Acyclic, star, and injective colouring: bounding the diameter*

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Abstract

We examine the effect of bounding the diameter for a number of natural and well-studied variants of the Colouring problem. A colouring is acyclic, star, or injective if any two colour classes induce a forest, star forest or disjoint union of vertices and edges, respectively. The corresponding decision problems are Acyclic Colouring, Star Colouring and Injective Colouring. The last problem is also known as $L(1,1)$-Labelling and we also consider the framework of $L(a,b)$-Labelling. We prove a number of (almost-)complete complexity classifications. In particular, we show that for graphs of diameter at most d, Acyclic 3-Colouring is polynomial-time solvable if $d \leqslant 2$ but NP-complete if $d \geqslant 4$, and Star 3-Colouring is polynomial-time solvable if $d \leqslant 3$ but NP-complete for $d \geqslant 8$. As far as we are

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aware, STAR 3-COLOURING is the first problem that exhibits a complexity jump for some $d \geqslant 3$. Our third main result is that $L(1,2)$-LABELLING is NP-complete for graphs of diameter 2; we relate the latter problem to a special case of HAMILTONIAN PATH.

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1 Introduction

A natural way of increasing our understanding of NP-complete problems is to put some restrictions on the input. For graph problems, this means that we may consider graphs from a class characterized by a special property or parameter. In particular, *hereditary* graph classes have been studied. These are the graph classes closed under vertex deletion. The framework of hereditary graph classes covers many well-known graph classes, including H-free graphs (graphs with no induced subgraph isomorphic to some fixed graph H), bipartite graphs, chordal graphs, planar graphs, and so on. However, not all natural graph classes studied in the literature are hereditary. Moreover, studying non-hereditary graph classes may also yield new insights in the computational complexity of NP-complete graph problems. The latter is the goal in this paper, and for this purpose we consider classes of graphs whose diameter is bounded by some constant $d \geqslant 1$. The diameter of a graph is a well-studied graph parameter; see, for instance, the survey [36]. For example, by first focussing on trees of bounded diameter, Omoomi, Roshanbin and Dastjerdi [26] were able to design a polynomial-time algorithm for determining an optimal star edge colouring of a tree (in our paper we consider, among others, star vertex colourings of graphs of bounded diameter).

1.1 Bounding the diameter

The *diameter* of a graph G is the maximum distance between any two vertices of G. For a positive integer d, the class of graphs of diameter at most d is hereditary if and only if $d \leqslant 1$; in order to see this, note that graphs of diameter 1 are the complete graphs, whereas the path P_3 on three vertices has diameter 2 but becomes disconnected after removing the middle vertex.

Many graph problems stay NP-complete if we bound the diameter, even if we set $d = 2$ (note that the case $d = 1$ is of limited interest in most problem settings). The reason for this hardness is usually the following: from a general problem instance we can obtain an equivalent instance of diameter 2 by adding a *dominating* vertex, that is, a vertex that is made adjacent to all the other vertices of the graph. For example, this reduction can be used for classical graph problems, such as those of deciding if for a given integer k, a graph has a clique of size at most k (CLIQUE) or an independent set of size at most k (INDEPENDENT SET). For the INDEPENDENT SET problem, bounding the diameter does not yield any new tractable classes even if the instance is also H-free for some graph H [10] (in this case adding a dominating vertex may violate the H-freeness condition).

The simple trick of adding a dominating vertex can also be used for graph partitioning problems. To give a well-known example, a vertex mapping $c : V \to \{1, 2, \ldots, k\}$ is a *colouring*, or more specifically, a k*-colouring* of a graph $G = (V, E)$ if for every edge $uv \in E$ it holds that $c(u) \neq c(v)$. The Colouring problem is to decide for a given graph G and integer k, if G has a k-colouring, or equivalently, if $V(G)$ can be partitioned into k independent sets. By using the trick, it is readily seen that Colouring stays NP-complete even for graphs of diameter 2.

However, the situation becomes less clear for graph partitioning problems if the upper bound k on the number of partitioning classes is *fixed*, that is, no longer part of the input. In this setting, adding a dominating vertex may increase the number of partition classes by one, as is the case for the Colouring problem (see [15] for a new bound on the diameter of a k-colourable graph). If k is fixed, we write k-Colouring instead. By using another (straightforward) gadget, it follows nevertheless that for $d \geqslant 2$ and $k \geqslant 3$, the k-Colouring problems for graphs of diameter at most d stays NP-complete for every pair $(d, k) \notin \{(2, 3), (3, 3)\}$. In addition, Mertzios and Spirakis [34] gave a highly non-trivial NP-hardness proof for the case $(3, 3)$. The case $(2, 3)$, that is, determining the computational complexity of 3-Colouring for graphs of diameter at most 2, is a notorious open problem [3, 11, 16, 32, 33, 34, 38] (which is not the focus of our paper).

The problem Near-Bipartiteness is to decide if a graph has a 3-colouring such that (only) two colour classes induce a forest. In contrast to the aforementioned problems, this problem is an example of a graph partitioning problem with fixed k, for which bounding the diameter to $d = 2$ gives us a positive result. Namely, the Near-Bipartiteness problem, on graphs of diameter at most d, is polynomial-time solvable if $d \leqslant 2$ [42] and NP-complete if $d \geqslant 3$ [7].

1.2 Our Focus

We consider a number of well-studied and closely related variants of graph colouring (in particular for fixed k) and ask:

How much does bounding the diameter help for obtaining polynomial-time algorithms for well-known graph colouring variants?

In order to define the variants, we first need to introduce some new terminology. For $i \in \{1, \ldots, k\}$, the ith *colour class* of a graph $G = (V, E)$ with a k-colouring c is the set

$$V_i = \{u \in V \mid c(u) = i\}.$$

For $i \neq j$, let $G_{i,j}$ be the (bipartite) subgraph of G induced by $V_i \cup V_j$. If every $G_{i,j}$ is a forest, then c is an *acyclic (k-)colouring*. For an integer $n \geqslant 1$, let P_n denote the n-vertex path. If every $G_{i,j}$ is a P_4-free forest, that is, a disjoint union of stars, then c is a *star (k-)colouring*. If every $G_{i,j}$ is P_3-free, that is, a disjoint union of vertices and edges, then c is an *injective (k-)colouring*. Note that an injective colouring is a star colouring and a star colouring is an acyclic colouring, but the reverse implications might not be true.

The three decision problems, which are to decide for a given graph G and integer $k \geqslant 1$, if G has an acyclic k-colouring, star k-colouring or injective k-colouring, respectively, are called Acyclic Colouring, Star Colouring and Injective Colouring, respectively.[1] If k is fixed, then we write Acyclic k-Colouring, Star k-Colouring and Injective k-Colouring.

In another well-studied framework, injective colourings are known as *distance-2 colourings* and as $L(1,1)$-*labelings*. Namely, a colouring of a graph G is injective if the neighbours of every vertex of G are coloured differently, that is, also vertices of distance 2 from each other must be coloured differently. More generally, for positive integers $a_1, \ldots, a_p$, a vertex mapping $c : V \to \{1, 2, \ldots, k\}$ is an $L(a_1, \ldots, a_p)$-(k-)labelling if for every two vertices u and v and every integer $1 \leqslant i \leqslant p$: if G contains a path of length i between u and v, then $|c(u) - c(v)| \geqslant a_i$; see also [12]. If $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_p$, the condition is equivalent to "if u and v are of distance i". For integers $a_1, \ldots, a_p$ ($p \geqslant 1$), the *distance constrained labelling* problem $L(a_1, \ldots, a_p)$-Labelling is to decide for a given graph G and integer k, if G has an $L(a_1, \ldots, a_p)$-k-labelling.

All the above problems are NP-complete, even for very restricted (hereditary) graph classes, see, for example, [1, 2, 4, 14, 13, 24, 27, 28, 29, 30, 31, 35, 37, 39, 41, 43] and more recent papers, such as [5, 6, 23, 40].[2] We refer to the survey paper of Calamoneri [12] for a large variety of complexity results on distance constrained labelling problems.

Recall from the aforementioned example of Near-Bipartiteness that bounding the diameter may yield a change in computational complexity from $d = 2$ to $d = 3$. To illustrate a possible complexity change even better, we consider $L(a_1, \ldots, a_p)$-k-Labelling. The degree of every vertex of a graph G with an $L(a_1, \ldots, a_p)$-k-labelling is at most k. Hence, $|V(G)| \leqslant 1 + k + \ldots + k^d$, where d is the diameter of G, and we can make the following observation:

Proposition 1. *For positive integers $a_1, \ldots, a_p, d$, and k, the $L(a_1, \ldots, a_p)$-k-*Labelling *problem is constant-time solvable for graphs of diameter at most d.*

Note that Proposition 1 implies that for every $k \geqslant 1$ and $d \geqslant 1$, Injective k-Colouring (the case where $p = 2$ and $a_1 = a_2 = 1$) is constant-time solvable for graphs of diameter at most d. However, if k is part of the input, then Injective Colouring is NP-complete even for graphs of diameter at most 2, and the same holds for Acyclic Colouring and Star Colouring. This follows immediately from the "dominating vertex" trick.

1.3 Our Results

Motivated by Proposition 1 we first consider the problems Acyclic k-Colouring and Star k-Colouring for graphs of bounded diameter. In Sections 2 and 3, respectively, we prove the following two almost-complete dichotomies; note that the case where $k \leqslant 2$ is trivial.

[1] In some papers (for example, [20, 21, 22]), injective colourings are not necessarily proper, that is, two adjacent vertices may be coloured alike. However, we do *not* allow this: as can be observed from the definitions, all colourings considered in our paper are proper.

[2] Some of the old and recent papers in this list also contain tractability results for hereditary graph classes. These classes are not the focus of our paper. However, these papers do illustrate that the colouring variants we study in the paper have a long history.

Theorem 2. *For $d \geqslant 1$ and $k \geqslant 3$,* Acyclic k-Colouring *on graphs of diameter at most d is polynomial-time solvable if $d = 1$, $k \geqslant 4$ or $d \leqslant 2$, $k = 3$ and* NP*-complete if $d \geqslant 2$, $k \geqslant 4$ or $d \geqslant 4$, $k = 3$.*

Theorem 3. *For $d \geqslant 1$ and $k \geqslant 3$,* Star k-Colouring *on graphs of diameter at most d is polynomial-time solvable if $d = 1$, $k \geqslant 4$ or $d \leqslant 3$, $k = 3$ and* NP*-complete if $d \geqslant 2$, $k \geqslant 4$ or $d \geqslant 8$, $k = 3$.*

Theorem 2 leaves only open the case where $d = k = 3$, that is, Acyclic 3-Colouring for graphs of diameter at most 3. Theorem 3 leaves only open four cases where $4 \leqslant d \leqslant 7$ and $k = 3$, that is, Star 3-Colouring for graph of diameter at most d, where $d \in \{4, 5, 6, 7\}$. The case $d = 3$, $k = 4$ in Theorem 2 follows from a stronger result that we prove. Namely, we show (in Section 2) that Acyclic 3-Colouring is NP-complete even for triangle-free 2-degenerate graphs of diameter at most 4 (a graph is 2-*degenerate* if every subgraph of it has a vertex of degree at most 2). This is a reduction using a new gadget. The other hardness results in Theorems 2 and 3 are obtained by straightforward reductions. Our new polynomial-time result in Theorem 2 is for the case where $d = 2$ and $k = 3$. We obtain this result by a careful analysis of the structure of the diameter-2 yes-instances of Acyclic 3-Colouring.

