A study of different interactions between solitary waves for fractional Korteweg-de Vries type equations

Arnaud Eychenne

Thesis for the degree of Philosophiae Doctor (PhD)
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Abstract

This thesis in composed by three articles. In the first one, we construct N-soliton solutions for the fractional Korteweg-de Vries (fKdV) equation

$$\partial_t u - \partial_x \left(|D|^\alpha u - u^2\right) = 0,$$

in the whole sub-critical range $\alpha \in (\frac{1}{2}, 2)$. More precisely, if Q_c denotes the ground state solution associated with fKdV evolving with velocity c, then given $0 < c_1 < \cdots < c_N$, we prove the existence of a solution U of (fKdV) satisfying

$$\lim_{t\to\infty} \|U(t,\cdot) - \sum_{j=1}^{N} Q_{c_j}(x - \rho_j(t))\|_{H^{\frac{\alpha}{2}}} = 0,$$

where $\rho_j'(t) \sim c_j$ as $t \to +\infty$.

The proof adapts the construction of Martel in the generalized KdV setting [Amer. J. Math. 127 (2005), pp. 1103-1140]) to the fractional case. The main new difficulties are the polynomial decay of the ground state Q_c and the use of local techniques (monotonicity properties for a portion of the mass and the energy) for a non-local equation. To bypass these difficulties, we use symmetric and non-symmetric weighted commutator estimates. The symmetric ones were proved by Kenig, Martel and Robbiano [Annales de l'IHP Analyse Non Linéaire 28 (2011), pp. 853-887], while the non-symmetric ones seem to be new.

In the second paper, we consider the fractional nonlinear Schrödinger equation in dimension 1:

$$|D|^\alpha u + u - f(u) = 0,$$

with $\alpha \in (0, 2)$, a prescribed coefficient $p^*(\alpha)$, and a non-linearity $f(u) = |u|^{p-1}u$ for $p \in (1, p^*(\alpha))$, or $f(u) = u^p$ with an integer $p \in [2; p^*(\alpha))$. Asymptotic developments of order 1 of the solutions at infinity are given, as well as second order developments for positive solutions, in terms of the coefficient of dispersion α and of the non-linearity p. The main tools are the kernel formulation introduced by Bona and Li [J. Math. Pures Appl. (9) 76 (1997), no. 5, 377-430], and an accurate description of the kernel by complex analysis theory.

In the last paper, we study one particular asymptotic behaviour of a solution of the fractional modified Korteweg-de Vries equation (also known as the dispersion generalised modified Benjamin-Ono equation):

$$\partial_t u + \partial_x(-|D|^\alpha u + u^3) = 0.$$

The dipole solution is a solution behaving in large time as a sum of two strongly interacting solitary waves with different signs. We prove the existence of a dipole for fmKdV. A novelty of this article is the construction of accurate profiles. Moreover, to deal with the non-local operator $|D|^\alpha$, we refine some weighted commutator estimates.

Sammendrag

Denne avhandlingen er satt sammen av tre artikler. I den første konstruerer vi N-soliton løsninger for den fractional Korteweg-de Vries (fKdV) ligningen

$$\partial_t u - \partial_x \left(|D|^\alpha u - u^2\right) = 0,$$

i hele det underkritiske tilfellet $\alpha \in (\frac{1}{2}, 2)$. Mer presist, hvis Q_c er grunntilstandsløsningen knyttet til fKdV som beveger seg med hastighet c, da gitt $0 < c_1 < \cdots < c_N$, beviser vi eksistensen av en løsning U av (fKdV) som tilfredstiller

$$\lim_{t\to\infty} \|U(t,\cdot) - \sum_{j=1}^{N} Q_{c_j}(x - \rho_j(t))\|_{H^{\frac{\alpha}{2}}} = 0,$$

hvor $\rho_j'(t) \sim c_j$ som $t \to +\infty$.

Beviset er basert på konstruksjonen gjort av Martel for den generaliserte KdV-ligningen [Amer. J. Math. 127 (2005), s. 1103-1140]) for ikke-lokale ligninger. De største utfordringene i dette arbeidet er knyttet til egenskapene av grunntilstanden Q_c. Mer presist, så avtar funksjonen som et algebraisk polynom. Samt, er det utfordringer knyttet til bruken av lokale teknikker (monotomiegenskaper for en del av massen og energien) for en ikke-lokal ligning. For å omgådisse vanskelighetene bruker vi symmetriske og ikke-symmetriske vektede kommutatorestimater. De symmetriske estimatene ble bevist av Kenig, Martel og Robbiano [Annales de l'IHP Analyze Non Linéaire 28 (2011), s. 853-887], mens de ikke-symmetriske estimatene ser ut til å være nye.

I den andre artikkelen studerer vi den fraksjonale ikke-lineære Schrödinger-ligningen i dimensjon en:

$$|D|^\alpha u + u - f(u) = 0,$$

med $\alpha \in (0, 2)$, en gitt koeffisient $p^*(\alpha)$, og en ikke-linæritet $f(u) = |u|^{p-1}u$ for $p \in (1, p^*(\alpha))$, eller $f(u) = u^p$ med et heltall $p \in [2; p^*(\alpha))$. Vi gir asymptotiske utviklinger av løsningen til første orden ved uendelig. Samt, gir vi andreordens utviklinger for positive løsninger. Disse asymptotiske utviklingene er avhenger av dispersjonskoeffisienten α og ikke-linæriteten p. Hovedverktøyene er kernelformuleringen introdusert av Bona og Li [J. Math. Pures Appl. (9) 76 (1997), no. 5, 377-430], og en nøyaktig beskrivelse av kernelen ved hjelp av kompleks analyse.

I den siste artikkelen studerer vi en spesiell asymptotisk oppførsel av en dipolløsning av den fractional modifiserte Korteweg-de Vries-ligningen:

$$\partial_t u + \partial_x(-|D|^\alpha u + u^3) = 0.$$

Dipolløsningen er en løsning som oppfører seg som en sum av to sterkt interaktive solitære bølger med forskjellige fortegn, når tiden er stor nok. Vi beviser eksistensen av en dipol for fmKdV. Et viktig bidrag i denne artikkelen er konstruksjonen av nøyaktige profiler, og dette er nytt for fmKdV ligningen. Dessuten, for å håndtere den ikke-lokale operatoren $|D|^\alpha$, må vi utbedre noen vektede kommutatorestimater.

List of publications

1. Eychenne, A. (2021). *Asymptotic N-soliton-like solutions of the fractional Korteweg-de Vries equation.* arXiv preprint arXiv:2112.11278.

2. Eychenne, A., and Valet, F. (2022). *Asymptotic behaviour of solutions to semi-linear fractional Schrödinger equations.* In preparation.

3. Eychenne, A., and Valet, F. (2022). *Strongly interacting solitary waves for the fractional modified Korteweg-de Vries equation.* arXiv preprint arXiv:2209.03841.

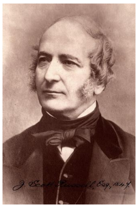
Contents

Chapter 1

Introduction

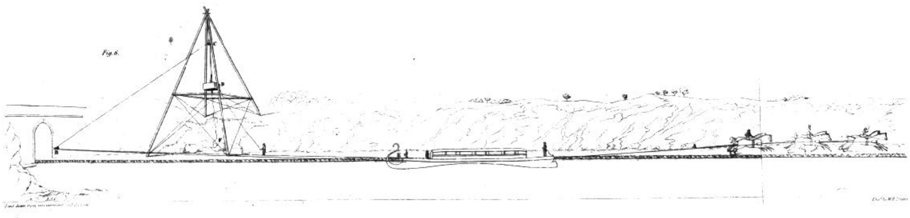
1.1 Soliton in shallow water

1.1.1 Discovery of solitons



J. S. Russell

In 1834, John Scott Russell, a Scottish civil engineer, was conducting experiments to determine the most efficient design for canal boats. He discovered a phenomenon that he described as the wave of translation or solitary wave. Nowadays this wave is called soliton or solitary wave. He observed the motion of a boat pulled by horses along the Edinburgh and Glasgow Union Canal. Suddenly the horses stopped, and thus, the boat also stopped, creating a wave in front of the ship. This wave was very irregular. However, rapidly, the wave become smooth and started to travel at a constant speed without any deformation. Then, Russell took his horse and followed the wave along the canal during some kilo-meters before losing track.



Edinburgh-Glasgow Union Canal

In 1844, he published his discovery in the British Association for the Advancement of Science [168].

His discovery was not well received by the scientists of his time. In particular, George Biddell Airy and George Gabriel Stokes each argued that Russell's wave theory was inaccurate [1], [176]. However, an equation was found independently by Boussinesq in 1877 [27] and Korteweg and de Vries in 1895 [106] to model the solitons

$$\partial_t u(t,x) + \partial_x^3 u(t,x) + \partial_x(u^2)(t,x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}. \tag{1.1.1}$$

This equation is now called the ***Korteweg-de Vries equation (KdV)*** and describes the unidirectional propagation on the shallow water of small amplitude waves. Despite the discovery of the KdV equation, the solitons have remained understudied for a long time.

In the 1950's, to get a better understanding of the non-linearity in partial differential equations, Enrico Fermi, John Pasta, Mary Tsingou and Stanislaw Ulam performed some numerical simulations [55]. They discovered that if we start with a nice ordered initial data, the non-linear effect would not necessarily distort and destroy the shape of the solution. Since the four scientists were working in Los Alamos, this discovery was not shared until the 1960's. In 1965, Zabusky-Kruskal published a paper [192] on the Femi-Pasta-Tsingou-Ulam (FPTU) problem. They discovered a relationship between the KdV solitons and the FPTU problem. In particular, they observed two interesting phenomena. First, if we start with a positive, localized solution, then this solution would eventually decompose into a sum of solitons and a dispersive term. Second, if we have two solitons moving to the right with two different constant speeds, the fastest on the left side, the fastest soliton would catch up the slowest, collide and afterward, the two solitons would return to their original shape, with the slowest soliton on the left[1]. This phenomena is called ***elastic collision*** of solitary waves. By analogy with the interaction of particles, these solitary waves were then renamed solitons.

The solitons were discovered in the context of fluid mechanics. However, they are universal physical objects. They appear in many different contexts. Indeed, the (KdV) equation is a model of shallow water but also used for plasma physic [185], [18]. Moreover, solitons have been obtained in quantum mechanics, for example, in the propagation of light in non-linear fiber optics described by the non-linear Schrödinger equation [178]. Nowadays, one of the highlights of the telecommunication systems is to use soliton in fiber optic to transfer information [77], [76]. More surprising, solitons were recently discovered in biology for the mass cell movement of non-chemotactic mutants [108], a motion of a biological cell [2], or in a model describing the invasion of cane toads in Australia [25].

1.1.2 Mathematical approach

The existence of solitons can be seen as a balance between the dispersive effect and the non-linearity of the equation. Indeed, the linear part of the KdV equation, the Airy equation

$$\partial_t u(t,x) + \partial_x^3 u(t,x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}, \tag{1.1.2}$$

is dispersive. One can observe the dispersive effect associated to a linear equation in a number of different ways. One (somewhat informal) way is to analyse plane wave solutions

$$u(t,x) = Ae^{i(t\tau + x\xi)}, \quad A \in \mathbb{R}, \tau \in \mathbb{R}, \xi \in \mathbb{R}.$$

To solve the Airy equation (1.1.2), the parameters must verify $\tau = \xi^3$, so that

$$u(t,x) = Ae^{i\xi(x-(-t\xi^2))}.$$

Thus, we see that for this equation, higher frequency plane waves have a much faster phase velocity than lower frequency ones, and the velocity is always in a leftward direction.

The second part of the KdV equation given by its non-linearity, corresponds to Burger's equation

$$\partial_t u(t,x) + \partial_x(u^2) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}. \tag{1.1.3}$$

Burger's equation is a transport equation with non-constant speed. The highest values of positive solution of (1.1.3) will travel faster to the right direction than the lowest values, creating possibly shocks in finite time.

Therefore, combining the non-linear and the dispersive effects, we can understand the existence of solitons as a balance between the non-linearity and the dispersive effects.

[1]For more detail on the history of solitons and the KdV equation we refer to [88].

Now let us give a more precise definition of a soliton. A solution u of (1.1.1) is called a ***soliton*** if there exist a speed $c > 0$ and a positive function Q_c vanishing at infinity such that

$$u(t,x) = Q_c(x-ct). \tag{1.1.4}$$

Let us suppose that we have a solution of (1.1.1) on the form (1.1.4). Then, by injecting this solution in (1.1.1), we get an equation for Q_c

$$Q_c''(x) - cQ_c(x) + Q_c^2(x) = 0, \quad x \in \mathbb{R}.$$

This equation admits a unique, up to translation, solution in $H^1(\mathbb{R})$ given by

$$Q_c(x) = \frac{3c}{2}\cosh^{-2}\left(\frac{\sqrt{c}x}{2}\right).$$

In this manuscript, we will only focus on the solitons belonging to the energy space. Furthermore, other types of coherent non-linear structures have been observed like kink, peakon, breathers for some non-linear dispersive equations.

1.1.3 Complete integrability of the Korteweg-de Vries equation

In 1968 Peter Lax fund a Lax pair for the KdV equation [113]. In other words, if u is a solution of the KdV equation, then

$$\partial_t \mathcal{L} + [\mathcal{L}, \mathcal{B}] = 0,$$

with $\mathcal{L} = -3\partial_x^2 - u$, $\mathcal{B} = -4\partial_x^3 - u\partial_x - \partial_x u$ and $\mathcal{L}$ and $\mathcal{B}$ act on a fixed Hilbert space.

The existence of the Lax pair allows us to get an infinite number of conserved quantities and then to use the inverse scattering method, which makes it possible to give a rigorous justification of the different former numerical observations. An equation is ***completely integrable*** if there exists an infinite number of conserved quantities.

The inverse scattering method has been used by Eckauss and Schuur in [46] to prove the ***soliton resolution conjecture***: any sufficiently smooth and decaying solution of KdV equation splits into two parts as $t \to +\infty$

$$u(t,x) = u_d(t,x) + u_c(t,x),$$

with u_d is an N-soliton solution and $u_c(t,x) \underset{t\to+\infty}{\longrightarrow} 0$ uniformly in $x > 0$.

A solution u of KdV is called a N-**soliton** (also N-soliton like solution or N-solitary waves) if u behaves at infinity like a sum of N decoupled solitons.

Moreover, the inverse scattering method provides a procedure to get an explicit formula for the N-soliton solutions and quantitative information about general solutions. For example, for $0 < c_1 < c_2$ two different speeds, we have the explicit formula for the 2-soliton solutions given by

$$\frac{2(c_2-c_1)\left(c_1\cosh^2\left(\frac{\sqrt{c_1}}{2}\xi_1\right) + c_2\cosh^2\left(\frac{\sqrt{c_1}}{2}\xi_2\right)\right)}{\left(\left(\sqrt{c_2}-\sqrt{c_1}\right)\cosh\left(\frac{\sqrt{c_1}}{2}\xi_1 + \frac{\sqrt{c_2}}{2}\xi_2\right) + \left(\sqrt{c_2}+\sqrt{c_1}\right)\cosh\left(\frac{\sqrt{c_1}}{2}\xi_1 - \frac{\sqrt{c_2}}{2}\xi_2\right)\right)^2}$$

with $\xi_1 = x - c_1 t$ and $\xi_2 = x - c_2 t$. Using the former formula, we obtain a better description of the interaction of two solitons.

As explained above, the soliton existence is an equilibrium between the non-linearity and the dispersion effect of the equation. What happens if one changes the non-linearity or the dispersion?

1.2 Generalized Korteweg-de Vries equation

The KdV equation can be naturally extended by increasing its non-linearity. For $p \in \mathbb{N}$, with $p \geqslant 2$, we have the ***generalized Korteweg-de Vries equation (gKdV)*** defined by

$$\partial_t u(t,x) + \partial_x^3 u(t,x) + \partial_x(u^p)(t,x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}. \tag{1.2.1}$$

1.2.1 Special case $p = 3$, the modified Korteweg-de Vries equation

For $p = 3$ the equation is called the modified Korteweg-de Vries equation (mKdV) and has many applications to physics, for example in traffic congestions [105], phonons in anharmonic lattices [160], meandering ocean jets [166], a subclass of hyperbolic surfaces [172] and for ion acoustic solitons [181].

For this equation the soliton is still explicit and it is obtained by direct computations on the equation

$$Q_c''(x) - cQ_c(x) + Q_c^3(x) = 0, \quad x \in \mathbb{R}.$$

The formula for the soliton Q of mKdV is

$$Q_c(x) = \sqrt{2c}\cosh^{-1}\left(\sqrt{c}x\right).$$

The mKdV equation has been studied from 1968 by Miura-Gardner and Kruskal. In [143] Miura found that if v solves mKdV then $u = v^2 + \partial_x v$ solves KdV. Since there is no formula to get a solution of the mKdV equation from KdV, it seems difficult to derive the complete integrability of mKdV from KdV. However, there exists an infinite number of conserved quantities for mKdV. They have been discovered simultaneously in 1968 by Miura-Gardner and Kruskal [145].

From the complete integrability, we can derive a formula for the N-soliton solution see [144]. However, no result has been obtained for the soliton resolution for the mKdV equation.

1.2.2 Solitons of the generalized Korteweg-de Vries equation

Henceforth, we consider $p \in \mathbb{N}$ with $p \geqslant 4$. For these values of p, the gKdV is not completely integrable. Even if the equation is not completely integrable, the gKdV equation possesses two conserved quantities. Let u be a solution of (1.2.1), then the following quantities are formally conserved

$$M(u)(t) = \int_{\mathbb{R}} u^2(t,x)dx, \quad E(u)(t) = \int_{\mathbb{R}} \frac{1}{2}(\partial_x u)^2(t,x) - \frac{1}{p+1}u^{p+1}(t,x)dx.$$

Furthermore, there exists an explicit formula for the solitons Q_c of gKdV given by

$$Q_c(x) = \left(\frac{(p+1)c}{2}\cosh^{-2}\left(\sqrt{c}\frac{p-1}{2}x\right)\right)^{\frac{1}{p-1}}.$$

where Q solves

$$Q_c''(x) - cQ_c(x) + Q_c^p(x) = 0, \quad x \in \mathbb{R}.$$

Note that the solitons for gKdV are, up to some transformations, the ground states of the 1 dimensional non-linear Schrödinger equation (NLS).

Remark 1.2.1. For the NLS equation it is natural to look at the equation in $\mathbb{R}^d$ with $d \in \mathbb{N}^+$. In the case $d > 1$ the question of the existence and the uniqueness of the soliton is non trivial. The equation of Q_c becomes

$$\Delta Q_c(x) - cQ_c(x) + Q_c^p(x) = 0, \quad x \in \mathbb{R}^d. \tag{1.2.2}$$

The solitons are in these cases not explicit. Their existence in $H^1(\mathbb{R}^d)$ was proved by Weinstein [189] by using a variational argument. The uniqueness of the soliton in the energy space is highly non trivial, and the proof is decomposed in two step.

First, using the moving plane method, we get that the solitons Q_c must be radial. The method of moving planes went back to Alexandrov [5] and was popularized by Serrin [173], and in particular, by Gidas-Ni-Nirenberg [63, 64].

The second step is an argument of shooting method due to Kwong [109]. The idea is the following. Since a positive solution Q_c of (1.2.2) must be radial by the first step, using the polar coordinate, we can replace (1.2.2) by

$$\begin{cases} \partial_r^2 Q_c + \frac{n-1}{r}\partial_r Q_c + Q_c^{p-1} = cQ_c \\ Q_c'(0) = 0, \lim\limits_{r\to+\infty} Q_c(r) = 0 \\ Q_c > 0, \end{cases}$$

We replace the condition $\lim\limits_{r\to+\infty} Q_c(r) = 0$ by $Q_c(0) = a$ with $a \in \mathbb{R}$. The goal is to prove that there exists a unique $a \in \mathbb{R}$ such that the solution Q_a associated to

$$\begin{cases} \partial_r^2 Q_a + \frac{d-1}{r}\partial_r Q_a + Q_a^{p-1} = cQ_a \\ Q_a'(0) = 0, Q_a(0) = a \\ Q_a > 0, \end{cases}$$

verifies $\lim\limits_{r\to+\infty} Q_a(r) = 0$. For more details of the proof, we refer to the lecture notes of Frank [59].

Since the equation is not completely integrable, we cannot anymore use the inverse scattering method to get the existence of N-soliton. A proof has been introduced by Martel [120] in 2005 inspired by the work of Merle for the blow-up of the non-linear Schrödinger equation [139]. We will come back later to the construction of N-soliton. First, we discuss the well-posedness of the gKdV equation as well as the existence and the classification of blow-up solutions.

1.2.3 Well-posedness results

Different results for the well posedness are expected for the different powers of the non-linearity depending on a scaling property of the solutions. More precisely, let u be a solution of (1.2.1) and $\lambda > 0$, then

$$u_\lambda(t,x) = \lambda^{\frac{1}{p-1}} u(\lambda^{\frac{3}{2}} t, \sqrt{\lambda} x)$$

is also a solution of (1.2.1). Let us look the L^2 norm of the solution u_λ. By changing the variable we get

$$\|u_\lambda(t,\cdot)\|_2 = \lambda^{\frac{1}{p-1}-\frac{1}{4}} \|u(t,\cdot)\|_2.$$

We say the gKdV equation is

- L^2***-sub-critical*** if $\frac{1}{p-1} - \frac{1}{4} > 0 \iff p < 5$,
- L^2***-critical*** if $\frac{1}{p-1} - \frac{1}{4} = 0 \iff p = 5$,
- and L^2***-super-critical*** if $\frac{1}{p-1} - \frac{1}{4} < 0 \iff p > 5$.

It is conjectured that if a partial differential equation is L^2-sub-critical then the solutions are globally well-posed in $L^2(\mathbb{R})$, and when the equation is L^2-critical or L^2-super-critical, blow-up in finite time may occur.

The problem of the well-posedness of gKdV equation has been studied by many authors Saut-Teman in 1976 [169], Kato in 1983 [89], Ginibre-Tsutsumi in 1989 [65], Bourgain in 1993 [26], Kenig-Ponce-Vega in 1993 [93] and in 1996 [94], Colliander-Kell-Staffilani-Takaoka-Tao in 2003 [33], Molinet-Ribaud in 2003 [149], Grünrock in 2005 [69], Guo in 2009 [70], Kishimoto in 2009 [99] and Killip-Kwon-Shao-Visan in 2009 [97]. A fundamental result needed here is due to Kenig-Ponce-Vega [93]. They proved that the equation is locally well-posed in the energy space $H^1(\mathbb{R})$ for all $p \in \mathbb{N}^*$, and if $2 \leqslant p < 5$ then the solutions are globally well-posed in $H^1(\mathbb{R})$ using the energy and mass conservation and the following Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}} u^{p+1}(x)dx \leqslant C_p \left(\int_{\mathbb{R}} u^2(x)dx\right)^{\frac{p+3}{4}} \left(\int_{\mathbb{R}} (\partial_x u)^2(x)dx\right)^{\frac{p-1}{4}}, \quad \forall u \in H^1(\mathbb{R}).$$

1.2.4 Stability of the solitons

Let $c > 0$. We define two types of stability. First, we have the **orbital stability**. We say Q_c is orbitally stable in $H^s(\mathbb{R})$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if

$$\|u_0 - Q_c\|_{H^s} \leqslant \delta,$$

then for all $t \in \mathbb{R}$ there exists $x(t)$ such that

$$\|u(t, x + x(t)) - Q_c\|_{H^s} \leqslant \varepsilon.$$

The first result on the $H^1(\mathbb{R})$ orbital stability of a soliton has been proved by Benjamin in 1972 [15], Bona in 1975 [21] and Weinstein in 1986 [188] for the subcritical case. However, for the critical and super-critical cases the solitons are not orbitally stable. The first proof of this result has been provided by Bona-Souganidis-Strauss in 1987. The case of N-soliton has been studied by Maddocks-Sachs in 1993 [118]. They obtained the orbital stability of N-soliton in $H^N(\mathbb{R})$ only for the KdV equation. The stability result of N-soliton has been improved in 2002 by Martel-Merle-Tsai [130]. They proved the orbital stability in $H^1(\mathbb{R})$ for the all subcritical range. Moreover, in [130], they proved another type stability of the N-soliton in $H^1(\mathbb{R})$. The ***asymptotic stability*** is defined has following:

Let $c > 0$, $x_0 \in \mathbb{R}$. We say Q_c is asymptotically stable in $H^s(\mathbb{R})$ if there exists $\varepsilon > 0$ such that if

$$\|u_0 - Q_c\|_{H^s} \leqslant \varepsilon,$$

then there exists $c_{+\infty}$ and for all $t \in \mathbb{R}$ there exists $x(t) \in \mathbb{R}$ such that

$$u(t, \cdot + x(t)) \rightharpoonup Q_{c_{+\infty}} \text{ in } H^s(\mathbb{R}).$$

The asymptotic stability of the solitons has been proved first by Pego-Weinstein in 1994 [163] and by Martel-Merle in 2001 [121] for an initial data in the energy space. Moreover Martel-Merle-Tsai proved in 2002 [130] the asymptotic stability of the N-soliton in $H^1(\mathbb{R})$.

1.2.5 Interactions of solitons

Let us come back to the N-soliton solution. Such solutions behave at infinity like a sum of N decoupled solitons. There are two types of interactions. The first one is when the relative distance between the different solitons increase linearly. In this case, we say the solitons are ***weakly interacting***. On the other hand, if the relative distance is sub-linear, like logarithmic or on the form t^β with $\beta < 1$, we say the solitons are ***strongly interacting***. In this thesis, we call the weak interaction by N-soliton solution and the strong interaction by strongly interacting N-soliton solution.

The first proof of the existence of N-soliton solution not based on the theory of complete integrability was done in 1998 by Feireisl [53] for non-linear damped wave equations. The proof is based on the existence of a Palais-Smale sequence related to some stationary problem.

New techniques have been introduced by Martel in 2005 [120] to construct N-soliton solutions for the sub-critical and critical case ($2 \leqslant p \leqslant 5$) of the gKdV equation. This pioneer result is inspired by a fundamental work of Merle [139] for the blow-up of the NLS equation. An outline of this construction is given in Section 2.1.4. Later, the existence of N-soliton solution for the super-critical case of the gKdV and NLS equation has been achieved by Cote-Martel-Merle in 2011 [39], see also Combet [35]. Several constructions of N-soliton solutions for a large variety of equations, like the NLS equation [39, 56, 124], Klein-Gordon equation [13, 38, 41], water waves equation [142], Zakharov-Kuznetsov equation [182], are based on this method.

A different method for the construction of the N-soliton solutions, based on a fixed point argument of Merle in [139], has been introduced by Le Coz, Li and Tsai in [114]. We refer also to Chen for the wave equation [31] and Van Tin for the derivative NLS [183] for other construction by fixed point.

Few results are known for the strong interaction. The construction is derived from an argument developed by Martel-Raphael in 2018 [135] for blow-up for the critical NLS equation. Nguyen in [156] constructed a 2-soliton solution interacting strongly with logarithmic distance for the gKdV equation, in [157] for the NLS equation, and Martel-Nguyen in [132] for the cubic NLS equation.

1.2.6 Blow-up in the critical and super-critical cases

The existence and the classification of blow-up solutions are difficult questions which have only been partially solved. Until now, the solitons play an essential role in the blow-up process in the critical case.

First, from the variational characterization of the solitons Q, we get the sharp Gagliardo-Nirenberg inequality [186]

$$\frac{1}{6}\|u\|_6^6 \leqslant \frac{1}{2\|Q\|_2^2}\|u\|_2^2\|\partial_x u\|_2^2, \quad \forall u \in \mathcal{S}(\mathbb{R}).$$

In particular, for the critical case $p = 5$, we have that if $\|u_0\|_2 < \|Q\|_{L^2}$, then the solution u, associated to u_0 and solving gKdV equation, is global and uniformly bounded in $H^1(\mathbb{R})$.

Secondly, in the case of the gKdV equation the solitons can be seen as the unique universal attractor of the flow in the singular regime. In other words, if u is a solution of (1.2.1) blows up in finite or infinite time T_{max}, then

$$\forall x \in \mathbb{R}, \quad \lim_{t\to T_{max}} \lambda^{\frac{1}{p-1}}(t)u(\lambda^{\frac{3}{2}}(t)t, \sqrt{\lambda(t)}(x - x(t)) = Q(x),$$

with $0 < \lambda(t)$, $\lim_{t\to T_{max}} \lambda(t) = 0$ and $x(t) \in \mathbb{R}$.

Numerical simulations by Bona-Dougalis-Karakashian-McKinney in 1995 [22] have indicated that blow-ups may occur in finite time. The first rigorous proofs of existence of blow-up for the critical case $p = 5$ were established by Merle in 2001 [140] and Martel-Merle in 2002 [122, 123]. Later a complete classification around Q was obtained by Martel-Merle-Raphael in 2014 [128] assuming some decay on the right, and leading in particular to a stable blow-up, in this topology, with a blow-up rate $\frac{1}{T-t}$. Note that exotic blow-up with different rates, $\frac{1}{(T-t)^\nu}$, in the neighbourhood of Q were also constructed by Martel-Merle-Raphael in 2015 [129] and Martel-Pilod in 2021 [134]

For the supercritical case, $p > 5$, few results are known. Pioneering numerical studies by Bona-Dougalis-Karakashian-McKinney in 1995 [22], Dix-McKinney in 1998 [45] and more recent works Klein-Peter in 2015 [101] exhibited self-similar blow-up for the supercritical equation which was confirmed rigorously by Koch in 2015 [103], Lan in 2017 [110] for $p > 5$ close to 5.

1.3 Fractional Korteweg-de Vries equation

The dispersion of the KdV equation is too strong in high frequencies when compared to the full water waves system. To solve this problem, one idea is to decrease the dispersion of the KdV equation. One possibility is to replace the dispersion of the KdV equation with the dispersion of the linearized water waves system. This idea was introduced by Whitham in [190] to derive the Whitham equation

$$\partial_t u + \sqrt{K_{mu}}(D)\partial_x u + \varepsilon\partial_x(u^2) = 0,$$

with $\mathcal{F}(\sqrt{K_{mu}}(D)u)(\xi) = \dfrac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}(1+\beta\mu\xi^2)\mathcal{F}(u)(\xi)$, $0 < \mu, \varepsilon \ll 1$ are small parameters related to the level of dispersion and nonlinearity, and $\beta > 0$ is related to the surface tension.

Another generalization of KdV is obtained by replacing the dispersion of KdV, given by $\partial_x^3 = -(-\partial_x^2)\partial_x$, by $-|D|^\alpha\partial_x$, with $\mathcal{F}(|D|^\alpha u)(\xi) = |\xi|^\alpha\mathcal{F}(u)(\xi)$. Then, one gets the ***fractional Korteweg-de Vries equation (fKdV)***

$$\partial_t u(t,x) - |D|^\alpha\partial_x u(t,x) + \partial_x\left(u^2\right)(t,x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}, \tag{1.3.1}$$

with $\alpha \in \mathbb{R}$. The fKdV recovers some well-known equation. When $\alpha = 2$, (1.3.1) becomes the KdV equation, when $\alpha = 1$, it becomes the Benjamin-Ono equation. For $\alpha = \frac{1}{2}$ and $\alpha = -\frac{1}{2}$ it is somehow reminiscent of the linear dispersion of the finite depth water waves equation with and without surface tension. In other words, the case $\alpha = \frac{1}{2}$ and $\alpha = -\frac{1}{2}$, for large frequencies, corresponds to the Whitham equations with and without surface tension.

In this thesis, we will only focus on the fKdV equation.

1.3.1 A particular case: Benjamin-Ono equation

In this subsection, we focus on the ***Benjamin-Ono equation (BO)***, corresponding to (1.3.1) with $\alpha = 1$

$$\partial_t u(t,x) - \mathcal{H}\partial_x^2 u(t,x) + \partial_x(u^2)(t,x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R},$$

with $\mathcal{H}$ denotes the Hilbert transform. The well-posedness has been extensively studied, and we cite only some results. Ionescu-Kenig in 2007 [82] proved the global well-posedness in $L^2(\mathbb{R})$ based on a gauge transformation introduced by Tao in 2004 [180].

Even if this equation is non-local, the BO equation is completely integrable, and admits therefore, an explicit formula for the soliton Q_c

$$Q_c(x) = \frac{4c}{1+c^2x^2},$$

with Q_c solves

$$\mathcal{H}Q' + cQ - Q^2 = 0.$$

An important difference with the soliton of KdV, is the decay of the soliton in the non local case. The soliton is algebraically decaying at infinity. This will play an important role in the construction of the N-soliton for a general value of α. We will come back on it in the part 1.4.1 Weak interactions.

Moreover, the soliton is unique. The proof of the uniqueness is based on a harmonic extension argument. Let Q be a soliton of the BO equation. By taking the convolution of Q with the Poisson kernel P, we get that $U(x,y) = Q *_x P(x,y)$ is the harmonic extension of Q on the upper half-plane. We recall that $\lim_{y\to 0}\partial_y U(x,y) = |D|^1 Q = \mathcal{H}\partial_x Q$ (see Stein chapter III [175]). Using this result we can replace the equation of Q by the following elliptic problem with Neumann condition

$$\begin{cases} \Delta U(x,y) = 0, \quad x \in \mathbb{R}, y > 0, \\ \partial_y U(x,0) = cU(x,0) - U^2(x,0), \quad x \in \mathbb{R}, \\ U(x,0) \to 0, \quad |x| \to +\infty. \end{cases}$$

The rest of the proof is based on complex analysis. This proof has been derived by Amick-Toland in 1991 [8] relying on an idea proposed by Benjamin in 1967 [14].

The solitons are known, for the BO equation, to be orbitally stable, see Bennett-Brown-Stansfield-Stroughair-Bona in 1983 [16] and Weinstein in 1987 [189]. More recently, Kenig-Martel in 2009 [90] proved the asymptotic stability of the soliton.

By using the complete integrability of the BO equation, Matsuno in 1979 [136] gave an explicit formula for the N-soliton. Moreover, Kenig-Martel also proved in 2009 [90] that the N-soliton are orbitally and asymptotic stable. The two main ingredients for the asymptotic stability are a monotonicity property and a Liouville type theorem. The Liouville type theorem was introduced first by Martel-Merle in 2001 [121] for the asymptotic stability of the solitons of KdV in the subcritical case. To obtain the monotonicity property, Kenig-Martel studied the evolution of a truncated mass

$$M_{\varphi}(u)(t) = \int_{\mathbb{R}} u^2(t,x)\varphi(x)dx,$$

with u solution of the BO equation and $\varphi(x) = \frac{\pi}{2} + \arctan(x)$.

Unlike the KdV equation, the BO equation is non local. We cannot use integration by part in order to get a sign for $\partial_t M(u)$. To overcome this difficulty, Kenig-Martel introduced the following weighted estimates for the Hilbert transform

$$\int_{\mathbb{R}} \left(\mathcal{H}\partial_x u\right) u\varphi' dx \leqslant C \int_{\mathbb{R}} u^2\varphi' dx, \tag{1.3.2}$$

$$\left| \int_{\mathbb{R}} \left(\mathcal{H}\partial_x u\right) \partial_x u\varphi dx \right| \leqslant C \int_{\mathbb{R}} u^2\varphi' dx. \tag{1.3.3}$$

The proof of the estimate (1.3.2) is based on a harmonic extension argument, whereas the estimate (1.3.3) is obtained by estimating directly the integral in different areas.

1.3.2 Well-posedness results

In this thesis, we focus on the case $\alpha \in (\frac{1}{2}, 2]$. Except for the case $\alpha = 1$ and $\alpha = 2$, the equation fKdV is not completely integrable. However, there exist two conserved quantities. The mass

$$M(u)(t) = \int_{\mathbb{R}} u^2(t,x)dx,$$

and the energy

$$E(u)(t) = \int_{\mathbb{R}} \frac{\left(|D|^{\frac{\alpha}{2}}u\right)^2(t,x)}{2} - \frac{u^3(t,x)}{3}dx.$$

The fKdV equation admits a scaling invariance, in other words, if u is a solution of (1.3.1), then

$$\forall x_0 \in \mathbb{R}, c > 0, \quad u_c(t,x) = cu\left(c^{\frac{1+\alpha}{\alpha}}t, c^{\frac{1}{\alpha}}(x - x_0)\right),$$

is also a solution. Therefore, the equation is

- L^2***-sub-critical*** if $\alpha > \frac{1}{2}$,
- L^2***-critical*** if $\alpha = \frac{1}{2}$,
- L^2***-super-critical and*** $H^{\frac{\alpha}{2}}$***-sub-critical*** if $\frac{1}{3} < \alpha < \frac{1}{2}$,
- $H^{\frac{\alpha}{2}}$***-critical*** if $\alpha = \frac{1}{3}$,
- and $H^{\frac{\alpha}{2}}$***-super-critical*** if $\alpha < \frac{1}{3}$,

Relying on numerical simulation by Klein-Saut in 2015 [101], it has been conjectured that in the sub-critical range $\frac{1}{2} < \alpha$, the Cauchy problem to (1.3.1) is globally well-posed in the energy space $H^{\frac{\alpha}{2}}(\mathbb{R})$. Herr-Ionescu-Kenig-Koch proved this result for $\alpha \in [1, 2)$. More recently, Molinet-Pilod-Vento proved the global well-posedness in the energy space $H^{\frac{\alpha}{2}}(\mathbb{R})$ for $\alpha > \frac{6}{7}$. For the rest of the range, for $\alpha \in (\frac{1}{2}, \frac{6}{7}]$, the global well-posedness in the energy space is still an open question.

1.3.3 Existence and properties of solitons

In this subsection, we summarize the properties of the soliton for the fKdV equation. Let $c > 0$. The soliton Q_c is a solution of the non-local ODE

$$|D|^\alpha Q_c + cQ_c - Q_c^2 = 0. \tag{1.3.4}$$

Except for the cases $\alpha = 1$ and $\alpha = 2$, no explicit formula for the soliton is known. The existence of soliton for (1.3.1) was proved by Weinstein in 1987 [189] and Albert-Bona-Saut in 1997 [4] by minimizing the functional

$$J^\alpha(u) = \frac{\left(\displaystyle\int ||D|^{\frac{\alpha}{2}}u|^2\right)^{\frac{1}{2\alpha}} \left(\displaystyle\int |u|^2\right)^{\frac{\alpha-1}{2\alpha+1}}}{\displaystyle\int |u|^3}.$$

A solution Q of the minimising problem is called a ***ground state***. From the decay of the soliton for the BO equation, algebraic decay of the ground state was expected in the whole range $(\frac{1}{2}, 2)$. Indeed it was proved by Frank-Lenzmann-Silvestre in 2016 [61] and Kenig-Martel-Robbiano in 2011 [91] that there exists $C_1, C_2 > 0$

$$\frac{C_1}{(1+x^2)^{\frac{1+\alpha}{2}}} \leqslant Q(x) \leqslant \frac{C_2}{(1+x^2)^{\frac{1+\alpha}{2}}}, \quad x \in \mathbb{R}.$$

However, in this thesis, we will derive a more precise asymptotic development of Q and its derivatives, for $\alpha \in [\frac{1}{2}, 2)$ (see Chapter 3).

1. (*First-order expansion*) The function Q verifies the following decay estimate

$$Q^{(j)}(x) - (-1)^j \frac{(\alpha+j)!}{\alpha!} \frac{a_1}{x^{1+\alpha+j}} = o_{+\infty}\left(\frac{1}{x^{1+\alpha+j}}\right), \quad j \in \mathbb{N},$$

for some $C_j > 0$, with $a_1 = k_1 \|Q\|_2^2 > 0$ and $k_1 \in \mathbb{R}$.

2. (*Higher order expansion*) There exists $C > 0$ such that

$$\begin{aligned} &Q(x) - \left(\frac{a_1}{x^{\alpha+1}} + \frac{a_2}{x^{2\alpha+1}}\right) = o_{+\infty}\left(\frac{1}{x^{2\alpha+1}}\right), \\ &Q'(x) + (\alpha+1)\frac{a_1}{x^{\alpha+2}} + (2\alpha+1)\frac{a_2}{x^{2\alpha+2}} = o_{+\infty}\left(\frac{1}{x^{2\alpha+2}}\right), \end{aligned}$$

with $a_2 = k_2 \|Q\|_2^2$, and $k_2 \in \mathbb{R}$.

Another important property of the ground state is its uniqueness. Since the uniqueness for the case $\alpha = 2$ is based on classical results on an ODE problem, like Sturm-Liouville's argument, it is not possible to adapt the proof from the case $\alpha = 2$ to the general case. Moreover, the uniqueness for $\alpha = 1$ is based on a harmonic extension process. Although, an extension problem for the operator $|D|^\alpha$ was introduced in the seminal paper of Cafarelli-Silvestre in 2007 [29], we do not know how to find a good Green function for (1.3.4), which does not allow us to use the same arguments than in the case $\alpha = 1$.

Recently, Frank-Lenzmann in 2013 [60] proved the uniqueness of the ground state, via the following argument. Let fix $\alpha_1 \in (\frac{1}{3}, 2]$. By using the implicit function theorem, the map $\Phi(Q, \lambda, \alpha) := \begin{pmatrix} Q - \frac{1}{|D|^\alpha + \lambda} Q^2 \\ \|Q\|_4^4 - c_0 \end{pmatrix}$ is C^1. In other words, one can follow the ground state continuously depending on the dispersion's value α. The implicit function theorem gives only the existence of Φ on a neighborhood of $\alpha_1 \in (\frac{1}{3}, 2]$. The goal is then to extend the function Φ until $\alpha = 2$. Once the function Φ has been extended, combining the uniqueness in the implicit function theorem and the uniqueness of the problem in the case $\alpha = 2$, allow to conclude the proof of the uniqueness in the general case. It is worth noticing that the result of uniqueness is only valid for the ground state. The uniqueness of the solution of (1.3.4) which does not minimise the functional J^α is still an open question. Furthermore, as a by-product of their proof, Frank-Lenzmann in [60] also obtained that the non-degeneracy of the kernel for the following unbounded operator on $L^2(\mathbb{R})$

$$Lu = |D|^\alpha u + u - 2Qu.$$

More precisely, they proved $\ker(L) = \operatorname{span}\{Q'\}$. To describe $\ker(L)$ they use the extension process introduced by Caffarelli-Silvestre [29] to see $|D|^\alpha$ as a Dirichlet-Neumann operator for a suitable elliptic problem on the upper-half plane.

1.4 Main results

1.4.1 Weak interactions

The first result of this thesis is the construction of N-soliton for the fKdV equation in the sub-critical range $\frac{1}{2} < \alpha < 2$.

Theorem 1.4.1. *We assume $\alpha \in]\frac{1}{2}, 2[$. Let $N \in \mathbb{N}$, $0 < c_1 < \cdots < c_N < +\infty$. Then, there exist some constants $T_0 > 0, C_0 > 0$, N functions $\rho_1, \cdots, \rho_N \in C^1([T_0, +\infty[)$ and $U \in C^0([T_0, +\infty[: H^{\frac{\alpha}{2}}(\mathbb{R}))$ solution of* (1.3.1) *such that, for all $t \geqslant T_0$,*

$$\left\| U(t, \cdot) - \sum_{j=1}^{N} Q_{c_j}(\cdot - \rho_j(t)) \right\|_{H^{\frac{\alpha}{2}}} \leqslant \frac{C_0}{t^{\frac{\alpha}{2}}},$$

$$|\rho_j(t) - c_j t| \leqslant t^{1-\frac{\alpha}{4}} \quad \textit{and} \quad |\rho_j'(t) - c_j| \leqslant \frac{C_0}{t^{\frac{\alpha}{2}}},$$

for all $j \in \{1, \cdots, N\}$.

The proof is given in Chapter 2. Below we explain the key ideas of the proof. The construction of the N-solitons for fKdV follows the line of the construction for KdV. We use the techniques introduced by Merle [139] and Martel [124]. The novelty of this construction is a deeper understanding of the non-local commutator estimates. For the construction of the N-solitons, we need to estimate the time evolution of the localized mass and the localized energy

$$M_\varphi(u)(t) = \int_{\mathbb{R}} u^2(t,x)\varphi(x)dx, \quad E_\varphi(u)(t) = \int_{\mathbb{R}} \left(\frac{u\left(|D|^\alpha u\right)^2(t,x)}{2} - \frac{u^3(t,x)}{3} \right) \varphi(x)dx.$$

To this end, we generalize the estimates

$$\left| \int \left(|D|^\alpha u\right) u|\varphi'| - \int \left(|D|^{\frac{\alpha}{2}} \left(u\sqrt{|\varphi'|}\right)\right)^2 \right| \leqslant C \int u^2 |\varphi'|, \tag{1.4.1}$$

and

$$\left| \int \left(|D|^\alpha u\right) \partial_x u \varphi + \frac{\alpha - 1}{2} \int \left(|D|^{\frac{\alpha}{2}} \left(u\sqrt{|\varphi'|}\right)\right)^2 \right| \leqslant C \int u^2 |\varphi'|, \tag{1.4.2}$$

introduced by Kenig-Martel-Robbiano in 2011 [91], to a non symmetric form

$$\left| \int \left(\left(|D|^\alpha u\right) v - \left(|D|^\alpha v\right) u\right) |\varphi'| \right| \leqslant \begin{cases} C \displaystyle\int \left(u^2 + v^2\right) |\varphi'|, & \text{if } \alpha \in]0, 1], \\ C \displaystyle\int \left(u^2 + v^2 + \left(|D|^{\frac{\alpha}{2}} u\right)^2\right) |\varphi'|, & \text{if } \alpha \in]1, 2[, \end{cases} \tag{1.4.3}$$

and

$$\left| \int \left(\left(|D|^{\alpha} u \right) \partial_x v + \left(|D|^{\alpha} v \right) \partial_x u \right) \varphi + (\alpha - 1) \int |D|^{\frac{\alpha}{2}} \left(u \sqrt{|\varphi'|} \right) |D|^{\frac{\alpha}{2}} \left(v \sqrt{|\varphi'|} \right) \right|$$
$$\leqslant \begin{cases} C \int \left(u^2 + v^2 \right) |\varphi'|, & \text{if } \alpha \in]0,1], \\ C \int \left(u^2 + v^2 + \left(|D|^{\frac{\alpha}{2}} u \right)^2 \right) |\varphi'|, & \text{if } \alpha \in]1,2[, \end{cases} \quad (1.4.4)$$

Note that for the case $\alpha = 1$, the estimates (1.4.1) and (1.4.2) were proved by Kenig-Martel in 2009 [90]. The estimate (1.4.1), for $\alpha = 1$, is proved by using a harmonic extension process, and (1.4.2), for $\alpha = 1$, is proved by estimating directly the integral on different areas.

Even with the extension result of Caffarelli-Silvestre in hand, it is not clear how to adapt the proof of (1.4.1) and (1.4.2) in the case $\alpha = 1$ by Kenig-Martel to the general case $\alpha \in (0, 2)$ by using similar arguments. To overcome this difficulty, Kenig-Martel-Robbiano relied on the pseudo-differential calculus to prove estimate (1.4.1) and (1.4.2) in the case $\alpha \in (1, 2)$.

Note that, since the multiplier associated with $|D|^{\alpha}$ is singular at zero, this will imply a strong restriction on the weight φ to localize the mass and the energy. Indeed, to obtain the different estimates, we look at these quantities in high and low frequency. Note that, a derivative corresponds to a polynomial weight in high frequency on the Fourier side. When we are looking at the low frequency, it becomes difficult to transfer derivative on the weight of the remainder term on the right-hand side of (1.4.2) and (1.4.4). The restriction on the weight is therefore a consequence of the singularity of the operator $|D|^{\alpha}$.

However, the estimates (1.4.1) and (1.4.2) are not enough to control the localized energy E_{φ}. For this purpose, we extend the result in [91] to a non symmetric one, see (1.4.3) and (1.4.4). More precisely, these non-symmetric weighted commutator estimates allow us to deal with non-symmetric cross terms involving the gradient term and the potential term.

Then, the proof of the construction is based on a classical bootstrap argument. However, the restriction on the weight makes it difficult to close the bootstrap argument by a direct integration of the differential inequalities on the different parameters. To bypass this difficulty, we use a topological argument to adapt carefully the initial data of the geometrical parameters ρ_j, with $j \in \{1, ..., N\}$. Therefore, in Theorem 1.4.1, the N-solitons constructed depend on the N functions ρ_j, with $j \in \{1, ..., N\}$, which are not explicitly given by $c_j t$ as in the other constructions . This is a consequence of the lack of decay of the weight φ. More details for this method of construction are given in Section 2.1.4.

A physical explanation of Theorem 1.4.1 could be that since the solitons are decaying algebraically, they are interacting too much to get a solution that will behave like

$$\sum_{j=1}^{N} Q_{c_j}(x - x_j - c_j t), \quad (x_j)_{j=1}^{N} \in \mathbb{R}^N, (c_j)_{j=1}^{N} \in \mathbb{R}_+^N.$$

Another possibility is that unlike in cases where the solitons are exponentially decaying, it is not possible to construct N-solitons without understanding more deeply the interactions between the different solitons. A solution may be to use a profile decomposition in order to understand this interaction. The profile decomposition method is explained in the section of the strong interactions for a cubic non-linearity.

1.4.2 Asymptotic behaviour of the ground states for a general non-linearity

The second result of this Ph.D. is the asymptotic development of solutions for some generalizations of the fractional semi-linear elliptic equation (1.3.4). The set of solutions for the equation (1.3.4) admits potentially a complex structure. For example, the existence and uniqueness of a solution of (1.3.4) not minimizing the functional J^{α} remain open problems. The existence and uniqueness of the ground state exposed in the previous section was an important step in the understanding of the elliptic equation (1.3.4). To continue the study of this fractional elliptic equation, we studied the asymptotic development of some solutions of (1.3.4) in a joint work with F. Valet.

The asymptotic expansion of Q relies on the operator $k(x) := (1+|D|^\alpha)^{-1} = \mathcal{F}^{-1}\left(\frac{1}{1+|\xi|^\alpha}\right)$, with $\mathcal{F}^{-1}$ denoting the inverse Fourier transform. The main tool to study the kernel $k(x)$ is a contour integration, introduced by Pólya in 1923 [165], to get the asymptotic development of k and k'. From the asymptotic of k and k', one can derive the development of Q, the asymptotic development of order 1 for the derivatives of Q, and also higher order expansion for Q. With F. Valet, we proved the following results.

Proposition 1.4.2. *Let $\alpha \in (0,2)$, $p \in (1, p^*(\alpha))$ and Q be a weak solution of $|D|^\alpha Q + Q - |Q|^{p-1}Q = 0$, satisfying:*

$$Q \in L^p(\mathbb{R}) \quad and \quad \exists l > 0, \quad |x|^l Q(x) \in L^\infty(\mathbb{R}). \tag{1.4.5}$$

Then, $Q \in C^0(\mathbb{R})$ and verifies:

$$Q(x) - \frac{a_1}{x^{\alpha+1}} = o_{+\infty}\left(\frac{1}{x^{\alpha+1}}\right),$$

with $a_1 \in \mathbb{R}$.

Futhermore, if $\alpha > 1$, then $Q \in C^{\lfloor p \rfloor + 1}(\mathbb{R})$ with $\lfloor p \rfloor$ the floor function of p, and verifies for $j \leqslant \lfloor p \rfloor$, that for all $x > 1$:

$$Q^{(j)}(x) - (-1)^j \frac{(\alpha+j)!}{\alpha!}\frac{a_1}{x^{\alpha+1+j}} = o_{+\infty}\left(\frac{1}{x^{\alpha+1+j}}\right).$$

Proposition 1.4.3. *Let $\alpha \in (0,2)$. Let Q satisfying the assumptions* (1.4.5) *of Proposition 1.4.2.*

- *If the coefficient of the non-linearity p is an integer then $Q \in H^\infty(\mathbb{R})$.*
- *If Q is positive, then $Q \in H^\infty(\mathbb{R})$, even (up to translation), decaying and verifies that:*

$$Q^{(j)}(x) - (-1)^j \frac{(\alpha+j)!}{\alpha!}\frac{a_1}{x^{\alpha+1+j}} = o_{+\infty}\left(\frac{1}{x^{\alpha+1+j}}\right), \quad \forall j \in \mathbb{N},$$

and the next order asymptotic expansion holds for some $a_2 \in \mathbb{R}$:

$$\begin{aligned}
&\textit{Case } p < \frac{2\alpha+1}{\alpha+1}: \quad Q(x) - \frac{a_1}{x^{\alpha+1}} - \frac{a_1^p}{x^{p(\alpha+1)}} = o_{+\infty}\left(\frac{1}{x^{p(\alpha+1)}}\right).\\
&\textit{Case } p = \frac{2\alpha+1}{\alpha+1}: \quad Q(x) - \frac{a_1}{x^{\alpha+1}} - \frac{a_1^p}{x^{2\alpha+1}} - \frac{a_2}{x^{2\alpha+1}} = o_{+\infty}\left(\frac{1}{x^{2\alpha+1}}\right).\\
&\textit{Case } p > \frac{2\alpha+1}{\alpha+1}: \quad Q(x) - \frac{a_1}{x^{\alpha+1}} - \frac{a_2}{x^{2\alpha+1}} = o_{+\infty}\left(\frac{1}{x^{2\alpha+1}}\right).
\end{aligned}$$

Proposition 1.4.4. *Let $p \in \mathbb{N}$, $p \geqslant 2$, $\alpha \in \left(\frac{p-1}{1+p}, 2\right)$, and Q be solution of $|D|^\alpha Q + Q - Q^p = 0$ verifying condition* (1.4.5)*.*

Then $Q \in H^\infty(\mathbb{R})$ and verifies:

$$Q^{(j)}(x) - (-1)^j \frac{(\alpha+j)!}{\alpha!}\frac{k_1}{x^{\alpha+1+j}} \int Q^p = o_{+\infty}\left(\frac{1}{x^{\alpha+1+j}}\right), \quad \forall j \in \mathbb{N},$$

with $k_1 \in \mathbb{R}$

Proposition 1.4.5 (Higher order expansion)**.** *Let $\alpha \in (1,2)$, $p = 3$, and Q be a solution of $|D|^\alpha Q + Q - Q^p = 0$ verifying condition* (1.4.5)*. Then, there exists a constant $C = C(\alpha, p) > 0$, $a_3 \in \mathbb{R}$:*

$$\begin{aligned}
&\left|Q(x) - \left(\frac{a_1}{x^{\alpha+1}} + \frac{a_2}{x^{2\alpha+1}} + \frac{a_3}{x^{\alpha+3}}\right)\right| \leqslant \frac{C}{x^{3\alpha+1}},\\
&\left|Q'(x) + (\alpha+1)\frac{a_1}{x^{\alpha+2}} + (2\alpha+1)\frac{a_2}{x^{2\alpha+2}}\right| \leqslant \frac{C}{x^{3\alpha+1}},.
\end{aligned}$$

The last proposition will be useful to study the interactions between solitons, and construct a strongly interacting 2-soliton solution for the **fractional modified KdV (fmKdV)**, as it will be seen in the next subsection.

1.4.3 Strong interactions for a cubic non-linearity

The last result of this PhD is the construction of a strongly interacting 2-soliton solution for the **fractional modified KdV (fmKdV)**

$$\partial_t u(t,x) - \partial_x |D|^\alpha u(t,x) + \partial_x(u^3)(t,x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}, \tag{1.4.6}$$

with $\alpha \in (1,2)$. This construction has been obtained in a joint work with F. Valet. We have proved the following result.

Theorem 1.4.6. *Let $\alpha \in (1,2)$. There exist some constants $T_0 > 0, C > 0$ and $U \in C^0([T_0,+\infty) : H^{\frac{\alpha}{2}}(\mathbb{R}))$ solution of* (1.4.6) *such that, for all $t \geqslant T_0$:*

$$\left\| U(t,\cdot) + Q\left(\cdot - t - \frac{a}{2} t^{\frac{2}{\alpha+3}}\right) - Q\left(\cdot - t + \frac{a}{2} t^{\frac{2}{\alpha+3}}\right) \right\|_{H^{\frac{\alpha}{2}}} \leqslant C t^{-\frac{\alpha-1}{4(\alpha+3)}},$$

where

$$a := \left(\frac{\alpha+3}{2} \sqrt{\frac{-4b_1}{\alpha+1}} \right)^{\frac{2}{\alpha+3}} \quad \text{and} \quad b_1 := -2\frac{(\alpha+1)^2}{\alpha-1} \frac{\sin(\frac{\pi}{2}\alpha)}{\pi} \int_0^{+\infty} e^{-\frac{1}{r^\alpha}} dr \frac{\|Q\|_{L^3}^6}{\|Q\|_{L^2}^2} < 0.$$

This construction follows the lines of the proof of the construction of the weakly interacting N-soliton solution (Theorem 1.4.1). The main difference here is the necessity to understand more deeply the interactions between the two solitons. The first construction of such solutions has been obtained by Nguyen in 2017 [156] for the gKdV equation and in 2019 [157] for the NLS equation and Nguyen-Martel [131] for the cubic NLS equation. The relative distance between the strongly interacting solitons is connected to the decay of the solitons. Since the solitons of the gKdV equation are exponentially decaying, Nguyen constructed a solution such that the relative distance is increasing logarithmically. However, in our construction, the distance between the 2 solitons is given by $t^{\frac{2}{\alpha+3}}$, since the solitons have an algebraic decay.

We explain now how to understand the interactions between two solitons R_1 and R_2. The natural approach would be to look for a solution on the form $u = R_1 + R_2 + \varepsilon$ as in the case of the weakly interacting solitons. However, in the case of strongly interacting solitons, the error created by the interaction between R_1 and R_2 will be too large to close the bootstrap on ε. For this reason we refine the ansatz and look for a solution on the form $u = R_1 + R_2 + P_1 + P_2 + bW + \widetilde{\varepsilon}$, where bW is localized between the two solitary waves R_1 and R_2. Furthermore bW has a plateau between R_1 and R_2. This function bW describes the interactions localized between the two solitary waves. The functions P_1, respectively P_2, are constructed to cancel the remainder terms which are localized close to R_1 respectively close to R_2. These two functions P_1, P_2 are called profiles. To construct P_1 and P_2, we use the properties of the linearized operator

$$Lu = |D|^\alpha u - u + 3Q^2 u,$$

to find a solution f, of some equations on the form

$$Lf = g + aQ,$$

where the term g will be given by the interaction between R_1 and R_2. The interaction between the 2 solitons can be understood by using the asymptotic development of Q. For example, let take $x > t$ and $R_1 = Q\left(\cdot - t - \frac{a}{2} t^{\frac{2}{\alpha+3}}\right)$ and $R_2 = Q\left(\cdot - t + \frac{a}{2} t^{\frac{2}{\alpha+3}}\right)$. Then, we can use the asymptotic development of Q to get "$R_1 R_2^2 \sim \left(\frac{1}{t^{\frac{2(\alpha+1)}{\alpha+3}}} + \frac{1}{t^{\frac{2(\alpha+1)}{\alpha+3}}} + \frac{1}{t^{\frac{2(2\alpha+1)}{\alpha+3}}} \right) R_2^2$". The function P_1, P_2 are constructed to absorb the term coming from the asymptotic expansion of Q, and to obtain a remainder term small enough to get a good control on $\widetilde{\varepsilon}$.

Since the equation is non-local, the proof makes use of the weighted commutator estimates (1.4.1)-(1.4.4), and some refinements introduced to track the constants with respect to the different parameters. More details of the proof are given in Chapter 4.

1.5 Open problem

In this PhD, we have proved the existence of N-soliton weakly and strongly interacting in a non-local context. However, some natural questions remain open. For the weakly interacting N-soliton, one could ask if it is possible in Theorem 1.4.1 to replace the ρ_j by $c_j t$. A possibility for that, would be to adapt the profiles construction introduced to construct strongly interacting 2-soliton in order to sharpen the estimate on the error.

The existence of weakly interacting N-soliton for fKdV equation in the critical and super-critical range, $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, remains open. The main difference, compare to the sub-critical case, will be the coercivity of the linearized operator L. For example, the coercivity of L is obtained up to two orthogonality conditions. In the sub-critical case, the orthogonality conditions are given by Q and Q'. This will give special cancellation and some parameters will be controlled by a quadratic term on the error. In the critical case, the orthogonality conditions can be driven by Q^3 and Q'. The special cancellation does not appear anymore. Since we are working with algebraic decaying solution, it is not clear whether the loss of this quadratic bound can be overcome.

The existence of weakly interacting N-soliton for the fractional non-linear Schrödinger equation is also an interesting question. The exponential decay of the soliton is an essential tool for the construction of the N-soliton for the non-linear Schrödinger. It is not clear in a context of algebraic decay how the proof could be adapted.

Another natural question is about the uniqueness of the solutions constructed in Theorem 1.4.1 and Theorem 1.4.6. The improvement of the understanding of (1.3.4) by the asymptotic development of some solutions, raises also the question of the uniqueness of this type of solutions.

In [91], Kenig-Martel-Robbiano proved by an argument of perturbation the existence of blow-up solutions, for the equation

$$\partial_t u(t,x) - \partial_x |D|^\alpha u(t,x) + |u|^{2\alpha}(t,x)\partial_x u(t,x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R},$$

for all $\alpha \in (\alpha_1, 2]$, for some $1 < \alpha_1 < 2$ close to 2. The question to extend this result for all $1 < \alpha < 2$ is still open. One strategy would be to adapt the result of Martel-Pilod [133] on the minimal mass blow-up on the cubic Benjamin-Ono equation, to get the blow-up on the full range $\alpha \in [1, 2]$.

A by-product of the paper [91] is the asymptotic stability around the solitons for all $\alpha \in (\alpha_1, 2]$ with $1 < \alpha_1 < 2$. The asymptotic stability around the solitons remains open the full range $\alpha \in \left(\frac{1}{2}, 2\right)$ of this equation and also the fKdV equation.

1.6 Outline of the thesis

The next chapter of this manuscript is dedicated of the construction of N-soliton like solutions for the fractional Korteweg-de Vries equation. In the third chapter we derive an asymptotic of non-linear ground states for fractional Laplacian, while the last chapter is devoted to the existence of strongly interacting solitary waves for the fractional modified Korteweg-de Vries equation.

Chapter 2

Construction of N-soliton like solution for the fractional Korteweg-de Vries equation

2.1 Introduction

2.1.1 The fractional Korteweg-de Vries equation

We consider the fractional Korteweg-de Vries equation (fKdV), also called the dispersion generalized Benjamin-Ono equation,

$$\partial_t u - |D|^\alpha \partial_x u + \partial_x(u^2) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{2.1.1}$$

where $\alpha \in \mathbb{R}$, $|D|^\alpha$ is the Riesz potential of order $-\alpha$, defined by $\mathcal{F}(|D|^\alpha u)(\xi) = |\xi|^\alpha \mathcal{F}(u)(\xi)$ and $\mathcal{F}$ is the Fourier transform.

In the cases $\alpha = 2$, respectively $\alpha = 1$, this equation corresponds to the well-known Korteweg de Vries (KdV), respectively Benjamin-Ono (BO) equations, which are completely integrable (see [57, 113]). In the case $\alpha = 0$, one recovers the inviscid Burgers' equation after a suitable change of variable, while the case $\alpha = -1$ corresponds to the Burgers-Hilbert equation. Finally, the cases $\alpha = \frac{1}{2}$ and $\alpha = -\frac{1}{2}$ are somehow reminiscent of the linear dispersion of the finite depth water waves equation with and without surface tension. In other words, for large frequencies, equation (2.1.1) corresponds in those cases to the Whitham equations with and without surface tension (see [100] for more details).

From a mathematical point of view, these equations are also useful to understand the "fight" between nonlinearity and dispersion. Instead of fixing the dispersion (*e.g.* that of the KdV equation) and increasing the nonlinearity (*e.g.* $u^p\partial_x u$ for the generalized KdV equation), one chooses to fix the nonlinearity $u\partial_x u$ and lower the dispersion, allowing then fractional dispersion of the form $|D|^\alpha$, $\alpha < 2$. As pointed out by Linares, Pilod and Saut in [115], this viewpoint is probably more physical since in many problems arising from physics or continuum mechanics the nonlinearity is quadratic with terms like $(u\cdot\nabla)u$ and the dispersion is in some sense weak. Here will focus on positive values of α's. Note however, that the dynamics for negative α's is quite different with the formation of shocks (see [81], [170], [159]).

Although equation (2.1.1) is not completely integrable outside of the cases $\alpha = 1$ and 2, it enjoys a hamiltonian structure. In particular, the mass

$$M(u)(t) := \int u^2(t,x)dx,$$

and the energy

$$E(u)(t) := \frac{1}{2}\int \left(|D|^{\frac{\alpha}{2}}u(t,x)\right)^2 dx - \frac{1}{3}\int u^3(t,x)dx.$$

are formally preserved by the flow of (2.1.1).

Moreover, we have the scaling-translation invariance of (2.1.1). Let u be a solution of (2.1.1) then

$$\forall x_0 \in \mathbb{R}, c > 0, u_c(t,x) = cu(c^{\frac{1+\alpha}{\alpha}}t, c^{\frac{1}{\alpha}}(x-x_0)),$$

is also a solution. A straightforward computation shows that $\|u_c\|_{\dot{H}^s} = c^{s+\alpha-\frac{1}{2}}\|u\|_{\dot{H}^s}$. In particular, equation (2.1.1) is *mass-critical* for $\alpha = \frac{1}{2}$ and *energy-critical* for $\alpha = \frac{1}{3}$.

