Master Thesis

An Introduction to Riemann Surfaces

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Spring 2023

Acknowledgements

First and foremost, I would like to thank my supervisor Erlend Grong for his feedback and support. He has guided me through every step of the writing process, and I have learned nearly everything I know about Riemann surfaces from him. I would also like to thank the "Lektorbois," especially Sondre, for keeping morale high throughout all our years of studying. Finally, I would like to thank my family and God for being there for me all my life.

Contents

Chapter 1

Introduction

Bernhard Riemann is arguably one of the greatest mathematicians who ever lived, and Riemann surfaces are considered among his most influential work. The goal of this thesis is to cover the basics of Riemann surfaces.

1.1 History

First, a brief history of Riemann surfaces, taken from Monastyrsky [Mon99]. The father of the Riemann surface is the German mathematician Bernhard Riemann(1826-1866), who, in his relatively short mathematical career, made vast contributions to nearly every area of mathematics. Riemann introduced Riemann surfaces in his doctoral dissertation *Foundations of a general theory of functions of a complex variable* in 1851. Although Riemann is probably best known for the Riemann integral, at least among many students, his introduction of Riemann surfaces laid the foundations of complex analysis and topology and is considered one of his greatest contributions to science.
In his dissertation, Riemann wanted to consider the behavior of a holomorphic function, but instead of considering the function on the plane, he considered it on a surface lying over the plane. To do this, he turned to what is now known as topology. This resulted in the introduction of the Riemann surface.

1.2 Motivations

Why are Riemann surfaces important? The primary source for this thesis, Otto Forster's *Lectures on Riemann Surfaces* [For99], states that Riemann surfaces originated as a solution to complex analysis problems with multi-valued functions. A multi-valued function is a function that, for each input, has multiple outputs. Classic examples of multi-valued functions are $\log z$ and $z^{\frac{1}{2}}$, studied in Section 3.1 and at the end of Section 4.2. To solve this problem, Riemann proposed a surface made of many layers of sheets, such that for each point in the plane, there are as many sheets as the function has values at that point. By using this surface as the domain of the function, the function now becomes single-valued, as the many values of the point are distributed across the sheets. The examples in Section 3.1 show how the Riemann surfaces of $\log z$ and $z^{\frac{1}{2}}$ can be constructed in this way.

The importance of Riemann surfaces is also apparent from their relevancy to other mathematical topics. According to McMullen [McM], Riemann surfaces are related to topology, manifolds, differential geometry, complex geometry, algebraic geometry, arithmetic geometry, Lie groups, number theory, and dynamics. These are essential topics from all over mathematics, which are also important in physics. For instance, differential geometry is critical to Albert Einstein's theory of general relativity.

1.3 Structure

This thesis is structured in the following way. Chapter 2 introduces notation and results from complex analysis and topology, which will be needed later. These topics are not merely assumed to be known because I had to learn or review them myself. Chapter 3 defines Riemann surfaces and considers some functions, maps, and other properties on them. Lastly, Chapter 4 looks at analytic continuation on Riemann surfaces, an important topic in studying Riemann surfaces, and finishes with some results on the torus.

Due to the relatively small scope of this thesis, many things are not touched upon, such as, among other things, integration on Riemann surfaces. The priority has been on covering the results needed later in the thesis and on covering the most central results of Riemann surfaces.

After the preliminaries, the contents and progression of the first seven sections of Forster [For99] are primarily followed. Still, other sources are used to supplement. All the figures are my work, but most of them are inspired by figures from the sources or pictures from the internet.

The reader is assumed to be familiar with complex numbers, complex functions, and real analysis. In addition, if the reader has some knowledge of topics such as topology, manifolds, algebra, and complex analysis, it will probably ease the reading.

Chapter 2

Preliminaries

The theory of Riemann surfaces is built on a foundation of topology and complex analysis. Hence, before starting the discussion of Riemann surfaces, it is necessary to cover some ideas and results from complex analysis and topology that are needed in the following chapters.