The *main result* of our paper is the new polynomial-time result in Theorem 3, which is for the case where $d = 3$ and $k = 3$. Namely, we prove that Star 3-Colouring can be solved in polynomial time for graphs of diameter at most 3. As we also show that Star 3-Colouring is NP-complete for graphs of diameter at most 8, we have a complexity jump between $d = 3$ and $d = 8$. We are not aware of any other graph problems exhibiting a jump after $d = 3$.

In order to prove our main result we deduce some structural and easy-to-verify properties of diameter 3 yes-instances of Star 3-Colouring. This analysis allows us to preprocess the input graph in order to make its structure simpler. Consequently, we can reduce a single instance of Star 3-Colouring to a polynomial number of instances of 2-List Colouring. This is a standard step in graph colouring, as 2-List Colouring is known to be polynomial-time solvable. Nevertheless some problem-specific technical analysis is needed in order to perform this step. Moreover, in contrast to classical graph colouring, we are not done yet as we need the star colouring property to hold as well. However, we show that this property can indeed be preserved by a small blow-up of the created instances of 2-List Colouring.

Finally, we consider $L(a, b)$-Labelling for the most studied values of (a, b), namely when $1 \leqslant a \leqslant b \leqslant 2$. Due to Proposition 1, we now assume that k is part of the input. Every two non-adjacent vertices in a graph G of diameter 2 have a common neighbour. Hence, an $L(1, 1)$-labelling of G colours each vertex uniquely. Therefore, $L(1, 1)$-Labelling is trivial for graphs of diameter at most 2. The L(1,1)-Labelling problem is still NP-complete for graphs of diameter at most 3, as it is NP-complete for split graphs [4] (a graph is *split* if its vertex set can be partitioned into a clique and independent set, and connected split graphs have diameter at most 3). Griggs and Yeh [19] proved that $L(2, 1)$-

Labelling is NP-complete for graphs of diameter at most 2 by pinpointing a relation with Hamiltonian Path.

The above leaves us with exactly one case, namely $L(1,2)$-Labelling for graphs of diameter at most 2. In Section 4, we prove that this case is NP-complete as well, by making a connection to Hamiltonian Path as well. That is, we observe that an n-vertex graph G of diameter 2 has an $L(1,2)$-n-labelling if and only if G has a Hamiltonian path, no edge of which is contained in a triangle. This observation allows us to adapt the construction of Krishnamoorthy [25] for proving that Hamilton Cycle is NP-complete for bipartite graphs.

To summarize, we obtained the following dichotomy for $L(a,b)$-Labelling with $(a,b) \in \{1,2\}$ restricted to graphs of bounded diameter:

Theorem 4. *For $a,b \in \{1,2\}$ and $d \geqslant 1$, $L(a,b)$-Labelling on graphs of diameter at most d is*

- *polynomial-time solvable if $a=b$ and $d \leqslant 2$, or $d=1$; and*
- NP*-complete if either $a=b$ and $d \geqslant 3$, or $a \neq b$ and $d \geqslant 2$.*

2 The Proof of Theorem 2

We first prove the following result. In the proof of this result we let the graph $2P_2$ denote the disjoint union of two 2-vertex paths, that is, $2P_2$ is the graph with vertices x_1, x_2, y_1, y_2 and edges x_1x_2 and y_1y_2. For a graph $G=(V,E)$, we write $N_G(u) = \{v \mid uv \in E\}$ for the *neighbourhood* of a vertex $u \in V$, and we write $N_G(U) = \bigcup_{u \in U} N_G(u) \setminus U$ for the *neighbourhood* of a set $U \subseteq V$ (we omit subscripts if there is no confusion possible).

Lemma 5. Acyclic 3-Colouring *is polynomial-time solvable for graphs of diameter at most* 2.

Proof. Let $G=(V,E)$ be a graph of diameter at most 2 with n vertices and m edges. If $n \leqslant 24$ or G has diameter 1, we check if G has an acyclic 3-colouring in constant time. We can check in $O(mn)$ time if there is a vertex u such that $G-u$ is a forest. If so, then G has an acyclic 3-colouring (give u colour 1 and use colours 2 and 3 for $G-u$). Now assume that G has at least 25 vertices and diameter 2 and $G-u$ is not a forest for every $u \in V$. In particular, the latter implies that if G is a yes-instance of Acyclic 3-Colouring, then every acyclic 3-colouring of G has three non-empty colour classes. We show a crucial claim:

Claim 1. If G is a yes-instance of Acyclic 3-Colouring*, then there exists a set $S \subseteq V$ with $|S| \leqslant 1$ such that $G-S$ contains an induced $2P_2$, say with edges u_1v_1 and u_2v_2, for which the following two conditions hold:*

- *$(N_G(\{u_1,v_1\}) \cap N_G(\{u_2,v_2\})) \setminus S$ is a colour class of an acyclic 3-colouring c of G, and*

- *the other two colour classes of c induce a subgraph of G with at most two connected components.*

Proof of Claim 1. Assume G has an acyclic 3-colouring c with colour classes X_1, X_2 and X_3 with $|X_1| \leqslant |X_2| \leqslant |X_3|$. If $|X_1| = 1$, say $X_1 = \{v\}$, then $V \setminus \{v\} = X_2 \cup X_3$ induces a forest, as c is acyclic. This contradicts our assumption that $G - u$ is not a forest for every $u \in V$. We obtain the same contradiction if $X_1 = \emptyset$; in that case we can pick v to be an arbitrary vertex of V. Hence, we find that $2 \leqslant |X_1| \leqslant |X_2| \leqslant |X_3|$.

Let $F = G - X_3$. Note that F is a forest, as c is acyclic, and that $|X_3| \geqslant 9$, as $|V| \geqslant 25$. Let U be the set of isolated vertices of F, and let M be a matching of maximum size in F such that each edge of M is incident to a leaf of F. As G has diameter 2, every vertex of X_3 is adjacent to every vertex of U. By the same reason, every vertex in X_3 must be adjacent to a leaf of F or else to its parent in F. Hence, every vertex of X_3 is also adjacent to at least one end-vertex of every edge of M.

As c is acyclic, the subgraphs induced by $X_1 \cup X_3$ and $X_2 \cup X_3$ are forests. Hence, there do not exist sets $T \subseteq X_1 \cup X_2$ with $|T| = 3$ and $X_3' \subseteq X_3$ with $|X_3'| = 2$ such that every vertex of T is adjacent to every vertex of X_3'. This observation leads to bounds on the sizes of U, M and $M \cup U$, as we show below.

First, as $|X_3| \geqslant 9 \geqslant 2$ and every vertex of X_3 is adjacent to every vertex of $U \subseteq X_1 \cup X_2$, the above observation implies that $|U| \leqslant 2$. We claim that also $|M| \leqslant 2$. For a contradiction, assume that a_1b_1, a_2b_2, a_3b_3 are three distinct edges of M where $\{a_1, a_2, a_3\} \subseteq X_1$ and $\{b_1, b_2, b_3\} \subseteq X_2$. Recall that $|X_3| \geqslant 9$ and that every vertex of X_3 is adjacent to at least one end-vertex of every edge of M. Then, at least five vertices of X_3 are adjacent to the same vertex in $\{a_1, b_1\}$, at least three out of these five vertices are adjacent to the same vertex in $\{a_2, b_2\}$ and at least two out of the latter three vertices are adjacent to the same vertex in $\{a_3, b_3\}$. Hence, we found a subset $X_3' \subseteq X_3$ with $|X_3'| = 2$ and a subset $T \subseteq X_1 \cup X_2$ with $|T| = 3$, such that every edge between X_3' and T exists, a contradiction. We conclude that $|M| \leqslant 2$ must hold indeed. By the same arguments we find that $|M| + |U| \leqslant 2$.

We now continue as follows. Recall that $2 \leqslant |X_1| \leqslant |X_2| \leqslant |X_3|$. First suppose that $|M| = 0$, so $X_1 \cup X_2 = U$. Then $4 \leqslant |X_1| + |X_2| = |U|$, a contradiction to the fact that $|U| \leqslant 2$. Hence, $|M| \geqslant 1$. As $|M| \leqslant 2$, this means that $|M| = 1$ or $|M| = 2$. As $|M| + |U| \leqslant 2$, we find that either $|M| = 1$ and $|U| = 0$; or $|M| = |U| = 1$; or $|M| = 2$ and $|U| = 0$.

First suppose that $|M| = 1$ and $|U| = 0$. Then F is a star, say with center u, on two vertices or more. As $|X_1| \leqslant |X_2|$, we find that $X_1 = \{u\}$, a contradiction to the fact that $|X_1| \geqslant 2$. Now suppose that $|M| = 1$ and $|U| = 1$. Let $U = \{u\}$ for some $u \in X_1 \cup X_2$. As $|M| = 1$, we find that $V(F) \setminus U$ induces a star with center a and leaves $b_1, \ldots, b_r$ for some $r \geqslant 1$. As $2 \leqslant |X_1| \leqslant |X_2|$, we find that $r \geqslant 2$ and that we may assume without loss of generality that $X_1 = \{a, u\}$ and $X_2 = \{b_1, \ldots, b_r\}$. As every vertex of X_3 is adjacent to u and c is acyclic, at most one vertex of X_3 is adjacent to a. Consequently, every other vertex of X_3 is adjacent to every b_i, as G has diameter 2. However, as c is acyclic and $r \geqslant 2$, at most one vertex of X_3 can be adjacent to every b_i. Then X_3 has at most two vertices, contradicting the fact that $|X_3| \geqslant 9$.

