Čech complex with an elementary geometric approach.

Given a finite subset X of $\mathbb{R}^d$ we define the Voronoi cell of $x \in X$ as

$$\mathrm{Vor}(X, x) = \{p \in \mathbb{R}^d \mid d(x, p) \le d(y, p) \text{ for all } y \in X\}.$$

Let $\mathbb{R}^d_d$ be Euclidean space with the discrete topology. The *discrete Delaunay dissimilarity* of X is defined as

$$\mathrm{del}^X\colon X \times \mathbb{R}^d_d \to [0, \infty], \quad \mathrm{del}^X(x, p) = \begin{cases} 0 & \text{if } p \in \mathrm{Vor}(X, x), \\ \infty & \text{if } p \notin \mathrm{Vor}(X, x). \end{cases}$$

The *Delaunay complex* $\mathrm{Del}(X)$ is the simplicial complex with vertex set X and with $\sigma \subseteq X$ a simplex of $\mathrm{Del}(X)$ if and only if there exists a point in $\mathbb{R}^d$ belonging to $\mathrm{Vor}(X, x)$ for every $x \in \sigma$. That is, $\mathrm{Del}(X) = N\,\mathrm{del}^X_t$ for $0 < t < \infty$.

Note that with respect to Euclidean topology, the discrete Delaunay dissimilarity is not continuous, and hence $\mathrm{del}^X\colon X \times \mathbb{R}^d \to [0, \infty]$ is not a dissimilarity. One way to deal with this is to use the Nerve Lemma for absolute neighbourhood retracts [5, Thm. 8.2.1]. Another way is to construct a continuous version of the Delaunay dissimilarity and to use Theorem 4.1 and Proposition 4.2, where the Voronoi cells are replaced by open sets as explained below.

Given a subset σ of X and $p \in \mathbb{R}^d$, let

$$d_{\mathrm{Vor}}(p, \sigma) = \max\{d(p, \mathrm{Vor}(X, x)) \mid x \in \sigma\},$$

where for any $A \subseteq \mathbb{R}^d$, we define $d(p, A) = \inf_{a \in A}\{d(p, a)\}$. Note that if $\sigma \notin \mathrm{Del}(X)$, the infimum ε_σ of the continuous function $d_{\mathrm{Vor}}(-, \sigma)\colon \mathbb{R}^d \to \mathbb{R}$ is strictly positive. Choose $\varepsilon > 0$ so that $2\varepsilon < \varepsilon_\sigma$ for every subset σ of X that is not in $\mathrm{Del}(X)$. Given $x \in X$ we define the ε-thickened Voronoi cell $\mathrm{Vor}(X, x)^\varepsilon$ by

$$\mathrm{Vor}(X, x)^\varepsilon = \{p \in \mathbb{R}^d \mid d(p, \mathrm{Vor}(X, x)) < \varepsilon\}.$$

By construction the nerve of the open cover $(\mathrm{Vor}(X, x)^\varepsilon)_{x \in X}$ of $\mathbb{R}^d$ is equal to $\mathrm{Del}(X)$. Let $h\colon [0, \infty] \to [0, \infty]$ be a continuous order preserving map with $h(0) = 0$ and $h(\varepsilon) = \infty$. In order to be specific we let

$$h(t) = \begin{cases} -\ln\left(1 - \dfrac{t}{\varepsilon}\right) & \text{if } t < \varepsilon, \\ \infty & \text{if } t \ge \varepsilon. \end{cases} \tag{5.1}$$

For $x \in X$, let $\mathrm{Del}_x\colon \mathbb{R}^d \to [0, \infty]$ be the function defined by $\mathrm{Del}_x(p) = h(d(p, \mathrm{Vor}(X, x)))$ so that $\mathrm{Del}_x(\mathrm{Vor}(X, x)) = 0$ and $\mathrm{Del}_x(\mathbb{R}^d \setminus \mathrm{Vor}(X, x)^\varepsilon) = \infty$.