In this paper, we focus on the *mass-subcritical* case $\alpha \in \left(\frac{1}{2}, 2\right)$. We assume that the initial value problem associated to (2.1.1) is globally well-posed in the energy space $H^{\frac{\alpha}{2}}(\mathbb{R})$ in the whole subcritical range $\frac{1}{2} < \alpha < 2$, in the sense that for all $u_0 \in H^{\frac{\alpha}{2}}(\mathbb{R})$ and $T > 0$, there exists a solution $u \in C([0,T] : H^{\frac{\alpha}{2}}(\mathbb{R}))$ of (2.1.1) satisfying $u(0,\cdot) = u_0$ which is unique in some class $X_T \subset C([0,T] : H^{\frac{\alpha}{2}}(\mathbb{R}))$, and that the flow : $u_0 \in H^{\frac{\alpha}{2}}(\mathbb{R}) \mapsto u \in C([0,T] : H^{\frac{\alpha}{2}}(\mathbb{R}))$ is continuous. Such a result has been proved by Herr, Ionescu, Kenig and Koch in [78] in the range $1 \leqslant \alpha < 2$, extending a previous result of Ionescu and Kenig for the BO equation [82]. For weaker dispersion, the global well-posedness in the energy space has been conjectured through numerical simulations by Klein and Saut [101] in the whole range $\frac{1}{2} < \alpha < 1$. Progress has been made in this direction: recently Molinet, Pilod and Vento proved in [148] global well-posedness in $H^{\frac{\alpha}{2}}(\mathbb{R})$ for $\frac{6}{7} < \alpha < 1$ (see also Linares, Pilod, Saut [115] for former results). Note however, that the problem is still open in the case $\frac{1}{2} < \alpha < \frac{6}{7}$.

Finally, we mention some other interesting results concerning the fractional KdV equation with positive dispersion α. In [47], Ehrnström and Wang proved long time existence for small initial data. Fonseca, Linares and Ponce in [58] proved some persistence results in weighted Sobolev spaces. Kenig, Ponce and Vega [95], Kenig, Ponce, Pilod and Vega in [92] and Riano in [167] proved some unique continuation results, while Mendez in [138], [137] proved propagation of regularity results. We also refer to Linares, Pilod and Saut [115] and Klein and Saut [101] for other results, conjectures and numerical simulations regarding the fractional KdV equation.

2.1.2 Solitary wave solutions

A fundamental property of this equation is the existence of solitary wave solutions of the form

$$u(t,x) = Q_c(x-ct) \quad \text{with} \quad Q_c(x) \underset{|x|\to+\infty}{\longrightarrow} 0,$$

for $c>0$, where $Q_c(x) = cQ(c^{\frac{1}{\alpha}}x)$ and Q is solution of the non-local ODE

$$|D|^\alpha Q + Q - Q^2 = 0. \tag{2.1.2}$$

In other words, Q_c satisfies

$$|D|^\alpha Q_c + cQ_c - Q_c^2 = 0. \tag{2.1.3}$$

For some particular values of α the solution of (2.1.2) is explicit and unique (up to translations). For $\alpha = 2$, $Q_{KdV}(x) = \dfrac{3}{2}\cosh^{-2}\left(\dfrac{x}{2}\right)$, while for $\alpha = 1$, $Q_{BO}(x) = 4(1+x^2)^{-1}$. The uniqueness result for BO is non-trivial and was proved by Benjamin [14] and Amick and Toland [8] by combining complex analysis techniques with properties of the harmonic extension of the Hilbert transform.

For the other values of α, there does not exist, as far as we know, any explicit formulation of Q. However, the existence of solutions of (2.1.2) minimising the functional

$$J^\alpha(u) = \frac{\left(\displaystyle\int ||D|^{\frac{\alpha}{2}}u|^2\right)^{\frac{1}{2\alpha}} \left(\displaystyle\int |u|^2\right)^{\frac{\alpha-1}{2\alpha+1}}}{\displaystyle\int |u|^3}. \tag{2.1.4}$$

is well-known since the work of Weinstein in [189] and Albert, Bona and Saut [4]. Such solutions are called *ground states* solutions of (2.1.2). They decay polynomially at infinity (see [91]), this property being related to the singularity at the origin of the symbol $|\xi|^\alpha$. Moreover, their uniqueness is delicate and was proved by Frank and Lenzmann in [60] relying on the non-degeneracy of the kernel of the linearized operator associated to Q. Below, we summarize the properties of the ground states of (2.1.2).

Theorem 2.1.1 ([4, 60, 91, 189]). *Let $\alpha \in \left(\frac{1}{3}, 2\right)$. There exists $Q \in H^{\frac{\alpha}{2}}(\mathbb{R}) \cap C^\infty(\mathbb{R})$ such that*

1. *(Existence) The function Q solves (2.1.2) and $Q = Q(|x|) > 0$ is even, positive and strictly decreasing in $|x|$. Moreover, the function Q is a minimizer of J^α in the sense that*

$$J^\alpha(Q) = \inf_{u\in H^{\frac{\alpha}{2}}(\mathbb{R})} J^\alpha(u).$$

2. *(Decay) The function Q verifies the following decay estimate*

$$\frac{1}{C(1+|x|)^{k+1+\alpha}} \leqslant Q^{(k)}(x) \leqslant \frac{C}{(1+|x|)^{k+1+\alpha}}, \quad k=0,1,2, \tag{2.1.5}$$

for some $C>0$.

3. *(Uniqueness) The even ground state solution $Q = Q(|x|) > 0$ of* (2.1.2) *is unique. Furthermore, every optimizer $v \in H^{\frac{\alpha}{2}}(\mathbb{R})$ for the Gagliardo-Nirenberg problem* (2.1.4) *is of the form $v = \beta Q(\gamma(\cdot + y))$ with some $\beta \in \mathbb{C}, \beta \neq 0, \gamma > 0$ and $y \in \mathbb{R}$.*

4. *(Linearized operator) Let L be the unbounded operator defined on $L^2(\mathbb{R})$ by*
$$Lu = |D|^{\alpha} u + u - 2Qu.$$
Then, the continuous spectrum of L is $[1, +\infty)$, L has one negative eigenvalue μ_0, associated to an even eigenfunction $W_0 > 0$, and $\ker L = span\,\{Q'\}$.

Remark 2.1.2. The uniqueness problem for the solutions of (2.1.2) which are not ground states is still an open question when $\alpha \neq 1$.

These solitary waves are orbitally stable under the flow of (2.1.1) (see Linares, Pilod and Saut [116] and [9, 10] for other proofs) in the mass sub-critical range $\alpha \in \left(\frac{1}{2}, 2\right)$. They were proven to be linearly unstable in the the mass super-critical range $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ (see [9]).

Sometimes, we also call these solutions *solitons* even though they are not known to have elastic interactions outside of the integrable case $\alpha = 1$.

2.1.3 N-soliton solution

An important conjecture for nonlinear dispersive equations is to prove the *soliton resolution property*, which states that arbitrary initial data eventually resolve over time into a finite sum of solitary waves and an oscillatory remainder of essentially linear type. It has been proved in the KdV case for sufficiently smooth and decaying initial data by using the complete integrable structure (see [46]). Note however that despite some recent progress (see [191], [179]), it is still an open problem for the Benjamin-Ono equation on the line.

For KdV type equations, we are still far from a complete understanding of this phenomenon. In this direction, an important question is to construct solutions behaving like a superposition of N solitary waves at infinity. Indeed, such objects are expected to be universal attractors in the region $x > 0$ for any smooth and decaying solutions at infinity. These solutions, also called N*-soliton solutions* by abuse of language, were first constructed by Martel in [120] for the sub-critical and critical gKdV equations by adapting the construction by Merle in [139] of solutions blowing up at k given points for the critical non-linear Schrödinger equation to the KdV setting, and by relying on the energy methods by Martel, Merle and Tsai [130]. This construction was extended to the super-critical gKdV equations by Côte, Martel and Merle [39] (see also Combet [34]).

For the fractional KdV equations, outside of the case $\alpha = 1$, no result concerning construction of N-solitary wave solutions at infinity seems to be known. Of course, in the case $\alpha = 1$, the N-soliton solutions of the Benjamin-Ono equation are explicit by using inverse scattering method [113], [46], [20], [153], [136]. These N-solitons were also proved to be orbitally stable by Neves and Lopes [155] and Gustafson, Takaoka and Tsai [73] and even asymptotically stable by Kenig and Martel [90].

The main result of this paper states the existence of such N-soliton solutions for any given set of velocities $0 < c_1 < c_2 < \cdots < c_N$.

Theorem 2.1.3. *We assume $\alpha \in \left(\frac{1}{2}, 2\right)$. Let $N \in \mathbb{N}$, $0 < c_1 < \cdots < c_N < +\infty$. Then, there exist some constants $T_0 > 0, C_0 > 0$, N functions $\rho_1, \cdots, \rho_N \in C^1([T_0, +\infty))$ and $U \in C^0([T_0, +\infty) : H^{\frac{\alpha}{2}}(\mathbb{R}))$ solution of* (2.1.1) *such that, for all $t \geqslant T_0$,*
$$\left\| U(t, \cdot) - \sum_{j=1}^{N} Q_{c_j}(\cdot - \rho_j(t)) \right\|_{H^{\frac{\alpha}{2}}} \leqslant \frac{C_0}{t^{\frac{\alpha}{2}}}, \tag{2.1.6}$$
$$|\rho_j(t) - c_j t| \leqslant t^{1-\frac{\alpha}{4}} \quad \text{and} \quad |\rho_j'(t) - c_j| \leqslant \frac{C_0}{t^{\frac{\alpha}{2}}}, \tag{2.1.7}$$
for all $j \in \{1, \cdots, N\}$.

Remark 2.1.4. Due to the polynomial decay of the error in (2.1.7), we are in the case of *strong interactions*, and thus the asymptotic of $\rho_j(t)$ in (2.1.6) may be more complicated than just $c_j t$. Related to this strong interactions phenomenon, the uniqueness of these N-soliton solutions is an interesting open problem.

Remark 2.1.5. The construction in the case $\alpha \in \left(\frac{1}{2}, \frac{6}{7}\right]$ is conditional to the well-posedness of the equation in the energy space, which is still an open problem for this range of α's.

Similar construction of N-soliton-like solutions have already been performed for other nonlinear dispersive equations. Outside of the gKdV equations commented above, we refer to Martel and Merle [124] and Côte, Martel and Merle [39] for the non-linear Schrödinger (NLS) equation, and more recently to Ferriere for the logarithmic-NLS equation [56]. We also refer to the works of Martel and Merle [127] and Jendrej [86] for the wave equation, to the works of Côte-Muñoz [41], Bellazzini, Ghimenti, Le Coz [13] and Côte, Martel [38] for the Klein-Gordon equation, to the work Rousset-Tzvetkov [142] for the water-waves equation, and to the work of Valet [182] for the Zakharov-Kuznetsov equation.

A different method of construction of multi-solitons, based on the fixed point argument of Merle in [139], has been introduced by Le Coz, Li and Tsai in [114] for the NLS equation. This strategy has also been used by Chen for the wave equation [31] and by Van Tin for the derivative NLS [183].

Recently Jendrej, Kowalczyk and Lawrie introduced in [84] a new version of the Liapunov-Schmidt reduction in the setting of dispersive equations to derive a complete classification of all kink-antikink pairs in the strongly interacting regime for the classical nonlinear scalar field models on the real line.

Finally, let us observe that the result of Theorem 2.1.3 would be the first step to study the collision of multi-soliton solutions in the cases $\alpha \in \left(\frac{1}{2}, 2\right)$, $\alpha \neq 1$. We refer to the works of Martel and Merle [125] and [126] for the study of the inelastic collision of two solitons of the quartic KdV equation.

2.1.4 Outline of proof of Theorem 2.1.3

The proof of Theorem 2.1.3 follows the strategy of [139], [120], [130]. After fixing a sequence of time $(S_n) \nearrow +\infty$, one considers the sequence (u_n) of solutions to (2.1.1) evolving from the initial data $\displaystyle\sum_{j=1}^{N} Q_{c_j}(\cdot - c_j S_n)$ at time S_n. As long as the solution remains sufficiently close to the sum of N solitary waves, one introduces modulated translation parameters $(\rho_{j,n}(t))_{j=1}^{N}$ allowing to satisfy suitable orthogonality conditions. The goal is to obtain backwards uniform estimates for the difference $u_n(t) - \displaystyle\sum_{j=1}^{N} Q_{c_j}(\cdot - \rho_j(t))$ on some time interval $[T_0, S_n]$, for some T_0 independent of n. The N-soliton is then obtained by letting $n \to +\infty$ and using a compactness argument. Moreover, it is worth to observe that the uniform estimate relies on monotonicity properties for suitable portions of the mass and the energy of the solution.

Compared to the previous constructions, we have to deal here with two major new difficulties. Firstly, due to the singularity at the origin of the symbol $|\xi|^\alpha$ related to the non-local operator $|D|^\alpha$, the solitary waves have only polynomial decay[1] of order $(1+|x|)^{-(1+\alpha)}$. As a consequence the uniform estimates on the parameters $\rho_{j,n}(t)$ are only polynomial and thus cannot be integrated directly. Relying on a topological argument introduced in [39], we need then to adapt carefully the initial data of the translation parameters $\rho_{j,n}$ at time S_n to be able to close the bootstrap estimates.

Secondly, observe that the monotonicity techniques introduced by Martel and Merle for gKdV are local in space, and are therefore tailored for differential but not integral (nonlocal) equations. To adapt these techniques to the fKdV equations, one need to use suitable weighted commutator estimates (see Lemma 2.3.1). Those estimates were introduced in the symmetric case by Kenig and Martel [90] in the case $\alpha = 1$ and Kenig, Martel and Robbiano [91] for the general case $0 < \alpha < 2$ (see also [133] for an application to the critical modified Benjamin-Ono equation). Note however that to derive the monotonicity property of the energy, one needs a non-symmetric version of these estimates (see estimates (2.3.6)-(2.3.7)), whose

[1] The decay of the solitary waves of gKdV is always exponential.

proof is based on pseudo-differential calculus and follows the one of Kenig, Martel and Robbiano for the symmetric case.

The paper is organised as follows: in Section 2.2, we modulate the geometrical translation parameters for a solution close to N solitary waves, set up the bootstrap setting and close the construction of the N-soliton solution after assuming the main bootstrap estimate. In Section 2.3, we state several weighted estimates whose proofs are given in the appendices. These weighted estimates are useful to derive the monotonicity properties and to prove the bootstrap estimate in Section 2.4.

2.1.5 Notation

1. From now on, C will denote a positive constant changing from line to line and independent of the different parameters. We also denote by C_* a positive constant changing from line to line and depending only on the parameters $\{c_1, \cdots, c_N\}$.
2. Unless stated otherwise, all the integrals will be over $\mathbb{R}$ with respect to the space variable.
3. For $x \in \mathbb{R}^N$, we recall the definition of the Japanese brackets $\langle x \rangle := \sqrt{1+|x|^2}$.
4. We denote by $\|f\|_{L^p} := \left(\int |f|^p\right)^{\frac{1}{p}}$ and $\|f\|_{H^s} := \|\langle \cdot \rangle^{\frac{s}{2}} \mathcal{F}(f)(\cdot)\|_{L^2}$, where $\mathcal{F}(f)$ is the Fourier transform of f. Finally, $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space of real-valued functions.
5. We fix $0 < c_1 < \cdots < c_N$ and we set $\beta = \frac{1}{2}\min(c_1, c_2 - c_1, \cdots, c_N - c_{N-1})$.

2.2 Construction of the asymptotic N-soliton

Notation 2.2.1. *1. For $L > 0$ and $N \in \mathbb{N}$, we define*

$$\mathbb{R}^N_L = \{(y_j)_{j=1}^N \in \mathbb{R}^N : y_j - y_{j-1} > L, \forall j \in \{2, \cdots, N\}\}. \tag{2.2.1}$$

2. For $Y = (Y_j)_{j=1}^N \in \mathbb{R}^N_L$, we denote

$$R_Y(x) = \sum_{j=1}^N R_{Y,j}(x) := \sum_{j=1}^N Q_{c_j}(x - Y_j). \tag{2.2.2}$$

3. Let $M = (m_{i,j})_{i,j=1}^N \in M_N(\mathbb{R})$, be a $N \times N$ matrix.

2.2.1 Modulation of the geometrical parameters

Proposition 2.2.2 (Modulation)**.** *There exist $L_1, \gamma_1, T_1 > 0$ such that the following is true. Assume that u is a solution of* (2.1.1) *satisfying that for $L > L_1$, $0 < \gamma < \gamma_1$, $S > t^* > T_1$,*

$$\sup_{t^* \leqslant t \leqslant S} \left(\inf_{(Y_j)_{j=1}^N \in \mathbb{R}^N_L} \left\| u(t,\cdot) - \sum_{j=1}^N Q_{c_j}(\cdot - Y_j) \right\|_{H^{\frac{\alpha}{2}}} \right) < \gamma. \tag{2.2.3}$$

Then, there exist N unique C^1 functions $\rho_j : [t^*, S] \longrightarrow \mathbb{R}$, $j \in \{1, ..., N\}$, such that

$$\eta(t,x) = u(t,x) - R(t,x), \tag{2.2.4}$$

where

$$R(t,x) = \sum_{j=1}^N R_j(t,x) := \sum_{j=1}^N Q_{c_j}(x - \rho_j(t)), \tag{2.2.5}$$

satisfies the following orthogonality conditions

$$\int (\partial_x R_j)\eta = 0, \quad \forall j \in \{1, ..., N\}, \forall t \in [t^*, S_n]. \tag{2.2.6}$$

Moreover, for all $t \in [t^*, S]$

$$\|\eta(t,\cdot)\|_{H^{\frac{\alpha}{2}}} \leqslant C\gamma, \tag{2.2.7}$$

$$\inf_{j\in\{1,\dots,N-1\}} (\rho_{j+1}(t) - \rho_j(t)) \geqslant \frac{L}{2}. \tag{2.2.8}$$

Remark 2.2.3. A solution u satisfying (2.2.3) lives for all time $t \in [t^*, S]$ in the tube

$$\mathcal{T}_{\gamma,L} := \Big\{ v \in H^{\frac{\alpha}{2}}(\mathbb{R}) : \inf_{Y_j \geqslant Y_{j-1}+L} \Big\| v - R_Y \Big\|_{H^{\frac{\alpha}{2}}} < \gamma \Big\}. \tag{2.2.9}$$

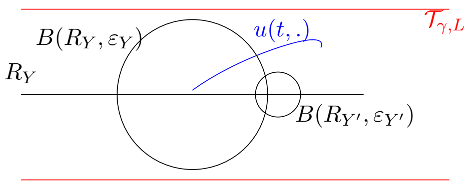
The proof of Proposition 2.2.2 is an application of the implicit function theorem to the functional

$$\begin{aligned} \Phi : \mathcal{T}_{\gamma,L} \times \mathbb{R}^N_L &\to \mathbb{R}^N \\ (v, Y) &\mapsto \left(\int (v - R_Y)\, Q'_{c_j}(\cdot - Y_j) \right)_{j=1}^N . \end{aligned} \tag{2.2.10}$$

Note that a direct application of the implicit function theorem at the point (R_Y, Y) for $Y \in \mathbb{R}^N_L$ would imply

$$\forall Y \in \mathbb{R}^N_L, \exists \varepsilon_Y > 0, \exists!\, (\rho_j)_{j=1}^N \in C^1(\mathcal{T}_{\gamma,L} \cap B(R_Y, \varepsilon_Y)) : \mathbb{R}), \tag{2.2.11}$$

such that $\Phi(v, (\rho_j(v))_{j=1}^N) = 0$ for $v \in B(R_Y, \varepsilon_Y)$. This would not be enough to conclude the proof of (2.2.6) due to the lack of control of $\varepsilon_{u(t)}$ uniformly in $[t^*, S]$. Indeed, an application of (2.2.11) to a solution u satisfying $u(S,\cdot) = \sum_{j=1}^N Q_{c_j}(\cdot - \rho^{\text{in}}_{j,n})$ for $(\rho^{\text{in}}_{j,n}) \in \mathbb{R}^N_L$, and a continuity argument would provide the existence of $\varepsilon > 0$ such that $u(t,\cdot) \in B := B(\sum_{j=1}^N Q_{c_j}(\cdot - \rho^{\text{in}}_{j,n}), \varepsilon)$ for all $t \in (t_1, S]$, where t_1 is the first time before S with $u(t_1,\cdot) \notin B$. Nevertheless, nothing would guarantee that $u(t_1,\cdot)$ belongs to a ball $B(R_Y, \varepsilon_Y)$ for some $Y \in R^N_L$.



To bypass this difficulty, we will use the following quantitative version of the implicit function theorem (see section 2.2 in [32]). We refer to [37] Lemma 3, [83] Lemme 3.3, [74] Proposition 3 , [141] Proposition 3.1 for applications of this theorem in a similar context.

Theorem 2.2.4. *Let X, Y and Z be Banach spaces, $x_0 \in X, y_0 \in Y$, $\gamma, \delta > 0$ and $\Phi : B(x_0,\gamma) \times B(y_0,\delta) \longrightarrow Z$ be continuous in x, continuously differentiable in y, satisfy $\Phi(x_0,y_0) = 0$, $M_0 := d_y\Phi(x_0,y_0)$ has a bounded inverse in $\mathcal{L}(Z,Y)$. Assume moreover that*

$$\|M_0 - d_y\Phi(x,y)\|_{\mathcal{L}(Y,Z)} \leqslant \frac{1}{3}\|M_0^{-1}\|^{-1}_{\mathcal{L}(Z,Y)}, \quad \forall x \in B(x_0,\gamma), y \in B(y_0,\delta), \tag{2.2.12}$$

$$\|\Phi(x,y_0)\|_Z \leqslant \frac{\delta}{3}\|M_0^{-1}\|^{-1}_{\mathcal{L}(Z,Y)}, \quad \forall x \in B(x_0,\gamma). \tag{2.2.13}$$

Then there exists $y \in C^1(B(x_0,\gamma) : B(y_0,\delta))$ such that for $x \in B(x_0,\gamma)$, $y(x)$ is the unique solution of the equation $\Phi(x,y(x)) = 0$ in $B(x_0,\gamma)$.

Before giving the proof of Proposition 2.2.2, we need the following lemma.

Lemma 2.2.5. *There exist $C > 0$, $L_2 > 0$, such that for all $L > L_2$, and all $Y = (Y_j)_{j=1}^N \in \mathbb{R}^N_L$ we have*

$$\left|\int (\partial_x R_{Y,j})(\partial_x R_{Y,k})\right| \leqslant \frac{C}{1+L^{2+\alpha}}, \quad j \neq k, \tag{2.2.14}$$

with $R_{Y,j}$ defined in (2.2.2) *.*

Moreover, let $(\rho_j)_{j=1}^N \in C^1([t^*,S] : \mathbb{R})$ satisfying $\rho_{j+1} - \rho_j \geqslant \frac{L}{2}$ for all $j,k \in \{1,\cdots,N-1\}$, with $j \neq k$, then

$$\left|\int (\partial_x R_j)(\partial_x R_k)\right| \leqslant \frac{C}{1+L^{2+\alpha}}, \tag{2.2.15}$$

$$\left|\int R_j(\partial_x^2 R_k)\right| \leqslant \frac{C}{1+L^{1+\alpha}}, \tag{2.2.16}$$

$$\left|\int R_j R_k(\partial_x^2 R_l)\right| \leqslant \frac{C}{1+L^{1+\alpha}}, \tag{2.2.17}$$

with R_j defined in (2.2.5)*.*

Furthermore, if the functions $(\rho_j)_{j=1}^N$ satisfy $|\rho_{j+1}(t) - \rho_j(t)| \geqslant \beta t$, with $\beta > 0$, then

$$\left|\int (\partial_x R_j)(\partial_x R_k)\right| \leqslant \frac{C}{(\beta t)^{2+\alpha}}, \tag{2.2.18}$$

$$\left|\int R_j(\partial_x^2 R_k)\right| \leqslant \frac{C}{(\beta t)^{1+\alpha}}. \tag{2.2.19}$$

$$\left|\int R_j R_k(\partial_x^2 R_l)\right| \leqslant \frac{C}{(\beta t)^{1+\alpha}}. \tag{2.2.20}$$

Proof of Lemma 2.2.5 . By symmetry, we can suppose $j < k$. Let $\Omega := \{x \in \mathbb{R} : x < \frac{Y_j+Y_k}{2}\}$. By (2.1.5) and $Y_k - Y_j > L$, we deduce that

$$\left|\int_\Omega (\partial_x R_{Y,j})(\partial_x R_{Y,k})\right| \leqslant \frac{C}{1+\left(Y_k - \frac{Y_j+Y_k}{2}\right)^{2+\alpha}}\int_\Omega |\partial_x R_{Y,j}| \leqslant \frac{C}{1+L^{2+\alpha}}.$$

On the other hand, by (2.1.5) and $Y_j - Y_k < -L$, we get on Ω^c

$$\left|\int_{\Omega^c} (\partial_x R_{Y,j})(\partial_x R_{Y,k})\right| \leqslant \frac{C}{1+\left(\frac{Y_j+Y_k}{2} - Y_j\right)^{2+\alpha}}\int_{\Omega^c} |\partial_x R_{Y,k}| \leqslant \frac{C}{1+L^{2+\alpha}},$$

which concludes (2.2.14). To prove the other estimates, we use the same argument on $\Omega := \{x \in \mathbb{R} : x < \frac{\rho_j+\rho_k}{2}\}$ with the estimates (2.1.5), $\rho_{j+1} - \rho_j \geqslant \frac{L}{2}$ for (2.2.15), (2.2.16), (2.2.17) and (2.1.5),$|\rho_{j+1}(t) - \rho_j(t)| \geqslant \beta t$ for (2.2.18), (2.2.19), (2.2.20). □

Proof of Proposition 2.2.2. We decompose the proof in two steps. First, by using Theorem 2.2.4, we show that we can find N unique functions ρ_j continuous on $\mathcal{T}_{\gamma,L}$, defined in (2.2.9), satisfying (2.2.6) - (2.2.8). To obtain the regularity of the functions, we use the Cauchy-Lipschitz theorem.

First step : existence of the functions ρ_j. We recall the definition of $\mathbb{R}^N_L$ and R_Y given respectively in (2.2.1) and (2.2.2). First, we check that the functional Φ defined in (2.2.10) satisfies the hypotheses of Theorem 2.2.4. It is clear that, for all $Y \in \mathbb{R}^N_L$,

$$\Phi\left(R_Y, Y\right) = 0.$$

Let us define

$$M_0 = M_0(R_Y, Y) := d_Y \Phi\left(R_Y, Y\right) = A + B,$$

where

$$A := \begin{pmatrix} \int (Q'_{c_1})^2 & 0 & \cdots & & 0 \\ 0 & \int (Q'_{c_2})^2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & 0 & \int (Q'_{c_N})^2 \end{pmatrix},$$

and

$$B = B(Y) := \begin{pmatrix} 0 & \mathcal{Q}_{1,2} & \cdots & & \mathcal{Q}_{1,N} \\ \mathcal{Q}_{2,1} & 0 & \mathcal{Q}_{2,3} & \cdots & \mathcal{Q}_{2,N} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \mathcal{Q}_{N,1} & \cdots & & \mathcal{Q}_{N,N-1} & 0 \end{pmatrix},$$

with $\mathcal{Q}_{j,k} := \displaystyle\int \partial_x R_{Y,j} \partial_x R_{Y,k}$. The matrix A is invertible, and by (2.2.14), we get for all $L > L_2$

$$|\mathcal{Q}_{j,k}| \leqslant \frac{C}{1 + L^{2+\alpha}}.$$

Then for $L > L_3$, with L_3 big enough, M_0 is invertible. Moreover the matrix A is independent of $Y \in \mathbb{R}^N_L$, and $\lim\limits_{L\to\infty} \|B\|_\infty \to 0$. Then, there exists κ independent of $L > 1$ such that for all $Y \in \mathbb{R}^N_L$

$$\|M_0\left(R_Y, Y\right)^{-1}\|_\infty = \|A^{-1}(Id + B(Y)A^{-1})^{-1}\|_\infty \leqslant \|A^{-1}\|_\infty \sum_{n=0}^{\infty} \left[C \frac{\|A^{-1}\|_\infty}{1 + L^{2+\alpha}}\right]^n \leqslant \kappa.$$

Thus, to verify the conditions (2.2.12),(2.2.13), since $\|\cdot\|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^N_L)} \leqslant \|\cdot\|_\infty$, it suffices to prove that

$$\|M_0 - d_Y \Phi(v, Z)\|_\infty \leqslant \frac{1}{3}\kappa^{-1}, \text{ for } v \in B(R_Y, \gamma), Z \in B(Y, C_1\gamma), \tag{2.2.21}$$

$$\|\Phi(v, Y)\|_\infty \leqslant \frac{C_1 \gamma}{3}\kappa^{-1}, \text{ for } v \in B(R_Y, \gamma), \tag{2.2.22}$$

for a positive constant C_1 to be chosen later. First, we show (2.2.22). Let $j \in \{1, ..., N\}$, by Cauchy-Schwarz and since $v \in B(R_Y, \gamma)$

$$|\Phi_j(v, Y)| \leqslant \int |v - R_Y||\partial_x R_{Y,j}| \leqslant \|v - R_Y\|_{L^2} \|\partial_x R_{Y,j}\|_{L^2} \leqslant \gamma \|\partial_x Q_{c_j}\|_{L^2},$$

which implies (2.2.22) by choosing $C_1 = 3\kappa \sup\limits_j \|\partial_x Q_{c_j}\|_{L^2}$. Since the constant C_1 does not play any role in the rest of the paper, we write C instead of C_1.

Now, let us verify (2.2.21). First we define

$$\mathcal{Q}^*_{j,k}(Z, Y) := \int \left((\partial_x R_{Z,j})(\partial_x R_{Z,k}) - (\partial_x R_{Y,j})(\partial_x R_{Y,k})\right),$$

with $Z = (Z_j)_{j=1}^N, Y = (Y_j)_{j=1}^N \in \mathbb{R}_L^N$, and

$$\mathcal{P}_k(v, Z) = \int (v(x) - R_Z(x))\, \partial_x^2 R_{Y,k}.$$

with $Z = (z_j) \in \mathbb{R}_L^N$, $v \in B(R_Y, \gamma)$. We have

$$\|M_0 - d_Y\Phi(v, Z)\|_\infty = \left\| \begin{pmatrix} \mathcal{P}_1(v,Z) & \mathcal{Q}^*_{2,1}(Z,Y) & \cdots & & \mathcal{Q}^*_{N,1}(Z,Y) \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \mathcal{Q}^*_{1,N}(Z,Y) & \cdots & & \mathcal{Q}^*_{N-1,N}(Z,Y) & \mathcal{P}_N(v,Z) \end{pmatrix} \right\|_\infty .$$

For $j \in \{1, \cdots, N\}$, by Cauchy-Schwarz since $v \in B(R_Y, \gamma)$

$$|\mathcal{P}_j(v, Z)| \leqslant \int |v - R_Z||\partial_x^2 R_{Y,j}| \leqslant \|v - R_Z\|_{L^2} \|\partial_x^2 R_{Y,j}\|_{L^2} \leqslant C\gamma. \tag{2.2.23}$$

For $j, k \in \{1, \cdots, N\}, j \neq k$, by (2.2.14)

$$|\mathcal{Q}^*_{j,k}(Z, Y)| \leqslant \frac{C}{1 + L^{2+\alpha}}. \tag{2.2.24}$$

Gathering (2.2.23) and (2.2.24), we get

$$\|M_0 - D_Y\Phi(v, Y)\|_\infty \leqslant C\gamma + \frac{C}{1 + L^{2+\alpha}} \leqslant \frac{1}{3}\kappa^{-1}.$$

for $\gamma < \gamma_2$ small enough and $L > L_4$ big enough.

Then for $L > \max(L_2, L_3, L_4)$ and $\gamma < \gamma_2$ we deduce from Theorem 2.2.4 the existence and uniqueness of $(\rho_j)_{j\in\{1,\ldots,N\}}$ in $C^1(B(R_Y, \gamma) : B(Y, C\gamma))$ satisfying (2.2.6). Moreover, since γ can be chosen independently of $Y \in \mathbb{R}_L^N$, we can extend by uniqueness $(\rho_j)_{j\in\{1,\ldots,N\}}$ to the whole tube $\mathcal{T}_{\gamma,L}$ defined in (2.2.9). Furthermore, for all $v \in \mathcal{T}_{\gamma,L}$, there exists $Y = (Y_j) \in \mathbb{R}_L^N$ such that $(\rho_j(v))_{j\in\{1,\ldots,N\}} \in B(Y, C\gamma)$. Therefore

$$|\rho_{j+1}(v) - \rho_j(v)| \geqslant |Y_{j+1} - Y_j| - |\rho_{j+1}(v) - Y_{j+1}| - |\rho_j(v) - Y_j| \geqslant L - 2C\gamma \geqslant \frac{L}{2}. \tag{2.2.25}$$

Now, by abuse of notation, we define $\rho_j(t) := \rho_j(u(t, \cdot))$. Then it is clear that ρ_j is $C^0([t^*, S] : \mathbb{R})$ since $u(t, \cdot) \in C^0([t^*, S] : H^{\frac{\alpha}{2}}(\mathbb{R}))$.

Let us prove the estimate (2.2.7). By construction of $(\rho_j(t))_{j=1}^N$, we have that for all $t \in [t^*, S]$, there exists $(Y_j(t))_{j=1}^N \in \mathbb{R}_L^N$ such that

$$|\rho_j(t) - Y_j(t)| \leqslant C\gamma,$$
$$\|u(t, \cdot) - R_{Y(t)}\|_{H^{\frac{\alpha}{2}}} \leqslant \gamma.$$

By the triangle inequality and mean value theorem, we deduce

$$\begin{aligned} \|\eta(t, \cdot)\|_{H^{\frac{\alpha}{2}}} &\leqslant \|u(t, \cdot) - R_{Y(t)}\|_{H^{\frac{\alpha}{2}}} + \|R_{Y(t)} - R\|_{H^{\frac{\alpha}{2}}} \\ &\leqslant \gamma + \sum_{j=1}^N |\rho_j(t) - Y_j(t)| \|\partial_x \langle D\rangle^{\frac{\alpha}{2}} Q\|_\infty \int |\langle D\rangle^{\frac{\alpha}{2}} (R_{Y(t),j} - R_j)| \\ &\leqslant C\gamma. \end{aligned}$$

This finishes the proof of (2.2.7). Note also that (2.2.8) is a direct consequence of (2.2.25).

Second step : regularity of the functions ρ_j. Assume that the N functions ρ_j are $C^1([t^*, S] : \mathbb{R})$. First, we compute the equation for η using (2.1.1) and (2.2.4)

$$\partial_t \eta = \partial_x \left(\mathcal{G}(\eta) + |D|^\alpha R - R^2 \right) + \sum_{0 \leqslant k \leqslant N} \rho_k' \partial_x R_k,$$

where

$$\mathcal{G}(\eta) := |D|^\alpha \eta - 2R\eta - \eta^2.$$

Moreover, since

$$R^2 = \sum_{1 \leqslant k \leqslant N} R_k^2 + 2 \sum_{1 \leqslant l < m \leqslant N} R_l R_m,$$

this implies by using (2.1.3) that

$$\partial_t \eta = \partial_x \left(\mathcal{G}(\eta) - \sum_{1 \leqslant k \leqslant N} c_k R_k - 2 \sum_{1 \leqslant l < m \leqslant N} R_l R_m \right) + \sum_{1 \leqslant k \leqslant N} \rho_k' \partial_x R_k. \tag{2.2.26}$$

Furthermore, we obtain differentiating in time the relation $\displaystyle\int (\partial_x R_j)\, \eta = 0$

$$0 = \frac{d}{dt} \int (\partial_x R_j)\, \eta = -\rho_j' \int \left(\partial_x^2 R_j\right) \eta + \int (\partial_x R_j)\, \partial_t \eta. \tag{2.2.27}$$

Replacing (2.2.26) in (2.2.27), and integrating by parts, we obtain that

$$\begin{aligned} 0 = &- \int \left(\partial_x^2 R_j\right) \left(\mathcal{G}(\eta) - \sum_{1 \leqslant k \leqslant N} c_k R_k - 2 \sum_{1 \leqslant l < m \leqslant N} R_l R_m \right) + \sum_{1 \leqslant k \leqslant N} \rho_k' \int (\partial_x R_k)(\partial_x R_j) \\ &- \rho_j' \int \left(\partial_x^2 R_j\right) \eta. \end{aligned} \tag{2.2.28}$$

Finally, we deduce that, for all $j \in \{1, \dots, N\}$,

$$\begin{aligned} &\sum_{1 \leqslant k \leqslant N} \rho_k' \int (\partial_x R_k)(\partial_x R_j) - \rho_j' \int \left(\partial_x^2 R_j\right) \eta \\ &\qquad = \int \left(\partial_x^2 R_j\right) \left(\mathcal{G}(\eta) - \sum_{1 \leqslant k \leqslant N} c_k R_k - 2 \sum_{1 \leqslant l < m \leqslant N} R_l R_m \right). \end{aligned}$$

We can rewrite this ODE system in the matrix form

$$AY' = B, \tag{2.2.29}$$

where $Y := (\rho_j)_{j=1}^N$ and $A := A_0 + A_\eta$ where

$$A_\eta := \begin{pmatrix} -\int \left(\partial_x^2 R_1\right) \eta & \int (\partial_x R_1)(\partial_x R_2) & \dots & \int (\partial_x R_1)(\partial_x R_N) \\ \int (\partial_x R_2)(\partial_x R_1) & -\int \left(\partial_x^2 R_2\right) \eta & \dots & \int (\partial_x R_2)(\partial_x R_N) \\ \vdots & & & \vdots \\ \int (\partial_x R_N)(\partial_x R_1) & \dots & \dots & -\int \left(\partial_x^2 R_N\right) \eta \end{pmatrix},$$

$$A_0 := \begin{pmatrix} \int (\partial_x Q_{c_1})^2 & 0 & \dots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & \int (\partial_x Q_{c_N})^2 \end{pmatrix},$$

$$B := \left(\int \left(\partial_x^2 R_j\right) \left(\mathcal{G}(\eta) - \sum_{1\leqslant k\leqslant N} c_k R_k - 2 \sum_{1\leqslant l<m\leqslant N} R_l R_m \right) \right)_{1\leqslant j\leqslant N}.$$

In order to prove that A is invertible, it suffices to prove that A_0 is invertible and $\|A_\eta\|_\infty$ can be taken small enough. By using the Cauchy-Schwarz inequality and (2.2.7), we have

$$\left| \int \left(\partial_x^2 R_j\right) \eta \right| \leqslant C\gamma.$$

By (2.2.15), we obtain

$$\left| \int (\partial_x R_k)(\partial_x R_j) \right| \leqslant \frac{C}{1+L^{2+\alpha}}, \quad k \neq j.$$

Taking $L > L_5$ big enough, and $\gamma < \gamma_3$ small enough, the matrix A is invertible and we can rewrite (2.2.29) as

$$Y' = A^{-1}B.$$

Now, we have to prove that $A^{-1}B$ is globally Lipschitz. Let us begin with the term $\int \left(\partial_x^2 R_1\right)\eta$. Let $(\rho_j)_{j=1}^N, (\widetilde{\rho}_j)_{j=1}^N \in \mathbb{R}^N$, by the Plancherel identity and the Cauchy-Schwarz inequality

$$\begin{aligned}
&\left| \int \partial_x^2 \left(Q_{c_1}(x-\rho_1)\right) \left(u(t,x) - \sum_{j=1}^N Q_{c_j}(x-\rho_j) \right) - \partial_x^2 \left(Q_{c_1}(x-\widetilde{\rho}_1)\right) \left(u(t,x) - \sum_{j=1}^N Q_{c_j}(x-\widetilde{\rho}_j) \right) dx \right| \\
&\leqslant \int |\xi|^2 |\widehat{Q_{c_1}}||u| \left| e^{i\xi\rho_1} - e^{i\xi\widetilde{\rho}_1} \right| d\xi + \sum_{j=1}^N \int |\xi|^2 |\widehat{Q_{c_1}}||\widehat{Q_{c_j}}| \left| e^{i\xi(\rho_1+\rho_j)} - e^{i\xi(\widetilde{\rho}_1+\widehat{\rho}_j)} \right| d\xi \\
&\leqslant |\rho_1 - \widetilde{\rho}_1| \int |\xi|^3 |\widehat{Q_{c_1}}||\widehat{u}| d\xi + \left(|\rho_1-\widetilde{\rho}_1| + \sum_{j=1}^N |\rho_j - \widetilde{\rho}_j| \right) \sum_{j=1}^N \int |\xi|^3 |\widehat{Q_{c_1}}||\widehat{Q_{c_j}}| d\xi \\
&\leqslant C \sum_{j=1}^N |\rho_j - \widetilde{\rho}_j| \|\partial_x^3 Q_{c_1}\|_{L^2} \left(\|u_0\|_{L^2} + \|Q_{c_j}\|_{L^2} \right),
\end{aligned}$$

where we have used for the last inequality that $\|u(t,\cdot)\|_{L^2} = \|u_0\|_{L^2}$. Using the same argument for the other term in A and B, we get $A^{-1}B$ is globally Lipschitz. Therefore, we obtain N unique C^1 functions $\widetilde{\rho}_j : [t^*, S] \longrightarrow \mathbb{R}$ satisfying (2.2.27) with $\widetilde{\rho}_j(S) = \rho_j(S)$ as initial condition, where $(\rho_j)_{j=1}^N$ is given by the first step. Since (2.2.6) is verified at time S with $\rho_j(S)$, we deduce that for all $t \in [t^*, S]$,

$$\int (\partial_x Q_{c_j}(x - \widetilde{\rho}_j(t)))(u - Q_{c_j}(x - \widetilde{\rho}_j(t))) = 0.$$

By the uniqueness statement of the first step, we conclude that the N functions ρ_j, constructed in the first step, are C^1 functions. This concludes the proof of Proposition 2.2.2 by taking $\gamma < \gamma_1 = \min(\gamma_2, \gamma_3)$ and $L > L_1 = \max(L_2, L_3, L_4, L_5)$. □

2.2.2 Bootstrap setting

Let $(S_n)_{n=0}^{+\infty}$ be a non-decreasing sequence of time going to infinity, with $S_n > T_0$, for $T_0 > 1$ large enough to be chosen later. We define by u_n the solution of (2.1.1) satisfying

$$u_n(S_n, \cdot) = \sum_{j=1}^{N} Q_{c_j}(\cdot - \rho_{j,n}^{\mathrm{in}}),$$

with

$$\rho_{j,n}^{\mathrm{in}} \in I_{j,n} := \left[c_j S_n - S_n^{1-\frac{\alpha}{4}}, c_j S_n + S_n^{1-\frac{\alpha}{4}}\right], \quad \text{for all } j \in \{1, \cdots, N\}, \tag{2.2.30}$$

to be fixed later.

For $t \leqslant S_n$, as long as the solution u_n exists and satisfies (2.2.3) for suitable $0 < \gamma_0 < \gamma_1$ and $L_0 > L_1$ (which will also be fixed later), we consider the C^1 functions $(\rho_{j,n})_{j=1}^N$ provided by Proposition 2.2.2 and satisfying (2.2.4)-(2.2.8). At S_n, the decomposition satisfies

$$\eta(S_n) = 0, \quad \rho_{j,n}(S_n) = \rho_{j,n}^{in}, \quad j = 1, \cdots, N. \tag{2.2.31}$$

We introduce the bootstrap estimates at $t \leqslant S_n$, assuming that u_n satisfies (2.2.3):

$$\|\eta(t,x)\|_{H^{\frac{\alpha}{2}}} < \gamma_0, \tag{2.2.32}$$

$$\sup_{j\in\{1,\cdots,N\}} |\rho_{j,n}(t) - c_j t| \leqslant t^{1-\frac{\alpha}{4}}, \tag{2.2.33}$$

with η defined in (2.2.4).

For $T_0 > 1$, to be chosen later, we define

$$t_n^* = \inf\left\{T_0 < \widetilde{t} \leqslant S_n : \exists \varepsilon_n > 0 \text{ such that (2.2.32)} - \text{(2.2.33) holds for all } t \in [\widetilde{t}, S_n + \varepsilon_n]\right\}.$$

Note by (2.2.31) and by continuity that there exists $\varepsilon_n > 0$ such that (2.2.32) holds on $[S_n - \varepsilon_n, S_n + \varepsilon_n]$. Moreover, if $\rho_{j,n} \in \mathring{I}_{j,n}$ for all $j \in \{1, \cdots, N\}$, then by possibly taking ε_n smaller, (2.2.33) holds also on $[S_n - \varepsilon, S_n + \varepsilon]$ so that t_n^* is well-defined. In the case where $\rho_{j_0,n} \in \partial I_{j_0,n}$ for some $j_0 \in \{1, \cdots, N\}$, it follows from the transversality property (see (2.2.40) below) that $t_n^* = S_n$.

The main result of this section states that there exists at least one choice of $(\rho_{j,n}^{in})_{j=1}^N \sim (c_j S_n)_{j=1}^N$ such that $t_n^* = T_0$. In other words, the bootstrap estimates (2.2.32)-(2.2.33) are valid up to a time T_0 independent of n.

Proposition 2.2.6. *Let $\alpha \in \left(\frac{1}{2}, 2\right)$. There exist $T_0 > 1$, $C_0 > 1$, $\gamma_0 > 0$ satisfying $\frac{C_0}{T_0^{\frac{\alpha}{2}}} < \frac{\gamma_0}{2}$ and $L_0 := \frac{\beta T_0}{2} > L_1$ such that the following is true. For all $n \in \mathbb{N}$, there exists $(\rho_{j,n}^{in})_{j=1}^N \in I_{j,n}$, with $I_{j,n}$ defined in* (2.2.30)*, satisfying*

$$|\rho_{j,n}^{in} - c_j S_n| \leqslant S_n^{1-\frac{\alpha}{4}}, \quad j \in \{1, \cdots, N\}, \tag{2.2.34}$$

and $t_n^* = T_0$

Subsections 2.2.3 and 2.2.4 are dedicated to the proof of Proposition 2.2.6. In every step of the proof, T_0 will be taken large enough and $\gamma_0 > 0$ small enough independently of n.

2.2.3 Modulation estimates

Proposition 2.2.7. *For all $t \in [t_n^*, S_n]$, and for all $\rho_{j,n}^{in} \in I_{j,n}$, we have*

$$\inf_{j\in\{1,\cdots,N\}} |\rho_{j+1,n}(t) - \rho_{j,n}(t)| \geqslant \beta t, \tag{2.2.35}$$

$$|\rho_{j,n}'(t) - c_j| \leqslant C_* \left(\frac{1}{(\beta t)^{\alpha+1}} + \left(\int \frac{1}{(1+|x-\rho_{j,n}(t)|)^{1+\alpha}} \eta^2 \right)^{\frac{1}{2}} + \|\eta\|_{L^2}^2 \right). \tag{2.2.36}$$

Proof. By the triangle inequality and (2.2.33), for t large enough, we deduce that

$$|\rho_{j+1,n}(t) - \rho_{j,n}(t)| \geqslant (c_{j+1} - c_j)t - |\rho_{j+1,n} - c_{j+1}t| - |\rho_{j,n} - c_j t| \geqslant 2\beta t - 2t^{1-\frac{\alpha}{4}} \geqslant \beta t.$$

Now, we prove (2.2.36). We deduce from (2.2.28) that

$$\begin{aligned}\left(\rho_{j,n}' - c_j\right) \int (\partial_x R_j)^2 = \int \left(\partial_x^2 R_j\right) \left(\mathcal{G}(\eta) - \sum_{1\leqslant k\neq j\leqslant N} c_k R_k - 2\sum_{1\leqslant l<m\leqslant N} R_l R_m \right) \\ - \sum_{1\leqslant k\neq j\leqslant N} \rho_{k,n}' \int (\partial_x R_k)(\partial_x R_j) + \rho_{j,n}' \int \left(\partial_x^2 R_j\right)\eta\end{aligned} \tag{2.2.37}$$

for all $j \in \{1, ..., N\}$. By using the fact the operator $|D|^\alpha$ is self adjoint and the Cauchy-Schwarz inequality, we deduce

$$\left| \int \left(\partial_x^2 R_j\right) \mathcal{G}(\eta) \right| + \left| \int \left(\partial_x^2 R_j\right) \eta \right| \leqslant C \left(\|\eta\|_{L^2} + \|\eta\|_{L^2}^2 \right).$$

Moreover, by (2.2.18), (2.2.19), (2.2.20), and (2.2.35) we get

$$\sum_{k\neq j} \left(c_k \left| \int \left(\partial_x^2 R_j\right) R_k \right| + \rho_j' \left| \int \partial_x R_k \partial_x R_j \right| \right) + 2\sum_{1\leqslant l<m\leqslant N} \left| \int \left(\partial_x^2 R_j\right) R_l R_m \right| \leqslant \frac{C}{(\beta t)^{1+\alpha}}.$$

Gathering the two former estimates, we deduce that for all $j \in \{1, \cdots, N\}$,

$$|\rho_{j,n}' - c_j| \int (\partial_x R_j)^2 \leqslant C \left(\gamma_0 (1 + |\rho_{j,n}'|) + \sum_{1\leqslant k\neq j\leqslant N} \frac{1 + |\rho_{k,n}'|}{(\beta T_0)^{1+\alpha}} \right),$$

which implies after by summing over j,

$$\sum_{k=1}^{N} |\rho_{j,n}'| \leqslant C_*.$$

Finally, by reinjecting the former estimate in (2.2.37), we conclude that for all $j \in \{1, \cdots, N\}$

$$|\rho_{j,n}'(t) - c_j| \leqslant C_* \left(\frac{1}{(\beta t)^{\alpha+1}} + \left(\int \frac{1}{(1+|x-\rho_{j,n}(t)|)^{1+\alpha}} \eta^2 \right)^{\frac{1}{2}} + \|\eta\|_{L^2}^2 \right),$$

which yields (2.2.36). □

2.2.4 Proof of Proposition 2.2.6

The proof of Proposition 2.2.6 relies on the following result which will be proved in Section 3.

Proposition 2.2.8 (Bootstrap estimate). *Let $\alpha \in \left(\frac{1}{2}, 2\right)$. There exist $C_0 > 1$, $0 < \gamma_2 < \gamma_1$ and $T_2 > T_1$ such that for all $t \in [t_n^*, S_n]$, for all $0 < \gamma_0 < \gamma_2$, $T_0 > T_2$ and for all $\rho_{j,n}^{in} \in I_{j,n}$*

$$\|u_n(t,\cdot) - \sum_{j=1}^N Q_{c_j}(\cdot - \rho_{j,n}(t))\|_{H^{\frac{\alpha}{2}}} \leqslant \frac{C_0}{t^{\frac{\alpha}{2}}}. \tag{2.2.38}$$

Proof of Proposition 2.2.6 assuming Proposition 2.2.8. Let $0 < \gamma_0 < \gamma_2$ and $T_0 > T_2$ such that $\frac{C_0}{T_0^{\frac{\alpha}{2}}} < \frac{\gamma_0}{2}$. First, we show that u_n satisfies (2.2.3) with $L_0 = \frac{\beta T_0}{2}$ and that (2.2.32) is strictly improved on $[t_n^*, S_n]$. Indeed, it follows from (2.2.38) that

$$\|u_n(t,\cdot) - \sum_{j=1}^N Q_{c_j}(\cdot - \rho_{j,n}(t))\|_{H^{\frac{\alpha}{2}}} \leqslant \frac{C_0}{T_0^{\frac{\alpha}{2}}} < \frac{\gamma_0}{2}.$$

Moreover, (2.2.35) implies that

$$\inf_{j\in\{1,\cdots,N-1\}\}} |\rho_{j+1,n}(t_n^*) - \rho_{j,n}(t_n^*)| \geqslant \beta T_0 = 2L_0.$$

Now, we prove that there exists $\rho_n^{\text{in}} = (\rho_{j,n}^{\text{in}})_{j=1}^N \in \mathbb{R}^N$, satisfying (2.2.34), such that $t_n^* = T_0$. Assume by contradiction that for all choices ρ_n^{in} satisfying (2.2.34), the associated maximal time $t_n^*(\rho_n^{\text{in}}) > T_0$.

First, we remark that $\rho_{j,n}^{\text{in}} = c_j S_n + \lambda_{j,n} S_n^{1-\frac{\alpha}{4}}$ for a unique $\lambda_{j,n} \in [-1,1]$ and we denote $t^*(\lambda_n) := t_n^*(\rho_n^{\text{in}})$ (which will also be denoted t^* when there is no risk of confusion), with $\lambda_n = (\lambda_{j,n})_{j=1}^N$. By definition of t^* and the fact that (2.2.3) and (2.2.32) are strictly improved on $[t^*, S_n]$, we have that

$$|\rho_{j_0,n}(t^*) - c_{j_0} t^*| = (t^*)^{1-\frac{\alpha}{4}}, \tag{2.2.39}$$

for at least one $j_0 \in \{1, \cdots, N\}$. Then, we define

$$\begin{aligned} \Phi : [-1,1]^N &\to \quad \partial[-1,1]^N \\ \lambda \quad &\mapsto \quad \Big((\rho_{j,n}(t^*(\lambda)) - c_j t^*(\lambda))(t^*)^{\frac{\alpha}{4}-1}(\lambda)\Big)_{j=1}^N. \end{aligned}$$

We set

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}^+ \\ s &\mapsto \sup_{j\in\{1,\cdots,N\}} \Big((\rho_{j,n}(s) - c_j s) s^{\frac{\alpha}{4}-1}\Big)^2. \end{aligned}$$

We claim that if for $s_0 \in [T_0, S_n]$, (2.2.39) is verified in s_0 for at least one $j \in \{1, \cdots, N\}$, then

$$f \text{ is a decreasing function in a neighborhood of } s_0, \tag{2.2.40}$$

and

$$\Phi \in C^0([-1,1]^N, \partial[-1,1]^N). \tag{2.2.41}$$

Let us assume (2.2.40) and (2.2.41) and finish the proof of Proposition 2.2.6. For any $\lambda \in \partial[-1,1]^N$, we have that

$$|\rho_{j_0,n}^{\text{in}} - c_{j_0} S_n| = S_n^{1-\frac{\alpha}{4}}, \quad \text{for at least one } j_0 \in \{1, \cdots, N\},$$

which implies by (2.2.40) that $t^*=S_n$. Hence, we deduce that $\Phi_{|\partial[-1,1]^N}=\mathrm{Id}$. However, it is a well-known topological result that no such continuous function $\Phi:[-1,1]^N\to\partial[-1,1]^N$ can exist (see Theorem 1.4, Chapter 3 in [79]). This concludes the proof of Proposition 2.2.6.

Now, we prove (2.2.40) and (2.2.41). Let $f_j(s)=\left((\rho_{j,n}(s)-c_js)s^{\frac{\alpha}{4}-1}\right)^2$ for $j\in\{1,\cdots,N\}$. Let $s\in\mathbb{R}$. Note that for all $j\in\{1,\cdots,N\}$ the functions f_j are continuously derivable. Then, to prove (2.2.40), it is enough to show that for a time s_0 verifying (2.2.39), for all $j\in\{1,\cdots,N\}$ such that $f_j(s_0)=f(s_0)$, we have that $f_j'(s_0)<0$.

By direct computations, we have that

$$\begin{aligned}f_{j_0}'(s)=&2\left((\rho_{j_0,n}(s)-c_{j_0}s)s^{\frac{\alpha}{4}-1}\right)\left((\rho_{j_0,n}'(s)-c_{j_0})s^{\frac{\alpha}{4}-1}+\left(\frac{\alpha}{4}-1\right)s^{\frac{\alpha}{4}-2}(\rho_{j_0,n}(s)-c_{j_0}s)\right)\\=&2\left(\frac{\alpha}{4}-1\right)(\rho_{j_0,n}(s)-c_{j_0}s)^2s^{\frac{\alpha}{2}-3}+2(\rho_{j_0,n}(s)-c_{j_0}s)(\rho_{j_0,n}'(s)-c_{j_0})s^{\frac{\alpha}{2}-2}.\end{aligned}$$

Moreover, inserting (2.2.38) in (2.2.36), we get for all $j\in\{1,\cdots,N\}$

$$|\rho_{j,n}'(s)-c_j|\leqslant\frac{C_*}{s^{\frac{\alpha}{2}}},\tag{2.2.42}$$

which implies, combined with (2.2.39) in s_0 that

$$f_{j_0}'(s_0)\leqslant2\left(\frac{\alpha}{4}-1\right)s_0^{-1}+2s_0^{\frac{\alpha}{4}-1}|\rho_{j_0,n}'(s_0)-c_{j_0}|\leqslant2\left(\frac{\alpha}{4}-1\right)s_0^{-1}+2C_*s_0^{-\frac{\alpha}{4}-1}.$$

Since $\alpha<2$, for T_0 large enough, we conclude that

$$f_{j_0}'(s_0)<0.$$

The same computations yield

$$f_{j_1}'(s_0)<0.$$

Then, we conclude that $f'(s_0^+)<0$ and $f'(s_0^-)<0$, in other words f is a decreasing function at s_0. Note that for $s_0=S_n$ and $\lambda\in\partial[-1,1]^N$, we get that f is a decreasing function at S_n.

To show (2.2.41), we prove that the map : $\lambda\in[-1,1]^N\mapsto t^*(\lambda)$ is continuous. The continuity of $t^*(\lambda)$ follows from the transversality property (2.2.40). Indeed, by (2.2.40), for all $\varepsilon>0$ there exists $\delta>0$ such that $f(t^*(\lambda)-\varepsilon)>f(t^*(\lambda))+\delta=1+\delta$ and for all $t\in[t^*(\lambda)+\varepsilon,S_n]$ (possibly empty), $f(t)<1-\delta$.

Note that f is depending on the parameter λ since $\rho_{j,n}(t)=\rho_{j,n}(u(t,\cdot))$. Moreover the functions $\rho_{j,n}$ are globally defined.

Then, by the continuity of the flow , there exists $\eta>0$ such that for all $|\lambda-\bar\lambda|<\eta$, with $\bar\lambda\in[-1,1]^N$, the corresponding $\bar f$ satisfies $|\bar f(s)-f(s)|<\frac{\delta}{2}$ for $s\in[t^*(\lambda)-\varepsilon,S_n]$. We deduce that for all $s\in[t^*(\lambda)+\varepsilon,S_n]$

$$\bar f(s+\varepsilon)<|\bar f(s+\varepsilon)-f(s+\varepsilon)|+f(s+\varepsilon)<1-\frac{\delta}{2}.$$

In other words, $t^*(\bar\lambda)<t^*(\lambda)+\varepsilon$. Furthermore,

$$\bar f(t^*(\lambda)-\varepsilon)>f(t^*(\lambda)-\varepsilon)-|\bar f(t^*(\lambda)-\varepsilon)-f(t^*(\lambda)-\varepsilon)|>1+\frac{\delta}{2}$$

Then, $t^*(\lambda)-\varepsilon<t^*(\bar\lambda)$. This finishes the proof of (2.2.41). □

2.2.5 Proof Theorem 2.1.3 assuming Proposition 2.2.6

First, we state the weak continuity property of the flow of (2.1.1). Relying on the well-posedness result in [148], this result is proved in the Appendix 2.5.1 in the case $\alpha>\frac{6}{7}$. It will admitted otherwise.

Lemma 2.2.9 (Weak continuity of flow). *Let $\alpha \in \left(\frac{1}{2}, 2\right)$. Suppose that $z_{0,n} \rightharpoonup z_0$ in $H^{\frac{\alpha}{2}}(\mathbb{R})$. We consider solutions z_n of* (2.1.1) *corresponding to initial data $z_{0,n}$ and satisfying $z_n \in C([0,T] : H^{\frac{\alpha}{2}}(\mathbb{R}))$ for any $T > 0$. Then, $z_n(t) \rightharpoonup z(t)$ in $H^{\frac{\alpha}{2}}(\mathbb{R})$, for all $t \geqslant 0$.*

By Proposition 2.2.6, there exist $C_* > 0, T_0 > 0$ independent of n, $\rho_{1,n}, \cdots, \rho_{N,n} \in C^1([T_0, S_n])$ satisfying (2.2.6), (2.2.33) and (2.2.38) for all $T_0 \leqslant t \leqslant S_n$. Then, for all $t \in [T_0, S_n]$,

$$\|u_n(t,\cdot)\|_{H^{\frac{\alpha}{2}}} \leqslant \|u_n(t,\cdot) - \sum_{j=1}^{N} Q_{c_j}(\cdot - \rho_{j,n}(t))\|_{H^{\frac{\alpha}{2}}} + \|\sum_{j=1}^{N} Q_{c_j}(\cdot - \rho_{j,n}(t))\|_{H^{\frac{\alpha}{2}}} \leqslant C_*.$$

Thus, up to a subsequence, there exists $U_0 \in H^{\frac{\alpha}{2}}(\mathbb{R})$ such that

$$u_n(T_0) \rightharpoonup U_0 \quad \text{in } H^{\frac{\alpha}{2}}(\mathbb{R}).$$

Now, we prove the convergence of the modulation parameters. Let $t \in [T_0, +\infty)$ and set T such that $T_0 < t < T < +\infty$. By (2.2.33), we find that for all $j \in \{1, \cdots, N\}$ and $n \in \mathbb{N}$

$$|\rho_{j,n}(t)| \leqslant T^{1-\frac{\alpha}{4}} + c_j T.$$

Moreover from (2.2.42), we see that $\rho'_{j,n}$ is uniformly bounded independently of time. Thus, by the Arzela-Ascoli theorem, there exists $r_j(t) \in C^0([T_0, T])$ such that, after extracting a subsequence if necessary, we have

$$\rho_{j,n}(t) \to r_j(t). \tag{2.2.43}$$

Let $U \in C^0([T_0, +\infty) : H^{\frac{\alpha}{2}}(\mathbb{R}))$ be the solution of (2.1.1) satisfying $U(T_0, \cdot) = U_0$. We set $R^* := \sum_{j=1}^{N} Q_{c_j}(x - r_j(t))$ and let $t \in [T_0, \infty)$. By Lemma 2.2.9, we know that

$$u_n(t) \rightharpoonup U(t) \quad \text{in } H^{\frac{\alpha}{2}}(\mathbb{R}), \tag{2.2.44}$$

for all $t \geqslant T_0$. We deduce then from (2.2.38) and (2.2.43) that

$$\begin{aligned} \|U(t,\cdot) - R^*(t,\cdot)\|_{H^{\frac{\alpha}{2}}} &\leqslant \liminf_n \|u_n(t,\cdot) - \sum_{j=1}^{N} Q_{c_j}(\cdot - \rho_{j,n}(t))\|_{H^{\frac{\alpha}{2}}} \\ &\quad + \liminf_n \sum_{j=1}^{N} \|Q_{c_j}(\cdot - \rho_{j,n}(t)) - Q_{c_j}(\cdot - r_j(t))\|_{H^{\frac{\alpha}{2}}} \leqslant \frac{C_0}{t^{\frac{\alpha}{2}}}. \end{aligned}$$

By Proposition 2.2.6, we have $\frac{C_0}{T_0^{\alpha}} \leqslant \frac{\gamma_0}{2}$ and $\beta T_0 > 2L_1$. Moreover since $|\rho_{j+1,n}(t) - \rho_{j,n}(t)| \geqslant \beta T_0$, then $r_{j+1}(t) - r_j(t) \geqslant \beta T_0 > 2L_1$. Therefore, $U(t,.) \in \mathcal{T}_{\gamma, 2L_1}$ for all $t \in [T_0, \infty)$. By Proposition 2.2.2, there exist N unique functions $\rho_1, \cdots, \rho_n \in C^1([T_0, +\infty) : \mathbb{R})$ such that $(\rho_j)_{j=1}^{N}$ verify (2.2.6). On the other hand, since the solution u_n satisfies also (2.2.6) with $(\rho_{j,n})_{j=1}^{N}$, we deduce passing to the limit and using (2.2.43)-(2.2.44) that r_j satisfies also (2.2.6). Hence, by the uniqueness statement in Proposition 2.2.2, we see that $r_j(t) = \rho_j(t)$ for all $t \in \mathbb{R}$. Therefore, $R^*(t,x) = \sum_{j=1}^{N} Q_{c_j}(x - \rho_j(t))$, which concludes the proof of (2.1.6). The first estimate in (2.1.7) follows passing to the limit in (2.2.33), while the second is derived arguing as Proposition 2.2.7 and using (2.1.6).

2.3 Weighted estimates

We define N functions to localize the information around each solitary waves. Let

$$\varphi(x) = 1 - C_\varphi \int_{-\infty}^{x} \frac{dy}{\langle y\rangle^{1+\alpha}}, \quad \text{where} \quad C_\varphi = \left(\int_{-\infty}^{+\infty} \frac{dy}{\langle y\rangle^{1+\alpha}}\right)^{-1}. \tag{2.3.1}$$

We have $0 \leqslant \varphi \leqslant 1$. Using the function φ, we set, for $A > 1$ to be fixed later,

$$\varphi_{j,A}(t,x) = \varphi\left(\frac{x - \frac{\rho_j(t)+\rho_{j+1}(t)}{2}}{A}\right) = \varphi\left(\frac{x - m_j(t)}{A}\right), \text{ for } j \in \{1, \cdots, N-1\}, \tag{2.3.2}$$

and $\varphi_{N,A} := 1$, where the ρ_j's are defined in Section 2.2 (in particular, they satisfy (2.2.35)). The function $\varphi_{j,A}$ follows the first j solitary waves. Finally, for $j \in \{1, \cdots, N\}$, the function $\psi_{j,A}$ is localised around the j^{th} solitary wave. Let

$$\psi_{1,A} = \varphi_{1,A}, \quad \psi_{j,A} = \varphi_{j,A} - \varphi_{j-1,A}, \quad \psi_{N,A} = 1 - \varphi_{N-1,A}. \tag{2.3.3}$$

In this section, we state some important estimates involving to the weight $\varphi_{j,A}$ and its derivative $\varphi'_{j,A}$. These estimates will be crucial in the proof of the monotonicity of a localised part of the mass and the energy (see Proposition 2.4.1 in Section 2.4).

2.3.1 Weighted commutator estimates

Lemma 2.3.1. *Let $\alpha \in (0,2)$. In the symmetric case, there exists $C > 0$ such that*

$$\left|\int (|D|^\alpha u)\, u|\varphi'_{j,A}| - \int \left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi'_{j,A}|}\right)\right)^2\right| \leqslant \frac{C}{A^\alpha}\int u^2|\varphi'_{j,A}|,$$

and

$$\left|\int (|D|^\alpha u)\, \partial_x u \varphi_{j,A} + \frac{\alpha-1}{2}\int \left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi'_{j,A}|}\right)\right)^2\right| \leqslant \frac{C}{A^\alpha}\int u^2|\varphi'_{j,A}|,$$

for any $u \in \mathcal{S}(\mathbb{R})$, $A > 1$ and $j \in \{1, \cdots, N\}$.

