Dowker Duality

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Contents

1 Introduction

The *Dowker complex* of a relation $R \subseteq X \times Y$ is a simplicial complex $\mathrm{Dow}(R)$ constructed from R by considering a subset $\sigma \subseteq X$ to be a simplex if all vertices of σ are related by R to a common element of Y. This construction was first introduced by *Clifford Hugh Dowker* in his 1952 paper *Homology Groups of Relations* [Dow52]. Dowker constructed in fact two simplicial complexes from the same relation R, where the other considers subsets of Y with elements related to a common element of X. However, this simplicial complex is equal to the Dowker complex $\mathrm{Dow}(R^T)$ of the transposed relation (see Definition 2.10). Thus we keep the convention of describing $\mathrm{Dow}(R)$ as *the* Dowker complex of R.

Considering relations $S \subseteq R \subseteq X \times Y$, Dowker proves [Dow52, Theorem 1a] that the relative homology groups $H_p(\mathrm{Dow}(R), \mathrm{Dow}(S))$ and $H_p(\mathrm{Dow}(R)^T, \mathrm{Dow}(S)^T)$ are isomorphic in every degree $p \geq 0$ (with a corresponding result for the relative cohomology groups [Dow52, Theorem 1]). This result has later been named *Dowker's theorem.*

Anders Björner improved Dowkers result in [Bjö95, Theorem 10.9] by defining a cover of $\mathrm{Dow}(R)$ by subcomplexes before applying the nerve theorem [Bjö95, Theorem 10.6(i)] to show that $\mathrm{Dow}(R)$ and $\mathrm{Dow}(R^T)$ induce homotopy equivalent topological spaces $|\,\mathrm{Dow}(R)| \simeq |\,\mathrm{Dow}(R^T)|$ on geometric realization. This improved version of Dowkers theorem is sometimes referred to as *Dowker Duality* [CM18, Vir21].

Simplicial complexes provides a (finite) combinatorial representation for topological spaces, a representation more suited for computations. An example of a nerve is the *Cech complex*, which arises as the nerve of a collection of closed balls covering a metric space X. *Nerves* are the most common way of replacing topological spaces with simplicial complexes [BKRR23], due to the existence of nerve theorems. The various nerve theorems state that all topological features of the space are encoded within the nerve, given some goodness condition on the space and the cover. This fact is particularly important in topological data analysis, where one tries to extract topological features from some (hidden) underlying manifold based upon simplicial complex representations.

With the rise of topological data analysis over the past decades, Dowker's theorem has received more attention. Dowker complexes generalize nerves in the sense that a covering $\mathcal{U}$ of a topological space X gives rise to a relation $R \subseteq \mathcal{U} \times X$ where the Dowker complex of R is equal to the nerve of $\mathcal{U}$. In fact, every simplicial complex is the Dowker complex of some relation (see Proposition 2.9).

Dowker complexes provide a functorial way of constructing topological spaces (via geometric realization) from related sets. Functoriality is important for persistent homology, where one encounters filtered simplicial complexes. Given a nested sequence $R_0 \subseteq R_1 \subseteq \cdots \subseteq R_t$ of relations, functoriality ensures that the inclusions are respected by the Dowker construction, that is, we get a filtered Dowker complex

$$\mathrm{Dow}(R_0) \subseteq \mathrm{Dow}(R_1) \subseteq \cdots \subseteq \mathrm{Dow}(R_t).$$

This motivates the question of whether or not the homotopy equivalences $|\operatorname{Dow}(R)| \simeq |\operatorname{Dow}(R^T)|$ behave nicely with respect to the inclusions $\operatorname{Dow}(R_i) \subseteq \operatorname{Dow}(R_j)$.

Samir Chowdhury and *Facundo Mémoli* formulate a functorial Dowker theorem [CM18, Theorem 3] which states that there exists homotopy equivalences $\Phi_{R_i} : |\operatorname{Dow}(R_i)| \to |\operatorname{Dow}(R_i^T)|$ and $\Phi_{R_j} : |\operatorname{Dow}(R_j)| \to |\operatorname{Dow}(R_j^T)|$ such that the following diagram commutes up to homotopy:

$$\begin{tikzcd} |\operatorname{Dow}(R_i)| \arrow[r, "\Phi_{R_i}"] \arrow[d, "|\iota|"] & |\operatorname{Dow}(R_i^T)| \arrow[d, "|\iota|"] \\ |\operatorname{Dow}(R_j)| \arrow[r, "\Phi_{R_j}"] & |\operatorname{Dow}(R_j^T)|, \end{tikzcd} \tag{1}$$

where $|\iota|$ is the canonical inclusion. The functorial Dowker theorem was improved by *Žiga Virk* who proves in [Vir21, Theorem 5.2] that a diagram like (1) exists for arbitrary morphisms $R \to R'$ of relations (see Definition 2.13).

In this thesis, we introduce relations of categories $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ (see Definition 3.9). From relations of categories, we construct a simplicial set called the *Dowker nerve* (see Definition 3.15). The Dowker nerve resembles the Dowker complex, and we prove in Corollary 5.4 that every Dowker complex is naturally homotopy equivalent on geometric realization to some Dowker nerve.

Dowker nerves of relations of categories satisfy the Dowker Duality theorem stated in [BFS23, Theorem 4.6], with a functorial simplicial Dowker theorem (Corollary 2.14), analogous to the one given in diagram (1), as an immediate consequence. As we explain in Section 3, relations of sets may be generalized to relations of categories. Using this generalization, we prove in Section 5 that the functorial Dowker theorem in [Vir21] is a special case of the functorial simplicial Dowker theorem.

There exists various ways of turning simplicial complexes into simplicial sets, and one is by simplicial barycentric subdivision (see Definition 4.5). The simplicial barycentric subdivision gives a functorial construction of a simplicial set from a simplicial complex. This simplicial set is homeomorphic to the original simplicial complex on geometric realization, and furthermore, this homeomorphism is natural (Theorem 4.8). However, we prove (Proposition 4.9) that the simplicial barycentric subdivision has no adjoint functor going the opposite way, which is a desired property.

Dealing with this, we define the singular simplicial set (see Definition 4.10) which resembles the singular set (see Definition C.5) of a topological space. The singular simplicial set defines a functor from simplicial complexes to simplicial sets. We introduce a functor (see Definition 4.13) in section 4 going the opposite way, and prove that this functor is left adjoint to the singular simplicial set (Theorem 4.14). The downside of the singular simplicial set, compared to the simplicial barycentric subdivision, is that on geometric realization, we *only* get a naturally homotopy equivalent space to the original simplicial complex (Theorem 4.12).

This thesis is built upon the article *Dowker Duality for Relations of Categories* [BFS23], which I have written in collaboration with my supervisors *Morten Brun* and

Lars Moberg Salbu. The article is to be considered as a part of this thesis, and is included in the appendices C.

We define in Section 2 the nerve of a covering, and construct the Dowker complex of a relation. We state nerve theorems for covered topological spaces and covered simplicial complexes by subcomplexes. We also introduce the functorial Dowker theorem, and explain how Brun and Salbu provides a new proof of this using the Rectangle complex [BS22, Definition 3.2]. In Section 3, we introduce relations of categories. We explain how the article was written and highlight some of my contributions. We also provide a brief summary of its contents, which we apply in Section 4 and Section 5. To justify the use of simplicial sets, we establish three ways of constructing simplicial sets from simplicial complexes in Section 4. Utilizing these constructions, we prove in Section 5 that the functorial Dowker theorem is a consequence of the functorial simplicial Dowker theorem. Definitions involving categories, simplicial complexes and simplicial sets may be found in the appendices A, B and C. The article is to be found in Appendix C.

2 The Dowker Complex

We associate to a collection of sets $\mathcal{U} = \{U_i\}_{i \in I}$ a simplicial complex called the *nerve complex*. This simplicial complex was originally introduced in 1928 by *Paul Alexandroff* in his paper [Ale28], where he consider collections of sets arising as a covering of a topological space.

There exists many different variants of the nerve theorem [BKRR23, p. 3]. What they all have in common, is that they relates the homotopy type of some underlying object to the nerve of a covering. Early version of the nerve theorem may be found in [Bor48] and [Wei52].

In this section, we state a nerve theorem for covered topological spaces (Theorem 2.3) and a nerve theorem for covered simplicial complexes (Theorem 2.5). We then introduce the Dowker complex (Definition 2.8) before stating Dowker's theorem (Theorem 2.11), which is equivalent to the nerve theorem for covered simplicial complexes by Remark 2.12.

Lastly, we introduce the functorial Dowker theorem (Theorem 2.14) and show how *Morten Brun* and *Lars Moberg Salbu* proves this in [BS22, Theorem 5.2] using a simplicial complex called the *rectangle complex*. A different proof is given in Section 5.

Definition 2.1 (The Nerve complex; [Bjö95, p. 1849])**.** Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a collection of sets. The *nerve complex* of $\mathcal{U}$ is the simplicial complex $\mathcal{N}(\mathcal{U})$ with simplices given by finite subsets $\sigma \subseteq I$ such that the intersection $\bigcap_{j \in \sigma} U_j$ is non-empty.

Definition 2.2. A *covering* of a topological space X is a collection $\mathcal{U} = \{U_i\}_{i \in I}$ of subsets of X such that $X = \bigcup U_i$. We say that $\mathcal{U}$ is an *open cover* (resp. *closed cover*) if all the subsets U_i are open (resp. closed).

Nerve theorems are important in topological data analysis, since they relate the homotopy type of the underlying space and nerve complex of a covering.

Theorem 2.3 (The Nerve Theorem for Openly Covered Topological Spaces; [Hat02, Corollary 4G.3])**.** *Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of a paracompact topological space X. If every finite intersection $U_{i_0} \cap \cdots \cap U_{i_k}$ is contractible or empty, then $X \simeq |\mathcal{N}(\mathcal{U})|$.*

Definition 2.4. Let K be a simplicial complex. A *cover of K by subcomplexes* is a collection $\mathcal{K} = \{K_i\}_{i \in I}$ of subcomplexes $K_i \subseteq K$ for all $i \in I$ such that $\bigcup K_i = K$. The cover $\mathcal{K}$ is said to be *good* if all finite intersections $K_{i_0} \cap \cdots \cap K_{i_k}$ is contractible or empty.

Theorem 2.5 (The Nerve Theorem for Covered Simplicial Complexes; [Bjö95, Theorem 10.6])**.** *Let K be a simplicial complex and let $\mathcal{K} = \{K_i\}_{i \in I}$ be a good cover of K by subcomplexes. Then $|K| \simeq |\mathcal{N}(\mathcal{K})|$.*

There are several functorial versions of the Nerve theorem for covered topological spaces, and one version is stated in [BKRR23, Theorem 5.9] under the name *unified nerve theorem*. This version uses an intermediate topological space $\mathrm{Blowup}(\mathcal{U})$, called the *Blowup complex* of the covering $\mathcal{U}$. A definition of the Blowup complex may be found in [BKRR23, Definition 2.7].

To understand the functoriality of the unified nerve theorem, we introduce the category of covered topological spaces. Let a *covered (topological) space* be a pair $(X, \mathcal{U})$, where X is a topological space and $\mathcal{U}$ is a cover of X. A *morphism of covered topological spaces* $(f, \varphi) : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a pair consisting of a continuous map $f : X \to Y$ and a function $\varphi : \mathcal{U} \to \mathcal{V}$ such that $f(U) \subseteq \varphi(U)$ for all $U \in \mathcal{U}$. Existence of identities and compositions are inherited from `Top`. Let `Cov` be the category of covered spaces and morphisms of these.

The nerve of a covered space defines a functor from `Cov` to `Top`. Let $(f, \varphi) : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a morphism of covered spaces. Note that a non-empty intersection $U_{i_0} \cap \cdots \cap U_{i_k}$ is sent to the intersection $\varphi(U_{i_0}) \cap \cdots \cap \varphi(U_{i_k})$, which is non-empty since

$$f(U_{i_0} \cap \cdots \cap U_{i_k}) \subseteq f(U_{i_j}) \subseteq \varphi(U_{i_j}).$$

Thus we get a simplicial map $\mathcal{N}(\varphi) : \mathcal{N}(\mathcal{U}) \to \mathcal{N}(\mathcal{V})$. It is straightforward to check that identities and compositions are preserved, so the nerve defines a functor $\mathcal{N}$: `Cov` $\to$ `Cpx` from covered spaces to simplicial complexes.

Before stating the unified nerve theorem, we introduce two more functors from the category of covered spaces to the category of topological spaces. The first functor is defined by forgetting the cover $\mathcal{U}$ of a covered space $(X, \mathcal{U})$. A morphism $(f, \sigma) : (X, \mathcal{U}) \to (Y, \mathcal{V})$ of covered spaces is then sent to the continuous map $f : X \to Y$, by forgetting the function $\phi : \mathcal{U} \to \mathcal{V}$. The second functor is defined by the blowup complex $\mathrm{Blowup}(\mathcal{U})$, of a covered space $(X, \mathcal{U})$. An argument for why the blowup complex defines a functor is given in [BKRR23, p. 17], where also the two maps $\rho_S : \mathrm{Blowup}(\mathcal{U}) \to X$ and $\rho_N : \mathrm{Blowup}(\mathcal{U}) \to |\mathcal{N}(\mathcal{U})|$ are defined. The unified nerve theorem states that ρ_S and ρ_N, which are natural in `Cov`, are homotopy equivalences under some conditions on X and $\mathcal{U}$.

Theorem 2.6 (Unified Nerve Theorem; [BKRR23, Theorem 5.9, 1a and 2a])**.** *Let $(f, \phi) : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a map of covered topological spaces.*

- *1a. If A is an open cover and X is paracompact and Hausdorff, then the natural map $\rho_S : \mathrm{Blowup}(\mathcal{U}) \to X$ is a homotopy equivalence.*
- *2a. If $\mathcal{U}$ is a good cover, then the natural map $\rho_N : \mathrm{Blowup}(\mathcal{U}) \to |\mathcal{N}(\mathcal{U})|$ is a homotopy equivalence.*

Relations

Definition 2.7 ([BS22, Definition 2.1])**.** Let X and Y be sets. Then a *relation R between X and Y* is a triple (R, X, Y) of sets such that $R \subseteq X \times Y$. We say that x

is related to y if $(x, y) \in R$.

We represent the relation (R, X, Y) by the inclusion $R \subseteq X \times Y$ and we fix R to always be a subset of the product $X \times Y$. Thus to ease notation, we represent the relation $R \subseteq X \times Y$ by simply R. We apply the same convention to the relation $R' \subseteq X' \times Y'$.

We now introduce the Dowker complex of a relation R. The simplices of the Dowker complex are subsets of X with all elements related to a common element of Y.

Definition 2.8 (Dowker Complex of a relation)**.** Let $R \subseteq X \times Y$ be a relation. The *Dowker complex of* R is the simplicial complex $\mathrm{Dow}(R)$ with simplices given by the set

$$\mathrm{Dow}(R) = \{\sigma \subseteq X \mid \exists\, y \in Y \text{ such that } \sigma \times \{y\} \subseteq R\}.$$

Every simplicial complex may be obtained as a Dowker complex of a relation.

Proposition 2.9. *Every simplicial complex* K *is the Dowker complex of some relation, that is, there exists a relation* R *such that* $\mathrm{Dow}(R) = K$*.*

Proof. Let K be a simplicial complex and let $R \subseteq V \times K$ be the relation where $(v, \sigma) \in R$ if $v \in \sigma$. We prove that $\mathrm{Dow}(R_K)$ is contained in K and vice versa.

Let $S \subseteq V$ be a simplex of $\mathrm{Dow}(R)$. Then there exists a simplex $\sigma \in K$ such that $(S, \sigma) \in R$. This implies that $S \subseteq \sigma$, thus S is a simplex of K.

Conversely, suppose σ is a simplex of K. Then $\sigma \subseteq V$ and $(\sigma, \sigma) \in R$. Thus σ is a simplex of $\mathrm{Dow}(R_K)$. □

Definition 2.10. Let $R \subseteq X \times Y$ be a relation. The *transpose relation* of R is the relation $R^T \subseteq Y \times X$ where $(y, x) \in R^T$ if $(x, y) \in R$.

We now state Dowker' theorem, which relates the homotopy type of the Dowker complexes obtained from a relation and its transpose. We give a proof of Dowker's theorem in Section 5 by proving the stronger functorial Dowker theorem (Corollary 2.14).

Theorem 2.11 (Dowker's Theorem; [Bjö95, Theorem 10.9])**.** *Let* $R \subseteq X \times Y$ *be a relation. Then* $|\mathrm{Dow}(R)| \simeq |\mathrm{Dow}(R^T)|$.

Remark 2.12. Dowker's theorem 2.11 and the nerve theorem for covered simplicial complexes 2.5 are equivalent, that is, one implies the other.

(Nerve theorem 2.5 $\Longrightarrow$ *Dowker's theorem 2.11).* Let $R \subseteq X \times Y$ be a relation, and consider the subcomplex $R_y \subseteq \mathrm{Dow}(R)$ consisting of all simplices $\sigma \in \mathrm{Dow}(R)$ such that $(\sigma, y) \in R$. Then $\mathcal{K} = \{R_y\}_{y \in Y}$ is a good cover of $\mathrm{Dow}(R)$ by subcomplexes. The nerve theorem then implies that $|\mathrm{Dow}(R)| \simeq |\mathcal{N}(\mathcal{K})|$, and when noting that $\mathcal{N}(\mathcal{K}) = \mathrm{Dow}(R^T)$, we get that $|\mathrm{Dow}(R)| \simeq |\mathrm{Dow}(R^T)|$. Thus the nerve theorem implies Dowker's theorem.

(Dowker's theorem 2.11 $\implies$ nerve theorem 2.5). Let K be a simplicial complex and let $\mathcal{K} = \{K_i\}_{i \in I}$ be a cover of K by subcomplexes. We define the relation $R \subseteq \mathcal{K} \times V$, where $(K_i, v) \in R$ if $v \in K_i$. Note that $\mathrm{Dow}(R) \cong \mathcal{N}(\mathcal{K})$ and $\mathrm{Dow}(R^T) = K$. Then the nerve theorem follows by Dowker's theorem

Definition 2.13. A *morphism of relations* $f : (R, X, Y) \to (R', X', Y')$ is a pair of functions $(f_1 : X \to X', f_2 : Y \to Y')$ such that $(x, y) \in R$ implies $(f_1(x), f_2(y)) \in R'$.

Relations and morphism of relations defines a category. The existence of identity morphisms and compositions are inherited from the category of sets $\texttt{Set}$. Let $\texttt{Rel}$ be the category of relations and morphisms of relations.

The transpose relation defines the transposition functor $T : \texttt{Rel} \to \texttt{Rel}$, sending a morphism of relations $f = (f_1, f_2) : R \to R'$ to the *transposed morphism* $f^T = (f_2, f_1) : R^T \to R'^T$.

The Dowker complex determines a functor from relations to simplicial complexes. For a relation R, the Dowker complex $\mathrm{Dow}(R)$ has vertex set $V(\mathrm{Dow}(R)) \subseteq X$. Given a morphism of relations $f = (f_1, f_2) : R \to R'$, we get a function $f_1 : X \to X'$ of sets. We prove that f_1 takes simplices in $\mathrm{Dow}(R)$ to simplices in $\mathrm{Dow}(R')$. Let $\sigma \subseteq X$ be a simplex of $\mathrm{Dow}(R)$, which means there exists a $y \in Y$ such that $\sigma \times \{y\} \subseteq R$. Then $(f_1(\sigma), f_2(\{y\})) \subseteq R'$ by definition, which implies that $f_1(\sigma)$ is a simplex of $\mathrm{Dow}(R)'$. Restricting f_1 to the elements of $V(\mathrm{Dow}(R))$ induce a simplicial map $\mathrm{Dow}(f) : \mathrm{Dow}(R) \to \mathrm{Dow}(R')$. It is trivial that identities and compositions are preserved, thus the Dowker complex defines a functor $\mathrm{dow} : \texttt{Rel} \to \texttt{Cpx}$, sending the morphism $f : R \to R'$ to the simplicial map $\mathrm{Dow}(f) : \mathrm{Dow}(R) \to \mathrm{Dow}(R')$.

We state a functorial version of Dowker's theorem, called the *functorial Dowker theorem*. Samir Chowdhury and Facundo Mémoli introduced a functorial Dowker theorem in [CM18, Theorem 3] for morphisms $i : R' \to R$ given by inclusions $R' \subseteq R \subseteq X \times Y$. Žiga Virk gives a functorial Dowker theorem for Dowker complexes $\mathrm{Dow}(R)$ of relations $R \subseteq X \times \mathcal{U}$ that arises from covered spaces. Morten Brun and Lars Moberg Salbu introduces a functorial Dowker theorem in [BS22, Theorem 5.2] for arbitrary relations R and morphisms $f : R \to R'$ of these. Brun and Salbu's functorial Dowker theorem is different from the one given by Virk, however, one can prove that the statements in these are equivalent.

The functorial Dowker theorem stated here only considers the *existence* of a homotopy equivalence, in contrast to Virk and Brun and Salbu, which includes the specific homotopy equivalence in the statement.

Theorem 2.14 ([BS22, Theorem 5.2])**.** *Let $f : R \to R'$ be a morphism of relations. There exists a homotopy equivalence $\Psi_R : |\mathrm{Dow}(R)| \to |\mathrm{Dow}(R^T)|$ for every relation R such that the diagram*

$$\begin{tikzcd} |\mathrm{Dow}(R)| \arrow[r, "\Psi_R"] \arrow[d, "|\mathrm{Dow}(f)|"'] & |\mathrm{Dow}(R^T)| \arrow[d, "|\mathrm{Dow}(f^T)|"] \\ |\mathrm{Dow}(R')| \arrow[r, "\Psi_{R'}"] & |\mathrm{Dow}(R'^T)| \end{tikzcd} \tag{2}$$

commutes up to homotopy, that is, $\Psi_R \circ |\operatorname{Dow}(f)|$ *and* $|\operatorname{Dow}(f)^T| \circ \Psi_{R'}$ *are homotopic.*

Brun and Salbu proves the functorial Dowker's theorem in *The Rectangle Complex of a Relation* [BS22] using *rectangles* of relations. A rectangle of the relation $R \subseteq X \times Y$ is a finite subset $U \times V \subseteq R$ such that $U \subseteq X$ and $V \subseteq Y$.

They construct the *Rectangle Complex* $E(R)$ of the relation R, which is a simplicial complex with vertex set R and $\sigma \subseteq R$ is a simplex of $E(R)$ if it is contained in a rectangle $\sigma \subseteq U \times V$ of R. If the rectangle $U \times V$ is not contained in another rectangle of R, we say that $U \times V$ is a *maximal rectangle*. The maximal simplices of the Rectangle Complex are precisely the maximal rectangles of R. These are called *formal concepts* [Ayz19, Definition 5.2] in formal concept analysis.

The rectangle complex $E(R)$ projects onto the Dowker complexes $\operatorname{Dow}(R)$ and $\operatorname{Dow}(R^T)$. Given a morphism of relations $f : R \to R'$, there is a commutative diagram

$$\begin{tikzcd}
\operatorname{Dow}(R) \arrow[d] & E(R) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d] & \operatorname{Dow}(R^T) \arrow[d] \\
\operatorname{Dow}(R') & E(R') \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \operatorname{Dow}(R'^T).
\end{tikzcd} \tag{3}$$

Brun and Salbu prove that the projections π_1 and π_2 induces homotopy equivalences on geometric realization. Choosing a homotopy inverse of $|\pi_1|$, we get homotopy equivalences $\Psi_R : |\operatorname{Dow}(R)| \to |\operatorname{Dow}(R^T)|$ and $\Psi_{R'} : |\operatorname{Dow}(R')| \to |\operatorname{Dow}(R'^T)|$ from the composition fitting into a square on the form of (2) in Theorem 2.14. Strict commutativity is lost in this process, however, the square still commutes up to homotopy, providing a new proof for the functorial Dowkers theorem.

3 Dowker Duality

I have written an article with my supervisors *Morten Brun* and *Lars Moberg Salbu*. Using simplicial sets, we have stated and proved a Dowker duality theorem for relations generalized from sets to categories. Our result gives a proof of Quillen's Theorem A [Qui73].This led us to writing the article *Dowker Duality for Relations of Categories* [BFS23] during the fall of 2022.