2.1 Topology

This section presents the basics of topology relevant to this review of Riemann surfaces. All these results on topology are from the book *Topology* by James Munkres [Mun00].
Topology is the study of topological spaces, which are spaces that have a topology. The first step is to define a topology.

Definition 2.1.1 (Topology)**.** *A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:*

(i) *$\emptyset$ and X are in $\mathcal{T}$.*

(ii) *The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$, that is if*

$$\{U_\alpha\} \subset \mathcal{T} \quad \textit{then} \quad \bigcup_\alpha U_\alpha \in \mathcal{T}$$

(iii) *The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$, that is if*

$$U_{\alpha_1}, ..., U_{\alpha_k} \in \mathcal{T} \quad \textit{then} \quad U_{\alpha_1} \cap ... \cap U_{\alpha_k} \in \mathcal{T}$$

We say that a *topological space* is an ordered pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X. The topological space X is often referred to without specific mention of $\mathcal{T}$. A subspace of a topological space is also a topological space with the following topology.

Definition 2.1.2 (Subspace Topology)**.** *Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection*

$$\mathcal{T}_Y = \{Y \cap U : \; U \in \mathcal{T}\}$$

is a topology on Y, called the subspace topology.

Example 2.1.3 (Topologies)**.** *Some examples of topologies are:*

(i) *Topology on $\mathbb{R}^n$; Let d be a metric on $\mathbb{R}^n$. Define*

$$B_r(x) = \{y : \; d(x, y) < r\}$$

then

$$\mathcal{T} = \{U : \; \textit{for all } x \in U, \; \textit{there is an } r > 0 \textit{ such that } \; x \in B_r(x) \subset U\}$$

is the topology on $\mathbb{R}^n$.

For any metric space M, we can construct a topology in the same way as for $\mathbb{R}^n$.

(ii) *Topology on $\mathbb{S}^1$; Let $\mathcal{T}$ be a topology on $\mathbb{R}^2$*

$$\mathcal{T}_{\mathbb{S}^1} = \{U \cap \mathbb{S}^1 : \; U \textit{ open in } \mathcal{T}\}$$

(iii) *Topology on $\mathbb{S}^2$; Let $\mathcal{T}$ be a topology on $\mathbb{R}^3$*

$$\mathcal{T}_{\mathbb{S}^2} = \{U \cap \mathbb{S}^2 : \; U \textit{ open in } \mathcal{T}\}$$

The two latter topologies are examples of subspace topologies.

We continue with an important definition closely related to the topology and the following definitions.

Definition 2.1.4 (Open and Closed Set)**.** *If X is a topological space with topology $\mathcal{T}$ and $U \subset X$, we say that U is an open set of X if U belongs to the collection $\mathcal{T}$. We say that U is closed if the set $X \setminus U$ is open.*

A frequently used open set is the neighborhood. Let $x \in X$ be a point. We say that U is a neighborhood of x if U is an open set containing x. The definition of open sets gives us some more definitions depending on them.

Definition 2.1.5. *Let X and Y be topological spaces.*

(i) *A function $f : X \to Y$ is said to be continuous if, for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.*

(ii) *A map $f : X \to Y$ is said to be an open map if, for each open set U of X, the set $f(U)$ is open in Y.*

(iii) *A map $f : X \to Y$ is said to be a closed map if, for each closed set A of X, the set $f(A)$ is closed in Y.*

(iv) *A surjective map $p : X \to Y$ is said to be a quotient map provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.*

From the quotient map, we define a topology.

Definition 2.1.6 (Quotient Topology)**.** *Let X be a space, A be a set, and $p : X \to A$ be a surjective map. The unique topology $\mathcal{T}$ on A, relative to which p is a quotient map, is called the quotient topology induced by p.*

In most cases, it is too hard to describe a topology $\mathcal{T}$ by specifying all its sets. It is, therefore, practical to define a basis we can work with instead.