In the non-symmetric case, there exists $C > 0$ such that

$$\left|\int \left((|D|^\alpha u)\, v - (|D|^\alpha v)\, u\right)|\varphi'_{j,A}|\right| \leqslant \begin{cases} \dfrac{C}{A^\alpha}\displaystyle\int (u^2+v^2)\,|\varphi'_{j,A}|, & \text{if } \alpha \in (0,1], \\ \dfrac{C}{A^{\frac{\alpha}{2}}}\displaystyle\int \left(u^2+v^2+\left(|D|^{\frac{\alpha}{2}}u\right)^2\right)|\varphi'_{j,A}|, & \text{if } \alpha \in (1,2), \end{cases} \tag{2.3.6}$$

and

$$\begin{aligned}&\left|\int \left((|D|^\alpha u)\, \partial_x v + (|D|^\alpha v)\, \partial_x u\right)\varphi_{j,A} + (\alpha-1)\int |D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi'_{j,A}|}\right)|D|^{\frac{\alpha}{2}}\left(v\sqrt{|\varphi'_{j,A}|}\right)\right| \\ &\qquad \leqslant \begin{cases} \dfrac{C}{A^\alpha}\displaystyle\int (u^2+v^2)\,|\varphi'_{j,A}|, & \text{if } \alpha \in (0,1], \\ \dfrac{C}{A^{\frac{\alpha}{2}}}\displaystyle\int \left(u^2+v^2+\left(|D|^{\frac{\alpha}{2}}u\right)^2\right)|\varphi'_{j,A}|, & \text{if } \alpha \in (1,2), \end{cases}\end{aligned} \tag{2.3.7}$$

for any $u, v \in \mathcal{S}(\mathbb{R})$, $A > 1$ and $j \in \{1, \cdots, N\}$.

Remark 2.3.2. Instead of (2.3.6), we can obtain that for $\alpha_1 + \alpha_2 = \alpha - 1$, with $0 \leqslant \alpha_1, \alpha_2 \leqslant \alpha - 1$ and $\alpha \in (1,2)$, there exists $C > 0$ such that for all $u, v \in \mathcal{S}(\mathbb{R})$

$$\left|\int \left((|D|^\alpha u)\, v - (|D|^\alpha v)\, u\right)|\varphi'_{j,A}|\right| \leqslant \frac{C}{A^{\frac{\alpha}{2}}}\int \left(u^2+v^2+(|D|^{\alpha_1}u)^2+(|D|^{\alpha_2}v)^2\right)|\varphi'_{j,A}|.$$

Moreover the estimates (2.3.6) and (2.3.7) are given with $|D|^{\frac{\alpha}{2}}$ instead of $|D|^{\alpha-1}$. This is done to simplify the computations, terms with $|D|^{\frac{\alpha}{2}}$ appear naturally in the proof of Proposition 2.4.1.

Let us explain why we choose to force a dissymmetry on the right hand side of (2.3.6) and (2.3.7). These two estimates will be applied with the function $v = |D|^\alpha u$. However, the natural quantities appearing to prove Proposition 2.4.1 are $\int u^2|\varphi'_{j,A}|$, $\int (|D|^{\frac{\alpha}{2}}u)^2|\varphi'_{j,A}|$ and $\int (|D|^\alpha u)^2|\varphi'_{j,A}|$. Therefore, to control the remainder terms in (2.3.6) and (2.3.7) we need to impose a dissymmetry to avoid an extra derivative on the function v.

The estimates (2.3.4), (2.3.5) are proved in Lemmas 6 and 7 in [91] for $\alpha \in [1,2]$. Observe however that their proofs extend easily to the case $\alpha \in (0,2)$. Note also that while only one side of the inequalities in (2.3.4)-(2.3.5) is stated in Lemmas 6 and 7 in [91] , both sides are actually proved.

While the estimates (2.3.6) and (2.3.7) seem to be new, their proofs follow the lines of the ones of Lemma 6 and 7 of [91]. For the sake of completeness, we will present them in Appendix 2.5.3.

Lemma 2.3.3. *Let $\alpha \in (0,2]$. There exists $C > 0$ such that*

$$\left| \int \left(|D|^\alpha \left(u\sqrt{|\varphi'_{j,A}|} \right) \right)^2 - \int (|D|^\alpha u)^2 |\varphi'_{j,A}| \right| \leqslant \begin{cases} \dfrac{C}{A^\alpha} \displaystyle\int \left(u^2 + (|D|^\alpha u)^2 \right) |\varphi'_{j,A}|, & \text{if } \alpha \in (0,1], \\ \dfrac{C}{A^{\frac{\alpha}{2}}} \displaystyle\int \left(u^2 + \left(|D|^{\frac{\alpha}{2}} u\right)^2 + (|D|^\alpha u)^2 \right) |\varphi'_{j,A}|, & \text{if } \alpha \in (1,2), \end{cases} \tag{2.3.8}$$

for all $u \in \mathcal{S}(\mathbb{R})$, $A > 1$ and $j \in \{1, \cdots, N\}$.

Lemma 2.3.4. *Let $\alpha \in (0,2)$. There exists $C > 0$ such that*

$$\left| \int |D|^\alpha \left(u\sqrt{|\varphi'_{j,A}|} \right) \left((|D|^\alpha u)\sqrt{|\varphi'_{j,A}|} \right) \right| \leqslant \int (|D|^\alpha u)^2 |\varphi'_{j,A}| + \frac{C}{A^{\frac{\alpha}{2}}} \left(\int \left(u^2 + \left(|D|^{\frac{\alpha}{2}} u\right)^2 + (|D|^\alpha u)^2 \right) |\varphi'_{j,A}| \right), \tag{2.3.9}$$

for all $u \in \mathcal{S}(\mathbb{R})$, $A > 1$ and $j \in \{1, \cdots, N\}$.

The proofs of Lemmas 2.3.3 and 2.3.4 are also given in Appendix 2.5.3.

2.3.2 Weighted estimates for the solitary waves

Lemma 2.3.5. *Let $p, q \geqslant 0$. Then, we have for all $j, k \in \{1, ..., N\}$ with $k \neq j$,*

$$\int R_j^p R_k^q \leqslant \frac{C}{(\beta t)^{(1+\alpha)\min(p,q)}}, \tag{2.3.10}$$

$$\int \partial_x R_j^p \partial_x R_k^q \leqslant \frac{C}{(\beta t)^{(2+\alpha)\min(p,q)}},$$

$$\int R_k^p \psi_{j,A}^q \leqslant \frac{C}{(\beta t)^{\min(p(1+\alpha),q\alpha)}}. \tag{2.3.11}$$

Moreover, we have for all $j, k \in \{1, \cdots, N\}$,

$$\int R_k^p |\varphi'_{j,A}|^q \leqslant \frac{C}{(\beta t)^{(1+\alpha)\min(p,q)}}, \tag{2.3.12}$$

$$\int \partial_x R_k^p |\varphi'_{j,A}|^q \leqslant \frac{C}{(\beta t)^{\min(q(1+\alpha),p(2+\alpha))}}, \tag{2.3.13}$$

$$\int R_j^p \left(1 - (\psi_{j,A})^q\right) \leqslant \frac{C}{(\beta t)^{\min(q\alpha,p(1+\alpha))}}, \tag{2.3.14}$$

$$\int \partial_x R_j^p \left(1 - (\psi_{j,A})^q\right) \leqslant \frac{C}{(\beta t)^{\min(\alpha q,p(2+\alpha)}}. \tag{2.3.15}$$

Lemma 2.3.5 is proven arguing exactly as in the proof of Lemma 2.2.5.

2.3.3 Weighted estimates for the non-linear terms

Lemma 2.3.6. *Let $\alpha \in (0,2)$ and let $\eta \in H^{\frac{\alpha}{2}}(\mathbb{R})$ be defined in* (2.2.4) *and verify* (2.2.3). *Then, we have*

$$\int |\eta|^3 |\varphi'_{j,A}| \leqslant C\gamma \left[\int u^2 |\varphi'_{j,A}| + \int \left(|D|^{\frac{\alpha}{2}} \left(u\sqrt{|\varphi'_{j,A}|} \right) \right)^2 \right] + \frac{C}{(\beta t)^{1+\alpha}}, \tag{2.3.16}$$

and

$$\int |\eta|^4 |\varphi'_{j,A}| \leqslant C\gamma^2 \left[\int u^2 |\varphi'_{j,A}| + \int \left(|D|^{\frac{\alpha}{2}} \left(u\sqrt{|\varphi'_{j,A}|} \right) \right)^2 \right] + \frac{C}{(\beta t)^{1+\alpha}}. \tag{2.3.17}$$

Lemma 2.3.7. *Let $\alpha \in (0,2)$ and let u verify the hypotheses of Theorem 2.2.2. Then there exists $C > 0$ such that*

$$\begin{aligned} &\left| \int |D|^{\frac{\alpha}{2}} \left(u\sqrt{|\varphi'_{j,A}|} \right) |D|^{\frac{\alpha}{2}} \left(u^2\sqrt{|\varphi'_{j,A}|} \right) \right| \\ \leqslant & C\left(\gamma^2 + \frac{1}{A^{\frac{\alpha}{2}}} \right) \left(\int \left(u^2 + \left(|D|^{\frac{\alpha}{2}} u \right)^2 \right) |\varphi'_{j,A}| \right) + \frac{1}{8}\int \left(|D|^{\alpha} u \right)^2 |\varphi'_{j,A}| + \frac{C}{(\beta t)^{1+\alpha}}, \end{aligned}$$

for all $A > 1$ and $0 < \gamma < \gamma_1$.

The proofs of these lemmas are also based on pseudo-differential estimates and are given in Appendix 2.5.4.

2.4 Proof of the bootstrap estimate

The goal of this section is to prove Proposition 2.2.8. We work in the bootstrap setting of Section 2.2.2. In particular, the solutions u_n admit the decomposition of Proposition 2.2.2 on the time interval $[t_n^*, S_n]$. We also recall the definitions of the weight functions $\varphi_{j,A}$ and $\psi_{j,A}$ in (2.3.2) and (2.3.3).

In every step of the proof, the values of T_0 and A will be taken large enough independently of n, while the value of γ will be chosen small enough independently of n. Moreover, for simplicity of notation, we drop the index n of the functions u_n and $(\rho_{j,n})_{j=1}^N$ and of the time t_n^*, and the index A of the weight functions $\varphi_{j,A}$ and $\psi_{j,A}$.

Finally, we define the part of the mass M_j and of the energy E_j localized around the j^{th} solitary wave R_j by

$$M_j(t) := \int u(t,x)^2 \psi_j(t,x)dx, \quad E_j(t) := \int \left(\frac{1}{2} u|D|^{\alpha} u - \frac{1}{3} u^3 \right)(t,x)\psi_j(t,x)dx,$$

so that for all $j \in \{1, \cdots, N\}$

$$\sum_{k=1}^{j} M_k(t) = \int u(t,x)^2 \varphi_j(t,x)dx, \quad \sum_{k=1}^{j} E_k(t) = \int \left(\frac{1}{2} u|D|^{\alpha} u - \frac{1}{3} u^3 \right)(t,x)\varphi_j(t,x)dx.$$

We also define

$$\widetilde{E}_k = E_k + \sigma_0 M_k. \tag{2.4.1}$$

with

$$\sigma_0 := \min_{j \in \{1, \cdots, N-1\}} \left(\frac{c_j}{4}, \frac{c_N}{4}, \frac{c_j c_{j+1}}{4(c_j + c_{j+1})} \right). \tag{2.4.2}$$

2.4.1 Monotonicity

Proposition 2.4.1 (Monotonicity)**.** *Under the bootstrap assumptions* (2.2.32)-(2.2.33)*, we have*

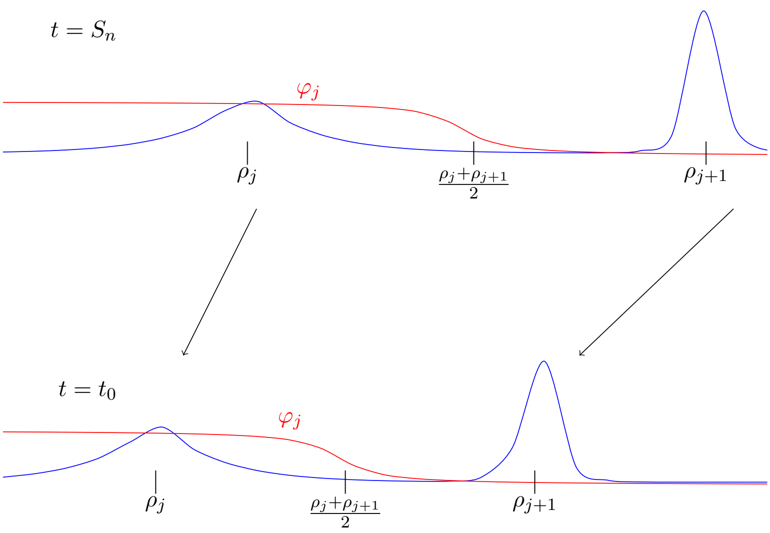
$$\sum_{k=1}^{j}(M_k(S_n)-M_k(t_0)) \geqslant -\frac{C}{(\beta t_0)^{\alpha}}, \tag{2.4.3}$$

and

$$\sum_{k=1}^{j}\left(\widetilde{E}_k(S_n)-\widetilde{E}_k(t_0)\right) \geqslant -\frac{C}{(\beta t_0)^{\alpha}}, \tag{2.4.4}$$

for all $j \in \{1,\cdots,N\}$, $t_0 \in [t^*, S_n]$.

Remark 2.4.2. From Proposition 2.4.1, we see that $M_j(S_n)$ is almost larger than $M_j(t_0)$ for $t_0 < S_n$. In other words, when the time decreases, the portion of the mass on the left of the $(j+1)^{th}$ solitary wave also decreases. A similar phenomenon occurs also for the energy. This can be seen as a manifestation of the dispersive character of *KdV-type* equations: if a wave moves to the right, then the dispersion effect pushes some mass to the left, see Figure 2.1. Moreover, if u is a solution of fKdV then $u(-t,-x)$ is also a solution. Therefore if a wave move to the left, then the dispersion effect pushes some mass to the right.



Monotonicity of the mass

Proof of Proposition 2.4.1. We remark that for $j = N$ the inequalities (2.4.3) and (2.4.4) are easily verified since M and E are preserved by the flow of (2.1.1). Then, we can always assume $1 \leqslant j \leqslant N-1$.

First, we give the proof of (2.4.3). By using (2.1.1), integration by parts and φ is non increasing function, we get

$$\frac{1}{2}\frac{d}{dt}\left(\sum_{k=1}^{j} M_k(t)\right) = \int \left(|D|^{\alpha}u\left(-\partial_x u\varphi_j + u|\varphi_j'|\right) - \frac{u^3}{3}|\varphi_j'| + \frac{m_j'}{2}u^2|\varphi_j'|\right).$$

Then, we deduce from (2.3.4), (2.3.5) that

$$\int \left(|D|^{\alpha}u\left(-\partial_x u\varphi_j + u|\varphi_j'|\right)\right) \geqslant -\frac{C}{A^{\alpha}}\int u^2|\varphi_j'| + \frac{\alpha+1}{2}\int \left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi_j'|}\right)\right)^2. \tag{2.4.5}$$

Observe from (2.2.36) that $m_j' \geqslant \frac{c_j+c_{j+1}}{4}$. Thus,

$$\frac{1}{2}\frac{d}{dt}\left(\sum_{k=1}^{j} M_k(t)\right) \geqslant -\int \frac{u^3}{3}|\varphi_j'| + \frac{c_j+c_{j+1}}{8}\int u^2|\varphi_j'| + \frac{\alpha+1}{2}\int \left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi_j'|}\right)\right)^2.$$

Now, we estimate the nonlinear term. With the notation of Proposition 2.2.2, we have

$$|u|^3 \leqslant C\left(\sum_{k=1}^{N} R_k^3 + |\eta|^3\right).$$

Therefore, by (2.3.12)

$$\sum_{k=1}^{N}\int |R_k^3||\varphi_j'| \leqslant \frac{C}{(\beta t)^{\alpha+1}},$$

and by (2.3.16)

$$\int |\eta|^3|\varphi_j'| \leqslant C\gamma\left[\int u^2|\varphi_j'| + \int \left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi_j'|}\right)\right)^2\right] + \frac{C}{(\beta t)^{\alpha+1}}.$$

Hence, we can conclude that

$$\frac{d}{dt}\left(\sum_{k=1}^{j} M_k(t)\right) \geqslant -\frac{C}{(\beta t)^{1+\alpha}}.$$

Thus, we have by integrating between t_0 and S_n

$$\sum_{k=1}^{j} M_k(S_n) - \sum_{k=1}^{j} M_k(t_0) \geqslant -\int_{t_0}^{S_n}\frac{C}{(\beta t)^{1+\alpha}}dt \geqslant -\frac{C}{(\beta t_0)^{\alpha}},$$

which proves (2.4.3).

Let us prove (2.4.4). We differentiate E_j with respect to time to find that

$$\begin{aligned}\frac{d}{dt}\left(\sum_{k=1}^{j} E_k(t)\right) =& \int\left[\left(\frac{1}{2}\partial_t u|D|^{\alpha}u + \frac{1}{2}u|D|^{\alpha}\partial_t u\right) - \partial_t u u^2\right]\varphi_j \\ &+ m_j'\int\left(\frac{1}{2}u|D|^{\alpha}u - \frac{1}{3}u^3\right)|\varphi_j'| \\ =& I_1 + m_j' I_2.\end{aligned}$$

Using (2.1.1), we obtain for I_1 that

$$\begin{aligned}I_1 =& \frac{1}{2}\int\left(|D|^{\alpha}\partial_x u - \partial_x(u^2)\right)\left(|D|^{\alpha}u\right)\varphi_j + \frac{1}{2}\int u|D|^{\alpha}\left(|D|^{\alpha}\partial_x u - \partial_x(u^2)\right)\varphi_j \\ &- \int\left(|D|^{\alpha}\partial_x u - \partial_x(u^2)\right)u^2\varphi_j \\ =& I_{1,1} + I_{1,2} + I_{1,3}.\end{aligned}$$

First we compute $I_{1,1}$ by integrating by parts

$$I_{1,1} = \frac{1}{4}\int\left(|D|^{\alpha}u\right)^2|\varphi_j'| - \frac{1}{2}\int \partial_x(u^2)\left(|D|^{\alpha}u\right)\varphi_j, \tag{2.4.6}$$

since the functions φ_j are non increasing. Now we decompose $I_{1,2}$ as

$$I_{1,2} = \frac{1}{2}\int u\left(|D|^{2\alpha}\partial_x u\right)\varphi_j - \frac{1}{2}\int u\left(|D|^{\alpha}\partial_x(u^2)\right)\varphi_j = I_{1,2,1} + I_{1,2,2}.$$

First we deal with $I_{1,2,1}$. By using integration by parts we get

$$I_{1,2,1} = -\frac{1}{2}\int \partial_x u\left(|D|^{2\alpha}u\right)\varphi_j + \frac{1}{2}\int u\left(|D|^{2\alpha}u\right)|\varphi_j'| = I^1_{1,2,1} + I^2_{1,2,1}.$$

On the one hand, using the estimate (2.3.7) with $v = |D|^{\alpha}u$ and integration by parts for the last integral, we get

$$\begin{aligned}
I^1_{1,2,1} = &-\frac{1}{2}\int \left(\partial_x u|D|^{2\alpha}u + \left(|D|^{\alpha}u\right)\partial_x|D|^{\alpha}u\right)\varphi_j + \frac{1}{2}\int \left(\partial_x|D|^{\alpha}u\right)\left(|D|^{\alpha}u\right)\varphi_j \\
&- \frac{\alpha-1}{2}\int |D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi_j'|}\right)|D|^{\frac{\alpha}{2}}\left(\left(|D|^{\alpha}u\right)\sqrt{|\varphi_j'|}\right) \\
&+ \frac{\alpha-1}{2}\int |D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi_j'|}\right)|D|^{\frac{\alpha}{2}}\left(\left(|D|^{\alpha}u\right)\sqrt{|\varphi_j'|}\right) \\
\geqslant &\ \frac{1}{4}\int \left(|D|^{\alpha}u\right)^2|\varphi_j'|dx + \frac{\alpha-1}{2}\int |D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi_j'|}\right)|D|^{\frac{\alpha}{2}}\left(\left(|D|^{\alpha}u\right)\sqrt{|\varphi_j'|}\right) \\
&- \frac{C}{A^{\frac{\alpha}{2}}}\int \left(u^2 + \left(|D|^{\frac{\alpha}{2}}u\right)^2 + \left(|D|^{\alpha}u\right)^2\right)|\varphi_j'|.
\end{aligned}$$

On the other hand, we deduce from (2.3.6) with $v = |D|^{\alpha}u$

$$\begin{aligned}
I^2_{1,2,1} &= \frac{1}{2}\int \left(u|D|^{2\alpha}u - \left(|D|^{\alpha}u\right)^2\right)|\varphi_j'| + \frac{1}{2}\int \left(|D|^{\alpha}u\right)^2|\varphi_j'| \\
&\geqslant \frac{1}{2}\int \left(|D|^{\alpha}u\right)^2|\varphi_j'| - \frac{C}{A^{\frac{\alpha}{2}}}\int \left(u^2 + \left(|D|^{\frac{\alpha}{2}}u\right)^2 + \left(|D|^{\alpha}u\right)^2\right)|\varphi_j'|.
\end{aligned}$$

Now, we deal with $I_{1,2,2}$. Using integration by parts

$$I_{1,2,2} = \frac{1}{2}\int \partial_x u|D|^{\alpha}(u^2)\varphi_j - \frac{1}{2}\int u|D|^{\alpha}(u^2)|\varphi_j'|.$$

Arguing similarly as for $I_{1,2,1}$, we get from (2.3.6), (2.3.7) with $v = u^2$ that

$$\begin{aligned}
I_{1,2,2} \geqslant &-\frac{\alpha-1}{2}\int |D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi_j'|}\right)|D|^{\frac{\alpha}{2}}\left(u^2\sqrt{|\varphi_j'|}\right) - \frac{1}{2}\int u^2\left(|D|^{\alpha}u\right)|\varphi_j'| \\
&- \frac{1}{2}\int \partial_x\left(u^2\right)\left(|D|^{\alpha}u\right)\varphi_j - \frac{C}{A^{\frac{\alpha}{2}}}\int \left(u^2 + u^4 + \left(|D|^{\frac{\alpha}{2}}u\right)^2\right)|\varphi_j'|.
\end{aligned}$$

Hence, we conclude gathering these estimates

$$\begin{aligned}
I_{1,2} \geqslant &\frac{3}{4}\int \left(|D|^{\alpha}u\right)^2|\varphi_j'| - \frac{1}{2}\int \partial_x\left(u^2\right)|D|^{\alpha}u\varphi_j - \frac{1}{2}\int u^2|D|^{\alpha}u|\varphi_j'| \\
&- \frac{\alpha-1}{2}\int |D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi_j'|}\right)|D|^{\frac{\alpha}{2}}\left(u^2\sqrt{|\varphi_j'|}\right) \\
&+ \frac{\alpha-1}{2}\int |D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi_j'|}\right)|D|^{\frac{\alpha}{2}}\left(\left(|D|^{\alpha}u\right)\sqrt{|\varphi_j'|}\right) \\
&- \frac{C}{A^{\frac{\alpha}{2}}}\int \left(u^2 + u^4 + \left(|D|^{\frac{\alpha}{2}}u\right)^2 + \left(|D|^{\alpha}u\right)^2\right)|\varphi_j'|.
\end{aligned} \tag{2.4.7}$$

Finally, we compute $I_{1,3}$ by integrating by parts

$$I_{1,3} = \int \left(|D|^{\alpha}u\right)\partial_x\left(u^2\right)\varphi_j - \int \left(|D|^{\alpha}u\right)u^2|\varphi_j'| + \frac{1}{2}\int u^4|\varphi_j'|. \tag{2.4.8}$$

Therefore, combining (2.4.6),(2.4.7) and (2.4.8), we deduce that

$$\begin{aligned} I_1 \geqslant & \int \left(|D|^\alpha u\right)^2 |\varphi_j'| - \frac{3}{2}\int u^2 \left(|D|^\alpha u\right)|\varphi_j'| + \frac{1}{2}\int u^4 |\varphi_j'| \\ & - \frac{\alpha-1}{2}\int |D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi_j'|}\right)|D|^{\frac{\alpha}{2}}\left(u^2\sqrt{|\varphi_j'|}\right) \\ & + \frac{\alpha-1}{2}\int |D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi_j'|}\right)|D|^{\frac{\alpha}{2}}\left(\left(|D|^\alpha u\right)\sqrt{|\varphi_j'|}\right) \\ & - \frac{C}{A^{\frac{\alpha}{2}}}\int \left(u^2 + u^4 + \left(|D|^{\frac{\alpha}{2}}u\right)^2 + \left(|D|^\alpha u\right)^2\right)|\varphi_j'|. \end{aligned}$$

By using the identity

$$\begin{aligned} & \int \left(|D|^\alpha u\right)^2 |\varphi_j'| + \frac{1}{2}\int u^4 |\varphi_j'| - \frac{3}{2}\int \left(|D|^\alpha u\right) u^2 |\varphi_j'| \\ = & \int \left(\frac{1}{4}|D|^\alpha u - 3u^2\right)^2 |\varphi_j'| + \frac{15}{16}\int \left(|D|^\alpha u\right)^2 |\varphi_j'| - \frac{17}{2}\int u^4 |\varphi_j'|, \end{aligned}$$

$\frac{\alpha-1}{2} \in [-\frac{1}{4}, \frac{1}{2}]$, and Lemmas 2.3.4, 2.3.7 we conclude that

$$I_1 \geqslant \frac{1}{4}\int \left(|D|^\alpha u\right)^2 |\varphi_j'| - 9\int u^4 |\varphi_j'| - C\left(\frac{1}{A^{\frac{\alpha}{2}}} + \gamma^2\right)\int \left(u^2 + \left(|D|^{\frac{\alpha}{2}}u\right)^2\right)|\varphi_j'| - \frac{C}{(\beta t)^{1+\alpha}}.$$

Using (2.3.4) to control I_2, we obtain that

$$\begin{aligned} \frac{d}{dt}\left(\sum_{k=1}^{j} E_k(t)\right) \geqslant & \frac{1}{4}\int \left(|D|^\alpha u\right)^2 |\varphi_j'| - C\left(\frac{1}{A^{\frac{\alpha}{2}}} + \gamma^2\right)\int \left(u^2 + \left(|D|^{\frac{\alpha}{2}}u\right)^2\right)|\varphi_j'| \\ & - \frac{m_j'}{3}\int |u|^3 |\varphi_j'| - 9\int u^4 |\varphi_j'| - \frac{C}{(\beta t)^{1+\alpha}}. \end{aligned}$$

We need to add the mass to the energy in order to control the remaining terms

$$\begin{aligned} \frac{d}{dt}\left(\sum_{k=1}^{j} E_k(t) + \sigma_0 M_k(t)\right) \geqslant & - 9\int u^4 |\varphi_j'| - C\left(\frac{1}{A^{\frac{\alpha}{2}}} + \gamma^2\right)\int \left(u^2 + \left(|D|^{\frac{\alpha}{2}}u\right)^2\right)|\varphi_j'| \\ & - \sigma_0 \int |D|^\alpha u \left(\partial_x u \varphi_j - u|\varphi_j'|\right) - \sigma_0 \int \frac{1}{3}|u|^3 |\varphi_j'| + \sigma_0 m_j' \int u^2 |\varphi_j'| \\ & - \frac{m_j'}{3}\int |u|^3 |\varphi_j'| - \frac{C}{(\beta t)^{1+\alpha}}. \end{aligned}$$

Thus, by using (2.4.5), we deduce

$$\begin{aligned} \frac{d}{dt}\left(\sum_{k=1}^{j} E_k(t) + \sigma_0 M_k(t)\right) \geqslant & - 9\int u^4 |\varphi_j'| + \left(\sigma_0 m_j' - \frac{C}{A^{\frac{\alpha}{2}}} - \gamma^2\right)\int u^2 |\varphi_j'| - \frac{m_j' + \sigma_0}{3}\int u^3 |\varphi_j'| \\ & + \sigma_0 \frac{3-\alpha}{2}\int \left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi_j'|}\right)\right)^2 - \left(\frac{C}{A^{\frac{\alpha}{2}}} + \gamma^2\right)\int \left(|D|^{\frac{\alpha}{2}}u\right)^2 |\varphi_j'| \\ & - \frac{C}{(\beta t)^{1+\alpha}}. \end{aligned}$$

Observe from (2.2.36) that $\frac{\sigma_0 m_j'}{2} - \frac{C}{A^{\frac{\alpha}{2}}} - \gamma^2 > 0$. Thus, by (2.3.8), we deduce that

$$\begin{aligned} \frac{d}{dt}\left(\sum_{k=1}^{j} E_k(t) + \sigma_0 M_k(t)\right) \geqslant & \frac{\sigma_0 m_j'}{2}\int u^2 |\varphi_j'| - 9\int u^4 |\varphi_j'| - \frac{m_j' + \sigma_0}{3}\int u^3 |\varphi_j'| \\ & + \sigma_0 \frac{3-\alpha}{4}\int \left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi_j'|}\right)\right)^2 - \frac{C}{(\beta t)^{1+\alpha}}. \end{aligned}$$

From Lemma 2.3.6 , we get

$$\begin{aligned}\frac{d}{dt}\left(\sum_{k=1}^{j} E_k(t)+\sigma_0 M_k(t)\right) &\geqslant \left(\frac{\sigma_0 m_j'}{4}-C\gamma\right)\int u^2|\varphi_j'|-\frac{C}{(\beta t)^{1+\alpha}}\\ &\quad+\left(\sigma_0\frac{3-\alpha}{2}-C\gamma\right)\int\left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi_j'|}\right)\right)^2\\ &\geqslant -\frac{C}{(\beta t)^{1+\alpha}}.\end{aligned}$$

Thus we have by integrating between t_0 and S_n

$$\sum_{k=1}^{j}\widetilde{E}_k(S_n)-\sum_{k=1}^{j}\widetilde{E}_k(t_0)\geqslant-\int_{t_0}^{S_n}\frac{C}{(\beta t)^{1+\alpha}}dt\geqslant-\frac{C}{(\beta t_0)^{\alpha}},$$

which proves (2.4.4). □

2.4.2 Mass and energy expansion

Lemma 2.4.3. *There exist $C>0$ such that the following hold:*

$$\left|M_j(t)-\left[\int Q_{c_j}^2+2\int\eta(t)R_j(t)+\int\eta^2(t)\psi_j(t)\right]\right|\leqslant\frac{C}{(\beta t)^{\alpha}},\tag{2.4.9}$$

$$\begin{aligned}\left|E_j(t)-\left[E(Q_{c_j})-c_j\int\eta(t)R_j(t)+\frac{1}{2}\int\left(\eta(t)|D|^{\alpha}\eta(t)-2R(t)\eta^2(t)\right)\psi_j(t)\right]\right|\\ \leqslant\frac{C}{(\beta t)^{\alpha}}+C\gamma\|\eta(t)\|_{H^{\frac{\alpha}{2}}}^2\end{aligned}\tag{2.4.10}$$

and

$$\begin{aligned}\left|\left(E_j(t)+\frac{c_j}{2}M_j(t)\right)-\left(E(Q_{c_j})+\frac{c_j}{2}M(Q_{c_j})\right)-\frac{1}{2}H_j(t)\right|\\ \leqslant\frac{C}{(\beta t)^{\alpha}}+C\gamma\|\eta(t)\|_{H^{\frac{\alpha}{2}}}^2,\end{aligned}\tag{2.4.11}$$

where

$$H_j(t):=H_j(\eta(t),\eta(t))=\int\left(\eta(t)|D|^{\alpha}\eta(t)+c_j\eta^2(t)-2R_j(t)\eta^2(t)\right)\psi_j(t).\tag{2.4.12}$$

Proof. Using $u=R+\eta$ in the mass, we get

$$M_j(t)=\int\left(R^2+2R\eta+\eta^2\right)\psi_j.$$

Thus by direct computations,

$$M_j(t)-\left[\int Q_{c_j}^2+2\int\eta R_j+\int\eta^2\psi_j\right]=\int\left(R^2\psi_j-Q_{c_j}^2\right)+2\int\eta\left(R\psi_j-R_j\right)=I_1+2I_2.$$

We use the translation invariance of the L^2 norm of Q_{c_j}, and (2.3.10), (2.3.11), (2.3.14) to deduce

$$|I_1|\leqslant\sum_{(k,i)\neq(j,j)}\int R_iR_k\psi_j+\int R_j^2\left(1-\psi_j\right)\leqslant\frac{C}{(\beta t)^{\alpha}}.$$

By Cauchy-Schwarz inequality, (2.3.11), (2.3.14), we obtain for I_2

$$|I_2| \leqslant 2\|\eta\|_{L^2}\left(\sum_{k\neq j}\|R_k\psi_j\|_{L^2}+\|R_j(1-\psi_j)\|_{L^2}\right)\leqslant \frac{C}{(\beta t)^{\alpha}}.$$

Combining these two inequalities we conclude the proof of (2.4.9).

To prove (2.4.10), we expand E_j as

$$\begin{aligned}E_j(t)=&\frac{1}{2}\int R\left(|D|^{\alpha}R\right)\psi_j+\frac{1}{2}\int \eta\left(|D|^{\alpha}\eta\right)\psi_j+\frac{1}{2}\int\left(R|D|^{\alpha}\eta+\eta|D|^{\alpha}R\right)\psi_j\\ &-\frac{1}{3}\int R^3\psi_j-\int R^2\eta\psi_j-\int R\eta^2-\frac{1}{3}\int\eta^3\psi_j.\end{aligned}$$

Hence,

$$\begin{aligned}&\left|E_j(t)-\left[E(Q_{c_j})-c_j\int\eta R_j+\frac{1}{2}\int\left(\eta|D|^{\alpha}\eta-2R\eta^2\right)\psi_j\right]\right|\\ &\leqslant\left|\int\left(\frac{1}{2}R|D|^{\alpha}R-\frac{1}{3}R^3\right)\psi_j-E(Q_{c_j})\right|+\frac{1}{3}\left|\int\eta^3\psi_j\right|\\ &\quad+\frac{1}{2}\left|\int\left(R|D|^{\alpha}\eta+\eta|D|^{\alpha}R\right)\psi_j-2\int R^2\eta\psi_j+2c_j\int\eta R_j\right|\\ &=J_1+J_2+J_3.\end{aligned}$$

We use the translation invariance of $E(Q_{c_j})$, (2.3.11), (2.3.14) and $\||D|^{\alpha}R_k\|_{L^\infty}\leqslant C$ to bound $|J_1|$ by

$$C\left(\int\left(|R_j||D|^{\alpha}R_j|+R_j^3\right)(1-\psi_j)+\sum_{(i,k)\neq(j,j)}\left|\int R_i\left(|D|^{\alpha}R_k\right)\psi_j\right|+\sum_{(i,k,l)\neq(j,j,j)}\int R_iR_kR_l\psi_j\right)$$

Thus, we get

$$|J_1|\leqslant\frac{C}{(\beta t)^{\alpha}}. \tag{2.4.13}$$

Replacing $|\varphi_j'|$ by ψ_j in (2.5.15), we have

$$|J_2|\leqslant C\gamma\|\eta\sqrt{\psi_j}\|^2_{H^{\frac{\alpha}{2}}}.$$

so that it follows arguing as (2.5.18) that

$$|J_2|\leqslant C\gamma\|\eta\|^2_{H^{\frac{\alpha}{2}}}. \tag{2.4.14}$$

By using (2.1.3), we get

$$\begin{aligned}2|J_3|\leqslant&\left|\int\left(R|D|^{\alpha}\eta+\eta|D|^{\alpha}R\right)\psi_j-2\int\left(|D|^{\alpha}R_j\right)\eta\right|+2\left|\int\left(R_j^2\eta-R^2\eta\psi_j\right)\right|\\ \leqslant&\left|\int R\left(|D|^{\alpha}\eta\right)\psi_j-\int\left(|D|^{\alpha}R_j\right)\eta\right|+\left|\int\eta\left(|D|^{\alpha}R\right)\psi_j-\int\left(|D|^{\alpha}R_j\right)\eta\right|\\ &+2\left|\int R_j^2\eta-R^2\eta\psi_j\right|\\ =&|J_{3,1}|+|J_{3,2}|+|J_{3,3}|.\end{aligned}$$

By using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|J_{3,2}|+|J_{3,3}|\leqslant&\|\eta\|_{L^2}\left(\|\left(1-\psi_j\right)|D|^{\alpha}R_j\|_{L^2}+\|R_j^2\left(1-\psi_j\right)\|_{L^2}\right)\\ &+\|\eta\|_{L^2}\left(\sum_{k\neq j}\|\psi_j|D|^{\alpha}R_k\|_{L^2}+\sum_{(k,l)\neq(j,j)}\|R_kR_l\psi_j\|_{L^2}\right)\end{aligned}$$

From (2.1.3), we rewrite $|D|^\alpha R_j = R_j^2 - c_j R_j$. Thus, it follows from (2.3.11) and (2.3.14)

$$|J_{3,2}| + |J_{3,3}| \leqslant \frac{C}{(\beta t)^\alpha}. \tag{2.4.15}$$

Now, we estimate $|J_{3,1}|$. By using the Cauchy-Schwarz inequality,

$$\begin{aligned} |J_{3,1}| &\leqslant \sum_{k\neq j} \left| \int R_k(|D|^\alpha \eta)\psi_j \right| + \left| \int R_j(|D|^\alpha \eta)\,(1-\psi_j) \right| \\ &\leqslant \||D|^{\frac{\alpha}{2}}\eta\|_{L^2} \left(\sum_{k\neq j} \||D|^{\frac{\alpha}{2}}\,(R_k\psi_j)\|_{L^2} + \||D|^{\frac{\alpha}{2}}\,(R_j\,(1-\psi_j))\|_{L^2} \right) \end{aligned}$$

By interpolation $\|u\|_{H^{\frac{\alpha}{2}}} \leqslant \|u\|_{L^2}^{1-\frac{\alpha}{2}} \|u\|_{H^1}^{\frac{\alpha}{2}}$. Thus, we deduce from (2.3.12), (2.3.14), (2.3.15),(2.3.13) that

$$|J_{3,1}| \leqslant \||D|^{\frac{\alpha}{2}}\eta\|_{L^2} \left(\sum_{k\neq j} \|R_k\psi_j\|_{L^2}^{1-\frac{\alpha}{2}} \|R_k\psi_j\|_{H^1}^{\frac{\alpha}{2}} + \|R_j\,(1-\psi_j)\|_{L^2}^{1-\frac{\alpha}{2}} \|R_j\,(1-\psi_j)\|_{H^1}^{\frac{\alpha}{2}} \right) \leqslant \frac{C}{(\beta t)^\alpha}. \tag{2.4.16}$$

Gathering (2.4.13), (2.4.14), (2.4.15) and (2.4.16) we conclude the proof of (2.4.10).

To prove (2.4.11), gathering (2.4.9) and (2.4.10) we get

$$\begin{aligned} &\left| \left(E_j(t) + \frac{c_j}{2} M_j(t) \right) - \left(E(Q_{c_j}) + \frac{c_j}{2} M(Q_{c_j}) \right) - \frac{1}{2} H_j(t) \right| \\ &\qquad\qquad \leqslant \frac{C}{(\beta t)^\alpha} + C\gamma \|\eta(t)\|_{H^{\frac{\alpha}{2}}}^2 + \left| \int \eta^2\,(R - R_j)\,\psi_j \right| \end{aligned}$$

Then by Cauchy-Schwarz, (2.3.11) and the Sobolev imbedding $L^4(\mathbb{R}) \hookrightarrow H^{\frac{1}{4}}(\mathbb{R})$, we obtain that

$$\left| \int \eta^2\,(R - R_j)\,\psi_j \right| \leqslant \|\eta\|_{L^4}^2 \sum_{k\neq j} \|R_k\psi_j\|_{L^2} \leqslant \frac{C}{(\beta t)^\alpha},$$

which concludes the proof of (2.4.11). ☐

2.4.3 Control of the R_j directions

We recall C_* is a positive constant changing from line to line and depending only on the parameters $\{c_1, \cdots, c_N\}$.

Proposition 2.4.4. *For all $j \in \{1, \cdots, N\}$, $t_0 \in [t^*, S_n]$,*

$$\sum_{k=1}^{j} \left| \int \eta(t_0) R_k(t_0) \right| \leqslant \frac{C_*}{t_0^\alpha} + C_* \|\eta(t_0)\|_{H^{\frac{\alpha}{2}}}^2. \tag{2.4.17}$$

Proof of Proposition 2.4.4. The proof is by induction. For $j = 1$, by (2.4.9) at time $t = t_0$ and $t = S_n$, we deduce that

$$\begin{aligned} 2\int \eta(t_0)R_1(t_0) &\leqslant \frac{C}{(\beta t_0)^\alpha} - \int \eta^2\psi_1 - \int Q_{c_1}^2 + M_1(t_0) \\ &\leqslant \frac{C}{(\beta t_0)^\alpha} - \int \eta^2\psi_1 - M_1(S_n) + M_1(t_0) + M_1(S_n) - \int Q_{c_1}^2 \\ &\leqslant \frac{C}{(\beta t_0)^\alpha} - \int \eta^2\psi_1 - M_1(S_n) + M_1(t_0). \end{aligned}$$

Moreover, by using estimate (2.4.3), we deduce that

$$2\int \eta(t_0)R_1(t_0) \leqslant \frac{C}{(\beta t_0)^{\alpha}} - \int \eta^2\psi_1 \leqslant \frac{C}{(\beta t_0)^{\alpha}}.$$

Now we want to obtain a lower bound of this scalar product. We recall that $\widetilde{E}_k$ and σ_0 are respectively defined in (2.4.1) and (2.4.2). By (2.4.9), (2.4.10) at time $t = t_0$ and $t = S_n$, we observe that

$$\begin{aligned}(c_1 - 2\sigma_0)\int \eta R_1 \geqslant & -\frac{C}{(\beta t_0)^{\alpha}} - C\gamma\|\eta\|^2_{H^{\frac{\alpha}{2}}} - E_1(t_0) + E(Q_{c_1}) \\ & + \frac{1}{2}\int \eta(|D|^{\alpha}\eta)\psi_1 - \int R\eta^2\psi_1 \\ & - \sigma_0 M_1(t_0) + \sigma_0\int Q_{c_1}^2 + \sigma_0\int \eta^2\psi_1 \\ \geqslant & -\frac{C_*}{t_0^{\alpha}} - C_*\|\eta\|^2_{H^{\frac{\alpha}{2}}} - \widetilde{E}_1(t_0) + \widetilde{E}_1(S_n) \\ & + \frac{1}{2}\int \eta(|D|^{\alpha}\eta)\psi_1 - \int R\eta^2\psi_1 + \sigma_0\int \eta^2\psi_1.\end{aligned}$$

Thus, we deduce from (2.4.4) and the fact $\|R\psi_1\|_{\infty} < C$

$$(c_1 - 2\sigma_0)\int \eta R_1 \geqslant -\frac{C_*}{t_0^{\alpha}} - C_*\|\eta\|^2_{H^{\frac{\alpha}{2}}} + \frac{1}{2}\int \eta(|D|^{\alpha}\eta)\psi_1.$$

Note that replacing φ_j' by ψ_j in (2.5.16) we deduce

$$\||D|^{\frac{\alpha}{2}}(\eta\psi_j)\|_{L^2} \leqslant C\|\eta\|_{H^{\frac{\alpha}{2}}},$$

so that

$$(c_1 - 2\sigma_0)\int \eta R_1 \geqslant -\frac{C_*}{t_0^{\alpha}} - C_*\|\eta\|^2_{H^{\frac{\alpha}{2}}}.$$

Combining the lower and upper bound, we conclude that

$$\left|\int \eta(t_0)R_1(t_0)\right| \leqslant \frac{C_*}{t_0^{\alpha}} + C_*\|\eta\|^2_{H^{\frac{\alpha}{2}}}.$$

Now, we prove the inductive step. We assume that (2.4.17) holds true for some $j \in \{1, \cdots, N-1\}$ and we prove it for $j+1$. Arguing similarly as in the case $j = 1$, we deduce by (2.4.9) at time $t = t_0$ and $t = S_n$, (2.4.3) and then the induction hypothesis (2.4.17) in j, that

$$\begin{aligned}2\int \eta(t_0)R_{j+1}(t_0) \leqslant & \frac{C}{(\beta t_0)^{\alpha}} + \sum_{k=1}^{j+1}(M_k(t_0) - M_k(S_n)) - \int \eta^2\psi_{j+1} \\ & - \left[\sum_{k=0}^{j}(M_k(t_0) - M_k(S_n)) - \sum_{k=1}^{j}\int \eta^2\psi_k\right] - \sum_{k=1}^{j}\int \eta^2\psi_k \\ \leqslant & \frac{C}{(\beta t_0)^{\alpha}} + 2\sum_{k=1}^{j}\left|\int \eta R_k\right| \\ \leqslant & \frac{C_*}{t_0^{\alpha}} + C_*\|\eta\|^2_{H^{\frac{\alpha}{2}}}.\end{aligned}$$

Arguing similarly as $j=1$ for the lower bound, we obtain from (2.4.9), (2.4.10) at time $t=t_0$ and $t=S_n$, that

$$
\begin{aligned}
(c_{j+1}-2\sigma_0)\int \eta R_{j+1} \geqslant & -\frac{C_*}{t_0^\alpha}-\widetilde{E}_{j+1}(t_0)+\widetilde{E}_{j+1}(S_n)+\sigma_0\int \eta^2\psi_{j+1} \\
& +\frac{1}{2}\int \left(\eta|D|^\alpha-2R\eta^2\right)\psi_{j+1}-C_*\|\eta\|^2_{H^{\frac{\alpha}{2}}} \\
\geqslant & -\frac{C_*}{t_0^\alpha}-C_*\|\eta\|^2_{H^{\frac{\alpha}{2}}}+\sigma_0\sum_{k=1}^{j+1}\int \eta^2\psi_k \\
& +\sum_{k=1}^{j+1}\left(\widetilde{E}_k(S_n)-\widetilde{E}_k(t_0)\right)+\frac{1}{2}\sum_{k=1}^{j+1}\int \left(\eta|D|^\alpha\eta-2R\eta^2\right)\psi_k \\
& -\left[\sum_{k=1}^{j}\left(\widetilde{E}_k(S_n)-\widetilde{E}_k(t_0)\right)+\frac{1}{2}\sum_{k=1}^{j}\int \left(\eta|D|^\alpha\eta-2R\eta^2\right)\psi_k\right. \\
& \left.+\sigma_0\sum_{k=1}^{j}\int \eta^2\psi_k\right].
\end{aligned}
$$

Thus, by using again (2.4.9), (2.4.10) at time $t=t_0$ and $t=S_n$, (2.4.4), and then the induction hypothesis (2.4.17) in j, it follows that

$$
\begin{aligned}
(c_{j+1}-2\sigma_0)\int \eta R_{j+1} \geqslant & -\frac{C_*}{t_0^\alpha}-C_*\|\eta\|^2_{H^{\frac{\alpha}{2}}}-\sum_{k=1}^{j}(c_k-2\sigma_0)\left|\int \eta R_k\right| \\
\geqslant & -\frac{C_*}{t_0^\alpha}-C_*\|\eta\|^2_{H^{\frac{\alpha}{2}}}.
\end{aligned}
$$

This concludes the proof of (2.4.17) in $j+1$, and thus the proof of Proposition 2.4.4 by induction. □

2.4.4 Proof of Proposition 2.2.8

Recalling the notation $\eta=u-R$, it suffices to prove that

$$
\|\eta\|^2_{H^{\frac{\alpha}{2}}}\leqslant \frac{C_*}{t_0^\alpha}. \tag{2.4.18}
$$

Proof of (2.4.18). The proof of the estimate (2.4.18) relies on the quadratic form $H_j(t)$ defined in (2.4.12) On the one hand, we have from (2.4.11),

$$
\begin{aligned}
\sum_{j=1}^{N}\frac{1}{c_j^2}H_j(t_0)\leqslant & \sum_{j=1}^{N}\frac{1}{c_j^2}\left(E_j(t_0)+\frac{c_j}{2}M_j(t_0)\right)-\sum_{j=1}^{N}\frac{1}{c_j^2}\left(E(Q_{c_j})+\frac{c_j}{2}M(Q_{c_j})\right) \\
& +\frac{C_*}{t_0^\alpha}+C_*\gamma\|\eta\|^2_{H^{\frac{\alpha}{2}}}.
\end{aligned} \tag{2.4.19}
$$

On the other hand, by a direct resummation argument, we observe that

$$
\sum_{j=1}^{N}\frac{1}{c_j^2}\widetilde{E}_j=\sum_{j=1}^{N-1}\left(\frac{1}{c_j^2}-\frac{1}{c_{j+1}^2}\right)\sum_{k=1}^{j}\widetilde{E}_k+\frac{1}{c_N^2}\sum_{k=1}^{N}\widetilde{E}_k
$$

and

$$\sum_{j=1}^{N}\frac{1}{c_j^2}\left(\frac{c_j}{2}-\sigma_0\right)M_j=\sum_{j=1}^{N-1}\left[\frac{1}{2}\left(\frac{1}{c_j}-\frac{1}{c_{j+1}}\right)\left(1-2\sigma_0\left(\frac{1}{c_j}+\frac{1}{c_{j+1}}\right)\right)\right]\sum_{k=1}^{j}M_k$$
$$+\frac{1}{2c_N}\left(1-2\frac{\sigma_0}{c_N}\right)\sum_{k=1}^{N}M_k.$$

Combining these two identities, since $\widetilde{E}_j=E_j+\sigma_0M_j$, we deduce that

$$\sum_{j=1}^{N}\frac{1}{c_j^2}\left(E_j+\frac{c_j}{2}M_j\right)=\sum_{j=1}^{N-1}\left(\frac{1}{c_j^2}-\frac{1}{c_{j+1}^2}\right)\sum_{k=1}^{j}\widetilde{E}_k+\frac{1}{c_N^2}\sum_{k=1}^{N}\widetilde{E}_k$$
$$+\sum_{j=1}^{N-1}\left[\frac{1}{2}\left(\frac{1}{c_j}-\frac{1}{c_{j+1}}\right)\left(1-2\sigma_0\left(\frac{1}{c_j}+\frac{1}{c_{j+1}}\right)\right)\right]\sum_{k=1}^{j}M_k$$
$$+\frac{1}{2c_N}\left(1-2\frac{\sigma_0}{c_N}\right)\sum_{k=1}^{N}M_k.$$

Note that all the coefficients in front of the partial sums on the right hand side of the above estimate are positive by definition of σ_0 in (2.4.2). Therefore, we deduce from (2.4.19), (2.4.9), (2.4.10) at time $t=t_0$ and $t=S_n$ and the monotonicity results (2.4.3) and (2.4.4) in Proposition 2.4.1, that

$$\sum_{j=0}^{N}\frac{1}{c_j^2}H_j(t_0)\leqslant\frac{C_*}{t_0^{\alpha}}+C_*\gamma\|\eta\|_{H^{\frac{\alpha}{2}}}^2. \tag{2.4.20}$$

On the other hand, by Corollary 2.5.8 and (2.2.6), there exists $\lambda_0>0$ such that

$$\sum_{j=0}^{N}\frac{1}{c_j^2}H_j(t_0)\geqslant\lambda_0\|\eta\|_{H^{\frac{\alpha}{2}}}^2-\frac{C_*}{t_0^{\alpha}}-\frac{1}{\lambda_0}\sum_{j=0}^{N}\left(\int\eta(t_0)R_j(t_0)\right)^2.$$

The control of the R_j directions derived in Proposition 2.4.4 yields

$$\sum_{j=0}^{N}\frac{1}{c_j^2}H_j(t_0)\geqslant\lambda_0\|\eta\|_{H^{\frac{\alpha}{2}}}^2-\frac{C_*}{t_0^{\alpha}}-\frac{1}{\lambda_0}\frac{C_*}{t_0^{2\alpha}}-\frac{1}{\lambda_0}C_*\|\eta\|_{H^{\frac{\alpha}{2}}}^4. \tag{2.4.21}$$

Therefore, we conclude the proof of (2.4.18) by combining (2.4.20) and (2.4.21), which finishes the proof of Proposition 2.2.8.

□

2.5 Appendix

2.5.1 Weak continuity of the flow

In this appendix, we give the proof of Lemma 2.2.9 in the case $\alpha>\frac{6}{7}$, where the IVP associated to (2.1.1) is globally well-posed in the energy space (see [148]). We follow a general argument given by L. Molinet [147] (see also [66]).

Proof of Lemma 2.2.9 in the case $\alpha > \frac{6}{7}$. Let $\alpha > \frac{6}{7}$. For $T > 0$, we denote by $Y_T := C^0([0,T] : H^{\frac{\alpha}{2}}(\mathbb{R})) \cap X_T^{\frac{\alpha}{2}-1,1} \cap L^2((0,T), W^{\frac{\alpha}{4}-\frac{1}{2}-,\infty}(\mathbb{R}))$ the resolution space (see Theorem 1.2 in [148]) and $\|\cdot\|_{Y_T}$ the norm associated to Y_T.

Assume $z_{0,n} \rightharpoonup z_0$ in $H^{\frac{\alpha}{2}}(\mathbb{R})$. By the Banach-Steinhaus theorem, we deduce that there exists $C > 0$ such that $\|z_{0,n}\|_{H^{\frac{\alpha}{2}}} \leqslant C$. Moreover, by the global well-posedness result in Theorem 1.2 in [148], there exists $C > 0$ such that for all $t \in [0,T]$, the solution z_n of (2.1.1), associated to $z_{0,n}$, verifies

$$\|z_n(t)\|_{Y_T} \leqslant C, \quad \forall t \in [-T,T]. \tag{2.5.1}$$

Thus, by Banach-Alaoglu's theorem, there exists $z \in X_T$ such that $z_n \overset{*}{\rightharpoonup} z$ in $L^\infty([0,T] : H^{\frac{\alpha}{2}}(\mathbb{R}))$, up to extracting a subsequence. By (2.5.1), we get

$$\|D^\alpha \partial_x z_n\|_{L^\infty_T H^{-\frac{\alpha}{2}-1}} \leqslant C,$$

and since $z_n^2 \in L^1(\mathbb{R}) \hookrightarrow H^{-\frac{1}{2}^-}(\mathbb{R})$, we have that

$$\|\partial_x(z_n^2)\|_{L^\infty_T H^{-\frac{3}{2}-}} \leqslant C\|z_n\|_{L^2}^2 \leqslant C.$$

Then, we obtain, by (2.1.1) that

$$\|\partial_t z_n\|_{L^\infty_T H^{\min(-\frac{3}{2}^-, -\frac{\alpha}{2}-1)}} \leqslant C. \tag{2.5.2}$$

Therefore, by the Aubin-Lions theorem (Theorem 1.71 in [158]), we deduce that $z_n \to z$ in $L^2([0,T] : L^2([-k,k])$, for all $k \in \mathbb{N}$. In particular, this implies $z_n^2 \to z^2$ in $L^1([0,T] : L^1([-k,k]))$.

Now, since z_n is a weak solution of (2.1.1) in the distributional sense satisfying $z_n(0,\cdot) = z_{0,n}$, we know that for all $\varphi \in C_c^\infty([-T,T] \times \mathbb{R})$,

$$\int_0^T \int_{\mathbb{R}} (\partial_t \varphi - \partial_x |D|^\alpha \varphi)\, z_n dx dt + \int_0^T \int_{\mathbb{R}} (\partial_x \varphi)\, z_n^2 dx dt - \int_{\mathbb{R}} \varphi(0,x) z_{0,n}(x) dx = 0.$$

Thus passing to the limit, we conclude that

$$\int_0^T \int_{\mathbb{R}} (\partial_t \varphi - \partial_x |D|^\alpha \varphi)\, z dx dt + \int_0^T \int_{\mathbb{R}} (\partial_x \varphi)\, z^2 dx dt - \int_{\mathbb{R}} \varphi(0,x) z_0(x) dx = 0,$$

which proves that z is a weak solution of (2.1.1) corresponding to the initial datum z_0.

Finally, let $\psi \in C_c^\infty(\mathbb{R})$. It follows from the Arzela-Ascoli theorem and the bounds (2.5.1)-(2.5.2) that the function $v_n : t \in [0,T] \mapsto \int_{\mathbb{R}} \psi(x) z_n(t,x) dx$ converges up to a subsequence in $C^0([-T,T] : \mathbb{R})$. Moreover, by uniqueness, this limit holds for the whole sequence and is equal to $\int_{\mathbb{R}} z(t,x)\psi(x)dx$, which implies that $z_n(t) \rightharpoonup z(t)$ in $H^{\frac{\alpha}{2}}(\mathbb{R})$ for all $t \in [0,T]$.

□

2.5.2 Pseudo-differential toolbox

First, we recall some well-known results on pseudo-differential operators (see [6], or [80] chapter 18). Let $D = -i\partial_x$. We define the symbolic class $\mathcal{S}^{m,q}$ by

$$a(x,\xi) \in \mathcal{S}^{m,q} \iff \begin{cases} a \in C^\infty(\mathbb{R}^2) \\ \forall k, \beta \in \mathbb{N}, \exists C_{k,\beta} > 0 \text{ such that } |\partial_x^k \partial_\xi^\beta a(x,\xi)| \leqslant C_{k,\beta} \langle x \rangle^{q-k} < \xi >^{m-\beta} \end{cases}$$

For all $u \in \mathcal{S}(\mathbb{R})$, we set the operator associated to the symbol $a(x,\xi) \in \mathcal{S}^{m,q}$ by

$$a(x,D)u := \frac{1}{2\pi} \int e^{ix\xi} a(x,\xi) \mathcal{F}(u)(\xi) d\xi.$$

We state the three following results

1. Let $a \in \mathcal{S}^{m,q}$, there exists $C > 0$, such that for all $u \in \mathcal{S}(\mathbb{R})$
$$\|a(x, D)u\|_{L^2} \leqslant C\|\langle x\rangle^q \langle D\rangle^m u\|_{L^2}. \tag{2.5.3}$$

2. Let $a \in \mathcal{S}^{m,q}$ and $b \in \mathcal{S}^{m',q'}$, then there exists $c \in \mathcal{S}^{m+m',q+q'}$ such that
$$a(x, D)b(x, D) = c(x, D). \tag{2.5.4}$$

3. If $a \in \mathcal{S}^{m,q}$ and $b \in \mathcal{S}^{m',q'}$ are two operators we define the commutator by $[a(xD), b(x, D)] := a(x, D)b(x, D) - b(x, D)a(x, D)$. Moreover there exists $c \in \mathcal{S}^{m+m'-1,q+q'-1}$ such that
$$[a(x, D), b(x, D)] = c(x, D). \tag{2.5.5}$$

As a consequence of (2.5.4), $\langle D\rangle^m \langle x\rangle^q \langle D\rangle^{-m} \in \mathcal{S}^{0,q}$. Therefore, by (2.5.3), we have
$$\begin{aligned}\|\langle D\rangle^m \langle x\rangle^q u\|_{L^2} &= \|\langle D\rangle^m \langle x\rangle^q \langle D\rangle^{-m} \langle D\rangle^m u\|_{L^2}\\ &\leqslant C_2\|\langle x\rangle^q \langle D\rangle^m u\|_{L^2},\end{aligned}$$
for $C_2 > 0$. By the same computations with $\langle x\rangle^q$ instead of $\langle D\rangle^m$, there exists $C_1 > 0$ such that
$$C_1\|\langle x\rangle^q \langle D\rangle^m u\|_{L^2} \leqslant \|\langle D\rangle^m \langle x\rangle^q u\|_{L^2}.$$
Gathering these two estimates, we conclude that
$$C_1\|\langle x\rangle^q \langle D\rangle^m u\|_{L^2} \leqslant \|\langle D\rangle^m \langle x\rangle^q u\|_{L^2} \leqslant C_2\|\langle x\rangle^q \langle D\rangle^m u\|_{L^2}. \tag{2.5.6}$$

2.5.3 Proof of the weighted commutator estimates

This section is devoted to the proofs of Lemmas 2.3.1, 2.3.3 and 2.3.4.

In this section u is a function in $H^{\frac{\alpha}{2}}(\mathbb{R})$. Let φ be defined as in (2.3.1) and for $A > 1$, $\varphi_{j,A}$ as in (2.3.2). Moreover, we define
$$\Phi(x) = \sqrt{|\varphi'(x)|} \sim \langle x\rangle^{-\frac{1+\alpha}{2}} \quad \text{and} \Phi_{j,A} = \sqrt{|\varphi'_{j,A}|}.$$

Finally, let $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi(\xi) = 1$ on $[-1, 1]$ and $\chi(\xi) = 0$ on $[-2, 2]^c$.

The proof of Lemma 2.3.1 is an extension of the proof of Lemmas 6 and 7 in [91]. Note that, while the estimates in Lemmas 6 and 7 in [91] are stated for $\alpha \in [1, 2]$, their proofs extend directly to the case $\alpha \in (0, 2)$. This yields the estimates (2.3.4) and (2.3.5). However, since the estimates (2.3.6) and (2.3.7) are not symmetric in u, we cannot use the Claim 3 in [91]. Instead, we use the following estimates (which are also derived from the techniques in [91]).

Lemma 2.5.1. *Let $\alpha \in (0, 2)$, $T = |D|^\alpha \varphi D - D\varphi|D|^\alpha$. Then, there exists $C > 0$ such that for all $u, v \in \mathcal{S}(\mathbb{R})$ we have*
$$i\int (Tu)v = (\alpha - 1)\int |D|^{\frac{\alpha}{2}}(u\Phi)\,|D|^{\frac{\alpha}{2}}(v\Phi) + R, \tag{2.5.7}$$
with
$$|R| \leqslant \begin{cases} C\displaystyle\int (u^2 + v^2)|\varphi'|, & \text{if } \alpha \in (0, 1],\\ C\left(\displaystyle\int u^2|\varphi'| + \frac{1}{A^{\frac{\alpha}{2}}}\int (|D|^{\frac{\alpha}{2}}u)^2|\varphi'| + A^{\frac{\alpha}{2}}\int v^2|\varphi'|\right), & \text{if } \alpha \in (1, 2),\end{cases} \tag{2.5.8}$$
for all $A > 1$.

Proof. The proof of (2.5.7) is a combination of the proofs of Claim 1 and Claim 2 in [91]. Following [91], we split $T = T_1 + T_2$, where

$$T_1 = |D|^\alpha (1-\chi(D))\varphi D - D\varphi(1-\chi(D))|D|^\alpha;$$
$$T_2 = |D|^\alpha \chi(D)\varphi D - D\varphi\chi(D)|D|^\alpha.$$

First, we have arguing exactly as in Claim 2 in [91] that, for any $\alpha \in (0,2)$,

$$i\int (T_2 u)v = (\alpha-1)\int |D|^{\frac{\alpha}{2}}\chi(D)\,(u\Phi)\,|D|^{\frac{\alpha}{2}}\,(v\Phi) + R_2,$$

with

$$|R_2| \leqslant C\int (u^2+v^2)|\varphi'|.$$

Now, we deal with the operator T_1. Let us define $a(x,\xi) = \varphi(x)|\xi|^\alpha(1-\chi(\xi)) \in \mathcal{S}^{\alpha,0}$. Then, following the computations in the proof of Claim 1 in [91], we have

$$i\int (T_1 u)v = (\alpha-1)\int |D|^{\frac{\alpha}{2}}(1-\chi(D))\,(u\Phi)\,|D|^{\frac{\alpha}{2}}\,(v\Phi) + \int (\widetilde{T_1}u)v + R_1,$$

with

$$\widetilde{T_1} = -\frac{i}{2}(\partial_x^2\partial_\xi^2 a)(x,D)D - \Phi[\Phi, |D|^\alpha(1-\chi(D))]$$
$$|R_1| \leqslant C\int (u^2+v^2)|\varphi'|.$$

To estimate $\left|\int (\widetilde{T_1}u)v\right|$, we cannot use Claim 3 in [91] due to the lack of symmetry. Instead, we use classical pseudo-differential calculus estimates. Observe that the symbol $t_1(x,\xi)$ of $\widetilde{T_1}$ belongs to the class $\mathcal{S}^{\alpha-1,-\alpha-2}$. In the case, $0<\alpha<1$, $t_1(x,\xi) \in \mathcal{S}^{0,-(\alpha+1)}$. Thus the Cauchy-Schwarz inequality and (2.5.3) yield

$$\left|\int (\widetilde{T_1}u)v\right| = \left|\int (\Phi^{-1}\widetilde{T_1}u)\Phi v\right| \leqslant \|\Phi^{-1}\widetilde{T_1}u\|_{L^2}\|\Phi v\|_{L^2} \leqslant C\int (u^2+v^2)|\varphi'|.$$

In the case $1<\alpha<2$, $t_1(x,\xi) \in \mathcal{S}^{\frac{\alpha}{2},-(1+\alpha)}$. By Cauchy-Schwarz' inequality, (2.5.3), and then Young's inequality, we get

$$\left|\int (\widetilde{T_1}u)v\right| \leqslant \|\Phi^{-1}\widetilde{T_1}u\|_{L^2}\|\Phi v\|_{L^2} \leqslant C\left(\frac{1}{A^{\frac{\alpha}{2}}}\int (\langle D\rangle^{\frac{\alpha}{2}}u)^2|\varphi'| + A^{\frac{\alpha}{2}}\int v^2|\varphi'|\right).$$

for any $A>1$. Moreover by pseudo-differential calculus (2.5.4), (2.5.3), and since the symbols of the operators $\Phi\chi(D)\langle D\rangle^{\frac{\alpha}{2}}\Phi^{-1}$ and $\Phi(1-\chi(D))\langle D\rangle^{\frac{\alpha}{2}}|D|^{-\frac{\alpha}{2}}\Phi^{-1}$ belong to $\mathcal{S}^{0,0}$,

$$\begin{aligned}\int \left(\langle D\rangle^{\frac{\alpha}{2}}u\right)^2|\varphi'| &\leqslant 2\int\left(\chi(D)\langle D\rangle^{\frac{\alpha}{2}}u\right)^2|\varphi'| + 2\int\left((1-\chi(D))\langle D\rangle^{\frac{\alpha}{2}}u\right)^2|\varphi'| \\ &\leqslant 2\int\left(\Phi\chi(D)\langle D\rangle^{\frac{\alpha}{2}}\Phi^{-1}\,(\Phi u)\right)^2 \\ &\quad + 2\int\left(\Phi(1-\chi(D))\langle D\rangle^{\frac{\alpha}{2}}|D|^{-\frac{\alpha}{2}}\Phi^{-1}\left(\Phi|D|^{\frac{\alpha}{2}}u\right)\right)^2 \\ &\leqslant C\left(\int u^2|\varphi'| + \int\left(|D|^{\frac{\alpha}{2}}u\right)^2|\varphi'|\right). \end{aligned} \tag{2.5.9}$$

Gathering these estimates concludes the proof of Lemma 2.5.1.