Dowker duality for relations of categories has the functorial Dowker theorem as a consequence. An exposition of this will be given in Section 5. Proving the functorial Dowker theorem was in fact our original motivation for writing the article.

The foundations for our article was laid in Brun and Salbu's article *The Rectangle Complex of a Relation* [BS22]. For a relation $R \subseteq X \times Y$ of sets, they constructed the rectangle complex $E(R)$, and used this as an intermediate object relating the Dowker complexes $\operatorname{Dow}(R)$ and $\operatorname{Dow}(R^T)$. We take on the same strategy, constructing an intermediate bisimplicial set ER relating two simplicial sets DR and DR^T via projections. To some extent, our article is a simplicial set version of Brun and Salbu's article.

In this section, we begin with an explanation of how the article came into life. We do this by introducing our first ideas and definitions, which later underwent several generalizations. Examples of these are the rectangle space (Definition 3.2) and the Dowker space (Definition 3.3), which may be regarded as predecessors of the bisimplicial rectangle nerve (Definition 3.13) and the Dowker nerve (Definition 3.15) respectively.

Since the article is a part of my thesis, we mention some of my contributions. One of them was writing a first draft.

Towards the end of this section, we give a brief summary of the article contents, highlighting the parts relevant for this thesis.

The Rectangle Complex in Terms of Simplicial Sets

The idea of writing an article was born during our last meeting before the summer break of 2022. I had recently completed a course in simplicial methods and my supervisors Morten and Lars were finishing their article on the rectangle complex. We were talking about possible projects for my master's thesis. All of a sudden Morten interrupts: "Is it possible to express the rectangle complex as a simplicial set?" To do this, we needed a new way of describing rectangles of relations.

Given a relation $R \subseteq X \times Y$, we may consider a rectangle of R as a pair (f, g) of functions

$$f : \{0, ..., m\} \to X, \quad g : \{0, ..., n\} \to Y, \tag{4}$$

for $m, n \geq 0$, such that the product of the images $\operatorname{im} f \times \operatorname{im} g$ is a subset of R.

Remark 3.1. For a fixed subset $A \subseteq X$, there exists several functions $f : \{0, \dots, m\} \to X$ with image equal to A. This implies that the set of rectangles of a relation $R \subseteq X \times Y$, regarded as pairs (f, g), is much larger than the set of rectangles on the form $U \times V \subseteq R$.

The advantage of considering rectangles as pairs of functions (f, g) is that it provides a bisimplicial structure. Considering the domains $\operatorname{dom} f = \{0, ..., m\}$ and $\operatorname{dom} g = \{0, ..., n\}$ suggests that the *bisimplicial dimension* of the rectangle (f, g) is $m \times n$, where the simplicial structure follows from the cosimplicial structure on the sets $\{0, \dots, m\}$. To define a "simplicial rectangle complex", we needed a bisimplicial set.

Definition 3.2 (Rectangle space)**.** Let $R \subseteq X \times Y$ be a relation. The *rectangle space* of the relation R is the bisimplicial set CR, with (m, n)-simplices given by the set

$$CR_{m,n} = \{(f : \{0, ..., m\} \to X, g : \{0, ..., n\} \to Y) \mid \operatorname{im} f \times \operatorname{im} g \subseteq R\}.$$

For two functions $\alpha : \{0, \dots, m'\} \to \{0, \dots, m\}$ and $\beta : \{0, \dots, n'\} \to \{0, \dots, n\}$, the structure map $\alpha^* \times \beta^* : CR_{m,n} \to CR_{m',n'}$ is defined by sending the pair (f, g) to the pair $(f\alpha, g\beta)$.

Having a definition of what looks like a simplicial version of the rectangle complex, we wondered if it was possible to do the same with the Dowker complex. Our main objective was now to state and prove a Dowker's theorem for simplicial sets. Fortunately, everything lined up exactly as we wanted.

Definition 3.3 (Dowker space)**.** Let $R \subseteq X \times Y$ be a relation. The *Dowker space* of R is the simplicial set BR with m-simplices given by the set

$$BR_m = \{f : \{0, \dots, m\} \to X \mid \exists y \in Y \text{ such that } \operatorname{im} f \times \{y\} \subseteq R\},$$

and for a function $\phi : \{0, \dots, n\} \to \{0, \dots, m\}$, the structure map $\phi^* : BR_m \to BR_n$ is defined by sending the simplex f to the composition $f \circ \phi$.

Proposition 3.4 (Dowker's theorem for simplicial sets)**.** *Let $R \subseteq X \times Y$ be a relation of sets. The Dowker spaces BR and BR^T are weakly equivalent, that is, $|BR| \simeq |BR^T|$.*

Proving the Dowker's theorem for simplicial sets involves the rectangle space as an intermediate object. Consider the projections of sets

$$BR_m \xleftarrow{\mathrm{pr}_1} CR_{m,n} \xrightarrow{\mathrm{pr}_2} BR^R_n$$

where $\mathrm{pr}_1 : (f, g) \mapsto f$ and $\mathrm{pr}_2 : (f, g) \mapsto g$. Recall that if pr_1 and pr_2 induce weak equivalences, we get homotopy equivalences on geometric realizations. We present a condition on R and R^T for when pr_1 and pr_2 induce weak equivalences in Theorem 3.18, and we prove in Lemma 5.5 that this condition is satisfied for all relations of sets.

The meeting got to an end. We were happy with our definitions and that it all seemed to fit together in a similar way as in [BS22]. All of a sudden, my "entire master's thesis" was standing on the blackboard. If the projections pr_1 and pr_2 indeed induce weak equivalences, we had successfully managed to prove a simplicial version of Dowkers theorem, an approach open for several generalizations and applications. We left the meeting that day convinced that this was the case (which is true, although our original proof was wrong).

Writing a First Draft

We decided that I was going to write a first draft of the article. I should give enough background on the theory as if I were writing to my fellow students. The idea was to trim it down, omitting the "obvious stuff", as we got closer to the final version.

We made a lot of imprecise notation when sketching our proofs on the blackboard. Writing a first draft involved introducing notation as much as reproducing our proofs and definitions. I made some choices regarding notation, which we kept until the end.

Defining the Dowker space as a simplicial set was not obvious. Let $R \subseteq X \times Y$ be a relation. Note that the rectangle space CR and the Dowker space BR are objects of different categories. Hence there exists no projection $\mathrm{pr}_1 : CR_{m,n} \to BR_m$ inducing a weak equivalence, as proposed earlier. This "problem" would have been avoided if we defined the Dowker space as a bisimplicial set, constant in one direction. However, adding an artificial dimension for notation would hide the "true nature" of the Dowker space. After all, the Dowker space *is* a simplicial set. To deal with the following consequences, I had to introduce some functors.

Let $\mathrm{P} : \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}} \to \Delta^{\mathsf{op}}$ be the projection functor sending the pair of maps $(f, g) : ([m], [n]) \to ([m'], [n'])$ to the map $f : [m] \to [m']$. Precomposition with the projection functor defines a functor $\mathrm{p} : \mathtt{sSet} \to \mathtt{ssSet}$ sending a simplicial set $X : \Delta^{\mathsf{op}} \to \mathtt{Set}$ to the composition $X \circ \mathrm{P}$, a bisimplicial set which is constant in the vertical simplicial degree, that is, $\mathrm{p}\, X_{m,n} = X_m$ for all $n \geq 0$. The simplicial map $f : X \to Y$ is sent to the bisimplicial map $\mathrm{p}\, f : \mathrm{p}\, X \to \mathrm{p}\, Y$ with component $\mathrm{p}\, f_{m,n} = f_m$ for all $n \geq 0$.

The diagonal of a bisimplicial set is a simplicial set. Let $\mathrm{diag} : \Delta^{\mathsf{op}} \to \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}}$ be the functor sending the map $f : [m] \to [n]$ to the pair $(f, f) : ([m], [m]) \to ([n], [n])$. Precomposition with diag defines a functor $d : \mathtt{ssSet} \to \mathtt{sSet}$, sending a bisimplicial set X to the diagonal $d(X)$ with m-simplices given by $d(X)_m = X_{m,m}$. The diagonal of a bisimplicial map $f : X \to Y$ is the simplicial map $d(f) : d(X) \to d(Y)$ with component $d(f)_m = f_{m,m}$.

Remark 3.5. The composition $\mathrm{P} \circ \mathrm{diag}$ induces the identity functor $d \circ \mathrm{p} = \mathrm{id}$ on simplicial sets since the following diagram commutes:

$$\begin{tikzcd}
\Delta^{\mathsf{op}} \arrow[r, "\mathrm{diag}"] \arrow[dr, "\mathrm{id}"'] & \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}} \arrow[d, "\mathrm{P}"] \\
& \Delta^{\mathsf{op}}.
\end{tikzcd}$$

We apply the functors p and d to make the projections pr_1 and pr_2 precise. The projections $\mathrm{pr}_1 : CR_{m,n} \to BR_m$ and $\mathrm{pr}_2 : CR_{m,n} \to BR^T_n$ induces bisimplicial projections $\pi_R : CR \to \mathrm{p}\, BR$ and $\hat{\pi}_R : CR \to \mathrm{p}\, BR^T$ defined by the mapping $\pi_R : (f, g) \mapsto f$ and $\hat{\pi}_R : (f, g) \mapsto g$. Applying the diagonal functor d to π and $\hat{\pi}$, we get simplicial maps

$$BR \xleftarrow{d(\pi_R)} d(CR) \xrightarrow{d(\hat{\pi}_R)} BR^T,$$

which are weak equivalences if π and $\hat{\pi}$ are pointwise weak equivalence by Proposition C.27.

At this point, we had not considered a functorial version of the simplicial Dowker's theorem. I decided to include this in the article while I was writing the first draft. To do this, I had to make the construction of the rectangle space and Dowker space functorial.

Let $f = (f_1, f_2) : R \to R'$ be a morphism of relations, and let (a, b) be an (m, n)-simplex of CR. Sending the pair (a, b) to the compositions $(f_1 a, f_2 b)$ defines a function $Cf_{m,n} : CR_{m,n} \to CR'_{m,n}$ which induce a bisimplicial map $Cf : CR \to CR'$. It is trivial that identities and compositions are preserved. Thus we get a functor $C : \mathtt{Rel} \to \mathtt{ssSet}$.

Similarly, let a be an m-simplex of BR. From the morphism $f = (f_1, f_2) : R \to R'$ we get a simplicial map $Bf : BR \to BR'$ defined by $a \mapsto f_1 a$. Identities and compositions are preserved, thus the Dowker space defines a functor $B : \mathtt{Rel} \to \mathtt{sSet}$.

Let $f = (f_1, f_2) : R \to R'$ be a morphism of relations. Recall that the transposition functor $T : \mathtt{Rel} \to \mathtt{Rel}$ sends the morphism $f = (f_1, f_2) : R \to R'$ of relations to the transposed morphism $f^T = (f_2, f_1) : R^T \to R'^T$. Having defined the functors C and B, we get a commutative diagram:

$$\begin{tikzcd}
\mathrm{p}\, BR \arrow[d, "\mathrm{p}\, Bf"'] & CR \arrow[l, "\pi_R"'] \arrow[r, "\hat{\pi}_R"] \arrow[d, "Cf"] & \mathrm{p}\, BR^T \arrow[d, "\mathrm{p}\, Bf^T"] \\
BR' & CR' \arrow[l, "\pi_{R'}"'] \arrow[r, "\hat{\pi}_{R'}"] & BR'^T.
\end{tikzcd} \tag{5}$$

Remark 3.6. For a bisimplicial set X, let $X_{\bullet,n}$ be the simplicial set $[m] \mapsto X_{m,n}$ obtained by fixing the vertical simplicial degree. Note that $(\mathrm{p}\, BR)_{\bullet,n} = BR$, thus $\pi_R : CR \to \mathrm{p}\, BR$ is a pointwise weak equivalence if the simplicial map $\pi_R : CR_{\bullet,n} \to BR$ is a weak equivalence for all $n \geq 0$.

Lemma 3.7 (Dowker Equivalence)**.** *Let $R \subseteq X \times Y$ be a relation. The simplicial map $d(\pi_R) : d(CR) \to BR$ is a weak equivalence.*

We give a proof of this later (as a consequence of the more general statement in Theorem 3.18). A direct proof can be made by defining an extra degeneracy on $CR_{\bullet,n}$ and applying [GJ09, Lemma III.5.1] to prove that $\pi_R : CR_{\bullet,n} \to BR$ is a weak equivalence for all $n \geq 0$. Then $d(\pi_R)$ is a weak equivalence by Proposition C.27.

Note that Lemma 3.7 does not in include the map $d(\hat{\pi}_R) : d(CR) \to BR^T$. However, the arguments given for $d(\pi_R) : d(CR) \to BR$ applies to the transposed version $d(\pi_{R^T}) : d(CR^T) \to BR^T$. By noting that $d(CR) \cong d(CR^T)$ and by an argument (which we leave out) that $d(\hat{\pi}_R)$ factors through this isomorphism, proves that $d(\hat{\pi}_R)$ is a weak equivalence.

Proposition 3.8 (Functorial simplicial Dowker theorem)**.** *Let $f : R \to R'$ be a morphism of relations. There exists homotopy equivalences $\Phi_R : |BR| \to |BR^T|$ and $\Phi_{R'} : |BR'| \to |BR'^T|$ such that the following diagram:*

$$\begin{tikzcd}
|BR| \arrow[r, "\Phi_R"] \arrow[d, "|Bf|"'] & |BR^T| \arrow[d, "|Bf^T|"] \\
|BR'| \arrow[r, "\Phi_{R'}"] & |BR'^T|,
\end{tikzcd}$$

commutes up to homotopy, that is, $|Bf^T| \circ \Phi_R \simeq \Phi_{R'} \circ |Bf|$.

Proof. Both maps $d(\pi_R)$ and $d(\hat{\pi}_R)$ are weak equivalences by Proposition 3.7. Thus we get homotopy equivalences $|d(\pi_R)|$ and $|d(\hat{\pi}_R)|$ on the geometric realization by definition. Let h_R be a homotopy inverse of $|d(\pi_R)|$. Then we get a homotopy equivalence from the composition $|d(\hat{\pi}_R)| \circ h_R : |BR| \to |BR^T|$. By letting $\Phi_R = |d(\hat{\pi}_R)| \circ h_R$ and $\Phi_{R'} = |d(\hat{\pi}_{R'})| \circ h_{R'}$ we get the desired maps. Commutativity follows functorially from diagram (5), though strict commutativity is lost by the choice of inverses h_R and $h_{R'}$. □

Even though it was I who chose to include a functorial version of the simplicial Dowker's theorem in the first draft, I believe it was the intention of Morten and Lars from the beginning. Doing this is a natural generalization once a first draft is completed, and it is in line with what they did in [BS22].

The first draft ended here. I had no experience writing scientific papers and little experience writing mathematics. Balancing precision and rigor against the flow and easy-to-read formulations was difficult. With guidance from Morten and Lars, I eventually completed the first draft, not knowing all the changes and generalizations this draft would go through.

Relations of Categories

By the order of Morten, Lars was tasked to generalize the rectangle space from pairs of functions to pairs of functors. This turned out to be a snowball rolling, growing bigger for each week. The end result was a new kind of relations, and corresponding generalizations of the rectangle space and Dowker space.

We will now present the various generalizations made, by giving a brief summary of the articles contents. Proofs and details may be found in the article.

Definition 3.9. Let $\mathcal{C}$ and $\mathcal{D}$ be small categories. A *relation of categories* from $\mathcal{C}$ to $\mathcal{D}$ is a small category $\mathcal{R}$ together with a functor $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$.

Definition 3.10. Let $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ and $R' : \mathcal{R}' \to \mathcal{C}' \times \mathcal{D}'$ be relations of categories. A morphism of relations of categories $f = (f_0, f_1, f_2) : R \to R'$ consists of three functors $f_0 : \mathcal{R} \to \mathcal{R}'$, $f_1 : \mathcal{C} \to \mathcal{C}'$ and $f_2 : \mathcal{D} \to \mathcal{D}'$ such that $R' \circ f_0 = (f_1 \times f_2) \circ R$.

The existence of identities and compositions are inherited from `Cat`. Thus relations of categories defines a category, which we denote `CatRel`.

Remark 3.11. In the article, this category is denoted by `Rel`. We have made this change of notation since we consider both kinds of relations in this thesis. There are several definitions in the rest of this section that are from the article. In these cases, we have made the necessary changes to fit the notation of this thesis.

Let $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ be a relation of categories. Then the transpose relation R^T is the composite functor

$$\mathcal{R} \xrightarrow{R} \mathcal{C} \times \mathcal{D} \xrightarrow{\mathrm{tw}} \mathcal{D} \times \mathcal{C},$$

where the twist isomorphism tw sends a pair (c, d) to the pair (d, c). For a morphism $f : R \to R'$ of relations of categories, we let $f^T : R^T \to R'^T$ be the morphism with $f_0^T = f_0$, $f_1^T = f_2$ and $f_2^T = f_1$.

Definition 3.12. Let X be a set. The *translation category* of X is the small category $\mathcal{X}$ with set of objects $\mathrm{ob}(\mathcal{X}) = X$ and exactly one arrow $x \to y$ for each ordered pair (x, y) of objects in X.

Translation categories are in one-to-one correspondence with sets. The functor $\mathrm{tr} : \mathtt{Set} \to \mathtt{Cat}$ sending the set X to its translation category $\mathcal{X}$ is monic and both full and faithful. Thus translation categories embeds the category of sets into the category of small categories.

Relations of categories generalize relations of sets. Let $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ be a relation of categories. If $\mathcal{C}$ and $\mathcal{D}$ are translation categories of some sets, we recover the definition of a relation of sets (up to isomorphism) from relations of categories if we require the functor R to be monic. Thus we will call *relation of categories* for just relations when there is no room for ambiguity.

We now give the definition of the bisimplicial rectangle nerve of a relation of categories.

Definition 3.13 (The bisimplicial rectangle nerve, [BFS23, Definition 3.1])**.** Let $R\colon \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ be a relation. The *bisimplicial rectangle nerve* ER is the bisimplicial set whose (m, n)-simplices are functors $r\colon [m] \times [n] \to \mathcal{R}$ such that there exist a necessarily unique pair of functors $(a : [m] \to \mathcal{C}, b : [n] \to \mathcal{D})$ such that $a \times b = R \circ r$, that is, so that the following diagram commutes

$$\begin{tikzcd}
{[m] \times [n]} \arrow[d, "r"'] \arrow[dr, "a \times b"] & \\
\mathcal{R} \arrow[r, "R"] & \mathcal{C} \times \mathcal{D}.
\end{tikzcd}$$

If $\alpha : [m'] \to [m]$ and $\beta : [n'] \to [n]$ are order-preserving maps, then $ER(\alpha, \beta) : ER_{m,n} \to ER_{m',n'}$ sends the (m, n)-simplex r to the (m', n')-simplex $r \circ (\alpha \times \beta)$.

The bisimplicial rectangle nerve defines a functor from relations of categories to the category of bisimplicial sets. Let $f : R \to R'$ be a morphism of relations. We define the bisimplicial map $Ef : ER \to ER'$ by sending a simplex r to the composition $Ef(r) = f_0 \circ r$. Let $E : \mathtt{CatRel} \to \mathtt{ssSet}$ be the functor given by $f \mapsto Ef$.

Remark 3.14. The pair $(a : [m] \to \mathcal{C}, b : [n] \to \mathcal{D})$ in Definition 3.13 is unique by the universal property of products. Let $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ be a relation and let $r : [m] \times [n] \to \mathcal{R}$ be a simplex in $ER_{m,n}$ with $a : [m] \to \mathcal{C}$ and $b : [n] \to \mathcal{D}$ such that $R \circ r = a \times b$. Consider the diagram

$$\begin{tikzcd}
{[m]} \arrow[dd, "a"'] & & {[m] \times [n]} \arrow[ll, "\mathrm{pr}_{[m]}"'] \arrow[rr, "\mathrm{pr}_{[n]}"] \arrow[d, "r"] \arrow[ddll, "f"'] \arrow[ddrr, "g"] & & {[n]} \arrow[dd, "b"] \\
& & \mathcal{R} \arrow[d, "R"] & & \\
\mathcal{C} & & \mathcal{C} \times \mathcal{D} \arrow[ll, "\mathrm{pr}_{\mathcal{C}}"'] \arrow[rr, "\mathrm{pr}_{\mathcal{D}}"] & & \mathcal{D}.
\end{tikzcd}$$

The maps f and g are uniquely determined by $R \circ r$. But $R \circ r = a \times b$ implies that $f = a \circ \mathrm{pr}_{[m]}$ and $g = b \circ \mathrm{pr}_{[n]}$. Thus a and b are uniquely determined by the composition $R \circ r$.

We define a "projection" of sets $\pi_R : ER_{m,n} \to N\mathcal{C}_m$ sending the simplex $r : [m] \times [n] \to \mathcal{R}$ of ER to the simplex $a : [m] \to \mathcal{C}$ of the nerve $N\mathcal{C}$. This function is well-defined by Remark 3.14, and we use the image to define the Dowker nerve.

Definition 3.15 (Dowker nerve of a relation, [BFS23, Definition 4.3])**.** Let $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ be a relation. The *Dowker Nerve* of R is the simplicial set DR whose set of m-simplices is given by the image of the map $\pi_R : ER_{m,0} \to N\mathcal{C}_m$.

Like the bisimplicial rectangle nerve, the Dowker nerve defines a functor from relations of categories to the category of simplicial sets. Let $D : \mathtt{CatRel} \to \mathtt{sSet}$ be the functor sending the morphism of relations $f : R \to R'$ to the simplicial map $Df : DR \to DR'$ where $Df(a) = f_1 \circ a$ for a simplex $a \in DR$.

Forgetting the categorical structure of a small category defines a functor to sets. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ consist of two functions, one which is defined on the objects $F^{\mathrm{ob}} : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$. The function F^{ob} is a function of sets if $\mathcal{C}$ and $\mathcal{D}$ are small categories. Thus mapping the functor $F : \mathcal{C} \to \mathcal{D}$ to the function $F^{\mathrm{ob}} : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$ defines a "forgetfull" functor $\mathrm{fg} : \mathtt{Cat} \to \mathtt{Set}$.

Proposition 3.16. *The forgetful functor* $\mathrm{fg} : \mathtt{Cat} \to \mathtt{Set}$ *is left adjoint to the translation functor* $\mathrm{tr} : \mathtt{Set} \to \mathtt{Cat}$*.*

Let $R \subseteq X \times Y$ be a relation of sets, and let $\mathcal{R}$, $\mathcal{X}$ and $\mathcal{Y}$ be the translation categories of R, X and Y respectively. We define the relation $S : \mathcal{R} \to \mathcal{X} \times \mathcal{Y}$ of categories from the inclusion of a full subcategory $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$ where $(x, y) \in \mathcal{R}$ if $(x, y) \in R$. From Proposition 3.16, we have a bijection $\phi_{[m],X}\mathtt{Cat}([m], \mathcal{X}) \to \mathtt{Set}(\{0, \dots, m\}, X)$, natural in $[m]$, inducing a simplicial isomorphism $\phi : DS \to BR$. Thus the Dowker nerve is a generalization of the Dowker space, since these are isomorphic for relations of categories that are constructed from relations of sets via translation categories.

Dowker Duality for Relations of Categories

Dowker equivalence does *not* hold for relations of categories in general. But proven true for translation categories (i. e. sets) suggests that it holds for some relations. We now introduce the notion of a *Dowker relation*, which provides the necessary condition for Dowker equivalence (Theorem 3.18) to be true.

Let $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ be a relation of categories, and let $a : [m] \to \mathcal{C}$ be an m-simplex of the nerve $N\mathcal{C}$. Further, let $ER_{m,\bullet}$ be the simplicial set $[n] \mapsto ER_{m,n}$ obtained by keeping the horizontal degree constant. The *fiber* of a under π_R is the simplicial subset $\pi_R^a \subseteq ER_{m,\bullet}$ with set of n-simplices given by all functors $r : [m] \times [n] \to \mathcal{R}$ such that $\pi_R(r) = a$, that is, there exists a functor $b : [n] \to \mathcal{D}$ such that $R \circ r = a \times b$.