We are now ready to introduce dissimilarities whose nerves give us the Delaunay complex and the filtered Čech, alpha and Delaunay–Čech complexes. The Delaunay dissimilarity of X is defined as

$$\mathrm{Del}^X\colon X \times \mathbb{R}^d \to [0,\infty], \quad \mathrm{Del}^X(x,p) = \mathrm{Del}_x(p).$$

By the above discussion we know that $N\,\mathrm{Del}^X_t = N\,\mathrm{del}^X_t = \mathrm{Del}(X)$ whenever $0 < t < \infty$. The Čech dissimilarity of X is defined as

$$d^X\colon X \times \mathbb{R}^d \to [0,\infty],$$

where $d^X(x,p)$ is the Euclidean distance between $x \in X$ and $p \in \mathbb{R}^d$. The alpha dissimilarity of X is defined as

$$A^X = \max\,(\mathrm{Del}^X, d^X)\colon X \times \mathbb{R}^d \to [0,\infty].$$

The Delaunay–Čech dissimilarity is defined as

$$\begin{aligned}\mathrm{Del\check{C}}^X\colon & X \times (\mathbb{R}^d \times \mathbb{R}^d) \to [0,\infty],\\ & \mathrm{Del\check{C}}^X(x,(p,q)) = \max\,(d^X(x,p), \mathrm{Del}^X(x,q)).\end{aligned}$$

Note the nerve of the dissimilarity

$$\begin{aligned}\mathrm{del\check{C}}^X\colon & X \times (\mathbb{R}^d \times \mathbb{R}^d_d) \to [0,\infty],\\ & \mathrm{del\check{C}}^X(x,(p,q)) = \max\,(d^X(x,p), \mathrm{del}^X(x,q)),\end{aligned}$$

with respect to the space $\mathbb{R}^d \times \mathbb{R}^d_d$, where one of the factors has the discrete topology, is identical to the nerve of $\mathrm{Del\check{C}}^X$. Moreover, the Dowker nerves of the Delaunay-, Čech-, alpha-, and Delaunay–Čech dissimilarities are the Delaunay-, Čech-, alpha-, and Delaunay–Čech complexes respectively. For all these dissimilarities, the corresponding balls are convex, so the geometric realizations are homotopy equivalent to the corresponding thickenings. In order to see that the inclusion morphism $A^X \to d^X$ induces homotopy equivalences $|NA^X_t| \xrightarrow{\simeq} |Nd^X_t|$ it suffices by Proposition 4.2 to note that the corresponding map $(A^X)^t \to (d^X)^t$ is the identity map. This holds because $B_{A^X}(x,t) = B_{d^X}(x,t) \cap B_{\mathrm{Del}^X}(x,t)$ and given $y \in B_{d^X}(x,t)$ we have that $y \in \mathrm{Vor}(X,x')$ for some $x' \in X$. Thus, $d^X(y,x')$ is minimal, so $d^X(y,x') \le d^X(y,x) < t$ and $y \in B_{d^X}(x',t) \cap B_{\mathrm{Del}^X}(x',t)$.

In order to see that the inclusion morphism $\mathrm{Del\check{C}}^X \to d^X$ induces homotopy equivalences $|N\,\mathrm{Del\check{C}}^X_t| \xrightarrow{\simeq} |Nd^X_t|$ we use the following lemma:

Lemma 5.1 *For every $(p,q) \in (\mathrm{Del\check{C}}^X)^t$, the entire line segment between (p,p) and (p,q) is contained in $(\mathrm{Del\check{C}}^X)^t$.*

Proof In order not to clutter notation we omit superscript X on dissimilarities. Let $\gamma\colon [0,1] \to \mathbb{R}^d$ be the function $\gamma(s) = (1-s)p + sq$. We claim that given $(p,q) \in \mathrm{Del\check{C}}^t$ and $s \in [0,1]$ the point $(p, \gamma(s)) = (p, (1-s)p + sq)$ is in $\mathrm{Del\check{C}}^t$.

If $(p,q) \in \mathrm{Del\check{C}}^t$, there exists a point $x \in X$, such that $p \in B_d(x,t)$ and $q \in B_{\mathrm{Del}}(x,t)$, that is, $d(q, \mathrm{Vor}(X,x)) < h^{\leftarrow}(t)$, where $h^{\leftarrow}$ is the generalized inverse of h. Pick $q' \in \mathrm{Vor}(X,x)$ so that $d(q,q') < h^{\leftarrow}(t)$. For the idea of the proof, it is convenient to first consider the case $q' = q$. In general, however, q may not be in $\mathrm{Vor}(X,x)$. Let $\gamma'\colon [0,1] \to \mathbb{R}^d$ be the function $\gamma'(s) = (1-s)p + sq'$. Given $s \in [0,1]$, suppose that the point $(p, \gamma'(s)) = (p, (1-s)p + sq')$ is in $\mathrm{del\check{C}}^t$. Then there exists $x' \in X$ such that $d(x', p) < t$ and $\gamma'(s) \in \mathrm{Vor}(X, x')$. In order to know that $(p, \gamma(s))$ is in $\mathrm{Del\check{C}}^t$ it suffices to observe that $h(d(\gamma(s), V(X,x'))) \le h(d(\gamma(s), \gamma'(s))) < t$ since $\gamma'(s) \in \mathrm{Vor}(X, x')$.