From the above we conclude that $|M| = 2$ and $|U| = 0$. Let $M = \{u_1v_1, u_2v_2\}$. Note that $\{u_1, v_1, u_2, v_2\}$ induces a $2P_2$, as c is acyclic and $\{u_1, v_1, u_2, v_2\} \subseteq X_1 \cup X_2$. As F is a forest, $N_G(\{u_1, v_1\}) \cap N_G(\{u_2, v_2\})$ contains at most one vertex s. Let $S = \{s\}$ if such a vertex s exists and let $S = \emptyset$ otherwise. Since every vertex of X_3 is adjacent to an end-vertex of every edge of M, we find that $X_3 = (N_G(\{u_1, v_1\}) \cap N_G(\{u_2, v_2\})) \setminus S$. Moreover, as $|M| \leqslant 2$ and $U = \emptyset$, we have that $F = G - X_3$ has at most two connected components. Hence, we have proven Claim 1. $\diamond$

We consider all possible $O(m^2n)$ selections of an induced $2P_2$ with edges u_1v_1 and u_2v_2 and a set of vertices $S \subseteq V(G) \setminus \{u_1, v_1, u_2, v_2\}$ of size at most 1. For each choice, we find, in $O(n+m)$ time, the set $X_3 = (N_G(\{u_1, v_1\}) \cap N_G(\{u_2, v_2\})) \setminus S$. Then we verify in $O(n+m)$ time if X_3 is an independent set and $F = G - X_3$ is a forest with at most two connected components. If this is not the case, we discard the current choice. Otherwise we continue. As F has at most two connected components, F has at most two 2-colourings (up to symmetry). For each 2-colouring of F, we check in $O(n+m)$ time if its two colour classes X_1 and X_2 together with X_3 yield an acyclic 3-colouring of G.

The correctness of the algorithm immediately follows from Claim 1. The running time is $O(m^3n)$. The latter can be improved to $O(n^4)$ as follows. We first check whether $m \leqslant 2n - 3$ holds. We are allowed to do so, as the latter is a necessary condition: if c is an acyclic 3-colouring of G with colour classes X_1, X_2 and X_3, then G is the union of the forests $G[X_1 \cup X_2]$, $G[X_1 \cup X_3]$ and $G[X_2 \cup X_3]$, and thus it holds that $m \leqslant |X_1 \cup X_2| - 1 + |X_1 \cup X_3| - 1 + |X_2 \cup X_3| - 1 = 2n - 3$. $\square$

We complement the previous, algorithmic result by a new hardness result. We prove this result by a reduction from the problem NEAR-BIPARTITENESS, which is known to be NP-complete [8]. Recall that this problem asks whether for a given graph G, it holds that $V(G) = V(F) \cup I$ for some forest F and independent set I such that $V(F) \cap I = \emptyset$. If so, we say that G is *near-bipartite* and we call the corresponding vertex partition an (I, F)*-partition*. We make the following observation.

Lemma 6. NEAR-BIPARTITENESS *is* NP*-complete even for graphs of minimum degree at least* 3.

Proof. We reduce from NEAR-BIPARTITENESS itself. Let G be a graph. We exhaustively delete vertices of degree at most 2 from G until we obtain a new graph G' in which every vertex has degree at least 3. Note that we obtained G' in polynomial time. We claim that G is near-bipartite if and only if G' is near-bipartite.

If G is near-bipartite, then G', being an induced subgraph of G, is near-bipartite as well. Now suppose that G' is near-bipartite. So G' has an (I', F')-partition for some forest F' and independent set I'. We restore each vertex $u \in V(G) \setminus V(G)$ back into the graph in the reverse order of deletion. So, at the moment we put a vertex u back, u is adjacent to at most two vertices in the graph. If one these vertices belong to I', then we can safely put u in F'; else we can safely put u in I'. In the end, we will have extended (I', F')-partition of G' into an (I, F)-partition of G. The latter means that G is near-bipartite as well. $\square$

Theorem 7. *The* ACYCLIC 3-COLOURING *problem is* NP*-complete for triangle-free* 2*-degenerate graphs of diameter at most* 4.

Proof. As mentioned, we reduce from the problem NEAR-BIPARTITENESS, which is known to be NP-complete [8]. Let G be an instance of NEAR-BIPARTITENESS. By Lemma 6 we may assume that G has minimum degree at least 3.

From G we construct a graph G' as follows. We subdivide every edge of G to obtain a bipartite graph containing the *old* vertices of degree at least 3 in one part and the *new* vertices of degree 2 in the other part. We add a vertex x. For every old vertex v_o, we add two vertices of degree 2 adjacent to v_o and x. For every new vertex v_n, we add three vertices of degree 2 adjacent to v_n and x. Figure 1 shows the graph G' if G is an edge uv.

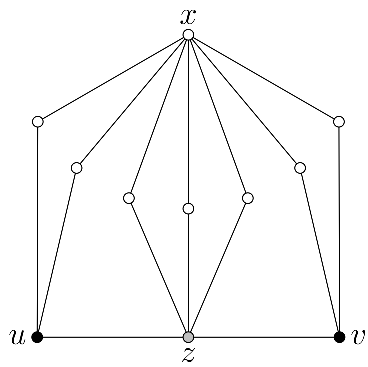

Figure 1: The graph G' if G consists of the edge uv. The vertices u and v are the old vertices of G', and their common neighbour z is the new vertex of G'.

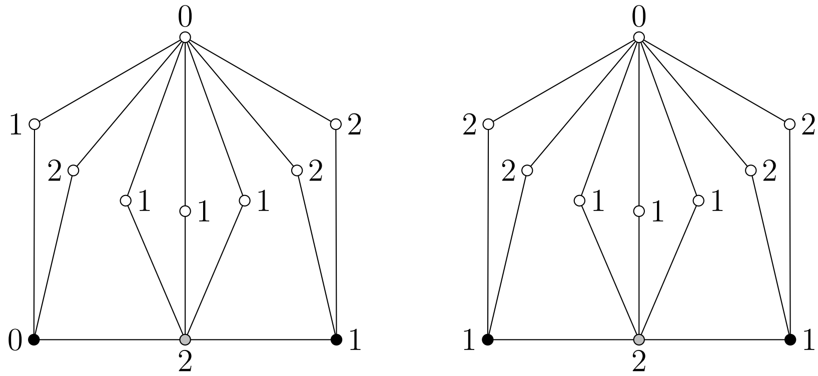

Figure 2: Two acyclic colourings of G', each resulting from a different (I, F)-partition of G. On the left: $(I, F) = (\{u\}, \{v\})$. On the right: $(I, F) = (\emptyset, \{u, v\})$.

By construction, G' is triangle-free, 2-degenerate, and its diameter is at most 4 since every vertex is at distance at most 2 from x. We claim that G has an (I, F)-partition if and only if G' has an acyclic 3-colouring.

First suppose that G has an (I, F)-partition. We assign colour 0 to x. We assign colour 2 to every new vertex. We assign colour 1 to every vertex adjacent to x and a new vertex. For every vertex in F, we assign colour 1 to the corresponding old vertex v_F and we assign colour 2 to the two vertices adjacent to x and v_F. For every vertex in I, we assign colour 0 to the corresponding old vertex v_I and we assign colours 1 and 2 to the two vertices adjacent to x and v_I. See also Figure 2.

We claim that this 3-colouring of G' is acyclic. In order to see this, let $G'_{i,j}$ be the induced subgraph of G' with vertices coloured i and j for $i \neq j$. We first consider $G'_{0,1}$. The connected component of $G'_{0,1}$ that contains x is a tree in which every vertex except x has degree 1 or 2 and is at distance at most 2 from x. Every other connected component of $G'_{0,1}$ consists of an isolated (old) vertex coloured 1.

Now consider $G'_{0,2}$. As I is an independent set of G, every new vertex in G' is adjacent to at most one vertex with colour 0. Hence, the connected component of $G'_{0,2}$ that contains x is a tree in which every vertex except x has degree 1 or 2 and is at distance at most 3 from x. Every other connected component of $G'_{0,2}$ consists of an isolated (new) vertex coloured 2.

Finally consider the graph $G'_{1,2}$. For contradiction, suppose $G'_{1,2}$ contains a cycle. This cycle cannot contain a vertex adjacent to x in G', since such a vertex has degree 1 in $G'_{1,2}$. So this cycle alternates between old vertices coloured 1 and new vertices coloured 2. These old and new vertices correspond respectively to the vertices and the edges of a cycle in F, a contradiction. So G' has an acyclic 3-colouring.

Now suppose that G' has an acyclic 3-colouring c. Say x is coloured 0. For every old vertex coloured 0, the corresponding vertex in G is assigned to I. Every other vertex of G is assigned to F.

For contradiction, suppose that I contains an edge uv. Then, G' contains a new vertex z that is adjacent to u and v. As c is acyclic, the two common neighbours of x and u are coloured 1 and 2, respectively, and the same holds for the two common neighbours of x and v. Then G' contains a bichromatic 6-cycle with colours 0 and $c(z)$, a contradiction. Hence, I is an independent set.

Now, for contradiction, suppose that F contains a cycle C. By construction, every old vertex corresponding to a vertex of C is not coloured 0 (as otherwise we would have placed it in I). We observe that new vertices are not coloured 0, as x is coloured 0 and every new vertex has three common neighbours with x, at least two of which are coloured alike. Hence, every new vertex corresponding to an edge of C is not coloured 0. Therefore G' contains a bichromatic cycle with colours 1 and 2, a contradiction. So G admits an (I, F)-partition. $\square$

We are now ready to prove Theorem 2.

Theorem 2 (restated). *For $d \geqslant 1$ and $k \geqslant 3$,* Acyclic k-Colouring *on graphs of diameter at most d is polynomial-time solvable if $d = 1$, $k \geqslant 4$ or $d \leqslant 2$, $k = 3$ and* NP*-complete if $d \geqslant 2$, $k \geqslant 4$ or $d \geqslant 4$, $k = 3$.*

Proof. The cases $d \leqslant 2$, $k = 3$ and $d \geqslant 4$, $k = 3$ follow from Lemma 5 and Theorem 7, respectively. The case $d = 1$, $k \geqslant 4$ is trivial: a graph of diameter 1 is a complete graph

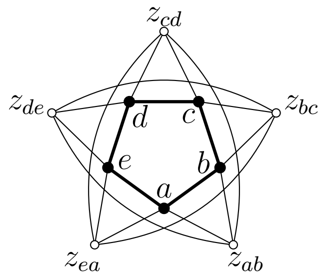

Figure 3: The edge-extension G_{ex} if G is the 5-vertex cycle with vertices a, b, c, d, e in that order.

and as such, it has an acyclic k-colouring if and only if it has at most k vertices. For the case $d \geqslant 2$, $k \geqslant 4$ we reduce from Acyclic 3-Colouring. To an instance G of Acyclic k-Colouring, we add a clique of $k-3$ vertices, which we make adjacent to every vertex of G. The resulting graph G' has diameter at most 2. Moreover, G' has an acyclic k-colouring if and only if G has an acyclic 3-colouring. □

3 The Proof of Theorem 3

A *list assignment* of a graph $G = (V, E)$ is a function L that gives each vertex $u \in V$ a *list of admissible colours* $L(u) \subseteq \{1, 2, \ldots\}$. A colouring c *respects* L if $c(u) \in L(u)$ for every $u \in V$. If $|L(u)| \leqslant 2$ for each $u \in V$, then L is a 2-*list assignment*. The 2-List Colouring problem is to decide if a graph G with a 2-list assignment L has a colouring that respects L. We need the following well-known result of Edwards.