Definition 3.17 ([BFS23, Definition 4.2]). A *Dowker relation* is a relation $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ of categories with the property that for every $a \in N\mathcal{C}_m$, the fiber π_R^a of a under π_R is contractible or empty.

Theorem 3.18 (Dowker Equivalence; [BFS23, Theorem 4.5]). *If $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ is a Dowker relation, then the projection maps*

$$\pi_R : ER_{m,n} \to DR_m$$

induce a weak equivalence $d(\pi_R) : d(ER) \to DR$ of simplicial sets.

We now state a functorial simplicial Dowker theorem for relations of categories. This result is not a part of the article, however, it is an direct consequence of the Dowker Duality theorem in [BFS23, Theorem 4.6]. This result generalizes the functorial simplicial Dowker theorem for relations of sets, since every Dowker space is isomorphic to a Dowker nerve of some relation.

Corollary 3.19 (Functorial simplicial Dowker theorem for relations of categories). *Let $f : R \to R'$ be a morphism of relations of categories. If R, R^T, R' and R'^T are Dowker relations, then there exists a square*

$$\begin{array}{ccc} |DR| & \xrightarrow{\Phi_R} & |DR^T| \\ {\scriptstyle |Df|}\downarrow & & \downarrow{\scriptstyle |Df^T|} \\ |DR'| & \xrightarrow{\Phi_R} & |DR'^T| \end{array}$$

that commutes up to homotopy and where Φ_R and $\Phi_{R'}$ are homotopy equivalences.

4 Simplicial Sets and Simplicial Complexes

Simplicial sets and simplicial complexes shares much of the same kind of structure. Both objects consists of simplices, and both have a notion of one simplex being a face of another. In this section we will investigate three different constructions of simplicial sets from simplicial complexes. Two of the constructions defines a functor, one with a left adjoint. We define the left adjoint functor, and prove the adjunction.

The Ordered Singular Simplicial Set

Definition 4.1. Let K be a simplicial complex with a total ordering $\leq$ on the vertex set V. The *ordered singular simplicial set of* K is the simplicial set $\mathrm{Sing}_{\leq}(K)$ with m-simplices given by order-preserving maps

$$\mathrm{Sing}_{\leq}(K)_m = \{f : [m] \to V \mid \operatorname{im} f \in K\},$$

such that the image is a simplex of K, and where the structure maps ϕ^* are defined by precomposition $f \circ \phi$.

In the following discussion we fix a simplicial complex K, and we fix a total order $\leq$ on the vertex set V. We use the symbol $V_{\leq}$ to mean the vertex set V with the total order $\leq$ defined on its elements.

The simplicial complex K and the corresponding ordered singular simplicial set $\mathrm{Sing}_{\leq}(K)$ induces homeomorphic topological spaces on geometric realization. We prove this by defining a continuous map $\varphi_{\leq} : |\mathrm{Sing}_{\leq}(K)| \to |K|$, which is a homeomorphism by Theorem 4.4.

An element of $|\mathrm{Sing}_{\leq}(K)|$ can be represented by a pair (a, t), where $a \in \mathrm{Sing}(K)_m$ and $t = (t_0, \ldots, t_m) \in \Delta^m$. The pair (a, t) induce a function $a_*(t) : V_{\leq} \to [0, 1]$, which sends the vertex v to the sum $\sum_{a(i)=v} t_i$. We prove in Lemma 4.2 that $a_*(t)$ is an element of $|K|$. For two related pairs $(\phi^* a, s) \sim (a, \phi_* s)$, we prove in Lemma 4.3 that the induced functions $(\phi^* a)_*(s) = a_*(\phi_* s)$ are equal. Thus mapping the class $[a, t]$ to the function $a_*(t)$ is a well defined continuous map $\varphi_{\leq} : |\mathrm{Sing}_{\leq}(K)| \to |K|$.

Lemma 4.2. *Let K be a simplicial set with a totally ordered vertex set $V_{\leq}$. For a map $a : [m] \to V_{\leq}$ in $|K|$ and an element $t = (t_0, \ldots, t_m) \in \Delta^m$, the function $a_*(t) : V_{\leq} \to [0, 1]$ is an element of $|K|$.*

Proof. To prove the statement of the Lemma, one need to verify that the support $\mathrm{supp}(a_*(t))$ is a simplex of K and that the sum of all $a_*(t)(v)$ equals one. We prove the former in part one and the latter in part two.

Part 1. We prove that $\mathrm{supp}(a_*(t))$ is a face of $\operatorname{im} a$. Let w be an element of the support $\mathrm{supp}(a_*(t))$. This implies that

$$\sum_{a(i)=w} t_i > 0,$$

which again implies that there exists an element $j \in [m]$ such that $a(j) = w$. Thus $w \in \operatorname{im} a$ for every element $w \in \mathrm{supp}(a_*(t))$, which implies $\mathrm{supp}(a_*(t)) \subseteq \operatorname{im} a$.

Part 2. Since every $i \in [m]$ takes a vertex $a(i)$ in $V_{\leq}$, we get that

$$\sum_{v \in V_{\leq}} \sum_{a(i)=v} t_i = \sum_{i \in [m]} t_i = 1,$$

where the last equality is by definition. $\square$

Lemma 4.3. *For two related pairs $(a, \phi_* s)$ and $(\phi^* a, s)$ of $\coprod_{[m]} \mathrm{Sing}_{\leq}(K)_m \times \Delta^m$, the induced functions $a_*(\phi_* s) = (\phi^* a)_*(s)$ are equal.*

Proof. Let $a : [m] \to V_{\leq}$ be an m-simplex of $\mathrm{Sing}_{\leq}(K)$ and let $s = (s_0, \dots, s_n)$ be an element of Δ^n. Then the lemma follows by the following sequence of equalities

$$a_*(\phi_* s)(v) = \sum_{a(i)=v} (\phi_* s)_i = \sum_{a(i)=v} \sum_{\phi(j)=i} s_j = \sum_{(a\circ\phi)(j)=v} s_j = (\phi^* a)(s)(v)$$

$\square$

We state in Theorem 4.4 that $\varphi_{\leq}$ is a homeomorphism. For a proof of this, we refer to [BFS23, Proposition 6.6].

Theorem 4.4. *Let K be a simplicial complex with a totally ordered the vertex set $V_{\leq}$. Then the continuous map $\varphi_{\leq} : |\,\mathrm{Sing}_{\leq}(K)| \to |K|$ is a homeomorphism.*

The map $\varphi_{\leq}$ is not natural in K, since the ordered singular simplicial set does not define a functor. This is because $\mathrm{Sing}_{\leq}(K)$ is dependant on the chosen total order $\leq$ on V. There is no canonical ordering on the vertex set V which implies that there is no functor $\mathtt{Cpx} \to \mathtt{sSet}$ from simplicial complexes to simplicial sets taking K to $\mathrm{Sing}_{\leq}(K)$. This fact prevents us from making the additional statement in Theorem 4.4 that $\varphi_{\leq}$ is natural in K.

There is another way of turning simplicial complexes into simplicial sets, a construction which resembles the barycentric subdivision. Unlike the ordered singular simplicial set, this construction does indeed define a functor, and induce homeomorphic topological spaces on geometric realization.

Simplicial Barycentric Subdivision

A simplicial complex give rise to a category. For a simplicial complex K, there is a canonical order on the simplices $\sigma \in K$ given by inclusion. From this ordering, we obtain the poset $K_{\subseteq}$ which we may consider as a category. Maps of simplicial complexes $\phi : K \to K'$ respects this order, implying that they induce functors $\phi_* : K_{\subseteq} \to K'_{\subseteq}$. Let $O : \mathtt{Cpx} \to \mathtt{Cat}$ be the functor sending a map of simplicial complexes $\phi : K \to K'$ to the functor $\phi_* : K_{\subseteq} \to K'_{\subseteq}$.

The nerve of a category provides a functorial way of constructing simplicial sets from simplicial complexes. Let $\phi : K \to L$ be a map of simplicial complex. Applying the nerve functor $N : \mathtt{Cat} \to \mathtt{sSet}$ to $\phi_* : K_{\subseteq} \to L_{\subseteq}$, we get a simplicial map $N(\phi_*) : NK_{\subseteq} \to NL_{\subseteq}$ defined by the composition $N(\phi_*)(a) = \phi_* \circ a$. Thus, the composition $N \circ O : \mathtt{Cpx} \to \mathtt{sSet}$ is a functor from simplicial complexes to simplicial sets.

Definition 4.5. Let K be a simplicial complex. The *simplicial barycentric subdivision* of K is the simplicial set $NK_{\subseteq}$.

Remark 4.6. Let K be a simplicial complex. The m-simplices of $NK_{\subseteq}$ may be considered as strings $\{f(0) \to f(1) \to \cdots \to f(m)\}$, where the arrows represent inclusions. Recall that the barycentric subdivision $sd\,K$ from Definition B.3 has simplices on the form $\{\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_m\}$. Thus there is a one-to-one correspondence between non-degenerate simplices of $NK_{\subseteq}$ and simplices of $sd\,K$.

The topological spaces $|K|$ and $|NK_{\subseteq}|$ are homeomorphic. To prove this, we define a map $|sd\,K| \to |NK_{\subseteq}|$ and prove that this is a homeomorphism. We then obtain a homeomorphism $|K| \to |NK_{\subseteq}|$ through a composition by using the homeomorphism $\eta : |K| \to |sd\,K|$ from Theorem B.10. Moreover, this composition is natural in K, which we prove in Theorem 4.8.

Let K be a simplicial complex. An element $\alpha : K \to [0,1]$ of $|sd\,K|$, give rise to a finite chain $\operatorname{supp}(\alpha) = \{\sigma_0 \subset \cdots \subset \sigma_{k_\alpha}\}$ of simplices of K and a tuple $t_\alpha = (\alpha(\sigma_0), \ldots, \alpha(\sigma_k))$. From Remark 4.6, we may consider $\operatorname{supp}(\alpha)$ as a map $f_\alpha : [k] \to K$, taking i to σ_i, which is a simplex of $NK_{\subseteq}$. Further, the tuple t_α is an element of Δ^k since

$$\sum_{\sigma_i \in \operatorname{supp}(\alpha)} \alpha(\sigma_i) = \sum_{\sigma_i \in K} \alpha(\sigma_i) = 1.$$

We define a continuous map $\psi_K : |sd\,K| \to |NK_{\subseteq}|$ by sending α to the class $[f_\alpha, t_\alpha]$, and claim that this is a bijection.

Lemma 4.7. *Let K be a simplicial complex. The map $\psi_K : |sd\,K| \to |NK_{\subseteq}|$ sending α to $[f_\alpha, t_\alpha]$ is a bijection.*

Proof. ψ_K is injective. Let α and β be two distinct elements of $|sd\,K|$, that is, there exists a simplex σ' of K such that $\alpha(\sigma') \neq \beta(\sigma')$. If the simplices are equal $f_\alpha = f_\beta$, then σ' is contained in both images $\operatorname{supp}(\alpha) = \operatorname{supp}(\beta)$ which implies that $t_\alpha \neq t_\beta$. On the other hand, if $f_\alpha \neq f_\beta$, then $(f_\alpha, t_\alpha) \not\sim (f_\beta, t_\beta)$ since both f_α and f_β is non-degenerate. Both cases implies that $[f_\alpha, t_\alpha] \neq [f_\beta, t_\beta]$, thus ψ_K is injective.

ψ_K is surjective. Let $[f,t]$ be an element of $|NK_{\subseteq}|$ represented by the pair (f,t) where $f(i) = \sigma_i$ and $t = (t_0, \ldots, t_m)$. We may assume that f is non-degenerate, that is, f is injective. Consider the map $\beta : K \to [0,1]$ sending σ to $\sum_{\sigma = f(j)} t_j$. Then $\operatorname{supp}(\beta) = \operatorname{im} f$ and $t_\beta = t$ since $\beta(\sigma_i) = t_i$. Thus $\beta \in |sd\,K|$ and $[f,t] = [f_\beta, t_\beta]$, and ψ_K is surjective. □

Theorem 4.8. *Let $\phi : K \to L$ be a simplicial map. Then the following diagram commutes*

$$\begin{tikzcd}
|K| \arrow[r, "\eta_K"] \arrow[d, "|\phi|"'] & |sd\,K| \arrow[r, "\psi_K"] \arrow[d, "|sd\,\phi|"'] & |NK_{\subseteq}| \arrow[d, "|N(\phi_*)|"] \\
|L| \arrow[r, "\eta_L"'] & |sd\,L| \arrow[r, "\psi_L"] & |NL_{\subseteq}|,
\end{tikzcd}$$

where all horizontal maps are homeomorphisms.

Proof. The maps ψ_K and ψ_L are bijections by Lemma 4.7. Assuming that K is finite, [RM14, Thm. 26.6] implies that ψ_K is a homeomorphism, since both $|sd\,K|$ and $|NK_\subseteq|$ are compact and Hausdorff (an argument can be given for infinite K, which we omit here). Commutativity of the left square and that η_K and η_L are homeomorphisms are proved in Theorem B.10. What is left to prove is that the right square commutes.

Let $\alpha : K \to [0,1]$ be an element of $|sd\,K|$, and suppose that the support $\mathrm{supp}(\alpha)$ has $k+1$ elements. Recall that $|\phi|(\alpha) : L \to [0,1]$ is the element of $|sd\,L|$ where

$$|\phi|(\alpha)(\tau) = \sum_{\phi(\sigma)=\tau} \alpha(\sigma).$$

We may suppose that $\mathrm{supp}(|\phi|(\alpha))$ has $l+1$ elements. To prove the theorem, we first construct a map $h : [k] \to [l]$ in Δ such that $h^* : (NL_\subseteq)_l \to (NL_\subseteq)_k$ takes $f_{|\phi|(\alpha)}$ to $\phi_* \circ f_\alpha$. We then prove that $h_* : \Delta^k \to \Delta^l$ takes t_α to $t_{|\phi|(\alpha)}$. This implies that $[\phi_* \circ f_\alpha, t_\alpha] = [f_{|\phi|(\alpha)}, t_{|\phi|(\alpha)}]$, which is an equivalent statement to commutativity in the right square.

A map $h : [k] \to [l]$ takes $f_{|\phi|(\alpha)}$ to $\phi_* \circ f_\alpha$ if the following diagram commutes

$$\begin{tikzcd}
{[k]} \arrow[r, hook, "f_\alpha"] \arrow[d, "h"'] & K_\subseteq \arrow[r, "\phi_*"] & L_\subseteq \\
{[l].} \arrow[urr, "f_{|\phi|(\alpha)}"'] & &
\end{tikzcd} \tag{6}$$

Let $I = \{i_0 \leq \cdots \leq i_{k-l}\}$ be the set of $i \in [k]$ such that $(\phi_* \circ f_\alpha)(i) = (\phi_* \circ f_\alpha)(i+1)$. We define the map $s^I : [k] \to [l]$ where $s^I := s^{i_0} \cdots s^{i_{k-l}}$. Then (6) commutes by letting $h = s^I$.

The ith coordinate of $t_{|\phi|(\alpha)}$ is the sum

$$\sum_{\phi(\sigma)=f_{|\phi|(\alpha)}(i)} \alpha(\sigma), \tag{7}$$

where the components are real numbers $\alpha(\sigma)$ obtained from evaluating a simplex σ of K by α. Components that equals zero does not contribute to the sum, so we may restrict the sum (7) to only simplices contained in the support $\mathrm{supp}(\alpha) = \mathrm{im}\, f_\alpha$, that is, simplices on the form $\sigma = f_\alpha(j)$ for some j in $[k]$. Further, such an element j is unique, since f_α is injective by definition. Thus summing in (7) over all simplices $\sigma \in K$ such that $\phi(\sigma) = f_{|\phi|(\alpha)}(i)$ is equivalent to summing over all $j \in [k]$ such that $\phi(f_\alpha(j)) = f_{|\phi|(\alpha)}(i)$.

Since diagram (6) commutes, then $\phi(f_\alpha(j)) = f_{|\phi|(\alpha)}(i)$ if and only if $h(j) = i$. Thus we get the following equalities

$$\sum_{\phi(\sigma)=f_{|\phi|(\alpha)}(i)} \alpha(\sigma) = \sum_{\phi(f_\alpha(j))=f_{|\phi|(\alpha)}(i)} \alpha(f_\alpha(j)) = \sum_{h(j)=i} \alpha(f_\alpha(j)), \tag{8}$$

where the rightmost sum is precisely the ith coordinate of $h_*(t_\alpha)$. Thus $t_{|\phi|(\alpha)} = h_*(t_\alpha)$ and $[\phi_* \circ f_\alpha, t_\alpha] = [f_{|\phi|(\alpha)}, t_{|\phi|(\alpha)}]$. □

Proposition 4.9. *The composite functor $N \circ O$: **Cpx** $\to$ **sSet** has no left or right adjoint*

Proof. The proposition can be proved by giving counterexamples where a colimit (resp. limit) is not preserved. This implies that $N \circ O$ cannot be a left (resp. right) adjoint.

We give an example where a colimit is not preserved. Consider the diagram:

$$\begin{array}{ccc} \{\{a\},\{b\}\} & \longrightarrow & \{\{a\},\{b\},\{a,b\}\} \\ \big\downarrow & & \\ \{a\}, & & \end{array}$$

of simplicial complexes where the maps are inclusions. Taking pushout in Cpx gives just a point $\{a\}$, and $N \circ O$ takes this simplicial complex to the constant simplicial set $\{*\}$. However, by first applying the functor $N \circ O$ and then taking pushout, we get the simplicial set with one 0-simplex and exactly one non-degenerate 1-simplex. Thus the colimit is not preserved by $N \circ O$.

A counterexample where a limit is not preserved is obtained by forming the categorical product of two simplicial complexes K and L and verifying that this is not preserved by $N \circ O$. □

The Singular Simplicial Set

We will now investigate a third way of constructing a simplicial set from a simplicial complex, which in all essence is about avoiding the choice of total order in the ordered singular simplicial set. The upside of this construction is that it defines a functor from simplicial complexes to simplicial sets. However, the downside is that the geometric realization is *only* homotopy equivalent to the original simplicial complex.

The advantage of this construction over the simplicial barycentric subdivision is that there exists a left adjoint functor from simplicial sets to simplicial complexes.

We start by defining a functor from simplicial complexes to simplicial sets, taking K to the *singular simplicial set* $\mathrm{Sing}(K)$, and then define a continuous map $|\mathrm{Sing}(K)| \to |K|$, which is natural in K by Lemma 4.11. We then prove that this map is a homotopy equivalence in Theorem 4.12, a proof which applies the Dowker duality theorem for relations of categories [BFS23, Theorem 4.6]. Lastly, we define a functor from simplicial sets to simplicial complexes and prove that this is left adjoint to the singular simplicial set in Theorem 4.14.

Definition 4.10. The *singular simplicial set* of a simplicial complex K is the simplicial set $\operatorname{Sing} K$ with m-simplices given by the set

$$\operatorname{Sing} K_m = \{f : \{0, \dots, m\} \to V \mid \operatorname{im} f \in K\}.$$

For a map $\phi : [m] \to [n]$, the structure map $\phi^* : \operatorname{Sing} K_n \to \operatorname{Sing} K_m$ sends a simplex $f : \{0, \dots, n\} \to V$ to $f \circ \phi$.

The singular simplicial set defines a functor from simplicial complexes to simplicial sets. Given a simplicial map $\phi : K \to L$, we get a map of vertex sets $\phi : V(K) \to V(L)$. Mapping the m-simplex $f : \{0, \dots, m\} \to V(K)$ to the composition $\phi \circ f$ defines a function $\operatorname{Sing}(\phi)_m : \operatorname{Sing} K_m \to \operatorname{Sing} L_m$ which commutes with structure maps (since these are define by precomposition). Thus we get a simplicial map $\operatorname{Sing}(\phi) : \operatorname{Sing} K \to \operatorname{Sing} L$. It is trivial that identities and compositions are preserved. Let $\operatorname{Sing} : \texttt{Cpx} \to \texttt{sSet}$ be the functor sending the map of simplicial complexes $\phi : K \to L$ to the map $\operatorname{Sing}(\phi) : \operatorname{Sing} K \to \operatorname{Sing} L$ of simplicial sets.

In the following discussion, we fix two simplicial complexes K and L with vertex sets V and U respectively. We also fix a total ordering $\leq$ on V, and use the symbol $V_\leq$ when we consider the vertex set *with* the total ordering.

We define a map $\varphi : |\operatorname{Sing}(K)| \to |K|$ by the same rule as $\varphi_\leq$, that is, φ maps the class $[a, t]$ to the function $a_*(t) : V \to [0, 1]$ where $a_*(t)(v) = \sum_{a(i)=v} t_i$. That $a_*(t) : V \to [0, 1]$ is an element of $|K|$ and that φ is well defined follows from arguments similar to the ones given in the proofs of Lemma 4.2 and Lemma 4.3.

Lemma 4.11. *Let K be a simplicial complex. The map $\varphi_K : |\operatorname{Sing}(K)| \to |K|$ is natural in K, that is, for a simplicial map $\phi : K \to L$, we get a commutative square:*

$$\begin{tikzcd}
|\operatorname{Sing}(K)| \arrow[r, "\varphi_K"] \arrow[d, "|\operatorname{Sing}(\phi)|"'] & |K| \arrow[d, "|\phi|"] \\
|\operatorname{Sing}(L)| \arrow[r, "\varphi_L"] & |L|.
\end{tikzcd} \tag{9}$$

Proof. Let $[a, t]$ be a class in $|\operatorname{Sing}(K)|$ and let u be a vertex of L. To prove that (9) commutes reduces to prove that

$$\sum_{(\phi \circ a)(i)=u} t_i = \sum_{\phi(v)=u} a_*(t)(v).$$

This is true, since $a_*(t)(v) = \sum_{a(i)=v} t_i$ so we get that

$$\sum_{\phi(v)=u} a_*(t)(v) = \sum_{\phi(v)=u} \sum_{a(i)=v} t_i = \sum_{(\phi \circ a)(i)=u} t_i.$$

$\square$

The map $\varphi_{\leq} : |\operatorname{Sing}_{\leq}(K)| \to |K|$ factors through $|\operatorname{Sing}(K)|$. There is an injective simplicial map $i : \operatorname{Sing}_{\leq}(K) \to \operatorname{Sing}(K)$ sending a simplex $f : [m] \to V_{\leq}$ to the function $i(f) : \{0, \ldots, m\} \to V$ where $i(f)(j) = f(j)$. Together with the map $\varphi : |\operatorname{Sing}(K)| \to |K|$ we get a diagram

$$\begin{tikzcd}
 & {|\operatorname{Sing}(K)|} \arrow[d, "\varphi"] \\
{|\operatorname{Sing}_{\leq}(K)|} \arrow[ur, "|i|"] \arrow[r, "\varphi_{\leq}"] & {|K|.}
\end{tikzcd}$$

This diagram commutes since $\operatorname{im} f = \operatorname{im} i(f)$ for every simplex $f \in \operatorname{Sing}_{\leq}(K)$. Thus, by the two-out-of-three rule, φ is a homotopy equivalence if i is a weak equivalence.

The Dowker duality theorem in [BFS23, Theorem 4.6] implies that i is a weak equivalence. A proof of this is given in Section 6.2 of [BFS23]. The approach is to define two Dowker relations R_1 and R_2, such that $\operatorname{Sing}_{\leq}(K) = DR_1$ and $\operatorname{Sing}(K) \cong DR_2$. Then Dowker duality implies that $|\operatorname{Sing}_{\leq}(K)| \simeq |\operatorname{Sing}(K)|$. An additional argument given in [BFS23, Proposition 5.4] implies that $i : \operatorname{Sing}_{\leq}(K) \to \operatorname{Sing}(K)$ is this weak equivalence.

Theorem 4.12 (Corollary 6.7, [BFS23])**.** *Let K be a simplicial complex. The map $\varphi : |\operatorname{Sing}(K)| \to |K|$ is a homotopy equivalence natural in K.*

We will now construct a simplicial complex MX from a simplicial set X. The idea is to express the simplices of X in terms of subsets of the 0-simplices X_0. More precisely, we consider the set $\sigma(x)$ of 0-dimensional faces of a simplex $x \in X_m$, and declare σ_x to be a simplex of $M(X)$. This construction defines a functor M, and we prove in Theorem 4.14 that M is left adjoint to Sing.