It remains to be shown that given any $s \in [0,1]$, the resulting point $(p, \gamma'(s)) = (p, (1-s)p + sq')$ is in $\mathrm{del\check{C}}^t$. Suppose $\gamma'(s) \in \mathrm{Vor}(X,y)$ for some $s \in [0,1)$ and some $y \in X$. We claim that then $p \in B_d(y,t)$. To see this, we may without loss of generality assume that $y \neq x$. Let H be the hyperplane in $\mathbb{R}^d$ between x and y, i.e.,

$$H = \{z \in \mathbb{R}^d \mid d(x,z) = d(y,z)\}.$$

Let

$$H_+ = \{z \in \mathbb{R}^d \mid d(x,z) \ge d(y,z)\} \quad \text{and} \quad H_- = \{z \in \mathbb{R}^d \mid d(x,z) \le d(y,z)\}.$$

Since $\gamma'(s) \in \mathrm{Vor}(X,y)$ we have $\gamma'(s) \in H_+$. Since $q' \in \mathrm{Vor}(X,x)$ we have $q' \in H_-$. Since the line segment between p and q' either is contained in H or intersects H at most once we must have $p \in H_+$. That is, $d(y,p) \le d(x,p) < t$, so $p \in B_d(y,t)$ as claimed. □

By Lemma 5.1, the inclusion

$$(d^X)^t = \bigcup_{x\in X} B_{d^X}(x,t) \to \bigcup_{x\in X} B_{\mathrm{Del\check{C}}^X}(x,t) = (\mathrm{Del\check{C}}^X)^t, \qquad p \mapsto (p,p),$$

is a deformation retract. In particular it is a homotopy equivalence and Proposition 4.2 implies that the inclusion morphism $\mathrm{Del\check{C}}^X \to d^X$ induces homotopy equivalences $|N\,\mathrm{Del\check{C}}^X_t| \xrightarrow{\simeq} |N d^X_t|$.

6 The Relative Delaunay–Čech Dissimilarity

In this section, we construct relative versions of the Čech and Delaunay–Čech dissimilarities defined in Sect. 5. Here we consider two subsets X_1 and X_2 of d-dimensional Euclidean space $\mathbb{R}^d$.

The *Voronoi diagram* of a finite subset X of $\mathbb{R}^d$ is the set of pairs of the form $(x, \mathrm{Vor}(X,x))$ for $x \in X$, that is,

$$\mathrm{Vor}(X) = \{(x, \mathrm{Vor}(X,x)) \mid x \in X\}.$$

This may seem overly formal since the projection on the first factor gives a bijection $\mathrm{Vor}(X) \to X$. However, when we work with Voronoi cells with respect to different subsets X_1 and X_2 of $\mathbb{R}^d$ it may happen that $\mathrm{Vor}(X_1, x_1) = \mathrm{Vor}(X_2, x_2)$ even when $x_1 \neq x_2$. The *Voronoi diagram* of the pair of subsets X_1 and X_2 of $\mathbb{R}^d$ is the union

$$\mathrm{Vor}(X_1, X_2) = \mathrm{Vor}(X_1) \cup \mathrm{Vor}(X_2).$$

The *discrete Delaunay dissimilarity* of X_1 and X_2 is defined as

$$\mathrm{del}^{X_1,X_2}\colon\ \mathrm{Vor}(X_1, X_2) \times \mathbb{R}^d_d \to [0,\infty],$$
$$\mathrm{del}^{X_1,X_2}((x,V),p) = \begin{cases} 0 & \text{if } p \in V, \\ \infty & \text{if } p \notin V. \end{cases}$$

The simplicial complex $N\,\mathrm{del}_t^{X_1,X_2}$ is independent of $0 < t < \infty$. It is the *Delaunay complex* $\mathrm{Del}(X_1, X_2)$ on X_1 and X_2. In order to describe the homotopy type of this simplicial complex we thicken the Voronoi cells like we did in the previous section: Given a subset σ of $\mathrm{Vor}(X_1, X_2)$ and $p \in \mathbb{R}^d$, let