□

Lemma 2.5.2. *Let $\alpha \in (0,2)$, $S = \Phi[\Phi, |D|^\alpha]$. Then, there exists $C > 0$ such that for all $u, v \in \mathcal{S}(\mathbb{R})$ we have*

$$\left|\int (Su)v\right| + \left|\int (Sv)u\right| \leqslant \begin{cases} C\displaystyle\int (u^2+v^2)|\varphi'|, & \textit{if } \alpha \in (0,1], \\ C\left(\displaystyle\int u^2|\varphi'| + \frac{1}{A^{\frac{\alpha}{2}}}\int (|D|^{\frac{\alpha}{2}}u)^2|\varphi'| + A^{\frac{\alpha}{2}}\int v^2|\varphi'|\right), & \textit{if } \alpha \in (1,2), \end{cases} \tag{2.5.10}$$

for all $A > 1$.

Proof. We split $S = S_1 + S_2$, where

$$S_1 = \Phi[\Phi, |D|^\alpha(1-\chi(D))];$$
$$S_2 = \Phi[\Phi, |D|^\alpha\chi(D)].$$

We first deal with the high frequency terms involving S_1. Since $1-\chi$ is supported outside 0, S_1 is a pseudo-differential operator of symbol $s_1(x,\xi)$, which belongs to the class $\mathcal{S}^{\alpha-1,-\alpha-2}$. In the case, $0 < \alpha \leqslant 1$, $s_1(x,\xi) \in \mathcal{S}^{0,-(\alpha+1)}$ and in the case, $1 < \alpha < 2$, $s_1(x,\xi) \in \mathcal{S}^{\frac{\alpha}{2},-(\alpha+1)}$. Thus by arguing as in the proof of Lemma 2.5.1, we deduce that

$$\left|\int (S_1u)v\right| \leqslant \begin{cases} C\displaystyle\int (u^2+v^2)|\varphi'|, & \text{if } 0 < \alpha \leqslant 1, \\ C\left(\displaystyle\frac{1}{A^{\frac{\alpha}{2}}}\int (\langle|D|\rangle^{\frac{\alpha}{2}}u)^2|\varphi'| + A^{\frac{\alpha}{2}}\int v^2|\varphi'|\right), & \text{if } 1 < \alpha < 2. \end{cases}$$

Observe that the same estimate also holds for $\left|\int (S_1v)u\right|$. Indeed, the proof is exactly the same in the case $0 < \alpha \leqslant 1$. In the case $1 < \alpha < 2$, $\Phi^{-1}\langle|\xi|\rangle^{-\frac{\alpha}{2}}s_1(x,\xi) \in \mathcal{S}^{0,-\frac{\alpha+1}{2}}$, so that

$$\left|\int (S_1v)u\right| \leqslant \|\Phi^{-1}\langle|D|\rangle^{-\frac{\alpha}{2}}S_1v\|_{L^2}\|\Phi\langle|D|\rangle^{\frac{\alpha}{2}}u\|_{L^2} \leqslant \left(\frac{1}{A^{\frac{\alpha}{2}}}\int (\langle D\rangle^{\frac{\alpha}{2}}u)^2|\varphi'| + A^{\frac{\alpha}{2}}\int v^2|\varphi'|\right)$$

Moreover, by using (2.5.9), we deduce that

$$\left|\int (S_1u)v\right| + \left|\int (S_1v)u\right| \leqslant \begin{cases} C\displaystyle\int (u^2+v^2)|\varphi'|, & \text{if } \alpha \in (0,1], \\ C\left(\displaystyle\int u^2|\varphi'| + \frac{1}{A^{\frac{\alpha}{2}}}\int (|D|^{\frac{\alpha}{2}}u)^2|\varphi'| + A^{\frac{\alpha}{2}}\int v^2|\varphi'|\right), & \text{if } \alpha \in (1,2), \end{cases}$$

Now we deal with the low frequency term involving S_2. We follow the proof given in [91] for the same type of operator. We remark that $|D|^\alpha\chi(D)u = k * u$, with $\widehat{k} = |\xi|^\alpha\chi(\xi)$. Then, we can rewrite

$$[\Phi, |D|^\alpha\chi(D)]u = \int k(x-y)\left(\Phi(x)-\Phi(y)\right)u(y)dy.$$

We want to prove that the operator defined by the kernel

$$\Lambda(x,y) = k(x-y)\left(\Phi(x)-\Phi(y)\right)\Phi^{-1}(y),$$

is bounded in $L^2(\mathbb{R})$. For this, we need the 3 following results.

Theorem 2.5.3 (Schur's test [75], Theorem 5.2)**.** *Let p, q be two non-negative measurable functions. If there exists $\alpha, \beta > 0$ such that*

1. $\displaystyle\int_Y |K(x,y)|q(y)dy \leqslant \alpha p(x)$ *a.e* $x \in \mathbb{R}$.

2. $\displaystyle\int_X |K(x,y)|p(x)dx \leqslant \beta q(y)$ *a.e* $y \in \mathbb{R}$.

Then $Tf := \int_{\mathbb{R}} K(x,y)f(y)dy$ *is a bounded operator on* $L^2(\mathbb{R})$.

Claim 2.5.4 ([91] Claim 8). *There exists* $C > 0$ *such that*

$$|\Phi(x) - \Phi(y)| \leqslant C \frac{|x-y|}{(\langle x\rangle + \langle y\rangle)^{\frac{\alpha+3}{2}}} \text{ if } |x-y| \leqslant \frac{1}{2}(\langle x\rangle + \langle y\rangle),$$

$$|\Phi(x) - \Phi(y)| \leqslant \frac{1}{\langle x\rangle^{\frac{1+\alpha}{2}}} + \frac{1}{\langle y\rangle^{\frac{1+\alpha}{2}}} \text{ if } |x-y| \geqslant \frac{1}{2}(\langle x\rangle + \langle y\rangle).$$

Lemma 2.5.5 ([91], Lemma A.2). *Let* p *be a homogeneous function of degree* $\beta > -1$. *Let* $\chi \in C_0^\infty(\mathbb{R})$ *such that* $0 \leqslant \chi \leqslant 1$, $\chi(\xi) = 1$ *if* $|\xi| < 1$ *and* $\chi(\xi) = 0$ *if* $|\xi| > 2$. *Let*

$$k(x) = \frac{1}{2\pi}\int e^{ix\xi}p(\xi)\chi(\xi)d\xi.$$

Then for all $q \in \mathbb{N}$, *there exists* $C_q > 0$ *such that, for all* $x \in \mathbb{R}$,

$$|\partial_x^q k(x)| \leqslant \frac{C_q}{\langle x\rangle^{\beta+q+1}}.$$

Let $\Lambda = \Lambda_1 + \Lambda_2$, where Λ_1 and Λ_2 are restricted respectively to the regions $|x-y| \leqslant \frac{1}{2}(\langle x\rangle + \langle y\rangle)$ and $|x-y| \geqslant \frac{1}{2}(\langle x\rangle + \langle y\rangle)$. By using Claim 2.5.4 and Lemma 2.5.5,

$$\begin{aligned}|\Lambda_1(x,y)| &\leqslant C\frac{1}{\langle x-y\rangle^{1+\alpha}}\frac{|x-y|}{(\langle x\rangle + \langle y\rangle)^{\frac{\alpha+3}{2}}}\langle y\rangle^{\frac{1+\alpha}{2}} \\ &\leqslant C\frac{1}{\langle x-y\rangle^{1+\alpha}}.\end{aligned}$$

Then, by Theorem 2.5.3, with $p = q = 1$, Λ_1 is the kernel of a bounded operator in $L^2(\mathbb{R})$. Now, we deal with Λ_2. By using Claim 2.5.4 and Lemma 2.5.5,

$$\begin{aligned}|\Lambda_2(x,y)| &\leqslant C\frac{1}{\langle x-y\rangle^{1+\alpha}}\left(\frac{1}{\langle x\rangle^{\frac{1+\alpha}{2}}} + \frac{1}{\langle y\rangle^{\frac{1+\alpha}{2}}}\right)\langle y\rangle^{\frac{1+\alpha}{2}} \\ &\leqslant C\frac{1}{\langle x-y\rangle^{1+\alpha}} + C\frac{\langle y\rangle^{\frac{1+\alpha}{2}}}{\langle x-y\rangle^{1+\alpha}\langle x\rangle^{\frac{1+\alpha}{2}}} \\ &\leqslant \Lambda_3(x,y) + \Lambda_4(x,y).\end{aligned}$$

Then, by Theorem 2.5.3, with $p = q = 1$, Λ_3 is the kernel of a bounded operator in L^2. We compute

$$\int \Lambda_4(x,y)\langle x\rangle^{-\frac{1}{2}}dx \leqslant C\langle y\rangle^{-\frac{1+\alpha}{2}}, \quad \int \Lambda_4(x,y)\langle y\rangle^{-\frac{1+\alpha}{2}}dy \leqslant C\langle x\rangle^{-\frac{1}{2}}.$$

Then by Theorem 2.5.3, we deduce that Λ_4 is the kernel of a bounded operator in $L^2(\mathbb{R})$. Gathering these estimates, we conclude that

$$\|[\Phi, |D|^\alpha\chi(D)]u\|_{L^2} \leqslant C\|u\Phi\|_{L^2}.$$

Therefore, by Young's inequality, we get

$$\left|\int (S_2u)v\right| = \left|\int v\Phi[\Phi, |D|^\alpha]u\right| \leqslant C\left(\int v^2|\varphi'| + \int ([\Phi, |D|^\alpha]u)^2\right) \leqslant C\int \left(u^2+v^2\right)|\varphi'|.$$

The estimate for $\left|\int (S_2v)u\right|$ is similar. This concludes the proof of Lemma 2.5.2. □

Proof of (2.3.7). By integration by parts, we get

$$\int \left(\left(|D|^{\alpha}u\right)\partial_x v + \left(|D|^{\alpha}v\right)\partial_x u\right)\varphi = \int \left(|D|^{\alpha}(\varphi\partial_x u) - \partial_x(\varphi|D|^{\alpha}u)\right)v = i\int Tuv,$$

with $T = |D|^{\alpha}\varphi D - D\varphi|D|^{\alpha}$. Hence, we deduce from (2.5.7) that

$$\left|\int \left(\left(|D|^{\alpha}u\right)\partial_x v + \left(|D|^{\alpha}v\right)\partial_x u\right)\varphi - (\alpha-1)\int |D|^{\frac{\alpha}{2}}\left(u\sqrt{|\varphi'|}\right)|D|^{\frac{\alpha}{2}}\left(v\sqrt{|\varphi'|}\right)\right| = |R|,$$

where $|R|$ satisfies (2.5.8). Therefore, we conclude the proof of (2.3.7) by performing the change of variable $x' = \frac{x-m_j}{A}$. □

Proof of (2.3.6). By direct computation we get

$$\int \left(\left(|D|^{\alpha}u\right)v - \left(|D|^{\alpha}v\right)u\right)|\varphi'| = \int v\Phi[\Phi,|D|^{\alpha}]u - \int u\Phi[\Phi,|D|^{\alpha}]v = \int (Su)v - \int (Sv)u,$$

where $S = \Phi[\Phi,|D|^{\alpha}]$. Therefore, we conclude the proof of (2.3.6) by using Lemma 2.5.2 and performing the change of variable $x' = \frac{x-m_j}{A}$. □

This finishes the proof of Lemma 2.3.1. Now we prove Lemma 2.3.3.

Proof of Lemma 2.3.3. By direct computations, we obtain

$$\begin{aligned}\int \left(|D|^{\alpha}\left(u\Phi_{j,A}\right)\right)^2 - \int \left(|D|^{\alpha}u\right)^2|\varphi'_{j,A}| &= \int (|D|^{2\alpha}u)u|\varphi'_{j,A}| - \int \left(|D|^{\alpha}u\right)^2|\varphi'_{j,A}| \\ &\quad - \int u\Phi_{j,A}[\Phi_{j,A},|D|^{2\alpha}]u \\ &= \int (|D|^{2\alpha}u)u|\varphi'_{j,A}| - \int \left(|D|^{\alpha}u\right)^2|\varphi'_{j,A}| \\ &\quad - \frac{1}{2}\int u\left[\Phi_{j,A},\left[\Phi_{j,A},|D|^{2\alpha}\right]\right]u. \end{aligned} \tag{2.5.11}$$

By applying (2.3.6) to $v = |D|^{\alpha}u$, we get that

$$\begin{aligned}&\left|\int (|D|^{2\alpha}u)u|\varphi'_{j,A}| - \int \left(|D|^{\alpha}u\right)^2|\varphi'_{j,A}|\right| \\ &\quad\leqslant \begin{cases}\dfrac{C}{A^{\alpha}}\displaystyle\int \left(u^2 + (|D|^{\alpha}u)^2\right)|\varphi'_{j,A}|, & \text{if } \alpha\in(0,1], \\ \dfrac{C}{A^{\frac{\alpha}{2}}}\displaystyle\int \left(u^2 + \left(|D|^{\frac{\alpha}{2}}u\right)^2 + (|D|^{\alpha}u)^2\right)|\varphi'_{j,A}|, & \text{if } \alpha\in(1,2).\end{cases}\end{aligned} \tag{2.5.12}$$

It remains then to estimate the term (2.5.11). By using the proof of Claim 7 in [91], we know that for any $\alpha > 0$, there exists $C > 0$ such that for all $u,\ v \in \mathcal{S}(\mathbb{R})$,

$$\left|\int \left(\left[\Phi,[\Phi,\chi(D)|D|^{\alpha}]\right]u\right)v\right| \leqslant C\int (u^2+v^2)|\varphi'|. \tag{2.5.13}$$

We observe that the symbol of the pseudo-differential operator $\Phi^{-1}\left[\Phi,\left[\Phi,\chi(D)|D|^{2\alpha}\right]\right]$ belongs to $\mathcal{S}^{2\alpha-2,-\frac{1+\alpha}{2}-2} \subset \mathcal{S}^{\alpha,-\frac{1+\alpha}{2}}$, since $\alpha\in(0,2)$. Thus it follows from (2.5.3), (2.5.13) and Young's inequality that

$$\begin{aligned}\left|\int u\left[\Phi,\left[\Phi,|D|^{2\alpha}\right]\right]u\right| &\leqslant \left|\int u\Phi\Phi^{-1}\left[\Phi,\left[\Phi,(1-\chi(D))|D|^{2\alpha}\right]\right]u\right| \\ &\quad + \left|\int u\left[\Phi,\left[\Phi,\chi(D)|D|^{2\alpha}\right]\right]u\right| \\ &\leqslant \frac{C}{A^{\alpha}}\int \left(\langle D\rangle^{\alpha}u\right)^2|\varphi'| + CA^{\alpha}\int u^2|\varphi'|.\end{aligned}$$

Moreover, arguing as in (2.5.9), we obtain that

$$\int \left(\langle D\rangle^{\alpha} u\right)^2 |\varphi'| dx \leqslant C \left(\int u^2 |\varphi'| + \int \left(|D|^{\alpha} u\right)^2 |\varphi'| \right).$$

By changing variable $x' = \frac{x-m_j}{A}$, we deduce that

$$\left| \int u\Phi_{j,A} \left[\Phi_{j,A}, |D|^{2\alpha}\right] u \right| \leqslant \frac{C}{A^{\alpha}} \int \left(u^2 + \left(|D|^{\alpha} u\right)^2\right) |\varphi'_{j,A}| \tag{2.5.14}$$

Therefore, by gathering (2.5.12), (2.5.14), we conclude the proof of (2.3.8).

□

Proof of Lemma 2.3.4. By Young's inequality and (2.3.8), we obtain that

$$\begin{aligned} \left| \int |D|^{\alpha} (u\Phi_{j,A}) (|D|^{\alpha} u) \Phi_{j,A} \right| &\leqslant \frac{1}{2} \left(\int \left(|D|^{\alpha} (u\Phi_{j,A})\right)^2 + \int \left(|D|^{\alpha} u\right)^2 |\varphi'_{j,A}| \right) \\ &\leqslant \int \left(|D|^{\alpha} u\right)^2 |\varphi'_{j,A}| + \frac{1}{2} \left| \int \left(|D|^{\alpha} (u\Phi_{j,A})\right)^2 - \int \left(|D|^{\alpha} u\right)^2 |\varphi'_{j,A}| \right| \\ &\leqslant \int \left(|D|^{\alpha} u\right)^2 |\varphi'_{j,A}| + \frac{C}{A^{\frac{\alpha}{2}}} \left(\int \left(u^2 + \left(|D|^{\frac{\alpha}{2}} u\right)^2 + \left(|D|^{\alpha} u\right)^2\right) |\varphi'_{j,A}| \right), \end{aligned}$$

which yields (4.4.28).

□

2.5.4 Proof of the non-linear weighted estimates

Proof of Lemma 2.3.6. Let $j \in \{1, ..., N\}$. First we prove estimate (2.3.16). By the Cauchy-Schwarz inequality, the Sobolev embedding $\dot{H}^{\frac{1}{4}}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$ and $\frac{1}{4} < \frac{\alpha}{2}$, we get that

$$\int |\eta|^3 |\varphi'_{j,A}| \leqslant \left(\int \eta^2\right)^{\frac{1}{2}} \left(\int \eta^4 |\varphi'_{j,A}|^2\right)^{\frac{1}{2}} \leqslant C\gamma \| |D|^{\frac{1}{4}} (\eta\Phi_{j,A}) \|_{L^2}^2 \leqslant C\gamma \|\eta\Phi_{j,A}\|_{H^{\frac{\alpha}{2}}}^2. \tag{2.5.15}$$

From the decomposition $\eta = u - R$, we have

$$\|\eta\Phi_{j,A}\|_{H^{\frac{\alpha}{2}}} \leqslant \|u\Phi_{j,A}\|_{H^{\frac{\alpha}{2}}} + \|R\Phi_{j,A}\|_{H^{\frac{\alpha}{2}}} \leqslant \|u\Phi_{j,A}\|_{H^{\frac{\alpha}{2}}} + \|R\Phi_{j,A}\|_{H^{\alpha}}. \tag{2.5.16}$$

To deal with the second term on the right-hand side of (2.5.16), we use pseudo-differential calculus. Observe that the symbols of $\Phi\chi(D)\langle D\rangle^{\alpha}\Phi^{-1}$ and $\Phi(1-\chi(D))\langle D\rangle^{\alpha}|D|^{-\alpha}\Phi^{-1}$ belong to $\mathcal{S}^{0,0}$. It follows then from (2.5.6), and then (2.5.4) that, for all $v \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} \| |D|^{\alpha} (v\Phi) \|_{L^2} &\leqslant \|v\Phi\|_{H^{\alpha}} \\ &\leqslant C \| \left(\langle D\rangle^{\alpha} v\right) \Phi \|_{L^2} \\ &\leqslant C \|\chi(D) \left(\langle D\rangle^{\alpha} v\right) \Phi\|_{L^2} + \|(1-\chi(D))\langle D\rangle^{\alpha} |D|^{-\alpha} \left(|D|^{\alpha} v\right) \Phi\|_{L^2} \\ &\leqslant C \left(\|v\Phi\|_{L^2} + \| \left(|D|^{\alpha} v\right) \Phi\|_{L^2}\right). \end{aligned} \tag{2.5.17}$$

Then, we obtain, by changing variable $x' = \frac{x-m_j}{A}$,

$$\| |D|^{\alpha} (v\Phi_{j,A}) \|_{L^2} \leqslant C \left(\frac{1}{A^{\alpha}} \|v\Phi_{j,A}\|_{L^2} + \| \left(|D|^{\alpha} v\right) \Phi_{j,A}\|_{L^2} \right),$$

so that

$$\|v\Phi_{j,A}\|_{H^{\alpha}} \leqslant C \left(\|v\Phi_{j,A}\|_{L^2} + \| \left(|D|^{\alpha} v\right) \Phi_{j,A}\|_{L^2}\right), \tag{2.5.18}$$

for all $v \in \mathcal{S}(\mathbb{R})$.

Therefore, we deduce combining (2.5.15), (2.5.16) and (2.5.18) with $v = R$ that

$$\int |\eta|^3 |\varphi'_{j,A}| \leqslant C\gamma \|u\Phi_{j,A}\|^2_{H^{\frac{\alpha}{2}}} + C\gamma \int \left(R^2 + (|D|^\alpha R)^2\right) |\varphi'_{j,A}|.$$

Moreover, by using the equation (2.1.3) and (2.3.12), we obtain

$$\int (|D|^\alpha R)^2 |\varphi'_{j,A}| \leqslant C \left(\int R^2 + R^4\right) |\varphi'_{j,A}| \leqslant \frac{C}{(\beta t)^{1+\alpha}},$$

which concludes the proof of (2.3.16).

Now we prove (2.3.17). Using the Cauchy-Schwarz inequality, the Sobolev embedding and the former estimates, we conclude that

$$\int \eta^4 |\varphi'_{j,A}| \leqslant \left(\int \eta^4\right)^{\frac{1}{2}} \left(\int \eta^4 |\varphi'_{j,A}|^2\right)^{\frac{1}{2}} \leqslant C\gamma^2 \left[\int u^2 |\varphi'_{j,A}| + \int \left(|D|^{\frac{\alpha}{2}} (u\Phi_{j,A})\right)^2\right] + \frac{C\gamma^2}{(\beta t)^{1+\alpha}},$$

which is exactly estimate (2.3.17). □

Proof of Lemma 2.3.7. By using Young's inequality and the decomposition $u = R + \eta$, we deduce that

$$\begin{aligned}\left|\int |D|^{\frac{\alpha}{2}} (u\Phi_{j,A})\, |D|^{\frac{\alpha}{2}} \left(u^2\Phi_{j,A}\right)\right| =& \left|\int |D|^\alpha (u\Phi_{j,A})\, u^2 \Phi_{j,A}\right| \\ \leqslant& 2\int u^4 |\varphi'_{j,A}| + \frac{1}{8}\int (|D|^\alpha (u\Phi_{j,A}))^2 \\ \leqslant& C\left(\int \eta^4 |\varphi'_{j,A}| + \int R^4 |\varphi'_{j,A}|\right) + \frac{1}{8}\int (|D|^\alpha (u\Phi_{j,A}))^2.\end{aligned}$$

Using (2.3.17), (2.3.12), (2.3.8), we have that

$$\begin{aligned}\left|\int |D|^{\frac{\alpha}{2}} (u\Phi_{j,A})\, |D|^{\frac{\alpha}{2}} \left(u^2\Phi_{j,A}\right)\right| \leqslant& C\left(\gamma^2 + \frac{1}{A^{\frac{\alpha}{2}}}\right) \left(\int \left(u^2 + \left(|D|^{\frac{\alpha}{2}} u\right)^2 + (|D|^\alpha u)^2\right) |\varphi'_{j,A}|\right) \\ &+ C\gamma^2 \int \left(|D|^{\frac{\alpha}{2}} (u\Phi_{j,A})\right)^2 + \frac{1}{8}\int (|D|^\alpha u)^2 |\varphi'_{j,A}| \\ &+ \frac{C}{(\beta t)^{1+\alpha}}.\end{aligned}$$

Furthermore, by using again (2.3.8) with $\frac{\alpha}{2} < 1$, we deduce that

$$\begin{aligned}\left|\int |D|^{\frac{\alpha}{2}} (u\Phi_{j,A})\, |D|^{\frac{\alpha}{2}} \left(u^2\Phi_{j,A}\right)\right| \leqslant& C\left(\gamma^2 + \frac{1}{A^{\frac{\alpha}{2}}}\right) \left(\int \left(u^2 + \left(|D|^{\frac{\alpha}{2}} u\right)^2 + (|D|^\alpha u)^2\right) |\varphi'_{j,A}|\right) \\ &+ \frac{1}{8}\int (|D|^\alpha u)^2 |\varphi'_{j,A}| + \frac{C}{(\beta t)^{1+\alpha}}.\end{aligned}$$

This concludes the proof of Lemma 2.3.7. □

2.5.5 Coercivity of the localized operator

To begin, we recall the definition of H_j given in (2.4.12)

$$H_j(u,u) = \int \left(u|D|^\alpha u + c_j u^2 - 2R_j u^2\right) \psi_{j,A},$$

where R_j is defined in (2.2.5), $\psi_{j,A}$ is defined in (2.3.3) and $u \in H^{\frac{\alpha}{2}}(\mathbb{R})$. Moreover, let

$$L_j u = |D|^\alpha u + c_j u - 2R_j u \quad \text{and} \quad Lu = |D|^\alpha u + u - 2Qu.$$

It has been proved in Theorem 2.3 in [60] that the spectrum of L is composed by one negative eigenvalue, the eigenvalue 0 and that the rest is the continuous spectrum $[1,+\infty)$. Moreover, the eigenspaces associated with the negative eigenvalue and 0 are one-dimensional vector spaces and the eigenspace of 0 is spanned by Q'.

Furthermore, from Lemma E.1 in [187], since we are in the subcritical case, we can replace the eigenfunction associated with the negative eigenvalue by Q to get the coercivity property stated in the following theorem.

Theorem 2.5.6. *Let $\alpha \in \left(\frac{1}{2},2\right)$. Then, there exists $\mu>0$ such that for all $u \in H^{\frac{\alpha}{2}}(\mathbb{R})$*

$$\int uLu \geqslant \mu\|u\|_{H^{\frac{\alpha}{2}}} - \frac{1}{\mu}\left(\int uQ\right)^2 - \frac{1}{\mu}\left(\int uQ'\right)^2$$

Remark 2.5.7. By using a scaling argument, the result of Theorem 2.5.6 still holds if one replaces L by L_j and Q by R_j, for $j \in \{1,\cdots,N\}$.

As a consequence of the former theorem, we deduce a coercivity property for the bilinear form H_j.

Corollary 2.5.8. *There exist $\nu>0$, $C>0$ such that for all $A>1$, $u \in H^{\frac{\alpha}{2}}(\mathbb{R})$*

$$\sum_{j=1}^N H_j(u,u) \geqslant \left(\nu - \frac{C}{(\beta t)^\alpha} - \frac{C}{A^{\frac{\alpha}{2}}}\right)\|u\|_{H^{\frac{\alpha}{2}}}^2 - \frac{1}{\nu}\sum_{j=1}^N\left(\left(\int uR_j\right)^2 + \left(\int u\partial_x R_j\right)^2\right).$$

Proof of Corollary 2.5.8. For all $j \in \{1,\cdots,N\}$, we have from Theorem 2.5.6 that

$$\begin{aligned}
H_j(u,u) =& \int u\sqrt{\psi_{j,A}}L(u\sqrt{\psi_{j,A}}) + \int u\left(|D|^\alpha u\right)\psi_{j,A} - \int\left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{\psi_{j,A}}\right)\right)^2 \\
\geqslant& \mu\|u\sqrt{\psi_{j,A}}\|_{H^{\frac{\alpha}{2}}}^2 - \frac{1}{\mu}\left(\int u\sqrt{\psi_{j,A}}R_j\right)^2 - \frac{1}{\mu}\left(\int u\sqrt{\psi_{j,A}}\partial_x R_j\right)^2 \\
&+ \int u\left(|D|^\alpha u\right)\psi_{j,A} - \int\left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{\psi_{j,A}}\right)\right)^2.
\end{aligned}$$

By (2.3.14), (2.3.15), and the Cauchy-Schwarz inequality we deduce that for all $j \in \{1,\cdots,N\}$

$$\begin{aligned}
\left(\int u\sqrt{\psi_{j,A}}R_j\right)^2 + \left(\int u\sqrt{\psi_{j,A}}\partial_x R_j\right)^2 \leqslant& 2\left(\int uR_j\right)^2 + 2\left(\int u\partial_x R_j\right)^2 \\
&+ 2\left(\int u(1-\sqrt{\psi_{j,A}})R_j\right)^2 + 2\left(\int u(1-\sqrt{\psi_{j,A}})\partial_x R_j\right)^2 \\
\leqslant& 2\left(\int uR_j\right)^2 + 2\left(\int u\partial_x R_j\right)^2 + \frac{C\|u\|_{L^2}^2}{(\beta t)^\alpha}.
\end{aligned}$$

Observe from $\langle D\rangle^{\frac{\alpha}{2}} \sim 1+|D|^{\frac{\alpha}{2}}$ that

$$\|u\sqrt{\psi_{j,A}}\|_{H^{\frac{\alpha}{2}}}^2 \geqslant c\int (u^2 + (|D|^{\frac{\alpha}{2}}u)^2)\psi_{j,A} - c\int (|D|^{\frac{\alpha}{2}}u)^2\psi_{j,A} + c\int\left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{\psi_{j,A}}\right)\right)^2,$$

for a small positive constant $0<c<1$. Since $\sum_{j=1}^N \psi_{j,A} = 1$, we have

$$\sum_{j=1}^N \int u\left(|D|^\alpha u\right)\psi_{j,A} = \int\left(|D|^{\frac{\alpha}{2}}u\right)^2 = \sum_{j=1}^N\int\left(\left(|D|^{\frac{\alpha}{2}}u\right)\sqrt{\psi_{j,A}}\right)^2. \qquad (2.5.19)$$

Hence, we deduce by summing over j that

$$\sum_{j=1}^{N} H_j(u,u) \geqslant N\left(c\mu - \frac{C}{(\beta t)^\alpha}\right)\|u\|^2_{H^{\frac{\alpha}{2}}} - \frac{2}{\mu}\sum_{j=1}^{N}\left(\left(\int uR_j\right)^2 + \left(\int u\partial_x R_j\right)^2\right)$$
$$+ (1-c\mu)\sum_{j=1}^{N}\left(\int u\left(|D|^\alpha u\right)\psi_{j,A} - \int\left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{\psi_{j,A}}\right)\right)^2\right).$$

It remains to estimate the last term on the right hand side of the former inequality. By using (2.5.19) and direct computations,

$$\sum_{j=1}^{N}\left(\int u\left(|D|^\alpha u\right)\psi_{j,A} - \int\left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{\psi_{j,A}}\right)\right)^2\right)$$
$$= \sum_{j=1}^{N}\int\left(\left(|D|^{\frac{\alpha}{2}}u\right)\sqrt{\psi_{j,A}} + |D|^{\frac{\alpha}{2}}\left(u\sqrt{\psi_{j,A}}\right)\right)\left(\left(|D|^{\frac{\alpha}{2}}u\right)\sqrt{\psi_{j,A}} - |D|^{\frac{\alpha}{2}}\left(u\sqrt{\psi_{j,A}}\right)\right)$$
$$= -\sum_{j=1}^{N}\int\left(\left(|D|^{\frac{\alpha}{2}}u\right)\sqrt{\psi_{j,A}} + |D|^{\frac{\alpha}{2}}\left(u\sqrt{\psi_{j,A}}\right)\right)[|D|^{\frac{\alpha}{2}}, \sqrt{\psi_{j,A}}]u.$$

By arguing as in (2.5.17) and using $0 \leqslant \psi_{j,A} \leqslant 1$, we deduce that

$$\sum_{j=1}^{N}\||D|^{\frac{\alpha}{2}}(u\sqrt{\psi_{j,A}})\|_{L^2} \leqslant C\sum_{j=1}^{N}\left(\|u\sqrt{\psi_{j,A}}\|_{L^2} + \|\left(|D|^{\frac{\alpha}{2}}u\right)\sqrt{\psi_{j,A}}\|_{L^2}\right) \leqslant C\|u\|_{H^{\frac{\alpha}{2}}}. \quad (2.5.20)$$

Then, it follows from the Cauchy-Schwarz inequality and (2.5.20) that

$$\sum_{j=1}^{N}\left|\int u\left(|D|^\alpha u\right)\psi_{j,A} - \int\left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{\psi_{j,A}}\right)\right)^2\right| \leqslant C\|u\|_{H^{\frac{\alpha}{2}}}\sum_{j=1}^{N}\|[|D|^{\frac{\alpha}{2}}, \sqrt{\psi_{j,A}}]u\|_{L^2}$$

Finally to estimate the commutator on the right-hand side of the former estimate, we will rely on pseudo-differential calculus and argue as in the previous subsection. By (2.5.5), we have that the symbol of $[|D|^{\frac{\alpha}{2}}(1-\chi(D)), \sqrt{\psi}]$ belongs to $\mathcal{S}^{\frac{\alpha}{2}-1,-1} \subset \mathcal{S}^{0,0}$, since $\alpha < 2$. Then, it follows from (2.5.3) that

$$\|[|D|^{\frac{\alpha}{2}}, \sqrt{\psi}]u\|_{L^2} \leqslant C\left(\|u\|_{L^2} + \|[|D|^{\frac{\alpha}{2}}\chi(D), \sqrt{\psi}]u\|_{L^2}\right).$$

We recall $|D|^{\frac{\alpha}{2}}\chi(D)u = k * u$, with $\widehat{k} = |\xi|^{\frac{\alpha}{2}}\chi(\xi)$, so that

$$[\chi(D)|D|^{\frac{\alpha}{2}}, \sqrt{\psi}]u = \int k(x-y)\left(\sqrt{\psi}(x) - \sqrt{\psi}(y)\right)u(y)dy.$$

We want to prove that the operator T defined by the kernel

$$R(x,y) = k(x-y)\left(\sqrt{\psi}(x) - \sqrt{\psi}(y)\right),$$

is bounded in $L^2(\mathbb{R})$. By Lemma 2.5.5, we obtain that $|k(x)| \leqslant \dfrac{C}{\langle x\rangle^{1+\frac{\alpha}{2}}}$. Since $\sqrt{\psi} \in L^\infty(\mathbb{R})$, we deduce by Lemma 2.5.3 that

$$\|[|D|^{\frac{\alpha}{2}}\chi(D), \sqrt{\psi}]u\|_{L^2} \leqslant C\|u\|_{L^2}.$$

By changing the variable $x' = \frac{x-m_j}{A}$, we get that

$$\|[|D|^{\frac{\alpha}{2}}, \sqrt{\psi_{j,A}}]u\|_{L^2} \leqslant \frac{C}{A^{\frac{\alpha}{2}}}\|u\|_{L^2}.$$

We finish the proof of Corollary 2.5.8 by combining all these estimates and by choosing ν small enough. $\square$

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Chapter 3

Asymptotic of non-linear ground states for fractional Laplacian

3.1 Introduction

3.1.1 Motivation

Consider the class of elliptic equations:

$$|D|^\alpha u + u - f(u) = 0, \quad u : \mathbb{R} \to \mathbb{R}, \quad 0 < \alpha < 2, \tag{3.1.1}$$

with the notation $|D|^\alpha$ standing for the Riesz potential of order $-\alpha$ given by the Fourier multiplier:

$$\mathcal{F}(|D|^\alpha u) := |\xi|^\alpha \mathcal{F}(u),$$

and with a non-linearity $f(u) = |u|^{p-1}u$ or $f(u) = u^p$, p integer, and $1 < p < p^*(\alpha)$ where:

$$p^*(\alpha) := \begin{cases} \dfrac{2\alpha}{1-\alpha} + 1 & \text{if } 0 < \alpha \leqslant 1, \\ +\infty & \text{if } 1 \leqslant \alpha < 2. \end{cases}$$

$p^*(\alpha)$ is defined such that the "equation is $H^{\frac{\alpha}{2}}(\mathbb{R})$-subcritical", where $H^s(\mathbb{R})$ stands for the Sobolev spaces.

Those fractional elliptic equations appear naturally when studying solitary waves of the following equations:

- the fractional generalized Korteweg-de Vries equation [171, 174]:

$$\partial_t u + \partial_x \left(-|D|^\alpha u + f(u)\right) = 0, \tag{fgKdV}$$

- the fractional generalized Benjamin-Bona-Mahony equation [48, 49, 161]:

$$\partial_t u + \partial_x u + \partial_x(f(u)) - |D|^\alpha \partial_t u - |D|^\alpha \partial_x u = 0 \tag{fgBBM}$$

- the fractional generalized nonlinear Schrödinger equation [112]:

$$i\partial_t u + |D|^\alpha u + f(u) = 0. \tag{fgNLS}$$

The particular case of $f(u) = u^p$ with an integer p is particularly relevant for (fgKdV) and (fgBBM), whereas the non-linearity $f(u) = |u|^{p-1}u$ naturally appears when studying the Schrödinger type equations.

A solitary wave for (fgKdV) is a wave moving in one direction with a constant velocity c, keeping its form along the time and decaying at infinity, and can thus be written as $u(t,x) = Q_c(x - ct)$. In the case $c = 1$, the function Q_1 has to be a solution of (3.1.1).

Up to adequate change of variables, the solitary waves of (fgBBM) satisfy the same equation. The counterparts of (fgNLS) are stationary waves of the form $e^{i\omega t}Q_\omega(x)$, where the solution Q_1 with phase $\omega = 1$ also satisfies (3.1.1).

Having a deeper understanding of the solutions of the fractional elliptic equation (3.1.1) is necessary to get more insights on the behaviour of solitary waves. It is of great interest to get more properties of those solutions, such as the existence and uniqueness of solutions, their regularity, the number of zeros or the asymptotic behaviour. The aim of this article to give the asymptotic development of the solutions to that fractional elliptic equations and some subsequent properties.

3.1.2 Survey of properties of the solutions.

In the two specific cases $\alpha = 2$ for general p, and $\alpha = 1$ with $f(u) = u^2$, the solutions of the elliptic equation (3.1.1) have been widely studied during the last fifty years. The main questions on those solutions are their existence, uniqueness, and some intrinsic properties.

Let us give a brief review of the previous results in the case $\alpha = 2$, for algebraic non-linearities $p \in (1, p^*(\alpha))$. We consider the set of solutions $Q : \mathbb{R}^d \to \mathbb{R}$ with $|Q(x)| \underset{x \longrightarrow +\infty}{\to} 0$. For the dimension $d = 1$, there exists a unique solution $Q(x) = \left(\frac{(p+1)}{2} \cosh^{-2}\left(\frac{p-1}{2}\right)\right)^{\frac{1}{p-1}}$ up to translations of the origin (see Theorem 5 of Berestycki-Lions [17]). For higher dimensions $d \geqslant 2$, there exists a unique positive solution (see Weinstein [189] and Kwong [109]), but also an infinite number of non-positive solutions as shown in Strauss [177], [17] and Musso-Pacard-Wei [152]. All the previous solutions have an exponential decay.

Concerning the equation associated with the Benjamin-Ono equation (BO), corresponding to $\alpha = 1$, $f(Q) = Q^2$ and $d = 1$, a solution is explicit $Q(x) = 4(1 + x^2)^{-1}$, see Benjamin [14]. In fact, any solution to this problem is equal, up to translation, to this solution, as proved in Amick-Toland [7] relying on former ideas of Benjamin [14] (see also Albert [3] for an alternative proof). Notice that the solution is even, positive and has a polynomial decay at infinity.

In the generic case $\alpha \in (0, 2)$, the existence of solutions relies on the existence of a minimizer for the functional J^α defined by:

$$J^\alpha(u) = \frac{\left(\displaystyle\int ||D|^{\frac{\alpha}{2}} u|^2\right)^{\frac{p+1}{2\alpha}} \left(\displaystyle\int |u|^2\right)^{\frac{(p+1)}{2\alpha}(\alpha-1)+1}}{\displaystyle\int |u|^{p+1}}. \tag{3.1.2}$$

A minimizer Q of the functional is called a ground state. The existence of a minimizer has been obtained by Weinstein [189] and Albert-Bona-Saut [4]. The structure of the set of solutions of (3.1.1) is complex, and it is not easy to know which elements compose this set. For example, the question of existence and uniqueness of solutions of (3.1.1) not minimizing the functional J^α remains open. However, a breakthrough was achieved by Frank-Lenzmann [60] by proving the uniqueness of ground states. Their proof relies on the non degeneracy of the linearized operator $L = |D|^\alpha + 1 - pQ^{p-1}$, in other words ker$(L)$ =span(Q'). The understanding of the kernel of L is based on a result by Caffarelli-Silvestre [29] to express $|D|^\alpha$ as a Dirichlet-to-Neumann operator for a local problem on the upper half-plane. Furthermore in higher dimensions, Felmer-Quaas-Tan in [54] derived the existence and some properties of the ground states and Frank-Lenzmann-Silvestre in [61] extended the uniqueness result.

In the following theorem, we summarize the well-known properties of the ground states.

Theorem 3.1.1 ([4, 60, 61, 189]). *Let $\alpha \in (0, 2)$ and $p \in (1, p^*(\alpha))$. There exists $Q \in H^s(\mathbb{R})$, for all $s \geqslant 0$, such that*

1. *(Existence) The function Q solves* (3.1.1) *and $Q = Q(|x|) > 0$ is even, positive and strictly decreasing in $|x|$. Moreover, the function Q is a minimizer of J^α in the sense that:*

$$J^\alpha(Q) = \inf_{u \in H^{\frac{\alpha}{2}}(\mathbb{R})} J^\alpha(u).$$

2. *(Uniqueness) The even ground state solution $Q = Q(|x|) > 0$ of* (3.1.1) *is unique. Furthermore, every optimizer $v \in H^{\frac{\alpha}{2}}(\mathbb{R})$ for the Gagliardo-Nirenberg problem* (3.1.2) *is of the form $v = \beta Q(\gamma(\cdot + y))$ with some $\beta \in \mathbb{R}, \beta \neq 0, \gamma > 0$ and $y \in \mathbb{R}$.*

3. *(Decay) The function Q verifies the following decay estimate:*

$$\frac{C_1}{\langle x \rangle^{\alpha+1}} \leqslant Q(x) \leqslant \frac{C_2}{\langle x \rangle^{\alpha+1}},$$

for some $C_1, C_2 > 0$.

As explained above, it is not clear whether other solutions of (3.1.1) which are not minimizers of (2.1.4) exist or not. The critical points of the functional J^α are called bound states and solve (3.1.1). In dimension 1 for other elliptic equations, we expect the ground state to be the unique solution vanishing at infinity. For some elliptic equations in higher dimensions, other bound states different from the ground state exist: they are called excited states. For non-local equations such as (3.1.1), except for $\alpha = 1$ and $f(u) = u^2$, the existence of a unique bound state in dimension 1 and of multiple excited states in higher dimensions is still an open problem. In this context, we propose in this article to sharpen the asymptotic behaviour of the ground-state and to address the issue of the asymptotic behaviour of any solution of (3.1.1) at $+\infty$.

3.1.3 Main results

In this paper, we give several results on the asymptotic expansion of a solution Q of (3.1.1) and of its derivatives, and extend this development in the case of a non-linearity $f(u) = u^3$. These results extend the ones of Cappiello-Gramchev-Rodino [30], where they proved that the solutions of a wider class of equations remain in some algebraic weighted spaces.

All the propositions stated in this article are given for $x > 1$. However the proofs can be adapted to get the asymptotic developments at $-\infty$.

To do so, we define the following explicit constants:

$$k_1 := \frac{\sin\left(\frac{\pi}{2}\alpha\right)}{\pi}\Gamma\left(\alpha+1\right), \quad k_2 := -\frac{\sin\left(\pi\alpha\right)}{\pi}\Gamma\left(2\alpha+1\right) \tag{3.1.3}$$

with Γ the Euler Γ-function, and fir a given function Q:

$$a_1 := k_1\int |Q|^{p-1}Q(x)dx, \quad a_2 := k_2\int |Q|^{p-1}Q(x)dx, \quad a_3 := \frac{(\alpha+1)(\alpha+2)}{2}k_1\int x^2|Q|^{p-1}Q(x)dx. \tag{3.1.4}$$

We recall the definition of a weak solution of (3.1.1):

Definition 3.1.2. *The function u is called a weak solution of* (3.1.1) *if for all φ in the Schwartz space $\mathcal{S}(\mathbb{R})$, we have that:*

$$\int_{\mathbb{R}} \left(u(x)|D|^\alpha\varphi(x) + u(x)\varphi(x) - f(u)(x)\varphi(x)\right)dx = 0.$$

The next proposition states the main order terms of the development of the derivatives.

Proposition 3.1.3. *Let $\alpha \in (0,2)$, $p \in (1, p^*(\alpha))$ and Q be a weak solution of $|D|^\alpha Q + Q - |Q|^{p-1}Q = 0$, satisfying:*

$$Q \in L^p(\mathbb{R}) \quad \textit{and} \quad \exists l > 0, \quad |x|^l Q(x) \in L^\infty(\mathbb{R}). \tag{3.1.5}$$

Then, $Q \in C^0(\mathbb{R})$ and verifies:

$$Q(x) - \frac{a_1}{x^{\alpha+1}} = o_{+\infty}\left(\frac{1}{x^{\alpha+1}}\right),$$

with a_1 dependent on Q and defined in (3.1.4).

Futhermore, if $\alpha > 1$, then $Q \in C^{\lfloor p\rfloor+1}(\mathbb{R})$ with $\lfloor p\rfloor$ the floor function of p, and verifies for $j \leqslant \lfloor p\rfloor$:

$$Q^{(j)}(x) - (-1)^j\frac{(\alpha+j)!}{\alpha!}\frac{a_1}{x^{\alpha+1+j}} = o_{+\infty}\left(\frac{1}{x^{\alpha+1+j}}\right).$$

Remark 3.1.4. Some comments on the previous result are in order.

1. Bona-Li in [24] studied the decay of solutions of elliptic equations similar to (3.1.1), with a set of assumptions different from (3.1.5). In the context of (3.1.1), if we ask for $\alpha > \frac{3}{2}$, $Q \in L^\infty(\mathbb{R})$ and vanishing at infinity, Theorem 3.1.2 of [24] implies the condition (3.1.5). Thus condition (3.1.5) is coherent with their article.

2. Note that if Q is not positive the coefficient a_1 is potentially null.

3. If $\alpha > 1$, our method does not give an asymptotic expansion of $Q^{(\lfloor p \rfloor+1)}$. Moreover, we do not know if $Q^{(\lfloor p \rfloor+2)}$ exists.

4. If the coefficient of the non-linearity p is an integer then $Q \in H^\infty(\mathbb{R})$. The proof is given in Appendix 3.6.1.

In the next proposition, we assume the function Q to be positive, so that Theorem 3.1.1 applies. We recall some important results of this theorem and give new asymptotic behaviours.

Proposition 3.1.5. *Let $\alpha \in (0,2)$. Let Q satisfying the assumptions* (3.1.5) *of Proposition 3.1.3. Suppose also that Q is positive. Then $Q \in H^\infty(\mathbb{R})$, even (up to translation), decaying and verifies that:*

$$Q^{(j)}(x) - (-1)^j \frac{(\alpha+j)!}{\alpha!} \frac{a_1}{x^{\alpha+1+j}} = o_{+\infty}\left(\frac{1}{x^{\alpha+1+j}}\right), \quad \forall j \in \mathbb{N},$$

and the next order asymptotic expansion holds, with a positive constant $\widetilde{a}_1$:

$$\begin{aligned}
&\textit{Case } p < \frac{2\alpha+1}{\alpha+1}: \quad Q(x) - \frac{a_1}{x^{\alpha+1}} - \frac{\widetilde{a}_1}{x^{p(\alpha+1)}} = o_{+\infty}\left(\frac{1}{x^{p(\alpha+1)}}\right).\\
&\textit{Case } p = \frac{2\alpha+1}{\alpha+1}: \quad Q(x) - \frac{a_1}{x^{\alpha+1}} - \frac{\widetilde{a}_1}{x^{2\alpha+1}} - \frac{a_2}{x^{2\alpha+1}} = o_{+\infty}\left(\frac{1}{x^{2\alpha+1}}\right).\\
&\textit{Case } p > \frac{2\alpha+1}{\alpha+1}: \quad Q(x) - \frac{a_1}{x^{\alpha+1}} - \frac{a_2}{x^{2\alpha+1}} = o_{+\infty}\left(\frac{1}{x^{2\alpha+1}}\right).
\end{aligned}$$

Remark 3.1.6. Let us notice that the constants involved in the asymptotic expansion are coherent with other situations. In the case $\alpha = 1$ and $f(u) = u^2$, thus for (BO), only the terms with an even power are present, since $a_2 = 0$, but $a_1 \neq 0$. On the other hand, the case $\alpha = 2$ and $f(u) = u^p$ with p an integer corresponds to the generalized Korteweg-de Vries equation whose solitons have an exponential decay. By replacing $\alpha = 2$ in the coefficients of the asymptotic expansion, we find $a_1 = a_2 = 0$, which is coherent with the exponential decay.

The next propositions refine the asymptotic development of Q in the case of a polynomial non-linearity.

Proposition 3.1.7. *Let $p \in \mathbb{N}$, $p \geqslant 2$, $\alpha \in \left(\frac{p-1}{1+p}, 2\right)$, and Q be solution of $|D|^\alpha Q + Q - Q^p = 0$ verifying condition* (3.1.5). *Then $Q \in H^\infty(\mathbb{R})$ and verifies that:*

$$Q^{(j)}(x) - (-1)^j \frac{(\alpha+j)!}{\alpha!} \frac{k_1}{x^{\alpha+1+j}} \int Q^p = o_{+\infty}\left(\frac{1}{x^{\alpha+1+j}}\right), \quad \forall j \in \mathbb{N}.$$

In the case of a cubic non-linearity, the next proposition provides a sharper asymptotic development.

Proposition 3.1.8 (Higher order expansion)**.** *Let $\alpha \in (1,2)$, $p = 3$, and Q be a solution of $|D|^\alpha Q + Q - Q^p = 0$ verifying condition* (3.1.5). *Then, there exists a constant $C = C(\alpha, p) > 0$ such that:*

$$\left| Q(x) - \left(\frac{a_1}{x^{\alpha+1}} + \frac{a_2}{x^{2\alpha+1}} + \frac{a_3}{x^{\alpha+3}}\right)\right| \leqslant \frac{C}{x^{3\alpha+1}}, \tag{3.1.6}$$

$$\left| Q'(x) + (\alpha+1)\frac{a_1}{x^{\alpha+2}} + (2\alpha+1)\frac{a_2}{x^{2\alpha+2}}\right| \leqslant \frac{C}{x^{3\alpha+1}}. \tag{3.1.7}$$

Remark 3.1.9. The constants obtained in the asymptotic development of $Q(x)$ are dependent of the functions $x^l k(x)$ and $x^l |Q|^{p-1}(x)Q(x)$, for $l \in \mathbb{N}$, which are dependent on p and α as for a_3 defined in (3.1.4). Thus, our method allows a further asymptotic development while $x^l |Q|^{p-1}(x)Q(x)$ and $x^l k(x)$ remain integrable.

Let us point out an application of these results. The authors of this article describe in [52] the long term interaction of two solitary waves with the same velocity for the fgKdV equation. To this aim, they used the asymptotic behaviour of those waves to quantify the distance between the two objects. Indeed, the asymptotic behaviour up to order 3 given in Proposition 3.1.7 was necessary to quantify the strong interaction.

A natural question is the generalization of those results to higher dimensions. The authors do not know how the computations introduced in this article can be generalized for higher dimensions, and if similar theorems can be stated in this new setting.

3.1.4 Ideas of the proof

Let us describe the main ideas to obtain the asymptotic developments. We use the kernel formulation, introduced first by Bona-Li [24], of the equation satisfied by Q: $Q = k \star f(Q)$. Let us explain formally why the asymptotic behaviour of Q is characterised by the one of k, as remarked by Bona-Li [24]. Consider the general formulation:

$$-cu + \mathcal{L}u + f(u) = 0, \quad u : \mathbb{R}^n \to \mathbb{R},$$

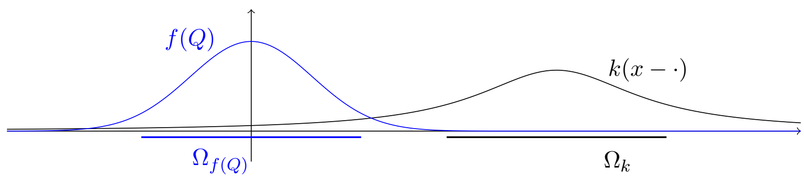
where $\mathcal{L}$ stands for a linear operator with symbol m, and f a general non-linearity satisfying $|f(u)| \simeq |u|^p$ for $p > 1$. Finding a solution to this equation is equivalent to find u of the form:

$$u(x) = \int_{\mathbb{R}^n} k(y-x) f(u(y)) dy,$$

where the kernel k is given by:

$$k(x) := \mathcal{F}^{-1}\left(\frac{1}{c - m(\xi)}\right)(x), \tag{3.1.8}$$

where $\mathcal{F}^{-1}$ stands for the inverse Fourier transform. Let us suppose that Q decays at infinity. The asymptotic behaviour of Q at infinity is given by the largest term in the asymptotic behaviours of k and of $f(Q)$ at infinity. Indeed, for x large, from the convolution formula, $k(x - \cdot)$ and $f(Q)$ are localized at different places.



One can notice that on $\Omega_{f(Q)}$, where the mass of f is located, the tail of $k(x-\cdot)$ is larger than the tail of $f(Q)$ on Ω_k, where the mass of $k(x-\cdot)$ is located. By the definition of $|f(u)| \simeq |u|^p$, with $p > 1$, the decay of $f(Q)$ is larger than the one Q. In other words, the main order term in the integral formulation is given by the decay of k on $\Omega_{f(Q)}$.

The first part of the article is dedicated to the study of integrability and of a potential differentiability of k. The key point of the analysis of k is the use of complex analysis techniques developed in the beginning of the twentieth century. Following the road map of Pólya [165] developed in 1923 (see also Blumenthal-Getoor [19]), we rewrite the function k with the help of an auxiliary function h. The asymptotic development of h is obtained by complex analysis tools : we consider the holomorphic extension of h in different regions to extract the terms at the main orders. We conclude that k is in $L^1(\mathbb{R})$ for any

$\alpha \in (0,2)$, and for $\alpha \in (1,2)$ the derivative of k exists and is also in $L^1(\mathbb{R})$. The bottleneck of the study of k stands in its behaviour at 0: having a L^1-bound of this operator at 0 is sufficient to get the asymptotic behaviour of Q.

Once the asymptotic development of k is established, we use this development to obtain the one of Q with the previous convolution formula. Since the assumption of Q only consists in a slow decay, we first improve the property of decay by injecting successively this bound into the convolution. The limit point of improvement of the decay is when the biggest term of the asymptotic expansion is coming from the asymptotic development of k. We obtain the main order term in the development of Q as being the same as the one of k. Then, to get the asymptotic development of the first derivative of Q, we need the differentiability of k, and our method applies only if $\alpha > 1$. By using the same arguments as before, we obtain the asymptotic at first order of the derivatives of Q. Similarly, we obtain the first order development of the derivatives of Q, while those derivatives exist.

The previous results can be extended if Q is a positive solution. Indeed, the previous issue of the existence of a finite number of derivatives was due to the non-regularity of $u \mapsto |u|^{p-1}u$ applied at Q, so potentially applied at 0. Nevertheless the assumption of Q positive circumvents this issue by studying the regularity of the function $u \mapsto |u|^{p-1}u$ on $\mathbb{R}_+^*$ only. This function is smooth, and so is Q. Then, we notice that having Q positive also implies that Q is an even function, and this property helps to get the next term in the asymptotic of Q. We distinguish the different cases of balance between the non-linearity and the dispersion. If the non-linearity is to large, the next order term comes from the development of k; if the non-linearity is to small, the next order term comes from the development of Q^p; in the case of exact balance, the two previous terms are at the same order and the sum of them furnishes the next order term.

Finally, in Proposition 3.1.8 we study the equation in the case of a particular non-linearity, and the asymptotic development at order 3, is obtained originating only from the asymptotic development of k.

3.1.5 Related results

For many equations, the solitary waves can be defined by a convolution formula, and their asymptotic expansion is a consequence of the regularity of the kernel. From the definition of k by the Fourier transform, see (3.1.8), the decay of k is given by the regularity of $(c-m)^{-1}$. Thus, if $(c-m)^{-1}$ is a smooth function, we expect to get Q exponentially decaying, whereas if $(c-m)^{-1}$ is not smooth the solution Q should vanish at infinity algebraically. Let us give an overview of those phenomena for diverse equations.

We begin with exponentially decaying solutions. For the local dispersion setting, the generalized KdV equation admits solitary waves which decay exponentially, with the kernel $k = \mathcal{F}^{-1}\left(\frac{1}{c+|\xi|^2}\right)$. Some non-local equations admit solitary waves with exponential decay, like the Whitham type equations as studied by Bruell-Ehrnström-Pei and Arnesen [11,28]. For solitary waves with velocity $c > 1$, the symbol of dispersion of the Whitham equation is given by $m(\xi) = \left(\frac{\tanh(\xi)}{\xi}\right)^{\frac{1}{2}}$, and thus $(c - m(\xi))^{-1}$ is not in $L^2(\mathbb{R})$. Even though k is not well-defined, by an elegant reformulation in [11,28], the study of the asymptotic behaviour of the solitary waves is equivalent to the study of the kernel operator $\widetilde{k}(x) := \mathcal{F}^{-1}(\frac{m}{c-m})(x)$. Because this last formulation is given by a smooth operator $\frac{m}{c-m}$ in $L^2(\mathbb{R})$, [28] proved the exponential decay of the solitary waves. Following the lines of [28], Pei in [164] proved the exponential decay of the solitary waves for the Degasperis-Procesi equation.

Other equations are known to own solitary waves with algebraic decay. In the local setting, the nonlinear wave equation in dimension $3 \leqslant d \leqslant 5$, with the kernel $k = \mathcal{F}^{-1}\left(|\xi|^{-2}\right)$ (defined in the weak sense), admits steady waves with algebraic decay proved by Gidas-Ni-Nirenberg [63]. For the non-local case one can cite the generalized Benjamin-Ono equation studied by Mariş [119], with the kernel $k = \mathcal{F}^{-1}\left(\frac{1}{c+|\xi|}\right)$, or the fKdV equation investigated by Frank-Lenzmann-Silvestre [61], where the kernel is given by $k = \mathcal{F}^{-1}\left(\frac{1}{c+|\xi|^\alpha}\right)$, with $0 < \alpha < 2$.

For non-radial solutions in dimensions larger than 2, the asymptotic behaviour when $|x|$ tend to $+\infty$ can depend on the direction. Indeed, in de Bouard-Saut [42], an algebraic bound of the asymptotic behaviour is obtained for a solitary wave of the generalized Kadomstev-Petviashvili equation; furthermore, there exists one direction for which this bound is optimal. This kind of asymptotic behaviour has also been observed by Gravejat [68]: by studying the solitary waves of the Gross-Pitaevskii equation, he gave an algebraic asymptotic behaviour of order one, whose coefficient at the main order depends explicitly on the angle. For the Benjamin-Ono–Zakharov-Kuznetsov equation, Esfahani-Pastor-Bona [50] proved that the solitary waves of this 2-dimensional equation decay at least polynomially in one direction, and faster than any polynomial in another direction. The question of asymptotic behaviour for several dimensions is thus more difficult to answer, in particular if the dispersion operator is not rotationally invariant.

When the non-linearity is not an algebraic function, the asymptotic behaviour can depend at the main order on the dispersion operator and on the non-linearity. For the non-local Gross-Pitaevskii equation, de Laire-López-Martínez [43] described the asymptotic behaviour of solitary waves v_c depending on a convolution function in the non-linearity. The asymptotic decay of $1-|v_c|^2$ can be algebraic or exponential at infinity depending of the choice of the convolution function.

3.1.6 Outline of the paper

The second section is dedicated to the study of the kernel k. Section 3 is dedicated to the proof, in the general case with no assumption on Q, α nor p, of the the first order asymptotic of Q, see Proposition 3.1.3. Section 4 deals with the proof of Proposition 3.1.5 with the case of more regularity. Finally, one particular case is dealt with in Section 5 where the asymptotic expansion is given at order 3, see Proposition 3.1.8. The Appendix recalls the proof of the regularity of Q if p is an integer.

3.1.7 Notations

The japanese bracket $\langle\cdot\rangle$ is defined on $\mathbb{R}$ by $\langle x\rangle := (1+|x|^2)^{\frac{1}{2}}$.

If Ω is a subset of $\mathbb{R}$, we denote by $\mathcal{C}^k(\Omega)$ the set of k-differentiable functions, with the usual generalization for $k=\infty$.

By denoting λ the Lebesgue measure, we define the generalized L^p space as:

$$\forall p \in (1,+\infty), \quad L^p(\mathbb{R}) := \left\{ f \text{ function on } \Omega, \quad \Omega \subset \mathbb{R} \text{ and } \lambda(\Omega^C)=0, \quad \|f\|_{L^p(\Omega)} < \infty \right\}.$$

By abuse of notations, we denote by $\|\cdot\|_{L^p}$ the L^p-norm over any subset Ω of $\mathbb{R}$ over which the function is well-defined, with $\lambda(\Omega^C)=0$.

We denote the Fourier transform by $\mathcal{F}$, defined by:

$$\forall u \in L^2(\mathbb{R}), \quad \mathcal{F}(u)(\xi) := \int_{\mathbb{R}} e^{-2i\pi x\xi} u(x)dx,$$

and its inverse by $\mathcal{F}^{-1}$. The Sobolev space for $s\in\mathbb{R}$ defined by the Fourier transform is thus:

$$H^s(\mathbb{R}) := \left\{ u \in L^2(\mathbb{R}); \int_{\mathbb{R}} \langle\xi\rangle^{2s} \left|\mathcal{F}(u)(\xi)\right|^2 d\xi < \infty \right\}.$$

The usual convolution operator $\star$ is defined by:

$$\forall f,g \in L^2(\mathbb{R}), \quad f\star g(x) := \int_{\mathbb{R}} f(x-y)g(y)dy.$$

For a fixed $x\in\mathbb{R}$, we define the following subset of $\mathbb{R}$:

$$\Omega_x := \left\{ |y| \leqslant \frac{x}{2} \right\}. \tag{3.1.9}$$

3.2 Asymptotic expansion of the kernel operator

Let us define the kernel associated to the operator $(1+|D|^{\alpha})^{-1}$:

$$k(x) := \mathcal{F}^{-1}\left(\frac{1}{1+|\xi|^{\alpha}}\right)(x).$$

For $\alpha \in (0,1]$, the inverse Fourier transform of $(1+|\xi|^{\alpha})^{-1}$ is understood as an improper integral for each $x \in \mathbb{R}^*$.

In order to give the asymptotic of the ground state, we establish an asymptotic development of k. Lemma 3.2.1 is dedicated to this asymptotic development. As an application, we prove in Corollary 3.2.6 that some decay properties are preserved under the convolution with k.

Lemma 3.2.1. *The function k is in $\mathcal{C}^{\infty}(\mathbb{R}^*)$, and in $L^1(\mathbb{R})$. Furthermore, there exists a sequence $(k_n)_n \in \mathbb{R}^{\mathbb{N}}$, such that for any $N>0$, there exists $C_N>0$ such that:*

$$\forall |x|>1, \quad \left|k(x) - \sum_{n=1}^{N} \frac{k_n}{|x|^{n\alpha+1}}\right| \leqslant \frac{C_N}{|x|^{(N+1)\alpha+1}}, \tag{3.2.1}$$

and k' admits the following development:

$$\forall |x|>1, \quad \left|k'(x) - sign(x)\sum_{n=1}^{N} \frac{(n\alpha+1)k_n}{|x|^{n\alpha+2}}\right| \leqslant \frac{C_N}{|x|^{(N+1)\alpha+2}}. \tag{3.2.2}$$

Furthermore, in the case $\alpha>1$ $k' \in L^1(\mathbb{R})$.

Notice that the values of k_1 and k_2 are defined in (3.1.3).

Remark 3.2.2. Frank, Lenzmann and Silvestre established in [61], Lemma C.1., the asymptotic of the kernel at the first order, and proved that the kernel k is positive.

Remark 3.2.3. The previous development only holds for fixed N, we ignore if the serie converges.

Let us first write this kernel with a more convenient formula, by defining the function h by:

$$h(y) = \int_0^{\infty} \cos(y\eta)\, e^{-\eta^{\alpha}}\, d\eta.$$

We claim the formula:

Claim 3.2.4. *The following equality holds:*

$$\forall x \neq 0, \quad k(x) = \frac{1}{\pi}\int_0^{\infty} \frac{e^{-s}}{s^{\frac{1}{\alpha}}} h\left(\frac{x}{s^{\frac{1}{\alpha}}}\right) ds.$$

One can notice that this formula is more convenient. Indeed, for any value of $\alpha \in (0,2)$, the asymptotic development of h at infinity induces that k is $\mathcal{C}^{\infty}$ on $\mathbb{R}^*$, whereas this property was not clear by the definition by the Fourier transform. Notice that Lemma C.1 of [61] used another formulation of k : by applying Bernstein's theorem on the fractional heat kernel, k is rewritten by the semi-group associated with the heat kernel and an adequate non-negative finite measure.