Given a m-simplex x of simplicial set X, let

$$S(x) := \{v \in X_0 \mid \exists \phi : [0] \to [m] \text{ such that } \phi^*(x) = v\}.$$

Definition 4.13. From a simplicial set X we construct the simplicial complex MX with vertex set $V(MX) = X_0$ and

$$MX := \{S(x) \mid x \in X\}.$$

Maps of simplicial sets $X \to Y$ induce maps of simplicial complexes on the form $MX \to MY$. Given a simplicial map $f : X \to Y$, we get a function $f_0 : X_0 \to Y_0$ on the 0-simplices. We show that a simplex $\sigma \subseteq X_0$ of MX is sent to a simplex $f_0(\sigma) \subseteq Y_0$ of MY. If $\sigma \in MX$, then there exists a simplex $x \in X$ such that $S(x) = \sigma$. We may assume that $S(x) = \{\phi_0^*(x), \ldots, \phi_k^*(x)\}$ The image of $S(x)$ under f_0 is the set $f_0(S(x)) = \{f_0(\phi_0^*(x)), \ldots, f_0(\phi_l^*(x))\}$ (the index has changed from k to l since f_0 may produce duplicates). Since simplicial maps commutes with structure maps, we get that

$$\{f_0(\phi_0^*(x)), ..., f_0(\phi_l^*(x))\} = \{\phi_0^*(f_m(x)), \ldots, \phi_l^*(f_m(x))\}.$$

Thus $f_0(S(x)) = S(f(x))$, which is a simplex in MY. Hence the function $f_0 : X_0 \to Y_0$ is a map of simplicial complexes, and we write $Mf : MX \to MY$ for this map.

It is trivial to verify that identities and compositions are preserved. Thus we get a functor $M : \texttt{sSet} \to \texttt{Cpx}$ sending a map $f : X \to Y$ of simplicial sets to the map $Mf : MX \to MY$ of simplicial complexes. We end this discussion by a theorem, proving that Sing is right adjoint to the functor M.

Theorem 4.14. *The functor* Sing : ***Cpx*** $\to$ ***sSet*** *is a right adjoint functor to* M : ***sSet*** $\to$ ***Cpx***.

Proof. Let X be a simplicial set and let x be an m-simplex of X. Further, let $\delta_i : [0] \to [m]$ be the map $0 \mapsto i$ and let $f_x : [m] \to X_0$ be the m-simplex of $\mathrm{Sing}(MX)$ taking i to $\delta_i^*(x)$. We prove that there exists a natural transformation $\eta : \mathrm{id}_{\texttt{sSet}} \implies \mathrm{Sing} \circ M$ such that the component map $\eta_X : X \to \mathrm{Sing}\, MX$, defined by $\eta_X(x) = f_x$, is a universal map form X to Sing.

To prove that η_X is a universal map form X to Sing, we need to prove that (i) for all simplicial maps $g : X \to \mathrm{Sing}\, K$, there exists a map $h : MX \to K$ such that the left triangle of (10) commutes and (ii) such a map h is unique. Further, to prove that η_X is a component in the natural transformation $\eta : \mathrm{id}_{\texttt{sSet}} \implies \mathrm{Sing} \circ M$ we must prove that (iii) η_X is natural in X.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "g"'] & \mathrm{Sing}\, MX \arrow[d, "\mathrm{Sing}(h)"] & MX \arrow[d, "h"] \\ & \mathrm{Sing}\, K & K \end{tikzcd} \tag{10}$$

Existence. Let $g : X \to \mathrm{Sing}(K)$ be a simplicial map.

For all $v \in X_0$, there exists a simplex $y \in X_k$, for some $k \geq 0$ and an order-preserving map $a : [0] \to [k]$ such that $a^*(y) = v$ (for example, let $y = v$ and let $a = \mathrm{id}_{[0]}$). We define a map $h : X_0 \to V$ of vertex sets, sending $v = a^*(y)$ to the vertex $g(y)(a(0))$. After proving that h is well defined, we prove that h is a map of simplicial complexes and that $\mathrm{Sing}(h) \circ \eta_X = g$.

Let $z \in X_l$ be a simplex for some $l \geq 0$, and let $b : [0] \to [l]$ be an order-preserving map such that $v = b^*(z)$. Since structure maps commutes with simplicial maps we get that

$$g(y) \circ a = g(a^* y) = g(b^* z) = g(z) \circ b,$$

which implies that $g(y)(a(0)) = g(z)(b(0))$. In other words, the mapping $v \mapsto g(y)(a(0))$ is independant of the choice of simplex y and order-preserving map a, as long as $a^*(y) = v$, thus h is a well defined map of vertex sets.

We now prove that h is a map of simplicial complexes, that is, given a simplex $\sigma \in MX$, the image $h(\sigma)$ is a simplex of K. If $\sigma \in MX$, then there exists a simplex $y \in X$ such that $S(y) = \sigma$. That is, all vertices of σ are on the form $\delta_i^*(y)$. Then $h(\sigma)$ is a face if the image $\mathrm{im}\, g(y) \in K$, since h sends the vertex $\delta_i^*(y)$ to $g(y)(\delta_i(0))$. This implies that $h(\sigma)$ is itself a simplex of K, thus $h : MX \to K$ is a map of simplicial complexes.

The triangle in diagram (10) is commutative, since $\mathrm{Sing}(h)$ is defined by post-composition. That is, an m-simplex $f : [m] \to X_0$ of $\mathrm{Sing}(MX)$ is mapped by $\mathrm{Sing}(h)$ to the m-simplex $h \circ f$. Then $h \circ f_x$ maps i to $g(x)(\delta_i(0))$ which is equal to $g(x)(i)$ by the definition of δ_i. Thus $\mathrm{Sing}(h) \circ \eta_X = g$.

Uniqueness. We now prove that h is unique, that is, if there exists a map h' such that $\mathrm{Sing}(h') \circ \eta_X = g$, then $h = h'$. Assume that such a map h' exits, and let v be a 0-simplex of X. Then $f_v : [0] \to X_0$ maps 0 to v, and $\mathrm{Sing}(h')$ maps f_v to the map $g(v)$, which implies that $(h' \circ f_v)(0) = h'(v) = g(v)(0)$. But $h(v) = g(v)(0)$, thus $h = h'$.

Natural in X. Let $\gamma : X \to Y$ be a map of simplicial sets. We show that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "\gamma"'] & \mathrm{Sing}\,(MX) \arrow[d, "\mathrm{Sing}(\gamma)"] \\ Y \arrow[r, "\eta_Y"] & \mathrm{Sing}\,(MY). \end{tikzcd} \tag{11}$$

Let x be an m-simplex of X. Proving that diagram (11) commutes reduces to prove that $f_{\gamma(x)} = \gamma \circ f_x$. Note that $f_{\gamma(x)}(i) = \delta_i^*(\gamma(x))$ and that $(\gamma \circ f_x)(i) = \gamma(\delta_i^*(x))$. But γ is a simplicial map and d_i^* is a structure map, thus we get that $f_{\gamma(x)}(i) = (\gamma \circ f_x)(i)$ for all $0 \leq i \leq m$, and the diagram commutes.

We conclude that $\eta : \mathrm{id}_{\mathtt{sSet}} \implies \mathrm{Sing} \circ M$ is a natural transformation with component map η_X, which is an universal morphism from X to Sing for every simplicial set X. Thus Sing is right adjoint to M by Proposition A.20. $\square$

5 Proof of the Functorial Dowker Theorem

We present in this subsection a proof of the functorial Dowker theorem 2.14, using the functorial simplicial Dowker theorem 3.19 together with Theorem 4.12. The approach is to connect the Dowker complex to the Dowker nerve of a relation and its transpose, with a homotopy equivalence that is natural with respect to relations. We do this by combining several commutative diagrams, where all horizontal maps are homotopy equivalences. The main part of the proof involves verifying that everything is natural with respect to relations.

Let $R \subseteq X \times Y$ be a relation of sets and let $\mathcal{R}$, $\mathcal{X}$ and $\mathcal{Y}$ be translation categories of the sets R, X and Y respectively. We define the functor $S : \mathtt{Rel} \to \mathtt{CatRel}$, sending the relation R to the relation $S(R) : \mathcal{R} \to \mathcal{X} \times \mathcal{Y}$ of categories given by the inclusion $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$ of a full subcategory. The morphism of relations $f = (f_1, f_2) : R \to R'$ is sent to the morphism $S(f) : S(R) \to S(R')$ where $S(f)_0 = \mathrm{tr}\,(f_1) \times \mathrm{tr}\,(f_2)$, $S(f)_1 = \mathrm{tr}\,(f_1)$ and $S(f)_2 = \mathrm{tr}\,(f_2)$.

In the following discussion, we fix the relation $R \subseteq X \times Y$. Without loss of generality, we may assume that the vertex set of the Dowker complex is the whole set X, that is, $V(\mathrm{Dow}(R)) = X$.

Proposition 5.1. *Let $m \geq 0$ and let $\gamma_m : DS(R)_m \to \mathrm{Sing}(\mathrm{Dow}(R))_m$ be the function sending a functor $a : [m] \to \mathcal{X}$ to the function $f_a : \{0, \dots, m\} \to X$ where $f_a(i) = a(i)$. The functions γ_m are bijections for all $m \geq 0$ and induce an isomorphism $\gamma : DS(R) \to \mathrm{Sing}(\mathrm{Dow}(R))$ of simplicial sets.*

Lemma 5.2. *The isomorphism $\gamma : DS(R) \to \mathrm{Sing}(\mathrm{Dow}(R))$ of Proposition 5.1 is natural in R, that is, for a morphism of relations $f = (f_1, f_2) : R \to R'$ of sets, we get a commutative diagram:*

$$\begin{tikzcd}
\mathrm{Sing}(\mathrm{Dow}(R)) \arrow[d, "\mathrm{Sing}(\mathrm{Dow}(f))"'] & DS(R) \arrow[l, "\gamma"'] \arrow[d, "DS(f)"] \\
\mathrm{Sing}(\mathrm{Dow}(R')) & DS(R'). \arrow[l, "\gamma"']
\end{tikzcd} \tag{12}$$

Lemma 5.3. *The homotopy equivalence $\varphi : |\,\mathrm{Sing}(\mathrm{Dow}(R))| \to |\,\mathrm{Dow}(R)|$ of Theorem 4.12 is natural in R, that is, for a morphism of relations $f : R \to R'$ we get a commutative diagram:*

$$\begin{tikzcd}
|\,\mathrm{Dow}(R)| \arrow[d, "|\,\mathrm{Dow}(f)|"'] & |\,\mathrm{Sing}(\mathrm{Dow}(R))| \arrow[l, "\varphi"'] \arrow[d, "|\,\mathrm{Sing}(\mathrm{Dow}(f))|"] \\
|\,\mathrm{Dow}(R')| & |\,\mathrm{Sing}(\mathrm{Dow}(R'))| \arrow[l, "\varphi"']
\end{tikzcd} \tag{13}$$

We summarize the preceding lemmas in the following corollary. This corollary may be seen as a justification of using the name "Dowker" in the Dowker nerve.

Corollary 5.4. *Let $R \subseteq X \times Y$ be a relation of sets. There exists a homotopy equivalence $\iota_R : |DS(R)| \to |\,\mathrm{Dow}(R)|$ that is natural in R, that is, for a morphism $f : R \to R'$ of relations, we get a commutative diagram:*

$$\begin{tikzcd}
|\,\mathrm{Dow}(R)| \arrow[d, "|\,\mathrm{Dow}(f)|"'] & |DS(R)| \arrow[l, "\iota_R"'] \arrow[d, "|DS(f)|"] \\
|\,\mathrm{Dow}(R')| & |DS(R')|. \arrow[l, "\iota_{R'}"']
\end{tikzcd} \tag{14}$$

Proof. Taking the geometric realization of diagram (12) and combining it with diagram (13), we get a larger diagram

$$\begin{tikzcd}
|\,\mathrm{Dow}(R)| \arrow[d, "|\,\mathrm{Dow}(f)|"'] & |\,\mathrm{Sing}(\mathrm{Dow}(R))| \arrow[l, "\varphi_R"'] \arrow[d, "|\,\mathrm{Sing}(\mathrm{Dow}(f))|"] & |DS(R)| \arrow[l, "|\gamma|"'] \arrow[d, "|DS(f)|"] \\
|\,\mathrm{Dow}(R')| & |\,\mathrm{Sing}(\mathrm{Dow}(R'))| \arrow[l, "\varphi_{R'}"] & |DS(R')|, \arrow[l, "|\gamma|"]
\end{tikzcd} \tag{15}$$

which is commutative and where all horizontal maps are homotopy equivalences. Taking horizontal compositions gives the desired homotopy equivalences $\iota_R = \varphi_R \circ |\gamma|$ and $\iota_{R'} = \varphi_{R'} \circ |\gamma|$. □

Lemma 5.5. *Let $R \subseteq X \times Y$ be a relation of sets. The relation of categories $S(R) : \mathcal{R} \to \mathcal{X} \times \mathcal{Y}$ is a Dowker relation.*

Proof. We prove in [BFS23, Corollary 6.4] that relations $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ of categories given by inclusion of a full subcategory $\mathcal{R} \subseteq \mathcal{C} \times \mathcal{D}$ are Dowker relations if $\mathcal{D}$ has the property that every full subcategory $\mathcal{D}' \subseteq \mathcal{D}$ have an initial or terminal object. Note that translation categories has this property (every object is an initial object). Thus by definition of the functor $S : \mathtt{Rel} \to \mathtt{CatRel}$, the relation $S(R)$ is a Dowker relation. □

Lemma 5.6. *Let $R \subseteq X \times Y$ be a relation of sets. The relations $S(R^T)$ and $S(R)^T$ are equal.*

Proof of the functorial Dowker theorem 2.14. We prove the theorem by combining the commutative diagrams obtained from (14) for the relations R and R^T with the diagram (16) (obtained from the functorial simplicial Dowker theorem 3.19) in the middle. All horizontal maps in this combined diagram are homotopy equivalences. Some of the homotopy equivalences do not have the desired direction, so we choose homotopy inverses in these cases. Taking horizontal composition gives the desired homotopy equivalence, proving the functorial Dowker theorem.

Let $f : R \to R'$ be a morphism of relations of sets. Since the relations $S(R)$, $S(R')$, $S(R^T) = S(R)^T$ and $S(R'^T) = S(R')^T$ are Dowker relations by Lemma 5.5, we get a diagram from Corollary 3.19 on the form:

$$\begin{tikzcd} {|DS(R)|} \arrow[r, "\Phi_{S(R)}"] \arrow[d, "|DS(f)|"'] & {|DS(R)^T|} \arrow[d, "|DS(f)^T|"] \\ {|DS(R')|} \arrow[r, "\Phi_{S(R')}"] & {|DS(R')^T|,} \end{tikzcd} \tag{16}$$

that commutes up to homotopy and where $\Phi_{S(R)}$ and $\Phi_{S(R')}$ are homotopy equivalences. Combining this diagram with the diagrams obtained from (14) for the relations R and R^T, we form a larger diagram:

$$\begin{tikzcd} {|\operatorname{Dow}(R)|} \arrow[d, "|\operatorname{Dow}(f)|"'] & {|DS(R)|} \arrow[l, "\iota_R"'] \arrow[r, "\Phi_{S(R)}"] \arrow[d, "DS(f)"'] & {|DS(R)^T|} \arrow[r, "\iota_{R^T}"] \arrow[d, "DS(f)^T"] & {|\operatorname{Dow}(R^T)|} \arrow[d, "\operatorname{Dow}(f^T)"] \\ {|\operatorname{Dow}(R')|} & {|DS(R')|} \arrow[l, "\iota_{R'}"] \arrow[r, "\Phi_{S(R')}"] & {|DS(R')^T|} \arrow[r, "\iota_{R'^T}"'] & {|\operatorname{Dow}(R'^T)|,} \end{tikzcd} \tag{17}$$

where all horizontal maps are homotopy equivalences.

Let $h : |\operatorname{Dow}(R)| \to |DS(R)|$ be a homotopy inverse of ι_R and let $h' : |\operatorname{Dow}(R')| \to |DS(R')|$ be a homotopy inverse of $\iota_{R'}$. Then the compositions:

$$(\iota_{R^T} \circ \Phi_{S(R)} \circ h) : |\operatorname{Dow}(R)| \to |\operatorname{Dow}(R^T)|,$$

$$(\iota_{R'^T} \circ \Phi_{S(R')} \circ h') : |\operatorname{Dow}(R')| \to |\operatorname{Dow}(R'^T)|,$$

are the desired homotopy equivalences, proving the functorial Dowker theorem 2.14. $\square$

6 Conclusion

We have in this thesis generalized relations from sets to categories (Definition 3.9). This generalized notion of relations defines new category `CatRel`, and we define a functor S : `Rel` $\to$ `CatRel` in Section 5 using that translation categories (Definition 3.12) are in one-to-one correspondence with sets. It can be shown that the functor S is monic and both full and faithful.

Proving the functorial simplicial Dowker theorem involves the bisimplicial rectangle nerve ER (Definition 3.13) of the relation of categories R, as an intermediate object. We introduced Dowker relations (Definition 3.17) in Section 3, which provides a necessary condition on the relation R for the simplicial map $d(\pi_R) : d(ER) \to DR$ to be a weak equivalence.

Relation of categories coming from relation of sets via the functor S are Dowker relations (Lemma 5.5). Thus the functorial Dowker theorem 2.14 is a consequence of the functorial simplicial Dowker theorem 3.19. We gave a proof of this in Section 5, were we used the fact that the Dowker complex is homotopy equivalent (Corollary 5.4) to the Dowker nerve on geometric realization for relations on the mentioned form (this also justifies the name “Dowker” nerve).

Further Research

We may consider a morphism $A \to B \times C$ in a category $\mathcal{C}$ as a relation. Let $F : \Delta \to \mathcal{C}$ be a cosimplicial object in $\mathcal{C}$, sending the order-preserving map $\phi : [m] \to [n]$ to the morphism $\phi_* : F([m]) \to F([n])$. The bisimplicial rectangle nerve has an easy generalization to relations in $\mathcal{C}$ on the form $R : A \to B \times C$ by letting the (m,n)-simplices be given by the set

$$EA(F, \mathcal{C})_{m,n} = \{F([m]) \times F([n]) \xrightarrow{r} A \mid \exists F([m]) \xrightarrow{a} B, F([n]) \xrightarrow{b} C \text{ such that } R \circ r = a \times b\}.$$

My supervisors and I have stated and proved a nerve theorem for covered simplicial sets, using this generalization. Unfortunately, there was not enough time to include this in my thesis.

Let $\mathcal{S}(X,\mathcal{U})$ be the simplicial set with m-simplices given by the set

$$\mathcal{S}(X,\mathcal{U})_m = \{f : \Delta^m \to X \mid \operatorname{im} f \subseteq U_i, \text{ for some } U_i \in \mathcal{U}\}.$$

Further let $E(X,\mathcal{U})$ be the bisimplicial set with (m,n)-simplices given by the set

$$E(X,\mathcal{U})_{m,n} = \{f : \Delta^m \to X, \sigma : \{0,\dots,n\} \to \mathcal{U} \mid f(\Delta^m) \subseteq \sigma(i) \forall i \in \{0,\dots,n\}\}.$$

We have a diagram

$$\mathcal{S}(X,\mathcal{U}) \longleftarrow E(X,\mathcal{U}) \longrightarrow N\mathcal{U},$$

where the horizontal maps are the canonical projections. If I had more time, I would have defined this properly, and given an argument for why the projections are weak equivalences, provided that the cover $\mathcal{U}$ is good. By an extra argument, proving that the inclusion $\mathcal{S}(X,\mathcal{U}) \subseteq \mathcal{S}(X)$ is a weak equivalence, we have related the singular set $\mathcal{S}(X)$ on X with the nerve $N\mathcal{U}$ of the covering, via a chain of weak equivalences. This offers a proof of the functorial nerve theorem for covered spaces 2.6, and I believe this can by "attached" to the bisimplicial rectangle nerve introduced in the article [BFS23].

Further research would include generalizations of relations to arbitrary categories, and defining corresponding bisimplicial rectangle nerves and Dowker nerves.

A Category Theory

We now introduce categories, and the theory needed for this thesis. The books [Lei14] and [ML98] are recommended if the reader is not familiar with category theory.

Categories

Definition A.1 ([Lei14, Definition 1.1.1])**.** A *category* $\mathcal{C}$ consists of the following data:

(i) a collection $\mathrm{ob}(\mathcal{C})$ of *objects*,

(ii) a set $\mathcal{C}(c, d)$ of *morphisms* for every two objects c, d.

(iii) an *identity morphism* $\mathrm{id}_c \in \mathcal{C}(c, c)$ for every object c,

(iv) and a composite morphism $g \circ f \in \mathcal{C}(c, e)$, called the *composition of f and g*, for every three objects c, d, e and two morphisms $f \in \mathcal{C}(c, d)$ and $g \in \mathcal{C}(d, e)$.

In addition, the following requirements has to be satisfied:

- *unitality*: for any morphism $f \in \mathcal{C}(c, d)$, composing with the identity does nothing, that is, $f \circ \mathrm{id}_c = f = \mathrm{id}_d \circ f$.
- *associativity*: for any three morphisms $f \in \mathcal{C}(c_1, c_2)$, $g \in \mathcal{C}(c_2, c_3)$ and $h \in \mathcal{C}(c_3, c_4)$, the composition is associative, that is, $(h \circ g) \circ f = h \circ (g \circ f)$.

Remark A.2. We make the following conventions:

- we write $f : c \to d$ for a morphism $f \in \mathcal{C}(c, d)$.
- we write $c \in \mathcal{C}$ for an object $c \in \mathrm{ob}(\mathcal{C})$.
- we sometimes write gf to mean $g \circ f$, and use these notations interchangeably.
- for a morphism $f : c \to d$, we say that c is the domain $\mathrm{dom}\, f = c$ of f and we say that d is the codomain $\mathrm{codom}\, f = d$ of f.

Definition A.3. Let $\mathcal{C}$ be a category.

- An object $c \in \mathcal{C}$ is said to be *initial* if $\mathcal{C}(c, d) = \{*\}$ for all objects $d \in \mathcal{C}$.
- An object $d \in \mathcal{C}$ is said to be *terminal* if $\mathcal{C}(c, d) = \{*\}$ for all objects $c \in \mathcal{C}$.

Example A.4. Examples of categories:

- The category of sets `Set` where objects are sets and morphisms are functions.

- The category of topological spaces `Top` where objects are topological spaces and morphisms are continuous maps.

Definition A.5 ([ML98, p. 19])**.** Let $\mathcal{C}$ be a category, and let c and d be objects of $\mathcal{C}$.

- A morphism $m : c \to d$ is *monic* if every pair of morphisms

$$a \overset{f}{\underset{g}{\rightrightarrows}} c \xrightarrow{m} d,$$

  where $m \circ f = m \circ g$ implies that $f = g$.

- A morphism $e : c \to d$ is *epi* if every pair of morphisms

$$c \xrightarrow{e} d \overset{f}{\underset{g}{\rightrightarrows}} b,$$

  where $f \circ e = g \circ e$ implies that $f = g$.

Definition A.6 ([ML98, p. 19])**.** Let $\mathcal{C}$ be a category, and let c and d be objects of $\mathcal{C}$. A morphism $f : c \to d$ is an *isomorphism* if there exists a morphism $g : d \to c$ such that $g \circ f = \mathrm{id}_c$ and $f \circ g = \mathrm{id}_d$. Two objects c and d are *isomorphic* $c \cong d$ if there exists an isomorphism between them.

Definition A.7. Let $\mathcal{C}$ be a category. The *opposite category* of $\mathcal{C}$ is the category $\mathcal{C}^{\mathsf{op}}$ with $\mathrm{ob}(\mathcal{C}^{\mathsf{op}}) := \mathrm{ob}(\mathcal{C})$ and $\mathcal{C}^{\mathsf{op}}(c, d) := \mathcal{C}(d, c)$ for all objects $c, d \in \mathcal{C}$.