Theorem 8 ([17])**.** *The* 2-List Colouring *problem is solvable in time* $O(n+m)$ *on graphs with* n *vertices and* m *edges.*

We will use Theorem 8 in the proof of Lemma 13, which is the main result of the section. In order to do this, we must first be able to modify an instance of Star 3-Colouring into an equivalent instance of 3-Colouring. We can do this as follows. Let $G = (V, E)$ be a graph. We define a supergraph G_{ex} of G as follows. For each edge $e = uv$ of G we add a vertex z_{uv} that we make adjacent to both u and v. We also add an edge between two vertices z_{uv} and $z_{u'v'}$ if and only if u, v, u', v' are four distinct vertices such that G has at least one edge with one end-vertex in $\{u, v\}$ and the other one in $\{u', v'\}$. We say that G_{ex} is the *edge-extension* of G; see Figure 3 for an example. Observe that G_{ex} can be constructed in $O(m^2)$ time.

The next lemma is not difficult to prove.

Lemma 9. *A graph G has a star* 3*-colouring if and only if its edge-extension G_{ex} has a* 3*-colouring.*

Proof. First suppose that G has a star 3-colouring c. We extend c from $V(G)$ to $V(G_{\text{ex}})$ by assigning each z_{uv} the unique colour from $\{1, 2, 3\}$ that is not used on its neighbours

u and v (which are coloured differently by c, as they are adjacent). Assume, for a contradiction, that the resulting mapping $c_{\mathrm{ex}} : V(G_{\mathrm{ex}}) \to \{1, 2, 3\}$ is not a 3-colouring of G_{ex}. Then there must exist two adjacent vertices z_{uv} and $z_{u'v'}$ that have been assigned the same colour by c_{ex}, say colour 1. Then, by definition of G_{ex}, the vertices u, v, u', v' are four different vertices. Moreover, we may assume without loss of generality that c_{ex}, and thus also c, has coloured u, v, u', v' with colours 2, 3, 2, 3, respectively, and moreover that vu' is an edge. However, as $\{u, v, u', v'\}$ induces either a C_4 or P_4, this means that c is not a star 3-colouring of G, a contradiction.

Now suppose that G_{ex} has a 3-colouring c_{ex}. We claim that the restriction c of c_{ex} to $V(G)$ is a star 3-colouring of G. For a contradiction, suppose that c is not a star 3-colouring of G. Then, G has a subgraph that is a 4-path $uvu'v'$ with alternating colours, say 2 and 3. However, then the vertices z_{uv} and $z_{u'v'}$, which by definition are adjacent in G_{ex}, are both coloured 1 by c_{ex}. This contradicts our assumption that c_{ex} is a 3-colouring of G_{ex}. □

Now suppose that G has a 2-list assignment L with $L(u) \subseteq \{1, 2, 3\}$ for every $u \in V$. We extend L to a list assignment L_{ex} of G_{ex}. We first set $L_{\mathrm{ex}}(u) = L(u)$ for every $u \in V(G)$. Initially, we set $L_{\mathrm{ex}}(z_e) = \{1, 2, 3\}$ for each edge $e \in E(G)$. We now adjust every list $L_{\mathrm{ex}}(z_e)$ as follows. Let $e = uv$.

1. If $L(u) = L(v)$ or $L(u)$ has size 1, then set $L_{\mathrm{ex}}(z_{uv}) = L_{\mathrm{ex}}(z_{uv}) \setminus L(u)$.
2. If $L(v)$ has size 1, then set $L_{\mathrm{ex}}(z_{uv}) = L_{\mathrm{ex}}(z_{uv}) \setminus L(v)$.
3. If $z_{u'v'}$ is adjacent to some z_{uv} with $|L'(z_{uv})| = 1$, then set $L_{\mathrm{ex}}(z_{u'v'}) = L_{\mathrm{ex}}(z_{uv}) \setminus L'(z_{uv})$.

We apply the three above rules exhaustively. We call the resulting list assignment L_{ex} of G_{ex} the *edge-extension* of L. We say that an edge uv of G is *unsuitable* if $|L(u)| = |L(v)| = 2$ but $L(u) \neq L(v)$, whereas uv is *list-reducing* if $|L(u)| = |L(v)| = 1$ and $L(u) \neq L(v)$. Note that in G_{ex}, we may have $|L_{\mathrm{ex}}(z_e)| = 3$ if e is unsuitable, whereas $|L_{\mathrm{ex}}(z_e)| = 1$ if e is list-reducing. We say that an end-vertex u of an unsuitable edge e is a *fixer* for e if u is adjacent to an end-vertex of a list-reducing edge $u'v'$ (note that $\{u, v\} \cap \{u', v'\} = \emptyset$). If an unsuitable edge e has a fixer, then $|L_{\mathrm{ex}}(z_e)| \leqslant 2$. More generally, by using Lemma 9, we can make the following straightforward observation.

Lemma 10. *Let $G = (V, E)$ be a graph on m edges with a 2-list assignment L such that $L(u) \subseteq \{1, 2, 3\}$ for every $u \in V$. Then we can construct in $O(m^2)$ time the edge-extension G_{ex} of G and the edge-extension L_{ex} of L. Moreover, G has a star 3-colouring that respects L if and only if G_{ex} has a 3-colouring that respects L_{ex}. Furthermore, L_{ex} is a 2-list assignment of G_{ex} if every unsuitable edge uv of G has a fixer.*

Let $d_G(u)$ denote the degree of a vertex u in G. We need two structural lemmas for the correctness proof of our algorithm for Star 3-Colouring on graphs of diameter at most 3.

Lemma 11. *If a graph G has a star 3-colouring, then*

1. *for every 4-cycle $v_0v_1v_2v_3v_0$ of G, $d_G(v_0) = d_G(v_2) = 2$ or $d_G(v_1) = d_G(v_3) = 2$, and*
2. *there is no 5-cycle in G.*

Proof. Assume that G has a star 3-colouring c. We first show Property 1. Let C be a (not necessarily induced) 4-cycle $v_0v_1v_2v_3v_0$. As c is a star colouring, it is not possible that $c(v_0) = c(v_2)$ and $c(v_1) = c(v_3)$. Hence, we may assume without loss of generality that $c(v_1) \neq c(v_3)$. As c is a 3-colouring, this means that $c(v_0) = c(v_2)$. Say, $c(v_0) = c(v_2) = 1$ and $c(v_1) = 2$ and $c(v_3) = 3$. As $c(v_0) = c(v_2)$, vertices v_0 and v_2 are not adjacent. So, if $d(v_0) \geqslant 3$, then v must have a neighbour u not on C. As $c(v_0) = 1$, we find that $c(u) = 2$ or $c(u) = 3$. In the first case, the vertices u, v_0, v_1, v_2 form a bichromatic P_4 or C_4. In the second case, the vertices u, v_0, v_3, v_2 form a bichromatic P_4 or C_4. However, both cases are not possible, as c is a star colouring. We conclude that $d_G(v_0) = 2$, and by the same arguments that $d_G(v_2) = 2$.

We now show Property 2. Let C be a (not necessarily induced) 5-cycle $v_0v_1v_2v_3v_4v_0$. Then, without loss of generality, $c(v_0) = c(v_2) = 1$ and $c(v_1) = 2$. As c is a star 3-colouring, $c(v_3) \neq 2$ and $c(v_4) \neq 2$. So, $c(v_3) = c(v_4) = 3$, a contradiction, as v_3 and v_4 are adjacent. □

Lemma 12. *Let G be a graph of diameter at most 3. Let u and v be two distinct vertices of G, such that $|N(u) \cap N(v)| \geqslant 3$ and moreover, each vertex in $N(u) \cap N(v)$ has degree 2. Let $w \in N(u) \cap N(v)$. Then G has a star 3-colouring if and only if $G - w$ has a star 3-colouring. Moreover, $G - w$ has diameter at most 3.*

Proof. As each vertex in $N(u) \cap N(v)$ has degree 2, it is only adjacent to u and v. As $|N(u) \cap N(v)| \geqslant 3$, this means that $G - w$ has diameter at most 3. Suppose that G has a star 3-colouring. Then the restriction of this colouring to $V(G) \setminus \{w\}$ is a star 3-colouring of $G - w$.

To prove the reverse implication, suppose that $G - w$ has a star 3-colouring. First assume that $V(G) = (N(u) \cap N(v)) \cup \{u, v\}$. Then we colour u and v with colours 1 and 3, respectively, whilst we give all other vertices of G, which are only adjacent to u and v, colour 2. This yields a star 3-colouring of G. If $V(G) \neq (N(u) \cap N(v)) \cup \{u, v\}$, then we claim that every star 3-colouring of $G - w$ assigns the same colour to all vertices of $(N(u) \cap N(v)) \setminus \{w\}$. Then, as $|(N(u) \cap N(v)) \setminus \{w\}| \geqslant 2$, we can safely assign this colour to w as well and obtain a star 3-colouring of G.

Let c be a star 3-colouring of $G - w$. As G is connected and every vertex in $N(u) \cap N(v)$ is only adjacent to u and v, there exists a vertex z that is adjacent to exactly one of u, v, say z is adjacent to u but not to v. As $|N(u) \cap N(v)| \geqslant 3$, there exist vertices w' and w'' in $N(u) \cap N(v)$. For a contradiction, assume that $c(w') = 1$ and $c(w'') = 2$. Then $c(u) = c(v) = 3$. If $c(z) = 1$, then the vertices z, u, w', v form a bichromatic P_4. If $c(z) = 2$, then the vertices z, u, w'', v form a bichromatic P_4. In both cases, c is not a star 3-colouring of $G - w$, a contradiction. □

Two non-adjacent vertices in a graph G that have the same neighbourhood are *false twins* of G. Our algorithm for Star List 3-Colouring in Lemma 13 that takes as input a graph G of diameter at most 3 can be summarized as follows:

Outline:

1. We modify G into a graph G' by removing all but at most two vertices from any set of false twins of degree 2 in $O(n^2)$ time; we prove that G has a star 3-colouring if and only if G' has a star 3-colouring.

2. We then construct $O(n)$ 2-list assignments L' of G', each of which in $O(n+m)$ time, such that G' has a star 3-colouring if and only if G' has a star 3-colouring respecting at least one of the constructed 2-list assignments L'.

3. For each (G', L'), we prove that the edge-extension L'_{ex} of L' is a 2-list assignment of the edge-extension G'_{ex} of G'. Hence, it remains to solve 2-List-Colouring for each of the $O(n)$ instances $(G'_{\text{ex}}, L'_{\text{ex}})$, which we can do in $O(m^2)$ time by Theorem 8 as the size of G'_{ex} is $O(m^2)$.

We are now ready to give our algorithm in detail.