Let us begin by proving Claim 3.2.4. To do so, we need the asymptotic development of h:

Claim 3.2.5. *The following expansions hold:*

$$\forall y>1, \quad \left|h(y) - \frac{\pi k_1}{y^{\alpha+1}} - \frac{\pi k_2}{2y^{2\alpha+1}}\right| \leqslant \frac{C}{y^{3\alpha+1}}, \tag{3.2.3}$$

and

$$\forall y > 1, \quad \left| h'(y) - \frac{k_1'}{y^{\alpha+2}} \right| \leqslant \frac{C}{y^{2\alpha+2}}, \quad k_1' \in \mathbb{R}. \tag{3.2.4}$$

Furthermore, for any $\beta \in (0,1]$, there is another constant $C = C(\beta)$ such that:

$$\forall |y| \leqslant 1, y \neq 0, \quad |yh(y)| \leqslant Cy^{\beta} \quad \textit{and} \quad |h'(y)| \leqslant \frac{C}{y}. \tag{3.2.5}$$

Proof of Claim 3.2.4. In the case $\alpha \in (1,2)$, the proof holds by a Fubini's argument, as in [19]. We give the general proof that also holds in the case $\alpha \in (0,1]$.

Using that $\frac{1}{t} = \int_{\mathbb{R}^+} e^{-st} ds$:

$$k(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \int_0^1 e^{-s|\xi|^\alpha} e^{-s} ds d\xi + \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \int_1^\infty e^{-s|\xi|^\alpha} e^{-s} ds d\xi.$$

We then apply Fubini's theorem for the first integral. By using Claim 3.2.5, it is also possible to use Fubini's theorem on the first integral: the asymptotic development of h is necessary, in particular if $\alpha \in (0,1]$. Finally, the change of variable $\eta^\alpha = \xi^\alpha s$ concludes Claim 3.2.4. □

Proof of Claim 3.2.5. The next ideas are inspired from Pólya [165]. By integration by parts and changing the variable $u^{\frac{1}{\alpha}} = y\eta$, we deduce that:

$$y^{1+\alpha} h(y) = y^\alpha \int_0^{+\infty} \frac{d}{d\eta} (\sin(y\eta)) e^{-\eta^\alpha} d\eta = \int_0^{+\infty} \sin\left(u^{\frac{1}{\alpha}}\right) e^{-\frac{u}{y^\alpha}} du = \operatorname{Im}\left(\int_0^{+\infty} e^{iu^{\frac{1}{\alpha}} - \frac{u}{y^\alpha}} du \right).$$

Note that the previous integral is not well defined for $y = +\infty$. To bypass this difficulty, we apply contour integration, see [165]. Let:

$$D_n := \{ re^{i\frac{\pi}{4}\alpha} : r \in (0,n] \}, \quad D := \{ re^{i\frac{\pi}{4}\alpha} : r \in (0,+\infty) \}, \quad C_n := \{ ne^{i\gamma} : \gamma \in \left[0, \frac{\pi}{4}\alpha\right] \}.$$

We set γ_n the curves that range D_n, C_n and then $[0,n]$ counterclockwise. Since $u \mapsto e^{iu^{\frac{1}{\alpha}} - \frac{u}{y^\alpha}}$ is holomorphic in $\mathbb{R}_+^* + i\mathbb{R}_+$, we deduce that:

$$0 = \lim_{n \to +\infty} \int_{\gamma_n} e^{iu^{\frac{1}{\alpha}} - \frac{u}{y^\alpha}} du = \int_D e^{iu^{\frac{1}{\alpha}} - \frac{u}{y^\alpha}} du - \int_0^{+\infty} e^{iu^{\frac{1}{\alpha}} - \frac{u}{y^\alpha}} du.$$

Furthermore, we get that:

$$y^{1+\alpha} h(y) = \operatorname{Im}\left(\int_0^{+\infty} \exp\left(\frac{\sqrt{2}}{2}(i-1) r^{\frac{1}{\alpha}} - \frac{re^{i\frac{\pi}{4}\alpha}}{y^\alpha} \right) e^{i\frac{\pi}{4}\alpha} dr \right).$$

To obtain the asymptotic expansion of $y^{1+\alpha}h(y)$ at $+\infty$, we split the former integral in two parts. We set:

$$J_1 := \operatorname{Im}\left(\int_0^{y^{\frac{\alpha}{2}}} \exp\left(\frac{\sqrt{2}}{2}(i-1) r^{\frac{1}{\alpha}} - \frac{re^{i\frac{\pi}{4}\alpha}}{y^\alpha} \right) e^{i\frac{\pi}{4}\alpha} dr \right),$$
$$J_2 := \operatorname{Im}\left(\int_{y^{\frac{\alpha}{2}}}^{+\infty} \exp\left(\frac{\sqrt{2}}{2}(i-1) r^{\frac{1}{\alpha}} - \frac{re^{i\frac{\pi}{4}\alpha}}{y^\alpha} \right) e^{i\frac{\pi}{4}\alpha} dr \right).$$

Since $\alpha \in (1,2)$, we have that:

$$|J_2| \leqslant \int_{y^{\frac{\alpha}{2}}}^{+\infty} \exp\left(-\frac{\sqrt{2}}{2} r^{\frac{1}{\alpha}} - \cos\left(\frac{\pi}{4}\alpha\right) \frac{r}{y^\alpha} \right) dr \leqslant C \int_{y^{\frac{\alpha}{2}}}^{+\infty} \exp\left(-\frac{\sqrt{2}}{2} r^{\frac{1}{\alpha}} \right) dr \leqslant Ce^{-\frac{\sqrt{y}}{4}}. \tag{3.2.6}$$

Now, we estimate J_1. First, we rewrite J_1 as:

$$\begin{aligned} J_1 =&\mathrm{Im}\left(\int_0^{y^{\frac{\alpha}{2}}} \exp\left(i\left(re^{i\frac{\pi}{4}\alpha}\right)^{\frac{1}{\alpha}} - \frac{re^{i\frac{\pi}{4}\alpha}}{y^\alpha}\right) e^{i\frac{\pi}{4}\alpha}dr\right) \\ =&\mathrm{Im}\left(\int_0^{y^{\frac{\alpha}{2}}} \exp\left(i\left(re^{i\frac{\pi}{4}\alpha}\right)^{\frac{1}{\alpha}} + i\frac{\pi}{4}\alpha\right)\left[\exp\left(-\frac{re^{i\frac{\pi}{4}\alpha}}{y^\alpha}\right) - 1 + \frac{re^{i\frac{\pi}{4}\alpha}}{y^\alpha}\right]dr\right) \\ &+\mathrm{Im}\left(\int_0^{y^{\frac{\alpha}{2}}} \exp\left(i\left(re^{i\frac{\pi}{4}\alpha}\right)^{\frac{1}{\alpha}} + i\frac{\pi}{4}\alpha\right)\left[1 - \frac{re^{i\frac{\pi}{4}\alpha}}{y^\alpha}\right]dr\right) =: J_{11} + J_{12}. \end{aligned} \tag{3.2.7}$$

From the Taylor expansion of e^z, we deduce that:

$$|J_{11}| \leqslant C\int_0^{y^{\frac{\alpha}{2}}} e^{-\frac{\sqrt{2}}{2}r^{\frac{1}{\alpha}}}\left(\frac{r}{y^\alpha}\right)^2 dr \leqslant \frac{C}{y^{2\alpha}}. \tag{3.2.8}$$

Let us rewrite J_{12}:

$$\begin{aligned} J_{12} = &\mathrm{Im}\left(\int_0^{+\infty} e^{\frac{\sqrt{2}}{2}(i-1)r^{\frac{1}{\alpha}}+i\frac{\pi}{4}\alpha}dr\right) - \mathrm{Im}\left(\int_{y^{\frac{\alpha}{2}}}^{+\infty} e^{\frac{\sqrt{2}}{2}(i-1)r^{\frac{1}{\alpha}}+i\frac{\pi}{4}\alpha}dr\right) \\ &- \frac{1}{y^\alpha}\mathrm{Im}\left(\int_0^{+\infty} re^{\frac{\sqrt{2}}{2}(i-1)r^{\frac{1}{\alpha}}+i\frac{\pi}{2}\alpha}dr\right) + \mathrm{Im}\left(\frac{1}{y^\alpha}\int_{y^{\frac{\alpha}{2}}}^{+\infty} re^{\frac{\sqrt{2}}{2}(i-1)r^{\frac{1}{\alpha}}+i\frac{\pi}{2}\alpha}dr\right). \end{aligned} \tag{3.2.9}$$

Arguing similarly as (3.2.6), we get that:

$$\left|\int_{y^{\frac{\alpha}{2}}}^{+\infty} e^{\frac{\sqrt{2}}{2}(i-1)r^{\frac{1}{\alpha}}+i\frac{\pi}{4}\alpha}dr\right| + \left|\frac{1}{y^\alpha}\int_{y^{\frac{\alpha}{2}}}^{+\infty} re^{\frac{\sqrt{2}}{2}(i-1)r^{\frac{1}{\alpha}}+i\frac{\pi}{2}\alpha}dr\right| \leqslant Ce^{-\frac{\sqrt{y}}{4}}. \tag{3.2.10}$$

In order to get the sign of the two integrals over $\mathbb{R}^+$ in (3.2.9), we use again a contour integration. We set:

$$\widetilde{D_n} := \{re^{i\frac{\pi}{2}\alpha} : r \in (0,n]\}, \quad \widetilde{D} := \{re^{i\frac{\pi}{2}\alpha} : r \in (0,+\infty)\}, \quad \widetilde{C_n} := \{ne^{i\gamma} : \gamma \in \left[\frac{\pi}{4}\alpha, \frac{\pi}{2}\alpha\right]\}.$$

Therefore, by using the curves who range $\widetilde{D_n}$, $\widetilde{C_n}$ and D_n counterclockwise, we have that:

$$\begin{aligned} &\mathrm{Im}\left(\int_0^{+\infty} e^{\frac{\sqrt{2}}{2}(i-1)r^{\frac{1}{\alpha}}+i\frac{\pi}{4}\alpha}dr\right) = \mathrm{Im}\left(\int_0^{+\infty} e^{-r^{\frac{1}{\alpha}}+i\frac{\pi}{2}\alpha}dr\right) = \sin(\frac{\pi}{2}\alpha)\Gamma(\alpha+1), \\ &\mathrm{Im}\left(\int_0^{+\infty} re^{\frac{\sqrt{2}}{2}(i-1)r^{\frac{1}{\alpha}}+i\frac{\pi}{2}\alpha}dr\right) = \mathrm{Im}\left(\int_0^{+\infty} re^{-r^{\frac{1}{\alpha}}+i\pi\alpha}dr\right) = \frac{1}{2}\sin(\pi\alpha)\Gamma(2\alpha+1). \end{aligned} \tag{3.2.11}$$

Gathering (3.2.6), (3.2.8), (3.2.10) and (3.2.11) we conclude (3.2.3), with k_1 and k_2 recalled in (3.1.3).

We continue by proving (3.2.4). The proof of this property is also obtained by contour argument as for h, Claim 3.2.5. Let us give some steps of the proof. By integration by part, and the change of variables $u = (y\eta)^2$ we obtain that:

$$y^2h'(y) = -\frac{1}{2}\int_0^\infty \sin\left(u^{\frac{1}{2}}\right)e^{-u^{\frac{\alpha}{2}}y^{-\alpha}}du.$$

First, using the contour defined by

$$D'_n := \left\{re^{i\frac{\pi}{2}} : r \in (0,n]\right\}, \quad D' := \left\{re^{i\frac{\pi}{2}} : r \in (0,+\infty)\right\}, \quad C'_n := \left\{ne^{i\gamma} : \gamma \in \left[0, \frac{\pi}{2}\right]\right\},$$

we have

$$y^2h'(y) = -\frac{1}{2}\mathrm{Im}\left(\int_0^{+\infty} \exp\left(\sqrt{r}e^{i\frac{3}{4}\pi} - \frac{r^{\frac{\alpha}{2}}}{y^\alpha}e^{i\frac{\pi}{4}\alpha} + i\frac{\pi}{2}\right)dr\right).$$

By using a decomposition as in (3.2.6) and (3.2.7) with the bound of the different terms, we have:

$$\left|y^2h'(y)+\frac{1}{2}\mathrm{Im}\left(\int_0^{+\infty}\exp\left((re^{i\frac{3}{2}\pi})^{\frac{1}{2}}+i\frac{\pi}{2}\right)\left(1-\frac{r^{\frac{\alpha}{2}}}{y^\alpha}e^{i\frac{\pi}{4}\alpha}\right)dr\right)\right|\leqslant\frac{C}{y^{2\alpha}}.$$

Using a second contour integration defined by

$$\widetilde{D'_n}:=\left\{re^{i\frac{3\pi}{2}}:r\in(0,n]\right\},\quad \widetilde{D}':=\{r:r\in(0,+\infty)\},\quad \widetilde{C'_n}:=\left\{ne^{i\gamma}:\gamma\in\left[\frac{3\pi}{2},2\pi\right]\right\},$$

and the fact

$$\Re\left(\int_0^{+\infty}e^{\frac{\sqrt{2}}{2}(i-1)\sqrt{u}}du\right)=2\Re\left(\int_0^{+\infty}ve^{\frac{\sqrt{2}}{2}(i-1)v}dv\right)=0,$$

we obtain the asymptotic development (3.2.4) of h'.

To prove (3.2.5), we multiply h by $y^{1-\beta}$ and integrate by part:

$$y^{1-\beta}h(y)=\int_0^{+\infty}\frac{\sin(y\eta)}{(y\eta)^\beta}\alpha\eta^{\alpha-1+\beta}e^{-\eta^\alpha}d\eta.$$

Since $\frac{\sin(z)}{z^\beta}$ is uniformly bounded in $\beta\leqslant 1$, we obtain the first inequality of (3.2.5). For the second inequality:

$$y^2h'(y)=-y-y\int_0^{+\infty}\frac{\sin(y\eta)}{y\eta}\alpha\left((\alpha-1)\eta^{\alpha-1}-\alpha\eta^{2\alpha-1}\right)e^{-\eta^\alpha}d\eta,$$

and a direct bound gives the second part of (3.2.5). □

We continue with the proof of Lemma (3.2.1). We focus on the case $N=2$, the general statement is obtained using the same proof.

Proof of Lemma 3.2.1. Concerning the regularity of k, our description of the kernel is not well-suited to get this regularity. However, in Lemma C.1 of [61], the kernel k is proved to be in $\mathcal{C}^\infty(\mathbb{R}^*)$.

We continue with the asymptotic expansion of k at infinity. By parity of k, we focus on the case $x>0$. By Claim 3.2.4, we decompose the integral according to the values of s; in other words, we need to study carefully when $\frac{x}{s^{\frac{1}{\alpha}}}$ tends to $+\infty$ as $x\to+\infty$. We thus decompose:

$$\pi k(x)=\int_0^{x^{\frac{\alpha}{2}}}\frac{e^{-s}}{s^{\frac{1}{\alpha}}}h\left(\frac{x}{s^{\frac{1}{\alpha}}}\right)ds+\int_{x^{\frac{\alpha}{2}}}^{+\infty}\frac{e^{-s}}{s^{\frac{1}{\alpha}}}h\left(\frac{x}{s^{\frac{1}{\alpha}}}\right)ds=I_1+I_2.$$

Let us first find a bound of I_2. Since h is a bounded function, we get:

$$|I_2|\leqslant Ce^{-x^{\frac{\alpha}{4}}}.$$

The asymptotic expansion of I_1 is now given by the the one of h, since $\frac{x}{s^{\frac{1}{\alpha}}}\geqslant x^{\frac{1}{2}}$ for $s\leqslant x^{\frac{\alpha}{2}}$. Since $\displaystyle\int_{\mathbb{R}}se^{-s}ds=1$, $\displaystyle\int_{\mathbb{R}}s^2e^{-s}ds=2$ and by using (3.2.3), we conclude the proof of the asymptotic development of k.

This development justifies the integrability of k at ∞. We now justify the integrability of k around 0:

$$xk(x)=\frac{1}{\pi}\int_0^{x^\alpha}e^{-s}\left(yh(y)\right)_{|y=xs^{-\frac{1}{\alpha}}}ds+\frac{1}{\pi}\int_{x^\alpha}^{+\infty}e^{-s}\left(yh(y)\right)_{|y=xs^{-\frac{1}{\alpha}}}ds.$$

For the first integral, we use (3.2.3) for $|y| \geqslant 1$, and the first integral is bounded by x^α. For the second integral, we use (3.2.5) for $|y| \leqslant 1$ with $\beta = \min(1, \alpha)$ and we compute the integral:

$$\text{if } \alpha \leqslant 1, \quad \int_{x^\alpha}^{+\infty} e^{-s} \left(\frac{x}{s^{\frac{1}{\alpha}}} \right)^\alpha ds \leqslant C x^\alpha |\ln(x)|; \quad \text{if } \alpha > 1, \quad \int_{x^\alpha}^{+\infty} e^{-s} \frac{x}{s^{\frac{1}{\alpha}}} ds \leqslant Cx.$$

and thus a bound on the behaviour of k at 0:

$$\text{if } \alpha \leqslant 1, \quad |k(x)| \leqslant C x^{\alpha - 1} |\ln(x)|; \quad \text{if } \alpha \geqslant 1, \quad |k(x)| \leqslant C.$$

This last inequality justifies $k \in L^1(\mathbb{R})$, and justifies that k is bounded when x goes to 0 for $\alpha > 1$.

In addition for $\alpha > 1$, we obtain more results on k'. We prove first the asymptotic development at $+\infty$, and its proof is along the same lines as the arguments for k. We have for any $x \neq 0$, by the development (3.2.4) for the integrability at 0:

$$k'(x) = \frac{1}{\pi} \int_0^{+\infty} \frac{e^{-s}}{s^{\frac{2}{\alpha}}} h' \left(\frac{x}{s^{\frac{1}{\alpha}}} \right) ds.$$

Performing as for k, we get the asymptotic development of k' for $\alpha > 1$. We investigate on the behaviour of k' at 0. As before, we split the integral into two parts: on $(0, x^\alpha)$, by (3.2.4), the first part of the integral is bounded by $x^{\alpha-2}$; on $(x^\alpha, +\infty)$, with (3.2.5), this second part of the integral is bounded by $x^{\alpha-2}$. With the two estimates, k' is in $L^1(\mathbb{R})$ for $\alpha > 1$.

This concludes the proof of Lemma 3.2.1.

□

Corollary 3.2.6. *Let $g \in L^1(\mathbb{R})$ satisfying $|g(x)| \leqslant C|x|^{-\alpha-1}$. There exists $C = C(g)$ such that:*

$$|k * g|(x) \leqslant \frac{C}{\langle x \rangle^{\alpha+1}}. \tag{3.2.12}$$

Furthermore, if $g \in \mathcal{C}^1(\mathbb{R})$ and $|g'(x)| \leqslant C|x|^{-2-\alpha}$, then there exists $C = C(g, g')$ such that:

$$|\partial_x (k \star g)|(x) \leqslant \frac{C}{\langle x \rangle^{\alpha+2}}.$$

Proof. For sake of simplicity, we focus on the case $x \geqslant 0$, the negative terms are obtained by parity of k. First, by the decomposition of (3.1.9), we have that:

$$k * g(x) = \int_{\Omega_x} k(x-y) g(y) dy + \int_{\Omega_x^c} k(x-y) g(y) dy = I_1 + I_2.$$

By the decay assumption on g, we get that:

$$|I_2| \leqslant \frac{C}{\langle x \rangle^{\alpha+1}}.$$

Moreover, from Lemma 3.2.1, we deduce that:

$$|k(x)| \leqslant \frac{C}{\langle x \rangle^{\alpha+1}}.$$

Furthermore, the inequality $|y| \leqslant \frac{x}{2}$ implies $|x - y| \geqslant \frac{x}{2}$. Thus, we deduce that:

$$|I_1| \leqslant \frac{C}{\langle x \rangle^{\alpha+1}}.$$

Gathering these estimates, we conclude (3.2.12) for $x > 0$. By arguing similarly as the case $x > 0$, we get (3.2.12) for $x \in \mathbb{R}$.

The second estimate is based on one step further on the asymptotic development of k:

$$\left| k(x) - \frac{k_1}{x^{\alpha+1}} \right| \leqslant \frac{C}{\langle x \rangle^{2\alpha+1}}. \tag{3.2.13}$$

By the same decomposition as previously, we have:

$$\partial_x (k \star g)(x) = I_1 + I_2,$$

where:

$$\begin{aligned} I_1 &= \int_{\Omega_x} \left(k(x-y) - \frac{k_1}{(x-y)^{\alpha+1}} \right) g'(y)dy + \int_{\Omega_x} \frac{k_1}{(x-y)^{\alpha+1}} g'(y)dy. \\ I_2 &= \int_{\Omega_x^c} k(x-y) g'(y)dy. \end{aligned}$$

It suffices to prove the bound with I_1, since I_2 is dealt with like the previous step. For the first term, due to (3.2.13), we have:

$$\left| \int_{\Omega_x} \left(k(x-y) - \frac{k_1}{|x-y|^{\alpha+1}} \right) g'(y)dy \right| \leqslant \frac{C}{\langle x \rangle^{2\alpha+1}}.$$

For the second term of I_1, by integration by part:

$$\left| \int_{\Omega_x} \frac{k_1}{(x-y)^{\alpha+1}} g'(y)dy \right| \leqslant \left| \frac{k_1}{(\frac{x}{2})^{\alpha+1}} g\left(-\frac{x}{2}\right) \right| + \left| \frac{k_1}{(\frac{3x}{2})^{\alpha+1}} g\left(\frac{x}{2}\right) \right| + \left| \int_{\Omega_x} \frac{k_1(\alpha+1)}{(x-y)^{\alpha+2}} g(y)dy \right| \leqslant \frac{C}{\langle x \rangle^{\alpha+2}}.$$

This achieves the proof of one derivative applied to $k \star g$ in Lemma 3.2.6. □

3.3 Proof of the asymptotic expansion of order 1

In this section we prove Proposition 3.1.3, the asymptotic development of order 1 of Q for $\alpha \in (0,2)$ and $p \in (1, p^*(\alpha))$. [61] proved the decay of a positive Q at infinity:

$$|Q(x)| \leqslant \frac{C}{\langle x \rangle^{\alpha+1}}. \tag{3.3.1}$$

We propose here to establish a more precise asymptotic development in the general case that Q is not necessarily positive, but satisfies (3.1.5). The proof is separated in two steps: the first consists in establishing the first order asymptotic, and to prove in particular that (3.3.1) is satisfied. Second, we get successively the asymptotic development of order 1 of the derivatives of $Q^{(j)}$, by using an induction process on j.

3.3.1 First order expansion of Q

Let us prove the asymptotic expansion of order 1 for $|x| > 1$. To begin with, if Q is a weak solution of (3.1.1), then Q is a weak solution of the following equation:

$$Q = \mathcal{F}^{-1}\left((|\xi|^\alpha + 1)^{-1} \right) \star \left(|Q|^{p-1} Q \right) = k \star \left(|Q|^{p-1} Q \right). \tag{3.3.2}$$

Before giving the asymptotic expansion, we improve the bound on Q:

$$|Q(x)| \leqslant \frac{C}{\langle x \rangle^{\alpha+1}}.$$

We split the integral (3.3.2) in high and low values, with Ω_x defined in (3.1.9):

$$K_1 := \int_{\Omega_x} k(x-y)|Q|^{p-1}Q(y)dy, \quad K_2 := \int_{\Omega_x^c} k(x-y)|Q|^{p-1}Q(y)dy. \tag{3.3.3}$$

Using (3.2.1) on K_1, and the hypothesis $Q \in L^p(\mathbb{R})$, we deduce that:

$$|K_1| \leqslant \frac{C}{\langle x\rangle^{\alpha+1}}.$$

Now we continue with K_2. We use the decay assumption on Q: $|Q(x)| \leqslant C\langle x\rangle^{-l}$, and $k \in L^1(\mathbb{R})$ to deduce:

$$|K_2| \leqslant \frac{C}{\langle x\rangle^{pl}}.$$

Then, we obtain that $|Q(x)| \leqslant C\langle x\rangle^{-pl}$. Since $p > 1$, by iterating the previous steps with the improved bound on the decay of Q, we conclude that:

$$|Q(x)| \leqslant \frac{C}{\langle x\rangle^{\alpha+1}}$$

From this inequality, we deduce that $Q \in L^q(\mathbb{R})$ for any $q \in [1, +\infty]$. By the equation (3.1.1), $|D|^\alpha Q$ is also in $L^q(\mathbb{R})$ and $Q \in H^\alpha(\mathbb{R})$. Thus Q is a solution of (3.1.1) and solves (3.3.2) in the usual sense.

We continue with the asymptotic expansion of Q. We find the equivalent at the first order, which comes from K_1. We have:

$$\begin{aligned}\left|K_1 - \int_{\Omega_x} \frac{k_1}{(x-y)^{1+\alpha}}|Q|^{p-1}Q(y)dy\right| &\leqslant \int_{\Omega_x}\left|k(x-y) - \frac{k_1}{(x-y)^{\alpha+1}}\right||Q|^p(y)dy \\ &\leqslant C\int_{\Omega_x}\frac{|Q|^p(y)}{|x-y|^{2\alpha+1}}dy \leqslant \frac{C}{\langle x\rangle^{2\alpha+1}}.\end{aligned}$$

We then get a development in x of the term in the left hand-side, where a_1 is defined in (3.1.4):

$$\left|\int_{\Omega_x}\frac{k_1}{(x-y)^{\alpha+1}}|Q|^{p-1}Q(y)dy - \frac{a_1}{x^{\alpha+1}}\right| \leqslant \frac{C}{\langle x\rangle^{\alpha+1}}\int_{\Omega_x^c}|Q|^p(y)dy + \frac{C}{\langle x\rangle^{\alpha+2}}\int_{\Omega_x}|y||Q|^p(y)dy.$$

The first term is dealt with the asymptotic bound $Q(y) \leqslant C\langle y\rangle^{-(1+\alpha)}$:

$$\frac{1}{\langle x\rangle^{\alpha+1}}\int_{\Omega_x^c}|Q|^p(y)dy \leqslant \frac{C}{\langle x\rangle^{\alpha+p(1+\alpha)}}.$$

For the second term, we need to split the different cases between α and p:

$$\frac{C}{\langle x\rangle^{\alpha+2}}\int_{\Omega_x}|y|^{1-\min(\frac{p\alpha}{2},1)}|y|^{\min(\frac{p\alpha}{2},1)}|Q|^p(y)dy \leqslant \frac{C}{\langle x\rangle^{\alpha+1+\min(\frac{p\alpha}{2},1)}}.$$

We continue with K_2:

$$|K_2| \leqslant C\int_{\Omega_x^c}\frac{|k(x-y)|}{\langle x\rangle^{p(\alpha+1)}}dy \leqslant \frac{C}{\langle x\rangle^{p(\alpha+1)}}.$$

Gathering the estimates on K_1 and on K_2 we obtain the asymptotic development of Q at the main order:

$$\forall x > 1, \quad \left|Q(x) - \frac{a_1}{x^{\alpha+1}}\right| \leqslant \frac{C}{\langle x\rangle^{2\alpha+1}} + \frac{C}{\langle x\rangle^{p(\alpha+1)}} + \frac{C}{\langle x\rangle^{\alpha+1+\min(\frac{p\alpha}{2},1)}}.$$

3.3.2 First order expansion of the derivatives of Q

This section is dedicated to the asymptotic expansion of any j-derivative of Q, for $\alpha > 1$ and $j \leqslant \lfloor p \rfloor$. We use the notation $f(u) := |u|^{p-1}u$. We prove the following statement by induction on $j \in [1, \lfloor p \rfloor]$:

Statement. The function Q is of class $\mathcal{C}^j(\mathbb{R})$. If $j \geqslant 2$, there exists a sequence of polynomials $R_{j,l}$ such that:

$$\frac{d^j}{dx^j}(f(Q)) = pQ^{(j)} f'(Q) + \sum_{l=2}^{j} R_{j,l}(Q', \cdots, Q^{(j-1)}) f^{(l)}(Q), \tag{3.3.4}$$

where $R_{j,l}$ satisfies:

$$\forall j, l, \quad |R_{j,l}(Q'(x), \cdots, Q^{(j-1)}(x)) f^{(l)}(Q(x))| \leqslant \frac{C}{\langle x \rangle^{p(\alpha+1)+j}}.$$

If $j \geqslant 1$, the following asymptotic expansion holds:

$$\forall x \geqslant 1, \quad Q^{(j)}(x) - (-1)^j \frac{(\alpha+j)!}{\alpha!} \frac{a_1}{x^{\alpha+1+j}} = o_{+\infty}\left(\frac{1}{x^{\alpha+1+j}}\right). \tag{3.3.5}$$

Notice that in this induction process, we need j such that $p - j > 0$ to ensure that $f^{(p-j)}(Q)$ or equivalently $|Q|^{p-j}$ is well-defined; this constraint does not occur if Q is positive.

We prove the statement for $j = 1$. The first derivative of Q is continuous since k' is in $L^1(\mathbb{R})$ (see Claim 3.2.4), $Q \in L^q(\mathbb{R})$, for any $q \in [1, +\infty]$ (see Proposition 3.1.3), and given by $Q' = k' \star (|Q|^{p-1}Q)$. Moreover, we have $Q' \in L^\infty(\mathbb{R})$, since $k' \in L^1(\mathbb{R})$ and $Q \in L^\infty(\mathbb{R})$. Now, we prove that:

$$|Q'(x)| \leqslant \frac{C}{\langle x \rangle^{\alpha+2}}.$$

We write, for $x > 1$:

$$J_1 := \int_{\Omega_x} k'(x-y)|Q(y)|^{p-1}Q(y)dy, \quad J_2 := \int_{\Omega_x^c} k'(x-y)|Q(y)|^{p-1}Q(y)dy. \tag{3.3.6}$$

On the one hand, by using $Q \in L^p(\mathbb{R})$ and the decay property of k' (3.2.2), we deduce that:

$$|J_1| \leqslant \frac{C}{\langle x \rangle^{\alpha+2}}.$$

On the other hand, from the bound $|Q(x)| \leqslant C\langle x \rangle^{-(1+\alpha)}$ and the fact $k' \in L^1(\mathbb{R})$ for $\alpha > 1$, we have that:

$$|J_2| \leqslant \frac{C}{\langle x \rangle^{p(1+\alpha)}}.$$

Then, we obtain that $|Q'(x)| \leqslant C\langle x \rangle^{-\min(p(1+\alpha), 2+\alpha)}$. By integrating by parts J_2, and using the new bound on Q', we get that:

$$|Q'(x)| \leqslant \frac{C}{\langle x \rangle^{\min((2p-1)(1+\alpha), 2+\alpha)}}.$$

Since $p > 1$, by iterating the previous steps, we conclude that:

$$|Q'(x)| \leqslant \frac{C}{\langle x \rangle^{2+\alpha}}.$$

Now, we derive the asymptotic development of order 1 of Q' in (3.3.5), with:

$$Q'(x) = k \star ((|Q|^{p-1}Q)')(x).$$

First, we estimate the convolution on Ω_x. From the asymptotic development of k (3.2.1), and the fact $Q'|Q|^{p-1} \in L^1(\mathbb{R})$ we obtain that:

$$\left|\int_{\Omega_x} \left(k(x-y) - \frac{k_1}{(x-y)^{\alpha+1}} - \frac{k_2}{(x-y)^{2\alpha+1}}\right)(|Q|^{p-1}Q)'(y)dy\right| \leqslant \frac{C}{\langle x\rangle^{3\alpha+1}}.$$

By integrating by parts, we deduce that:

$$\left|\int_{\Omega_x} \frac{k_2}{(x-y)^{2\alpha+1}}(|Q|^{p-1}Q)'(y)dy\right| \leqslant \frac{C}{\langle x\rangle^{2\alpha+2}}.$$

After an integration by parts on the term with k_1, using the fact $|y|^{\min\left(1,\frac{p\alpha}{2}\right)}Q \in L^1(\mathbb{R})$, we get that:

$$\left|\int_{\Omega_x} \frac{k_1}{(x-y)^{\alpha+1}}(|Q|^{p-1}Q)'(y)dy + \frac{(\alpha+1)k_1}{x^{\alpha+2}}\int_{\Omega_x}|Q|^{p-1}(y)Q(y)dy\right| \leqslant \frac{C}{\langle x\rangle^{\alpha+2+\min\left(1,\frac{p\alpha}{2}\right)}} + \frac{C}{\langle x\rangle^{2\alpha+2}}.$$

Furthermore, using the decay assumption on Q, we obtain that:

$$\left|\frac{(\alpha+1)k_1}{x^{\alpha+2}}\int_{\Omega_x^c}|Q|^{p-1}(y)Q(y)dy\right| \leqslant \frac{C}{\langle x\rangle^{(p+1)(\alpha+1)+1}}.$$

To summarize, we have proved that:

$$\int_{\Omega_x} k(x-y)(|Q|^{p-1}(y)Q(y))'dy + \frac{(\alpha+1)k_1}{x^{\alpha+2}}\int_{\mathbb{R}}|Q|^{p-1}(y)Q(y)dy = o_{+\infty}\left(\frac{1}{x^{\alpha+2}}\right).$$

To finish the proof of the asymptotic development of Q', we have to estimate $Q'(x) = k \star ((|Q|^{p-1}Q)')(x)$ on Ω_x^c. By using the decay assumption on Q and Q', we obtain that:

$$\left|\int_{\Omega_x^c} k(x-y)|Q|^{p-1}(y)Q'(y)dy\right| \leqslant \frac{C}{\langle x\rangle^{p(1+\alpha)+1}}.$$

Therefore, gathering these estimates, we conclude that:

$$Q'(x) + \frac{(\alpha+1)a_1}{x^{\alpha+2}} = o_{+\infty}\left(\frac{1}{x^{\alpha+2}}\right).$$

To prove the statement for the other j, we proceed by induction. The regularity of $Q^{(j+1)}$ is obtained by $k' \in L^1(\mathbb{R})$, the formula (3.3.4) at j, and the asymptotic expansion of $Q^{(i)}$ for $i \leqslant j$. Thus, we have:

$$Q^{(j+1)} = k \star (|Q|^{p-1}Q)^{(j+1)}.$$

The proof of the existence of the polynomials $R_{j+1,l}$ in (3.3.4), for $l \in [2, j+1]$, is given by direct computations. Before giving the bound on the polynomials, we remark that for all $l \in \{0, \cdots, \lfloor p\rfloor\}$ there exists $C_l > 0$ such that:

$$f^{(l)}(x) = \begin{cases} C_l x|x|^{p-l-1}, & \text{if } l \text{ is even,} \\ C_l|x|^{p-l}, & \text{if } l \text{ is odd} \end{cases}.$$

We detail the bound satisfied by the polynomial $R_{j+1,l}$. If $l = 2$, we have that:

$$|Q^{(j)}f^{(2)}(Q)| \leqslant \frac{C}{\langle x\rangle^{p(1+\alpha)+j+1}} \quad \text{and} \quad \left|\frac{d}{dx}(R_{j,2})f^{(2)}(Q)\right| \leqslant \frac{C}{\langle x\rangle^{p(1+\alpha)+j+1}}.$$

Moreover, for a fixed number $l \in [3, j+1]$, the polynomial $R_{j+1,l}$ is at most composed of two terms:

$$\left|R_{j,l-1} f^{(l)}(Q)\right| \leqslant \frac{C}{\langle x\rangle^{p(1+\alpha)+j+1}} \quad \text{and} \quad \left|\frac{d}{dx}(R_{j,l}) f^{(l)}(Q)\right| \leqslant \frac{C}{\langle x\rangle^{p(1+\alpha)+j+1}},$$

which concludes the decay of the term with $f^{(l)}$ for $j+1$.

Now, we prove (3.3.5) for $Q^{(j+1)}$. Let us sketch the proof of the rough bound of $Q^{(j+1)}$:

$$|Q^{(j+1)}(x)| \leqslant \frac{C}{\langle x\rangle^{\alpha+2+j}}. \tag{3.3.7}$$

As for the first derivative, we define $J_{1,j+1}$ and $J_{2,j+1}$ as in (3.3.6):

$$J_{1,j+1} := \int_{\Omega_x} k'(x-y)(|Q|^{p-1}Q)^{(j)}(y)dy, \quad J_{2,j+1} := \int_{\Omega_x^c} k'(x-y)(|Q|^{p-1}Q)^{(j)}(y)dy.$$

For the first term with the asymptotic development (3.2.2) of k', for any N and by successive integration by parts:

$$\begin{aligned} |J_{1,j+1}| \leqslant & \left|\int_{\Omega_x} \left(k'(x-y) - \sum_{n=1}^{N} \frac{(n\alpha+1)k_n}{(x-y)^{n\alpha+2}}\right)(|Q|^{p-1}Q)^{(j)}(x)dx\right| \\ & + \left|\int_{\Omega_x} \sum_{n=1}^{N} \frac{(n\alpha+1)k_n}{(x-y)^{n\alpha+2}}(|Q|^{p-1}Q)^{(j)}(x)dx\right| \\ \leqslant & \frac{C}{\langle x\rangle^{(N+1)\alpha+2}} + \frac{C}{\langle x\rangle^{\alpha+2+p(\alpha+1)+j}} + \frac{C}{\langle x\rangle^{\alpha+2+j}} \leqslant \frac{C}{\langle x\rangle^{\alpha+2+j}}, \end{aligned}$$

where the last inequality holds for N large enough, for instance $N > \frac{j}{\alpha}$.

For the second term, by Lemma 3.2.1 and the Statement, we have:

$$|J_{2,j+1}| \leqslant \frac{C}{\langle x\rangle^{p(\alpha+1)+j}}.$$

Once we have the bound $|Q^{(j+1)}(x)| \leqslant C\langle x\rangle^{-(\alpha+2+j)} + C\langle x\rangle^{-(p(\alpha+1)+j)}$, we improve this bound by injecting it successively in $J_{2,j+1}$ after an integration by part and with (3.3.4):

$$|J_{2,j+2}| \leqslant C\|(|Q|^{p-1}Q)^{(j+1)}\|_{L^\infty(\Omega_x^C)} \leqslant \frac{C}{\langle x\rangle^{(p-1)(\alpha+1)}}\|Q^{(j+1)}\|_{L^\infty(\Omega_x^C)} + \frac{C}{\langle x\rangle^{p(\alpha+1)+j+1}}.$$

By doing the injection enough times, we get the bound (3.3.7).

To obtain the exact asymptotic development, it suffices to use the same proof as the one of Q' and the decomposition used in $J_{1,j+1}$ to extract the term of main order. We get:

$$\left|J_{1,j+1} - \frac{k_1}{x^{\alpha+j+2}}\frac{(\alpha+j+1)!}{\alpha!}\int |Q|^p Q\right| = o_{+\infty}\left(\frac{1}{x^{\alpha+j+2}}\right)$$

By injecting (3.3.7) in $J_{2,j+1}$, we conclude the asymptotic expansion (3.3.5) for $j+1$.

By finishing to prove the Statement for any $j \leqslant \lfloor p \rfloor$, we conclude the proof of Proposition 3.1.3.

Notice that by the same method, we obtain the existence of the next derivative:

$$Q^{(\lfloor p\rfloor+1)} = \int k'(x-y)\left(|Q|^{p-1}Q\right)^{(\lfloor p\rfloor)}(y)dy.$$

However, an integration by parts is not justified since Q can be equal to 0 at some points, therefore the function $\left(|Q|^{p-1}Q\right)^{(\lfloor p\rfloor+1)}$ may not be defined. Our method does not provide an asymptotic expansion of $Q^{(\lfloor p\rfloor+1)}$.

3.4 Asymptotic expansion for positive solutions

To start this section dedicated to the proof of Proposition 3.1.5, we recall that, on the one hand, if $Q > 0$ then $Q \in H^\infty(\mathbb{R})$, see [61], the third point in the Remarks after Proposition 1.1, and Q is even (up to translation) see Proposition 1.1 of [61] and [117].

Since $Q \in H^\infty(\mathbb{R})$ and even, we give the asymptotic expansion of $Q^{(j)}$ for any j. The proof of this result follows the line of the one of Proposition 3.1.3. We use the induction on the statements (3.3.4) and (3.3.5) as for Proposition 3.1.3. For $Q > 0$ the induction process given in Proposition 3.1.3 does not stop anymore. Therefore, in the case of $Q > 0$ we get the asymptotic development of order 1 for all the derivatives.

We continue by giving the second order expansion of Q.

The proof is based on the same arguments as the ones of subsection 3.3.1, where we have obtained:

$$Q(x) - \frac{a_1}{x^{\alpha+1}} = o_{+\infty}\left(\frac{1}{x^{\alpha+1}}\right).$$

We prove in this section the expansion at the next order. We study separately the different cases.

Let K_1 and K_2 be defined as in (3.3.3), we have the following decomposition. First, with K_1:

$$K_1 = K_{11} + K_{12} + K_{13} + K_{14}$$

with

$$\begin{aligned}
K_{11} &= \int_{\Omega_x} \left(k(x-y) - \frac{k_1}{(x-y)^{\alpha+1}} - \frac{k_2}{(x-y)^{2\alpha+1}}\right) Q^p(y)dy,\\
K_{12} &= \int_{\Omega_x} \left(\frac{k_1}{(x-y)^{\alpha+1}} + \frac{k_2}{(x-y)^{2\alpha+1}} - \left(\frac{k_1}{x^{\alpha+1}} + \frac{k_2}{x^{2\alpha+1}}\right)\right) Q^p(y)dy,\\
K_{13} &= \left(\frac{k_1}{x^{\alpha+1}} + \frac{k_2}{x^{2\alpha+1}}\right)\int_{\mathbb{R}} Q^p(y)dy \quad \text{and} \quad K_{14} = -\left(\frac{k_1}{x^{\alpha+1}} + \frac{k_2}{x^{2\alpha+1}}\right)\int_{\Omega_x^c} Q^p(y)dy.
\end{aligned}$$

By the asymptotic expansion (3.2.1) of k, we have $|K_{11}| \leqslant C\langle x\rangle^{-(3\alpha+1)}$. For K_{12}, we use the parity of Q to get $\int_{\Omega_x} yQ^p(y)dy = 0$, therefore we obtain:

$$|K_{12}| \leqslant \frac{C}{\langle x\rangle^{3+\alpha}}\int y^2 Q^p(y)dy + \frac{C}{\langle x\rangle^{3+\alpha}}\int |y|Q^p(y)dy \leqslant \Theta(x)$$

where

$$\Theta(x) = \begin{cases} \dfrac{1}{\langle x\rangle^{\alpha+3}} & \text{if} \quad p(1+\alpha) > 3,\\[2mm] \dfrac{\ln(|x|)}{\langle x\rangle^{\alpha+3}} & \text{if} \quad p(1+\alpha) = 3,\\[2mm] \dfrac{1}{\langle x\rangle^{\alpha+p(1+\alpha)}} & \text{if} \quad p(1+\alpha) < 3. \end{cases}$$

From the asymptotic of Q on Ω_x^c, we have:

$$|K_{14}| \leqslant \frac{C}{x^{p(1+\alpha)+\alpha}}.$$

K_{13} is the only remaining term that could potentially give the next order term in the asymptotic expansion of Q.

We decompose K_2 as:

$$K_2 = \int_{\Omega_x^c \cap (\frac{x}{2}, \frac{3x}{2})} k(x-y)Q^p(y)dy + \int_{\Omega_x^c \cap (\frac{x}{2}, \frac{3x}{2})^c} k(x-y)Q^p(y)dy = K_{2,1} + K_{2,2}.$$

Using the decay assumptions on k (3.2.1) and Q given in Proposition 3.1.3 we obtain that:

$$|K_{2,2}| = \left| \int_{\Omega_x^c \cap (\frac{x}{2}, \frac{3x}{2})} k(x-y) Q^p(y) dy \right| \leqslant \frac{C}{\langle x \rangle^{p(1+\alpha)+\alpha}}.$$

Furthermore, by using again the decay estimate on k (3.2.1) and the asymptotic expansion of Q given in Proposition 3.1.3, we deduce that:

$$K_{2,1} = \int_{\Omega_x^c \cap (\frac{x}{2}, \frac{3x}{2})} k(x-y) \frac{a_1^p}{y^{p(\alpha+1)}} \left(1 + \frac{Q(y) - \frac{a_1}{y^{\alpha+1}}}{\frac{a_1}{y^{\alpha+1}}} \right)^p dy = \frac{\widetilde{a}_1}{x^{p(1+\alpha)}} + o_{+\infty}\left(\frac{1}{x^{p(1+\alpha)}} \right),$$

with $\widetilde{a}_1 := a_1^p \int_{\mathbb{R}} k(x) dx$, and let us decompose the proof into three cases, depending on p with respect to $\frac{2\alpha+1}{\alpha+1}$.

First case : $p < \frac{2\alpha+1}{\alpha+1}$

This case corresponds to a low non-linearity compared to the influence of the dispersion, and the biggest error term comes from K_{21}. Gathering the estimates on K_1 and on K_2, we obtain:

$$Q(x) - \frac{a_1}{x^{\alpha+1}} - \frac{\widetilde{a}_1}{x^{p(\alpha+1)}} = o_{+\infty}\left(\frac{1}{x^{p(1+\alpha)}} \right)$$

Second case: $p = \frac{2\alpha+1}{\alpha+1}$

In this particular case of balance between dispersion and non-linearity, the next order is given by two different terms, from K_{13} and K_{21}:

$$Q(x) - \frac{a_1}{x^{\alpha+1}} - \frac{\widetilde{a}_1}{x^{2\alpha+1}} - \frac{a_2}{x^{2\alpha+1}} = o_{+\infty}\left(\frac{1}{x^{2\alpha+1}} \right).$$

where a_2 is given in (3.1.4).

Third case: $p > \frac{2\alpha+1}{\alpha+1}$

When the non-linearity is above $\frac{2\alpha+1}{\alpha+1}$, the tail of Q is negligible compared to the next order term given by the dispersion:

$$Q(x) - \frac{a_1}{x^{\alpha+1}} - \frac{a_2}{x^{2\alpha+1}} = o_{+\infty}\left(\frac{1}{x^{2\alpha+1}} \right).$$

This concludes the proof of Proposition 3.1.5.

3.5 Asymptotic expansion for polynomial non-linearities

The proof of Proposition 3.1.7 follows the arguments given in the proof of the Proposition 3.1.3 and of Proposition 3.1.5.

We now continue with the proof of Proposition 3.1.8. In this section the non-linearity is fixed at $p = 3$ and the dispersion $\alpha \in (1, 2)$.

To begin with, using (3.3.2), we obtain the decomposition in high and low values:

$$Q(x) = \int_{\Omega_x} k\,(x-y)\, Q^3\,(y)\, dy + \int_{\Omega_x^c} k\,(x-y)\, Q^3\,(y)\, dy =: K_1 + K_2.$$

Using the asymptotic development of k (3.2.1), we get that:

$$\begin{aligned} &\left| K_1 - \int_{\Omega_x} \left(\frac{k_1}{(x-y)^{\alpha+1}} + \frac{k_2}{(x-y)^{2\alpha+1}} \right) Q^3(y) dy \right| \\ &= \left| \int_{\Omega_x} \left(k(x-y) - \frac{k_1}{(x-y)^{\alpha+1}} - \frac{k_2}{(x-y)^{2\alpha+1}} \right) Q^3(y) dy \right| \\ &\leqslant C \int_{\Omega_x} \frac{1}{|x-y|^{3\alpha+1}} Q^3(y) dy \leqslant \frac{C}{\langle x \rangle^{3\alpha+1}}. \end{aligned}$$

From the asymptotic expansion of $\frac{1}{(1+x)^\beta}$ for $\beta \in \mathbb{R}$, with $\int |y|^3 Q^3(y)dy < \infty$ for $\alpha > 1$, we deduce that:

$$\begin{aligned}\Bigg|\int_{\Omega_x} \Bigg(&\frac{k_1}{(x-y)^{\alpha+1}} - k_1 \left(\frac{1}{x^{\alpha+1}} + (\alpha+1)\frac{y}{x^{\alpha+2}} + \frac{(\alpha+1)(\alpha+2)}{2}\frac{y^2}{x^{\alpha+3}} + \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{3!}\frac{y^3}{x^{\alpha+4}} \right) \\ &+ \frac{k_2}{(x-y)^{2\alpha+1}} - k_2\left(\frac{1}{x^{2\alpha+1}} + (2\alpha+1)\frac{y}{x^{2\alpha+2}} \right) \Bigg) Q^3(y)dy \Bigg| \\ &\leqslant \frac{C}{\langle x\rangle^{\alpha+5}} + \frac{C}{\langle x\rangle^{2\alpha+3}} \leqslant \frac{C}{\langle x\rangle^{2\alpha+3}}.\end{aligned}$$

Since $\displaystyle\int_{\Omega_x} yQ^3(y)dy = \int_{\Omega_x} y^3 Q^3(y)dy = 0$, we deduce that the terms for the asymptotic development of Q are $\dfrac{a_1}{x^{\alpha+1}} + \dfrac{a_2}{x^{2\alpha+1}} + \dfrac{a_3}{x^{\alpha+3}}$, with a_1, a_2, a_3 defined in (3.1.4). Now, we have to verify that the estimates on the asymptotic development hold on Ω_x^c.

From the decay of Q (3.3.1), we obtain that:

$$|K_2| \leqslant \int_{\Omega_x^c} |k(x-y)|dy \|Q\|^3_{L^\infty(|y|\geqslant \frac{x}{2})} \leqslant \frac{C}{\langle x\rangle^{3(1+\alpha)}} \quad \text{and} \quad \int_{\Omega_x^c} \frac{Q^3(y)}{x^{3(\alpha+1)}} + \frac{y^2 Q^3(y)}{x^{\alpha+3}} dy \leqslant \frac{C}{\langle x\rangle^{4\alpha+3}}.$$

This concludes the proof of the estimate (3.1.6).

The proof of the estimate (3.1.7) is similar as the proof of the estimate (3.1.6).

3.6 Appendix

3.6.1 Regularity result for polynomial non-linearities

Let $p \in \mathbb{N}$. We prove in this appendix that if Q verifies (3.1.5), then:

$$\forall \beta \in \mathbb{R}^+, \quad \|Q\|_{H^\beta} < \infty.$$

We prove this statement by induction. Since Q is in $L^q(\mathbb{R})$ for any $q \in [1, +\infty]$ by Proposition 3.1.3, we obtain the result for $\beta = 0$. To prove that $Q \in H^{(n+1)\alpha}(\mathbb{R})$ with the assumption $Q \in H^{n\alpha}(\mathbb{R})$, it suffices to study $|D|^{n\alpha}(Q^p)$. By the fractional Leibniz rule (also called Kato-Ponce commutator estimate, see [67] for the endpoint), and $\gamma > 0$, we have:

$$\begin{aligned}\||D|^\gamma (Q^p)\|_{L^2} &\leqslant \|\langle |D|\rangle^\gamma (Q^p)\|_{L^2} \leqslant C\left(\|Q\|_{L^\infty} \|\langle |D|\rangle^\gamma (Q^{p-1})\|_{L^2} + \|\langle |D|\rangle^\gamma Q\|_{L^2} \|Q^{p-1}\|_{L^\infty} \right) \\ &\leqslant C\|Q\|^{p-1}_{L^\infty} \|\langle |D|\rangle^\gamma Q\|_{L^2},\end{aligned}$$

where the last step is obtained by induction. Thus we obtain:

$$\||D|^{(n+1)\alpha} Q\|_{L^2} \leqslant \||D|^{n\alpha} Q\|_{L^2} + \||D|^{n\alpha}(Q^p)\|_{L^2} \leqslant C\left(1 + \|Q\|^{p-1}_{L^\infty}\right) \|Q\|_{H^{n\alpha}} < \infty.$$

Remark 3.6.1. Instead of the set of assumptions (3.1.5) on Q, one can ask $Q \in L^2(\mathbb{R}) \cap L^{p+1}(\mathbb{R})$ to obtain the same result (see Lemma B.1 of [60]).

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Chapter 4

Strongly interacting solitary waves for the fractional modified Korteweg-de Vries equation

4.0.1 Introduction of the equation

This article is dedicated to the fractional-modified Korteweg-de Vries equation (also known as the dispersion generalized modified Benjamin-Ono equation):

$$\partial_t u + \partial_x\left(-|D|^\alpha u + u^3\right) = 0, \quad u : I_t \times \mathbb{R}_x \to \mathbb{R}, \quad 1 < \alpha < 2, \tag{fmKdV}$$

where I_t is a time interval, ∂_x (respectively ∂_t) denotes the space (respectively time) derivative, and the symbol $|D|^\alpha$ is defined by the Fourier transform as an operator acting on the space of distributions:

$$\mathcal{F}(|D|^\alpha u)(\xi) := |\xi|^\alpha \mathcal{F}(u)(\xi).$$

For the purposes of motivating the equation, let us introduce the more generalized equation:

$$\partial_t u + \mathcal{L}\partial_x u + \partial_x(f(u)) = 0. \tag{4.0.1}$$

The operator L represents the dispersion of the equation, and $f(u)$ stands for the non-linearity.

In the case of a quadratic non-linearity $f(u) = u^2$ and a dispersion $\mathcal{L} = -|D|^\alpha$, we get respectively the Benjamin-Ono equation (BO) and the Korteweg-de Vries equation (KdV) for $\alpha = 1$ and $\alpha = 2$. Shrira and Voronovich, in [174], introduced the equation of coastal waves, where the parameter is the evolution of the depth of the coast. If the evolution of the depth is algebraic and given by $-(1+X)^{\alpha-1}$, for $\alpha \in (1,2)$, then the dispersion operator is approximated, for waves with a small wave number, by $-c|D|^\alpha$. Notice that other dispersions are justified by Klein, Linares, Pilod and Saut [100].

While the change of dispersion in the quadratic case models different phenomena, the change of non-linearity helps to understand the balance between non-linearity and dispersion. Indeed, studying equations with a cubic non-linearity $f(u) = u^3$ and different dispersions give new insights of the competition between those two terms. The case $\mathcal{L} = \partial_x^2 = -|D|^2$ corresponds to the modified Korteweg-de Vries equation (mKdV), while the case $\mathcal{L} = -|D|$ corresponds to the modified Benjamin-Ono equation (mBO). We chose in this article to focus on the case of a non-local dispersion $\mathcal{L} = -|D|^\alpha$, with $1 < \alpha < 2$.

Since, for $1 < \alpha < 2$, fmKdV does not enjoy a Lax pair as KdV, BO or mKdV, no tools from complete integrability can be applied to this equation. On the other hand, fmKdV possesses 3 conserved quantities (at least formally):

$$\int_{\mathbb{R}} u(t,x)dx, \quad \frac{1}{2}\int_{\mathbb{R}} u^2(t,x)dx, \quad \int_{\mathbb{R}} \left(\frac{(|D|^\alpha u(t,x))^2}{2} - \frac{u^4(t,x)}{4}\right) dx.$$

We define the scaling operators by:

$$\forall \lambda \in \mathbb{R}_+^*, \quad u \mapsto u_\lambda, \quad \text{with} \quad u_\lambda(t,x) := \lambda^{\frac{\alpha}{2(1+\alpha)}} u(\lambda t, \lambda^{\frac{1}{1+\alpha}} x). \tag{4.0.2}$$

The set of solutions of fmKdV is fixed under the scaling operations. The mBO equation is mass-critical in the sense that the L^2-norm is preserved under any scaling operation. Meanwhile, fmKdV is mass-subcritical since the conserved space under the operator of scaling is the homogeneous Sobolev space $\dot{H}^s(\mathbb{R})$ with $s = \frac{1-\alpha}{2} < 0$ as soon as $\alpha > 1$. The equation fmKdV has been proved to be locally well-posed in $H^s(\mathbb{R})$ for $s \geqslant \frac{3-\alpha}{4}$ by Guo [71], and the flow is locally continuous on that space. As a consequence, the equation is globally well-posed in the energy space $H^{\frac{\alpha}{2}}(\mathbb{R})$ (see Appendix 4.5.1). We also refer to Guo and Huang [72], Kim and Schippa [98], Molinet and Tanaka [150] for other well-posedness results. Moreover, in the case $\alpha = 1$ the problem is locally well-posed in the energy space, see Kenig and Takaoka [96].

4.0.2 Ground states and solitary waves.

Different coherent structures may appear in the study of non-linear dispersive equations, and solitary waves are one of them. A solitary wave is a solution $u(t,x) = Q_c(x - ct)$ moving at a velocity c in one direction, decaying at infinity and keeping its form along the time. The function Q_c satisfies the elliptic equation:

$$-|D|^{\alpha}Q_c - cQ_c + Q_c^3 = 0. \tag{4.0.3}$$

A remarkable point is the existence of those objects for any velocity $c > 0$. Unlike the mKdV equation the function Q_c is not explicit. The existence of a such solution of the elliptic problem (4.0.3) relies on the existence of a minimizer of an adequate functional. Such a minimizer is called a ground state, and the existence of a ground state has been proved by Weinstein in [189] and Albert-Bona-Saut in [4]. Moreover, the ground state is positive. For now, the notation Q_c will refer to the ground-state of the functional.

If we denote by Q the positive ground state associated to $c = 1$, all the other ground states Q_c can be expressed in terms of the ground state Q by the operation of scaling (4.0.2):

$$Q_c(x) = (Q_1)_c(x).$$

The question of the uniqueness of the ground state of (4.0.3) is difficult and has been solved by Frank-Lenzmann in [60]. Note however that no result seems to be known for the uniqueness of solutions to (4.0.3) which do not minimize the Euler-Lagrange functional. The non-locality of the operator $|D|^{\alpha}$ does not allow to use classical ODE's tools for this equation. The uniqueness of the solution of the non-local elliptic problem (4.0.3) is derived from the non-degerenency of the linearized operator

$$L = |D|^{\alpha} + 1 - 3Q^2,$$

by proving that $\ker(L) = \mathrm{span}(Q')$. This result was obtained by Frank-Lenzmann in [60]. The proof is based on an extension process to the upper half-plane, introduced by Caffarelli-Silvestre [29], which allows to look at the operator $|D|^{\alpha}$ as a Dirichlet-Neumann operator.

Furthermore, as soon as $\alpha < 2$, the function Q has a algebraic decay (see (4.1.4) for a more precise expansion):

$$Q(x) \simeq_{+\infty} \frac{1}{x^{1+\alpha}}.$$

The question of stability of a solitary wave in this case has been done by Pava [9], see also [154].

One conjecture in the field of dispersive equation states that any solution decomposes, at large time, into different dispersive objects (such as the solitary waves) plus a radiation term. Whereas the solitary waves move to the right, the radiation term moves to the left. This conjecture has been proved for the KdV equation using the tools of complete integrability, but remains open in the non-integrable cases. It is then natural to introduce multi-solitary waves, which are solutions u that in large time $[T_0, +\infty)$ are close to a sum of K decoupled solitary waves:

Definition 4.0.1. *Let $K > 0$, and K different velocities $0 < c_1 < \cdots < c_K$. A function u is called a multi-solitary waves associated to the previous velocities (or pure multi-solitary waves) if there exist $T_0 > 0$, K functions $v_k : (T_0, +\infty) \to \mathbb{R}$ such that:*

$$\lim_{t\to+\infty} \left\| u(t) - \sum_{k=1}^{K} Q_{c_k}(\cdot - v_k(t)) \right\|_{H^{\frac{\alpha}{2}}} = 0 \quad \textit{and} \quad \forall k \in (1, K), \quad |v_k(t) - c_k t| = o_{+\infty}(t).$$

Notice that the definition of a multi-solitary waves may depend on the information one can get from those objects. For example, in a recent result by the first author [51], the proof of the existence of the multi-solitary waves has been established for the equation fKdV with a dispersion $\alpha \in (\frac{1}{2}, 2)$ and an explicit rate of convergence of the solution to the sum of the K-decoupled solitary waves. Notice that the proof can easily be adapted to fmKdV, establishing then the existence of multi-solitary waves for this equation. The proof of existence of those objects is a first step toward the soliton resolution conjecture for this equation in the case $1 < \alpha < 2$.

4.0.3 Dipoles and main theorem

Notice that in the previous definition of multi-solitary waves, all the velocities are distinct. One can wonder if there exist solutions u behaving at infinity as a sum of two solitary waves with the same velocity c and different signs. A solution satisfying this definition is called a dipole. In particular, if the two solitary waves have the same velocity, they interact in large time one with each other, and the velocity of the different solitary waves is thus expected to be of the form $v_k(t) \sim_{+\infty} ct - g_k(t)$, with $g_k(t) = o_{+\infty}(t)$.

This object has first been observed on the mKdV equation using the complete integrability of the equation [184]. For an odd non-linearity $f(u) = |u|^{p-1}u$, $p \in (2,5)$ and a dispersion $\mathcal{L} = \partial_x^2$, Nguyen in [156] proved the existence of dipoles for those equations that are not completely integrable.

In this paper, we prove the existence of a dipole for the fmKdV in the L^2-subcritical case:

Theorem 4.0.2. *Let $\alpha \in (1,2)$. There exist some constant $T_0 > 0, C > 0$ and $U \in C^0([T_0,+\infty) : H^{\frac{\alpha}{2}}(\mathbb{R}))$ solution of* (fmKdV) *such that, for all $t \geqslant T_0$:*

$$\left\| U(t,\cdot) + Q\left(\cdot - t - \frac{a}{2}t^{\frac{2}{\alpha+3}}\right) - Q\left(\cdot - t + \frac{a}{2}t^{\frac{2}{\alpha+3}}\right)\right\|_{H^{\frac{\alpha}{2}}} \leqslant Ct^{-\frac{\alpha-1}{4(\alpha+3)}},$$

where

$$a := \left(\frac{\alpha+3}{2}\sqrt{\frac{-4b_1}{\alpha+1}}\right)^{\frac{2}{\alpha+3}} \quad \textit{and} \quad b_1 := -2\frac{(\alpha+1)^2}{\alpha-1}\frac{\sin(\frac{\pi}{2}\alpha)}{\pi}\int_0^{+\infty} e^{-\frac{1}{r^\alpha}}dr\frac{\|Q\|_{L^3}^6}{\|Q\|_{L^2}^2} < 0. \quad (4.0.4)$$

This result sheds new light on the relation between the dispersion $\mathcal{L}$ and the distance between two solitary waves of a dipole. Indeed, Nguyen in [156, 157] studied the case of a dispersion $\mathcal{L} = -|D|^2 = \partial_x^2$ and different non-linearities, which corresponds to the generalized Korteweg-de Vries equation. Since the ground states Q have an exponential decay $e^{-|x|}$, the distance between the two solitary waves of a dipole is logarithmic in time $2\ln(tc)$, with c depending on the non-linearity. A second example is the recent preprint of Lan and Wang [111], where they studied the generalized Benjamin-Ono equation with a dispersion $\mathcal{L} = -|D| = -\mathcal{H}\partial_x$ with $\mathcal{H}$ the Hilbert transform and different non-linearities. For this equation, since the ground states have a prescribed algebraic decay x^{-2}, the solitary waves of the dipoles they studied have a distance $\alpha\sqrt{t} + \beta\ln(t) + \gamma$, where α, β and γ are constants dependent only on the non-linearity. Theorem 4.0.2 emphasises how the dispersion influences the distance between the two solitary waves, that is $at^{\frac{2}{\alpha+3}}$. One can conjecture that the dipoles for an equation $\mathcal{L} = -|D|^\alpha$, for $\alpha \in (1,2)$ and a non-linearity $f(u) = |u|^{p-1}u$ with various values of p, are composed of two solitary waves at a distance $ct^{\frac{2}{\alpha+3}}$, with a constant c dependent on p.

4.0.4 Related results

As explained in the introduction, the behaviour of a solution of (4.0.5) is determined by the balance between the non-linearity and the dispersion, therefore blow-ups are expected in the critical and super-critical cases. An important result for blow-up, in finite or infinite time, in a non-local setting has been obtained by Kenig-Martel-Robbiano in [91] for:

$$\partial_t u - \partial_x|D|^\alpha u + |u|^{2\alpha}u = 0.$$

This equation is critical for all the values of α. For $\alpha = 2$ in the former equation, which corresponds to the critical general Korteweg-de Vries equation, Merle [140] proved the existence of blow-up solutions in finite or infinite time. Using this result, [91] proved by a perturbative argument the existence of blow-up for all $\alpha \in (\alpha_1, 2]$, for some $1 < \alpha_1 < 2$. The proof is based on the existence of a Liouville property and localized energy estimates. Those localized estimates generalize the pioneering work of Kenig and Martel [90] for the asymptotic stability of the soliton of the Benjamin-Ono equation.

In the case $\alpha = 1$ in fmKdV, the equation is L^2-critical and blow-up phenomena occur. Bona-Kalisch [23], and Klein-Saut-Wang [102] studied numerically the critical fmKdV and conjecture a blow-up in finite time for this equation. In [133] Martel-Pilod proved rigorously the existence of minmial mass blow-up solution for mBO. We mention also the result by Kalisch-Moldabayev-Verdier in [87], where they observed that two solitary waves may interact in such a way that the smaller wave is annihilated.

For the super-critical case we refer to the work of Saut-Wang in [171], where they proved the global well-posedness for small initial data and [102] for numerical simulation of blow-up in finite time.

The phenomenon of strong interaction between two different objects also occurs in different situations. Let us enumerate the different families of equations and results (this list may not be exhaustive) by beginning with the KdV family. By using the integrable structure of mKdV, Wadati and Ohkuma [184] exhibited the existence of a dipole. More recently, Koch and Tataru [104] characterized the set of complex two-solitons as a 8-dimensional symplectic submanifold of H^s for $s > -\frac{1}{2}$. The explicit formula of a dipole holds for the mKdV equation only. In the non-integrable case Nguyen [156] proved the existence of a dipole for (4.0.1) for a dispersion $\mathcal{L} = \partial_x^2$ and a non-linearity $f(u) = |u|^{p-1}u$, with $p \in (2, 5)$. Moreover, he discovered that for each super-critical equation with a non-linearity $p > 5$, there exists a dipole formed by two solitary waves with same signs, and the distance between the two objects is also logarithmic in time. Inspired by this result, Lan and Wang [111] looked for the phenomenon of dipoles for a dispersion $\mathcal{L} = -|D|$ and a non-linearity $f(u) = |u|^{p-1}u$, with various values of $p \neq 3$. We also list some results in the setting of the strong interaction of two non-linear objects in the non-linear Schrödinger setting. Ovchinnikov and Sigal [162] for the time-dependent Ginzburg Landau equation, with two vortices with different signs; Krieger, Martel and Raphaël [107] for the three dimensional gravitational Hartree equation with two solitons; Nguyen [157] for the subcritical non-linear Schrodinger with two solitary waves with different signs, and the same signs for the super-critical case; Nguyen and Martel [132] for coupled non-linear Schrödinger, for two solitary waves with different velocities. The phenomenon of dipole also appears in the family of wave equations: Gérard, Lenzmann, Pocovnicu and Raphaël [62] for the cubic half-wave equation; Côte, Martel, Yuan and Zhao [40] for the damped Klein-Gordon equation; Aryan [12] for the Klein-Gordon equation; Jendrej and Lawrie [85] for the wave maps equation.