Definition 2.1.7 (Basis for a Topology)**.** *If X is a set, a basis for a topology on X is a collection $\mathscr{B}$ of subsets of X, called basis elements, such that*

(i) *For each $x \in X$, there is at least one basis element B containing x.*

(ii) *If $x \in B_1 \cap B_2$, then there is a basis element B_3 containing x such that $x \in B_3 \subset B_1 \cap B_2$.*

Definition 2.1.8 (Topology Generated by $\mathscr{B}$)**.** *If $\mathscr{B}$ is a basis, we define the topology $\mathcal{T}$ generated by $\mathscr{B}$ as follows: A subset $U \subset X$ is said to be open in X, that is $U \in \mathcal{T}$, if for each $x \in U$, there is a basis element $B \in \mathscr{B}$ such that $x \in B$ and $B \subset U$.*

We now look at some definitions that will be important when we define Riemann surfaces.

Definition 2.1.9 (Hausdorff space)**.** *A topological space X is called a Hausdorff space if for each pair of points $x \neq y \in X$ there exist neighborhoods U and V of x and y respectively, that are disjoint, that is*

$$U \cap V = \emptyset$$

Definition 2.1.10 (Connected Space)**.** *Let X be a topological space. A separation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be connected if there does not exist a separation of X. Equivalently, X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.*

A space X is said to be *simply connected* if every closed curve in X can be shrunk to a point in X. That is, the space does not have any unavoidable holes. For example, the torus, studied in Example 3.3.4 and Section 4.3, is not simply connected.

Definition 2.1.11 (Homeomorphism)**.** *Let X and Y be topological spaces and let $f : X \to Y$ be a bijection. If both f and $f^{-1} : Y \to X$ are continuous then f is called a homeomorphism.*

If $f : X \to Y$ is a homeomorphism, then we say that X is homeomorphic to Y.

Definition 2.1.12 (Manifold)**.** *An m-manifold is a Hausdorff topological space X with a countable basis such that each point $x \in X$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}^m$.*

For our purposes, we are interested in surfaces, which are 2-manifolds. When we define Riemann surfaces in Section 3.2, we will see that they are connected 2-manifolds.

2.2 Complex Analysis

Similarly to the section on topology, this section presents the necessary ideas and results of complex analysis. The Results are here taken from Brown and Churchill's *Complex Variables and Applications* [BC14] unless stated otherwise.

Holomorphic functions and maps are important concepts for the rest of the thesis. In Section 3.4, holomorphic functions and maps on Riemann surfaces are defined. Those definitions, rely on the classical definitions given here.

Definition 2.2.1 (Holomorphic Function)**.** *A function f of the complex variable z is holomorphic in an open set S if it has a derivative everywhere in that set. It is holomorphic at a point z_0 if it is holomorphic in some neighborhood of z_0.*

Some sources use the equivalent term analytic, or to be more precise, complex analytic, but here the term holomorphic will be used in all cases except in names, such as analytic continuation. Some functions, closely related to holomorphic functions, have their own names.

(i) A function that is holomorphic throughout the whole complex plane is called an entire function.
(ii) A function that is holomorphic and bijective is called biholomorphic.
(iii) A function that is holomorphic throughout a domain D except for poles is called meromorphic in D.

If f is biholomorphic, then f^{-1} is holomorphic, as shown by letting g be bijective in the following proposition.

Proposition 2.2.2. *[Con78, Proposition 3.2.20] Let G and Ω be open subsets of $\mathbb{C}$. Suppose $f : G \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ are continuous functions such that $f(G) \subset \Omega$ and $g(f(z)) = z$ for all $z \in G$. If f and g are differentiable and $f'(z) = \frac{1}{g'(f(z))}$, with $g'(z) \neq 0$ and if g is holomorphic, then f is holomorphic.*

From the rules of differentiation, it follows that the combinations of holomorphic functions are also holomorphic.