Lemma 13. Star 3-Colouring *is polynomial-time solvable for graphs of diameter at most* 3.

Proof. Let G be a graph of diameter 3. We first determine in $O(nm^2)$ time all 4-cycles and all 5-cycles in G. If G has a 4-cycle with two adjacent vertices of degree at least 3 in G or if G has a 5-cycle, then G is not star 3-colourable by Lemma 11. We continue by assuming that G satisfies the two properties of Lemma 11. We reduce G by applying Lemma 12 exhaustively. Let G' be the resulting graph, which has diameter at most 3 (by Lemma 12). We can determine in $O(n)$ time all vertices of degree 2 in G. For each vertex of degree 2 we can compute in $O(n)$ time all its false twins. Hence, we found G' in $O(n^2)$ time. As we only removed vertices, G' also satisfies the two properties of Lemma 11.

If G' has maximum degree at most 4, then $|V(G')| \leqslant 53$, as G' has diameter at most 3. We can check in constant time if $|V(G')| \leqslant 53$ and if so, in constant time, if G' has a star 3-colouring. Otherwise, we found a vertex v of degree at least 5 in G'.

Let N_i be the set of vertices of distance i from v. Note that $N_1 = N(v)$ and $V(G') = \{v\} \cup N_1 \cup N_2 \cup N_3$, as G' has diameter at most 3. We assume without loss of generality that if G' has a star 3-colouring, then v will be coloured 1. We will now detect in polynomial time whether G' has a star 3-colouring c with $c(v) = 1$, such that exactly one of the following situations hold: c gives each vertex in N_1 colour 3; or c gives at least one vertex of N_1 colour 2 and at least three vertices of N_1 colour 3. As v has degree at least 5, at least one of colours 2, 3 must occur three times on $N(v)$, and we may assume without loss of generality that this colour is 3. Hence, G' has a star 3-colouring if and only if one of these two situations holds.

Case 1. Check if G' has a star 3-colouring that gives every vertex of N_1 colour 3.

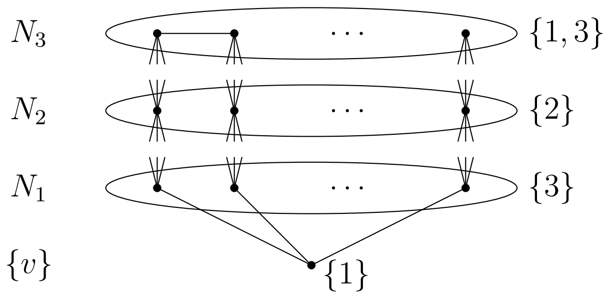

Figure 4: The pair (G', L') in Case 1.

As $|N_1| \geqslant 5$, such a star 3-colouring c must assign each vertex of N_2 colour 2. This means that every vertex of N_3 gets colour 1 or 3. Hence, we obtained, in $O(n)$ time, a 2-list assignment L' of G'. We construct the pair $(G'_{\text{ex}}, L'_{\text{ex}})$. By Lemma 10 this takes $O(m^2)$ time. As every list either has size 1 or is equal to $\{1, 3\}$, we find that the edge-extension L'_{ex} of L' is a 2-list assignment of G'_{ex}. By Lemma 10, it remains to solve 2-List-Colouring on $(G'_{\text{ex}}, L'_{\text{ex}})$. We can do this in $O(m^2)$ time using Theorem 8 as the size of G'_{ex} is $O(m^2)$. Hence, the total running time for dealing with Case 1 is $O(m^2)$. See also Figure 4.

Case 2. Check if G' has a star 3-colouring that gives at least one vertex of N_1 colour 2 and at least three vertices of N_1 colour 3.

We set $L'(v) = \{1\}$. This gives us the following property.

P0. $N_0 = \{v\}$ and $L'(v) = \{1\}$.

We now select four arbitrary vertices of $N(v)$. We consider all possible colourings of these four vertices with colours 2 and 3, where we assume without loss of generality that colour 3 is used on these four vertices at least as many times as colour 2. For the case where colour 2 is not used we consider each of the $O(n)$ options of colouring another vertex from $N(v)$ with colour 2. For the cases where colour 3 is used exactly twice, we consider each of the $O(n)$ options of colouring another vertex from $N(v)$ with colour 3. Hence, the total number of options is $O(n)$, and in each option we have a neighbour x_v of v with colour 2 and a set

$$W = \{w_1, w_2, w_3\}$$

of three distinct neighbours of v with colour 3. So we set $L'(x_v) = \{2\}$ and $L'(w_i) = \{3\}$ for $1 \leqslant i \leqslant 3$.

For each set $\{x_v\} \cup W$ we do as follows. We first check if W is independent; otherwise we discard the option. If W is independent, then initially we set $L'(u) = \{1, 2, 3\}$ for each $u \notin \{x_v, v\} \cup W$. We now show that we can reduce the list of every such vertex u by at least 1. As an *implicit step*, we will discard the instance (G', L') if one of the lists has become empty. In doing this we will use the following *Propagation Rule*:

Whenever a vertex has only one colour in its list, we remove that colour from the list of each of its neighbours.

By the Propagation Rule, we obtain the following property, in which we updated the set W:

P1. N_1 can be partitioned into sets W, X, Y with $|W| \geqslant 3$, $|X| \geqslant 1$ and $|Y| \geqslant 0$, such that no vertex of Y is adjacent to any vertex of $X \cup W$, and moreover, X is an independent set with $x_v \in X$ and W is an independent set with $\{w_1, w_2, w_3\} \subseteq W$, such that

- every vertex $w \in W$ has list $L'(w) = \{3\}$,
- every vertex $x \in X$ has list $L'(x) = \{2\}$, and
- every vertex $y \in Y$ has list $L'(y) = \{2, 3\}$.

Note that by the Propagation Rule, we removed colour 3 from the list of every neighbour of a vertex of W in N_2. We now also remove colour 1 from the list of every neighbour of a vertex of W in N_2; the reason for this is that if a neighbour y of, say, w_1 is coloured 1, then the vertices y, w_1, v, w_2 form a bichromatic P_4. Hence, any neighbour of every vertex in W in N_2 has list $\{2\}$.

Now consider a vertex $z \in N_2$ that still has a list of size 3. Then z is not adjacent to any vertex in N_1 with a singleton list (as otherwise we applied the Propagation Rule), but by definition z still has a neighbour z' in N_1. This means that $z' \in Y$ and thus z' has list $\{2, 3\}$. Hence, z cannot be coloured 1: if z' gets colour 2, the vertices x_v, v, z', z will form a bichromatic P_4, and if z' gets colour 3, the vertices w_1, v, z', z will form a bichromatic P_4. Hence, we may remove colour 1 from $L'(z)$, so $L'(z)$ will have size at most 2.

We make another observation. Namely, no vertex in N_2 has a neighbour both in W and in X. For a contradiction, suppose $z \in N_2$ is adjacent to a vertex in W and to a vertex in X. Recall that every neighbour of every vertex in W in N_2 has list $\{2\}$. Hence, z has list $\{2\}$. Recall also that every vertex in X has list $\{2\}$ as well. This is a contradiction, as z must now have an empty list by the Propagation Rule and we would have discarded this option.

Due to the above, we can *partition* N_2 into sets W^*, X^*, and Y^* such that the vertices of W^* are the neighbours of W and the vertices of X^* are the neighbours of X, whereas $Y^* = N_2 \setminus (X^* \cup W^*)$. Consequently, the neighbours in N_1 of every vertex of Y^* belong to Y. Furthermore, every vertex $w^* \in W^*$ has list $L'(w^*) = \{2\}$, every vertex $x^* \in X^*$ has list $L'(x^*) = \{1, 3\}$, and every vertex $y^* \in Y^*$ has list $L'(y^*) = \{2, 3\}$.

So far, we have that the neighbours in N_1 of every vertex of W^* belong to $W \cup Y$. However, we can apply the Propagation Rule, and afterwards we can update the sets W and Y. Then no vertex in W^* (which has list $\{2\}$) has a neighbour in Y (which has list $\{2, 3\}$), as we moved such neighbours from Y tot W. Hence, the neighbours in N_1 of every vertex of W^* now only belong to W.

So far, we have that the neighbours in N_1 of every vertex of X^* belong to $X \cup Y$. Now suppose that a vertex $x^* \in X^*$ has a neighbour $y \in Y$. Then vxx^*yv is a 4-cycle where x is a neighbour of x^* in X. We claim that x and y must be coloured alike. This can be seen as follows. Suppose x and y are coloured differently, so one of them is coloured 2 and the other one is coloured 3. Then, as x^* and v are each adjacent to both x and y, we have that x^* must get colour 1 just like v. Recall that v has degree at least 5. We consider a third neighbour s of v, which will have colour 2 or 3, so the same colour as either x or y. However, now s, v, x or y, z^* form a bichromatic P_4. This proves the claim. Consequently, as x and y are coloured alike and x has list $\{2\}$, we can update the list of y from $\{2,3\}$ to $\{2\}$. Then, afterwards, we can move y to X. Now, after updating the sets X and Y in this way, we now also have that the neighbours in N_1 of every vertex of X^* only belong to X.

Note that in the above two paragraphs we only updated the sets W, X and Y. That is, the sets W^*, X^* and Y^* have remained the same. The reason is that every vertex y that we moved from Y to either W or X has degree 2 in G'. This follows from the fact that the neighbour z^* of y in $W^* \cup X^*$ is also adjacent to a vertex z in W (if $z^* \in W^*$) or in X (if $z^* \in X^*$). Consequently, vzz^*yv is a 4-cycle in G'. Recall that G' satisfies the properties of Lemma 11. As v has degree at least 5 in G', this means that y has degree 2 in G'.

We apply the Propagation Rule to find that every vertex $x^* \in X^*$ has list $L'(x^*) \subseteq \{1,3\}$, and every vertex $y^* \in Y^*$ has list $L'(y^*) \subseteq \{2,3\}$. We note that applying the Propagation Rule may also result in updated lists of vertices in N_3, but we will deal with the vertices in N_3 next. Recall that G' has no 5-cycles. Hence, there is no edge between vertices from two different sets of $\{W^*, X^*, Y^*\}$. We summarize the above in the following property (see also Figure 5):

P2. N_2 can be partitioned into sets W^*, X^* and Y^*, such that

- every vertex $w^* \in W^*$ has list $L'(w^*) = \{2\}$ and all its neighbours in N_1 belong to W,
- every vertex $x^* \in X^*$ has list $L'(x^*) \subseteq \{1,3\}$ and all its neighbours in N_1 belong to X,
- every vertex $y^* \in Y^*$ has list $L'(y^*) \subseteq \{2,3\}$ and all its neighbours in N_1 belong to Y, and
- there is no edge between vertices from two different sets of $\{W^*, X^*, Y^*\}$.