$$d_{\mathrm{Vor}}(p,\sigma) = \max\{d(p,V) \mid (x,V) \in \sigma\}.$$

Note that if $\sigma \notin \mathrm{Del}(X_1, X_2)$, the infimum ε_σ of the continuous function $d_{\mathrm{Vor}}(-,\sigma)\colon \mathbb{R}^d \to \mathbb{R}$ is strictly positive. Choose $\varepsilon > 0$ so that $2\varepsilon < \varepsilon_\sigma$ for every subset σ of $\mathrm{Vor}(X_1, X_2)$ that is not in $\mathrm{Del}(X_1, X_2)$. Given $(x, V) \in \mathrm{Vor}(X_1, X_2)$ we define the ε-thickening V^ε of V by

$$V^\varepsilon = \{p \in \mathbb{R}^d \mid d(p,V) < \varepsilon\}.$$

By construction, the nerve of the open cover $\{V^\varepsilon\}_{(x,V)\in\mathrm{Vor}(X_1,X_2)}$ is equal to $\mathrm{Del}(X_1, X_2)$. The Delaunay dissimilarity Del^{X_1,X_2} of X_1 and X_2 is defined as

$$\mathrm{Vor}(X_1, X_2) \times \mathbb{R}^d \xrightarrow{\mathrm{Del}^{X_1,X_2}} [0,\infty], \qquad \mathrm{Del}^{X_1,X_2}((x,V),p) = h(d(p,V)),$$

for $h\colon [0,\infty] \to [0,\infty]$ the order preserving map defined in Sect. 5.1. The inclusion $X_1 \to \mathrm{Vor}(X_1, X_2)$ taking $x \in X_1$ to $(x, \mathrm{Vor}(X_1, x))$ induces a morphism of dissimilarities $\mathrm{Del}^{X_1} \to \mathrm{Del}^{X_1,X_2}$ and an inclusion of nerves $N\,\mathrm{Del}_t^{X_1} \subseteq N\,\mathrm{Del}_t^{X_1,X_2}$ for $t > 0$.

Next, we construct the dissimilarity A^{X_1,X_2} as

$$\mathrm{Vor}(X_1, X_2) \times \mathbb{R}^d \xrightarrow{A^{X_1,X_2}} [0,\infty],$$
$$((x,V),p) \mapsto \max\,(d(x,p), \mathrm{Del}^{X_1,X_2}((x,V),p)).$$

Also here we have an obvious inclusion $NA_t^{X_1} \to NA_t^{X_1,X_2}$, and the A^{X_1,X_2}-balls are convex so the Nerve Lemma yields a homotopy equivalence

$$|NA_t^{X_1,X_2}| \simeq \bigcup_{(x,V)\in \mathrm{Vor}(X_1,X_2)} B_{A^{X_1,X_2}}((x,V),t)$$
$$= \bigcup_{x\in X_1\cup X_2} B_{d^{X_1\cup X_2}}(x,t) = (X_1\cup X_2)^t$$

between $|NA_t^{X_1,X_2}|$ and the t-thickening $\bigcup_{x\in X_1\cup X_2} B(x,t)$ of $X_1\cup X_2$. Finally, we construct the dissimilarity $\mathrm{Del}\check{\mathrm{C}}^{X_1,X_2}$:

$$\mathrm{Vor}(X_1,X_2)\times(\mathbb{R}^d\times\mathbb{R}^d) \xrightarrow{\mathrm{Del}\check{\mathrm{C}}^{X_1,X_2}} [0,\infty],$$
$$((x,V),(p,q)) \mapsto \max\big(d(x,p), \mathrm{Del}^{X_1,X_2}((x,V),q)\big).$$

Here again we have an obvious inclusion $N\,\mathrm{Del}\check{\mathrm{C}}_t^{X_1} \to N\,\mathrm{Del}\check{\mathrm{C}}_t^{X_1,X_2}$, and the $\mathrm{Del}\check{\mathrm{C}}^{X_1,X_2}$-balls are convex so the Nerve Lemma yields a homotopy equivalence

$$|N\,\mathrm{Del}\check{\mathrm{C}}_t^{X_1,X_2}| \simeq (\mathrm{Del}\check{\mathrm{C}}^{X_1,X_2})^t.$$

The following variant of Lemma 5.1 implies that $(\mathrm{Del}\check{\mathrm{C}}^{X_1,X_2})^t$ is a deformation retract of $(X_1\cup X_2)^t$.