Definition 2.2.3 (Combination of Holomorphic Functions are Holomorphic)**.** *If f and g are holomorphic in a domain D, then $f + g$, $f \cdot g$ and $f \circ g$ are holomorphic in D. Also if $g \neq 0$, for all $z \in D$, then $\frac{f}{g}$ is holomorphic in D.*

An important property of holomorphic functions is that they can be represented as a power series. We will see the use of these series in Chapter 4.

Theorem 2.2.4 (Taylor's Theorem)**.** *Suppose that a function f is holomorphic on the disk $|z - z_0| < R_0$, centered at z_0 and with radius R_0. Then $f(z)$ has the following power series representation, which we call the Taylor series expansion of $f(z)$ about the point z_0;*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (|z - z_0| < R_0)$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad (n = 0, 1, 2, ...)$$

The series converges to $f(z)$ when z lies in the open disk $|z - z_0| < R_0$. We call R_0 the radius of convergence of the Taylor series.

Any function which is holomorphic at a point z_0 has a Taylor series about the point because the function is necessarily holomorphic in some neighborhood $|z - z_0| < \varepsilon$ of z_0. Even if the function is not holomorphic on the whole disc $|z - z_0| < R_0$, we can still find a series representation.

Theorem 2.2.5 (Laurent's Theorem)**.** *Suppose that a function f is holomorphic on an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ has the following series representation, which we call the Laurent series;*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad (R_1 < |z - z_0| < R_2)$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}} \qquad (n = 0, \pm 1, \pm 2, ...)$$

If f is holomorphic everywhere on the disk $|z - z_0| < R_2$ except the point z_0, the expression is valid for $R_1 = 0$ since we can choose R_1 arbitrarily small. We will later see that the Laurent series on this form($R_1 = 0$) are useful for meromorphic functions.

The part of the Laurent series with negative coefficients is called the principal part of f and can be written as

$$h(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

We can identify the singular point at z_0 from the coefficients of the principal part of f as follows.

(i) Removable singular point: Every b_n is zero. That is $h(z) = 0$. If we let $f(z_0) = a_0$, then f is holomorphic on the disc $|z - z_0| < R_2$, and the singularity is therefore removed.

(ii) Pole of order m: There are finitely many nonzero coefficients b_n. We have that $b_n = 0$ for $n > m$, while $b_m \neq 0$.

(iii) Essential singular point: There are infinitely many nonzero coefficients b_n.

The following two theorems will be revisited for Riemann surfaces in Section 3.4.

Theorem 2.2.6 (Riemann's Removable Singularities Theorem)**.** *[Ahl79, Theorem 4.3.7][BC14, Theorem 6.84.2] Suppose that f(z) is holomorphic in the region $U' = U \setminus \{a\}$. Then the following statements are equivalent.*

(i) *There exist a unique holomorphic function $\tilde{f}$, in U, which coincides with f(z) in U'.*

(ii) $\lim_{z \to a}(z - a)f(z) = 0$.

(iii) *f is bounded in a neighborhood around a.*

Theorem 2.2.7 (Identity theorem)**.** *Let f and g be two holomorphic functions on a domain D. If $f(z) = g(z)$ on some subset $G \subset D$ with a limit point z_0 in D, then $f(z) = g(z)$ on D.*

Analytic continuation is the topic of Chapter 4, but it will help our understanding to first take a look at how it works on the complex plane.

Definition 2.2.8 (Analytic Continuation)**.** *Let f_1 and f_2 be holomorphic functions in the domains D_1 and D_2, respectively. We call f_2 the analytic continuation of f_1 into D_2 if $f_1 = f_2$ in $D_1 \cap D_2$. The function*

$$F(z) = \begin{cases} f_1(z) & z \in D_1 \\ f_2(z) & z \in D_2 \end{cases}$$

is holomorphic in $D_1 \cup D_2$ and is called the analytic continuation into $D_1 \cup D_2$.