Remark A.8. Let $\mathcal{C}$ be a category and led c be an object of $\mathcal{C}$. An object c is initial in $\mathcal{C}$ if and only if c is terminal in $\mathcal{C}^{\mathsf{op}}$ and vice versa. A morphism $f : c \to d$ is monic in $\mathcal{C}$ if and only if f is epi in $\mathcal{C}^{\mathsf{op}}$ and vice versa.

Definition A.9. A category $\mathcal{C}$ is said to be small of $\mathrm{ob}(\mathcal{C})$ is a set.

Functors

Definition A.10 ([Lei14, Definition 1.2.1])**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of

(i) a function $F^{\mathrm{ob}} : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$ written as $c \mapsto F(c)$,

(ii) and a function $\mathcal{C}(c, d) \to \mathcal{D}(F(c), F(d))$, written as $f \mapsto F(f)$, for all pairs c and d of objects,

such that the following axioms are satisfied

- $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$ for all $c \in \mathcal{C}$,

- $F(g \circ f) = F(g) \circ F(f)$ for all composable morphisms $f, g \in \mathcal{C}$.

Definition A.11. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *faithful* (resp. *full*) if $\mathcal{C}(c,d) \to \mathcal{D}(F(c), F(d))$ is injective (resp. surjective) for all $c, d \in \mathcal{C}$.

Definition A.12. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then $\mathcal{C}$ is a *subcategory* of $\mathcal{D}$ if $\mathrm{ob}(\mathcal{C}) \subseteq \mathrm{ob}(\mathcal{D})$ and $\mathcal{C}(c,d) \subseteq \mathcal{D}(c,d)$ for all $c, d \in \mathcal{C}$. Moreover, $\mathcal{C}$ is a *full* subcategory if $\mathcal{C}(c,d) = \mathcal{D}(c,d)$ for all $c, d \in \mathcal{C}$.

Small categories with functors as morphisms define a category. Since a functor of small categories consists of two functions of sets, existence of identities and compositions are inherited from the category of sets `Set`. We let `Cat` denote the category of small categories.

Definition A.13 ([ML98, Definition, p. 80])**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An *adjunction* from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, G, \phi) : \mathcal{C} \to \mathcal{D}$ where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors and ϕ is a function assigning to each pair of objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$ a bijection

$$\phi_{c,d} : \mathcal{D}(F(c), d) \to \mathcal{C}(c, G(d)),$$

which is natural in c and d, this is, such that both diagrams commutes

$$\begin{tikzcd}
\mathcal{D}(F(c),d) \arrow[r, "\phi_{c,d}"] \arrow[d, "F(f)^*"] & \mathcal{C}(c,G(d)) \arrow[d, "f^*"] \\
\mathcal{D}(F(c'),d) \arrow[r, "\phi_{c',d}"] & \mathcal{C}(c',G(d))
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}(F(c),d) \arrow[r, "\phi_{c,d}"] \arrow[d, "g_*"] & \mathcal{C}(c,G(d)) \arrow[d, "G(g)_*"] \\
\mathcal{D}(F(c),d') \arrow[r, "\phi_{c,d'}"] & \mathcal{C}(c,G(d'))
\end{tikzcd}$$

for every pair of morphisms $f : c' \to c$ and $g : d \to d'$.

We say that $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to $G : \mathcal{D} \to \mathcal{C}$, or equivalently, G is right adjoint to F, if there exists and adjunction $(F, G, \phi) : \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$. Let $c \in \mathcal{C}$ and $d \in \mathcal{D}$ be objects and let $f : F(c) \to d$ be a morphism in $\mathcal{D}$. The morphism $\phi_{c,d}(f)$ is said to be the *right adjunct* of f.

Natural Transformations

Definition A.14 ([ML98, p. 16])**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\tau : F \implies G$ is a collection $\{\tau_c : F(c) \to G(c)\}_{c \in \mathcal{C}}$ of morphisms in $\mathcal{D}$ such that the following square commutes

$$\begin{tikzcd}
c \arrow[d, "f"'] \\
d
\end{tikzcd}
\qquad
\begin{tikzcd}
F(c) \arrow[r, "\tau_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\
F(d) \arrow[r, "\tau_d"] & G(d)
\end{tikzcd}
\tag{18}$$

for every morphism $f : c \to d$ in $\mathcal{C}$.

Definition A.15. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural isomorphism* $\varphi : F \implies G$ is a natural transformation where φ_c is an isomorphism for every object $c \in \mathcal{C}$.

Definition A.16. Let $\mathcal{C}$ be a category and let a and b be objects of $\mathcal{C}$. The *categorical product* of a and b is a triple $(a \times b, \pi_1, \pi_2)$

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[d, dashed, "u"] \arrow[dr, "g"] & \\ a & a \times b \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & b \end{tikzcd} \tag{19}$$

such that for any other (c, f, g) as in diagram (19), there exists a unique morphism $u : c \to a \times b$ such that $f = \pi_1 \circ u$ and $g = \pi_2 \circ u$.

Definition A.17. Let $\mathcal{C}$ be a category and let (f, g) be the diagram consisting of the two morphisms $f : a \to b$ and $g : a \to b$ in C. The *coequalizer* of the diagram (f, g) is a pair (c, h) consisting of an object c and a morphism $h : b \to c$ such that $h \circ f = h \circ g$,

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "h"] \arrow[dr, "h'"'] & c \arrow[d, dashed, "u"] \\ & & c', \end{tikzcd} \tag{20}$$

and such that for any other pair (c', h'), like in diagram (20), where $h' \circ f = h' \circ g$, there exists an unique morphism $u : c \to c'$ making the triangle commute $u \circ h = h'$.

By an argument given in [ML98, p. 64-65], a coequalizer (c, h) of a diagram (f, g) can be described by a universal morphism from (f, g) to the constant functor on c. With this convention in mind, we name the morphism h in the coequalizer (c, h) as *the universal morphism of* c. We name the property of the existence of a unique morphism u in a coequalizer diagram like (20), such that the triangle commutes, by *universal property of coequalizers.*

Definition A.18 ([ML98, Definition, p. 55])**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and let d be an object of $\mathcal{D}$. An *universal morphism from* d *to* F is a pair (c, u) consisting of an object $c \in \mathcal{C}$ and a morphism $u : d \to F(c)$ in $\mathcal{D}$ such that for every pair (e, g) with $e \in \mathcal{C}$ and $g : d \to F(e)$ a morphism of $\mathcal{D}$,

$$\begin{tikzcd} d \arrow[r, "u"] \arrow[dr, "g"'] & F(c) \arrow[d, dashed, "F(f)"] & c \arrow[d, dashed, "f"] \\ & F(e) & e, \end{tikzcd}$$

there exists an unique morphism $f : c \to e$ in $\mathcal{C}$ where $F(f) \circ u = g$.

Proposition A.19 ([ML98, Theorem 1.(i), p. 82]). *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An adjunction $(F, G, \phi) : \mathcal{C} \to \mathcal{D}$ uniquely determines a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \implies GF$ where component morphism η_c is an universal morphism from c to G for all objects $c \in \mathcal{C}$ and the right adjunct of $f : F(c) \to d$ is the morphism*

$$\phi_{c,d}(f) : G(f) \circ \eta_c : c \to G(d).$$

Proposition A.20 ([ML98, Theorem 2.(i), p. 83]). *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Each adjunction $(F, G, \phi) : \mathcal{C} \to \mathcal{D}$ is completely determined by a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \implies GF$ where component morphism η_c is an universal morphism from c to G for all objects $c \in \mathcal{C}$.*

B Simplicial Complexes

We now introduce simplicial complexes.

Definition B.1. An *(abstract) simplicial complex* is a collection K of finite non-empty sets such that $\sigma \in K$ and $\tau \subseteq \sigma$ implies $\tau \in K$.

An element of K is called a *simplex*. For a simplex $\sigma \in K$ with $k+1$ elements, we say that σ is a simplex of dimension k or a *k-simplex*. A 0-simplex $\{v\}$ is also referred to as a *vertex*. We say that τ is a *face* of σ if σ and τ are simplices of K and $\tau \subseteq \sigma$.

We say that L is a *subcomplex* of K if L is a simplicial complex and $L \subseteq K$. The *n-skeleton* of K is the subcomplex $K_n \subseteq K$ consisting of all simplices $\sigma \in K$ with dimension less than or equal to n.

Definition B.2. The *vertex set* $V(K)$ of a simplicial complex K is the union $\bigcup_{\sigma \in K} \sigma$ of all the simplices of K.

For simplicial complexes K and K', we make the convention denoting $V(K)$ and $V(K')$ as simply V and V'. The elements v of V are in one-to-one correspondence with the vertices $\{v\}$ of K. Thus we refer to an element $v \in V$ as a vertex of K.

Definition B.3. The *barycentric subdivision* of a simplicial complex K is the simplicial complex $sd\,K$ with $\sigma \in \mathcal{P}(K)$ a simplex if inclusion $\subseteq$ defines a total order on σ.

Every singleton subset of the power set $\mathcal{P}(K)$ is totally ordered under inclusion, thus $V(sd\,K) = K$.

Definition B.4. A *simplicial map* from the simplicial complex K to the simplicial complex K' is a map of the vertex sets $\phi : V(K) \to V(K')$ such that $\sigma \in K$ implies $\phi(\sigma) \in K'$.

We write $\phi : K \to K'$ for the simplicial map given by $\phi : V(K) \to V(K')$.

The composition of two simplicial maps is a simplicial map. Let $\phi : K \to L$ and $\psi : L \to M$ be maps of simplicial complexes and let σ be a simplex of K. Then $\phi(\sigma)$ is a simplex in L and $\psi\phi(\sigma)$ is a simplex in M. Thus the composition $\psi \circ \phi$ is a simplicial map.

Simplicial complexes and simplicial maps defines a category. It is trivial that the identity map $\mathrm{id} : K \to K$ exists. Thus we get a category $\mathtt{Cpx}$, with objects given by simplicial complexes and morphisms given by simplicial maps.

The barycentric subdivision defines a functor from simplicial complexes to itself. Given a simplicial map $\phi : K \to K'$, we get a map $sd\,\phi : sd\,K \to sd\,K'$ sending a vertex σ of $sd\,K$ to the vertex $\phi(\sigma)$ of $sd\,K'$. This is a simplicial map since a simplex $\{\sigma_0 \subset \cdots \subset \sigma_m\}$ of $sd\,K$ is sent to the simplex $\{\phi(\sigma_0) \subset \cdots \subset \phi(\sigma_n)\}$ of $sd\,K'$. It is trivial that identities and compositions are preserved. Thus the barycentric subdivision defines a functor $sd : \mathtt{Cpx} \to \mathtt{Cpx}$ sending $\phi : K \to K'$ to $sd\,\phi : sd\,K \to sd\,K'$.

Geometric Realization

Definition B.5. Let A be a set and let $h : A \to \mathbb{R}$ be a function. The support of h is the subset $\sup(h) \subseteq A$ with elements $a \in \operatorname{supp}(h)$ if $h(a) \neq 0$.

For a fixed set S, consider the set $[0,1]^S$ of functions $\alpha : S \to [0,1]$ from S to the closed interval of real numbers between 0 and 1. We define a topology on $[0,1]^A$ making it a topological space. If S is finite, then $[0,1]^S$ is isomorphic to the product $\prod_S [0,1]$. Thus for finite S we give $[0,1]^S$ the product topology. If S is infinite, we give $[0,1]^S$ the topology where $U \subseteq [0,1]^S$ is open if, for every finite subset $W \subseteq S$, the subset $U \cap [0,1]^W$ is open in $[0,1]^W$. [BFS23]

A simplicial complex induce a topological space. We define the *geometric realization* of a simplicial complex K as a subspace of $[0,1]^V$.

Definition B.6. Let K be a simplicial complex. The *geometric realization* of K is the set

$$|K| = \left\{ \alpha : V \to [0,1] \,\middle|\, \operatorname{supp}(\alpha) \in K, \sum_{v \in V} \alpha(v) = 1 \right\},$$

given the subspace topology of $[0,1]^V$.

With the assignment of a topological space to each simplicial complex comes homotopy types. We say that the homotopy type of a simplicial complex K is the homotopy type of the geometric realization $|K|$.

Simplicial maps induce continuous maps on geometric realization. Let $\phi : K \to K'$ be a simplicial map, and let $\alpha : V \to [0,1]$ be an element of $|K|$. We define $|\phi|(\alpha) : V' \to [0,1]$ to be the map sending a vertex v' to the sum

$$|\phi|(\alpha)(v') = \sum_{\phi(v)=v'} \alpha(v).$$

The map $|\phi|(\alpha)$ is an element of $|K'|$ by Lemma B.7. Thus sending α to $|\phi|(\alpha)$ defines a continuous map $|\phi| : |K| \to |K'|$.

Lemma B.7. *Let $\phi : K \to K'$ be a simplicial map and let $\alpha : V \to [0,1]$ be an element of $|K|$. Then*

1. *the support of $|\phi|(\alpha)$ is a simplex of K',*
2. *and $\sum_{v' \in V'} |\phi|(\alpha)(v') = 1$.*

Proof. Part 1. We show that $\operatorname{supp}(|\phi|(\alpha))$ is a face of the simplex $\phi(\operatorname{supp}(\alpha))$. Let v' be a vertex of V'. If $v' \in \operatorname{supp}(|\phi|(\alpha))$, then the sum

$$\sum_{\phi(v)=v'} \alpha(v) > 0$$

is greater than zero. This implies that there exists a vertex $v \in V$ such that $f(v) = v'$ and $\alpha(v) > 0$. Then $v \in \operatorname{supp}(\alpha)$ and $v' \in \phi(\operatorname{supp}(\alpha))$. Thus $\operatorname{supp}(|\phi|(\alpha)) \subseteq \phi(\operatorname{supp}(\alpha))$ and $\phi(\operatorname{supp}(\alpha))$ is a simplex of K'.

Part 2. For a map $\phi : V \to V'$, the inverse image ϕ^{-1} defines a partition of V with a class $\phi^{-1}(v')$ for all $v' \in V'$. Summation over all elements $v \in \phi^{-1}(v')$ of a given partition class, and then over all partition classes $\phi^{-1}(v') \subseteq V$ is the same as summing over all elements $v \in V$. Thus we get the following equation

$$\sum_{v' \in V'} |\phi|(\alpha)(v') = \sum_{v' \in V'} \sum_{\phi(v)=v'} \alpha(v) = \sum_{v \in V} \alpha(v) = 1.$$

□

Proposition B.8. *The geometric realization preserves compositions, that is, for every composable pair ϕ and ψ of simplicial maps, $|\psi \circ \phi| = |\psi| \circ |\phi|$.*

Proof. Let $\phi : K \to L$ and $\psi : L \to M$ be simplicial maps and let $\alpha : V(K) \to [0,1]$ be an element of $|K|$. We show that $|\psi \circ \phi|(\alpha) = |\psi| \circ |\phi|(\alpha)$. Fix a vertex v'' in $V(M)$ and recall that $|\psi \circ \phi| : |K| \to |M|$ takes α to the map $|\psi \circ \phi|(\alpha) : V(M) \to [0,1]$ defined by the sum

$$|\psi \circ \phi|(\alpha)(v'') = \sum_{(\psi \circ \phi)(v)=v''} \alpha(v). \tag{21}$$

We trace α through first $|\phi|$ and then $|\psi|$. The map $|\phi|$ takes α to $|\phi|(\alpha) : V(L) \to [0,1]$, which is defined by

$$|\phi|(\alpha)(v') = \sum_{\phi(v)=v'} \alpha(v)$$

for $v' \in V(L)$. Applying $|\psi|$ to $|\phi|(\alpha)$ gives the map $|\psi||\phi|(\alpha)$ in $|M|$ which maps v'' to the sum

$$|\psi||\phi|(\alpha)(v'') = \sum_{\psi(v')=v''} |\phi|(\alpha)(v') = \sum_{\psi(v')=v''} \sum_{\phi(v)=v'} \alpha(v) \tag{22}$$

The combined sum of (22) ranges over all vertices $v \in V(K)$ such that there exists a $v' \in V(L)$ with $\phi(v) = v'$ and $\psi(v') = v''$. This is the same range as all vertices v in $V(K)$ such that $(\psi \circ \phi)(v) = v''$, which is the range of the sum in (21). Thus the sums in (21) and (22) are equal and $|\psi \circ \phi|(\alpha) = |\psi||\phi|(\alpha)$. □

Identity maps are preserved by geometric realization. Let $\mathrm{id} : K \to K$ be the identity simplicial map. Then $|\mathrm{id}| : |K| \to |K|$ sends the map $\alpha : V \to [0,1]$ to the map $|\mathrm{id}_K|(\alpha)$, which maps a vertex v' to the sum $\sum_{\mathrm{id}(v)=v'} \alpha(v)$. Thus $|\mathrm{id}_K|(\alpha)(v') = \alpha(v')$, and identities are preserved by geometric realization.

Geometric realization defines a functor from simplicial complexes to topological spaces. Compositions are preserved by Lemma B.8. We define the functor $|\cdot| : \texttt{Cpx} \to \texttt{Top}$ sending the simplicial map $\phi : K \to K'$ to the continuous map $|\phi| : |K| \to |K'|$.

Definition B.9. A simplicial map $f : K \to K'$ is a *weak equivalence* if the induced map $|f| : |K| \to |K'|$ on geometric realization is a homotopy equivalence of topological spaces.

There exists simplicial complexes with the same homotopy type, but with no weak equivalence between them. Let K be a simplicial complex. Theorem B.10 states that K and the barycentric subdivision $sd\,K$ are homeomorphic on geometric realization. However, this homeomorphism is not induced by any weak equivalence $K \to sd\,K$, which is proven by contradiction in Proposition B.11. Thus K and $sd\,K$ have different combinatorial structure, yet the same homotopy type.

We do not provide any proof of the following theorem. However, this result is well known, and a proof may be found in [Spa94, Chapter 3.3, p. 121-123]. We assert in addition that the homeomorphism is natural, which is straightforward to check with a diagram chase.

Theorem B.10. *Let K be a simplicial complex. Then there exists a homeomorphism $\eta_K : |K| \to |sd\,K|$, which is natural in K.*

Proposition B.11. *Let K be a simplicial complex, and let $h : |K| \to |sd\,K|$ be a homeomorphism. Then there exists no simplicial map $\phi : K \to sd\,K$ such that $|\phi| = h$.*

Proof. The proof is by contradiction. We assume that there exists a map $\phi : K \to sd\,K$ such that $|\phi| = h$, and then show that h is not surjective as a consequence.

Note that $V \subset K$, which implies that there exists a simplex $\sigma' \in K$ (that is, a vertex of $sd\,K$) with no $v \in V$ such that $\phi(v) = \sigma'$. The map $\beta : K \to [0,1]$ with support $\mathrm{supp}(\beta) = \{\sigma'\}$ is not contained in the image $\mathrm{im}\,h$, since

$$\sum_{\phi(v)=\sigma'} \alpha(v) = 0$$

for all $\alpha \in |K|$. Thus there exists no $\alpha \in |K|$ such that $|\phi|(\alpha) = \beta$ which implies that h is not surjective. This contradicts the assumption of h being a homeomorphism. $\square$

C Simplicial Sets

Given an integer $m \geq 0$, let $[m]$ be the totally ordered set of $m + 1$ elements,

$$[m] = \{0 < 1 < \cdots < m\}.$$

The *simplex category* Δ has objects given by $[m]$ for each integer $m \geq 0$ and arrows given by order-preserving maps.

Among all maps in Δ, there are special ones, namely the *coface-* and *codegeneracy maps*. For $0 \leq i \leq m$, the ith coface map is the only order-preserving injection $d^i : [m] \to [m+1]$ omitting i in its image, that is, for $0 \leq k \leq m$:

$$d^i(k) = \begin{cases} k, & \text{if } k < i, \\ k+1, & \text{if } k \geq i. \end{cases}$$

Similarly, for $0 \leq i \leq m > 0$, the ith codegeneracy map is the only order-preserving surjection $s^i : [m] \longrightarrow [m-1]$ taking the value i twice, that is, for $0 \leq k \leq m$:

$$s^i(k) = \begin{cases} k, & \text{if } k \leq i, \\ k-1, & \text{if } k > i. \end{cases}$$

Proposition C.1. *The coface and codegeneracy maps in Δ satisfy the* cosimplicial identities*:*

$$\begin{cases} d^j d^i = d^i d^{j-1}, & \text{if } i < j, \\ s^j d^i = d^i s^{j-1}, & \text{if } i < j, \\ s^i d^i = \mathrm{id} = s^i d^{i+1}, & \\ s^j d^i = d^{i-1} s^j, & \text{if } i > j+1, \\ s^j s^i = s^i s^{j+1}, & \text{if } i \leq j. \end{cases} \tag{23}$$

Proposition C.2 ([ML95, Lemma 5.2, p. 234])**.** *Every order-preserving map $\phi : [m] \to [n]$ in Δ can be factored through a unique sequence of coface and codegeneracy maps*

$$\phi = d^{i_k} \cdots d^{i_0} s^{j_0} \cdots s^{j_l},$$

such that $i_0 < \cdots < i_k$ and $j_0 < \cdots < j_l$.

Proof. Let $\{i_0 < \cdots < i_k\} \subseteq [n]$ be the subset of elements in $[n]$ not lying in the image $\operatorname{im} \phi$, and let $\{j_0 < \cdots < j_l\} \subseteq [m]$ be the subset of elements such that $\phi(j) = \phi(j+1)$. It is straightforward to check that

$$\phi = d^{i_k} \cdots d^{i_0} s^{j_0} \cdots s^{j_l},$$

and this factorization is indeed unique. $\square$

Definition C.3. A *simplicial set* is a functor $X : \Delta^{\mathtt{op}} \to \mathtt{Set}$.

Let X be a simplicial set. The set $X_m := X([m])$ is called the set of *m-simplices* of X and an element $x \in X_m$ is called an m-simplex. The order-preserving map $\phi : [m] \to [n]$ in Δ is sent to the *structure map* $\phi^* := X(\phi) : X_n \to X_m$.

The *simplicial structure* on a simplicial set X is decided by the structure maps, and some of the structure maps are more essential than other. The coface maps $d^i : [m-1] \to [m]$ are sent to the *face maps* $d_i := X(d^i) : X_m \to X_{m-1}$. Let x be an m-simplex and let y be an $m-1$-simplex of X. Then y is the *ith face* of x if $d_i(x) = y$. The codegeneracy maps $s^i : [m+1] \to [m]$ are sent to the *degeneracy maps* $s_i := X(s^i) : X_m \to X_{m+1}$. We say that x is a *degenerate* simplex if there exists a degeneracy map s_j, for $0 \leq j \leq m$, and an $m-1$-simplex z such that $s_j(z) = x$.

Definition C.4. The *topological standard m-simplex* $\Delta^m \subseteq \mathbb{R}^{m+1}$ is the set

$$\Delta^m := \{(t_0, t_1, \ldots, t_m) \,|\, t_i \geq 0 \text{ for all } 0 \leq i \leq m, \sum_{i=0}^{m} t_i = 1\}$$

given the subspace topology.

Example C.5. Let X be a topological space. The *singular set* $\mathcal{S}(X)$ is the simplicial set with m-simplices given by the set $\mathcal{S}(X)_m := \mathtt{Top}(\Delta^m, X)$, consisting of all continuous maps from Δ^m to X. An order-preserving map $\phi : [m] \to [n]$ induce a continuous map $\phi_* : \Delta^m \to \Delta^n$, mapping the tuple $t = (t_0, \ldots, t_m)$ to the tuple $\phi_*(t) = (\phi_*(t)_0, \ldots, \phi_*(t)_n)$ where

$$\phi_*(t)_i = \sum_{\phi(j)=i} t_j.$$

The structure map ϕ^* is defined by sending the an n-simplex $x : \Delta^n \to X$ to the composition $x \circ \phi_*$.

Definition C.6. Let X and Y be simplicial sets. A simplicial map $f : X \to Y$ is a natural transformation of the functors.

In line with the convention for simplicial sets, we denote the component map $f_{[m]} : X([m]) \to Y([m])$ by $f_m : X_m \to Y_m$.