We now consider the set N_3. We let T_1 be the set consisting of all vertices in N_3 that have at least two neighbours in W^*. We let T_2 be the set consisting of all vertices in N_3 that have exactly one neighbour in W^*. Moreover, we let S_1 be the set of vertices of $N_3 \setminus (T_1 \cup T_2)$ that have at least one neighbour in T_1. We let S_2 be the set of vertices of $N_3 \setminus (T_1 \cup T_2)$ that have no neighbours in T_1 but at least two neighbours in T_2. If for a vertex $s \in N_3$, there is a vertex $w \in W$ and a 4-path from s to w whose internal vertices are in X and X^*, then we let $s \in R$.

We note that the sets S_1, S_2, T_1 and T_2 are pairwise disjoint by definition, whereas the set R may intersect with $S_1 \cup S_2 \cup T_1 \cup T_2$. We now show that $N_3 = R \cup S_1 \cup S_2 \cup T_1 \cup T_2$. For contradiction, assume that s is a vertex of N_3 that does not belong to any of the five

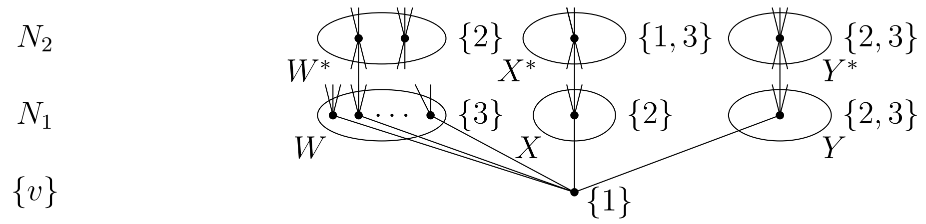

Figure 5: Vertex v with list $\{1\}$; the set $N_1 = W \cup X \cup Y$ satisfying **P1**; and the set $N_2 = W^* \cup X^* \cup Y^*$ satisfying **P2**.

sets R, S_1, S_2, T_1, T_2. As $s \notin T_1 \cup T_2$, we find that the distance from s to every vertex of W is at least 3. Then, as G' has diameter 3, there exists a 4-path P_i from s to each $w_i \in W$ (by **P1** we can write $W^* = \{w_1, \ldots, w_a\}$ for some $a \geqslant 3$). Every P_i must be of one of the following forms: $s - N_2 - N_1 - w_i$ or $s - N_2 - N_2 - w_i$ or $s - N_3 - N_2 - w_i$.

First assume that there exists some P_i that is of the form $s - N_2 - N_1 - w_i$, that is, $P_i = szz'w_i$ for some $z \in N_2$ and $z' \in N_1$. As z' is a neighbour of both w_i and v, we find that $z' \in X$ and $z' \in X^*$, and consequently, $s \in R$, a contradiction.

Now assume that there exists some P_i that is of the form $s - N_2 - N_2 - w_i$, that is, $P_i = szz'w_i$ for some z and z' in N_2. By definition, z must have a neighbour in N_1. As G' has no 5-cycle, this is only possible if z is adjacent to w_i. However, now s is no longer of distance 3 from w_i in G', a contradiction.

Finally, assume that no path from s to any w_i is of one of the two forms above. Hence, every P_i is of the form $s - N_3 - N_2 - w_i$. We write $P_i = st_iw_i^*w_i$ where $t_i \in T_1 \cup T_2$ and $w_i^* \in W^*$. We consider the paths P_1, P_2, P_3, which exist as $|W| \geqslant 3$. As $s \notin S_1$, we find that $t_i \notin T_1$. Moreover, as $s \notin S_2$, we find that $t_1 = t_2 = t_3$, and so $w_1^* = w_2^* = w_3^*$. In particular, the latter implies that w_1^* is adjacent to w_1, w_2 and w_3 and thus has degree at least 3. Recall that G' satisfies Property 1 of Lemma 11. As w_1^* and v each have degree at least 3 in G', this means that each w_i must only be adjacent to v and w_1^*. However, then w_1, w_2 and w_3 are three false twins of degree 2 in G', and by construction of G' we would have removed one of them, a contradiction. We conclude that $N_3 = R \cup S_1 \cup S_2 \cup T_1 \cup T_2$.

We now reduce the lists of the vertices in N_3. Let $s \in N_3$. First suppose that $s \in T_1 \cup T_2$, that is, s is adjacent to a vertex $w^* \in W^*$. Then, as $L'(w^*) = \{2\}$, we find that $L'(s) \subseteq \{1,3\}$. If $s \in T_1$, then we can reduce the list of s further as follows. By the definition of T_1, we have that s is adjacent to a second vertex $w' \neq w^*$ in W^*. By **P2**, we find that w' has a neighbour $w \in W$. We find that $L'(w^*) = L'(w') = \{2\}$ and $L'(w) = \{3\}$. Then s cannot be assigned colour 3, as otherwise w^*, s, w', w would form a bichromatic P_4. Hence, we can reduce the list of s from $\{1,3\}$ to $\{1\}$.

Now suppose that $s \in S_1$. Then, by the definition of the set S_1, we have that s has a neighbour $t \in T_1$. We deduced above that t has list $L'(t) = \{1\}$. Consequently, we can delete colour 1 from the list of s by the Propagation Rule, so $L'(s) \subseteq \{2,3\}$.

Now suppose that $s \in S_2$. Then, by the definition of S_2 and **P2**, there exist two paths $P_1 = st_1w_1^*w_1$ and $P_2 = st_2w_2^*w_2$ where $t_1, t_2 \in T_2$, $w_1^*, w_2^* \in W^*$, $w_1, w_2 \in W$, and

$t_1 \neq t_2$. We claim that s cannot be assigned colour 2. For contradiction, suppose that s has colour 2. Then t_1, which has list $\{1,3\}$, must receive colour 1, as otherwise t_1 will have colour 3 and s, t_1, w_1^*, w_1 is a bichromatic P_4 (recall that w_1^* and w_1 can only be coloured with colours 2 and 3, respectively). For the same reason, t_2 must get colour 1 as well. However, now w_1^*, t_1, s, t_2 is a bichromatic P_4, a contradiction. Hence, we can remove colour 2 from $L'(s)$. Afterwards, $L'(s) \subseteq \{1,3\}$.

Finally, suppose that $s \in R$. By the definition of R, there is some path $P_i = sx^*xw$ where $x^* \in X^*$, $x \in X$, and $w \in W$. By **P1** and **P2**, respectively, it holds that $L'(x) = \{2\}$ and $L'(x^*) \subseteq \{1,3\}$. Hence, s cannot be coloured 2: if x^* gets colour 1, then the vertices v, x, x^*, s will form a bichromatic P_4, and if x^* gets colour 3, then the vertices w_1, x, x^*, s will form a bichromatic P_4. In other words, we may remove colour 2 from $L'(s)$, so $L'(s) \subseteq \{1,3\}$.

As $N_3 = R \cup S_1 \cup S_2 \cup T_1 \cup T_2$, we found that every vertex of N_3 has a list of size at most 2 and more specifically we obtained the following property:

P3. N_3 only consists of vertices whose lists are a subset of $\{1,3\}$ or $\{2,3\}$, and N_3 can be split into sets R, S_1, S_2, T_1, T_2, such that S_1, S_2, T_1 and T_2 are pairwise disjoint, and

- every vertex $r \in R$ has list $L'(r) \subseteq \{1,3\}$ and there is a 4-path from r to a vertex in W that has its two internal vertices in X^* and X, respectively,
- every vertex $t \in T_1$ has list $L'(t) = \{1\}$ and has at least two neighbours in W^*,
- every vertex $t \in T_2$ has list $L'(t) \subseteq \{1,3\}$ and has exactly one neighbour in W^*,
- every vertex $s \in S_1$ has list $L'(s) \subseteq \{2,3\}$, has no neighbours in W^* but is adjacent to at least one vertex in T_1, and
- every vertex $s \in S_2$ has list $L'(s) \subseteq \{1,3\}$ and has no neighbours in $T_1 \cup W^*$ but at least two neighbours in T_2.

We conclude that we constructed a set $\mathcal{L}'$ of 2-list assignments of G', such that $\mathcal{L}'$ is of size $O(n)$ and G' has a star 3-colouring if and only if G' has a star 3-colouring that respects L' for some $L' \in \mathcal{L}'$. Moreover, we can find each $L' \in \mathcal{L}$ in $O(m+n)$ time by a breadth-first search for detecting the 4-paths. For each $L' \in \mathcal{L}$, we do as follows.

We must still construct the edge-extension G'_{ex} of G'. However, the edge-extension L'_{ex} of L' might not be a 2-list assignment. The reason is that G' may have an edge ss' with $L'(s) = \{2,3\}$ and $L'(s') = \{1,3\}$ such that $L'_{\text{ex}}(z_{ss'}) = \{1,2,3\}$. We distinguish between two cases. See Figure 6 for the situation in Case 2a and Figure 7 for the situation of Case 2b. Recall that x_v was the vertex in N_1 whose list we set as $\{2\}$ at the start of our algorithm.

Case 2a. Check if G' has a star 3-colouring that gives x_v colour 2 and every other vertex of N_1 colour 3.

As every vertex in X has list $\{2\}$, we only need to consider this case if $|X| = 1$, that is, $X = \{x_v\}$. We give every vertex in Y list $\{3\}$. Then, by the Propagation Rule, we can

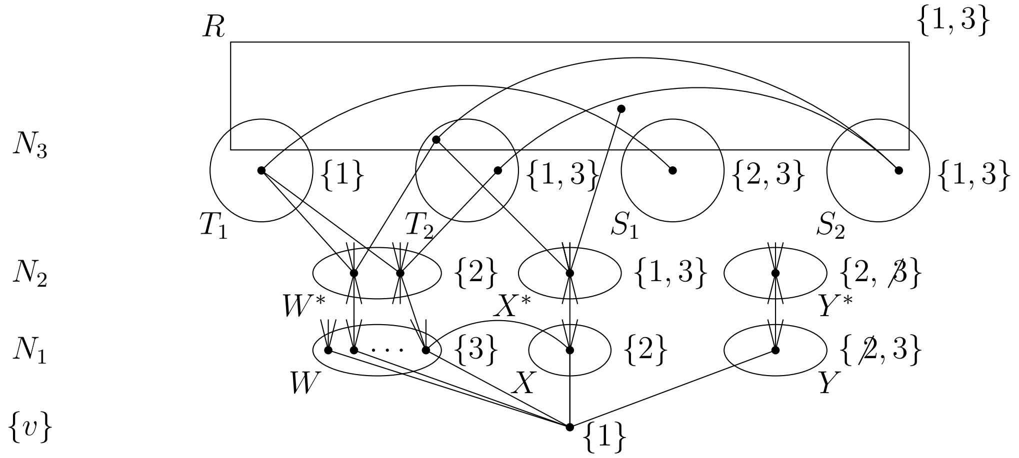

Figure 6: An example of a pair (G', L') in Case 2a. The colours crossed out show the difference between the general situation in Case 2 and what we show holds in Case 2a.

delete colour 3 from every list of a vertex in Y^*. We construct G'_{ex} and L'_{ex} in $O(m^2)$ time by Lemma 10.