The strong interaction between different objects also gives rise to exotic behaviours. For example, the existence of strongly interacting objects has been proved with multi-solitary waves for the mass-critical non-linear Schrödinger equation by Martel and Raphaël [135] and with bubbles for the critical gKdV equation by Combet and Martel [36].

Even if the question of dipoles occur at infinity, one can wonder what happens on the real line to a solution that behaves like a two soliton at $-\infty$. The problem of inelastic collision of two solitary waves has been investigated by Mizumachi [146], Martel and Merle [125,126] and Muñoz [151] for non-integrable equations in the KdV family. Indeed, only the completely integrable equations exhibit an elastic collision, that is a solution that can be decomposed at $+\infty$ with the same decomposition as at $-\infty$ (up to phase shift).

We end this part with open questions related to the dipoles of fmKdV. We begin with the particular case of the critical equation mBO: we do not know if the dipole phenomenon exists for this equation. For a fixed dispersion $\mathcal{L} = -|D|^\alpha$, one can also wonder about the importance of the non-linearity $f(u) = |u|^{p-1}u$: if p is close to 1, does the structure of a dipole still make sense, or does the non-linearity breaks the structure? Concerning the fmKdV equation, if a solution behaves at time $-\infty$ as a sum of two different solitary waves, what will be the behaviour of this solution at $+\infty$? Even though this article does not answer those questions, it gives insights and tools to tackle those problems with non-local dispersion.

4.0.5 Ideas of the proof

Let us perform the following change of variables. Let $y := x - t$, then $v(t, y) := u(t, x)$ verifies

$$\partial_t v + \partial_y \left(-v - |D|^\alpha v + v^3\right) = 0. \tag{4.0.5}$$

This equation is better suited than fmKdV for the phenomenon of strong interaction, since most of the objects considered here are moving at a velocity close to 1. Theorem 4.0.2 can be rewritten in this new setting:

Theorem 4.0.3. *Let $\alpha \in (1,2)$. There exist some constant $T_0 > 0, C > 0$ and $w \in C^0([T_0,+\infty) : H^{\frac{\alpha}{2}}(\mathbb{R}))$ solution of* (4.0.5) *such that, for all $t \geqslant T_0$:*

$$\left\| w(t,\cdot) + Q(\cdot - \frac{a}{2}t^{\frac{2}{\alpha+3}}) - Q(\cdot + \frac{a}{2}t^{\frac{2}{\alpha+3}}) \right\|_{H^{\frac{\alpha}{2}}} \leqslant Ct^{-\frac{\alpha-1}{4(\alpha+3)}},$$

with the constant a defined in (4.0.4).

From now on, we focus on proving the existence of the function w. We provide some ideas for the proof of Theorem 4.0.3.

The first important point is the construction of a good approximation. We look for a solution closed to the sum of two solitary waves $-R_1 + R_2$ modulated by a set of parameters $\Gamma = (z_1, z_2, \mu_1, \mu_2)$, where $z_i(t)$ correspond to the centres of the solitary waves moving along the time, whereas $1 + \mu_i(t)$ correspond to their size. To this aim, we search for an accurate description of $w + R_1 - R_2$, and we introduce the approximation V of the form $V(t,x) = -R_1(t,x) + R_2(t,x) + b(t)W(t,x) - P_1(t,x) + P_2(t,x)$. The goal is to adapt the four other functions such that V almost solves (fmKdV), in the sense that the quantity $\mathcal{E}_V$ is close to 0, with:

$$\mathcal{E}_V := \partial_t V + \partial_y(-|D|^\alpha V - V + V^3).$$

By computing the time derivative of R_1 and R_2, four intrinsic directions appear: $\partial_y R_1$, $\partial_y R_2$, ΛR_1 and ΛR_2. For convenience, we will write them under a vector form by $\overrightarrow{MV}$. They go hand in hand with the derivatives of the modulation parameters $\dot{z}_1$, $\dot{z}_2$, $\dot{\mu}_1$ and $\dot{\mu}_2$. Then, the function W is inherent to the problem : it compensates two of those specific directions, and has a plateau between z_2 and z_1. Even if the previous constructions of strong interactions ([125, 132, 156, 157]) used this function, it seems to be the first time that it is understood as an intrinsic part of the evolution of the solitary waves, and not only as a part of the profiles P_i. With this function we understand how the dispersion of the first solitary wave $-R_1$ on the front influences the second solitary wave in the back, and vice-versa. Once this function W is defined, we fix the functions P_1 and P_2 with algebraic decay to cancel the remainder terms with algebraic decay too, concentrated around the solitary waves. As a conclusion of this construction, the error can be decomposed into:

$$\mathcal{E}_V = \vec{m} \cdot \overrightarrow{MV} + \partial_y S + T,$$

with $\overrightarrow{MV}$ containing the four peculiar directions cited above, $\vec{m}$ gives a system of ODEs that is satisfied by Γ and adapted from the interaction terms. The two other source terms, S and T are error terms coming from the rough approximation and are bounded by functions depending on Γ. If one wants to go further in the development of the approximation, it suffices to extract from S and/or T the terms at the next order to build more precise profiles.

Once the approximation V is constructed, the second step is to estimate the error between the approximation and a solution, and to find a set of equations satisfied by $\mu := \mu_1 - \mu_2$ and $z := z_1 - z_2$. Fix $S_n >> 0$, and v_n the solution of fmKdV with final condition $v_n(S_n) = V(S_n)$. We estimate the $H^{\frac{\alpha}{2}}$-norm of the error backward in time by using an adequate weighted functional, mostly composed of quadratic terms in the error. Whereas studying the error by the energy is quite classic, we adapt in this article the energy functional used by Nguyen [156] by adding a source term $\int S\varepsilon$, linear in ε. This trick has been used by Martel and Nguyen [132], by mixing the source term S in the functional, and allows to get rid of the term $\int \partial_y LS\varepsilon$ in the functional. It generally helps to get a better approximation of the functional, but in our case, the use of the modified energy enables us not to compute the high Sobolev norms of the source term S. It means in particular that the influence of S on the error of the approximation is lower than the one of T.

One technical issue of this functional, as opposed to the ones previously used in this context, is the appearance of the non-local operator $|D|^\alpha$: two of the difficulties are the singularity of this operator for low frequencies, and the lack of an explicit Leibniz rule for this operator and the weight φ. To bypass those difficulties, we generalize the weighted commutator estimates given in Lemma 6 and Lemma 7 of Kenig-Martel-Robbiano [107] and of the first author [51].

These estimates rely on the understanding of the operator $|D|^\alpha$. Since the operator is singular at frequency 0, we need to localize in high and low frequencies : for the high frequencies, we use the pseudo-differential calculus, and the low frequency part is dealt with the theory of bounded operators on L^2. In particular, this method implies important restrictions on the choice of the weight.

When orthogonality conditions are imposed to the error, we get a system of ODEs ruling the behaviour of z and μ in $\overrightarrow{m}$. Roughly speaking, the system is the following:

$$\dot{\mu}(t) \sim \frac{2b_1}{z^{\alpha+2}(t)}, \quad \text{and} \quad \mu(t) \sim \dot{z}(t).$$

Notice that it is the solution of this system that gives the distance between the two solitary waves in Theorem 4.0.3.

To obtain a suitable bound on the different unknowns, we use a bootstrap argument. The more important ones are the error, the parameters z and μ. The error is dealt with the previous functional and μ by the bootstrap argument. Notice that a bootstrap argument alone would not have been sufficient to close the estimates: because of the algebraic decay in time of the different parameters, several integrations in time can not close the estimates. A topological argument, as introduced by Côte, Martel and Martel in [39], is necessary to conclude the estimate on z: roughly speaking, this argument of connectedness asserts that there exists at least one initial data z^{in}, chosen in a fixed interval of initial data, such that the estimates hold on the all time interval. Once this initial data is chosen, the all set of estimates is proved to hold on $[T_0, S_n]$.

With these estimates in hand, a classical argument of extraction by compactness allows to get an adequate initial data. By weak-continuity of the flow, we prove that the chosen initial data is close at any time to the sum of the two decoupled solitary waves. Furthermore, we obtain the algebraic decay in time of the error between the final solution and the two solitary waves.

4.0.6 Outline of the paper

The paper is organised as follow. Section 4.1 is dedicated to the properties related to the ground-state Q. It contains in particular the more recent results on those objects, the properties on the linearized operator and various lemmas related to this operator. Section 4.2 contains the construction of an approximation of the solution. Notice that the proof of the main theorem of this part can be skipped at first lecture. In section 4.3, we give the modulation theorem to describe a solution close to the multi-solitary waves with strong interaction. Section 4.4 provides the proof of the existence of the solution. The appendices recall satellite results used in this article : well-posedness, the pseudo-differential calculus, proofs of various lemmas based on pseudo-differential calculus, and the coercivity of the localised linearized operator.

4.0.7 Notations

Throughout the article, we use the following notations.

We denote by C a positive constant, changing from lines to lines independent of the different parameters.

We say $x \sim y$ if there exists $0 < c_1 < c_2 < +\infty$ such that $c_1 x \leqslant y \leqslant c_2 y$.

The japanese bracket $\langle \cdot \rangle$ is defined on $\mathbb{R}$ by $\langle x \rangle := (1 + x^2)^{\frac{1}{2}}$.

$L^2(\mathbb{R})$ is the set of square integrable functions. We denote the scalar product on $L^2(\mathbb{R})$ by $\langle u, v \rangle := \int_{\mathbb{R}} u(x)v(x)dx$ with $u, v \in L^2(\mathbb{R})$. The Fourier transform is defined by:

$$\forall f \in L^2(\mathbb{R}), \quad \widehat{f}(\xi) := \int_{\mathbb{R}} e^{ix\xi} f(x)dx.$$

We define the following spaces:

- the Sobolev space, for $s \in \mathbb{R}$: $H^s(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1+|\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi) d\xi < +\infty \right\}$,
- the Schwartz space : $\mathcal{S}(\mathbb{R}) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}); \forall \alpha \in \mathbb{N}, \forall \beta \in \mathbb{N}, \exists C_{\alpha,\beta}, |f^\alpha(x)| \leqslant C_{\alpha,\beta} \langle x \rangle^{-\beta} \right\}$,
- the set of functions with enough decay:

$$X^s(\mathbb{R}) := \left\{ f \in H^s(\mathbb{R}) : \exists C > 0, \forall x \in \mathbb{R}, |f(x)| \leqslant \frac{C}{\langle x \rangle^{1+\alpha}} \right\}, \text{ and } X^\infty(\mathbb{R}) = \bigcap_{s \in \mathbb{N}} X^s(\mathbb{R}). \quad (4.0.6)$$

Let $f, g \in L^2(\mathbb{R})$. We say that f is orthogonal to g if $\int_{\mathbb{R}} f(x)g(x)dx = 0$, and is sometimes shortened by $f \perp g$.

Q is the ground-state associated to the elliptic problem (4.1.1), and for $c > 0$, we set $Q_c(x) := c^{\frac{\alpha}{2(\alpha+1)}} Q(c^{\frac{1}{1+\alpha}} x)$. Moreover, let us define:

$$\begin{aligned} \Lambda Q_c &:= \frac{d}{dc'} Q_{c'_{|c'=c}} = \frac{1}{c} \left[\frac{\alpha}{2(\alpha+1)} Q + \frac{1}{\alpha+1} x Q' \right]_c, \\ \Lambda^2 Q_c &:= \frac{d^2}{dc'^2} Q_{|c'=c} = \frac{1}{c^2} \left(-\frac{\alpha(\alpha+2)}{4(\alpha+1)^2} Q + \frac{x^2 Q''}{(\alpha+1)^2} \right)_c. \end{aligned} \quad (4.0.7)$$

The parameters of the approximation are z_1, z_2, μ_1 and μ_2. We denote by $\Gamma = (z_1, z_2, \mu_1, \mu_2)$ the set of those parameters. z and μ are defined in (4.2.1), and $\bar{z}$ and $\bar{\mu}$ in (4.4.4). The two solitary waves are defined by:

$$R_1(\Gamma, y) := Q_{1+\mu_1}(y - z_1), R_2(\Gamma, y) := Q_{1+\mu_2}(y - z_2), \text{ and } \Lambda R_i(t, y) := (\Lambda Q_{1+\mu_i(t)})(y - z_i(t)).$$

Along the article, the functions z_1, z_2, μ_1, μ_2 and Γ can depend on the time, and it is precised when needed. The asset of this notation is to remark that the two solitary waves depend on the time through the parameter Γ. For purposes of notations, we can denote the solitary waves by $R_i(t)$ to emphasize on the time dependency. The solitary waves dependent only on the translation parameters are denoted by:

$$\widetilde{R}_i(t, y) := Q(y - z_i(t)) \quad \text{and} \quad \Lambda \widetilde{R}_i(t, y) := (\Lambda Q)(y - z_i(t)).$$

The derivatives are denoted by ∂_y and ∂_t. The notation ∇_Γ holds for the gradient along the four directions of Γ. When no confusion is possible, we denote by prime (as in Q') the space derivative, and by a dot (as in $\dot{\mu}$) the time derivative.

4.1 Ground state

This part recalls the properties known on Q: existence, uniqueness and the recently proved asymptotic expansion. We emphasize that the asymptotic expansion is composed of terms with algebraic decay, and is thus different from the one of the (gKdV) family $-cQ_c + \Delta Q_c + Q_c^p = 0$, with exponential decay. Next, we focus our attention on the linearized operator L.

4.1.1 Ground state properties

Considering the equation (4.0.5), the existence of solitary waves is related to the existence of solutions to the following elliptic time-independent equation:

$$-|D|^\alpha Q - Q + Q^3 = 0, \quad 1 < \alpha < 2. \quad (4.1.1)$$

The previous elliptic equation is related to a calculus of variation problem. If Q is a minimizer of the following functional J^α:

$$J^\alpha(v) = \frac{\left(\displaystyle\int ||D|^{\frac{\alpha}{2}} v|^2\right)^{\frac{1}{\alpha}} \left(\displaystyle\int |v|^2\right)^{2-\frac{1}{\alpha}}}{\displaystyle\int |v|^4}. \tag{4.1.2}$$

then it is a solution to the elliptic problem.

We now sum up the previous known results on the ground states, which are the minimizer of J^α.

Theorem 4.1.1 ([4, 60, 61, 189]). *Let $\alpha \in (1,2)$. There exists $Q \in H^s(\mathbb{R})$ for all $s \geqslant 0$ such that*

1. *(Existence) The function Q solves* (4.1.1) *and $Q = Q(|x|) > 0$ is even, positive and strictly decreasing in $|x|$. Moreover, the function Q is a minimizer of J^α in the sense that:*

$$J^\alpha(Q) = \inf_{v \in H^{\frac{\alpha}{2}}(\mathbb{R})} J^\alpha(v).$$

2. *(Uniqueness) The even ground state solution $Q = Q(|x|) > 0$ of*(4.1.2) *is unique, up to the multiplication by a constant, scaling and translation.*

3. *(Decay) The function Q verifies the following decay estimate:*

$$\frac{C_1}{(1+|x|)^{1+\alpha}} \leqslant Q(x) \leqslant \frac{C_2}{(1+|x|)^{1+\alpha}},$$

for some $C_1, C_2 > 0$.

4. *(Gagliardo-Niremberg inequality) There exists a constant $C = C(\alpha)$ such that:*

$$\|v\|_{L^4} \leqslant C \|v\|_{L^2}^{1-\frac{1}{2\alpha}} \|v\|_{H^{\frac{\alpha}{2}}}^{\frac{1}{2\alpha}}.$$

Remark 4.1.2. Notice that since the non-linearity is cubic, the function Q in the theorem and $-Q$ are both solutions of the elliptic equation (4.1.1).

Proof. We give some classic ideas to prove the Gagliardo-Niremberg inequality. The proof of this inequality relies on finding a universal constant C which bounds $(J^\alpha)^{-1}$. Indeed, by denoting the following 2-parameters transformation for $(\lambda, \gamma) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$:

$$v_{\lambda,\gamma}(x) := \lambda v\left(\frac{x}{\gamma}\right),$$

we notice that $J^\alpha(v_{\lambda,\gamma}) = J^\alpha(v)$. As a consequence, if for any v, the inequality is proved for some $v_{\lambda,\gamma}$ for some values of λ and γ, then the inequality is proved for any function. In particular, if we choose $\lambda = \|v\|_{L^2}^{-1}$ and $\gamma = \|v\|_{\dot{H}^{\frac{\alpha}{2}}}^{\frac{2}{\alpha-1}}$, we have $\|v_{\lambda,\gamma}\|_{L^2} = \|v_{\lambda,\gamma}\|_{\dot{H}^{\frac{\alpha}{2}}} = 1$. Thus it suffices to prove that $(J^\alpha)^{-1}(v_{\lambda,\gamma}) = \|v_{\lambda,\gamma}\|_{L^4}^4$ is uniformly bounded with the constraints on λ and γ. By the Sobolev embedding for a certain constant C (see for example [44]):

$$\|v\|_{L^4} \leqslant C\|v\|_{H^{\frac{\alpha}{2}}},$$

and thus we have $(J^\alpha)^{-1}(v_{\lambda,\gamma}) \leqslant 2C$ independently of $v_{\lambda,\gamma}$. This last inequality concludes the proof of the Gagliardo-Niremberg inequality. $\square$

Remark 4.1.3. As from [4, 91, 189], the optimal constant in the Gagliardo-Niremberg inequality can be given explicit in terms of Q.

Recently, the asymptotic expansions of the ground states have been improved, see [52]. We recall the results applied to our case:

Theorem 4.1.4 ([52]). *Let $\alpha \in (1,2)$ and $x > 1$. The positive, even function Q defined in Theorem 4.1.1 verifies:*

1. *(First-order expansion) The function Q verifies the following decay estimate:*

$$\left| Q^{(j)}(x) - (-1)^j \frac{(\alpha+j)!}{\alpha!} \frac{a_1}{x^{1+\alpha+j}} \right| \leqslant \frac{C_j}{(1+x)^{2+\alpha+j}}, \quad j \in \mathbb{N}, \tag{4.1.3}$$

for some $C_j > 0$, with $a_1 := k_1 \|Q\|_{L^3}^3 > 0$ and $k_1 := \dfrac{\sin\left(\frac{\pi}{2}\alpha\right)}{\pi} \displaystyle\int_0^{+\infty} e^{-r^{\frac{1}{\alpha}}} dr$.

2. *(Higher order expansion) There exists $C > 0$ such that:*

$$\left| Q(x) - \left(\frac{a_1}{x^{\alpha+1}} + \frac{a_2}{x^{2\alpha+1}} + \frac{a_3}{x^{\alpha+3}} \right) \right| \leqslant \frac{C}{x^{3\alpha+1}}, \tag{4.1.4}$$

$$\left| Q'(x) + (\alpha+1)\frac{a_1}{x^{\alpha+2}} + (2\alpha+1)\frac{a_2}{x^{2\alpha+2}} \right| \leqslant \frac{C}{x^{3\alpha+1}}, \tag{4.1.5}$$

$$\left| \Lambda Q(x) + \frac{a_1(\alpha+2)}{2(\alpha+1)} \frac{1}{x^{\alpha+1}} + \frac{a_2(3\alpha+2)}{2(\alpha+1)} \frac{1}{x^{2\alpha+1}} \right| \leqslant \frac{C}{x^{\alpha+3}}.$$

with $a_2 := k_2 \|Q\|_{L^3}^3$, $k_2 := -\dfrac{2\sin(\pi\alpha)}{\pi} \displaystyle\int_0^{+\infty} r e^{-r^{\frac{1}{\alpha}}} dr$, and $a_3 \in \mathbb{R}$.

We also recall some results of regularity given by convolution with the kernel k associated to the dispersion:

$$k(x) := \int_{\mathbb{R}} \frac{e^{ix\xi}}{1+|\xi|^\alpha} d\xi.$$

Lemma 4.1.5 ([52]). *Let $g \in X^0(\mathbb{R})$. There exists $C = C(g)$ such that:*

$$|k * g|(x) \leqslant \frac{C}{\langle x \rangle^{1+\alpha}}.$$

Furthermore, if $g \in \mathcal{C}^1(\mathbb{R})$, and $|g'(x)| \leqslant C\langle x\rangle^{-2-\alpha}$, then there exists $C = C(g, g')$ such that:

$$|\partial_x (k \star g)|(x) \leqslant \frac{C}{\langle x \rangle^{2+\alpha}}.$$

We set the expansion of the translated ground state $Q(x+z)$ at $+\infty$ in x by:

$$Q_{app}(x,z) := \frac{a_1}{z^{\alpha+1}} - (\alpha+1)a_1\frac{x}{z^{\alpha+2}} + \frac{a_2}{z^{2\alpha+1}} + \left(a_1 \frac{(\alpha+1)(\alpha+2)}{2} x^2 + a_3 \right) \frac{1}{z^{\alpha+3}}.$$

Lemma 4.1.6. *Let z be large enough. We have for all $|x| \leqslant \frac{z}{2}$:*

$$|Q_{app}(x,z) - Q(x+z)| + |Q_{app}(-x,z) - Q(x-z)| \leqslant C\left(\frac{|x|^3}{z^{\alpha+4}} + \frac{|x|}{z^{2\alpha+2}} + \frac{1}{z^{3\alpha+1}} \right) \tag{4.1.6}$$

$$|\partial_x Q_{app}(x,z) - Q'(x+z)| + |\partial_x Q_{app}(-x,z) - Q'(x-z)| \leqslant C\left(\frac{x^2}{z^{\alpha+4}} + \frac{1}{z^{2\alpha+2}} \right)$$

$$\left| \Lambda Q(x+z) + \frac{a_0(\alpha+2)}{2(\alpha+1)} \frac{1}{z^{\alpha+1}} \right| + |\partial_x \Lambda Q(x+z)| \leqslant C\left(\frac{|x|}{z^{2+\alpha}} + \frac{1}{z^{2\alpha+1}} \right)$$

Proof. From the asymptotic of Q in (4.1.4) and the asymptotic expansions:

$$\left|\frac{a_1}{|x-z|^{\alpha+1}}-\left(\frac{a_1}{z^{\alpha+1}}-a_1(\alpha+1)\frac{x}{z^{\alpha+2}}+a_1\frac{(\alpha+1)(\alpha+2)}{2}\frac{x^2}{z^{\alpha+3}}\right)\right|\leqslant C\frac{|x|^3}{z^{\alpha+4}}$$

and the ones of $\frac{a_2}{|x-z|^{2\alpha+1}}$ and $\frac{a_3}{|x-z|^{\alpha+3}}$, we get the development of $Q(x+z)$. The proof is similar for Q' with (4.1.5).

The proof of ΛQ is a combination of the two previous asymptotic expansions. □

Proposition 4.1.7. *Let $\mu^* > 0$ be small enough. There exists a constant $C > 0$, such that for any $\mu \leqslant \mu^*$, we have:*

$$|Q_{1+\mu}-Q-\mu\Lambda Q|+\left|Q_{1+\mu}^2-Q^2-2\mu Q\Lambda Q\right|\leqslant C\frac{\mu^2}{\langle x\rangle^{1+\alpha}}. \tag{4.1.7}$$

The following terms are also bounded in terms of μ:

$$\|Q_{1+\mu}-Q-\mu\Lambda Q\|_{H^2}\leqslant C\mu^2 \quad \text{and} \quad \|\Lambda Q_{1+\mu}-\Lambda Q\|_{H^1}\leqslant C\mu. \tag{4.1.8}$$

Moreover, the scalar product of Q with ΛQ is:

$$\langle Q,\Lambda Q\rangle=\frac{\alpha-1}{2(\alpha+1)}\|Q\|_{L^2}^2. \tag{4.1.9}$$

Proof. By the Taylor formula in μ, we have, with (4.0.7), (4.1.4) and (4.1.3) for the second derivative:

$$\begin{aligned} Q_{1+\mu}-Q-\mu\Lambda Q &= \int_1^{1+\mu}(1+\mu-s)\Lambda^2 Q_s ds,\\ |Q_{1+\mu}-Q-\mu\Lambda Q| &\leqslant \int_0^{\mu}\frac{\mu-s}{(1+s)^2}\frac{1}{\langle x\rangle_{1+s}^{\alpha+1}}ds\leqslant C\frac{\mu^2}{\langle x\rangle^{\alpha+1}}. \end{aligned}$$

The proof is similar for $Q_{1+\mu}^2$.

Notice that the previous bound still holds for two more derivatives, and the integral gives the first part of (4.1.8). The second part is similar. □

4.1.2 Properties of the linearized operator

We recall some results on the spectrum of the linearized operator L and establish new inversion lemma on L.

Theorem 4.1.8 ([4, 60, 91, 189]). *Let $\alpha \in]1,2[$. There exists $Q \in H^{\frac{\alpha}{2}}(\mathbb{R})\cap C^\infty(\mathbb{R})$ such that*

1. *(Linearized operator) Let L be the unbounded operator defined on $L^2(\mathbb{R})$ by:*

   $$Lv=|D|^\alpha v+v-3Q^2v.$$

   Then, the continuous spectrum of L is $[1,+\infty[$, L has one negative eigenvalue μ_0, associated to an even eigenfunction $v_0 > 0$, and ker L = span $\{Q'\}$.

2. *(Invertibility) For any $g \in L^2(\mathbb{R})$ orthogonal to v_0 and Q', there exists a unique $f \in L^2(\mathbb{R})$ such that $Lf = g$ and $f \perp Q'$. Furthermore, if $g \in H^k(\mathbb{R})$, then $f \in H^{k+\alpha}(\mathbb{R})$.*

Proof. We give the proof of the second point. By the Lax-Milgram theorem on $H^{\frac{\alpha}{2}}(\mathbb{R})$, we obtain the existence of f in the same space. Because f satisfies $|D|^\alpha f = g - f + 3Q^2 f$, we have $f \in H^\alpha(\mathbb{R})$.

Concerning the higher regularity of g, if f is solution of $Lf = g$ with $g \in H^k(\mathbb{R})$, then, since $[\partial_y, L]v = 3\partial_y(Q^2)v$ for all $v \in \mathcal{S}(\mathbb{R})$, we obtain that $f \in H^{k+\alpha}(\mathbb{R})$. □

Remark 4.1.9. From Theorem 4.1.8 we have the operator L verifies that there exists $\kappa > 0$ such that for all $f \in H^{\frac{\alpha}{2}}(\mathbb{R})$, with $f \perp v_0, Q'$ then:

$$\langle Lf, f\rangle \geqslant \kappa \|f\|^2_{H^{\frac{\alpha}{2}}}.$$

However, it is not convenient to work with v_0. An argument of Weinstein will allow us to replace the orthogonality on v_0 by an orthogonality on Q to get the coercivity. Indeed, from Lemma $E.1$ in [187] and since

$$\langle L^{-1}Q, Q\rangle = -\langle \Lambda Q, Q\rangle = -\frac{\alpha-1}{2(\alpha+1)}\|Q\|^2_{L^2} < 0.$$

we obtain the coercivity of L up to the orthogonality condition on Q and Q':

$$\forall f \in H^{\frac{\alpha}{2}}(\mathbb{R}), \quad f \perp Q, Q' \quad \text{implies} \quad \langle Lf, f\rangle \geqslant \kappa \|f\|_{\dot{H}^{\frac{\alpha}{2}}} \tag{4.1.10}$$

We continue this section with two lemmas on the characterisations on the inverse of particular functions by L on specific directions.

Lemma 4.1.10. *Let $k > 0$ and $g \in X^k(\mathbb{R})$ with $g \perp Q'$, then there exist a unique $f \in X^{k+\alpha}(\mathbb{R})$, $a \in \mathbb{R}$ such that:*

$$\begin{cases} Lf = g + aQ \\ f \perp Q, \quad f \perp Q' \end{cases}.$$

Proof. Since $g + aQ \perp Q'$, we apply the invertibility property of Theorem 4.1.8 and there exists a unique $f \in H^{k+\alpha}(\mathbb{R})$ such that:

$$\begin{cases} Lf = g + aQ \\ f \perp \ker(L) = \operatorname{span}(Q') \end{cases}.$$

To obtain the second orthogonality condition, since $L\Lambda Q = -Q$, with (4.1.9) we deduce that:

$$\langle f, Q\rangle = 0 \iff \langle g + aQ, \Lambda Q\rangle = 0 \iff a = -\frac{\langle g, \Lambda Q\rangle}{\langle Q, \Lambda Q\rangle} = -\frac{2(\alpha+1)}{\alpha-1}\frac{\langle g, \Lambda Q\rangle}{\|Q\|^2_{L^2}},$$

We finish with the decay in $\langle x\rangle^{-1-\alpha}$ from the definition (4.0.6) of $X^{k+\alpha}(\mathbb{R})$. Since $g + aQ + 3Q^2 f \in X^k(\mathbb{R})$, we obtain by Lemma 4.1.5 that $f = (|D|^\alpha + 1)^{-1}(g + aQ + 3Q^2 f) \in X^{k+\alpha}(\mathbb{R})$. This concludes the proof of Lemma 4.1.10. □

Lemma 4.1.11. *Let $g \in X^k(\mathbb{R})$. There exist a unique $a, \widetilde{a} \in \mathbb{R}$ and a unique function $f \in X^{k+\alpha}(\mathbb{R})$ such that:*

$$\begin{cases} \partial_y L(f - \widetilde{a}S_0) = \partial_y g + aQ' + \widetilde{a}\Lambda Q \\ f - \widetilde{a}S_0 \perp Q, \quad f - \widetilde{a}S_0 \perp Q' \end{cases}$$

with

$$a = -\frac{2(\alpha+1)}{\alpha-1}\frac{\langle g - \widetilde{a}(|D|^\alpha+1)S_0), \Lambda Q\rangle}{\|Q\|^2_{L^2}} \quad \textit{and} \quad \widetilde{a} = \frac{2(\alpha+1)}{\alpha-1}\frac{\langle g, Q'\rangle}{\|Q\|^2_{L^2}}. \tag{4.1.11}$$

Similarly, there exist a unique $a, \widetilde{a} \in \mathbb{R}$ and a unique function $f \in X^{k+\alpha}(\mathbb{R})$ such that:

$$\begin{cases} \partial_y L\left(f + \widetilde{a}(l - S_0)\right) = \partial_y g + aQ' + \widetilde{a}\Lambda Q \\ f + \widetilde{a}(l - S_0) \perp Q, \quad f + \widetilde{a}(l - S_0) \perp Q' \end{cases}$$

with

$$a = -\frac{2(\alpha+1)}{\alpha-1}\frac{\langle g + \widetilde{a}(|D|^\alpha+1)(l - S_0)), \Lambda Q\rangle}{\|Q\|^2_{L^2}} \quad \textit{and} \quad \widetilde{a} = \frac{2(\alpha+1)}{\alpha-1}\frac{\langle g, Q'\rangle}{\|Q\|^2_{L^2}}.$$

Proof. We denote by $\mathcal{H}$ the Hilbert transform. Since $|D|^\alpha = |D|^{\alpha-1}\mathcal{H}\partial_y$, we deduce that:

$$|D|^\alpha S_0 = |D|^{\alpha-1}\mathcal{H}(|D|^\alpha+1)^{-1}\Lambda Q = \int_y^{+\infty} |D|^\alpha(|D|^\alpha+1)^{-1}\Lambda Q.$$

Then, we get that:

$$\partial_y L S_0(y) = -\Lambda Q(y) - 3\partial_y\left(Q^2(y)S_0(y)\right).$$

Therefore, it is enough to prove that the following problem has a unique solution:

$$\begin{cases} Lf = g + aQ - \widetilde{a}3Q^2S_0 \\ f - \widetilde{a}S_0 \perp Q, \quad f - \widetilde{a}S_0 \perp Q' \end{cases}.$$

We choose $\widetilde{a}$ such that $g + aQ + \widetilde{a}3Q^2S_0$ is orthogonal to Q', and then arguing as in the proof of Lemma 4.1.10, we conclude the proof the first identity of Lemma 4.1.11. The second identity is similar.

□

4.2 Construction of the approximation

The approximation V of the expected solution u is built in this section. The purpose is to minimise the flow $\mathcal{E}_V$ associated to the approximation, by detailing V. By taking the time derivative of the sum of two solitons $-R_1 + R_2$, a particular direction intrinsic to the problem appears and is compensated by the use of a function W. This term possesses a tail at $-\infty$. We also define a time-dependent variable $b(z(t))$. We then minimise the flow associated to $-R_1 + R_2 + bW$ by adding localised profiles $-P_1$ and P_2 in the approximation to cancel the source term coming from the non-linearity.

4.2.1 Notation

Let us consider four $\mathcal{C}^1$ functions μ_1, μ_2, z_1 and z_2 on a time interval $I \subset \mathbb{R}$, and

$$\Gamma(t) = (\mu_1(t), \mu_2(t), z_1(t), z_2(t)).$$

We define the distance between the different functions by:

$$\mu(t) := \mu_1(t) - \mu_2(t), \quad z(t) := z_1(t) - z_2(t). \tag{4.2.1}$$

For a fixed constant $C_0 > 0$, we use the following set of assumptions on the interval I:

$$-z(t) \leqslant z_2(t) \leqslant -\frac{1}{8}z(t), \quad \frac{1}{8}z(t) \leqslant z_1(t) \leqslant z(t), \tag{4.2.2}$$

$$|\mu_1(t)| + |\mu_2(t)| + |\mu(t)| + |\dot{z}_1(t)| + |\dot{z}_2(t)| + |\dot{z}(t)| \leqslant \frac{C_0}{z(t)^{\frac{1+\alpha}{2}}}, \tag{4.2.3}$$

$$|\dot{\mu}_1(t)| + |\dot{\mu}_2(t)| \leqslant \frac{C_0}{z(t)^{2+\alpha}}. \tag{4.2.4}$$

Remark 4.2.1. The constant C_0 is used to fix the set of assumptions on Γ. The computations of this section involve the constant C_0, but it does not have any influence on the final constant C in Theorem 4.0.3. For the sake of simplicity, we omit the presence of this constant in the computations. To close the bootstrap, we will fix the constant C_0 to be large enough so that the set assumptions on Γ is satisfied.

We define a function

$$b(z(t)) := \frac{b_1}{z^{2+\alpha}(t)} \tag{4.2.5}$$

with $b_1 = -2a_1\dfrac{(\alpha+1)^2}{\alpha-1}\dfrac{\|Q\|_{L^3}^3}{\|Q\|_{L^2}^2} < 0$ and $a_1 > 0$ defined in Theorem 4.1.4.

4.2.2 Approximate solution

We define a function S_0 such that the $\partial_y L S_0$ is close to ΛQ, in the sense that the remaining terms are of the form $\partial_y(g)$ for some function g:

$$S_0(y) := \int_y^{+\infty} \left(|D|^\alpha + 1\right)^{-1} \Lambda Q(\widetilde{y}) d\widetilde{y}, \quad \text{and} \quad W(\Gamma(t), y) := S_0(y - z_1(t)) - S_0(y - z_2(t)). \tag{4.2.6}$$

S_0 is a well-defined function. It has a limit at $-\infty$, which may be different from 0 and is denoted by l:

$$l := \lim_{y \to -\infty} S_0(y). \tag{4.2.7}$$

See Appendix 4.5.2 for the justification of S_0.

Theorem 4.2.2. *Let $I \subset \mathbb{R}$ an interval such that the assumptions (4.2.2)-(4.2.4) on Γ are satisfied.*

There exist two constants β_0 and δ_0 in $\mathbb{R}$, two functions $\beta(\Gamma)$ and $\delta(\Gamma)$ and two functions $P_1(\Gamma, y)$ and $P_2(\Gamma, y)$ such that the following holds:

- *Asymptotic of β and δ. The functions β and δ have the following expansion:*

$$\left|\beta(\Gamma) - \frac{\beta_0}{z^{1+\alpha}}\right| + \left|\delta(\Gamma) - \frac{\delta_0}{z^{1+\alpha}}\right| \leqslant \frac{C}{z^{2+\alpha}}. \tag{4.2.8}$$

- *Orthogonality conditions and limits. The profiles $P_i(\Gamma) \in \mathcal{C}(I, X^{2+\alpha}(\mathbb{R}))$ satisfy:*

$$-P_1 + b(z) S_0(\cdot - z_1) \perp \widetilde{R}_1, \partial_y \widetilde{R}_1, \quad P_2 + b(z)(l - S_0(\cdot - z_2)) \perp \widetilde{R}_2, \partial_y \widetilde{R}_2.$$

  We then define the approximation V of a solution by:

$$V(\Gamma, y) := \sum_{i=1}^{2} (-1)^i \left(R_i(\Gamma, y) + P_i(\Gamma, y)\right) + b(z) W(\Gamma, y), \tag{4.2.9}$$

  and for simplicity we will write $V(t, y) := V(\Gamma(t), y)$.

- *Decomposition and estimate of the flow. The flow $\mathcal{E}_V$ of the approximation*

$$\mathcal{E}_V := \partial_t V + \partial_y \left(-|D|^\alpha V - V + V^3\right) \tag{4.2.10}$$

  can be decomposed into:

$$\mathcal{E}_V = \vec{m} \cdot \overrightarrow{MV} + \partial_y S + T \tag{4.2.11}$$

  with

$$\vec{m}(t) = \begin{pmatrix} -\dot{\mu}_1(t) + b(z(t)) \\ \dot{z}_1(t) - \mu_1(t) + \beta(\Gamma(t)) \\ \dot{\mu}_2(t) + b(z(t)) \\ -\dot{z}_2(t) + \mu_2(t) - \delta(\Gamma(t)) \end{pmatrix}, \quad \overrightarrow{MV}(t, y) = \begin{pmatrix} \Lambda R_1(t, y) \\ \partial_y R_1(t, y) \\ \Lambda R_2(t, y) \\ \partial_y R_2(t, y) \end{pmatrix}, \tag{4.2.12}$$

  and the source term S and the approximation due to the flow T are in $\mathcal{C}^1(I, X^{2+\alpha}(\mathbb{R}))$ and satisfy the set of inequalities :

$$\|S\|_{H^1} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}, \tag{4.2.13}$$

$$\|\partial_t S\|_{L^2} \leqslant \frac{C}{z^{3+2\alpha}}, \tag{4.2.14}$$

$$\|T\|_{H^1} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}} + \frac{C}{z^{1+\alpha}} \sum_{i=1}^{2} |\dot{z}_i - \mu_i|. \tag{4.2.15}$$

We add some estimates related to the previously defined functions. We recall the definition of φ in (4.4.19) and Φ in (4.4.20).

Proposition 4.2.3. *With the previous notations, the following estimates hold:*

- *Estimates on the solitary waves:*

$$\|R_i(\varphi-\delta_{2i})\|_{H^1}+\|\partial_y R_i\,(\varphi-\delta_{2i})\|_{H^1}+\|(1-\sqrt{|\delta_{1i}-\varphi|})R_i)\|_{L^2}\leqslant\frac{C}{z^{\alpha}},\quad i=1,2,\tag{4.2.16}$$

$$\|\partial_y R_i\Phi\|_{L^2}+\|\Lambda R_i\Phi\|_{L^2}\leqslant\frac{C}{z^{\frac{1+\alpha}{2}}},\quad i=1,2,\tag{4.2.17}$$

where δ_{ij} holds for Kronecker delta.

- *Estimates on the profiles:*

$$\|\left(P_1-P_2-bW\right)\|_{L^\infty}+\|\partial_y\left(P_1-P_2-bW\right)\|_{L^\infty}\leqslant\frac{C}{z^{1+\alpha}},\tag{4.2.18}$$

$$\|\partial_t\left(P_1-P_2-bW\right)\|_{L^\infty}\leqslant\frac{C}{z^{\frac{3+3\alpha}{2}}},\tag{4.2.19}$$

- *Estimates on the approximation:*

$$\|V\|_{L^\infty}+\|\partial_y V\|_{L^\infty}\leqslant C,\tag{4.2.20}$$

$$\|V^k\Phi^2\|_{L^\infty}+\|(V^2-R_1^2)\partial_y R_1\|_{L^2}\leqslant\frac{C}{z^{1+\alpha}}\quad k\in\mathbb{N}\tag{4.2.21}$$

$$\|\partial_t V\|_{L^\infty}\leqslant\frac{C}{z^{\frac{1+\alpha}{2}}}.\tag{4.2.22}$$

The next subsections are dedicated to the proof of the theorem on the approximation V. We begin with the expansion of $\mathcal{E}_V$ defined in (4.2.10). Let us first compute the different time derivatives:

$$\partial_t(-R_1)=\dot z_1\partial_y R_1-\dot\mu_1\Lambda R_1\quad\text{and}\quad\partial_t R_2=-\dot z_2\partial_y R_2+\dot\mu_2\Lambda R_2.$$

By the definition of V in (4.2.9), we get the development:

$$\mathcal{E}_V=\sum_{i=1}^{2}(-1)^i\left(\dot\mu_i\Lambda R_i-\dot z_i\partial_y R_i\right)+\sum_{i=1}^{2}(-1)^i\partial_y\left(-|D|^\alpha R_i-R_i+R_i^3\right)\tag{4.2.23}$$

$$+\sum_{i=1}^{2}\partial_y\left(\left(-|D|^\alpha-1+3R_i^2\right)\left((-1)^iP_i\right)\right)+\partial_y\left((-|D|^\alpha-1)(bW)\right)$$

$$+\partial_y(V^3+R_1^3-R_2^3+3R_1^2P_1-3R_2^2P_2)\tag{4.2.24}$$

$$+\frac{d}{dt}(-P_1+P_2+bW).\tag{4.2.25}$$

Notice also the following identities:

$$\partial_y(-|D|^\alpha(-R_1)-(-R_1)-R_1^3)=-\mu_1\partial_y R_1\quad\text{and}\quad\partial_y(-|D|^\alpha R_2-R_2+R_2^3)=\mu_2\partial_y R_2.$$

We extract from (4.2.23) and (4.2.25) the higher orders terms, and for sake of clarity we denote:

$$\forall i\in\{1,2\},\quad\mathscr{T}(i,\beta_0):=\frac{\beta_0}{z^{1+\alpha}}\partial_y\left(\mu_i\Lambda\widetilde{R_i}\right)-\mu_i\partial_y P_i,$$

and the remaining term outside of ∂_y by:

$$T := \sum_{i=1}^{2} b(z)\left(\Lambda\widetilde{R}_i - \Lambda R_i\right) + \beta(\Gamma)\left(-\partial_y R_1 + \partial_y \widetilde{R}_1\right) - \delta(\Gamma)\left(-\partial_y R_2 + \partial_y \widetilde{R}_2\right)$$
$$+ \mathscr{T}(1,\beta_0) - \mathscr{T}(2,\delta_0) + \frac{d}{dt}\left(-P_1 + P_2 + bW\right). \tag{4.2.26}$$

Note that the directions ΛR_i and $\partial_y R_i$ involved in $\overrightarrow{MV}$ are switched by T to $\Lambda\widetilde{R}_i$ and $\partial_y \widetilde{R}_i$.

We continue by decomposing the term (4.2.24). First, from the interaction term $V^3 + R_1^3 - R_2^3$ we extract the higher orders terms, and denote by S_{V^3} the rest:

$$\begin{aligned} S_{V^3} :=& V^3 + R_1^3 - R_2^3 + 3R_1^2 P_1 - 3R_2^2 P_2 - 3R_1^2 b S_0(y - z_1) - 3R_2^2 b(l - S_0(y - z_2)) \\ & - 3R_1^2 R_2 + 3R_1 R_2^2 \\ =& 3R_1^2(P_2 - bS_0(y - z_2)) + 3R_2^2(-P_1 + b(S_0(y - z_1) - l)) - 6R_1R_2(-P_1 + P_2 + bW) \\ & + 3(-R_1 + R_2)(-P_1 + P_2 + bW)^2 + (-P_1 + P_2 + bW)^3. \end{aligned} \tag{4.2.27}$$

The profiles P_i are built to remove the main orders of the interaction terms and some added terms in T. We want to inverse the following terms to get a better approximation of the solution:

$$\begin{aligned} \forall i \neq j \in \{1,2\}, \quad \mathscr{S}(i,j) :=& 3\widetilde{R}_i^2 \widetilde{R}_j - 6\mu_i \widetilde{R}_i \Lambda \widetilde{R}_i P_i + 6\mu_i \widetilde{R}_i \Lambda \widetilde{R}_i \widetilde{R}_j + 3\mu_j \widetilde{R}_i^2 \Lambda \widetilde{R}_j \\ & + \mu_i P_i - \beta_0 \frac{\mu_i}{z^{1+\alpha}} \Lambda \widetilde{R}_i, \end{aligned} \tag{4.2.28}$$

and thus the quantities that we want to be close to 0 are the functions S_1 and S_2, equal to 0 at $+\infty$ and satisfying:

$$\partial_y S_1 := \partial_y\left(\left(-|D|^\alpha - 1 + 3\widetilde{R}_1^2\right)(-P_1 + bS_0(\cdot - z_1)) + \mathscr{S}(1,2)\right) - b(z)\Lambda\widetilde{R}_1 - \beta(z)\partial_y \widetilde{R}_1, \tag{4.2.29}$$
$$\partial_y S_2 := \partial_y\left(\left(-|D|^\alpha - 1 + 3\widetilde{R}_2^2\right)(P_2 + b(l - S_0(\cdot - z_2))) - \mathscr{S}(2,1)\right) - b(z)\Lambda\widetilde{R}_2 + \delta(z)\partial_y \widetilde{R}_2. \tag{4.2.30}$$

By an adequate choice of P_i, b, β and δ, the functions S_1 and S_2 will not have a tail at $-\infty$.

We finally gather the previous approximations and find the remaining terms in ∂_y, by setting:

$$\forall i \neq j \in \{1,2\}, \quad \widetilde{\mathscr{S}}(i,j) := 6\mu_i \widetilde{R}_i \Lambda \widetilde{R}_i P_i + 3R_i^2 R_j - 3\widetilde{R}_i^2 \widetilde{R}_j - 6\mu_i \widetilde{R}_i \Lambda \widetilde{R}_i \widetilde{R}_j - 3\mu_j \widetilde{R}_i^2 \Lambda \widetilde{R}_j,$$

and thus:

$$\widetilde{S} := 3(R_1^2 - \widetilde{R}_1^2)(-P_1 + bS_0(\cdot - z_1)) + \widetilde{\mathscr{S}}(1,2) + 3(R_2^2 - \widetilde{R}_2^2)(P_2 + b(l - S_0(\cdot - z_2))) - \widetilde{\mathscr{S}}(2,1). \tag{4.2.31}$$

We finally get the following decomposition:

$$\mathcal{E}_V = \overrightarrow{m} \cdot \overrightarrow{MV} + \partial_y S + T,$$

where the coefficients $\overrightarrow{m}$ and $\overrightarrow{MV}$ are defined in (4.2.12), T is defined in (4.2.26), and S by

$$S = S_1 + S_2 + S_{V^3} + \widetilde{S}.$$

Let us continue the construction in the next subsection by the choices of P_1 and P_2.

4.2.3 Construction of the profiles

This part is dedicated to the construction of the profiles P_1 and P_2. The goal is to minimise the quantities S_1 and S_2 by exploiting the intrinsic directions of the problems $\partial_y R_1$, $\partial_y R_2$, ΛR_1 and ΛR_2. In particular, the coefficient b defined in (4.2.5) is central in the study of the interaction. The profiles P_i are established term by term in $\mathscr{S}$ in (4.2.28), and using the expansion of the interaction terms, given by Q_{app}.

Due to the definition of $\mathscr{S}$ in (4.2.28), for any $i \neq j \in \{1, 2\}$, we define an approximate value of the function $\mathscr{S}$, where it is located, by:

$$\mathscr{F}(i,j,\beta_0,B_0) := 3\widetilde{R}_i^2 Q_{app}((-1)^j(\cdot - z_i), z) - 6\frac{\mu_i}{z^{1+\alpha}}\widetilde{R}_i\Lambda\widetilde{R}_i B_0(\cdot - z_i) + 6\frac{\mu_i}{z^{1+\alpha}}\widetilde{R}_i\Lambda\widetilde{R}_i a_1$$
$$- 3\frac{\mu_j}{z^{1+\alpha}}\widetilde{R}_i^2\frac{a_1(\alpha+2)}{2(\alpha+1)} + \frac{\mu_i}{z^{1+\alpha}}B_0(\cdot - z_i) - \beta_0\frac{\mu_i}{z^{1+\alpha}}\Lambda\widetilde{R}_i.$$

The definitions of b_1 and b are given in (4.2.5), and S_0 and l respectively in (4.2.6) and (4.2.7).

Proposition 4.2.4. *There exist two constants β_0 and δ_0, two functions $\beta(\Gamma)$ and $\delta(\Gamma)$ in $\mathcal{C}^1(I)$ satisfying (4.2.8), two even functions $B_0, D_0 \in X^\infty(\mathbb{R})$ and two profile functions $P_1(\Gamma, y)$ and $P_2(\Gamma, y)$ in $\mathcal{C}(I, X^\infty(\mathbb{R}))$ satisfying:*

$$\left|P_1(\Gamma, y+z_1) - \frac{\beta_0}{z^{1+\alpha}}B_0(y)\right| + \left|P_2(\Gamma, y+z_2) - \frac{\delta_0}{z^{1+\alpha}}D_0(y)\right| \leqslant \frac{C}{z^{2+\alpha}}\frac{1}{\langle y\rangle^{1+\alpha}},$$
$$\partial_y\left(\left(-|D|^\alpha - 1 + 3\widetilde{R}_1^2\right)(-P_1 + b(z)S_0(\cdot - z_1)) + \mathscr{F}(1,2,\beta_0,B_0)\right) = b(z)\Lambda\widetilde{R}_1 + \beta(\Gamma)\partial_y\widetilde{R}_1, \tag{4.2.32}$$
$$\partial_y\left(\left(-|D|^\alpha - 1 + 3\widetilde{R}_2^2\right)(P_2 + b(z)(l - S_0(\cdot - z_2))) - \mathscr{F}(2,1,\delta_0,D_0)\right) = b(z)\Lambda\widetilde{R}_2 - \delta(\Gamma)\partial_y\widetilde{R}_2, \tag{4.2.33}$$

with the orthogonality conditions:

$$P_1 - b(z)S_0(\cdot - z_1) \perp \widetilde{R}_1, \partial_y\widetilde{R}_1, \quad \textit{and} \quad P_2 + b(z)\left(l - S_0(\cdot - z_1)\right) \perp \widetilde{R}_2, \partial_y\widetilde{R}_2.$$

Moreover, the profiles P_1, P_2 verify:

$$|P_i(\Gamma, y)| + |\partial_y P_i(\Gamma, y)| \leqslant \frac{C}{z^{1+\alpha}}\frac{1}{\langle y - z_i\rangle^{1+\alpha}}, \tag{4.2.34}$$
$$\left|\frac{d}{dt}P_i(\Gamma, y)\right| \leqslant \frac{C}{z^{\frac{3+3\alpha}{2}}}\frac{1}{\langle y - z_i\rangle^{1+\alpha}}, \tag{4.2.35}$$
$$\left|\frac{d}{dt}P_i(\Gamma, y) + \dot{z}_i\partial_y P_i(\Gamma, y)\right| \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}\frac{1}{\langle y - z_i\rangle^{1+\alpha}}. \tag{4.2.36}$$

The ***profiles*** P_1 and P_2 are defined by:

$$P_1(\Gamma(t), y) := \overrightarrow{f}(\Gamma(t)) \cdot \overrightarrow{B}(y - z_1(t)), \quad P_2(\Gamma(t), y) := \overrightarrow{f}(\Gamma(t)) \cdot \overrightarrow{D}(y - z_2(t)),$$

where the functions $\overrightarrow{f}$, $\overrightarrow{B}$ and $\overrightarrow{D}$ are established in the next proposition, and the quantities $\mathscr{F}$ are translated to be centered at 0 and correspond to $\overrightarrow{f} \cdot \overrightarrow{F}$. The proof of Proposition 4.2.4 is postponed after the proof of the next proposition.

Proposition 4.2.5. *Let us define the vector functions:*

$$\overrightarrow{f}(\Gamma) := \left(\frac{1}{z^{1+\alpha}}, \frac{1}{z^{2+\alpha}}, \frac{\mu_1}{z^{1+\alpha}}, \frac{\mu_2}{z^{1+\alpha}}, \frac{1}{z^{2\alpha+1}}, \frac{1}{z^{3+\alpha}}\right),$$

and for all $i \in \{1,2\}$*:*

$$\overrightarrow{F}(i,\beta_0,B_0) := \big(3Q^2a_1, \quad 3Q^2a_1(-1)^i(\alpha+1)y, \quad -6Q\Lambda QB_0 + 6Q\Lambda Qa_0 + B_0 - \beta_0\Lambda Q, \\ -3a_1\frac{\alpha+2}{\alpha+1}Q^2, \quad 3Q^2a_2, \quad 3Q^2\left(a_1\frac{(\alpha+1)(\alpha+2)}{2}y^2 + a_3\right)\Big).$$

There exist unique $\beta_0 \in \mathbb{R}$*,* $\beta(\Gamma) \in C^1(I)$ *satisfying* (4.2.8)*,* B_0, B_1, B_2, B_3, B_4 *and* $B_5 \in X^\infty(\mathbb{R})$*, with* B_0 *an even function, and* $B_0, B_1 + b_1S_0, B_2, B_3, B_4, B_5 \perp Q, Q'$ *such that, :*

$$\partial_y\left(L\left(\overrightarrow{f}(\Gamma)\cdot\overrightarrow{B} - b(z)S_0\right) + \overrightarrow{f}(\Gamma)\cdot\overrightarrow{F}(1,\beta_0,B_0)\right) = b(z)\Lambda Q + \beta(\Gamma)Q, \tag{4.2.37}$$

with $\overrightarrow{B} := (B_0, B_1, B_2, B_3, B_4, B_5)$*.*

Similarly, there exist unique $\delta_0 \in \mathbb{R}$*,* $\delta(\Gamma) \in \mathcal{C}^1(I)$*, satisfying* (4.2.8)*,* D_0, D_1, D_2, D_3, D_4 *and* $D_5 \in X^\infty(\mathbb{R})$ *with* D_0 *an even function,* $D_0, D_1 + b_1(l - S_0), D_2, D_3, D_4, D_5 \perp Q, Q'$ *such that:*

$$\partial_y\left(L\left(\overrightarrow{f}(\Gamma)\cdot\overrightarrow{D} + b(z)(l-S_0)\right) + \overrightarrow{f}(\Gamma)\cdot\overrightarrow{F}(2,\delta_0,D_0)\right) = -b(z)\Lambda Q + \delta(\Gamma)Q', \tag{4.2.38}$$

with $\overrightarrow{D} := (D_0, D_1, D_2, D_3, D_4, D_5)$*.*

Notice that in the previous decomposition, the tail of the profile of the first solitary wave, given by B_1, has an influence on the profiles around the second solitary wave, on D_1. However, this tail does not change the coefficient $-b(z)\Lambda Q$, which is of great importance in the system of ODEs ruling the equations of μ and of z.

To prove Proposition 4.2.5, we need Lemma 4.1.10 and 4.1.11 to find the adequate profiles.

Proof. We define from Lemma 4.1.10 the unique function $B_0 \in X^\infty(\mathbb{R})$ and the unique coefficient $\beta_0 \in \mathbb{R}$ satisfying:

$$\begin{cases} LB_0(y) = -3a_1Q^2(y) + \beta_0Q(y), \\ B_0 \perp Q, \quad B_0 \perp Q'. \end{cases} \tag{4.2.39}$$

Notice that since L keeps stable the parity of the functions, B_0 is an even function.

For the second term, we use Lemma 4.1.11 by defining the function B_1, and the coefficients β_1 and b_1 as the unique solution of the following problem:

$$\begin{cases} \partial_yL(B_1(y) - b_1S_0(y)) = \partial_y\left(3(\alpha+1)a_1yQ^2(y)\right) + \beta_1Q'(y) + b_1\Lambda Q, \\ B_1 - b_1S_0 \perp Q, \quad B_1 - b_1S_0 \perp Q'. \end{cases}$$

Notice in particular that b_1 is defined by the formula (4.1.11):

$$b_1 = -\frac{2(\alpha+1)^2a_1\|Q\|_{L^3}^3}{(\alpha-1)\|Q\|_{L^2}^2} < 0, \tag{4.2.40}$$

since the sign of $a_1 > 0$ is given in Lemma 4.1.4. This justifies the choice of definition of $b(z) := \frac{b_1}{z^{2+\alpha}}$, as stated in (4.2.5).

The third, fourth, fifth and sixth terms are defined as for B_0 and β_0. With Lemma 4.1.10, we define B_2, B_3, B_4, B_5 in $X^\infty(\mathbb{R})$, and the coefficients $\beta_2, \beta_3, \beta_4$ and β_5 as the solutions of the following problems:

$$\begin{cases} LB_2(y) = -6Q\Lambda QB_0 - 6Q\Lambda Qa_1 - B_0 + \beta_0\Lambda Q + \beta_2Q(y) \\ B_2 \perp Q, \quad B_2 \perp Q' \end{cases}, \quad \begin{cases} LB_3(y) = 3a_1\frac{\alpha+2}{\alpha+1}Q^2 + \beta_3Q(y) \\ B_3 \perp Q, \quad B_3 \perp Q' \end{cases}$$

and

$$\begin{cases} LB_4(y) = -3a_2Q^2(y) + \beta_4 Q(y) \\ B_4 \perp Q, \quad B_4 \perp Q'. \end{cases}, \begin{cases} LB_5(y) = -3\left(a_1 \dfrac{(\alpha+1)(\alpha+2)}{2} y^2 + a_3\right) Q^2(y) + \beta_5 Q(y) \\ B_5 \perp Q, \quad B_5 \perp Q'. \end{cases}$$

Therefore, we set:

$$\beta(\Gamma) = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5) \cdot \overrightarrow{f}(\Gamma).$$

Now, we continue with the construction of $\overrightarrow{D}$. Since the first, forth, fifth and sixth coordinates in $\overrightarrow{F}(2, \delta_0, D_0)$ are respectively equal to the first, forth, fifth and sixth terms in $\overrightarrow{F}(1, \beta_0, D_0)$, the functions D_0, D_3, D_4, D_5 will solve respectively the same problem as B_0, B_3, B_4 and B_5. Then, we take:

$$D_0 = B_0, \quad D_3 = B_3, \quad D_4 = B_4, \quad D_5 = B_5,$$

and

$$\delta_0 = \beta_0, \quad \delta_3 = \beta_3, \quad \delta_4 = \beta_4, \quad \delta_5 = \beta_5.$$

The situation is similar for $D_2 = B_2$ and for $\beta_2 = \delta_2$. To construct D_1, as for the function B_1, we use Lemma 4.1.11. Since $z^{2+\alpha}b(z) = b_1$, there exist a unique function $D_1 \in X^\infty(\mathbb{R})$ and coefficients $\delta_1, d_1 \in \mathbb{R}$ such that:

$$\begin{cases} \partial_y L\left(D_1(y) + d_1(l - S_0(y))\right) = \partial_y\left(-(\alpha+1)a_1 y 3Q^2(y)\right) + \delta_1 Q'(y) + d_1 \Lambda Q(y), \\ D_1 + d_1(l - S_0) \perp Q, \quad D_1 + d_1(l - S_0) \perp Q'. \end{cases}$$

Moreover, Q^2 is orthogonal to Q'. Therefore by the formula (4.1.11), we obtain that:

$$d_1 = -b_1.$$

Thus, we conclude the proof of Proposition 4.2.5 by defining:

$$\delta(\Gamma) := (\delta_0, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5) \cdot \overrightarrow{f}(\Gamma).$$

□

Proof of Proposition 4.2.4. The two identities (4.2.32) and (4.2.33) are deduced from the one of $\overrightarrow{B}$ and $\overrightarrow{D}$ in (4.2.37) and (4.2.38), as well as the orthogonality conditions.

We continue with the estimate (4.2.34) and (4.2.35). First, we deal with the term $\partial_y B_0$. From (4.2.39), we deduce that:

$$\partial_y B_0 = (|D|^\alpha + 1)^{-1} \partial_y \left(3Q^2 B_0 + 3a_1 Q^2 + \beta_0 Q\right).$$

Since $B_0 \in X^{2+\alpha}(\mathbb{R})$, we have that $\partial_y B_0 \in L^\infty(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$. Then, by Lemma 4.1.5, we obtain that $\partial_y B_0 \in X^{2+\alpha}(\mathbb{R})$. By a similar argument on B_1, B_2, B_3, B_4 and B_5 with (4.2.3), we conclude that:

$$|\partial_y P_1(\Gamma, y)| = |\overrightarrow{f}(\Gamma) \cdot (\partial_y \overrightarrow{B})(y - z_1)| \leqslant \frac{C}{z^{1+\alpha}} \frac{1}{\langle y - z_1 \rangle^{1+\alpha}}.$$

The same estimate holds for P_2.

Now, we estimate $\partial_t P_i$ for $i \in \{1, 2\}$. Note that the profiles $P_1(\Gamma)$ and $P_2(\Gamma)$ are $C^1(I)$, since $\Gamma \in C^1(I)$. By direct computation, we obtain that:

$$\frac{d}{dt} P_1(\Gamma(t), y) = \left(\frac{d}{dt}\Gamma(t) \cdot \nabla_\Gamma\right) \overrightarrow{f}(\Gamma(t)) \cdot \overrightarrow{B}(y - z_1(t)) - \dot{z}_1(t) \partial_y P_1(\Gamma(t), y).$$

By Proposition 4.2.5, we have that $B_j \in X^\infty$ for $j \in \{0, \cdots, 5\}$. Therefore, we deduce with (4.2.3) and (4.2.4), that:

$$\left|\left(\dot{\Gamma} \cdot \nabla_\Gamma\right) \overrightarrow{f}(\Gamma) \cdot \overrightarrow{B}(y - z_1)\right| \leqslant C\left(\frac{|\dot{z}|}{z^{2+\alpha}} + \frac{|\dot{\mu}_1| + |\dot{\mu}_2|}{z^{1+\alpha}}\right) \frac{1}{\langle y - z_1\rangle^{1+\alpha}} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}} \frac{1}{\langle y - z_1\rangle^{1+\alpha}}.$$

We conclude that:

$$\left|\frac{d}{dt} P_1(\Gamma) + \dot{z}_1 \partial_y P_1(\Gamma)\right| \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}} \langle y - z_1\rangle^{1+\alpha}}.$$

The same arguments hold to estimate the profile P_2. This finishes the proof of Proposition 4.2.4. □

4.2.4 Proof of Proposition 4.2.3 and Theorem 4.2.2

Once the construction of the profiles is finished we continue with the estimates of the different terms involved in the error.

Proof of Proposition 4.2.3. To obtain (4.2.16), we have the decomposition on $\partial_y R_1$:

$$\|\partial_y R_1 \varphi\|_{L^\infty(\{y \leqslant \frac{z_1}{2}\})} \leqslant \left\|\frac{C}{\langle y - z_1\rangle^{1+\alpha}} \frac{1}{\langle y\rangle^\alpha}\right\|_{L^\infty} \leqslant \frac{C}{z^\alpha},$$
$$\|\partial_y R_1 \varphi\|_{L^\infty(\{y \leqslant \frac{z_1}{2}\})} \leqslant \|\partial_y R_1\|_{L^\infty(\{y \leqslant \frac{z_1}{2}\})} \leqslant \frac{C}{z^{1+\alpha}}.$$

The same estimate holds for R_2. Applying the same argument for the H^1-norm, we deduce (4.2.16). We can replace $\partial_y R_i$ by ΛR_i in the former estimates and we get (4.2.17).

The estimate (4.2.18) and (4.2.19) are direct consequences of Proposition 4.2.4 and the definition of b.

By Proposition 4.2.5 the profiles P_i and $\partial_y P_i$ for $i = 1, 2$ belong to $L^\infty(\mathbb{R})$. Moreover, by definition, W and $\partial_y W$ are also in $L^\infty(\mathbb{R})$. Then we deduce (4.2.20).

By Proposition 4.2.5 for the profiles, and since $b(z) = \dfrac{b_1}{z^{\alpha+2}}$, we deduce that :

$$\|P_1\|_{L^\infty} + \|P_2\|_{L^\infty} + \|bW\|_{L^\infty} \leqslant \frac{C}{z^{1+\alpha}}.$$

Furthermore, using $\Omega_i = \{y \in \mathbb{R} : y \leqslant \dfrac{z_i}{2}\}$, we get for $i = 1, 2$ that:

$$\|R_i^k \Phi^2\|_{L^\infty} \leqslant \|R_i^k \Phi^2\|_{L^\infty(\Omega_i)} + \|R_i^k \Phi^2\|_{L^\infty(\Omega_i^c)} \leqslant \frac{C}{z^{1+\alpha}}.$$

Gathering these estimates, we conclude the first part of (4.2.21). Concerning the second term:

$$\begin{aligned}&\|(V^2 - R_1^2)\partial_y R_1\|_{L^2}\\ &\quad \leqslant \|2R_1(R_2 - P_1 + P_2 + bW)\partial_y R_1\|_{L^2} + \|R_2^2 \partial_y R_1\|_{L^2} + C\| - P_1 + P_2 + bW\|_{L^\infty}^2 \leqslant \frac{C}{z^{1+\alpha}}.\end{aligned}$$

By differentiating V and using Proposition 4.2.4, therefore we obtain the estimate (4.2.22). □

Proof of Theorem 4.2.2. We continue with the inequalities (4.2.13), (4.2.14) and (4.2.15).