Lemma 6.1 *For every $(p,q)\in(\mathrm{Del}\check{\mathrm{C}}^{X_1,X_2})^t$, the entire line segment between (p,p) and (p,q) is contained in $(\mathrm{Del}\check{\mathrm{C}}^{X_1,X_2})^t$.*

Proof Given $(p,q)\in(\mathrm{Del}\check{\mathrm{C}}^{X_1,X_2})^t = (\mathrm{Del}\check{\mathrm{C}}^{X_1})^t\cup(\mathrm{Del}\check{\mathrm{C}}^{X_2})^t$, we have $(p,q)\in(\mathrm{Del}\check{\mathrm{C}}^{X_i})^t$ for some $i\in\{1,2\}$. Then also (p,p) lies in $(\mathrm{Del}\check{\mathrm{C}}^{X_i})^t$, and the claim follows by Lemma 5.1. □

As for the nonrelative version, Proposition 4.2 implies that the inclusion morphism $\mathrm{Del}\check{\mathrm{C}}^{X_1,X_2} \to d^{X_1\cup X_2}$ induces homotopy equivalences $|N\,\mathrm{Del}\check{\mathrm{C}}^{X_1,X_2}| \xrightarrow{\simeq} |Nd^{X_1\cup X_2}|$.

7 Nerve of the Relative Delaunay–Čech Dissimilarity

In this section we introduce the relative Delaunay dissimilarity and show that its nerve is level homotopy equivalent to the relative Delaunay–Čech complex. We fix some notation used in this section: $X_1\subseteq\mathbb{R}^d$ and $X_2\subseteq\mathbb{R}^d$ are finite subsets. We let s be a positive real number, we let $Z = X_1\times\{s\}\cup X_2\times\{-s\}$ and we let $\mathrm{pr}\colon\mathbb{R}^{d+1}\to\mathbb{R}^d$ be the projection omitting the last coordinate.

Lemma 7.1 *The projection* $\mathrm{pr}\colon\mathbb{R}^{d+1}\to\mathbb{R}^d$ *induces a surjection*

$$\mathrm{Vor}(Z) \xrightarrow{g} \mathrm{Vor}(X_1,X_2), \qquad \begin{cases}((x,s),V)\mapsto(x,\mathrm{Vor}(X_1,x)),\\ ((x,-s),V)\mapsto(x,\mathrm{Vor}(X_2,x)),\end{cases}$$

with $\mathrm{pr}(V) \subseteq \mathrm{Vor}(X_i, x)$ *for* $x \in X_i$. *Given* $(x, \mathrm{Vor}(X_i, x)) \in \mathrm{Vor}(X_1, X_2)$ *for* $i \in \{1, 2\}$, *the fiber* $g^{-1}((x, \mathrm{Vor}(X_i, x)))$ *consists of all elements of* $\mathrm{Vor}(Z)$ *of the form* $((x, a), V)$ *for* $a \in \{\pm s\}$.

Proof We show that $\mathrm{pr}(V) \subseteq \mathrm{Vor}(X_1, x_1)$ for $((x_1, s), V) \in \mathrm{Vor}(Z)$ such that $x_1 \in X_1$. Given $(p, r) \in V$ we have for all points of the form (x_1', s) for $x_1' \in X_1$ that $d((p, r), (x_1, s)) \le d((p, r), (x_1', s))$. This implies that $d(p, x_1) \le d(p, x_1')$, and thus $p \in \mathrm{Vor}(X_1, x_1)$. We conclude that $\mathrm{pr}(V) \subseteq \mathrm{Vor}(X_1, x_1)$. An analogous argument applies for elements of the form $((x_2, -s), V)$ in $\mathrm{Vor}(Z)$.

Clearly every element of the form $((x, a), V)$ for $a \in \{\pm s\}$ is in the fiber $g^{-1}((x, \mathrm{Vor}(X_i, x)))$. Conversely, if $g(((y, a), V)) = (x, \mathrm{Vor}(X_i, x))$, then $y = x$ and $a \in \{\pm s\}$. Since there exists a Voronoi cell of Z containing (x, a), for every $(x, a) \in \mathbb{R}^{d+1}$, the fiber is non-empty, so g is surjective. If $x \in X_1 \cap X_2$ and $\mathrm{Vor}(X_1, x) = \mathrm{Vor}(X_2, x)$, then the fiber $g^{-1}((x, \mathrm{Vor}(X_i, x)))$ consists of two elements, so in this situation g is not injective. □

Let s_1 be the diameter of X_1 and $s > s_1$. Given a $\sigma \in \mathrm{Del}(X_1)$, we can choose a $p \in \mathbb{R}^d$ so that $d(p, x) \le s$ for every $(x, V) \in \sigma$. Then $d((p, s), (x_2, -s)) \ge d((p, s), (x_1, s))$ for every $x_1 \in X_1$ and $x_2 \in X_2$. This implies that (p, s) is in the intersection of the Voronoi cells $((x_1, s), \mathrm{Vor}(Z, (x_1, s)))$ for $(x_1, V) \in \sigma$. It follows that the function

$$j_1\colon \mathrm{Vor}(X_1) \to \mathrm{Vor}(Z), \qquad j_1(x_1, V) = ((x_1, s), \mathrm{Vor}(Z, (x_1, s))),$$

induces a simplicial map of nerves $\mathrm{Del}(X_1) \to \mathrm{Del}(Z)$. Similarly, given $s > s_2$ where s_2 is the diameter of X_2, there is a simplicial map $\mathrm{Del}(X_2) \to \mathrm{Del}(Z)$. Let $s(X_1, X_2) = \max(s_1, s_2)$.