We claim that L'_{ex} is a 2-list assignment of G'_{ex}. This can be seen as follows. Let $e = ss'$ be an unsuitable edge of G'. That is, $|L'(s)| = |L'(s')| = 2$ but $L'(s) \neq L'(s')$. As G' has no vertices with list $\{1, 2\}$, we find that $L'(s) = \{2, 3\}$ and $L'(s') = \{1, 3\}$. Then s must be in S_1. By the definitions of the sets S_1 and T_1, it follows that there exist vertices $t \in T_1$ and $w^* \in W^*$, respectively, such that st and tw^* are edges of G'. As $L'(t) = \{1\}$ and $L'(w^*) = \{2\}$, the edge tw^* is list-reducing. Hence, s is a fixer for the edge ss'. The claim now follows from Lemma 10, and by the same lemma, it remains to check if G'_{ex} has a 3-colouring that respects L'_{ex}. We can do the latter in $O(m^2)$ time by Theorem 8.

Case 2b. Check if G' has a star 3-colouring that gives at least one other vertex of N_1, besides x_v, colour 2.

If $|X| \geqslant 2$, then we found a vertex of $N_1 \setminus \{x_v\}$ that gets colour 2. If $X = \{x_v\}$, we will not try to find this vertex; for our algorithm its existence will suffice.

By Property **P2**, every vertex $x^* \in X^*$ has list $L'(x^*) \subseteq \{1, 3\}$. By **P2**, we find that x^* has a neighbour $x \in X$, which has $L'(x) = \{2\}$. By the assumption of Case 2b, there exists at least one other vertex $\bar{x}$ in N_1 that gets colour 2. Then we cannot give x^* colour 1, as otherwise $\bar{x}, v, x, x^*$ would form a bichromatic P_4.

Due to the above, we can remove colour 1 from the list of every vertex of X^* and afterwards we have $L'(x^*) = \{3\}$ for every $x^* \in X^*$. We now remove colour 3 from the list of every neighbour of a vertex of X^*. As L' is a 2-list assignment that does not assign any vertex of G' the list $\{1, 2\}$, we find afterwards that every neighbour of every vertex of X^* in N_3 has list $\{1\}$ or $\{2\}$. Moreover, it follows that X^* is an independent set (as otherwise we discard (G', L')). No vertex of $W^* \cup Y^*$ is adjacent to any vertex in X^* (by Property **P2**). Hence, every vertex in X^* has no neighbours in N_2.

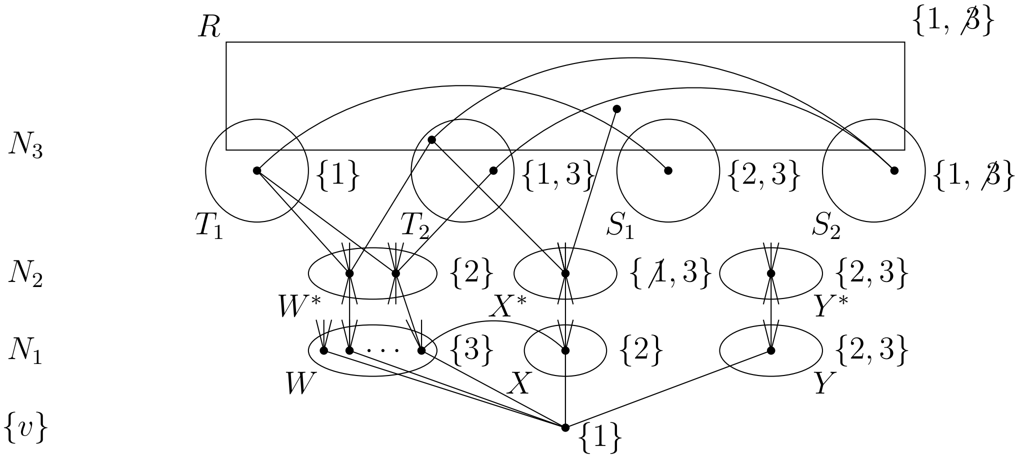

Figure 7: An example of a pair (G', L') in Case 2b. The colours crossed out show the difference between the general situation in Case 2 and what we show holds in Case 2b.

We now prove that no vertex in S_2 can receive colour 3. For contradiction, assume that c is a star 3-colouring of G that respects L' and that assigns a vertex $s \in S_2$ colour $c(s) = 3$. As G' has diameter 3, there is a path P from s to $x_v \in X$ of length at most 3. Then P is of the form $s - N_2 - x_v$ or $s - N_3 - N_2 - x_v$ or $s - N_2 - N_2 - x_v$ or $s - N_2 - N_1 - x_v$. If P is of the form $s - N_2 - x_v$, then s has a neighbour in X^*, which has list $\{3\}$. Hence, as s received colour 3, this is not possible. We show that the other three cases are not possible either.

First suppose that P is of the form $s - N_3 - N_2 - x_v$, say $P = szx^*x_v$ for some $z \in N_3$ and $x^* \in N_2$. As no vertex of $W^* \cup Y^*$ is adjacent to any vertex in X, we find that $x^* \in X^*$. This means that z must receive colour 1, as otherwise the vertices x_v, x^*, z, s would form a bichromatic P_4. As $s \in S_2$, we find that s has two neighbours t_1 and t_2 in T_2. Both t_1 and t_2 have list $\{1, 3\}$, so they must receive colour 1. At least one of them, say t_1, is not equal to z. However, now x^*, z, s, t_1 form a bichromatic P_4, a contradiction. Hence, this case cannot happen.

Now suppose that P is of the form $s - N_2 - N_2 - x_v$, say $P = szx^*x_v$ for some $z, x^* \in N_2$. As no vertex of $W^* \cup Y^*$ is adjacent to any vertex in X, it follows that $x^* \in X^*$. However, no vertex in X^* has a neighbour in N_2. Hence, this case cannot happen.

Finally, suppose that P is of the form $s - N_2 - N_1 - x_v$, say $P = sw^*wx_v$ for some $w^* \in N_2$ and $w \in N_1$. As X is independent and no vertex of Y is adjacent to a vertex of X, we find that $w \in W$ and thus $w^* \in W^*$. However, this is not possible, as $s \in S_2$ is not adjacent to any vertex in W^* by definition. Hence, this case cannot happen either, so we have proven the claim. So, we can remove colour 3 from the list of every vertex $s \in S_2$. Hence, $L'(s) = \{1\}$ for every $s \in S_2$.

We construct G'_{ex} and L'_{ex} in $O(m^2)$ time by Lemma 10. We claim that L'_{ex} is a 2-list assignment of G'_{ex}. This can be seen as follows. Let $e = ab$ be an unsuitable edge of G'. As G' has no vertices with list $\{1, 2\}$, we may assume that $L'(a) = \{1, 3\}$ and

$L'(b) = \{2, 3\}$. As every vertex in R is adjacent to a vertex in X^* with list $\{3\}$, no vertex in R has list $\{1, 3\}$. We just deduced that no vertex in S_2 has list $\{1, 3\}$ either. Hence, the only vertices with list $\{1, 3\}$ belong to T_2, so $a \in T_2$. Then, by definition, we find that a has a neighbour $w \in W^*$, which has a neighbour $w \in W$. As w^* has list $\{2\}$ and w has list $\{3\}$, the edge w^*w is list-reducing. Hence, a is a fixer for the edge ab. The claim now follows from Lemma 10, and by the same lemma, it remains to check if G'_{ex} has a 3-colouring that respects L'_{ex}. We can do the latter in $O(m^2)$ time by Theorem 8.

This concludes the description of our algorithm. Its correctness follows from the correctness of our branching steps. The total running time is $O(nm^2)$, as there are $O(n)$ branches, and we can deal with each branch in $O(m^2)$ time. □

We complement the previous, algorithmic result by a hardness result, which is just an observation on a known construction [1].

Lemma 14. Star 3-Colouring *is* NP*-complete for graphs of diameter at most* 8.

Proof. We recall the Albertson et al. [1] proved that Star 3-Colouring is NP-complete by making the following reduction from 3-Colouring, which is NP-complete even for graphs of diameter 3 [34]. Let G be a graph of diameter 3. For each uv do as follows. Remove uv and make both u and v adjacent to three new vertices x_{uv}, y_{uv} and z_{uv}. Then G has a 3-colouring if and only if the new graph G' has a star 3-colouring [1]. It remains to observe that G' has diameter at most 8. □

We are now ready to prove Theorem 3.

Theorem 3 (restated). *For $d \geqslant 1$ and $k \geqslant 3$,* Star k-Colouring *on graphs of diameter at most d is polynomial-time solvable if $d = 1$, $k \geqslant 4$ or $d \leqslant 3$, $k = 3$ and* NP*-complete if $d \geqslant 2$, $k \geqslant 4$ or $d \geqslant 8$, $k = 3$.*

Proof. The cases $d \leqslant 3$, $k = 3$ and $d \geqslant 8$, $k = 3$ follow from Lemmas 13 and 14, respectively. The case $d = 1$, $k \geqslant 4$ is trivial. For the case $d \geqslant 2$, $k \geqslant 4$ we reduce from Star 3-Colouring: to an instance G of Star k-Colouring, we add a clique of $k - 3$ vertices, which we make adjacent to every vertex of G. □

4 L(1, 2)-Labelling for Graphs of Diameter 2

In this section we prove the missing case in Theorem 4, namely that $L(1, 2)$-Labelling is NP-complete even for graphs of diameter 2. We need three lemmas. We first present, as Lemma 15 and 16, two hardness results for Hamiltonian Cycle. We use Lemma 15 to prove Lemma 16, and the latter to prove Lemma 17.

The *eccentricity* of a vertex u in a graph is the maximum distance of u to some other vertex of G. The *radius* of G is the minimum eccentricity of G.

Lemma 15. Hamiltonian Cycle *is* NP*-complete even for connected bipartite graphs of minimum degree* 2 *and maximum degree* 5 *that have the following three additional properties:*

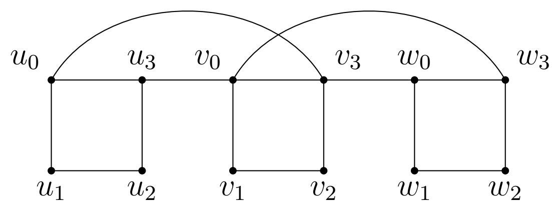

Figure 8: The graph G' from the proof of Lemma 15, when G is the 3-vertex path uvw.