Example C.7. Given a continuous map $f : X \to Y$ of topological spaces, we get a simplicial map $f_* : \mathcal{S}(X) \to \mathcal{S}(Y)$. Let $f_m : \mathcal{S}(X)_m \to \mathcal{S}(Y)_m$ be defined by sending a simplex $x : \Delta^m \to X$ to the composition $f \circ x$. The component maps f_m commutes with the structure maps ϕ^*, since these are defined by precomposition. Thus $f_* : \mathcal{S}(X) \to \mathcal{S}(Y)$ is indeed a simplicial map.

Simplicial sets and simplicial maps inherits the category structure from the functor category $\mathtt{Set}^{\Delta^{\mathrm{op}}}$. We reserve the symbol $\mathtt{sSet}$ to be the category for simplicial sets and simplicial maps.

Example C.8 (Constant Simplicial Set)**.** The simplicial set C where $C_i = C_j$ for all $i \geq 0$ and $j \geq 0$, and all structure maps are identities, is said to be *constant*.

Example C.9 (Products)**.** Let X and Y be simplicial sets. The *product* of X and Y is the simplicial set $X \times Y$ with m-simplices given by the product $(X \times Y)_m := X_m \times Y_m$, where structure maps are defined component wise $\phi^* = X_n \times Y_m \to X_m \times Y_m$, taking (x, y) to $(X(\phi)(x), Y(\phi)(y))$.

Remark C.10. Every set X defines a constant simplicial set $X : \Delta^{\mathrm{op}} \to \mathtt{Set}$ with $X_m = X$ for all $m \geq 0$. For instance, let $\{*\}$ be the constant simplicial set with simplices given by the singleton set $\{*\}$ in every degree. For every simplicial set X, we have a canonical isomorphism $X \times \{*\} \cong X$.

Example C.11 (Simplicial Subsets)**.** Let X be a simplicial set. A *simplicial subset* $A \subseteq X$ is a simplicial set A and a simplicial map $i : A \to X$ such that $i_m : A_m \to X_m$ is the inclusion of subsets for all $m \geq 0$.

Example C.12 (Standard Simplex)**.** The *standard n-simplex* $\Delta[n]$ is the simplicial set where the set of m-simplices are

$$(\Delta[n])_m := \Delta([m], [n]).$$

Let $f : [n] \to [k]$ be an n-simplex in $\Delta[k]$. For a map $\phi : [m] \to [n]$, the structure map $\phi^* : \Delta([n], [k]) \to \Delta([m], [k])$ is defined by the composition $\phi^*(f) = f \circ \phi$.

Remark C.13. The standard 0-simplex $\Delta[0]$ has only one simplex in each degree, since the object $[0]$ is terminal in Δ. Thus $\Delta[0] \cong \{*\}$.

Example C.14 (Nerve of a category)**.** Let $\mathcal{C}$ be a small category. The *simplicial nerve* of $\mathcal{C}$ is the simplicial set $N\mathcal{C}$ with m-simplices given by the set $N\mathcal{C}_m := \mathtt{Cat}([m], \mathcal{C})$. The structure maps ϕ^* are defined by precomposition with ϕ.

Remark C.15. Let $\mathcal{C}$ be a small category. An m-simplex $f : [m] \to \mathcal{C}$ of the nerve $N\mathcal{C}$ may be regarded as a string

$$\{f(m) \to f(m-1) \to \cdots \to f(0)\}$$

of composable morphisms in $\mathcal{C}$. The face map d_i removes $f(i)$ from the string by composing $f(i-1) \to f(i)$ and $f(i) \to f(i+1)$ (if $i = 0, m$ the corresponding morphism is simply removed) such that we get a string with one less morphism. The degeneracy map s_i inserts an extra object $f(i)$ to the string using the identity morphism.

Given a functor $F : \mathcal{C} \to \mathcal{D}$ of categories $\mathcal{C}$ and $\mathcal{D}$, we get a simplicial map $NF : N\mathcal{C} \to N\mathcal{D}$ defined by sending the simplex $f : [m] \to \mathcal{C}$ to the composition $F \circ f$. It is trivial that identity and composition are preserved. Thus the simplicial nerve defines a functor $N : \mathtt{Cat} \to \mathtt{sSet}$.

Remark C.16. We have an isomorphism $\Delta([m], [n]) \to \mathtt{Cat}([m], [n])$ for all integers $m \geq 0$. This collection of isomorphisms commutes with face and degeneracies, thus we get an isomorphism on simplicial sets $\Delta[n] \to N[n]$ for all $n \geq 0$.

Definition C.17. Let $f, g : X \to Y$ be simplicial maps from X to Y. A *simplicial homotopy* from f to g is a simplicial map $h : X \times \Delta[1] \to Y$ such that the following diagram commutes:

$$
\begin{tikzcd}
X \times \Delta[0] \arrow[d, "\mathrm{id} \times d^1"'] \arrow[dr, "f"] & \\
X \times \Delta[1] \arrow[r, "h"] & Y. \\
X \times \Delta[0] \arrow[u, "\mathrm{id} \times d^0"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

If there exists a homotopy from f to g, we say f and g are *homotopic* and write $f \simeq g$. [GJ09, p. 24]

Definition C.18 (Homotopy Equivalence)**.** Let X and Y be simplicial sets. A simplicial map $f : X \to Y$ is a *homotopy equivalence* if there exists a *homotopy inverse* of f, that is, a simplicial map $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

If there is a homotopy equivalence between X and Y, we say that X and Y are *homotopy equivalent* and write $X \simeq Y$.

Definition C.19. A simplicial set X is *contractible* if $X \simeq \{*\}$.

Simplicial and Cosimplicial Objects

Simplicial sets are instances of something more general. For a category $\mathcal{C}$, the functor category $\mathcal{C}^{\Delta^{\mathrm{op}}}$ suggests that there exists objects in different categories with a simplicial structure. This leads to the definition of a simplicial object in a category $\mathcal{C}$, which leaves simplicial sets as the special case where $\mathcal{C} = \mathtt{Set}$.

Definition C.20. Let $\mathcal{C}$ be a category. A *cosimplicial object* Y in $\mathcal{C}$ is a functor $Y : \Delta \to \mathcal{C}$ and a *simplicial object* X in $\mathcal{C}$ is a functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$.

For a cosimplicial object X in a category $\mathcal{C}$, we make the convention of denoting $X(d^i)$ and $X(s^i)$ by simply d^i and s^i respectively.

Example C.21. The topological standard simplices $\{\Delta^n\}$ determines a cosimplicial object in the category topological space. We define a functor $X : \Delta \to \mathtt{Top}$ by taking the order-preserving map $\phi : [m] \to [n]$ to the induced map $\phi_* : \Delta^m \to \Delta^n$ of Example C.5.

Example C.22. The construction of the singular set in Example C.5 utilizes the cosimplicial structure of the topological standard simplices. This can be generalized to other cosimplicial objects in a different categories. Let $\mathcal{C}$ be a category, and let $F : \Delta \to \mathcal{C}$ be the cosimplicial object sending $\phi : [m] \to [n]$ to $\phi_* : \mathbf{m} \to \mathbf{n}$. Given an object X in $\mathcal{C}$, we get a simplicial set $\sin X$ with m-simplices given by the set

$$\sin X_m = \{f : \mathbf{m} \to X\}.$$

The simplicial structure on $\sin X$ is inherited from the cosimplicial object F. For an order-preserving map $\phi : [m] \to [n]$ in Δ, the structure map $\phi^* : \sin X_n \to \sin X_m$ is defined by precomposition with $\phi_* : \mathbf{m} \to \mathbf{n}$.

Definition C.23. A *bisimplicial set* is a functor on the form $X : \Delta^{\mathtt{op}} \times \Delta^{\mathtt{op}} \to \mathtt{Set}$.

For a bisimplicial set X, we denote the set $X([m],[n])$ by $X_{m,n}$ and call it the *set of (m,n)-simplices*. For order-preserving maps $\phi_1 : [m'] \to [m]$ and $\phi_2 : [n'] \to [n]$, we get the structure map $\phi_1^* \times \phi_2^* : X_{m,n} \to X_{m',n'}$. In particular, we have *horizontal* face $d^i \times \mathrm{id}$ and degeneracy $s^i \times \mathrm{id}$ maps (resp. *vertical* face $\mathrm{id} \times d^i$ and degeneracy $\mathrm{id} \times s^i$ maps) that satisfy a dual vertical (resp. horizontal) version of the cosimplicial identities (23). Thus in each horizontal (resp. vertical) simplicial degree m of X, we get a simplicial set $X_{m,\bullet}$ (resp. $X_{\bullet,m}$) with n-simplices given by the set $X_{m,n}$ (resp. $X_{n,m}$) and structure maps given by the vertical (resp. horizontal) face and degeneracy maps.

Remark C.24. There is a bijection $\mathtt{Func}(\Delta^{\mathtt{op}}, \mathtt{Set}^{\Delta^{\mathtt{op}}}) \cong \mathtt{Func}(\Delta^{\mathtt{op}} \times \Delta^{\mathtt{op}}, \mathtt{Set})$. Thus a bisimplicial set X may be seen as a $\Delta^{\mathtt{op}} \to \mathtt{sSet}$ simplicial object in the category of simplicial sets.

Definition C.25 (Bisimplicial maps)**.** Let X and Y be bisimplicial sets. A *bisimplicial map $f : X \to Y$* is a natural transformation of the functors X and Y.

The functor category $\mathtt{Set}^{\Delta^{\mathtt{op}} \times \Delta^{\mathtt{op}}}$ is the category of bisimplicial sets and bisimplicial maps. We reserve the symbol $\mathtt{ssSet}$ to denote this category.

Definition C.26. A bisimplicial map $f : X \to Y$ is a *pointwise weak equivalence* if $f_m : X_m \to Y_m$ is a weak equivalence of simplicial sets for all $m \geq 0$.

Proposition C.27 ([GJ09, Proposition IV.1.7])**.** *Let $f : X \to Y$ be a bisimplicial map. If f is a pointwise weak equivalence, then f induce a weak equivalence $d(f) : d(X) \to d(Y)$ of simplicial sets on the diagonal.*

Geometric Realization of Simplicial Sets

We now construct a topological space $|X|$ from a simplicial set X. One may think of this construction as a collection of topological standard simplices glued together in the way that the simplicial structure of X demands. For example, given two m-simplices $x, y \in X_m$ such that $d_i x = d_j y$ for some $i \geq m$ and $j \geq m$, we identify the ith face and jth face of two standard m-simplices Δ^m. We formalize this identification in terms of a coequalizer.

Definition C.28. Let X be a simplicial set. The *geometric realization* of X is the coequalizer of the following diagram

$$\coprod_{[m],[n]} X_n \times \Delta([m],[n]) \times \Delta^m \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \coprod_{[m]} X_m \times \Delta^m \xrightarrow{\eta_X} |X|,$$

where $\alpha : (x, \phi, t) \mapsto (\phi^* x, t)$ and $\beta : (x, \phi, t) \mapsto (x, \phi_* t)$.

Proposition C.29 ([GJ09, Proposition I.2.3]). *The geometric realization $|X|$ of a simplicial set X is a CW-complex.*

Geometric realization defines a functor from simplicial sets to topological spaces. In order to define this functor, we first construct a continuous map using the universal property of coequalizers. We then show that identities and compositions are preserved, such that this construction indeed defines a functor.

Given a simplicial map $f : X \to Y$ we get a commutative diagram on the form

$$\begin{tikzcd} \coprod_{[m],[n]} X_n \times \Delta([m],[n]) \times \Delta^m \arrow[r, shift left, "\alpha_X"] \arrow[r, shift right, "\beta_X"'] \arrow[d, "f_n \times \mathrm{id} \times \mathrm{id}"] & \coprod_{[m]} X_m \times \Delta^m \arrow[r, "\eta_X"] \arrow[d, "f_n \times \mathrm{id}"'] \arrow[dr, "\gamma"] & {|X|} \\ \coprod_{[m],[n]} Y_n \times \Delta([m],[n]) \times \Delta^m \arrow[r, shift left, "\alpha_Y"] \arrow[r, shift right, "\beta_Y"'] & \coprod_{[m]} Y_m \times \Delta^m \arrow[r, "\eta_Y"'] & {|Y|,} \end{tikzcd} \tag{24}$$

where η_Y is the coequalizer map of Y and γ is defined to be the composition. We prove in Lemma C.30 that $\gamma \circ \alpha_X = \gamma \circ \beta_X$ implying the existence of a universal map $|f| : |X| \to |Y|$ such that the following diagram commutes

$$\begin{tikzcd} \coprod X_m \times \Delta^m \arrow[r, "\eta_X"] \arrow[d, "f \times \mathrm{id}"'] \arrow[dr, "\gamma"] & {|X|} \arrow[d, dashed, "|f|"] \\ \coprod Y_m \times \Delta^m \arrow[r, "\eta_Y"'] & {|Y|.} \end{tikzcd} \tag{25}$$

Thus given a simplicial map $f : X \to Y$ we get a continuous map $|f| : |X| \to |Y|$ of the respective geometric realizations.

Lemma C.30. *Let $f : X \to Y$ be a simplicial map, and let γ, α_X and β_X be the maps from diagram* (24)*. Then $\gamma \circ \alpha_X = \gamma \circ \beta_X$.*

Proof. Let x be a simplex of X, let ϕ be a map of Δ and let t be an element of Δ^m for some $m \geq 0$. Recall that γ is defined to be the composition $\eta_Y \circ f \times \mathrm{id}$.

The element (x, ϕ, t) is mapped by $\gamma\alpha_X$ to the class $[f\phi^* x, t]$ by the following diagram chase:

$$\begin{tikzcd} (x,\phi,t) \arrow[r, mapsto, "\alpha_X"] & (\phi^* x, t) \arrow[d, mapsto, "f \times \mathrm{id}"] & \\ & (f\phi^* x, t) \arrow[r, mapsto, "\eta_Y"] & {[f\phi^* x, t].} \end{tikzcd} \tag{26}$$

The following diagram chase shows that (x, ϕ, t) is mapped by $\gamma\beta_X$ to the class $[fx, \phi_* t]$:

$$\begin{tikzcd} (x,\phi,t) \arrow[r, mapsto, "\beta_X"] & (x, \phi_* t) \arrow[d, mapsto, "f \times \mathrm{id}"] & \\ & (fx, \phi_* t) \arrow[r, mapsto, "\eta_Y"] & {[fx, \phi_* t].} \end{tikzcd}$$

But $[f\phi^* x, t] = [\phi^* f x, t]$ since structure maps commutes with simplicial maps, and $[\phi^* f x, t] = [f x, \phi_* t]$ by definition. Thus $\gamma\alpha_X = \gamma\beta_X$. □

Having defined a continuous map $|f| : |X| \to |Y|$, from a simplicial map $f : X \to Y$, we will now proceed to prove that identities are preserved. For the identity map $\mathrm{id} : X \to X$, we get a commutative diagram

$$\begin{tikzcd} \coprod X_m \times \Delta^m \arrow[r, "\eta_X"] \arrow[d, "\mathrm{id}_X \times \mathrm{id}"'] \arrow[dr, "\gamma"] & {|X|} \arrow[d, "|\mathrm{id}_X|"] \\ \coprod X_m \times \Delta^m \arrow[r, "\eta_X"'] & {|X|,} \end{tikzcd}$$

where the commutativity of lower triangle implies $\gamma = \eta_X$. By commutativity of upper triangle we get that $|\mathrm{id}_X| = \mathrm{id}_{|X|}$, thus identities are preserved.

Proposition C.31. *Let $f : X \to Y$ and $g : Y \to Z$ be simplicial maps. Then the geometric realization preserves compositions, that is, $|g| \circ |f| = |g \circ f|$.*

Proof. Note that $|g| \circ |f| = |g \circ f|$ is equivalent to commutativity of outer square of the diagram

$$\begin{tikzcd} \coprod X_m \times \Delta^m \arrow[r, "\eta_X"] \arrow[d, "f\times\mathrm{id}"'] \arrow[dr, "\gamma_1"] & {|X|} \arrow[d, "|f|"] \\ \coprod Y_m \times \Delta^m \arrow[r, "\eta_Y"] \arrow[d, "g\times\mathrm{id}"'] \arrow[dr, "\gamma_2"] & {|Y|} \arrow[d, "|g|"] \\ \coprod Z_m \times \Delta^m \arrow[r, "\eta_Z"'] & {|Z|,} \end{tikzcd}$$

which indeed is true since all triangles in the diagram commutes. □

We let $|\cdot| : \texttt{sSet} \to \texttt{Top}$ be the *geometric realization functor*, sending a simplicial map $f : X \to Y$ the the continuous map $|f| : |X| \to |Y|$.

Proposition C.32. *The geometric realization functor $|\cdot| : \texttt{sSet} \to \texttt{Top}$ is left adjoint to the singular set functor $\mathcal{S}(:)\texttt{Top} \to \texttt{sSet}$. [GJ09, Prop 2.2, p. 7]*

Definition C.33. A simplicial map $f : X \to Y$ is a *weak equivalence* if the induced map $|f| : |X| \to |Y|$ on geometric realization is a homotopy equivalence.

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Dowker Duality for Relations of Categories

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Abstract

We propose a categorification of the Dowker duality theorem for relations. Dowker's theorem states that the Dowker complex of a relation $R \subseteq X \times Y$ of sets X and Y is homotopy equivalent to the Dowker complex of the transpose relation $R^T \subseteq Y \times X$. Given a relation R of small categories $\mathcal{C}$ and $\mathcal{D}$, that is, a functor of the form $R\colon \mathcal{R} \to \mathcal{C} \times \mathcal{D}$, we define the *bisimplicial rectangle nerve* ER and the *Dowker nerve* DR. The diagonal $d(ER)$ of the bisimplicial set ER maps to the simplicial set DR by a natural projection $d(\pi_R)\colon d(ER) \to DR$.

We introduce a criterion on relations of categories ensuring that the projection from the diagonal of the bisimplicial rectangle nerve to the Dowker nerve is a weak equivalence. Relations satisfying this criterion are called *Dowker relations*. If both the relation R of categories and its transpose relation R^T are Dowker relations, then the Dowker nerves DR and DR^T are weakly equivalent simplicial sets.

In order to justify the abstraction introduced by our categorification we give two applications. The first application is to show that Quillen's Theorem A can be considered as an instance of Dowker duality. In the second application we consider a simplicial complex K with vertex set V and show that the geometric realization of K is naturally homotopy equivalent to the geometric realization of the simplicial set with the set of n-simplices given by functions $\{0, 1, \dots, n\} \to V$ whose image is a simplex of K.

1 Introduction

In the paper "Homology Groups of Relations"[6] from 1952, C.H. Dowker associates an abstract simplicial complex $D(R)$ to a relation $R \subseteq X \times Y$ from a set X to a set Y. The vertex set of $D(R)$ is the set X, and a subset σ of X is a simplex in $D(R)$ if and only if there exists an element $y \in Y$ such that $\sigma \times \{y\} \subseteq R$. Dowker's theorem [6, Theorem 1a, p. 89] states that the homology groups of the *Dowker complex* $D(R)$ are isomorphic to the homology groups of the Dowker complex $D(R^T)$ of the transposed relation $R^T \subseteq Y \times X$ consisting of pairs (y, x) with $(x, y) \in R$.

Before introducing our categorification of the Dowker duality theorem, we give a short summary of its history. In [1, Theorem 10.9] Björner shows that the simplicial complexes $D(R)$ and $D(R^T)$ have homotopy equivalent geometric realizations by constructing an explicit homotopy equivalence $\varphi_R\colon |D(R)| \to |D(R^T)|$. Given an inclusion $R \subseteq S$ of relations from X to Y, Chowdhury and Mémoli [4, Theorem 3] shows that the diagram

$$\begin{array}{ccc} |D(R)| & \xrightarrow{\varphi_R} & |D(R^T)| \\ \downarrow & & \downarrow \\ |D(S)| & \xrightarrow{\varphi_S} & |D(S^T)| \end{array}$$

commutes up to homotopy, giving a *functorial* Dowker theorem. In [10, Theorem 5.2] Virk extends this result to morphisms $f\colon R \to R'$ of relations $R \subseteq X \times Y$ and $R' \subseteq X' \times Y'$ given by a pair (f_1, f_2) of functions $f_1\colon X \to X'$ and $f_2\colon Y \to Y'$ such that the image of R under the function $f_1 \times f_2\colon X \times Y \to X' \times Y'$ is contained in R'. Brun and Salbu give an alternative proof of the functorial Dowker theorem in [2] by introducing the *rectangle complex* $E(R)$ of the relation $R \subseteq X \times Y$. The assignment $R \mapsto E(R)$ is a functor from the category of relations with morphisms of the above form to the category of simplicial complexes. The projection $X \times Y \to X$ induces a natural map $E(R) \to D(R)$ and the projection $X \times Y \to Y$ induces a natural map $E(R) \to D(R^T)$. The functorial Dowker theorem is proven by showing that the geometric realizations of these maps are homotopy equivalences.

In this paper we consider relations R from a small category $\mathcal{C}$ to a small category $\mathcal{D}$, that is, functors of the form $R\colon \mathcal{R} \to \mathcal{C} \times \mathcal{D}$. Such functors are usually called a *span*, but guided by the work of Dowker we call them *relations of categories*. The aim of this paper is to propose a version of Dowker's Theorem for relations of this form.

Given a relation $R\colon \mathcal{R} \to \mathcal{C} \times \mathcal{D}$, we introduce the *bisimplicial rectangle nerve* ER. It is a bisimplicial set whose set $ER_{m,n}$ of (m,n)-simplices consists of functors of the form $r\colon [m] \times [n] \to \mathcal{R}$ with the property that there exist functors $a\colon [m] \to \mathcal{C}$ and $b\colon [n] \to \mathcal{D}$ such that $R \circ r = a \times b$. Here $[m]$ is the totally ordered set $\{0 < 1 < \cdots < m\}$ considered as a category. If such functors a and b exist, they are uniquely determined. This implies that there is a map $\pi_R\colon ER_{m,n} \to N\mathcal{C}_m$, into the m-simplices $N\mathcal{C}_m$ of the nerve of $\mathcal{C}$, taking r as above to $\pi_R(r) = a$. In this context, the *Dowker nerve* DR is the simplicial subset of $N\mathcal{C}$ with m-simplices given by the image of the map $\pi_R\colon ER_{m,0} \to N\mathcal{C}_m$. The bisimplicial rectangle nerve is our categorification of the rectangle complex of [2]. In Section 4 we prove our main results. In order to state them we need two definitions from that section.

Definition 4.1. Given $a \in N\mathcal{C}_m$, that is, a functor $a\colon [m] \to \mathcal{C}$, the *fiber* π_R^a *of* a *under* π_R is the simplicial subset of the simplicial set $[n] \mapsto ER_{m,n}$ consisting of functors $r\colon [m] \times [n] \to \mathcal{R}$ such that there exist a functor $b\colon [n] \to \mathcal{D}$ with $R \circ r = a \times b$.

Definition 4.2. A *Dowker relation* is a relation $R\colon \mathcal{R} \to \mathcal{C}\times\mathcal{D}$ with the property that for every $a \in N\mathcal{C}_m$, the fiber π_R^a of a under π_R is contractible or empty.

The *transpose relation* R^T of a relation $R\colon \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ is the composite $\mathrm{tw}\circ R\colon \mathcal{R} \to \mathcal{D} \times \mathcal{C}$ of R and the twist isomorphism $\mathrm{tw}\colon \mathcal{C} \times \mathcal{D} \to \mathcal{D} \times \mathcal{C}$. There is an isomorphism $\mathrm{tw}^*\colon ER_{m,n} \to ER^T_{n,m}$ taking $r\colon [m]\times[n] \to \mathcal{R}$ to the composition $r\circ\mathrm{tw}$ of $\mathrm{tw}\colon [n]\times[m] \to [m]\times[n]$ and r. The diagonal simplicial set $d(ER)$ of ER is the simplicial set with n-simplices given by the set $d(ER)_n = ER_{n,n}$.

Theorem 4.5 (Dowker Equivalence)**.** *If R is a Dowker relation, then the projection maps*

$$\pi_R\colon ER_{m,n} \to DR_m$$

induce a weak equivalence $d(\pi_R)\colon d(ER) \to DR$ of simplicial sets.