Recall, from the previous two sections, that ε_σ is the infimum of the continuous function $d_{\mathrm{Vor}}(-, \sigma)\colon \mathbb{R}^d \to \mathbb{R}$. Choose $\varepsilon > 0$ satisfying the following two criteria:

- $2\varepsilon < \varepsilon_\sigma$ for every subset σ of $\mathrm{Vor}(X_1, X_2)$ that is not in $\mathrm{Del}(X_1, X_2)$.
- $2\varepsilon < \varepsilon_\sigma$ for every subset σ of $\mathrm{Vor}(Z)$ that is not in $\mathrm{Del}(Z)$.

Let $h\colon [0, \infty] \to [0, \infty]$ be the order preserving map defined in (5.1), and let Del^Z and Del^{X_1, X_2} be constructed using h. We define the relative Delaunay dissimilarity

$$D\colon \mathrm{Vor}(Z) \times (\mathbb{R}^d \times \mathbb{R}^{d+1}) \to [0, \infty],$$
$$D((z, V), (p, q)) = \max(d(\mathrm{pr}(z), p), \mathrm{Del}^Z((z, V), q)).$$

Note that the underlying simplicial complex $\bigcup_{0<t<\infty} ND_t$ of the nerve of D is the Delaunay complex $\mathrm{Del}(Z)$. The filtration value of $\sigma \in \mathrm{Del}(Z)$ in the nerve of D is the filtration value of $g(\sigma)$ in the nerve of $\mathrm{Del\check{C}}^{X_1, X_2}$. In order to see this, note that $N\,\mathrm{Del}^Z = N\,\mathrm{del}^Z$ and $N\,\mathrm{Del}^{X_1, X_2} = N\,\mathrm{del}^{X_1, X_2}$.

Proposition 7.2 *Let* $X_1 \subseteq \mathbb{R}^d$ *and* $X_2 \subseteq \mathbb{R}^d$ *be finite. Choose* $s > s(X_1, X_2)$. *Then* $\mathrm{Vor}(Z) \xrightarrow{g} \mathrm{Vor}(X_1, X_2)$ *and* $\mathrm{id} \times \mathrm{pr}\colon \mathbb{R}^d \times \mathbb{R}^{d+1} \to \mathbb{R}^d \times \mathbb{R}^d$ *form a morphism*

$$f = (g, \mathrm{id} \times \mathrm{pr})\colon D \to \mathrm{Del\check{C}}^{X_1, X_2}$$

of dissimilarities inducing a homotopy equivalence

$$f_t \colon ND_t \to N\,\mathrm{Del\check{C}}_t^{X_1,X_2}$$

for every $t > 0$.

Proof For $i = 1, 2$ the inclusion $\mathrm{pr}(V) \subseteq \mathrm{Vor}(X_i, x)$ for $((x, (-1)^{i-1}s), V) \in \mathrm{Vor}(Z)$ implies that

$$\mathrm{Del}^{X_1,X_2}(g(z, V), \mathrm{pr}(q)) \le \mathrm{Del}^{Z}((z, V), q)$$

for all $(z, V) \in \mathrm{Vor}(Z)$ and $q \in \mathbb{R}^{d+1}$. So $f = (g, \mathrm{id} \times \mathrm{pr}) \colon D \to \mathrm{Del\check{C}}^{X_1,X_2}$ is a morphism.

In order to show that g induces a homotopy equivalence $|f_t|$ of geometric realizations, by the Nerve Lemma, it suffices to show that given a simplex σ of $N\,\mathrm{Del\check{C}}_t^{X_1,X_2}$, the inverse image $g^{-1}(\sigma)$ is a simplex of ND_t. Let p be a point in the intersection of the Voronoi cells in σ. Write $g^{-1}(\sigma) = \tau_1 \cup \tau_2$, where τ_1 consists of Voronoi cells with centers at height s and τ_2 consists of Voronoi cells with centers at height $-s$.