1. *for every two vertices x, y that belong to the same partition class and that have no common neighbour, there exists a vertex in the same partition class as x, y that is of distance greater than 2 from both x and y;*

2. *for every two non-adjacent vertices x, y that belong to different partition classes, either x has a neighbour of distance greater than 2 from y, or y has a neighbour of distance greater than 2 from x, and*

3. *no two vertices of degree 2 have the same neighbourhood.*

Proof. We reduce from Hamiltonian Cycle, which is NP-complete even for graphs of maximum degree 3 [18]. As graphs of bounded maximum degree and bounded radius have constant size, the problem remains NP-complete if in addition we assume that the input graph $G = (V, E)$ of maximum degree 3 has radius at least 10.

We follow the construction used in [25]. That is, from G we construct a graph $G' = (V', E')$ as follows. We replace each $v \in V$ by a 4-cycle v_0, v_1, v_2, v_3. Moreover, for each $uv \in E$, we do as follows. Let u_0, u_1, u_2, u_3 and v_0, v_1, v_2, v_3 be the 4-cycles that are associated with u and v, respectively. We add the two edges u_0v_3 and u_3v_0. This gives us the graph G'. See also Figure 8. It is readily seen that G has a Hamiltonian cycle if and only if G' has a Hamiltonian cycle. Moreover, G' is bipartite with one part $A = \{v_i : i = 0, 2\}$ and the other $B = \{v_i : i = 1, 3\}$, and G' has minimum degree 2 and maximum degree 5; the latter holds as every vertex v_i has two more neighbours than v and v has degree at most 3 (as G has maximum degree 3). We now prove properties 1–3.

We first show Property 1. Let x and y be in the same partition class, say A, and assume that x and y have no common neighbour. If every vertex of A is of distance 2 from either x or y then, as G is connected, x and y are of distance at most 6 from each other. Consequently, the distance from x to any other vertex is at most $6 + 2 + 1 = 9$. Hence, G' has radius at most 9. As the distance between any two vertices u_i and v_i in G' is at least the distance between u and v in G, we find that G also has radius at most 9, a contradiction.

We now show Property 2. Let $x \in A$ and $y \in B$ be non-adjacent. Then $x = u_i$ for some $i \in \{0, 2\}$ and $y = v_j$ for some $j \in \{1, 3\}$ for vertices $u, v \in V$ with $u \neq v$. First suppose that $x = u_0$. If $y = v_1$, then u_1 is adjacent to u_0 and shares no neighbour with v_1,

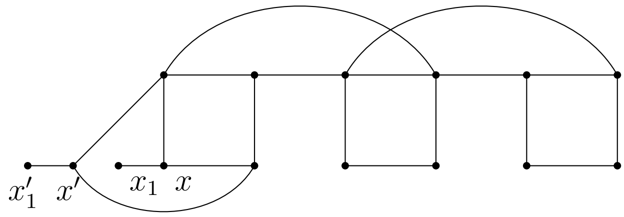

Figure 9: The graph G'' from the proof of Lemma 16.

since $u \neq v$. If $y = v_3$ then v_2 is adjacent to v_3 and shares no neighbour with u_0, since $x = u_0$ and $y = v_3$ are non-adjacent. Now suppose that $x = u_2$. If $y = v_1$, then u_1 is adjacent to u_2 and shares no neighbour with v_1. Finally, if $x = u_2$ and $y = v_3$ then v_2 is adjacent to v_3 and shares no neighbour with u_2.

Finally, Property 3 holds since the set of vertices of degree 2 is $\{v_1, v_2 : v \in V\}$, and no pair of vertices from this set has the same neighbours. $\square$

Lemma 16. Hamiltonian Path *is* NP*-complete even for connected bipartite graphs that satisfy Properties 1 and 2 of Lemma 15.*

Proof. We reduce from Hamiltonian Cycle, which is NP-complete even for the graphs $G' = (V', E')$ constructed in the proof of Lemma 15. We modify a given graph G' into a graph G'' as follows. We take some vertex x of degree 2 and add a new vertex x' with the same neighbourhood as x. We then add two further new vertices, x_1 and x'_1 such that x_1 is pendant on x and x'_1 is pendant on x'. See also Figure 9. We observe that G' has a Hamiltonian cycle if and only if G'' has a Hamiltonian path, which must start in u_1 and end in u'_1. As G' is bipartite, G'' is also bipartite. Hence, it remains to prove Properties 1 and 2.

We first show that Property 1 holds for G''. As Property 1 holds for G' by Lemma 15 and the three new vertices x', x_1, x'_1 do not decrease the distance between any two vertices of G', we only need to consider pairs of vertices involving at least one of $\{x', x_1, x'_1\}$. Vertices x_1 and x'_1 belong to the same partition class of G'' and have no common neighbour. Any non-neighbour z of x in G' is of distance greater than 2 from both x_1 and x'_1, and we can choose z such that z belongs to the same partition class of G'' as x_1 and x'_1. Now consider one of x_1, x'_1, say x_1, and a vertex y of G' that belongs to the same partition class as x_1 in G'', such that x_1 and y do not have a common neighbour. Then x'_1 is of distance greater than 2 from y in G'', and we can take x'_1. Vertices x and x' also belong to the same partition class of G'', but their neighbourhood is the same. Therefore, as Property 1 holds with respect to x in G', Property 1 also holds with respect to x' in G''.

We now show that Property 2 holds for G''. Again we need only to verify pairs involving at least one of $\{x', x_1, x'_1\}$. We first consider the pair (x', x_1); note that x' and x_1 are non-adjacent and belong to different partition classes of G''. We can take x'_1 as the desired vertex, as x'_1 is adjacent to x' but of distance greater than 2 from x_1 in G''. By symmetry, Property 2 holds for the pair (x, x'_1).

We now consider a pair (x', y) where $y \notin \{x_1, x'_1\}$ belongs to a different partition class of G'' than x' and is not adjacent to x'. As x and x' have the same neighbourhood in G'', we find that y and x are non-adjacent vertices in different partition classes as well. As Property 2 holds for G', there exists a vertex z that is a neighbour of one of $\{x, y\}$ but that is of distance greater than 2 from the other vertex of $\{x, y\}$. As the distance between two vertices of G' is the same in G'', we can take z as the desired vertex for the pair (x', y).

Finally, we consider a pair (x_1, y) or (x'_1, y), say (x_1, y) (by symmetry), where y is a non-neighbour of x_1 in G'' such that x_1 and y belong to different partition classes of G''. Note that y must be a vertex of G'. For contradiction, assume that every neighbour of y is of distance 2 from x_1 in G''. Then every neighbour of y in G'' is a neighbour of x. As y belongs to G', we find that y has degree at least 2 in G'. As x has degree 2 in G', this means that in G', both x and y have the same neighbourhood. The latter is a contradiction, as G' satisfies Property 3 of Lemma 16. We conclude that G'' has Property 2. ☐

Lemma 17. *It is* NP*-complete to decide if a graph has a Hamiltonian path, no edge of which is contained in a triangle, even for graphs of diameter* 2.

Proof. We reduce from Hamiltonian Path, which is NP-complete even for the graphs $G'' = (V'', E''')$ constructed in the proof of Lemma 16. We modify a given graph G'' into a graph G^* by adding an edge between any two vertices u, v that belong to the same partition class and that are of distance greater than 2 from each other in G''. By our construction, the distance between any two vertices that belong to the same partition class of G'' is at most 2 in G^*. As G'' has Property 2, the distance between any two vertices in different partition classes of G'' is at most 2 in G^* as well. Hence, G^* has diameter at most 2.

It remains to prove that G'' has a Hamiltonian path if and only if G^* has a Hamiltonian path, no edge of which is contained in a triangle. For showing this it suffices to prove that for every edge e of G^*, it holds that e does not belong to a triangle in G^* if and only if e is an edge of G''.

First suppose that e is not an edge of G''. Say e is an edge between x and y, where x and y are two vertices of distance greater than 2 that belong to the same partition class of G''. As G'' has Property 1, there exists a vertex z that also belongs to the same partition class as x and y and that is of distance greater than 2 from both x and y. Hence, we have added the edges xz and yz as well, thus e belongs to a triangle in G^*.

Now suppose that e is an edge of G''. Let $e = xy$ for two vertices x and y (which belong to different bipartition classes of G''). For contradiction, assume that x and y are contained in a triangle xyz where z belongs to the same partition class as x, so we added the edge xz. Note that x and z have a common neighbour in G'', namely y. This means that their distance is not greater than 2 in G''. Hence, we would not have added the edge xz, a contradiction. ☐

We can now prove our main result. For doing this, we show that an n-vertex graph G of diameter 2 has an $L(1,2)$-n-labelling if and only if G has a Hamiltonian path, no edge of which is contained in a triangle.

Theorem 18. *The $L(1,2)$-*Labelling *problem is* NP*-complete even for graphs of diameter at most* 2.

Proof. Let G be an n-vertex graph of diameter 2. It suffices to prove that G has an $L(1,2)$-n-labelling if and only if G has a Hamiltonian path, no edge of which is contained in a triangle. Then, afterwards, we can apply Lemma 17.

First suppose that G has an $L(1,2)$-n-labelling c. Since G has diameter 2, any two non-adjacent vertices have a common neighbour. Hence, colours of non-adjacent vertices must differ by at least 2. Consequently, two vertices with consecutive colours must be adjacent. As colours of adjacent vertices differ by at least 1, we also find that no two vertices have the same colour. Consequently, every colour i with $1 \leqslant i \leqslant n$ is used. Therefore we have a Hamiltonian path $P = v_1 \dots v_n$ where v_i is the vertices with colour $c(v_i) = i$. No edge $v_i v_{i+1}$ is contained in a triangle since there can be no path of length 2 between v_i and v_{i+1}.

Now suppose that G contains a Hamiltonian path $P = v_1 \dots v_n$, no edge of which is contained in a triangle. The latter means that there is no path of length 2 between v_i and v_{i+1} for $i \in \{1, \dots, n-1\}$. Then we obtain an $L(1,2)$-n-labelling c by defining $c(v_i) = i$. □

5 Conclusions

We obtained (almost) complexity dichotomies for classical variants of the graph colouring problem by bounding the diameter of the graph. In particular, we proved that Acyclic 3-Colouring is polynomially solvable for graphs of diameter at most 2 and that for Star 3-Colouring this holds even for graphs of diameter at most 3. We are not aware of any other problems that are polynomial-time solvable on graphs of diameter at most 3 but NP-complete on graphs of diameter d for some $d > 3$.

In light of the above it would be interesting to close the gaps in Theorems 2 (one open case) and 3 (four open cases). This seems challenging. The NP-hardness construction of Mertzios and Spirakis [34] for 3-Colouring of graphs of diameter 3 does lead to NP-hardness for Near-Bipartiteness for graphs of diameter 3, as observed by Bonamy et al. [7]. However, the construction of [34] cannot be used for Acyclic 3-Colouring and Star 3-Colouring. Hence, new techniques are required.

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