In Section 2 we introduce morphisms of relations. The following is our version of Dowker's duality theorem:

Theorem 4.6 (Dowker Duality)**.** *Given a morphism $f : R \to R'$ of relations of categories, there is a commutative diagram of the form*

$$\begin{tikzcd} DR \arrow[d, "Df"'] & d(ER) \arrow[l, "d(\pi_R)"'] \arrow[r, "d(\mathrm{tw}^*)"] \arrow[d, "d(Ef)"] & d(ER^T) \arrow[r, "d(\pi_{R^T})"] \arrow[d, "d(Ef^T)"] & DR^T \arrow[d, "Df^T"] \\ DR' & d(ER') \arrow[l, "d(\pi_{R'})"] \arrow[r, "d(\mathrm{tw}^*)"] & d(ER'^T) \arrow[r, "d(\pi_{R'^T})"] & DR'^T. \end{tikzcd}$$

If the relations R, R^T, R' and ${R'}^T$ are Dowker relations, then all horizontal maps in this diagram are weak equivalences of simplicial sets.

We end the paper with two applications of Theorems 4.5 and 4.6. Given functors of the form $F\colon \mathcal{C} \to \mathcal{A}$ and $G\colon \mathcal{D} \to \mathcal{A}$, the projection $R\colon F\downarrow G \to \mathcal{C}\times\mathcal{D}$ taking an object (c, d, f) of the comma category $F \downarrow G$ to (c, d) is a relation. We show that if the nerve of the category $F \downarrow d$ is contractible for every object d of $\mathcal{D}$, then R is a Dowker relation. Using this we basically recover Quillen's original proof of his Theorem A [9, Theorem A].

As a second application we show that the geometric realization of a simplicial complex K is naturally homotopy equivalent to the geometric realization of the *singular complex* $\mathrm{Sing}(K)$, a simplicial set defined as follows: Let V be the vertex set of K. The set of m-simplices of $\mathrm{Sing}(K)$ is the set of functions $\{0, 1, \ldots, m\} \to V$ whose image is a simplex in K.

The assignment $K \mapsto \mathrm{Sing}(K)$ is a functor from the category of simplicial complexes to the category of simplicial sets. The geometric realization of $\mathrm{Sing}(K)$ is much bigger than the geometric realization of K. There are other smaller simplicial sets that capture the homotopy type of the geometric realization of K. One example is the nerve $N(K_\subseteq)$ of the category $K_\subseteq$ given by K considered as a partially ordered set under inclusion. The assignment $K \mapsto N(K_\subseteq)$ is also a functor, with the convenient property that the geometric realizations of K and $N(K_{\subseteq})$ are naturally homeomorphic. This functor has neither a left- nor a right adjoint functor. In contrast, the singular complex $K \mapsto \mathrm{Sing}(K)$ has a left adjoint functor.

The fact that the geometric realizations of K and $\mathrm{Sing}(K)$ are homotopy equivalent is a well-known fact in topology, but to the best of our knowledge it has not yet been published in a peer-reviewed paper. Two proofs of this fact have been published on the personal web page of Omar Antolín Camarena [3], but the naturality of the homotopy equivalence is lacking as both proofs use a chosen order on the vertex set of K.

The paper is organized as follows: In Section 2 we give preliminary definitions concerning (bi)simplicial sets and we introduce relations of categories. In Section 3 we define the bisimplicial rectangle nerve, which in Section 4 we use to prove our Dowker Equivalence and Dowker Duality theorems. In Section 5 we look at homotopies of Dowker nerves, and finally, in Section 6 we present the two applications of our main results presented in the two preceding paragraphs.

2 Bisimplicial Sets and Relations of Categories

In this section we recall the definition of simplicial and bisimplicial sets (for details we refer to [7]), and introduce the concept of a relation of categories.

Let $[n]$ be the category with object set $\{0, 1, ..., n\}$ and a unique morphism $i \to j$ if $i \leq j$. Note that a functor $[m] \to [n]$ is the same as an order-preserving map from the set $\{0, 1, ..., m\}$ to the set $\{0, 1, ..., n\}$. The object set of the *simplex category* Δ consists of the categories $[n]$, for $n \geq 0$. Morphisms in Δ are functors between these categories.

Consider the interval $[0, 1]$ as a subspace of $\mathbb{R}$. Given an integer $n \geq 0$, the *geometric n-simplex* is the subspace Δ^n of $[0, 1]^{n+1}$ consisting of tuples $t = (t_0, \dots, t_n)$ with sum equal to 1. Denoting the standard basis for $\mathbb{R}^{n+1}$ by $e_0, \dots, e_n$, we may write $t = t_0 e_0 + \cdots + t_n e_n$. Let $\mathtt{Top}$ denote the category of topological spaces. There is a functor $\Delta \to \mathtt{Top}$, $[n] \mapsto \Delta^n$ taking an order-preserving map $f \colon [m] \to [n]$ to the affine map $f_* \colon \Delta^m \to \Delta^n$ with $f_*(e_i) = e_{f(i)}$ for $i = 0, 1, \dots, m$.

A *simplicial set* is a functor $Y : \Delta^{\mathtt{op}} \to \mathtt{Set}$, from the opposite category of the simplex category to the category of sets, sending $[n]$ to the set Y_n of n-simplices. Morphisms in the category of simplicial sets, called *simplicial maps*, are natural transformation of functors $\Delta^{\mathtt{op}} \to \mathtt{Set}$. We write $\mathcal{S}$ for the category of simplicial sets and simplicial maps.

The *geometric realization* $|Y|$ of a simplicial set Y is the topological space given by the coequalizer diagram

$$\coprod_{[m],[n]} Y_n \times \Delta([m], [n]) \times \Delta^m \rightrightarrows \coprod_{[n]} Y_n \times \Delta^n \to |Y|,$$

where the two parallel horizontal maps take $(y, f, t) \in Y_n \times \Delta([m], [n]) \times \Delta^m$ to $(Y(f)y, t)$ and $(y, f_*(t))$ respectively. A simplicial map $\phi : Y \to Y'$ is called a *weak equivalence* if the induced map $|\phi| : |Y| \to |Y'|$ on geometric realization (see [7, I.2]) is a homotopy equivalence.

A *bisimplicial set* is a functor $X : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathtt{Set}$, sending the tuple $([m], [n])$ to the set $X_{m,n}$ of (m, n)-simplices. As with simplicial sets, *bisimplicial maps* $\phi : X \to X'$ are natural transformations. We write $\mathcal{S}^2$ for the category of bisimplicial sets and bisimplicial maps.

A *relation (of categories)* from a small category $\mathcal{C}$ to a small category $\mathcal{D}$ is a functor of the form $R\colon \mathcal{R} \to \mathcal{C} \times \mathcal{D}$. A *morphism of relations of categories* $f\colon R \to R'$ from $R\colon \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ to $R'\colon \mathcal{R}' \to \mathcal{C}' \times \mathcal{D}'$ consists of functors $f_0\colon \mathcal{R} \to \mathcal{R}'$, $f_1\colon \mathcal{C} \to \mathcal{C}'$ and $f_2\colon \mathcal{D} \to \mathcal{D}'$ so that $(f_1 \times f_2) \circ R = R' \circ f_0$. We write $\mathtt{Rel}$ for the category of relations of categories.

Given two categories $\mathcal{C}$ and $\mathcal{D}$ the *twist isomorphism*

$$\mathrm{tw}\colon \mathcal{C} \times \mathcal{D} \to \mathcal{D} \times \mathcal{C}$$

is the functor sending objects (c, d) to (d, c), and morphisms (γ, δ) to (δ, γ). For a relation of categories $R\colon \mathcal{R} \to \mathcal{C} \times \mathcal{D}$, its *transposed relation* $R^T\colon \mathcal{R} \to \mathcal{D} \times \mathcal{C}$ is the composition $R^T = R \circ \mathrm{tw}$ of R and the twist isomorphism tw. The *transposition functor* $T : \mathtt{Rel} \to \mathtt{Rel}$ is the functor $R \mapsto R^T$.

3 The Bisimplicial Rectangle Nerve

In this section we introduce the bisimplicial rectangle nerve of a relation. This is a bisimplicial set that is in a sense symmetric under transposition.

Definition 3.1. Let $R\colon \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ be a relation. The *bisimplicial rectangle nerve* ER is the bisimplicial set whose (m, n)-simplices are functors $r\colon [m] \times [n] \to \mathcal{R}$ such that there exist a necessarily unique pair of functors $(a : [m] \to \mathcal{C}, b : [n] \to \mathcal{D})$ with $a \times b = R \circ r$, that is, so that the following diagram commutes

$$\begin{tikzcd}
{[m] \times [n]} \arrow[d, "r"'] \arrow[dr, "a\times b"] & \\
\mathcal{R} \arrow[r, "R"] & \mathcal{C} \times \mathcal{D}.
\end{tikzcd}$$

If $\alpha : [m'] \to [m]$ and $\beta : [n'] \to [n]$ are order-preserving maps, then $ER(\alpha, \beta) : ER_{m,n} \to ER_{m',n'}$ sends the (m, n)-simplex r to the (m', n')-simplex $r \circ (\alpha \times \beta)$.

A simplex $r\colon [m] \times [n] \to \mathcal{R}$ in the bisimplicial rectangle nerve can then be considered as a lift of a map of rectangles $a \times b\colon [m] \times [n] \to \mathcal{C} \times \mathcal{D}$ to $\mathcal{R}$. This is the motivation for the name "bisimplicial rectangle nerve".

Let $f\colon R \to R'$ be a morphism of relations given by relations $R\colon \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ and $R'\colon \mathcal{R}' \to \mathcal{C}' \times \mathcal{D}'$, and functors $f_0\colon \mathcal{R} \to \mathcal{R}'$, $f_1\colon \mathcal{C} \to \mathcal{C}'$ and $f_2\colon \mathcal{D} \to \mathcal{D}'$. There is a bisimplicial map $Ef\colon ER \to ER'$ taking an (m, n)-simplex r of ER to the (m, n)-simplex $f_0 \circ r$ of ER'. It is straightforward to check that the assignment $f \mapsto Ef$ gives us a functor $E : \mathtt{Rel} \to \mathcal{S}^2$.

Pre-composition with the twist isomoprhism $\mathrm{tw} : \Delta^{\mathtt{op}} \times \Delta^{\mathtt{op}} \to \Delta^{\mathtt{op}} \times \Delta^{\mathtt{op}}$ gives a functor $\tau : \mathcal{S}^2 \to \mathcal{S}^2$. Specifically, for $X \in \mathcal{S}^2$ the bisimplicial set τX is the composite functor

$$\Delta^{\mathtt{op}} \times \Delta^{\mathtt{op}} \xrightarrow{\mathrm{tw}} \Delta^{\mathtt{op}} \times \Delta^{\mathtt{op}} \xrightarrow{X} \mathtt{Set},$$

and $\tau X_{m,n} = X_{n,m}$.

If R is a relation, then $r : [m] \times [n] \to \mathcal{R}$ is a simplex in $ER_{m,n} = \tau ER_{n,m}$ if and only if the composition

$$[n] \times [m] \xrightarrow{\mathrm{tw}} [m] \times [n] \xrightarrow{r} \mathcal{R}$$

is in $ER^T_{n,m}$. These maps give us a bijective bisimplicial map $\mathrm{tw}^* \colon \tau ER \to ER^T$ taking $r \in ER_{m,n}$ to $r \circ \mathrm{tw}$. It is natural in the sense that given a morphism $f : R \to R'$, we have a commutative diagram

$$\begin{tikzcd}
\tau ER \arrow[r, "\mathrm{tw}^*"] \arrow[d, "\tau E f"] & ER^T \arrow[d, "Ef^T"] \\
\tau ER' \arrow[r, "\mathrm{tw}^*"] & ER'^T.
\end{tikzcd}$$

We sum up this discussion in the following lemma:

Lemma 3.2. *The map* $\mathrm{tw}^* \colon \tau E \to ET$ *is a natural isomorphism.* □

Next, we consider the *diagonal functor*

$$\mathrm{diag} : \Delta^{\mathtt{op}} \to \Delta^{\mathtt{op}} \times \Delta^{\mathtt{op}}$$

where $\mathrm{diag}([n]) = ([n], [n])$ on objects and $\mathrm{diag}(\alpha) = (\alpha, \alpha)$ on morphisms [7, p.197]. Pre-composing with diag gives a functor $d : \mathcal{S}^2 \to \mathcal{S}$ sending a bisimplicial set X to its *diagonal simplicial set* $d(X) := X \circ \mathrm{diag}$ whose n-simplices $d(X)_n$ are $X_{n,n}$.

Note that the diagram

$$\begin{tikzcd}
\Delta^{\mathtt{op}} \arrow[r, "\mathrm{diag}"] \arrow[dr, "\mathrm{diag}"'] & \Delta^{\mathtt{op}} \times \Delta^{\mathtt{op}} \arrow[d, "\mathrm{tw}"] \\
& \Delta^{\mathtt{op}} \times \Delta^{\mathtt{op}}
\end{tikzcd}$$

commutes. Since the diagonal functor d and the functor τ are defined by precomposition of diag and tw respectively, we have:

Lemma 3.3. *If R is a relation of categories, then $d(ER) = d(\tau ER)$.* □

Combining Lemma 3.3 with Lemma 3.2 we get:

Lemma 3.4. *The map* $\mathrm{tw}^* \colon \tau E \to ET$ *induces a natural isomorphism* $d(\mathrm{tw}^*) \colon dE \to dET$.

4 The Functorial Dowker Duality Theorem

In this section we finally introduce the Dowker nerve of a relation of categories, and we state our version of the functorial Dowker duality theorem. We have defined the bisimplicial rectangle nerve by

$$ER_{m,n} = \{r\colon [m] \times [n] \to \mathcal{R} \mid R \circ r \text{ is of the form } a \times b\colon [m] \times [n] \to \mathcal{C} \times \mathcal{D}\}\,.$$

The *nerve* of a small category $\mathcal{C}$ is the simplicial set $N\mathcal{C}$ whose m-simplices are functors from $[m]$ to $\mathcal{C}$, that is, $N\mathcal{C}_m = \mathtt{Cat}([m], \mathcal{C})$.

Note that given $r \in ER_{m,n}$ with $R \circ r = a \times b\colon [m] \times [n] \to \mathcal{C} \times \mathcal{D}$, the functors $a\colon [m] \to \mathcal{C}$ and $b : [n] \to \mathcal{D}$ are uniquely determined by the universal property of products. In particular, there is a function $\pi_R\colon ER_{m,n} \to N\mathcal{C}_m$ given by $\pi_R(r) = a$ for $a \in N\mathcal{C}_m$ with $R \circ r = a \times b\colon [m] \times [n] \to \mathcal{C} \times \mathcal{D}$.

Definition 4.1. Given $a \in N\mathcal{C}_m$, that is, a functor $a\colon [m] \to \mathcal{C}$, the *fiber* π_R^a *of* a *under* π_R is the simplicial subset of the simplicial set $[n] \mapsto ER_{m,n}$ consisting of functors $r\colon [m] \times [n] \to \mathcal{R}$ such that there exist a functor $b\colon [n] \to \mathcal{D}$ with $R \circ r = a \times b$.

In order to state our version of Dowker duality we introduce the concept of a Dowker relation.

Definition 4.2. A *Dowker relation* is a relation $R\colon \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ with the property that for every $a \in N\mathcal{C}_m$, the fiber π_R^a of a under π_R is contractible or empty.

In Section 6 we look at concrete Dowker relations, one class of which is described in Corollary 6.4.

Definition 4.3. The *Dowker nerve* of the relation $R\colon \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ is the simplicial set DR whose set of m-simplices DR_m is the image of the map $\pi_R\colon ER_{m,0} \to N\mathcal{C}_m$.

Let $f\colon R \to R'$ be a morphism of relations given by relations $R\colon \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ and $R'\colon \mathcal{R}' \to \mathcal{C}' \times \mathcal{D}'$, and functors $f_0\colon \mathcal{R} \to \mathcal{R}'$, $f_1\colon \mathcal{C} \to \mathcal{C}'$ and $f_2\colon \mathcal{D} \to \mathcal{D}'$. The assignment $a \mapsto Df(a) = f_1 \circ a$ defines a simplicial map $Df\colon DR \to DR'$, so we have a functor $D : \mathtt{Rel} \to \mathcal{S}$.

Remark 4.4. The m-simplices DR_m contain the image of $\pi_R\colon ER_{m,n} \to N\mathcal{C}_m$ for all $n \geq 0$. We also write $\pi_R\colon ER_{m,n} \to DR_m$ for the map π_R with DR_m as codomain instead of $N\mathcal{C}_m$. Fixing m we obtain the simplicial sets X and Y where $X_n := ER_{m,n}$ and $Y_n := DR_m$ as a constant simplicial set. Since the connected components[1] of a constant simplicial set are given by degenercies of zero simplices, the relation R is a Dowker relation if and only if the simplicial map $\pi : X \to Y$, which on n-simplices is $\pi_R\colon ER_{m,n} \to DR_m$, is a weak equivalence.

[1] The connected components of the simplicial set X are the graph-components of the multigraph $X_1 \rightrightarrows X_0$.

Theorem 4.5 (Dowker Equivalence)**.** *If R is a Dowker relation, then the projection maps*

$$\pi_R\colon ER_{m,n} \to DR_m$$

induce a weak equivalence $d(\pi_R)\colon d(ER) \to DR$ of simplicial sets.

Proof. Fixing $m \geq 0$, let A be the simplicial set with $A_n = ER_{m,n}$, and consider DR_m as a constant simplicial set. By our assumption on R the simplicial map $A \to DR_m$, which on n-simplices is $\pi_R\colon ER_{m,n} \to DR_m$, is a weak equivalence. Let B be the bisimplicial set $B_{m,n} := DR_m$ constant in one direction and consider $\pi_R\colon ER_{m,n} \to DR_m$ as a bisimplicial map $\pi_R\colon ER \to B$. By [7, Prop. IV.1.7], attributed to Tornehave in [9], the projection map π_R induces a weak equivalence of diagonals. That is, the map $d(\pi_R) : d(ER) \to d(B) = DR$ is a weak equivalence. □

We now use Theorem 4.5 to prove a functorial Dowker duality theorem for relations of categories. Consider the *left projection functor*

$$\mathrm{P}\colon \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}} \to \Delta^{\mathsf{op}}$$

sending objects $([m],[n])$ to $[m]$ and morphisms (α,β) to α. We consider the functor $\mathrm{p}\colon \mathcal{S} \to \mathcal{S}^2$, sending a simplicial set $Y : \Delta^{\mathsf{op}} \to \mathtt{Set}$ to the composite functor

$$\Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}} \xrightarrow{\mathrm{P}} \Delta^{\mathsf{op}} \xrightarrow{Y} \mathtt{Set}.$$

Note that $\mathrm{P} \circ \mathrm{diag}$ is the identity, so we have $d(\mathrm{p}(Y)) = Y$.

There is a natural transformation $\pi : E \to \mathrm{p}D$, so that $\pi_R : ER \to \mathrm{p}DR$ takes $r\colon [m] \times [n] \to \mathcal{R}$ with $R \circ r = a \times b$ to a. This means that for each morphism of relations $f : R \to R'$ we have a commutative square

$$\begin{array}{ccc} ER & \xrightarrow{\pi_R} & \mathrm{p}DR \\ {\scriptstyle Ef}\downarrow & & \downarrow{\scriptstyle \mathrm{p}Df} \\ ER' & \xrightarrow{\pi_{R'}} & \mathrm{p}DR'. \end{array} \tag{1}$$

For the transposed morphism $f^T : R^T \to R'^T$, we get a commutative square

$$\begin{array}{ccc} ER^T & \xrightarrow{\pi_{R^T}} & \mathrm{p}DR^T \\ {\scriptstyle Ef^T}\downarrow & & \downarrow{\scriptstyle \mathrm{p}Df^T} \\ ER'^T & \xrightarrow{\pi_{R'^T}} & \mathrm{p}DR'^T. \end{array} \tag{2}$$

In this way, we may regard π as a natural transformation $\pi\colon ET \to \mathrm{p}DT$ as well.

Theorem 4.6 (Dowker Duality)**.** *Given a morphism $f : R \to R'$ of relations of categories, there is a commutative diagram of the form*

$$\begin{array}{ccccccc} DR & \xleftarrow{d(\pi_R)} & d(ER) & \xrightarrow{d(\mathrm{tw}^*)} & d(ER^T) & \xrightarrow{d(\pi_{R^T})} & DR^T \\ {\scriptstyle Df}\downarrow & & \downarrow{\scriptstyle d(Ef)} & & \downarrow{\scriptstyle d(Ef^T)} & & \downarrow{\scriptstyle Df^T} \\ DR' & \xleftarrow{d(\pi_{R'})} & d(ER') & \xrightarrow{d(\mathrm{tw}^*)} & d(ER'^T) & \xrightarrow{d(\pi_{R'^T})} & DR'^T. \end{array}$$

If the relations R, R^T, R' and ${R'}^T$ are Dowker relations, then all horizontal maps in the diagram are weak equivalences of simplicial sets.

Proof. Applying the diagonal d to the commutative squares (1) and (2), together with Lemma 3.4, we see the diagram commutes. By Lemma 3.4, the maps labeled $d(\mathrm{tw}^*)$ are isomorphisms. The statement about weak equivalences is a now consequence of Theorem 4.5. $\square$

5 Dowker Nerves and Homotopies

We look at mophisms of relations of categories that induce homotopies when taking the Dowker nerve.

To talk about homotopies, we need to define the product of relations. Let $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ and $R' : \mathcal{R}' \to \mathcal{C}' \times \mathcal{D}'$ be relations of categories. We have projections $\pi_{\mathcal{C}} : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $\pi_{\mathcal{D}} : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$. The product $R \times R'$ in $\mathtt{Rel}$ is the relation $\mathcal{R} \times \mathcal{R}' \to (\mathcal{C} \times \mathcal{C}') \times (\mathcal{D} \times \mathcal{D}')$ sending (x, x') to $((\pi_{\mathcal{C}} R(x), \pi_{\mathcal{C}'} R'(x')), (\pi_{\mathcal{D}} R(x), \pi_{\mathcal{D}'} R'(x')))$. Projections to first and second factors give the two structure maps for the product.

Lemma 5.1. *The functor $D : \mathtt{Rel} \to \mathcal{S}$ preserves products, so given two relations $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ and $R' : \mathcal{R}' \to \mathcal{C}' \times \mathcal{D}'$ of categories, the projections onto R and R' induce an isomorphism $D(R \times R') \xrightarrow{\cong} D(R) \times D(R')$.*

Proof. A simplex $r \in DR_m$ is a map $r : [m] \times [0] \to \mathcal{R}$ with the property that there exist $a : [m] \to \mathcal{C}$ and $b : [0] \to \mathcal{D}$ so that $R \circ r = a \times b$. It is uniquely defined by a map $\tilde{r} : [m] \to \mathcal{R}$ such that the composition

$$[m] \xrightarrow{\tilde{r}} \mathcal{R} \xrightarrow{R} \mathcal{C} \times \mathcal{D} \xrightarrow{\pi_{\mathcal{D}}} \mathcal{D}$$

is constant. Explicitly, given r, a and b, we define $\tilde{r}$ by $\tilde{r}(i) = r(i, 0)$. Conversely, given $\tilde{r}$, the maps r, a and b are given by letting $r(i, 0) = \tilde{r}(i)$, $a = \pi_{\mathcal{C}} \circ R \circ \tilde{r}$ and $b(0) = \pi_{\mathcal{D}} \circ R \circ \tilde{r}(i)$.

A simplex in $D(R \times R')_m$ is uniquely defined by a map $(\tilde{r}, \tilde{r}') : [m] \to \mathcal{R} \times \mathcal{R}'$ where $\pi_{\mathcal{D} \times \mathcal{D}'} \circ (R \times R') \circ (\tilde{r}, \tilde{r}')$ is constant, which is equivalent to $\pi_{\mathcal{D}} \circ R \circ \tilde{r}$ and $\pi_{\mathcal{D}'} \circ R' \circ \tilde{r}'$ both being constant. Thus, under the isomorphism $\mathtt{Cat}([m], \mathcal{R} \times \mathcal{R}') \xrightarrow{\cong} \mathtt{Cat}([m], \mathcal{R}) \times \mathtt{Cat}([m], \mathcal{R}')$, m-simplices of $D(R \times R')$ are taken bijectively to m-simplices of $D(R) \times D(R')$. $\square$

The following is a consequence of the fact that the Dowker nerve of a relation of the form $\mathbb{1}_{\mathcal{C} \times \mathcal{D}} \colon \mathcal{C} \times \mathcal{D} \to \mathcal{C} \times \mathcal{D}$ is equal to the nerve of the category $\mathcal{C}$.