Suppose that τ_2 is empty. Then actually $\sigma \in \mathrm{Del\check{C}}_t^{X_1}$, and since $s > s_1$ we know that $j_1(\sigma) \in \mathrm{Del}(Z)$ by the above discussion. Since $g \circ j_1$ is the inclusion of $\mathrm{Vor}(X_1)$ in $\mathrm{Vor}(X_1, X_2) = \mathrm{Vor}(X_1) \cup \mathrm{Vor}(X_2)$ we know that $j_1(\sigma) \subseteq g^{-1}(\sigma) = \tau_1$ and that $j_1(\sigma) \in ND_t$. On the other hand, since τ_2 is empty, by Lemma 7.1 we know that $g^{-1}(\sigma)$ is contained in $j_1(\sigma)$, so they must be equal. We conclude that $g^{-1}(\sigma)$ is a simplex of ND_t. A similar argument applies when τ_1 is empty.

In the remaining case where both τ_1 and τ_2 are nonempty, recall that $p \in \bigcap_{(x,V)\in\sigma} V$. Let $\sigma_1 = \{(x_1, s) \mid (x_1, \mathrm{Vor}(X_1, x_1)) \in \sigma\}$ and $\sigma_2 = \{(x_2, -s) \mid (x_2, \mathrm{Vor}(X_2, x_2)) \in \sigma\}$. For every $\alpha \in \mathbb{R}$ the point (p, α) has the same distance $u(\alpha)$ to all points in σ_1, and it has the same distance $v(\alpha)$ to all points in σ_2. The continuous function

$$k \colon \mathbb{R} \to \mathbb{R}, \qquad k(\alpha) = u(\alpha) - v(\alpha),$$

has $k(-s) > 0$ and $k(s) < 0$. By the intermediate value theorem there exists $\alpha_0 \in [-s, s]$ with $k(\alpha_0) = 0$. Then (p, α_0) has the same distance to all elements of σ_1 and also has the same distance to all elements of σ_2. Moreover, given a point $(x_1, s) \in Z$, its distance to (p, α_0) is at least $u(\alpha_0) = v(\alpha_0)$. Similarly, given a point $(x_2, s) \in Z$, its distance to (p, α_0) is at least $u(\alpha_0) = v(\alpha_0)$. We conclude that (p, α_0) is in the intersection of the Voronoi cells in $g^{-1}(\sigma) = \tau_1 \cup \tau_2$. Thus $\mathrm{Del\check{C}}^Z((z, V), (p, \alpha_0)) = 0$ and $d(\mathrm{pr}(z), p) < t$ for all $(z, V) \in g^{-1}(\sigma)$. In particular, $g^{-1}(\sigma) \in ND_t$. □

We are now ready to compute persistent homology of $X_1 \cup X_2$ relative to X_1. The relative Delaunay–Čech complex $\mathrm{Del\check{C}}(X_1 \cup X_2, X_1)$ is the filtered simplicial complex with $\mathrm{Del\check{C}}(X_1 \cup X_2, X_1)_t = j_1(\mathrm{Del}(X_1)) \cup ND_t$. Note that this is consistent with Definition 3.1. Also note that $\mathrm{Del\check{C}}(X_1 \cup X_2, X_1)$ implicitly depends on s through the definition of Z. In order not to clutter notation we omit this dependency in the notation.

Theorem 7.3 *Let $X_1 \subseteq \mathbb{R}^d$ and $X_2 \subseteq \mathbb{R}^d$ be finite. Choose $s > s(X_1, X_2)$. Then the geometric realization of the filtered simplicial complex* $\mathrm{Del\check{C}}(X_1 \cup X_2, X_1)$ *is level homotopy equivalent to the filtered space $((X_1 \cup X_2)^t / X_1^t)_{t>0}$. In particular, there is an isomorphism*

$$(H_*(\mathrm{Del\check{C}}(X_1 \cup X_2, X_1)_t))_{t>0} \cong (H_*((X_1 \cup X_2)^t, X_1^t))_{t>0}$$

of persistence modules.