We first begin with the estimate on the L^2-norm of the term $S = S_{V^3} + S_1 + S_2 + \widetilde{S}$ with $S_{V^3}, S_1, S_2, \widetilde{S}$ are respectively defined in (4.2.27), (4.2.29), (4.2.30) and (4.2.31).

We begin with S_{V^3}, by decomposing the different terms. We have, using the decomposition of Proposition 4.1.7:

$$\left\|\left(R_1^2 - \widetilde{R}_1^2\right) P_2\right\|_{L^2} \leqslant C|\mu_1| \left\|\frac{1}{\langle y - z_1\rangle^{1+\alpha}} P_2\right\|_{L^2}.$$

Let $\Omega := \left\{ y \leqslant \frac{z_1+z_2}{2} \right\}$. By (4.2.34) and (4.2.3), we obtain that:

$$\begin{aligned} &\left\| \left(R_1^2 - \widetilde{R}_1^2 \right) P_2 \right\|_{L^2} \\ &\qquad \leqslant C \frac{|\mu_1|}{z^{1+\alpha}} \left(\left\| \frac{1}{\langle y - z_1 \rangle^{1+\alpha}} \frac{1}{\langle y - z_2 \rangle^{1+\alpha}} \right\|_{L^2(\Omega)} + \left\| \frac{1}{\langle y - z_1 \rangle^{1+\alpha}} \frac{1}{\langle y - z_2 \rangle^{1+\alpha}} \right\|_{L^2(\Omega^C)} \right) \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}. \end{aligned}$$

By similar computations, we have that:

$$\left\| R_1^2 P_2 \right\|_{L^2} \leqslant \left\| \widetilde{R}_1^2 P_2 \right\|_{L^2} + \left\| \left(R_1^2 - \widetilde{R}_1^2 \right) P_2 \right\|_{L^2} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}.$$

Similarly:

$$\left\| 3R_1^2(-bS_0(\cdot - z_2)) + 3R_2^2 \left(-P_1 + b(S_0(y - z_1) - l) \right) \right\|_{L^2} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}.$$

For the third and forth terms of S_{V^3}, by (4.2.18), we have:

$$\| R_1 R_2 (-P_1 + P_2 + bW) \|_{L^2} \leqslant \| R_1 R_2 \|_{L^2} \| -P_1 + P_2 + bW \|_{L^\infty} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}$$

and

$$\left\| (-R_1 + R_2) \left(-P_1 + P_2 + bW \right)^2 \right\|_{L^2} \leqslant \| -R_1 + R_2 \|_{L^2} \| -P_1 + P_2 + bW \|_{L^\infty}^2 \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}.$$

Finally, we compute the L^2-norm of the bump function W:

$$\| bW \|_{L^2} \leqslant \frac{C}{z^{2+\alpha}} \sqrt{z} = \frac{C}{z^{\frac{3}{2}+\alpha}},$$

and therefore:

$$\left\| (-P_1 + P_2 + bW)^3 \right\|_{L^2} \leqslant \| (-P_1 + P_2 + bW) \|_{L^\infty}^2 \, \| (-P_1 + P_2 + bW) \|_{L^2} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}.$$

With the previous computations, we conclude that :

$$\| S_{V^3} \|_{L^2} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}.$$

We continue with S_1. Notice that by definition of P_i, another formulation of S_1 and S_2 is available:

$$\begin{aligned} S_1 &= \mathscr{S}(1,2) - \mathscr{F}(1,2,\beta_0,B_0) \\ S_2 &= -\mathscr{S}(2,1) + \mathscr{F}(2,1,\delta_0,D_0). \end{aligned}$$

We focus on S_1, the computations are similar for S_2. We separate each term of $\mathscr{S}(1,2) - \mathscr{F}(1,2,\beta_0,B_0)$. First we look at $\left\| 3\widetilde{R}_1^2(\widetilde{R}_2 - Q_{app}(\cdot - z_2, z)) \right\|_{L^2}$. The approximation of $Q(\cdot + z)$ by $Q_{app}(\cdot, z)$ in (4.1.6) holds on a certain region, thus we begin with $\left\{ y \in \mathbb{R}; |y - z_1| \leqslant \frac{z}{2} \right\}$. In this region, we have:

$$\begin{aligned} \left\| 3\widetilde{R}_1^2(\widetilde{R}_2 - Q_{app}(\cdot - z_1, z)) \right\|_{L^2(|y - z_1| \leqslant \frac{z}{2})} &\leqslant C \left\| 3\widetilde{R}_1^2 \left(\frac{1}{z^{3\alpha+1}} + \frac{\langle y - z_1 \rangle}{z^{2\alpha+2}} + \frac{\langle y - z_1 \rangle^3}{z^{\alpha+4}} \right) \right\|_{L^2} \\ &\leqslant \frac{C}{z^{2\alpha+2}} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}. \end{aligned}$$

In the other part, we get:

$$\left\|3\widetilde{R}_1^2(\widetilde{R}_2 - Q_{app}(\cdot - z_1, z))\right\|_{L^2(|y-z_1|\geqslant\frac{z}{2})} \leqslant \left\|3\widetilde{R}_1^2\widetilde{R}_2\right\|_{L^2(|y-z_1|\geqslant\frac{z}{2})} + \left\|3\widetilde{R}_1^2 Q_{app}(\cdot - z_2, z)\right\|_{L^2(|y-z_1|\geqslant\frac{z}{2})}.$$

The first term on the right hand side of the former estimate is bounded by:

$$\left\|3\widetilde{R}_1^2\widetilde{R}_2\right\|_{L^2(|y-z_1|\geqslant\frac{z}{2})} \leqslant C\|\widetilde{R}_1^2\|_{L^\infty(|y-z_1|\geqslant\frac{z}{2})}\|\widetilde{R}_2\|_{L^2} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}.$$

We estimate the second term on the right hand side by:

$$\begin{aligned}&\left\|\widetilde{R}_1^2 Q_{app}(\cdot - z_1, z)\right\|_{L^2(|y-z_1|\geqslant\frac{z}{2})}\\ &\quad\leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}} + \frac{C}{z^{2+\alpha}}\left\|\widetilde{R}_1^2(y-z_1)\right\|_{L^2(|y-z_1|\geqslant\frac{z}{2})} + \frac{C}{z^{3+\alpha}}\left\|\widetilde{R}_1^2(y-z_1)^3\right\|_{L^2(|y-z_1|\geqslant\frac{z}{2})} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}.\end{aligned}$$

Thus we conclude:

$$\left\|3\widetilde{R}_1^2(\widetilde{R}_2 - Q_{app}(\cdot - z_1, z))\right\|_{L^2} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}.$$

The estimates on the other terms of S_1 are obtained by similar computations:

$$\begin{aligned}&\left\|6\mu_1\widetilde{R_1}\Lambda\widetilde{R}_1\left(P_1 - \frac{B_0(\cdot - z_1)}{z^{1+\alpha}}\right)\right\|_{L^2} + \left\|6\mu_1\widetilde{R_1}\Lambda\widetilde{R}_1\left(\widetilde{R}_2 - \frac{a_0}{z^{1+\alpha}}\right)\right\|_{L^2}\\ &\quad+\left\|3\mu_2\widetilde{R}_1^2\left(\Lambda\widetilde{R}_2 + \frac{1}{z^{1+\alpha}}\frac{a_0(\alpha+2)}{2(\alpha+1)}\right)\right\|_{L^2} + \left\|\mu_1\left(P_1 - \frac{B_0(\cdot - z_1)}{z^{1+\alpha}}\right)\right\|_{L^2} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}\end{aligned}$$

then we conclude:

$$\|S_1\|_{L^2} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}.$$

To finish the proof on S, we have to estimate $\widetilde{S}$. We focus on the first part of $\widetilde{S}$, which contains $\widetilde{\mathscr{S}}(1,2)$:

$$3\left(R_1^2 - \widetilde{R}_1^2\right)(-P_1 + bS_0(y-z_1)) + 6\mu_1\widetilde{R}_1\Lambda\widetilde{R}_1 P_1 + 3R_1^2R_2 - 3\widetilde{R}_1^2\widetilde{R}_2 - 6\mu_1\widetilde{R}_1\Lambda\widetilde{R}_1\widetilde{R}_2 - 3\mu_2\widetilde{R}_1^2\Lambda\widetilde{R}_2$$

since the computations are similar for the other part. By using Proposition 4.1.7, (4.2.3) and (4.2.5) we deduce:

$$\|3(R_1^2 - \widetilde{R}_1^2)bS_0(y-z_1)\|_{L^2} + \|3(R_1^2 - \widetilde{R}_1^2 - 2\mu_1\widetilde{R}_1\Lambda\widetilde{R}_1)P_1\|_{L^2} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}.$$

To estimate the next terms in $\widetilde{S}$, we remark:

$$R_1^2R_2 - \widetilde{R}_1^2\widetilde{R}_2 = R_1^2(R_2 - \widetilde{R}_2) + \widetilde{R}_2(R_1^2 - \widetilde{R}_1^2).$$

From Proposition 4.1.7 and (4.2.3), we obtain that:

$$\begin{aligned}\|3R_1^2(R_2 - \widetilde{R}_2) - 3\mu_2\Lambda\widetilde{R}_2\widetilde{R}_1^2\|_{L^2} \leqslant&\|3R_1^2(R_2 - \widetilde{R}_2) - 3\mu_2\Lambda\widetilde{R}_2R_1^2\|_{L^2} + \|3\mu_2\Lambda\widetilde{R}_2\left(R_1^2 - \widetilde{R}_1^2\right)\|_{L^2})\\ \leqslant& C\left(\frac{\mu_2^2}{z^{1+\alpha}} + \frac{|\mu_1||\mu_2|}{z^{1+\alpha}}\right) \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}.\end{aligned}$$

Arguing similarly, we obtain:

$$\|3\widetilde{R}_2(R_1^2 - \widetilde{R}_1^2) - 6\mu_1\widetilde{R}_1\Lambda\widetilde{R}_1\widetilde{R}_2\|_{L^2} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}.$$

This concludes the estimate on $\widetilde{S}$.

We continue with the estimate on T. We decompose each term of its definition in (4.2.26). First, we have with (4.1.7) and (4.2.3) :

$$\left\|b(z)\left(\Lambda R_1 - \Lambda\widetilde{R}_1\right)\right\|_{L^2} \leqslant C\frac{|\mu_1|}{z^{2+\alpha}}\left\|\frac{1}{\langle x-z_1\rangle^{\alpha+1}}\right\|_{L^2} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}.$$

Second, we use the inequality (4.2.8) as used in Proposition 4.2.4, and from the asymptotic development of $\partial_y Q$, by (4.1.8) and (4.2.4):

$$\begin{aligned}
&\left\|\beta(\Gamma)\left(-\partial_y R_1 + \partial_y\widetilde{R}_1\right) + \frac{\beta_0}{z^{1+\alpha}}\partial_y(\mu_i\Lambda\widetilde{R}_i)\right\|_{L^2} \\
&\leqslant \left\|\left(\beta(\Gamma) - \frac{\beta_0}{z^{1+\alpha}}\right)\partial_y(-R_1+\widetilde{R}_1)\right\|_{L^2} + \left\|\frac{\beta_0}{z^{1+\alpha}}\partial_y\left(-R_1+\widetilde{R}_1+\mu_1\Lambda\widetilde{R_1}\right)\right\|_{L^2} \\
&\leqslant C\frac{|\mu_1|}{z^{2+\alpha}} + C\frac{\mu_1^2}{z^{1+\alpha}} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}.
\end{aligned}$$

Then, we consider the case of the time derivative on $-P_1$. We have, by (4.2.36):

$$\left\|\frac{d}{dt}(-P_1) - \mu_1\partial_y P_i\right\|_{L^2} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}} + \frac{C}{z^{1+\alpha}}\left|\dot{z}_1 - \mu_1\right|$$

We continue with the the term $\frac{d}{dt}W$:

$$\frac{d}{dt}W(\Gamma(t)) = (|D|^\alpha+1)^{-1}\left(\dot{z}_1(t)\Lambda\widetilde{R}_1 - \dot{z}_2\Lambda\widetilde{R}_2\right), \tag{4.2.41}$$

which with (4.2.3) and (4.2.41) give:

$$\begin{aligned}
\left\|\frac{d}{dt}\left(b(z(t))W(\Gamma(t))\right)\right\|_{L^2} &\leqslant C\frac{|\dot{z}_1|+|\dot{z}_2|}{z^{2+\alpha}} + C\frac{|\dot{z}|}{z^{3+\alpha}}\|W\|_{L^2} \\
&\leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}} + C\frac{\sqrt{z}}{z^{\frac{7+3\alpha}{2}}} \leqslant \frac{C}{z^{\frac{5+3\alpha}{2}}}.
\end{aligned}$$

Those previous estimates conclude the bound (4.2.15) on T.

Since all the estimates have been established in L^2, we need to continue with the first derivative to establish the bound in H^1. We can notice that all the estimates are based on two main arguments:

- An argument of localisation : if two functions are located at a distance z large, and if the two functions have an explicit decay at infinity, then the product of the two functions can be quantified in terms of z. The spatial derivative either leaves unchanged the decay property in terms of z of this product or improves it.
- An argument of smallness of the objects: the objects already have a quantified bound in terms of z, see for example the L^∞-norm of P_i in (4.2.34).

Therefore the computations made on the L^2-norm are similar to those on the H^1-norm.

Concerning the time derivative of S in (4.2.14), let us deal with a generic example of a function $\frac{1}{z(t)^{1+\alpha}}g_{1+\mu(t)}(y-z(t))$, since all the involved functions, except W, are on this form. Either the time derivative applies to $\frac{1}{z(t)^{1+\alpha}}$, or to the scaling parameter $1+\mu(t)$ of the function g or to the translation parameter $-z(t)$. However, we get in each case either $\dot{\mu}(t)$ or $\dot{z}(t)$, which by (4.2.3) and (4.2.4) are bounded by $z^{-\frac{1+\alpha}{2}}$. Notice also that the time derivative of the considered functions leaves unchanged or improves the space decay at infinity, and from the remark on the space derivative above, the bound in z still holds. The time derivative of W has been developed in (4.2.41), and $\partial_t W$ fits in the previous discussion. As a result, the estimate on $\|\partial_t S\|_{L^2}$ is reduced to the product of two terms: one whose bound is the one of $\|S\|_{L^2}$, and one bounded by $z^{-\frac{1+\alpha}{2}}$. □

4.3 Modulation

The previous section was dedicated to the expected approximate solution. Here, we prove that if a solution is close to the approximation V, for two solitary waves far enough one to each other, then the solution stays close to this approximation on a certain time interval. Furthermore, we can impose some orthogonality conditions to the error between the solution and the approximation.

Let us define some conditions (Cond_Z) on a vector $\Gamma = (z_1, z_2, \mu_1, \mu_2) \in \mathbb{R}^4$ dependent on a parameter Z:

$$z_1 > \frac{Z}{4}, \quad z_2 < -\frac{Z}{4}, \quad 0 < -\mu_1 < \frac{1}{Z}, \quad \text{and} \quad 0 < \mu_2 < \frac{1}{Z}, \tag{Cond_Z}$$

and the tube:

$$\mathcal{U}(Z,\nu) := \left\{ u \in H^{\frac{\alpha}{2}}(\mathbb{R}); \inf_{\Gamma \text{ satisfying } (\mathrm{Cond}_Z)} \|u - V(\Gamma)\|_{H^{\frac{\alpha}{2}}} \leqslant \nu \right\}.$$

We also shorten the notations by:

$$R_i(y) = R_i(\Gamma, y) := Q_{1+\mu_i}(y - z_i). \tag{4.3.1}$$

This proposition is time-dependent, and can be found, for example, in [51, 130].

Proposition 4.3.1. *There exist $Z^* > 0$, $\nu^* > 0$ and a constant $K^* > 0$ such that the following holds. Let v be a solution of* (4.0.5) *in $\mathcal{C}(\mathbb{R}, H^{\frac{\alpha}{2}})$. Let us define a time interval I. If for $Z > 2Z^*$ and $\nu \in (0, \frac{\nu^*}{2})$, we have :*

$$\sup_{t\in I} \left(\inf_{\Gamma \text{ satisfying } (\mathrm{Cond}_Z)} \|v(t,\cdot) - V(\Gamma,\cdot)\|_{H^{\frac{\alpha}{2}}} \right) < \nu,$$

then there exists a unique $\mathcal{C}^1$-function $\Gamma : I \to \mathbb{R}^4$ such that:

$$\varepsilon(t,\cdot) := v(t,\cdot) - V(\Gamma(t),\cdot)$$

satisfies for any $i \in \{1,2\}$ and for any $t \in I$:

$$\varepsilon(t,\cdot) \perp R_i(t,\cdot) \quad \text{and} \quad \varepsilon(t,,\cdot) \perp \partial_y R_i(t,\cdot). \tag{4.3.2}$$

Moreover, for any $t \in I$:

$$\|\varepsilon(t,\cdot)\|_{H^{\frac{\alpha}{2}}} + |\mu_1(t)| + |\mu_2(t)| \leqslant K^*\nu, \tag{4.3.3}$$

$$|\dot z_1(t)| + |\dot z_2(t)| + |\dot\mu_1(t)| + |\dot\mu_2(t)| \leqslant K^*, \tag{4.3.4}$$

$$z_1(t) > \frac{Z}{8}, \quad z_2(t) \leqslant -\frac{Z}{8}. \tag{4.3.5}$$

Proof. We give here some insights of the proof. The proof is composed of two steps. The first part involves a qualitative version of the implicit function theorem, see section 2.2 in [32], to obtain the existence of the continuous function Γ. To this end, we study the functional

$$\begin{aligned} g : \mathcal{U}(Z,\nu) \times \mathbb{R}_+^* \times \mathbb{R}_-^* \times \mathbb{R} \times \mathbb{R} \quad &\longrightarrow \qquad\qquad \mathbb{R}^4 \\ (w, z_1, z_2, \mu_1, \mu_2) \qquad &\longmapsto \begin{pmatrix} \int (w - V(\Gamma))R_1, & \int (w - V(\Gamma))\partial_y R_1 \\ \int (w - V(\Gamma))R_2, & \int (w - V(\Gamma))\partial_y R_2 \end{pmatrix}, \end{aligned}$$

at the point $(V(\widetilde\Gamma), \widetilde\Gamma)$ with V defined in (4.2.9) and $\widetilde\Gamma$ satisfying (Cond_Z). Note that the estimates obtained on g and $d_\Gamma g$ used to verify the implicit function theorem, are uniform in $\widetilde\Gamma$ satisfying (Cond_Z), for $Z > 2Z^*$ with Z^* large enough, and $\nu < \frac{\nu^*}{2}$ with ν^* small enough. In other words, for all $\widetilde\Gamma$, the function Γ associated with $\widetilde\Gamma$ given by the implicit function theorem is defined on a ball $B(V(\widetilde\Gamma), \nu)$, with ν independent of the point $V(\widetilde{\Gamma})$. Since ν is chosen independently of $\widetilde{\Gamma}$ satisfying (Cond$_Z$), we can extend by uniqueness the parameters to the whole tube $\mathcal{U}(Z,\nu)$. Therefore, we get $\Gamma \in C^1(\mathcal{U}(Z,\nu))$.

However, the solution u of (4.0.5) is only continuous, then we obtain that the function $\Gamma(t) := \Gamma(v(t,\cdot))$ is only continuous. To get more regularity, we use the Cauchy-Lipischtz theorem. By differentiating the orthogonality condition, we have that the parameters verify an ODE system. By using the Cauchy-Lipischtz theorem, we obtain the regularity of the parameters even though u is only continuous.

$\square$

Remark 4.3.2. The parameters z_1, z_2, μ_1, μ_2 defined in Proposition 4.3.1, verify an ODE system which is globally Lipschitz. In other words, the function Γ is well-defined and $C^1(\mathbb{R})$. However, the conclusion of the Proposition 4.3.1 are only verified for $t \in I$.

4.4 Proof of the Theorem 4.0.3

4.4.1 Bootstrap setting

Let $(S_n)_{n=0}^{+\infty}$ be a increasing sequence of times going to infinity, with $S_n > T_0$, for $T_0 > 1$ large enough to be chosen later. Recall that V is defined in (4.2.9). For all $n \in \mathbb{N}$, we define u_n as being the solution of (4.0.5) verifying

$$v_n(S_n,\cdot) = V(\Gamma_n^{in},\cdot), \tag{4.4.1}$$

with

$$\begin{aligned}&\Gamma_n^{in} := (z_{1,n}^{in}, z_{2,n}^{in}, \mu_{1,n}^{in}, \mu_{2,n}^{in}),\\ &z_{1,n}^{in} = -z_{2,n}^{in} := \frac{z_n^{in}}{2}, \quad \mu_{1,n}^{in} = -\mu_{2,n}^{in} := \frac{\mu_n^{in}}{2}, \quad \mu_n^{in} := \sqrt{\frac{-4b_1}{\alpha+1}}\left(z_n^{in}\right)^{-\frac{\alpha+1}{2}}, \qquad (4.4.2)\\ \left(z_n^{in}\right)^{\frac{\alpha+3}{2}} &\in [a^{\frac{\alpha+3}{2}}S_n - S_n^{\frac{1}{2}+r}, a^{\frac{\alpha+3}{2}}S_n + S_n^{\frac{1}{2}+r}], \qquad (4.4.3)\end{aligned}$$

with b_1 defined in (4.2.40), $a = \left(\frac{\alpha+3}{2}\sqrt{\frac{-4b_1}{\alpha+1}}\right)^{\frac{2}{\alpha+3}}$ and $r = \frac{\alpha-1}{4(\alpha+3)}$. The constant z_n^{in} will be fixed later.

By choosing T_0 large enough and $C_0 = 2\sqrt{\frac{-4b_1}{\alpha+1}}$, we can suppose that (4.2.2)-(4.2.4) and (Cond$_Z$) are satisfied by Γ_n^{in} for any $n \in \mathbb{N}$. By (4.4.1), $v_n(S_n) \in \mathcal{U}(Z,\nu)$ and $V(\Gamma_n^{in})$ satisfies the assumption of theorem 4.2.2. By continuity of v_n (see Corollary 4.5.2), on an open time interval $I_n \ni S_n$, $\{v_n(t); t \in I_n\}$ is in $\mathcal{U}(Z,\nu)$. By applying Proposition 4.3.1, we define a unique function $\Gamma_n = (z_{1,n}, z_{2,n}, \mu_{1,n}, \mu_{2,n},)$ on I_n such that the conditions (4.3.2), (4.3.3) and (4.3.5) are satisfied and $\Gamma_n(S_n) = \Gamma_n^{in}$ by construction. Γ_n also satisfies (4.2.2)-(4.2.4), which justifies the setting of Theorem 4.2.2.

By sake of clarity, we drop the index n, and denote v, Γ, z_1, z_2, μ_1, μ_2 instead of v_n, Γ_n, $z_{1,n}$, $z_{2,n}$, $\mu_{1,n}$, $\mu_{2,n}$ for the subsections 4.4.2 and 4.4.3.

As in Section 4.2, we denote:

$$z := z_1 - z_2, \quad \mu := \mu_1 - \mu_2, \quad \bar{z} := z_1 + z_2, \quad \bar{\mu} := \mu_1 + \mu_2 \quad \text{and} \quad \varepsilon := v - V(\Gamma). \tag{4.4.4}$$

We introduce the bootstrap estimates

$$\begin{cases} \|\varepsilon(t)\|_{H^{\frac{\alpha}{2}}}^2 \leqslant t^{-\frac{3\alpha+5}{\alpha+3}}, \\ |z^{\frac{\alpha+3}{2}}(t) - a^{\frac{\alpha+3}{2}}t| \leqslant t^{\frac{1}{2}+r}, \\ \left|\mu(t) - \sqrt{\frac{-4b_1}{\alpha+1}}\frac{t^{-\frac{\alpha+1}{\alpha+3}}}{a^{\frac{\alpha+1}{2}}}\right| \leqslant C^* t^{-\frac{5\alpha+11}{4(\alpha+3)}}, \\ |\bar{z}(t)| \leqslant C^* t^{-\frac{\alpha-1}{2(\alpha+3)}}, \\ |\bar{\mu}(t)| \leqslant C^* t^{-2\frac{\alpha+1}{\alpha+3}}, \end{cases}$$

with $C^* > 1$ to be fixed later. Note that the condition (4.4.6) implies

$$|z(t) - at^{\frac{2}{\alpha+3}}| \leqslant Ct^{-r}. \tag{4.4.10}$$

We define

$$t^*(z_n^{in}) = \inf\{t \in [T_0, S_n] : \forall \widetilde{t} \in [t, S_n], \text{(4.4.5)} - \text{(4.4.9) is true }\}.$$

We want to prove that for an adequate choice of z_n^{in} in (4.4.3), $t^*(z_n^{in}) = T_0$.

By the previous choice of C_0, the assumptions (4.2.2)-(4.2.4) on the approximation and the condition (Cond$_Z$) on the modulation are satisfied on $(t^*(z_n^{in}), S_n]$, increasing T_0 if necessary.

The section 4.4.2 provides the tools to get a bound of z_1, z_2, μ_1 and μ_2, and the section 4.4.3 the bound on $\|\varepsilon\|_{H^{\frac{\alpha}{2}}}$. Next, in the section 4.4.4, we prove that we can choose z_n^{in} to close the bootstrap. We finish the proof of Theorem 4.0.3 in the section 4.4.5.

Remark 4.4.1. Notice that different parameters are involved along this section. We clarify the order in which they are fixed. First, we fix the parameter A, introduced in subsection 4.4.3; then the parameter C^* involved in the bootstrap dependently of A, and finally, the initial time T_0 dependently of A and C^*.

4.4.2 System of ODE

We now continue with the system of ODEs ruling the parameters z_1, z_2, μ_1 and μ_2. To do so, we compute the time derivative of the orthogonality conditions.

Proposition 4.4.2. *The functions z_1, z_2, μ_1 and μ_2 satisfy that for all $i \in \{1, 2\}$:*

$$\sum_{i=1}^{2} |\dot{\mu}_i(t) + (-1)^i b(z(t))| \leqslant C \left(\frac{1}{z^{\frac{5+3\alpha}{2}}(t)} + \frac{1}{z^{1+\alpha}(t)} \|\varepsilon(t)\|_{H^{\frac{\alpha}{2}}} + \|\varepsilon(t)\|^2_{H^{\frac{\alpha}{2}}} \right), \tag{4.4.11}$$

and

$$|\dot{z}_1(t) - \mu_1(t) + \beta(\Gamma(t))| + |\dot{z}_2(t) - \mu_2(t) + \delta(\Gamma(t))| \leqslant C \left(\frac{1}{z^{\frac{5+3\alpha}{2}}(t)} + \|\varepsilon(t)\|_{H^{\frac{\alpha}{2}}} \right). \tag{4.4.12}$$

Proof. We begin with the first orthogonality condition $\int \varepsilon R_1$. Since $\varepsilon = v - V$ and v solves (4.0.5), we deduce that:

$$\partial_t \varepsilon + \partial_y \left(-|D|^\alpha \varepsilon - \varepsilon + (\varepsilon + V)^3 - V^3 \right) = -\mathcal{E}_V.$$

By differentiating in time the equality $0 = \int \varepsilon R_1$ and using the fact $\int \varepsilon \partial_y R_1 = 0$, we obtain that:

$$\begin{aligned} 0 = \frac{d}{dt} \int \varepsilon R_1 = \int \left(-|D|^\alpha \varepsilon - \varepsilon + 3R_1^2 \varepsilon \right) \partial_y R_1 + \int \left((V + \varepsilon)^3 - V^3 - 3R_1^2 \varepsilon \right) \partial_y R_1 \\ - \int \overrightarrow{m} \cdot \overrightarrow{MV} R_1 - \int \partial_y S R_1 - \int T R_1 + \dot{\mu}_1 \int \varepsilon \Lambda R_1. \end{aligned}$$

By using the equation of R_1 and the condition $\varepsilon \perp \partial_y R_1$, we deduce that:

$$\int \left(-|D|^\alpha \varepsilon - \varepsilon + 3R_1^2 \varepsilon \right) \partial_y R_1 = \int \left(-|D|^\alpha \varepsilon - (1 + \mu_1) \varepsilon + 3R_1^2 \varepsilon \right) \partial_y R_1 = 0.$$

Now, we continue with $\int \left((V + \varepsilon)^3 - V^3 - 3R_1^2 \varepsilon \right) \partial_y R_1$. First, note that:

$$(V + \varepsilon)^3 - V^3 - 3R_1^2 \varepsilon = 3V\varepsilon^2 + \varepsilon^3 + 3\varepsilon \left(-2R_1 \left(R_2 - P_1 + P_2 + bW \right) + \left(R_2 - P_1 + P_2 + bW \right)^2 \right).$$

We recall $\|V\|_{L^\infty} + \|\partial_y R_1\|_{L^\infty} \leqslant C$. Therefore, using the Sobolev embedding $H^{\frac{1}{6}}(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$, we have that:

$$\left| \int \left(3\varepsilon^2 V + \varepsilon^3\right) \partial_y R_1 \right| \leqslant C \left(\|\varepsilon\|_{L^2}^2 + \|\varepsilon\|_{H^{\frac{\alpha}{2}}}^3 \right).$$

Furthermore, $|R_2 \partial_y R_1| \leqslant \frac{C}{z^{1+\alpha}}$ and $|P_1| + |P_2| + |bW| \leqslant \frac{C}{z^{1+\alpha}}$, we conclude that:

$$\left| \int \left((V+\varepsilon)^3 - V^3 - 3R_1^2 \varepsilon \right) \partial_y R_1 \right| \leqslant C \left(\frac{\|\varepsilon\|_{L^2}}{z^{1+\alpha}} + \|\varepsilon\|_{L^2}^2 + \|\varepsilon\|_{H^{\frac{\alpha}{2}}}^3 \right).$$

Let us estimate $\int \overrightarrow{m} \cdot \overrightarrow{MV} R_1$. Using the set $\{y \in \mathbb{R} : y \leqslant \frac{z_1+z_2}{2}\}$, we get that:

$$\left| \int \Lambda R_2 R_1 \right| + \left| \int \partial_y R_2 R_1 \right| \leqslant \frac{C}{z^{1+\alpha}}.$$

Moreover, with $R_1 \perp \partial_y R_1$, we obtain that:

$$\left| \int \overrightarrow{m} \cdot \overrightarrow{MV} R_1 - (-\dot{\mu}_1 + b(z)) \int \Lambda R_1 R_1 \right| \leqslant \frac{C}{z^{1+\alpha}} \left(|\dot{\mu}_2 + b(z)| + |\dot{z}_2 - \mu_2 + \delta(\Gamma)| \right).$$

Finally, using Cauchy-Schwarz inequality, (4.2.13) and (4.2.15) we get that:

$$\left| \int \partial_y S R_1 \right| + \left| \int T R_1 \right| + |\dot{\mu}_1| \left| \int \varepsilon \Lambda R_1 \right| \leqslant C \left(\frac{1}{z^{\frac{5+3\alpha}{2}}} + |\dot{\mu}_1| \|\varepsilon\|_{L^2} \right).$$

Gathering these estimates, and thanks to the facts $\|\varepsilon\|_{H^{\frac{\alpha}{2}}} \leqslant C\kappa$ and $|\dot{\mu}_i| + |\dot{z}_i| \leqslant C$ from (4.3.3) and (4.3.4), we obtain that:

$$\begin{aligned} \frac{\alpha-1}{2(\alpha+1)} \|Q\|_{L^2}^2 |\dot{\mu}_1 - b(z)| \leqslant & \frac{C}{z^{1+\alpha}} \left(|\dot{\mu}_2 + b(z)| + |\dot{z}_2 - \mu_2 + \delta(\Gamma)| + \|\varepsilon\|_{L^2} \right) + \frac{C}{z^{\frac{5+3\alpha}{2}}} \\ & + C \left(|\dot{\mu}_1| \|\varepsilon\|_{H^{\frac{\alpha}{2}}} + \|\varepsilon\|_{H^{\frac{\alpha}{2}}}^2 \right). \end{aligned} \tag{4.4.13}$$

By similar computations, we also deduce that:

$$\begin{aligned} \frac{\alpha-1}{2(\alpha+1)} \|Q\|_{L^2}^2 |\dot{\mu}_2 + b(z)| \leqslant & \frac{C}{z^{1+\alpha}} \left(|\dot{\mu}_1 - b(z)| + |\dot{z}_1 - \mu_1 + \beta(\Gamma)| + \|\varepsilon\|_{L^2} \right) + \frac{C}{z^{\frac{5+3\alpha}{2}}} \\ & + C \left(|\dot{\mu}_1| \|\varepsilon\|_{H^{\frac{\alpha}{2}}} + \|\varepsilon\|_{H^{\frac{\alpha}{2}}}^2 \right). \end{aligned} \tag{4.4.14}$$

Therefore, by adding (4.4.13) and (4.4.14) we obtain:

$$\begin{aligned} \sum_{i=1}^{2} |\dot{\mu}_i + (-1)^i b(z)| \leqslant & \frac{C}{z^{1+\alpha}} \left(|\dot{z}_1 - \mu_1 + \beta(\Gamma)| + |\dot{z}_2 - \mu_2 + \delta(\Gamma)| \right) \\ & + C \left(\frac{1}{z^{\frac{5+3\alpha}{2}}} + \left(|\dot{\mu}_1| + |\dot{\mu}_2| + \frac{1}{z^{1+\alpha}} \right) \|\varepsilon\|_{H^{\frac{\alpha}{2}}} + \|\varepsilon\|_{H^{\frac{\alpha}{2}}}^2 \right). \end{aligned} \tag{4.4.15}$$

Let us continue with the second orthogonality condition:

$$\begin{aligned} 0 = \frac{d}{dt} \int \varepsilon \partial_y R_1 = & \int \left(-|D|^\alpha \varepsilon - \varepsilon + (V+\varepsilon)^3 - V^3 \right) \partial_y^2 R_1 - \int \overrightarrow{m} \cdot \overrightarrow{MV} \partial_y R_1 \\ & + \int S \partial_y^2 R_1 - \int T \partial_y R_1 + \dot{\mu}_1 \int \varepsilon \partial_y \Lambda R_1 - \dot{z}_1 \int \varepsilon \partial_y^2 R_1. \end{aligned}$$

Since $|V| + \left|(\partial_y^2 + |D|^\alpha \partial_y^2) R_1\right| \leqslant C$ and using the Sobolev embedding , $H^{\frac{1}{6}}(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$, we deduce that:

$$\left| \int \left(-|D|^\alpha \varepsilon - \varepsilon + (V+\varepsilon)^3 - V^3 \right) \partial_y^2 R_1 \right| \leqslant C \left(\|\varepsilon\|_{H^{\frac{\alpha}{2}}} + \|\varepsilon\|_{H^{\frac{\alpha}{2}}}^2 + \|\varepsilon\|_{H^{\frac{\alpha}{2}}}^3 \right).$$

By developing $\overrightarrow{m} \cdot \overrightarrow{MV}$ and using the facts $\left|\int \partial_y R_1 \left(\partial_y R_2 + \Lambda R_2\right)\right| \leqslant \frac{C}{z^{1+\alpha}}$ and $\int \partial_y R_1 \Lambda R_1 = 0$ since $\partial_y R_1$ is odd, we get that:

$$\left|\int \overrightarrow{m} \cdot \overrightarrow{MV} \partial_y R_1 - (\dot{z}_1 - \mu_1 + \beta(\Gamma)) \int (\partial_y R_1)^2\right| \leqslant \frac{C}{z^{1+\alpha}} \left(|\dot{\mu}_2 + b(z)| + |\dot{z}_2 - \mu_2 + \delta(\Gamma)|\right).$$

We estimate the last terms by applying Cauchy-Schwarz inequality, (4.2.13) and (4.2.15). We have that:

$$\left|\int S\partial_y^2 R_1\right| + \left|\int T\partial_y R_1\right| + \left|\dot{\mu}_1 \int \varepsilon \partial_y \Lambda R_1\right| + \left|\dot{z}_1 \int \varepsilon \partial_y^2 R_1\right| \leqslant C\left(\frac{1}{z^{\frac{5+3\alpha}{2}}} + (|\dot{\mu}_1| + |\dot{z}_1|) \|\varepsilon\|_{L^2}\right).$$

Gathering these estimates and using $\|\varepsilon\|_{H^{\frac{\alpha}{2}}} \leqslant C\kappa$ and the fact $|\dot{\mu}_i| + |\dot{z}_i| \leqslant C$ (4.3.4), we conclude that:

$$|\dot{z}_1 - \mu_1 + \beta(\Gamma)| \int (\partial_y Q)^2 \leqslant C\left(\frac{1}{z^{1+\alpha}} |\dot{\mu}_2 + b(z)| + |\dot{z}_2 - \mu_2 + \delta(\Gamma)| + \frac{1}{z^{\frac{5+3\alpha}{2}}} + \|\varepsilon\|_{H^{\frac{\alpha}{2}}}\right). \quad (4.4.16)$$

By similar arguments, we deduce that:

$$|\dot{z}_2 - \mu_2 + \delta(\Gamma)| \int (\partial_y Q)^2 \leqslant C\left(\frac{1}{z^{1+\alpha}} |\dot{\mu}_1 - b(z)| + |\dot{z}_1 - \mu_1 + \beta(\Gamma)| + \frac{1}{z^{\frac{5+3\alpha}{2}}} + \|\varepsilon\|_{H^{\frac{\alpha}{2}}}\right). \quad (4.4.17)$$

Then, by adding (4.4.16) and (4.4.17), we obtain:

$$|\dot{z}_1 - \mu_1 + \beta(\Gamma)| + |\dot{z}_2 - \mu_2 + \delta(\Gamma)| \leqslant C\left(\frac{1}{z^{1+\alpha}} \left(|\dot{\mu}_1 - b(z)| + |\dot{\mu}_2 + b(z)|\right) + \frac{1}{z^{\frac{5+3\alpha}{2}}} + \|\varepsilon\|_{H^{\frac{\alpha}{2}}}\right). \quad (4.4.18)$$

Gathering (4.4.15) and (4.4.18), we obtain (4.4.12), and

$$\sum_{i=1}^{2} |\dot{\mu}_i + (-1)^i b(z)| \leqslant C\left(\frac{1}{z^{\frac{5+3\alpha}{2}}} + \left(|\dot{\mu}_1| + |\dot{\mu}_2| + \frac{1}{z^{1+\alpha}}\right) \|\varepsilon\|_{H^{\frac{\alpha}{2}}} + \|\varepsilon\|_{H^{\frac{\alpha}{2}}}^2\right).$$

Since $|\dot{\mu}_i| \leqslant |\dot{\mu}_i + (-1)^i b(z)| + b(z)$, by applying the former inequality and (4.2.5), we conclude (4.4.11). □

4.4.3 Monotonicity

We define:

$$\varphi(y) = \left(\int_{-\infty}^{+\infty} \frac{ds}{\langle s\rangle^{1+\alpha}}\right)^{-1} \int_y^{+\infty} \frac{ds}{\langle s\rangle^{1+\alpha}}, \quad (4.4.19)$$

and

$$\varphi_1(t,y) := \frac{1-\varphi(y)}{(1+\mu_1(t))^2} + \frac{\varphi(y)}{(1+\mu_2(t))^2} \quad \text{and} \quad \varphi_2(t,y) := \frac{\mu_1(t)}{(1+\mu_1(t))^2}(1-\varphi(y)) + \frac{\mu_2(t)}{(1+\mu_2(t))^2}\varphi(y).$$

Let $A > 0$, we define the rescaled functions:

$$\varphi_A(y) = \varphi\left(\frac{y}{A}\right), \quad \varphi_{1,A}(t,y) := \varphi_1\left(t, \frac{y}{A}\right), \quad \varphi_{2,A}(t,y) := \varphi_2\left(t, \frac{y}{A}\right),$$

the derivatives by:

$$\Phi(y) = \sqrt{|\varphi'(y)|}, \quad \Phi_i(t,y) = \sqrt{|\varphi_i'(t,y)|}, \quad \Phi_{i,A}(t,y) = \Phi_i\left(t, \frac{y}{A}\right). \quad (4.4.20)$$

By direct computation, we have:

$$\Phi_1(y) = \frac{c}{\langle y \rangle^{\frac{1+\alpha}{2}}} \frac{\mu^{\frac{1}{2}}(2+\bar{\mu})^{\frac{1}{2}}}{(1+\mu_1)(1+\mu_2)} \quad \text{and} \quad \Phi_2(y) = \frac{c}{\langle y \rangle^{\frac{1+\alpha}{2}}} \frac{\mu^{\frac{1}{2}}(1-\mu_1\mu_2)^{\frac{1}{2}}}{(1+\mu_1)(1+\mu_2)}. \tag{4.4.21}$$

We also define the functional:

$$F(t) = \int \left(\frac{\varepsilon |D|^\alpha \varepsilon}{2} + \frac{\varepsilon^2}{2} - \frac{(V+\varepsilon)^4}{4} + \frac{V^4}{4} + V^3 \varepsilon - S\varepsilon \right) \varphi_{1,A} + \frac{\varepsilon^2}{2} \varphi_{2,A}. \tag{4.4.22}$$

We claim the following theorem that will help us to get the estimate (4.4.5) on the error ε.

Theorem 4.4.3. *The following bound on the functional holds:*

$$F(t) \leqslant C t^{-\frac{7\alpha+9}{2(\alpha+3)}}.$$

Preliminary results

To get the monotinicity properties of the modified energy, we need to recall a result from Lemma 6 and Lemma 7 from [91] and Lemma 3.2 from [51].

Lemma 4.4.4. *Let $\alpha \in]0,2[$. In the symmetric case, there exists $C > 0$ such that:*

$$\left| \int (|D|^\alpha u)\, u \Phi_{j,A}^2 - \int \left(|D|^{\frac{\alpha}{2}} (u \Phi_{j,A}) \right)^2 \right| \leqslant \frac{C}{A^\alpha} \int u^2 \Phi_{j,A}^2,$$

and

$$\left| \int (|D|^\alpha u)\, \partial_x u \varphi_{j,A} + (-1)^{j+1} \frac{\alpha - 1}{2} \int \left(|D|^{\frac{\alpha}{2}} (u \Phi_{j,A}) \right)^2 \right| \leqslant \frac{C}{A^\alpha} \int u^2 \Phi_{j,A}^2,$$

for any $u \in \mathcal{S}(\mathbb{R})$, $A > 1$ and $j \in \{1, \cdots, N\}$.

In the non-symmetric case, there exists $C > 0$ such that:

$$\left| \int \left((|D|^\alpha u)\, v - (|D|^\alpha v)\, u \right) \Phi_{j,A}^2 \right| \leqslant \begin{cases} \dfrac{C}{A^\alpha} \displaystyle\int \left(u^2 + v^2 \right) \Phi_{j,A}^2, & \text{if } \alpha \in]0,1], \\ \dfrac{C}{A^{\frac{\alpha}{2}}} \displaystyle\int \left(u^2 + v^2 + \left(|D|^{\frac{\alpha}{2}} u \right)^2 \right) \Phi_{j,A}^2, & \text{if } \alpha \in]1,2[, \end{cases} \tag{4.4.25}$$

and

$$\left| \int \left((|D|^\alpha u)\, \partial_x v + (|D|^\alpha v)\, \partial_x u \right) \varphi_{j,A} + (-1)^{j+1} (\alpha - 1) \int |D|^{\frac{\alpha}{2}} (u \Phi_{j,A})\, |D|^{\frac{\alpha}{2}} (v \Phi_{j,A}) \right|$$
$$\leqslant \begin{cases} \dfrac{C}{A^\alpha} \displaystyle\int \left(u^2 + v^2 \right) \Phi_{j,A}^2, & \text{if } \alpha \in]0,1], \\ \dfrac{C}{A^{\frac{\alpha}{2}}} \displaystyle\int \left(u^2 + v^2 + \left(|D|^{\frac{\alpha}{2}} u \right)^2 \right) \Phi_{j,A}^2, & \text{if } \alpha \in]1,2[, \end{cases} \tag{4.4.26}$$

for any $u, v \in \mathcal{S}(\mathbb{R})$, $A > 1$ and $j \in \{1, \cdots, N\}$.

The estimates (4.4.23)-(4.4.24) are proved in Lemmas 6 and 7 in [91] for $\alpha \in [1,2]$. Observe however that their proofs extend easily to the case $\alpha \in]0,2[$. Note also that while only one side of the inequalities in (4.4.23)-(4.4.24) is stated in Lemmas 6 and 7 in [91] , both sides are actually proved.

Lemma 4.4.5 ([51], Lemma 3.3)**.** *Let $0 \leqslant \alpha \leqslant 2$. For all $u \in \mathcal{S}(\mathbb{R})$, we have that:*

$$\left| \int \left(|D|^{\frac{\alpha}{2}} (u \Phi_{1,A}) \right)^2 - \left(|D|^{\frac{\alpha}{2}} u \right)^2 \Phi_{1,A}^2 \right| \leqslant \frac{C}{A^{\frac{\alpha}{2}}} \int \left(u^2 + (|D|^{\frac{\alpha}{2}} u)^2 \right) \Phi_{1,A}^2. \tag{4.4.27}$$

The following estimates are proved in Appendix 4.5.3.

Lemma 4.4.6. *For $\alpha \in]0,2[$, then for all $u \in \mathcal{S}(\mathbb{R})$ we have that:*

$$\left\| [|D|^{\alpha}, \Phi_{j,A}]\, u \right\|_{L^2}^2 \leqslant \begin{cases} \dfrac{C}{A^{2\alpha}} \displaystyle\int u^2 \Phi_{j,A}^2, & \text{if} \quad \alpha \in]0,1] \\ \dfrac{C}{A^{\alpha}} \displaystyle\int \left(u^2 + \left(|D|^{\frac{\alpha}{2}} u \right)^2 \right) \Phi_{j,A}^2, & \text{if} \quad \alpha \in]1,2] \end{cases}$$

Lemma 4.4.7. *Let $\alpha \in]0,2[$, then for all $u \in \mathcal{S}(\mathbb{R})$ there exists $C > 0$ such that:*

$$\begin{aligned} &\left| \int |D|^{\alpha} (u\Phi_{j,A}) \left((|D|^{\alpha} u) \Phi_{j,A} \right) - \int (|D|^{\alpha} u)^2 \Phi_{j,A}^2 \right| \\ &\qquad \leqslant \frac{C}{A^{\frac{\alpha}{2}}} \int \left(u^2 + \left(|D|^{\frac{\alpha}{2}} u \right)^2 + (|D|^{\alpha} u)^2 \right) \Phi_{j,A}^2, \end{aligned} \tag{4.4.28}$$

for all $u \in \mathcal{S}(\mathbb{R})$, $A > 1$ and $j \in \{1, \cdots, N\}$.

Lemma 4.4.8. *Let $1 \leqslant \alpha \leqslant 2$. For all $u \in \mathcal{S}(\mathbb{R})$, we have that:*

$$\| [|D|^{\alpha}, \varphi_{1,A}] u \|_{L^2} \leqslant C \left| \frac{1}{(1+\mu_1)^2} - \frac{1}{(1+\mu_2)^2} \right|^{\frac{1}{2}} \| u \Phi_{1,A} \|_{H^1}.$$

Remark 4.4.9. Notice that the scaling in A is not coherent with the previous inequality. In the proof in the appendix, we establish this inequality in $H^{\frac{\alpha}{2}}(\mathbb{R})$ and use at the very end the embedding $H^{\frac{\alpha}{2}}(\mathbb{R}) \subset H^1(\mathbb{R})$.

Lemma 4.4.10. *Let $0 \leqslant \alpha \leqslant 2$. For all $u \in \mathcal{S}(\mathbb{R})$, we have that:*

$$\| [|D|^{\alpha}, \sqrt{\varphi_A}] u \|_{L^2} + \| [|D|^{\alpha}, \sqrt{1 - \varphi_A}] u \|_{L^2} \leqslant \begin{cases} \dfrac{C}{A^{\alpha}} \|u\|_{L^2}, & \alpha \in (0,1] \\ \dfrac{C}{A^{\frac{\alpha}{2}}} \|u\|_{H^{\frac{\alpha}{2}}}, & \alpha \in (1,2] \end{cases}.$$

Proof of the Theorem 4.4.3

In this part, we study the functional F defined in (4.4.22), dependent on the two functions $\varphi_{1,A}$ and $\varphi_{2,A}$. For sake of clearness, we drop the indices A in this part only and denote those functions by φ_1 and φ_2. The parameter A will appear explicitly when needed.

We recall the equation satisfied by ε:

$$\partial_t \varepsilon + \partial_y \left(-|D|^{\alpha} \varepsilon - \varepsilon + (\varepsilon + V)^3 - V^3 \right) = -\mathcal{E}_V.$$

We differentiate in time the functional F defined in (4.4.22), by using (4.2.11) we deduce that:

$$\begin{aligned} \frac{d}{dt} F(t) = &\int (\partial_t \varepsilon) \left(|D|^{\alpha} \varepsilon + \varepsilon - (\varepsilon + V)^3 + V^3 - S \right) \varphi_1 + \frac{1}{2} \int \left(\varepsilon |D|^{\alpha} \partial_t \varepsilon - (\partial_t \varepsilon) |D|^{\alpha} \varepsilon \right) \varphi_1 \\ &+ \int \left(-(\partial_t V) \left((V+\varepsilon)^3 - V^3 - 3V^2 \varepsilon \right) \varphi_1 + (\partial_t \varepsilon) \varepsilon \varphi_2 \right) - \int (\partial_t S) \varepsilon \varphi_1 \\ &+ \int \left(\frac{\varepsilon |D|^{\alpha} \varepsilon}{2} + \frac{\varepsilon^2}{2} - \frac{(V+\varepsilon)^4}{4} + \frac{V^4}{4} + V^3 \varepsilon - S\varepsilon \right) \partial_t \varphi_1 + \int \frac{\varepsilon^2}{2} \partial_t \varphi_2 \\ = &I_1 + \cdots + I_6. \end{aligned}$$

Estimate on I_1***:*** Using integration by parts and the definition of $\mathcal{E}_V$, we deduce that:

$$\begin{aligned}
I_1 &= \frac{1}{2}\int \left(|D|^\alpha\varepsilon+\varepsilon-(V+\varepsilon)^3+V^3\right)^2\Phi_1^2 - \int \partial_y\left(|D|^\alpha\varepsilon+\varepsilon-(V+\varepsilon)^3+V^3\right)S\varphi_1\\
&\quad -\int \mathcal{E}_V\left(|D|^\alpha\varepsilon+\varepsilon-(V+\varepsilon)+V^3-S\right)\varphi_1\\
&= \frac{1}{2}\int \left(|D|^\alpha\varepsilon+\varepsilon-(V+\varepsilon)^3+V^3\right)^2\Phi_1^2 - \int \Big(\Big(\overrightarrow{m}\cdot\overrightarrow{MV}+T\Big)\left(|D|^\alpha\varepsilon+\varepsilon-(V+\varepsilon)^3+V^3\right)-TS\Big)\varphi_1\\
&\quad -\int S\Big(|D|^\alpha\varepsilon+\varepsilon-(V+\varepsilon)^3+V^3\Big)\Phi_1^2+\int \overrightarrow{m}\cdot\overrightarrow{MV}S\varphi_1+\frac{1}{2}\int S^2\Phi_1^2\\
&= I_{1,1}+\cdots+I_{1,5}.
\end{aligned}$$

We start with $I_{1,1}$. By direct computations, we get that:

$$\begin{aligned}
&\frac{1}{2}\int(|D|^\alpha\varepsilon+\varepsilon-(V+\varepsilon)^3+V^3)^2\Phi_1^2-\frac{1}{2}\int\Big((|D|^\alpha\varepsilon)^2+\varepsilon^2+\left(-(V+\varepsilon)^3+V^3\right)^2\Big)\Phi_1^2\\
&\quad=\int(|D|^\alpha\varepsilon)\,\varepsilon\Phi_1^2+\int(|D|^\alpha\varepsilon+\varepsilon)\Big(V^3-(V+\varepsilon)^3\Big)\Phi_1^2=I_{1,1,1}+I_{1,1,2}.
\end{aligned}$$

By using the estimate (4.4.23), we obtain that:

$$\left|I_{1,1,1}-\int\Big(|D|^{\frac{\alpha}{2}}(\varepsilon\Phi_1)\Big)^2\right|\leqslant\frac{C}{A^\alpha}\int\varepsilon^2\Phi_1^2.$$

Since $V^3-(V+\varepsilon)^3=-3V^2\varepsilon-3V\varepsilon^2-\varepsilon^3$, by applying Young's inequality, the bound on V (4.2.20) and Cauchy-Schwarz' inequality, we have that:

$$|I_{1,1,2}|\leqslant\frac{C}{A^\alpha}\int(|D|^\alpha\varepsilon)^2\Phi_1^2+CA^\alpha\int\left(V^4\varepsilon^2+\varepsilon^4+\varepsilon^6\right)\Phi_1^2+C\int\left(V^4\varepsilon^2+\varepsilon^3+\varepsilon^4\right)\Phi_1^2.$$

We recall that $2\mu_1=\mu+\bar{\mu}$, $2\mu_2=\bar{\mu}-\mu$ and $\alpha>1$. Therefore, using the bootstrap estimates (4.4.5), (4.4.7) and (4.4.9) and (4.2.21), we conclude for $I_{1,1}$ that:

$$I_{1,1}-\frac{1}{2}\int(\varepsilon^2+(|D|^\alpha\varepsilon)^2)\Phi_1^2-\int\Big(|D|^{\frac{\alpha}{2}}(\varepsilon\Phi_1)\Big)^2\geqslant-\frac{C}{A^\alpha}\int\left(\varepsilon^2+(|D|^\alpha\varepsilon)^2\right)\Phi_1^2-CA^\alpha t^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.$$

Let us estimate $I_{1,2}$. By using the definition of $\overrightarrow{m}\cdot\overrightarrow{MV}$ in (4.2.12), we obtain that:

$$\begin{aligned}
&\int\overrightarrow{m}\cdot\overrightarrow{MV}\left(|D|^\alpha\varepsilon+\varepsilon-(V+\varepsilon)^3+V^3\right)\varphi_1\\
&\quad=\sum_{i=1}^{2}\int\left((-1)^i\dot{\mu}_i-b(z)\right)\Lambda R_i\left(|D|^\alpha\varepsilon+\varepsilon-(V+\varepsilon)^3+V^3\right)\varphi_1\\
&\qquad+\int\left((\dot{z}_1-\mu_1+\beta(\Gamma))\partial_yR_1-(\dot{z}_2-\mu_2+\delta(\Gamma))\partial_yR_2\right)\left(|D|^\alpha\varepsilon+\varepsilon-(V+\varepsilon)^3+V^3\right)\varphi_1\\
&\quad=J_1+J_2.
\end{aligned}$$

Since $\frac{1}{1+\mu_i}\leqslant C$, we deduce that:

$$\begin{aligned}
|J_1|&\leqslant\sum_{i=1}^{2}|(-1)^i\dot{\mu}_i-b(z)|\left|\int\Lambda R_i\left(|D|^\alpha\varepsilon+\varepsilon-(V+\varepsilon)^3+V^3\right)\Big(\varphi_1+(-1)^i\frac{1}{(1+\mu_i)^2}\Big)\right|\\
&\quad+C\sum_{i=1}^{2}|(-1)^i\dot{\mu}_i-b(z)|\left|\int\Lambda R_i\left(|D|^\alpha\varepsilon+(1+\mu_i)\varepsilon-R_i^2\varepsilon\right)\right|\\
&\quad+C\sum_{i=1}^{2}|(-1)^i\dot{\mu}_i-b(z)|\left|\int\Lambda R_i\left(-\mu_i\varepsilon-(V+\varepsilon)^3+V^3-3R_i^2\varepsilon\right)\right|=J_{1,1}+J_{1,2}+J_{1,3}.
\end{aligned}$$

Thanks to the identity $(V+\varepsilon)-V^3=\varepsilon^3+3\varepsilon^2V+3\varepsilon V^2$, the fact $\alpha<2$, and by Cauchy-Schwarz' inequality, we get that:

$$J_{1,1}\leqslant C\sum_{i=1}^{2}|(-1)^i\dot{\mu}_i-b(z)|\left|\frac{1}{(1+\mu_1)^2}-\frac{1}{(1+\mu_2)^2}\right|\|\Lambda R_i(\varphi-\delta_{2i})\|_{H^1}\|\varepsilon\|_{H^{\frac{\alpha}{2}}}.$$

Moreover, we recall $L\Lambda Q=-Q$ and since $\varepsilon\perp R_i$, we deduce that:

$$J_{1,2}=C\sum_{i=1}^{2}|(-1)^i\dot{\mu}_i-b(z)|\left|\int R_i\varepsilon\right|=0.$$

Applying Cauchy-Schwarz' inequality, and Sobolev embedding $H^{\frac{1}{3}}(\mathbb{R})\hookrightarrow L^6(\mathbb{R})$, we have that:

$$J_{1,3}\leqslant C\sum_{i=1}^{2}|(-1)^i\dot{\mu}_i-b(z)|\left(\left(|\mu_i|+\left\|(V^2-R_i^2)\Lambda R_i\right\|_{L^2}\right)\|\varepsilon\|_{L^2}+\|\varepsilon\|_{L^2}^2+\|\varepsilon\|_{H^{\frac{\alpha}{2}}}^3\right).$$

Now, let us estimate J_2. We focus on the first term of J_2 with ∂_yR_1, the second is similar. We decompose this term into:

$$\begin{aligned}&(\dot{z}_1-\mu_1+\beta(\Gamma))\int\partial_yR_1\left(|D|^\alpha\varepsilon+\varepsilon-(V+\varepsilon)^3+V^3\right)\left(\varphi_1-\frac{1}{(1+\mu_1)^2}\right)\\&\quad+\frac{\dot{z}_1-\mu_1+\beta(\Gamma)}{(1+\mu_1)^2}\left(\int\partial_yR_1\left(|D|^\alpha\varepsilon+\varepsilon-3R_1^2\varepsilon\right)+\int\partial_yR_1\left(V^3-(V+\varepsilon)^3+3R_1^2\varepsilon\right)\right)\\&\quad=J_{2,1}+J_{2,2}+J_{2,3}.\end{aligned}$$

By applying the Cauchy-Schwarz' inequality and Sobolev embedding $H^{\frac{1}{3}}(\mathbb{R})\hookrightarrow L^6(\mathbb{R})$, we obtain that:

$$|J_{2,1}|\leqslant C|\dot{z}_1-\mu_1+\beta(\Gamma)|\left|\frac{1}{(1+\mu_1)^2}-\frac{1}{(1+\mu_2)^2}\right|\left(\|\partial_yR_1\varphi\|_{H^1}\|\varepsilon\|_{H^{\frac{\alpha}{2}}}+\|\varepsilon\|_{L^2}^2+\|\varepsilon\|_{H^{\frac{\alpha}{2}}}^3\right).$$

Since $\varepsilon\perp\partial_yR_i$ and $LQ'=0$, we deduce that:

$$J_{2,2}=0.$$

Moreover, by Cauchy-Schwarz inequality and Sobolev embedding , $H^{\frac{1}{6}}(\mathbb{R})\hookrightarrow L^3(\mathbb{R})$ we have that:

$$|J_{2,3}|\leqslant C|\dot{z}_1-\mu_1+\beta(\Gamma)|\left(\|\varepsilon\|_{L^2}\|(V^2-R_1^2)\partial_yR_1\|_{L^2}+\|\varepsilon\|_{L^2}^2+\|\varepsilon\|_{H^{\frac{\alpha}{2}}}^3\right).$$

By Cauchy-Schwarz' inequality, we have that:

$$\left|\int T\left(|D|^\alpha\varepsilon+\varepsilon-(V+\varepsilon)^3+V^3-S\right)\varphi_1\right|\leqslant C\|T\varphi_1\|_{H^{\frac{\alpha}{2}}}\left(\|\varepsilon\|_{H^{\frac{\alpha}{2}}}+\|\varepsilon\|_{H^{\frac{\alpha}{2}}}^2+\|\varepsilon\|_{H^{\frac{\alpha}{2}}}^3+\|S\|_{L^2}\right).$$

Gathering those identities, and using the estimate on T (4.2.15), the estimates on the solitary waves (4.2.16), (4.2.17), (4.2.21), the bootstrap estimates (4.4.5), (4.4.7), (4.4.9) and the equation on $\dot{\mu}_i$ (4.4.11) and $\dot{z}_i$ (4.4.12), we get that:

$$|I_{1,2}|\leqslant Ct^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.$$

Let us estimate $I_{1,3}$. By Cauchy-Schwarz inequality, the estimate on S (4.2.13) and the bootstrap estimate on ε (4.4.5), we obtain that:

$$|I_{1,3}|\leqslant C\left\|S\Phi_1^2\right\|_{H^{\frac{\alpha}{2}}}\left(\|\varepsilon\|_{H^{\frac{\alpha}{2}}}+\left\|(V+\varepsilon)^3-V^3\right\|_{L^2}\right)\leqslant Ct^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.$$

Using the definition of $\overrightarrow{m} \cdot \overrightarrow{MV}$, the estimate on $\dot{\mu}_i$ (4.4.11), $\dot{z}_i$ (4.4.12) and the estimate on S (4.2.13), we deduce that:

$$|I_{1,4}| \leqslant C\|S\|_{L^2}\left(\sum_{i=1}^{2}|(-1)^i\dot{\mu}_i - b(z)| + |\dot{z}_1-\mu_1-\beta(\Gamma)| + |\dot{z}_2-\mu_2-\delta(\Gamma)|\right) \leqslant Ct^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.$$

Finally, by the estimate on S (4.2.13):

$$|I_{1,5}| \leqslant Ct^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.$$

Conclusion:

$$I_1 - \frac{1}{2}\int(\varepsilon^2 + (|D|^\alpha\varepsilon)^2)\Phi_1^2 - \int\left(|D|^{\frac{\alpha}{2}}(\varepsilon\Phi_1)\right)^2 \geqslant -\frac{C}{A^\alpha}\int\left(\varepsilon^2+(|D|^\alpha\varepsilon)^2\right)\Phi_1^2 - CA^\alpha t^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.$$

Estimate on I_2***:*** From the equation of ε, since φ_1 is decreasing and integration by parts, we deduce that:

$$\begin{aligned}2I_2 &= -\frac{1}{2}\int(|D|^\alpha\varepsilon)^2\,\Phi_1^2 - \int\partial_y\varepsilon|D|^\alpha\left(|D|^\alpha\varepsilon+2\varepsilon\right)\varphi_1 + \int\varepsilon|D|^\alpha\left(|D|^\alpha\varepsilon+\varepsilon\right)\Phi_1^2\\ &+\int\left(\mathcal{E}_V|D|^\alpha\varepsilon - \varepsilon|D|^\alpha\mathcal{E}_V\right)\varphi_1 + \int\left(\varepsilon|D|^\alpha\partial_y\left(-(V+\varepsilon)^3+V^3\right) - \partial_y\left(-(V+\varepsilon)^3+V^3\right)(|D|^\alpha\varepsilon)\right)\varphi_1\\ &= I_{2,1}+\cdots I_{2,5}.\end{aligned}$$

Let us estimate $I_{2,2}$ and $I_{2,3}$. Using the commutator estimates in the non-symmetric case (4.4.25), (4.4.26) with $v = |D|^\alpha\varepsilon$, the commutator estimates in the symmetric case (4.4.23), (4.4.24) , and Lemma 4.4.7 we get that:

$$\left|I_{2,2}+I_{2,3} - \alpha\int\left(|D|^{\frac{\alpha}{2}}(\varepsilon\Phi_1)\right)^2 - \left(\alpha+\frac{1}{2}\right)\int(|D|^\alpha\varepsilon)^2\,\Phi_1^2\right| \leqslant \frac{C}{A^{\frac{\alpha}{2}}}\int\left(\varepsilon^2 + \left(|D|^{\frac{\alpha}{2}}\varepsilon\right)^2 + (|D|^\alpha\varepsilon)^2\right)\Phi_1^2$$

From Cauchy-Schwarz inequality and Lemma 4.4.8, we get that:

$$\begin{aligned}|I_{2,4}| &= \left|\int\varepsilon\left(|D|^\alpha\left(\mathcal{E}_V\varphi_1\right) - |D|^\alpha\left(\mathcal{E}_V\right)\varphi_1\right)\right| \leqslant \|\varepsilon\|_{L^2}\|[|D|^\alpha,\varphi_1]\mathcal{E}_V\|_{L^2}\\ &\leqslant C\left|\frac{1}{(1+\mu_1)^2} - \frac{1}{(1+\mu_2)^2}\right|\|\varepsilon\|_{L^2}\left\|\mathcal{E}_V\sqrt{\varphi'}\right\|_{H^1}\\ &\leqslant C\left|\frac{1}{(1+\mu_1)^2} - \frac{1}{(1+\mu_2)^2}\right|\|\varepsilon\|_{L^2}\left(\left\|\overrightarrow{m}.\overrightarrow{MV}\sqrt{\varphi'}\right\|_{H^1} + \|S\|_{H^1} + \|T\|_{H^1}\right).\end{aligned}$$

Therefore, by using the estimates on $\dot{\mu}_i$ (4.4.11), on $\dot{z}_i$ (4.4.12), the estimates on S (4.2.13), T (4.2.15), the interaction between $\partial_y R_i$ or ΛR_i and Φ (4.2.17) and the bootstrap estimates (4.4.5)-(4.4.9), we have that:

$$|I_{2,4}| \leqslant t^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.$$

Now, we estimate $I_{2,5}$. Note that:

$$(V+\varepsilon)^3 - V^3 = 3V^2\varepsilon + 3V\varepsilon^2 + \varepsilon^3.$$

Then, we decompose $I_{2,5}$ as:

$$\begin{aligned}I_{2,5} &= \int\left(\partial_y\left(3V^2\varepsilon\right)(|D|^\alpha\varepsilon) - \varepsilon|D|^\alpha\partial_y\left(3V^2\varepsilon\right)\right)\varphi_1 + \int\left(\partial_y\left(3V\varepsilon^2\right)(|D|^\alpha\varepsilon) - \varepsilon|D|^\alpha\partial_y\left(3V\varepsilon^2\right)\right)\varphi_1\\ &+\int\left(\partial_y\left(\varepsilon^3\right)(|D|^\alpha\varepsilon) - \varepsilon|D|^\alpha\partial_y\left(\varepsilon^3\right)\right)\varphi_1.\end{aligned}$$