Lemma 5.2. *Given $n \geq 0$, the Dowker nerve of the relation $\mathbb{1}_{[n] \times [0]} \colon [n] \times [0] \to [n] \times [0]$ is the simplicial n-simplex $\Delta[n]$.* $\square$

For $i = 0, 1$ we have the morphism of relations $d^i : \mathbb{1}_{[0] \times [0]} \to \mathbb{1}_{[1] \times [0]}$ where the map $d^i_0 : [0] \times [0] \to [1] \times [0]$ does not hit $(i, 0)$, the map $d^i_1 : [0] \to [1]$ does not hit i and $d^i_2 = \mathbb{1}_{[0]}$.

Definition 5.3. Given two morphisms of relations $f^0, f^1 : R \to R'$, a *transformation* H from f^1 to f^0 is a morphism $H : \mathbb{1}_{[1]\times[0]} \times R \to R'$ of relations such that the diagram

$$\begin{array}{ccc} \mathbb{1}_{[0]\times[0]} \times R & \xrightarrow{d^i \times \mathbb{1}_R} & \mathbb{1}_{[1]\times[0]} \times R \\ \cong\uparrow & & \downarrow H \\ R & \xrightarrow{\quad f^i \quad} & R' \end{array} \tag{3}$$

commutes for $i = 0, 1$.

Proposition 5.4. *Given two morphisms of relations $f^0, f^1 : R \to R'$ and a transformation H from f^1 to f^0, the maps $Df^0, Df^1 : DR \to DR'$ are homotopic.*

Proof. Taking the Dowker nerve of diagram (3) using Lemma 5.1 and Lemma 5.2, for $i = 0, 1$, we get a commutative diagram of the form

$$\begin{array}{ccc} \Delta[0] \times DR & \xrightarrow{d_i \times \mathbb{1}_{DR}} & \Delta[1] \times DR \\ \cong\uparrow & & \downarrow \widehat{DH} \\ DR & \xrightarrow{\quad Df^i \quad} & DR'. \end{array}$$

The map $\widehat{DH}$ is the desired homotopy. □

6 Applications

In order to justify our categorification of the Dowker Theorem, we show how it is related to Quillen's Theorem A and different versions of the singular simplicial set of a simplicial complex.

6.1 Quillen's Theorem A

We first apply Theorem 4.5 to prove Quillen's Theorem A. Given functors $F\colon \mathcal{C} \to \mathcal{A}$ and $G\colon \mathcal{D} \to \mathcal{A}$, the *comma category* $F \downarrow G$ has objects given by triples (c, d, f), where $c \in \mathcal{C}$, $d \in \mathcal{D}$ and f is a morphism $f\colon Fc \to Gd$ in $\mathcal{A}$. A morphism in $F \downarrow G$, of the form $(c, d, f) \to (c', d', f')$, consists of morphisms $\alpha_L\colon c \to c'$ and $\alpha_R\colon d \to d'$ such that the following diagram commutes:

$$\begin{array}{ccc} Fc & \xrightarrow{f} & Gd \\ \downarrow F\alpha_L & & \downarrow G\alpha_R \\ Fc' & \xrightarrow{f'} & Gd'. \end{array}$$

The projection $R\colon F \downarrow G \to \mathcal{C} \times \mathcal{D}$ sending (c, d, f) to (c, d) is a relation. Let $r\colon [m] \times [n] \to F \downarrow G$ be a (m, n)-simplex of ER with $a\colon [m] \to \mathcal{C}$ and

$b\colon [n] \to \mathcal{D}$ satisfying the equation $R \circ r = a \times b$. Given $(i,j) \in [m] \times [n]$, we write $r(i,j) = (a(i), b(j), f_{ij}\colon Fa(i) \to Gb(j))$. Note that $f_{ij} = f_{mj} \circ Fa(i \to m)$, and that $f_{mj} = Gb(0 \to j) \circ f_{m0}$, so the morphism $f_{ij}\colon Fai \to Gbj$ is equal to the composition

$$Fa(i) \xrightarrow{Fa(i \to m)} Fa(m) \xrightarrow{f_{m0}} Gb(0) \xrightarrow{Gb(0 \to j)} Gb(j).$$

This means that r is uniquely determined by a, b and $f_{m0}\colon Fa(m) \to Gb(0)$. With this in mind we see that the fiber of $a \in DR_m$ under $\pi_R : ER_{m,n} \to DR_m$ is isomorphic to the nerve $N(Fa(m) \downarrow G)$ of the comma category $Fa(m) \downarrow G$ for the functors $Fa(m)\colon * \to \mathcal{D}$ and G. Similarly, the fiber of $b \in DR_n^T$ under $ER_{m,n} \to DR_n^T$ is isomorphic to nerve $N(F \downarrow Gb(0))$ of the comma category $F \downarrow Gb(0)$ for the functors F and $Gb(0)\colon * \to \mathcal{C}$.

Specializing to $\mathcal{A} = \mathcal{D}$ and $G = \mathbb{1}_{\mathcal{D}}$ we can prove the following:

Corollary 6.1 (Quillen's Theorem A [9])**.** *Consider a functor $F : \mathcal{C} \to \mathcal{D}$. If $N(F \downarrow d)$ is contractible for every object $d \in \mathcal{D}$, then the map $NF : N\mathcal{C} \to N\mathcal{D}$ is a weak equivalence.*

Proof. Consider the comma category $F \downarrow \mathbb{1}_{\mathcal{D}}$. Note that the category $Fa(m) \downarrow \mathbb{1}_{\mathcal{D}}$ has initial object $(*, Fa(m), \mathbb{1}_{Fa(m)})$, so the nerve $N(Fa(m) \downarrow \mathbb{1}_{\mathcal{D}})$ is contractible, implying that the projection $R : F \downarrow \mathbb{1}_{\mathcal{D}} \to \mathcal{C} \times \mathcal{D}$ is a Dowker relation. Furthermore, by the preceding discussion, if all $N(F \downarrow d)$ are contractible, then also the transpose R^T is a Dowker relation. Since $r \in ER_{m,0}$ is uniquely determined by $a\colon [m] \to \mathcal{C}$, $d \in \mathrm{ob}(\mathcal{D})$ and $f\colon Fa(m) \to d$, the set of m-simplices of the nerve DR consists of functors $a : [m] \to \mathcal{C}$ such that there exists an object d in $\mathcal{D}$ and a morphism $f : Fa(m) \to d$. We can always choose $d = Fa(m)$ and $f = \mathbb{1}_{Fa(m)}$, so $DR = N\mathcal{C}$. The simplices in DR_n^T similarly are functors $b : [n] \to \mathcal{D}$ such that there is an object c in $\mathcal{C}$ and a functor $f : Fc \to b(0)$, but such a triple $(c, *, f)$ is an object in $F \downarrow b(0)$ which is non-empty by the assumption that $N(F \downarrow b(0))$ is contractible. So we get that $DR^T = N\mathcal{D}$ and $N\mathcal{C} \simeq N\mathcal{D}$. We still need to show that NF is a weak equivalence.

Consider the projections $\pi_{\mathcal{C}} : F \downarrow \mathbb{1}_{\mathcal{D}} \to \mathcal{C}$ and $\pi_{\mathcal{D}} : F \downarrow \mathbb{1}_{\mathcal{D}} \to \mathcal{D}$ sending (c, d, f) to c and d respectively. We have the commuting diagram

$$\begin{tikzcd}
 & d(ER) \arrow[r, "d(\mathrm{tw}^*)"] \arrow[dl, "d(\pi_R)"'] \arrow[d, "\mathrm{diag}^*"] & d(ER^T) \arrow[d, "d(\pi_{R^T})"] \\
N\mathcal{C} & N(F \downarrow \mathbb{1}_{\mathcal{D}}) \arrow[l, "N\pi_{\mathcal{C}}"] \arrow[r, "N\pi_{\mathcal{D}}"'] & N\mathcal{D},
\end{tikzcd} \tag{4}$$

where $\mathrm{diag}^*\colon d(ER)_m = ER_{m,m} \to N(F \downarrow \mathbb{1}_{\mathcal{D}})_m$ is precomposition with the diagonal functor $\mathrm{diag} : [m] \to [m] \times [m]$. Furthermore, there is a natural map $\eta : F \circ \pi_{\mathcal{C}} \to \pi_{\mathcal{D}}$ with components $\eta_{(c,d,f)} = f : Fc \to d$ for objects (c, d, f) in $F \downarrow \mathbb{1}_{\mathcal{D}}$. This induces a homotopy on nerves $NF \circ N\pi_{\mathcal{C}} \simeq N\pi_{\mathcal{D}}$. Using diagram (4) we get $NF \circ d(\pi_R) \simeq d(\pi_{R^T}) \circ d(\mathrm{tw}^*)$. The map $d(\mathrm{tw}^*)$ is an isomorphism. Since R is a Dowker relation, the maps $d(\pi_R)$ and $d(\pi_{R^T})$ are weak equivalences, and therefore so is NF. $\square$

The above argument is very close to the proof in [9]. Arguably, the proof in [9] is more elegant than the proof presented here. The point we are making is that Quillen's Theorem A and the Dowker duality of Theorem 4.6 are closely connected.

6.2 Simplicial Sets from Simplicial Complexes

In this subsection we look at two ways of turning simplicial complexes into simplicial sets resembling the singular complex of a topological space. We use the Dowker duality of Theorem 4.6 to prove that one of these singular complex constructions is functorial and that it is of the correct homotopy type. We begin by investigating relations given by inclusions of full subcategories.

Definition 6.2. Let $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ be a relation with R an inclusion of a full subcategory. Given $a \in N\mathcal{C}_m$, we let $\mathcal{D}_R^a \subseteq \mathcal{D}$ be the full subcategory consisting of all objects $d \in \mathcal{D}$ such that $(a(i), d) \in \mathcal{R}$ for $i = 0, 1, \dots, m$.

Lemma 6.3. *Let $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ be a relation with R an inclusion of a full subcategory. Given $a \in N\mathcal{C}_m$, the projection $\pi_R^a \to N\mathcal{D}$ taking $r \colon [m] \times [n] \to \mathcal{R}$ to the uniquely determined $b \colon [n] \to \mathcal{D}$ such that $R \circ r = a \times b$ induces a bijection $\pi_R^a \to N\mathcal{D}_R^a$ of simplicial sets.*

Proof. By construction, the given a and r as in the statement, the uniquely determined $b \colon [n] \to \mathcal{D}$ takes values in $\mathcal{D}_R^a$. Thus we have an induced function $\pi_R^a \to N\mathcal{D}_R^a$ of simplicial sets. Since R is an inclusion, the assignment $r \mapsto b$ is injective. For surjectivity, note that by construction, given $b \in N\mathcal{D}_R^a$, the functor $a \times b \colon [m] \times [n] \to \mathcal{C} \times \mathcal{D}$ factors through $\mathcal{R}$. ☐

The nerve of a category with either initial or terminal object is contractible [9, p.8], so we have the following corollary.

Corollary 6.4. *Let $R : \mathcal{R} \to \mathcal{C} \times \mathcal{D}$ be the inclusion of a full subcategory. If $\mathcal{D}$ has the property that all full subcategories have an initial or terminal object, then R is a Dowker relation.* ☐

We see two examples of such categories below, namely categories that are totally ordered sets and the translation category of a set.

A *simplicial complex* (K, V) is a set V and a set K of finite subsets of V such that $\sigma \in K$ and $\tau \subseteq \sigma$ implies $\tau \in K$. We follow standard terminology and say that K is a simplicial complex, leaving the vertex set V implicit. Note that inclusion $\subseteq$ is a partial order on K making it a partially ordered set $K_{\subseteq}$.

Consider the topological space $[0,1]^S$ whose elements are functions from a set S to the interval $[0,1] \subseteq \mathbb{R}$. If S is finite, then $[0,1]^S = \prod_S [0,1]$ is given the product topology. If S is infinite, then $[0,1]^S$ is given the topology where $U \subseteq [0,1]^S$ is open if and only if for every finite subset W of S the set $U \cap [0,1]^W$ is open in $[0,1]^W$. The *support* of a function $S \to [0,1]$ is the subset of S consisting of the elements that give non-zero values of the given function.

The *geometric realization* $|K|$ of a simplicial complex (K,V) is the subspace of $[0,1]^V$ consisting of functions $\alpha\colon V \to [0,1]$ satisfying firstly that its support is a simplex in K and secondly that the sum of its values is equal to 1, that is $\sum_{v\in V} \alpha(v) = 1$.

We consider two ways of constructing a simplicial set from a simplicial complex (K,V). The *singular complex* on K is the simplicial set $\mathrm{Sing}(K)$ whose set of m-simplices are

$$\mathrm{Sing}(K)_m = \{a : \{0,1,\dots,m\} \to V \mid \{a(0),a(1),\dots,a(m)\} \in K\}.$$

The simplicial structure on $\mathrm{Sing}(K)$ is induced from the cosimplicial set $[m] \mapsto \{0,\dots,m\}$ given by forgetting the order on $[m]$.

Suppose that the simplicial complex (K,V) has a total order $\le$ on V. The *ordered singular complex* on K is the simplicial set $\mathrm{Sing}_{\le}(K)$ with set of m-simplices given by order-preserving maps, that is,

$$\mathrm{Sing}_{\le}(K)_m = \{a : [m] \to V_{\le} \mid a([m]) \in K\}.$$

This simplicial set is a simplicial subset of the nerve of the category $V_{\le}$.

Remark 6.5. The functor $K \mapsto \mathrm{Sing}(K)$ from simplicial complexes to simplicial sets is right adjoint to a functor $X \mapsto MX$. Here MX is the simplicial complex with vertex set X_0 and with simplices given by vertex sets of simplices of the simplicial set X. Note that by the vertex set of $x \in X_n$ we mean the set of zero-dimensional faces of x. Moreover, the functor $K \mapsto \mathrm{Sing}_{\le}(K)$ from ordered simplicial complexes to the category of simplicial sets with a total order on the set of 0-simplices also has a right adjoint functor.

We now explain how $\mathrm{Sing}_{\le}(K)$ and $\mathrm{Sing}(K)$ can be considered as Dowker nerves of relations.

1. Assume that V has a total order $\le$ making it a totally ordered set $V_{\le}$. Consider the full subcategory $\mathcal{R}_1 \subseteq V_{\le} \times K_{\subseteq}$ where $(v,\sigma) \in \mathrm{ob}(\mathcal{R}_1)$ if and only if $v \in \sigma$. The inclusion $R_1 : \mathcal{R}_1 \to V_{\le} \times K_{\subseteq}$ is a relation, and $DR_1 = \mathrm{Sing}_{\le}(K)$.
2. The *translation category* $\mathcal{V}$ of V has object set $\mathrm{ob}(\mathcal{V}) = V$ and a unique morphism $v \to w$ between any pair of objects $v,w \in V$. Consider the full subcategory $\mathcal{R}_2 \subseteq \mathcal{V} \times K_{\subseteq}$ where $(v,\sigma) \in \mathrm{ob}(\mathcal{R}_2)$ if and only if $v \in \sigma$. The inclusion $R_2 : \mathcal{R}_2 \to \mathcal{V} \times K_{\subseteq}$ is a relation, and $DR_2 = \mathrm{Sing}(K)$.

Note that for any choice of order $\le$ on the vertex set V of a simplicial complex (K,V) we have an injective map $\mathrm{Sing}_{\le}(K) \hookrightarrow \mathrm{Sing}(K)$ induced by the inclusion $V_{\le} \hookrightarrow \mathcal{V}$.

We define a map $\varphi\colon |\mathrm{Sing}(K)| \to |K|$. Every element in $|\mathrm{Sing}(K)|$ is represented by a pair $(a,t) \in \mathrm{Sing}(K)_m \times \Delta^m$. Given such a pair (a,t) with $t = (t_0,\dots,t_m)$, let $a_*(t)\colon V \to [0,1]$ be the element of $|K|$ with $a_*(t)(v) = \sum_{a(i)=v} t_i$. It is straight-forward to verify that $(a,t) \mapsto \varphi(a,t) = a_*(t)$ defines a natural continuous map $\varphi\colon |\mathrm{Sing}(K)| \to |K|$. Given a total order $\leq$ on the vertex set V of K, we denote by $\varphi_{\leq} : |\mathrm{Sing}_{\leq}(K)| \to |K|$ the map given by the composition

$$|\mathrm{Sing}_{\leq}(K)| \hookrightarrow |\mathrm{Sing}(K)| \xrightarrow{\varphi} |K|.$$

The following is well-known (stated by Milnor in [8, p.358] and Curtis in [5, p.118]).

Proposition 6.6. *Let K be a simplicial complex with a total order $\leq$ on the vertex set V. The map $\varphi_{\leq}\colon |Sing_{\leq}(K)| \to |K|$ is a homeomorphism.*

Proof. We first consider the situation where V is finite. If V has cardinality $m+1$, then $V_{\leq}$ is isomorphic to the ordinal $[m]$ by an order-preserving bijection $\gamma\colon [m] \to V$. Given an element $\alpha\colon V \to [0,1]$ of $|K|$, the composition $\alpha \circ \gamma\colon [m] \to [0,1]$ is an element of Δ^m, so the pair $(\gamma, \gamma \circ \alpha) \in \mathrm{Sing}_{\leq}(K)_m \times \Delta^m$ represents an element $\psi(\alpha) \in |\mathrm{Sing}_{\leq}(K)|$. This defines a continuous map $\psi\colon |K| \to |\mathrm{Sing}_{\leq}(K)|$. A direct verification yields that $\varphi_{\leq}$ and ψ are inverse of each other, and thus they are homeomorphisms.

If V is not finite, given a finite subset W of V we let $W_{\leq}$ be the total order induced from $V_{\leq}$, and we let K_W be the simplicial complex on the vertex set W consisting of subsets of W contained in K. Then K is the union of the simplicial complexes K_W for W a finite subset of V and $\mathrm{Sing}_{\leq}(K) = \bigcup_{W\subseteq V} \mathrm{Sing}_{\leq}(K_W)$, where the union is taken over all finite subsets of V. That $\varphi_{\leq}\colon |\mathrm{Sing}_{\leq}(K)| \to |K|$ is a homeomorphism now follows from the fact that both kinds of geometric realization are given a topology that commutes with unions, and that $\varphi_{\leq} = \bigcup_{W\subseteq V} \varphi^W_{\leq}$, where the union is taken over all finite subsets of V and $\varphi^W_{\leq}\colon |\mathrm{Sing}_{\leq}(K_W)| \to |K_W|$ is the restriction of $\varphi_{\leq}$ to $|\mathrm{Sing}_{\leq}(K_W)|$. □

The geometric realization of the simplicial set $\mathrm{Sing}_{\leq}(K)$ is homeomorphic to the geometric realization of the simplicial complex (K, V) but choosing an order on V breaks functoriality. The simplicial set $\mathrm{Sing}(K)$ is functorial in (K, V), but its geometric realization is not homeomorphic to the geometric realization of (K, V). However, we proceed to show that they are homotopy equivalent.

Consider the full subcategory $\mathcal{R}_0 \subseteq (\mathcal{V} \times V_{\leq}) \times K_{\subseteq}$ consisting of pairs $((v,w),\sigma)$, where both v and w are vertices of the simplex σ. Let $R_0\colon \mathcal{R}_0 \to (\mathcal{V} \times V_{\leq}) \times K_{\subseteq}$ be the inclusion relation. The projections $\mathcal{V} \times V_{\leq} \to V_{\leq}$ and $\mathcal{V} \times V_{\leq} \to \mathcal{V}$ induce morphisms of relations $R_0 \to R_1$ and $R_0 \to R_2$ giving, by Theorem 4.6, a commutative diagram of the form

$$\begin{tikzcd}
DR_1 & d(ER_1) \arrow[l, "d(\pi_{R_1})"'] \arrow[r, "d(\mathrm{tw}^*)"] & d(ER_1^T) \arrow[r, "d(\pi_{R_1^T})"] & DR_1^T \\
DR_0 \arrow[u] \arrow[d] & d(ER_0) \arrow[l, "d(\pi_{R_0})"'] \arrow[r, "d(\mathrm{tw}^*)"] \arrow[u] \arrow[d] & d(ER_0^T) \arrow[r, "d(\pi_{R_0^T})"] \arrow[u] \arrow[d] & DR_0^T \arrow[u] \arrow[d] \\
DR_2 & d(ER_2) \arrow[l, "d(\pi_{R_2})"'] \arrow[r, "d(\mathrm{tw}^*)"] & d(ER_2^T) \arrow[r, "d(\pi_{R_2^T})"] & DR_2^T.
\end{tikzcd} \tag{5}$$

We show that all relations appearing in diagram (5) are Dowker relations so that all horizontal maps are weak equivalences. The categories $V_{\leq}$, $\mathcal{V}$ and $\mathcal{V} \times V_{\leq}$ have the property that every full subcategory has an initial object, so by Corollary 6.4 we conclude that that R_1^T, R_2^T and R_0^T are Dowker relations. Next, let $a : [m] \to V_{\leq}$ be a functor whose image is a simplex in K. By Lemma 6.3 the fiber $\pi_{R_1}^a$ is isomorphic to the nerve of the category $(K_{\subseteq})_{R_1}^a$ consisting of all simplices that contain the image of a. The simplex $a([m])$ is an initial object in $(K_{\subseteq})_{R_1}^a$, so the fiber is contractible and R_1 is a Dowker Relation. Similarly, the nerve $N(K_{\subseteq})_{R_2}^a$ is contractible for every $a \in (DR_2)_m$ and the nerve $N(K_{\subseteq})_{R_0}^a$ is contractible for every $a \in (DR_0)_m$, making R_2 and R_0 Dowker relations as well.

The rightmost vertical maps in diagram (5) are identity maps, thus we can conclude that the maps $\mathrm{Sing}_{\leq}(K) = DR_1 \leftarrow DR_0 \to DR_2 = \mathrm{Sing}(K)$ are weak equivalences. Finally, consider the (non-commutative) diagram

$$\begin{tikzcd} R_0 \arrow[r] \arrow[d] & R_2 \\ R_1 \arrow[ur, hook] & \end{tikzcd} \tag{6}$$

The functor $([1] \times [0]) \times \mathcal{R}_0 \to \mathcal{R}_2$ defined on objects by

$$((i,0),((v,w),\sigma)) \mapsto \begin{cases} (v,\sigma) & \text{if } i = 0 \\ (w,\sigma) & \text{if } i = 1 \end{cases}$$

induces a transformation $H : \mathbb{1}_{[1]\times[0]} \times R_0 \to R_2$ from the top path in diagram (6) to the bottom path, and so by Proposition 5.4 the two paths after taking the Dowker nerve are homotopic. In particular, the top-left triangle in the diagram

$$\begin{tikzcd} {|DR_0|} \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & {|\mathrm{Sing}(K)|} \arrow[d, "\varphi"] \\ {|\mathrm{Sing}_{\leq}(K)|} \arrow[ur, hook] \arrow[r, "\varphi_{\leq}", "\cong"'] & {|K|.} \end{tikzcd}$$

commutes up to homotopy. By construction the triangle at the bottom-right commutes, so we can conclude the following:

Corollary 6.7. *Let K be a simplicial complex. The map $\varphi\colon |Sing(K)| \to |K|$ is a homotopy equivalence and it is natural in K.* □

In [3] it is proven that the inclusion $|\mathrm{Sing}_{\leq}(K)| \to |\mathrm{Sing}(K)|$ is a homotopy equivalence. However the lack of functoriality of $\mathrm{Sing}_{\leq}(K)$ prevents [3] from stating the result about naturality in Corollary 6.7.

This result can be related to topological data analysis since given a filtered simplicial complex $\{K_\alpha\}_{\alpha \in A}$ we obtain a filtered simplicial set $\{\mathrm{Sing}(K_\alpha)\}_{\alpha \in A}$. The filtered topological spaces obtained by taking geometric realizations of these two filtrations are of the same homotopy type. In particular, they have isomorphic persistent homology.

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