Proof Since $j_1(\mathrm{Del}(X_1))$ is contractible, the geometric realization of the relative Delaunay–Čech complex $\mathrm{Del\check{C}}(X_1 \cup X_2, X_1)_t = j_1(\mathrm{Del}(X_1)) \cup ND_t$ is homotopy equivalent to the quotient space $|\mathrm{Del\check{C}}(X_1 \cup X_2, X_1)_t|/|j_1(\mathrm{Del}(X_1))|$. This quotient space is homeomorphic to $|ND_t|/|ND_t \cap j_1(\mathrm{Del}(X_1))|$. The homotopy equivalence $f_t \colon ND_t \to N\,\mathrm{Del\check{C}}_t^{X_1,X_2}$ from Proposition 7.2 induces an isomorphism $ND_t \cap j_1(\mathrm{Del}(X_1)) \to N\,\mathrm{Del\check{C}}_t^{X_1}$. So we obtain a homotopy equivalence $|ND_t|/|ND_t \cap j_1(\mathrm{Del}(X_1))| \to |N\,\mathrm{Del\check{C}}_t^{X_1,X_2}|/|N\,\mathrm{Del\check{C}}_t^{X_1}|$. By Lemma 6.1, the space $|N\,\mathrm{Del\check{C}}_t^{X_1,X_2}|$ is homotopy equivalent to the Euclidean t-thickening $(X_1 \cup X_2)^t$ of $X_1 \cup X_2$ and by Lemma 5.1, $|N\,\mathrm{Del\check{C}}_t^{X_1}|$ is homotopy equivalent to the Euclidean t-thickening X_1^t of X_1. □

In order to prove Theorem 3.2, it remains to be shown that for each $t \geq 0$, the geometric realization of the inclusion $\mathrm{Del\check{C}}_t(X, A) \subseteq \check{C}_t(X, A)$ is a homotopy equivalence. Consider the diagram

$$\begin{array}{ccccccc}
|Nd_t^A| & \longleftarrow & (d^A)^t & \longleftarrow & (\mathrm{Del\check{C}}^A)^t & \longrightarrow & |N\,\mathrm{Del\check{C}}_t^A| \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
|Nd_t^X| & \longleftarrow & (d^X)^t & \longleftarrow & (\mathrm{Del\check{C}}^{X,A})^t & \longrightarrow & |N\,\mathrm{Del\check{C}}_t^{X,A}|,
\end{array}$$

where the right hand square is the diagram of Proposition 4.2 for the morphism $\mathrm{Del\check{C}}^A \to \mathrm{Del\check{C}}^{X,A}$ induced by the inclusion $\mathrm{Vor}(A) \subseteq \mathrm{Vor}(X, A)$, the middle square is induced by the inclusion of Delaunay–Čech complexes in Čech complexes, and the left hand square is the square of Proposition 4.2 for the morphism $d^A \to d^X$. By Lemmas 5.1 and 6.1 the horizontal maps in the middle square are homotopy equivalences, and by Theorem 4.1 the horizontal maps in the left- and right squares are homotopy equivalences, in the sense that they induce homotopy equivalences of geometric realizations. Proposition 4.2 also implies that the diagrams

$$\begin{array}{ccc}
(\mathrm{Del\check{C}}^A)^t & \longrightarrow & |N\,\mathrm{Del\check{C}}_t^A| \\
\downarrow & & \downarrow \\
(d^A)^t & \longrightarrow & |Nd_t^A|
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
(\mathrm{Del\check{C}}^{X,A})^t & \longrightarrow & |N\,\mathrm{Del\check{C}}_t^{X,A}| \\
\downarrow & & \downarrow \\
(d^X)^t & \longrightarrow & |Nd_t^X|
\end{array}$$

commute up to homotopy. We conclude that all the maps in these diagrams induce homotopy equivalences of geometric realizations and that there is a homotopy equivalence $|N\,\mathrm{Del}\check{\mathrm{C}}_t^{X,A}|/|N\,\mathrm{Del}\check{\mathrm{C}}_t^{A}| \to |Nd_t^X|/|Nd_t^A|$ induced by inclusions of Delaunay–Čech complexes in Čech complexes, and that this map induces an isomorphism of persistence modules. The map $\mathrm{Del}\check{\mathrm{C}}(X,A)_t \to N\,\mathrm{Del}\check{\mathrm{C}}_t^{X,A}/N\,\mathrm{Del}\check{\mathrm{C}}_t^{A}$ collapsing $j_1(\mathrm{Del}(A)) \cap \mathrm{Del}\check{\mathrm{C}}(X,A)_t$ to a point induces a homotopy equivalence of geometric realizations. Similarly, the map $\check{\mathrm{C}}(X,A) \to Nd_t^X/Nd_t^A$ collapsing Nd_∞^A to a point induces a homotopy equivalence. We conclude the proof of Theorem 3.2 by noting that this implies that the inclusion $\mathrm{Del}\check{\mathrm{C}}_t(X,A) \subseteq \check{\mathrm{C}}_t(X,A)$ is a homotopy equivalence.

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