Let $v \in \{3\varepsilon V^2, 3\varepsilon^2 V, \varepsilon^3\}$. Using integration by parts, the commutator estimates in the non-symmetric case (4.4.25) and (4.4.26), we get that:

$$\begin{aligned}\left|\int (\partial_y v(|D|^\alpha\varepsilon) - \varepsilon|D|^\alpha\partial_y v)\,\varphi_1\right| &= \left|\int \partial_y v(|D|^\alpha\varepsilon)\varphi_1 + \int \partial_y\varepsilon(|D|^\alpha v)\varphi_1 - \int \varepsilon(|D|^\alpha v)\Phi_1^2\right| \\ &\leqslant (\alpha-1)\left|\int |D|^{\frac{\alpha}{2}}(v\Phi_1)\,|D|^{\frac{\alpha}{2}}(\varepsilon\Phi_1)\right| + \left|\int (|D|^\alpha\varepsilon)v\Phi_1^2\right| \\ &+ \frac{C}{A^{\frac{\alpha}{2}}}\int \left(\varepsilon^2 + v^2 + (|D|^{\frac{\alpha}{2}}\varepsilon)^2\right)\Phi_1^2.\end{aligned}$$

Moreover, from Young's inequality, we obtain that:

$$\left|\int (|D|^\alpha\varepsilon)v\Phi_1^2\right| \leqslant \frac{C}{A^\alpha}\int (|D|^\alpha\varepsilon)^2\Phi_1^2 + CA^\alpha\int v^2\Phi_1^2.$$

By using Young's inequality and (4.4.27), we deduce that:

$$\left|\int |D|^{\frac{\alpha}{2}}(v\Phi_1)\,|D|^{\frac{\alpha}{2}}(\varepsilon\Phi_1)\right| \leqslant \frac{C}{A^\alpha}\int (\varepsilon^2 + (|D|^\alpha\varepsilon)^2)\Phi_1^2 + CA^\alpha\int v^2\Phi_1^2$$

By the Sobolev's embeddings, $H^{\frac{1}{4}}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$, $H^{\frac{1}{3}}(\mathbb{R}) \hookrightarrow L^6(\mathbb{R})$, we obtain that:

$$|I_{2,5}| \leqslant CA^\alpha\left(\|V^4\Phi_1^2\|_{L^\infty}\|\varepsilon\|_{L^2}^2 + \|V^2\Phi_1^2\|_{L^\infty}\|\varepsilon\|_{H^{\frac{\alpha}{2}}}^4 + \|\varepsilon\|_{H^{\frac{\alpha}{2}}}^6\right) + \frac{C}{A^{\frac{\alpha}{2}}}\int\left(\varepsilon^2 + (|D|^{\frac{\alpha}{2}}\varepsilon)^2 + (|D|^\alpha\varepsilon)^2\right)\Phi_1^2.$$

Moreover, applying the estimate (4.2.21), (4.4.5) and (4.4.6), we get that:

$$|I_{2,5}| \leqslant CA^\alpha t^{-\frac{3(3\alpha+5)}{2(\alpha+3)}} + \frac{C}{A^{\frac{\alpha}{2}}}\int\left(\varepsilon^2 + \left(|D|^{\frac{\alpha}{2}}\varepsilon\right)^2 + (|D|^\alpha\varepsilon)^2\right)\Phi_1^2.$$

Conclusion:

$$\left|I_2 - \frac{\alpha}{2}\int (|D|^\alpha\varepsilon)^2\,\Phi_1^2 - \frac{\alpha}{2}\int\left(|D|^{\frac{\alpha}{2}}(\varepsilon\Phi_1)\right)^2\right| \leqslant CA^\alpha t^{-\frac{3(3\alpha+5)}{2(\alpha+3)}} + \frac{C}{A^{\frac{\alpha}{2}}}\int\left(\varepsilon^2 + (|D|^{\frac{\alpha}{2}}\varepsilon)^2 + (|D|^\alpha\varepsilon)^2\right)\Phi_1^2.$$

Estimate on I_3**:** We decompose I_3 as:

$$\begin{aligned}I_3 = -3\int \partial_y(V^2\varepsilon)\varepsilon\varphi_2 + \partial_t VV\varepsilon^2\varphi_1 - \int \partial_y\left(3V\varepsilon^2 + \varepsilon^3\right)\varepsilon\varphi_2 + \partial_t V\varepsilon^3\varphi_1 \\ + \int \partial_y\left(|D|^\alpha\varepsilon + \varepsilon\right)\varepsilon\varphi_2 - \int \mathcal{E}_V\varepsilon\varphi_2 = I_{3,1} + I_{3,2} + I_{3,3} + I_{3,4}.\end{aligned}$$

By adding 0 and integrating by part, we deduce that:

$$\begin{aligned}I_{3,1} = 3\int \partial_y R_1(\varphi_2 - \dot{z}_1\varphi_1)V\varepsilon^2 + 3\int \partial_y R_2(\dot{z}_2\varphi_1 - \varphi_2)V\varepsilon^2 - 3\int\left(\partial_y V + \partial_y R_1 - \partial_y R_2\right)V\varepsilon^2\varphi_2 \\ - 3\int\left(\partial_t V - \dot{z}_1\partial_y R_1 + \dot{z}_2\partial_y R_2\right)V\varepsilon^2\varphi_1 + \frac{3}{2}\int V^2\varepsilon^2\Phi_2^2 = I_{3,1,1} + \cdots + I_{3,1,5}.\end{aligned}$$

Using the definition of φ_1 and φ_2, we obtain that:

$$\begin{aligned}|I_{3,1,1}| &= \left|3\frac{\mu_1 - \dot{z}_1}{1+\mu_1}\int \partial_y R_1 V\varepsilon^2(1-\varphi) + 3\frac{\dot{z}_1 - \mu_2}{1+\mu_2}\int \partial_y R_1 V\varepsilon^2\varphi\right| \\ &\leqslant C\|\varepsilon\|_{L^2}^2\left(|\dot{z}_1 - \mu_1| + (|\dot{z}_1| + |\mu_2|)\,\|\partial_y R_1 V\varphi\|_{L^\infty}\right).\end{aligned}$$

Using the same argument, we deduce that:

$$|I_{3,1,2}| \leqslant C\|\varepsilon\|_{L^2}^2 \left(|\dot{z}_2 - \mu_2| + (|\dot{z}_2| + |\mu_1|) \|\partial_y R_2 V(1-\varphi)\|_{L^\infty}\right).$$

Using the definition of V and φ_2:

$$|I_{3,1,3}| \leqslant C\|\varepsilon\|_{L^2}^2 \left(|\mu_1| + |\mu_2|\right) \|\partial_y \left(P_2 - P_1 + bW\chi\right)\|_{L^\infty}$$

and

$$|I_{3,1,4}| \leqslant C\|\varepsilon\|_{L^2}^2 \left(\left(|\dot{\mu_1}| + |\dot{\mu_2}|\right) + \left(|\mu_1| + |\mu_2|\right) \|\partial_t \left(P_2 - P_1 + bW\chi\right)\|_{L^\infty}\right).$$

Gathering these identities, and using the bootstrap hypothesis, the time estimate of the different terms and (4.2.18) and (4.2.19), we conclude that:

$$|I_{3,1}| \leqslant Ct^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.$$

For $I_{3,2}$, using integration by parts and Sobolev embedding, and the bootstrap hypothesis, we deduce that:

$$\begin{aligned}|I_{3,2}| &\leqslant C\left(\|\varepsilon\|_{H^{\frac{\alpha}{2}}}^3 \|\partial_t V\|_{L^\infty} + (|\mu_1| + |\mu_2|)(\|V\|_{L^\infty} + \|\partial_y V\|_{L^\infty})\|\varepsilon\|_{H^{\frac{\alpha}{2}}}^3 + (|\mu_1| + |\mu_2|)\|\varepsilon\|_{H^{\frac{\alpha}{2}}}^4\right)\\ &\leqslant Ct^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.\end{aligned}$$

Using integration by part, the commutator estimates in the symmetric case (4.4.23) and (4.4.24), and since $\partial_y \varphi_2 < 0$, we obtain that:

$$I_{3,3} \geqslant -\frac{\alpha+1}{2}\int \left(|D|^{\frac{\alpha}{2}}\left(\varepsilon\Phi_2\right)\right)^2 - \left(\frac{1}{2} + \frac{C}{A^\alpha}\right)\int \varepsilon^2\Phi_2^2.$$

Moreover with (4.4.21), we have:

$$\Phi_2^2 = \left|\frac{\mu_1\mu_2 - 1}{2 + \mu_1 + \mu_2}\right|\Phi_1^2.$$

Then, we get that:

$$I_{3,3} \geqslant -\frac{\alpha+1}{2}\frac{1-\mu_1\mu_2}{2+\mu_1+\mu_2}\int \left(|D|^{\frac{\alpha}{2}}\left(\varepsilon\Phi_1\right)\right)^2 - \left(\frac{1}{2} + \frac{C}{A^\alpha}\right)\frac{1-\mu_1\mu_2}{2+\mu_1+\mu_2}\int \varepsilon^2\Phi_1^2.$$

Since $\frac{1-\mu_1\mu_2}{2+\mu_1+\mu_2} \leqslant \frac{3}{4}$ by (4.4.7) and (4.4.9), we deduce that:

$$I_{3,3} \geqslant -\frac{3(\alpha+1)}{8}\int \left(|D|^{\frac{\alpha}{2}}\left(\varepsilon\Phi_1\right)\right)^2 - \left(\frac{3}{8} + \frac{C}{A^\alpha}\right)\int \varepsilon^2\Phi_1^2.$$

Let us estimate the last term of I_3. Using the definition of $\mathcal{E}_V$ and Cauchy-Schwarz inequality, we have that:

$$|I_{3,4}| \leqslant C(|\mu_1| + |\mu_2|)\|\varepsilon\|_{L^2}\left(\|\partial_y S\|_{L^2} + \|T\|_{L^2}\right) + \left|\int \overrightarrow{m}\cdot\overrightarrow{MV}\varepsilon\varphi_2\right|.$$

Using the definition of $\overrightarrow{m}\cdot\overrightarrow{MV}$ and the orthogonality condition $\varepsilon \perp \partial_y R_i$, we deduce that:

$$\left|\int \overrightarrow{m}\cdot\overrightarrow{MV}\varepsilon\varphi_2\right| \leqslant C\|\varepsilon\|_{L^2}(|\mu_1| + |\mu_2|)\left(\sum_{i=1}^{2}|(-1)^i\dot{\mu_i} - b(z)| + |\dot{z}_i - \mu_i|\,\|\partial_y R_i(\varphi - \delta_{2,i})\|_{L^2}\right).$$

Therefore with (4.2.16), we get that:

$$|I_{3,4}| \leqslant Ct^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.$$

Conclusion:

$$I_3 \geqslant -\frac{3(\alpha+1)}{8}\int\left(|D|^{\frac{\alpha}{2}}(\varepsilon\Phi_1)\right)^2-\left(\frac{3}{8}+\frac{C}{A^{\alpha}}\right)\int\varepsilon^2\Phi_1^2-Ct^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.$$

Estimate on I_4: Applying Cauchy-Schwarz inequality and the estimate on the time derivative of S (4.2.14), we obtain that:

$$|I_4|\leqslant C\|\partial_t S\|_{L^2}\|\varepsilon\|_{L^2}\leqslant Ct^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.$$

Estimate on I_5: First, note by direct computation, we have:

$$|\partial_t\varphi_1|=\left|\frac{2\dot{\mu}_1}{(1+\mu_1)^3}(1-\varphi)+\frac{2\dot{\mu}_2}{(1+\mu_2)^3}\varphi\right|\leqslant C\left(|\dot{\mu}_1|+|\dot{\mu}_2|\right).$$

Then, by the Sobolev embedding $H^{\frac{1}{3}}(\mathbb{R})\hookrightarrow L^6(\mathbb{R})$ and $H^{\frac{1}{4}}(\mathbb{R})\hookrightarrow L^4(\mathbb{R})$, we deduce that:

$$\left|\int\left(\frac{\varepsilon^2}{2}-\frac{(V+\varepsilon)^4}{4}+\frac{V^4}{4}+V^3\varepsilon\right)\partial_t\varphi_1\right|\leqslant C\left(|\dot{\mu}_1|+|\dot{\mu}_2|\right)\left(\|\varepsilon\|^2_{H^{\frac{\alpha}{2}}}+\|\varepsilon\|^3_{H^{\frac{\alpha}{2}}}+\|\varepsilon\|^4_{H^{\frac{\alpha}{2}}}\right).$$

Moreover, by Cauchy-Schwarz inequality, we get:

$$\left|\int S\varepsilon\partial_t\varphi_1\right|\leqslant C\left(|\dot{\mu}_1|+|\dot{\mu}_2|\right)\|S\|_{L^2}\|\varepsilon\|_{L^2}.$$

Now, let us estimate the first term in I_5. By direct computations, we have that:

$$\begin{aligned}\int\varepsilon|D|^{\alpha}\varepsilon\partial_t\varphi_1=&-\frac{2\dot{\mu}_1}{(1+\mu_1)^3}\left(\int D^{\frac{\alpha}{2}}\varepsilon[|D|^{\frac{\alpha}{2}},(1-\varphi)]\varepsilon+\int\left(|D|^{\frac{\alpha}{2}}\varepsilon\right)^2(1-\varphi)\right)\\&-\frac{2\dot{\mu}_2}{(1+\mu_2)^3}\left(\int D^{\frac{\alpha}{2}}\varepsilon[|D|^{\frac{\alpha}{2}},\varphi]\varepsilon+\int\left(|D|^{\frac{\alpha}{2}}\varepsilon\right)^2\varphi\right).\end{aligned}$$

Using Lemma 4.4.10, we deduce that:

$$\left|\int\varepsilon|D|^{\alpha}\varepsilon\partial_t\varphi_1\right|\leqslant C(|\dot{\mu}_1|+|\dot{\mu}_2|)\|\varepsilon\|^2_{H^{\frac{\alpha}{2}}}.$$

Conclusion:

$$|I_5|\leqslant C(|\dot{\mu}_1|+|\dot{\mu}_2|)\left(\|S\|_{L^2}\|\varepsilon\|_{H^{\frac{\alpha}{2}}}+\|\varepsilon\|^2_{H^{\frac{\alpha}{2}}}+\|\varepsilon\|^3_{H^{\frac{\alpha}{2}}}+\|\varepsilon\|^4_{H^{\frac{\alpha}{2}}}\right)\leqslant Ct^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.$$

Estimate on I_6: By definition of φ_2, we obtain that:

$$|\partial_t\varphi_2|\leqslant C\left(|\dot{\mu}_1|+|\dot{\mu}_2|\right).$$

then, by using the estimate on $\dot{\mu}_i$ (4.4.11), the bootstrap estimates (4.4.5), (4.4.6), we have that:

$$|I_6|\leqslant Ct^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.$$

Gathering the estimates on $I_1,...,I_6$, we obtain that:

$$\begin{aligned}\frac{d}{dt}F(t)\geqslant&\frac{\alpha+1}{2}\int\left(|D|^{\alpha}\varepsilon\right)^2\Phi_1^2+\left(1+\frac{\alpha}{2}-\frac{3(\alpha+1)}{8}\right)\int(|D|^{\frac{\alpha}{2}}(\varepsilon\Phi_1))^2+\left(\frac{1}{2}-\frac{3}{8}\right)\int\varepsilon^2\Phi_1^2\\&-\frac{C}{A^{\frac{\alpha}{2}}}\int(\varepsilon^2+(|D|^{\frac{\alpha}{2}}\varepsilon)^2+(|D|^{\alpha}\varepsilon)^2)\Phi_1^2-CA^{\alpha}t^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.\end{aligned}$$

To compare the quantities $\int (|D|^\alpha \varepsilon)^2 \Phi_1^2$ and $\int (|D|^\alpha \varepsilon \Phi_1)^2$ we use (4.4.27), thus we have:

$$\frac{d}{dt}F(t) \geqslant \frac{\alpha+1}{2}\int (|D|^\alpha \varepsilon)^2 \Phi_1^2 + \left(1+\frac{\alpha}{2}-\frac{3(\alpha+1)}{8}\right)\int (|D|^{\frac{\alpha}{2}}\varepsilon)^2\Phi_1^2 + \left(\frac{1}{2}-\frac{3}{8}\right)\int \varepsilon^2\Phi_1^2 - \frac{C}{A^{\frac{\alpha}{2}}}\int (\varepsilon^2 + (|D|^{\frac{\alpha}{2}}\varepsilon)^2 + (|D|^\alpha\varepsilon)^2)\Phi_1^2 - CA^\alpha t^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.$$

By taking $A > A_1$ large enough, T_0 large enough, we deduce that:

$$\frac{d}{dt}F(t) \geqslant -CA^\alpha t^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}$$

However, the choice of A is independent of parameters. We set $A > \max(A_1, A_2)$, with A_2 defined in Claim 4.5.7 for the coercivity of the localized linearized operator. For now, A is a constant. Then, integrating in time from t to S_n we conclude that:

$$F(t) \leqslant Ct^{-\frac{7\alpha+9}{2(\alpha+3)}},$$

with the constant C independent of the different parameters.

4.4.4 Topological argument

We argue by contradiction. Let suppose for all z_n^{in} in (4.4.3), we have $t^*(z_n^{in}) > T_0$.

Suppose first that one of the bootstrap estimates (4.4.5), (4.4.7), (4.4.8) or (4.4.9) is saturated, in the sense that the equality is achieved.

1) Closing bootstrap for ε. First we start to show we can improve (4.4.5). We recall that the notations φ, φ_1 and φ_2 holds respectively for φ_A, $\varphi_{1,A}$ and $\varphi_{2,A}$. Using the Cauchy-Schwarz inequality, (4.2.13), (4.4.5) and the definition of φ_1 , we get that:

$$F(t) \geqslant -Ct^{\frac{3(3\alpha+5)}{2(\alpha+3)}} + \frac{1}{2}\int \left(\varepsilon|D|^\alpha\varepsilon + \varepsilon^2 - 3\widetilde{R}_1^2\varepsilon^2\right)\frac{1-\varphi}{(1+\mu_1)^2} + \frac{1}{2}\int \left(\varepsilon|D|^\alpha\varepsilon + \varepsilon^2 - 3\widetilde{R}_2^2\varepsilon^2\right)\frac{\varphi}{(1+\mu_2)^2} + \int \frac{\varepsilon^2}{2}\varphi_2 + \int \left(\frac{V^4}{4} + V^3\varepsilon - \frac{(V+\varepsilon)^4}{4}\right)\varphi_1 + \frac{3}{2}\widetilde{R}_1^2\varepsilon^2\frac{1-\varphi}{(1+\mu_1)^2} + \frac{3}{2}\widetilde{R}_2^2\varepsilon^2\frac{\varphi}{(1+\mu_2)^2}. \quad (4.4.29)$$

First of all, we estimate the last term on the right hand side. We get that, by straight forward computations:

$$\frac{V^4}{4} + V^3\varepsilon - \frac{(V+\varepsilon)^4}{4} = -\frac{3}{2}V^2\varepsilon^2 - \varepsilon^3 V - \frac{\varepsilon^4}{4}.$$

Using the Sobolev embedding and the bootstrap estimates on ε (4.4.5), we deduce that:

$$\left|\int (\varepsilon^3 V + \frac{1}{4}\varepsilon^4)\varphi_1\right| \leqslant Ct^{-\frac{3(3\alpha+5)}{2(\alpha+3)}}.$$

Moreover, we have that:

$$\widetilde{R}_1^2\frac{1-\varphi}{(1+\mu_1)^2} + \widetilde{R}_2^2\frac{\varphi}{(1+\mu_2)^2} - V^2\varphi_1 = \left(\widetilde{R}_1^2 - R_1^2\right)\varphi_1 + \left(\widetilde{R}_2^2 - R_2^2\right)\varphi_1 - \widetilde{R}_1^2\frac{\varphi}{(1+\mu_2)^2} - \widetilde{R}_2^2\frac{1-\varphi}{(1+\mu_1)^2} + 2R_1R_2\varphi_1 - 2(-R_1+R_2)(-P_1+P_2+bW)\varphi_1 - (-P_1+P_2+bW)^2\varphi_1$$

Therefore, by applying the bootstrap estimate on ε (4.4.5), the estimate on the profile P_i (4.2.34), the estimate on the solitary waves (4.2.16), the estimate on ΛQ (4.1.7) and finally the bootstrap estimate on z (4.4.6), we get that:

$$\left|\int \left(\frac{V^4}{4} + V^3\varepsilon - \frac{(V+\varepsilon)^4}{4}\right)\varphi_1 + \frac{3}{2}\widetilde{R}_1^2\varepsilon^2\frac{1-\varphi}{(1+\mu_1)^2} + \frac{3}{2}\widetilde{R}_2^2\varepsilon^2\frac{\varphi}{(1+\mu_2)^2}\right| \leqslant Ct^{-\frac{4\alpha+6}{\alpha+3}}.$$

Moreover, from the bootstrap estimates on μ (4.4.7) and $\bar{\mu}$ (4.4.9) we have that:

$$\left|\int \varepsilon^2\varphi_2\right| \leqslant Ct^{-\frac{\alpha+1}{\alpha+3}}\|\varepsilon\|_{L^2}^2.$$

Now, we estimate the two first integrals in (4.4.29). We claim the following:

$$\int \left(\varepsilon|D|^\alpha\varepsilon + \varepsilon^2 - 3\widetilde{R}_i^2\varepsilon^2\right)\frac{1-\varphi}{(1+\mu_1)^2} + \left(\varepsilon|D|^\alpha\varepsilon + \varepsilon^2 - 3\widetilde{R}_i^2\varepsilon^2\right)\frac{\varphi}{(1+\mu_2)^2} \geqslant \kappa\|\varepsilon\|_{H^{\frac{\alpha}{2}}}^2, \quad i=1,2.$$

The proof of this inequality is given in Claim 4.5.7 in the Appendix 4.5.4. The proof is based on the coercivity of the linearized operator L. By combining the former inequalities and using Theorem 4.4.3, we deduce that:

$$\kappa\|\varepsilon\|_{H^{\frac{\alpha}{2}}}^2 - Ct^{-\frac{4\alpha+6}{\alpha+3}} - Ct^{-\frac{\alpha+1}{\alpha+3}}\|\varepsilon\|_{L^2}^2 \leqslant F(t) \leqslant Ct^{-\frac{7\alpha+9}{2(\alpha+3)}}.$$

Therefore for T_0 large enough, we conclude that:

$$\|\varepsilon\|_{H^{\frac{\alpha}{2}}}^2 \leqslant Ct^{-\frac{7\alpha+9}{2(\alpha+3)}}.$$

Therefore, we strictly improved the bound (4.4.5) on ε. This concludes the proof for ε.

2) Closing bootstrap for μ,$\bar{\mu}$ and $\bar{z}$. Now, we improve the bound on μ (4.4.7). We recall $\mu = \mu_1 - \mu_2$ and $z = z_1 - z_2$. Combining the bootstrap estimate on ε (4.4.5) and z (4.4.6) on the right hand side of the estimate of $\dot{\mu}_i$ in (4.4.11) we deduce that:

$$\left|\dot{\mu} - \frac{2b_1}{z^{\alpha+2}}\right| \leqslant Ct^{-\frac{3\alpha+5}{\alpha+3}}.$$

Because $b_1 < 0$ and by the equivalent of z in (4.4.10), we have $\dot{\mu} < 0$. By the initial condition $\mu(S_n) > 0$, see (4.4.2), μ is positive on $(t^*, S_n]$.

Then, multiplying by μ, using the estimate on $\dot{z}_i$ (4.4.12) and the bootstrap on z (4.4.6) and μ (4.4.7), we obtain that:

$$\left|\dot{\overbrace{\frac{\mu^2}{2}}} + \frac{2b_1}{\alpha+1}\dot{\overbrace{\frac{1}{z^{\alpha+1}}}}\right| \leqslant Ct^{-\frac{4\alpha+6}{\alpha+3}}.$$

By the choice of the initial data, we have that:

$$\mu^2(S_n) = -\frac{4b_1}{\alpha+1}\frac{1}{z^{\alpha+1}(S_n)}.$$

Therefore, by integrating from t to S_n, we get that:

$$\left|\frac{\mu^2}{2} + \frac{2b_1}{\alpha+1}\frac{1}{z^{\alpha+1}}\right| \leqslant Ct^{-\frac{3(\alpha+1)}{\alpha+3}}. \tag{4.4.30}$$

With the bootstrap hypothesis on z (4.4.6), we deduce that:

$$\left|\mu - \sqrt{\frac{-4b_1}{\alpha+1}}\frac{t^{-\frac{\alpha+1}{\alpha+3}}}{a^{\frac{\alpha+1}{2}}}\right| \leqslant C_1t^{-\frac{5\alpha+11}{4(\alpha+3)}},$$

with the constant $C_1 > 0$.

Let us compute the bound on $\bar{\mu}$. From the estimate on $\dot{\mu}_i$ (4.4.11) and the bootstrap estimate on ε (4.4.5) and z (4.4.6), we obtain that:

$$|\dot{\bar{\mu}}| \leqslant Ct^{-\frac{3\alpha+5}{\alpha+3}}.$$

By the choice of the initial data, we have that $\mu_1(S_n) = -\mu_2(S_n)$. Thus, by integrating we deduce that:

$$|\bar{\mu}| \leqslant C_2 t^{-\frac{2(\alpha+1)}{\alpha+3}} \tag{4.4.31}$$

with the constant $C_2 > 0$.

Let us get a bound on $\bar{z}$. Using the fact that $|\beta(\Gamma)|+|\delta(\Gamma)| \leqslant \frac{2(\beta_0+\delta_0)}{z^{\alpha+1}}$, the bound obtain for $\bar{\mu}$ (4.4.31) and the estimate on $\dot{z}_i$ (4.4.12), we deduce that:

$$\begin{aligned}|\dot{\bar{z}}| \leqslant & |\dot{\bar{z}} - \bar{\mu} + \beta(\Gamma) + \delta(\Gamma)| + |\bar{\mu}| + |\beta(\Gamma) + \delta(\Gamma)| \quad leq C_3 t^{-\frac{3\alpha+5}{2(\alpha+3)}} + (2(\beta_0+\delta_0) + C_2) t^{-\frac{2(\alpha+1)}{\alpha+3}} \\ & \leqslant 2C_3 t^{-\frac{3\alpha+5}{2(\alpha+3)}}.\end{aligned}$$

Therefore by integrating, we conclude that:

$$|\bar{z}| \leqslant \frac{2C_3(2(\alpha+3))}{\alpha-1} t^{-\frac{\alpha-1}{2(\alpha+3)}}$$

Hence, by taking the constant $C^* > \max\left(C_1, C_2, \dfrac{2C_3(2(\alpha+3))}{\alpha-1}\right)$, we can close the bootstrap estimate on μ, $\bar{\mu}$ and $\bar{z}$. Then, none of the previous inequalities on $\dot{\mu}$, $\dot{\bar{\mu}}$ and $\dot{\bar{z}}$ can saturate independently of the initial condition z_n^{in}.

3) Closing bootstrap for z. Subsequently, the inequality (4.4.6) saturates for any z_n^{in}. We now prove that this equality is the source of a contradiction on $t^*(z_n^{in})$.

First, we remark $z_n^{in} = \left(a^{\frac{\alpha+3}{2}} S_n + \lambda_n S_n^{\frac{1}{2}+r}\right)^{\frac{2}{\alpha+3}}$, for some $\lambda_n \in [-1,1]$. Therefore, we can write $t^*(z_n^{in}) = t^*(\lambda_n)$. We set:

$$\begin{aligned}\Phi : [-1,1] &\longrightarrow \{-1,1\} \\ \lambda &\longmapsto \left(z^{\frac{\alpha+3}{2}}(t^*(\lambda)) - a^{\frac{\alpha+3}{2}} t^*(\lambda)\right)(t^*(\lambda))^{-\frac{1}{2}-r},\end{aligned}$$

and

$$\begin{aligned}f : \mathbb{R} &\longrightarrow \mathbb{R}^+ \\ s &\longmapsto \left(z^{\frac{\alpha+3}{2}}(s) - a^{\frac{\alpha+3}{2}} s\right)^2 s^{-1-2r}.\end{aligned}$$

By assumption, we have for any $\lambda \in [-1,1]$, $t^*(\lambda) > T_0$ and thus:

$$|z^{\frac{\alpha+3}{2}}(t^*(\lambda)) - a^{\frac{\alpha+3}{2}} t^*(\lambda)| = (t^*(\lambda))^{\frac{1}{2}+r}. \tag{4.4.32}$$

We claim:

Claim 4.4.11. *1. Transversality condition: Let $s_0 > T_0$ such that* (4.4.32) *is verified at s_0, then:*

$$f \textit{ is decreasing on a neighbourhood of } s_0. \tag{4.4.33}$$

2. Continuity: $\Phi \in C^0([-1,1] : \{-1,1\})$.

Let us assume the claim and finish the proof. The transversality condition (4.4.33) implies that $t^*(\pm 1) = S_n$. Moreover, $\Phi(\pm 1) = \pm 1$. This contradicts (2) of the former claim. Now, we prove the claim. First, we prove the transversality condition (4.4.33). By direct computations, we have that:

$$f'(s) = 2\left(\dot{\widehat{z^{\frac{\alpha+3}{2}}}}(s) - a^{\frac{\alpha+3}{2}}\right)\left(z^{\frac{\alpha+3}{2}}(s) - a^{\frac{\alpha+3}{2}} s\right) s^{-1-2r} - (1+2r)\left(z^{\frac{\alpha+3}{2}}(s) - a^{\frac{\alpha+3}{2}} s\right)^2 s^{-2-2r}.$$

From the estimate obtain on μ^2 (4.4.30) and the estimate on $\dot{z}_i$ (4.4.12), we obtain that:

$$\left|\dot{\widehat{z^{\frac{\alpha+3}{2}}}}(t) - \frac{\alpha+3}{2}\sqrt{\frac{-4b_1}{\alpha+1}}\right| \leqslant Ct^{-\frac{\alpha+1}{\alpha+3}}. \tag{4.4.34}$$

Therefore, by using (4.4.32) and (4.4.34), and since $a^{\frac{\alpha+3}{2}} = \frac{\alpha+3}{2}\sqrt{\frac{-4b_1}{\alpha+1}}$, we deduce that:

$$f'(s_0) < Cs_0^{-1-3r} - (1+2r)\, s_0^{-1}.$$

Since $r > 0$ and for T_0 large enough, we conclude that:

$$f'(s_0) < 0.$$

To prove the second part of the former claim, it is enough to show that $\lambda \mapsto t^*(\lambda)$ is continuous. Let us fix $\lambda \in [-1, 1]$. From the transversality condition, there exists $\varepsilon_\lambda > 0$ such that $\forall \varepsilon \in (0, \varepsilon_\lambda)$, $\exists \delta > 0$ and the two following conditions are verified: $f(t^*(\lambda) - \varepsilon) > 1 + \delta$, and for all $t \in [t^*(\lambda) + \varepsilon, S_n]$ (possibly empty), $f(t) < 1 - \delta$.

Note that the function is well defined, since the function z is globally well defined, see Remark (4.3.2). Then by the continuity of the flow, there exists $\eta > 0$ such that for all $|\lambda - \bar{\lambda}| < \eta$, with $\bar{\lambda} \in [-1, 1]$, the corresponding $\bar{f}$ verifies $|\bar{f}(s) - f(s)| < \frac{\delta}{2}$ for $s \in [t^*(\lambda) - \varepsilon, S_n]$. Therefore, we obtain that for all $s \in [t^*(\lambda) + \varepsilon, S_n]$:

$$\bar{f}(s) < |\bar{f}(s) - f(s)| + f(s) < 1 - \frac{\delta}{2}.$$

Thus, $t^*(\bar{\lambda}) < t^*(\lambda) + \varepsilon$. Furthermore,

$$\bar{f}(t^*(\lambda) - \varepsilon) > f(t^*(\lambda) - \varepsilon) - |\bar{f}(t^*(\lambda) - \varepsilon) - f(t^*(\lambda) - \varepsilon)| > 1 + \frac{\delta}{2}.$$

In other words, $t^*(\lambda) - \varepsilon < t^*(\bar{\lambda})$, and then Φ is continuous.

This contradicts the fact $t^*(\lambda) > T_0$ and implies the existence of z_n^{in} such that (4.4.5)-(4.4.9) are true for all $t \in [T_0, S_n]$.

4.4.5 Conclusion

In this section we have proved that there exists $(z_n^{in})^{\frac{\alpha+3}{2}} \in [a^{\frac{\alpha+3}{2}} S_n^{\frac{1}{2}+r} - S_n, a^{\frac{\alpha+3}{2}} S_n + S_n^{\frac{1}{2}+r}]$ such that the bootstrap estimates (4.4.5)-(4.4.9) are true for all $t \in [T_0, S_n]$. Let us show this implies Theorem 4.0.3. From (4.4.5), we obtain that:

$$\|v_n(T_0, \cdot)\|_{H^{\frac{\alpha}{2}}} \leqslant \|\varepsilon_n(T_0, \cdot)\|_{H^{\frac{\alpha}{2}}} + \|V(\Gamma_n(T_0), \cdot)\|_{H^{\frac{\alpha}{2}}} \leqslant C.$$

Therefore, by Banach-Alaoglu, there exists $w_0 \in H^{\frac{\alpha}{2}}(\mathbb{R})$ and a sub-sequence also denoted by $(v_n)_n$ such that:

$$v_n(T_0) \rightharpoonup w_0.$$

Thus, we denote by w the solution of (4.0.5) such that $w(T_0) = w_0$. Let $t > T_0$. From the weak continuity of the flow of Theorem 4.5.3, we have that:

$$\begin{aligned}&\left\| w(t, \cdot) + Q\left(\cdot - \frac{a}{2} t^{\frac{2}{\alpha+3}}\right) - Q\left(\cdot + \frac{a}{2} t^{\frac{2}{\alpha+3}}\right)\right\|_{H^{\frac{\alpha}{2}}} \\ &\quad \leqslant \liminf_{n\to\infty} \|\varepsilon_n(t, \cdot)\|_{H^{\frac{\alpha}{2}}} + \liminf_{n\to\infty} \left\| V(\Gamma_n(t), \cdot) + Q\left(\cdot - \frac{a}{2} t^{\frac{2}{\alpha+3}}\right) - Q\left(\cdot + \frac{a}{2} t^{\frac{2}{\alpha+3}}\right)\right\|_{H^{\frac{\alpha}{2}}}.\end{aligned}$$

Then, by using (4.4.5)-(4.4.8), we conclude that:

$$\left\| w(t, \cdot) + Q\left(\cdot - \frac{a}{2} t^{\frac{2}{\alpha+3}}\right) - Q\left(\cdot + \frac{a}{2} t^{\frac{2}{\alpha+3}}\right)\right\|_{H^{\frac{\alpha}{2}}} \leqslant Ct^{-\frac{\alpha-1}{4(\alpha+3)}}.$$

4.5 Appendix

4.5.1 Local well-posedness

We recall the results of well-posedness of (4.0.5).

Theorem 4.5.1 ([71], Theorem 1.5). *Let $\alpha \in (1,2)$, and $u_0 \in H^s(\mathbb{R})$, with $s \geqslant \frac{1}{2} - \frac{\alpha}{4}$. There exists a time $T = T(\|u_0\|_{H^{\frac{1}{2}-\frac{\alpha}{4}}(\mathbb{R})}) > 0$, and a unique solution $u \in \mathcal{C}([-T,T], H^s(\mathbb{R}))$ of (4.0.5). Furthermore, the flow $u_0 \mapsto u$ is locally Lipschitz continuous from $H^s(\mathbb{R})$ to $\mathcal{C}([-T,T], H^s(\mathbb{R}))$.*

Because the equation is subcritical, we obtain as a corollary the global well-posedness.

Corollary 4.5.2 ([71], Corollary 1.6). *For any initial condition $u_0 \in H^{\frac{\alpha}{2}}(\mathbb{R})$, there exists a unique global solution of 4.0.5 in $\mathcal{C}(\mathbb{R}, H^{\frac{\alpha}{2}}(\mathbb{R}))$.*

We continue with another property of the flow, which is the weak-continuity in $H^{\frac{\alpha}{2}}(\mathbb{R})$.

Theorem 4.5.3 (Weak continuity of the flow). *Let $\alpha \in (1,2)$. Suppose that $u_{0,n} \rightharpoonup u_0 \in H^{\frac{\alpha}{2}}(\mathbb{R})$. We consider u_n solutions of (4.0.5) corresponding to the initial data $u_n(0) = u_{n,0}$ and satisfying $u_n \in C([0,T] : H^{\frac{\alpha}{2}}(\mathbb{R}))$ for any $T > 0$. Then, $u_n(t) \rightharpoonup u(t)$ in $H^{\frac{\alpha}{2}}(\mathbb{R})$, for all $t \geqslant 0$.*

The proof of the weak continuity of the flow relies on the well-posedness result given in the Corollary 4.5.2. We refer to [51] Appendix A, [66] for a proof of this result.

4.5.2 Justification of the definition of S_0

First, we recall some well-known results on pseudo-differential operators (see [6], or [80] chapter 18). Let $D = -i\partial_x$. We define the symbolic class $\mathcal{S}^{m,q}$ by

$$\mathcal{S}^{m,q} := \left\{ a \in C^\infty(\mathbb{R}_x \times \mathbb{R}_\xi); \quad \forall k, \beta \in \mathbb{N}, \exists C_{k,\beta} > 0 \text{ such that } |\partial_x^k \partial_\xi^\beta a(x,\xi)| \leqslant C_{k,\beta} \langle x \rangle^{q-k} \langle \xi \rangle^{m-\beta} \right\}.$$

For all u in the Schwartz space $\mathcal{S}(\mathbb{R})$, we set the operator associated to the symbol $a(x,\xi) \in \mathcal{S}^{m,q}$ by

$$a(x,D)u := \frac{1}{2\pi} \int e^{ix\xi} a(x,\xi) \mathcal{F}(u)(\xi) d\xi.$$

We state the three following results

1. Let $a \in \mathcal{S}^{m,q}$, there exists $C > 0$, such that for all $u \in \mathcal{S}(\mathbb{R})$
$$\|a(x,D)u\|_{L^2} \leqslant C \|\langle x \rangle^q \langle D \rangle^m u\|_{L^2}. \tag{4.5.1}$$

2. Let $a \in \mathcal{S}^{m,q}$ and $b \in \mathcal{S}^{m',q'}$, then there exists $c \in \mathcal{S}^{m+m',q+q'}$ such that
$$a(x,D)b(x,D) = c(x,D). \tag{4.5.2}$$

3. If $a \in \mathcal{S}^{m,q}$ and $b \in \mathcal{S}^{m',q'}$ are two operators, we define the commutator by $[a(xD), b(x,D)] := a(x,D)b(x,D) - b(x,D)a(x,D)$. Moreover there exists $c \in \mathcal{S}^{m+m'-1,q+q'-1}$ such that
$$[a(x,D), b(x,D)] = c(x,D). \tag{4.5.3}$$

4. Let $a \in \mathcal{S}^{m,q}$, we have the following development for the adjoint a^* of a. Let $k \in \mathbb{N}$, then
$$a^*(x,\xi) = \sum_{\beta \leqslant k} \frac{1}{\beta!} \partial_\xi^\beta D_x^\beta \bar{a}(x,\xi) + R_k(x,\xi)$$
with $\partial_\xi^\beta D_x^\beta \bar{a} \in \mathcal{S}^{m-\beta,q-\beta}$ and $R_k \in \mathcal{S}^{m-\beta-1,q-\beta-1}$. Moreover the rest R_k is given by
$$R_k(x,\xi) = \frac{1}{2\pi} \int_0^1 (1-t)^{2k+1} dt \int e^{-iy\eta} \sum_{\beta+\gamma=2k+2} \frac{2k+2}{\beta!\gamma!} \partial_y^\beta \partial_\eta^\gamma \bar{a}(x-ty, \xi-t\eta) y^\beta \eta^\beta dy d\eta.$$

As a consequence of (4.5.2), $\langle D\rangle^m\langle x\rangle^q\langle D\rangle^{-m}\in\mathcal{S}^{0,q}$. Therefore, by (4.5.1), we have

$$\begin{aligned}\|\langle D\rangle^m\langle x\rangle^q u\|_{L^2}&=\|\langle D\rangle^m\langle x\rangle^q\langle D\rangle^{-m}\langle D\rangle^m u\|_{L^2}\\&\leqslant C_2\|\langle x\rangle^q\langle D\rangle^m u\|_{L^2},\end{aligned}$$

for $C_2>0$. By the same computations with $\langle x\rangle^q$ instead of $\langle D\rangle^m$, there exists $C_1>0$ such that

$$C_1\|\langle x\rangle^q\langle D\rangle^m u\|_{L^2}\leqslant\|\langle D\rangle^m\langle x\rangle^q u\|_{L^2}.$$

Gathering these two estimates, we conclude that

$$C_1\|\langle x\rangle^q\langle D\rangle^m u\|_{L^2}\leqslant\|\langle D\rangle^m\langle x\rangle^q u\|_{L^2}\leqslant C_2\|\langle x\rangle^q\langle D\rangle^m u\|_{L^2}.$$

We recall also the Schur's test.

Theorem 4.5.4 (Schur's test [75], Theorem 5.2)**.** *Let p,q be two non-negative measurable functions. If there exists $\alpha,\beta>0$ such that*

1. $\displaystyle\int_{\mathbb{R}}|K(x,y)|q(y)dy\leqslant\alpha p(x)$ *a.e.* $x\in\mathbb{R}$.
2. $\displaystyle\int_{\mathbb{R}}|K(x,y)|p(x)dx\leqslant\beta q(y)$ *a.e.* $y\in\mathbb{R}$.

Then $Tf:=\displaystyle\int_{\mathbb{R}}K(x,y)f(y)dy$ *is a bounded operator on* $L^2(\mathbb{R})$.

We recall two other lemmas useful for the rest of the appendix. The definition of φ is given in (4.4.19).

Lemma 4.5.5 ([91] Claim 5)**.** *There exists $C>0$ such that*

$$\begin{aligned}&|\varphi(x)-\varphi(y)|\leqslant C\frac{|x-y|}{(\langle x\rangle\langle y\rangle)^{\frac{\alpha+1}{2}}}+C\frac{|x-y|^2}{(\langle x\rangle+\langle y\rangle)^{\alpha+2}}\quad\text{if}\quad|x-y|\leqslant\frac{1}{2}\left(\langle x\rangle+\langle y\rangle\right),\\&|\varphi(x)-\varphi(y)|\leqslant C\quad\text{if}\quad|x-y|\geqslant\frac{1}{2}\left(\langle x\rangle+\langle y\rangle\right).\end{aligned}$$

Lemma 4.5.6 ([91], Lemma A.2)**.** *Let p be a homogeneous function of degree $\beta>-1$. Let $\chi\in C_0^\infty(\mathbb{R})$ such that $0\leqslant\chi\leqslant1$, $\chi(\xi)=1$ if $|\xi|<1$ and $\chi(\xi)=0$ if $|\xi|>2$. Let*

$$k(x)=\frac{1}{2\pi}\int e^{ix\xi}p(\xi)\chi(\xi)d\xi.$$

Then for all $q\in\mathbb{N}$, there exists $C_q>0$ such that, for all $x\in\mathbb{R}$,

$$|\partial_x^q k(x)|\leqslant\frac{C_q}{\langle x\rangle^{\beta+q+1}}.$$

Now, we can start the proof of the justification of the definition of S_0.

Proof. We recall the definition of ΛQ, and estimate on Q from [61]:

$$\Lambda Q=\frac{\alpha}{2(\alpha+1)}Q+\frac{1}{\alpha+1}x\partial_x Q,\quad|Q|+|x\partial_x Q|\leqslant\frac{1}{1+|x|^{1+\alpha}}.$$

Since $\Lambda Q\in L^2(\mathbb{R})$, we can define by the Fourier transform $(1+|D|^\alpha)^{-1}\Lambda Q\in H^\alpha(\mathbb{R})$:

$$\left\|(1+|D|^\alpha)^{-1}\Lambda Q\right\|_{H^\alpha}^2=\left\|\frac{(1+|\xi|^2)^{\frac{\alpha}{2}}}{1+|\xi|^\alpha}\widehat{\Lambda Q}\right\|_{L^2}^2\lesssim\|\Lambda Q\|_{L^2}<\infty.$$

The integral of $(1+|D|^{\alpha})^{-1}\Lambda Q$ on a finite interval is well-defined since it is in $L^2(\mathbb{R})$. However, it is not clear that the integral over an infinite interval is finite. We use the pseudo-differential theory to prove that the limit is finite. Let us define χ, a cut-off function equal to 1 in a neighbourhood of 0, with compact support. Let I be a compact interval. By the Cauchy-Schwarz inequality :

$$\begin{aligned}\int_I \left|(1+|D|^{\alpha})^{-1}\Lambda Q\right| &\leqslant \int_I \left|(1-\chi(D))\,(1+|D|^{\alpha})^{-1}\Lambda Q\right| + \int_I \left|\chi(D)\,(1+|D|^{\alpha})^{-1}\Lambda Q\right| \\ &\leqslant C\left\|\langle x\rangle^{\frac34}\,(1-\chi(D))\,(1+|D|^{\alpha})^{-1}\Lambda Q\right\|_{L^2(I)} + C\left\|\langle x\rangle^{\frac34}\chi(D)\,(1+|D|^{\alpha})^{-1}\Lambda Q\right\|_{L^2(I)} \\ &= \mathcal{I}_1 + \mathcal{I}_2.\end{aligned}$$

Note that the previous constant can be chosen independently of I. We have from (4.5.3) that the symbol $\langle x\rangle^{\frac34}(1-\chi(\xi))(1+|\xi|^{\alpha})^{-1}$ belongs to $\mathcal{S}^{-\alpha,\frac34}\subset\mathcal{S}^{0,\frac34}$. Thus, since $\langle x\rangle^{\frac34}\Lambda Q\in L^2(\mathbb{R})$:

$$\mathcal{I}_1 \lesssim \|\langle x\rangle^{\frac34}\Lambda Q\|_{L^2(\mathbb{R})} < \infty.$$

We can not deal with the integral $\mathcal{I}_2$ with symbols only, because $\chi(\xi)(1+|\xi|^{\alpha})^{-1}$ is not smooth around 0. We use the commutator to bring the decay in x close to ΛQ (notice the integral is over $\mathbb{R}$):

$$\mathcal{I}_2^2 \lesssim \int_{\mathbb{R}}\left(\left[\langle x\rangle^{\frac34},\chi(D)(1+|D|^{\alpha})^{-1}\right]\Lambda Q\right)^2 + \int_{\mathbb{R}}\left(\chi(D)(1+|D|^{\alpha})^{-1}\langle x\rangle^{\frac34}\Lambda Q\right)^2.$$

By the Plancherel formula, the second term can be bounded by $\left\|\langle x\rangle^{\frac34}\Lambda Q\right\|_{L^2}^2<\infty$. The first term needs to develop the commutator. First, let us define the kernel k satisfying:

$$\chi(D)(1+|D|^{\alpha})^{-1}u(x) = \frac{1}{2\pi}\int e^{i\xi x}\frac{\chi(\xi)}{1+|\xi|^{\alpha}}\widehat{u}(\xi)d\xi = k\star u(x), \quad \text{so} \quad \widehat{k}(\xi) = \frac{\chi(\xi)}{1+|\xi|^{\alpha}}.$$

The kernel k is well-defined as the inverse Fourier transform of a function in L^2. We thus get:

$$\begin{aligned}\left[\langle x\rangle^{\frac34},\chi(D)(1+|D|^{\alpha})^{-1}\right]u &= \langle x\rangle^{\frac34}k\star u(x) - k\star\left(\langle x\rangle^{\frac34}u\right)(x) \\ &= \int k(x-y)\left(\langle x\rangle^{\frac34}-\langle y\rangle^{\frac34}\right)u(y)dy.\end{aligned}$$

By Lemma 4.5.4 and the symmetry of k, it is enough to prove that $y\mapsto k(x-y)\left(\langle x\rangle^{\frac34}-\langle y\rangle^{\frac34}\right)\in L^1(\mathbb{R})$. First, we have to estimate k. By integrating by parts twice, we deduce that :

$$\frac{1}{1+x^2}(1-\partial_\xi^2)e^{ix\xi} = e^{ix\xi} \quad \text{and} \quad |k(x)| = \left|\frac{1}{2\pi}\int e^{ix\xi}\frac{\chi(\xi)}{1+|\xi|^{\alpha}}d\xi\right| \leqslant \frac{C_\alpha}{\langle x\rangle^2}. \tag{4.5.4}$$

Let $A_1 := \{y\in\mathbb{R} : |x-y|\leqslant\frac12(\langle x\rangle+\langle y\rangle)\}$, and $A_2 := \{y\in\mathbb{R} : |x-y|>\frac12(\langle x\rangle+\langle y\rangle)\}$. Notice the following equivalences :

$$|x-y|\leqslant\frac12(\langle x\rangle+\langle y\rangle)\Rightarrow\langle x\rangle\sim\langle y\rangle, \tag{4.5.5}$$

and

$$|x-y|>\frac12(\langle x\rangle+\langle y\rangle)\Rightarrow\langle x-y\rangle\sim|x-y|\sim\langle x\rangle+\langle y\rangle. \tag{4.5.6}$$

Then, from (4.5.4) and (4.5.6), we deduce that

$$\left|\int_{A_2} k(x-y)\left(\langle x\rangle^{\frac34}-\langle y\rangle^{\frac34}\right)dy\right| \leqslant \int_{A_2}\frac{\langle x\rangle^{\frac34}+\langle y\rangle^{\frac34}}{(\langle x\rangle+\langle y\rangle)^{\frac34}}\frac{1}{\langle x-y\rangle^{\frac54}}dy \leqslant C. \tag{4.5.7}$$

Moreover by (4.5.5), we obtain that $A_1 \subset [-c_2|x|, -c_1|x|] \cup [c_1|x|, c_2|x|]$, for some $0 < c_1 < c_2 < +\infty$ independent of x. Moreover, by the mean value theorem and $\ln(x + \langle x \rangle)' = \frac{1}{\langle x \rangle}$, we get that

$$\left| \int_{A_1} k(x-y) \left(\langle x \rangle^{\frac{3}{4}} - \langle y \rangle^{\frac{3}{4}} \right) dy \right| \leqslant C \langle x \rangle^{-\frac{1}{4}} \int_{A_1} \frac{1}{\langle x-y \rangle} dy \leqslant C \langle x \rangle^{-\frac{1}{4}} \ln(C(|x| + \langle x \rangle)) \leqslant C \tag{4.5.8}$$

Gathering (4.5.7) and (4.5.8), we conclude that k defines a bounded operator on $L^2(\mathbb{R})$. It implies that $\mathcal{I}_2$ is bounded, and thus $\int_I |(1 + |D|^\alpha)\Lambda Q|$ is bounded independently of I. This achieves the proof of the well-posedness of S_0, and that S_0 has a finite limit at $-\infty$. □

4.5.3 Proof of the preliminary results

Proof of Lemma 4.4.6. Let χ be a smooth cut-off function supported around 0. To estimate this commutator we split the norm in low and high frequency. For the low frequency we use the Schur's Lemma (Lemma 4.5.4), and the pseudo-differential calculus for the high frequency. To get an explicit dependence in A we prove the estimate

$$\left\| [|D|^\alpha, \Phi]\, u \right\|_{L^2}^2 \leqslant \begin{cases} C \int u^2 \Phi^2, & \text{if} \quad \alpha \in]0,1] \\ C \int \left(u^2 + \left(|D|^{\frac{\alpha}{2}} u \right)^2 \right) \Phi^2, & \text{if} \quad \alpha \in]1,2] \end{cases}$$

Then, we conclude Lemma 4.4.6 by changing the variable $x = \frac{x'}{A}$ and multiplying by $\left| \frac{1}{(1+\mu_1)^2} - \frac{1}{(1+\mu_2)^2} \right|$.

Let us start the proof. By the Schur's lemma (Lemma 4.5.4), we deduce that

$$\left\| [\chi(D))|D|^\alpha, \Phi]\, u \right\|_{L^2}^2 \leqslant C \int u^2 \Phi^2$$

From pseudo-differential calculus, and $\langle x \rangle^{\frac{\alpha}{2}} \sim 1 + x^{\frac{\alpha}{2}}$, we get that

$$\left\| [(1-\chi(D))|D|^\alpha, \Phi]\, u \right\|_{L^2}^2 \leqslant \begin{cases} C \int u^2 \Phi^2, & \text{if} \quad \alpha \in]0,1] \\ C \int u^2 \Phi^2 + C \int \left(|D|^{\frac{\alpha}{2}} (u\Phi) \right)^2, & \text{if} \quad \alpha \in]1,2] \end{cases}.$$

Again, by applying the pseudo-differential calculus, we deduce that

$$\begin{aligned} \int \left(|D|^{\frac{\alpha}{2}} (u\Phi) \right)^2 &\leqslant C \left(\int \left(\chi(D) |D|^{\frac{\alpha}{2}} (u\Phi) \right)^2 + \int \left((1-\chi(D)) |D|^{\frac{\alpha}{2}} (u\Phi) \right)^2 \right) \\ &\leqslant C \left(\int u^2 \Phi^2 + \int (|D|^{\frac{\alpha}{2}} u)^2 \Phi^2 \right). \end{aligned}$$

Then, by changing the variable $x = \frac{x'}{A}$ and multiplying by $\left| \frac{1}{(1+\mu_1)^2} - \frac{1}{(1+\mu_2)^2} \right|$, we conclude the proof of Lemma 4.4.6. □

Proof of Lemma 4.4.7. By direct computations and Young's inequality, we have that

$$\begin{aligned} \int |D|^\alpha (u\Phi_{j,A}) \left((|D|^\alpha u) \Phi_{j,A} \right) - \int \left(|D|^\alpha u \right)^2 \Phi_{j,A}^2 &= \int |D|^\alpha u \Phi_{1,A} [|D|^\alpha, \Phi_{1,A}] u \\ &\leqslant \frac{C}{A^{\frac{\alpha}{2}}} \int \left(|D|^\alpha u \right)^2 \Phi_{1,A}^2 + C A^{\frac{\alpha}{2}} \left\| [|D|^\alpha, \Phi_{1,A}] u \right\|_{L^2}^2. \end{aligned} \tag{4.5.9}$$

and by the change of variable $x' = \frac{x}{A}$ and $v(x') = u(x)$:

$$\left\| [|D|^\alpha, \Phi_{1,A}] u \right\|_{L^2}^2 = \frac{1}{A^{2\alpha - 1}} \left\| [|D|^\alpha, \Phi_1] v \right\|_{L^2}^2.$$

We write

$$\|[|D|^\alpha,\Phi_1]v\|_{L^2}^2 \leqslant C\left(\|[|D|^\alpha\chi(D),\Phi_1]v\|_{L^2}^2+\|[|D|^\alpha(1-\chi(D)),\Phi_1]v\|_{L^2}^2\right).$$

Using Theorem 4.5.4, we deduce that

$$\|[|D|^\alpha\chi(D),\Phi_1]v\|_{L^2}^2 \leqslant C\int v^2\Phi_1^2.$$

Moreover, using pseudo-differential calculus, we deduce that

$$\|[|D|^\alpha(1-\chi(D)),\Phi_1]v\|_{L^2}^2 \leqslant C\int\left(v^2+\left(|D|^{\frac{\alpha}{2}}v\right)^2\right)\Phi_1^2.$$

Gathering those estimates and coming back to the initial data, we get:

$$\|[|D|^\alpha(1-\chi(D)),\Phi_{1,A}]u\|_{L^2}^2 \leqslant \frac{C}{A^\alpha}\int\left(u^2+\left(|D|^{\frac{\alpha}{2}}u\right)\right)^2\Phi_{1,A}.$$

Using this last inequality in (4.5.9), we conclude the lemma. □

Proof of Lemma 4.4.8. We recall that if A,B are two pseudo-differential operators then the commutator $[A,B]$ is also a pseudo-differential C. Moreover the principal symbol of C is given by

$$\{a,b\}=\partial_\xi a\partial_y b-\partial_y a\partial_\xi b, \tag{4.5.10}$$

with a,b respectively symbol of A and B. Therefore, $[(1-\chi(D))|D|^\alpha,\varphi_1]\in\mathcal{S}^{\alpha-1,-\alpha-1}\subset\mathcal{S}^{\frac{\alpha}{2},-\alpha-1}$. Then, by applying the pseudo-differential calculus and the fact $\partial_y\varphi_1=\left(\frac{1}{(1+\mu_2)^2}-\frac{1}{(1+\mu_1)^2}\right)\partial_y\varphi$, we have that

$$\|[(1-\chi(D))|D|^\alpha,\varphi_1]u\|_{L^2}\leqslant C\left|\frac{1}{(1+\mu_1)^2}-\frac{1}{(1+\mu_2)^2}\right|^{\frac{1}{2}}\|u\Phi_1\|_{H^{\frac{\alpha}{2}}}.$$

Now, we estimate the low frequency. Let k be the operator defined by $\mathcal{F}(k(u))(\xi)=\chi(\xi)|\xi|^\alpha\mathcal{F}(u)(\xi)$. Then, we have that

$$[\chi(D)|D|^\alpha,\varphi_1]u=\left(\frac{1}{(1+\mu_2)^2}-\frac{1}{(1+\mu_1)^2}\right)\int k(x-y)(\varphi(y)-\varphi(x))u(y)dy.$$

To prove that $[\chi(D)|D|^\alpha,\varphi_1]$ defines an operator bounded on $L^2(\mathbb{R})$, we use the Schur's lemma (Lemma 4.5.4) on $x\mapsto\displaystyle\int k(x-y)(\varphi(y)-\varphi(x))u(y)dy$ and by using Lemma 4.5.5 and 4.5.6. Notice that this process gives us an explicit constant in term of μ_1 and μ_2. By changing the variable $x=\dfrac{x'}{A}$, we deduce that:

$$\|[|D|^\alpha,\varphi_{1,A}]u\|_{L^2}\leqslant\frac{C}{A^{\frac{\alpha-1}{2}}}\left|\frac{1}{(1+\mu_1)^2}-\frac{1}{(1+\mu_2)^2}\right|^{\frac{1}{2}}\|u\Phi_{1,A}\|_{H^{\frac{\alpha}{2}}}.$$

We obtain by definition of the Sobolev space:

$$\|[|D|^\alpha,\varphi_{1,A}]u\|_{L^2}\leqslant C\left|\frac{1}{(1+\mu_1)^2}-\frac{1}{(1+\mu_2)^2}\right|^{\frac{1}{2}}\|u\Phi_{1,A}\|_{H^1}.$$

This concludes the proof of Lemma 4.4.8. □

Proof of Lemma 4.4.10. The proof is based on the same arguments as the former lemmas. For the high frequency we use the pseudo-differential calculus, except that we use the function $\sqrt{\varphi}$ instead of φ. Using the Poisson bracket in (4.5.10), we deduce that the commutator satisfies $[(1-\chi(D))|D|^\alpha,\sqrt{\varphi}]\in\mathcal{S}^{\alpha-1,-1-\frac{\alpha}{2}}\subset\mathcal{S}^{\frac{\alpha}{2},0}$, and we can use the same arguments as above. For the low frequency we use the Schur's lemma (Lemma 4.5.4). □

4.5.4 Proof of the coercivity property

We prove the following result of coercivity which is time-independent, with R_1, R_2, $\widetilde{R}_1$ and $\widetilde{R}_2$ defined in (4.3.1) and dependent on Γ satisfying the condition (Cond$_Z$):

Claim 4.5.7. *Let $\varepsilon \in H^{\frac{\alpha}{2}}(\mathbb{R})$ satisfying the four orthogonality conditions:*

$$0 = \int \varepsilon R_1 = \int \varepsilon \partial_y R_1 = \int \varepsilon R_2 = \int \varepsilon \partial_y R_2,$$

and $\Gamma = (z_1, z_2, \mu_1, \mu_2)$ satisfying (Cond$_Z$). *Then, there exists A_2, Z_1^*, $\kappa > 0$ such that for all $A > A_2$ and Γ satisfying (Cond$_{Z_1^*}$):*

$$\sum_{i=1}^{2} \int \left(\varepsilon |D|^\alpha \varepsilon + \varepsilon^2 - 3\widetilde{R}_i^2 \varepsilon^2\right) \psi_{i,A} \geqslant \kappa \|\varepsilon\|^2_{H^{\frac{\alpha}{2}}}, \quad i = 1, 2,$$

with $\psi_{1,A} := \dfrac{1-\varphi_A}{(1+\mu_1)^2}$ *or* $\psi_{2,A} := \dfrac{\varphi_A}{(1+\mu_2)^2}$.

Proof. Since $\psi_{i,A} > 0$, and L is coercive, see (4.1.10), we deduce that:

$$\begin{aligned}
&\int \left(\varepsilon |D|^\alpha \varepsilon + \varepsilon^2 - 3\widetilde{R}_i^2 \varepsilon^2\right) \psi_{i,A} \\
&\quad = \int \left(|D|^{\frac{\alpha}{2}} \left(\varepsilon \sqrt{\psi_{i,A}}\right)\right)^2 + \left(\varepsilon \sqrt{\psi_{i,A}}\right)^2 - 3\widetilde{R}_i^2 \left(\varepsilon \sqrt{\psi_{i,A}}\right)^2 + \int \varepsilon \sqrt{\psi_{i,A}} \left[|D|^\alpha, \sqrt{\psi_{i,A}}\right] \varepsilon \\
&\quad \geqslant \kappa_1 \left\|\varepsilon \sqrt{\psi_{i,A}}\right\|^2_{H^{\frac{\alpha}{2}}} + \int \varepsilon \sqrt{\psi_{i,A}} \left[|D|^\alpha, \sqrt{\psi_{i,A}}\right] \varepsilon - \frac{1}{\kappa_1} \left(\int \varepsilon \sqrt{\psi_{i,A}} \widetilde{R}_i\right)^2 - \frac{1}{\kappa_1} \left(\int \varepsilon \sqrt{\psi_{i,A}} \partial_y \widetilde{R}_i\right)^2.
\end{aligned}$$

Since $\langle \xi \rangle^{\frac{\alpha}{2}} \geqslant \kappa_2 (1 + |\xi|^{\frac{\alpha}{2}})$, we obtain that:

$$\left\|\varepsilon \sqrt{\psi_{i,A}}\right\|^2_{H^{\frac{\alpha}{2}}} \geqslant \kappa_2 \int \left(\varepsilon^2 + (|D|^{\frac{\alpha}{2}} \varepsilon)^2\right) \psi_{i,A} + \kappa_2 \int (|D|^{\frac{\alpha}{2}} (\varepsilon \sqrt{\psi_{i,A}}))^2 - (|D|^{\frac{\alpha}{2}} \varepsilon)^2 \psi_{i,A}.$$

Notice that:

$$\int (|D|^{\frac{\alpha}{2}} (\varepsilon \sqrt{\psi_{i,A}}))^2 - (|D|^{\frac{\alpha}{2}} \varepsilon)^2 \psi_{i,A} = 2 \int \left(|D|^{\frac{\alpha}{2}} (\varepsilon \sqrt{\psi_{i,A}})\right) [|D|^{\frac{\alpha}{2}}, \sqrt{\psi_{i,A}}] \varepsilon - \int \left([|D|^{\frac{\alpha}{2}}, \sqrt{\psi_{i,A}}] \varepsilon\right)^2.$$

Using Lemma 4.4.10 and Young's inequality, we obtain that:

$$\kappa_1 \left\|\varepsilon \sqrt{\psi_{i,A}}\right\|^2_{H^{\frac{\alpha}{2}}} + \int \varepsilon \sqrt{\psi_{i,A}} \left[|D|^\alpha, \sqrt{\psi_{i,A}}\right] \varepsilon \geqslant \kappa_1 \kappa_2 \int \left(\varepsilon^2 + (|D|^{\frac{\alpha}{2}} \varepsilon)^2\right) \psi_{i,A} - \frac{C}{A^{\frac{\alpha}{2}}} \int \varepsilon^2 + (|D|^{\frac{\alpha}{2}} \varepsilon)^2.$$

Note that since $\varepsilon \perp R_i$, we have that:

$$\int \varepsilon \sqrt{\psi_{i,A}} \widetilde{R}_i = \int \varepsilon \left(\sqrt{\psi_{i,A}} - 1\right) R_i + \int \varepsilon \sqrt{\psi_{i,A}} \left(\widetilde{R}_i - R_i\right)$$

Then, by using the Cauchy-Schwarz' inequality, (4.2.16), we get that:

$$\left(\int \varepsilon \sqrt{\psi_{i,A}} \widetilde{R}_i\right)^2 + \left(\int \varepsilon \sqrt{\psi_{i,A}} \partial_y \widetilde{R}_i\right)^2 \leqslant C \|\varepsilon\|^2_{L^2} \left(\frac{1}{z^\alpha} + \|R_i - \widetilde{R}_i\|^2_{H^1}\right).$$

Moreover, we have that $\psi_{1,A} + \psi_{2,A} \geqslant \kappa_3 > 0$. Therefore, we can conclude, with (4.1.7):

$$\|R_i - \widetilde{R}_i\|_{H^1} \leqslant C \mu_i^2,$$

by taking Z and $A > A_2$ large enough, that there exists $\kappa > 0$ such that:

$$\sum_{i=1}^{2} \int \left(\varepsilon |D|^\alpha \varepsilon + \varepsilon^2 - 3\widetilde{R}_i^2 \varepsilon^2\right) \psi_{i,A} \geqslant \kappa \|\varepsilon\|^2_{H^{\frac{\alpha}{2}}}.$$

□

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