By the Identity Theorem [Theorem 2.2.7], the analytic continuation is unique if it exists.

The last result we need from complex analysis is the following theorem, which lets us factor a holomorphic function.

Theorem 2.2.9. *Let f denote a function that is holomorphic at a point z_0. The following two statements are equivalent:*

(i) *f has a zero of order k at z_0;*

(ii) *there is a function g, which is holomorphic and nonzero at z_0, such that*

$$f(z) = (z - z_0)^k g(z)$$

Chapter 3

Riemann Surfaces

Moving on to the core of the thesis, this chapter introduces Riemann surfaces. When studying Riemann surfaces, it is natural to look at examples. For this reason, Section 3.1 and Section 3.3 are entirely devoted to examples of Riemann surfaces. Between these blocks of examples, Section 3.2 defines Riemann surfaces. The following three sections of the chapter examine holomorphic and meromorphic functions and maps on Riemann surfaces. The final two sections consider coverings of Riemann surfaces and Riemann surfaces as coverings. The results in this chapter are from Forster [For99] unless stated otherwise.

3.1 Introductory Examples

This first section considers two examples of Riemann surfaces to better understand why we need them and what they can look like. The construction of these surfaces will resemble the idea given in Section 1.2, but it aims to be more explicit.

Example 3.1.1 (Riemann Surface of $\log z$)**.** *[BC14, Example 8.110.1] The complex logarithm is written as*

$$\log z = \ln r + i\theta, \quad z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\},$$

where $z = re^{i\theta}$. This function is multi-valued, with infinitely many values, since we can use $\theta + 2n\pi$ for any $n \in \mathbb{Z}$. We want the function to be single-valued, so we construct a new surface, X, to define the function on. Imagine infinitely many copies of $\mathbb{C}^*$ as thin sheets stacked on top of each other. Slice these sheets along the positive real axis and connect the cut's upper edge to the sheet above and the lower edge to the upper edge below. This gives us a spiral, as shown in Figure 3.1.

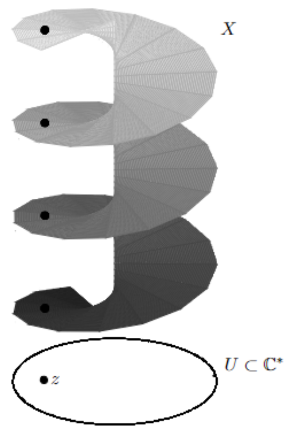

Figure 3.1: Riemann surface of $\log z$

Now we have that for any $n \in \mathbb{Z}$, $\theta \in [2(n-1)\pi, 2n\pi)$ on the n-th sheet of X. The function $\log z$ is single-valued on X since each value at the point z now lies on a different sheet of X, as shown in the figure. The surface X is the Riemann surface for $\log z$, and $\log z$ defined on X represents the analytic continuation of $\log z = \ln r + i\theta \quad (0 < \theta < 2\pi)$.

Example 3.1.2 (Riemann Surface of $z^{\frac{1}{2}}$)**.** *[BC14, Example 8.110.2] The complex square root is written as*

$$z^{\frac{1}{2}} = \sqrt{r}e^{i\frac{\theta}{2}}$$

This function has two values for each z other than the origin. We can construct a surface where the function is single-valued, similar to the last example, but we only need two sheets this time. We connect one sheet's lower edge to the other's upper edge. The surface Y that we end up with is closed and connected, and it is such that where we, in the last example, would have moved to the third sheet, we here go back to the first. On this surface Y, the complex square root function is single-valued and analogous to the last example Y is the Riemann surface of $z^{\frac{1}{2}}$, and $z^{\frac{1}{2}}$ defined on Y represents the analytic continuation of $z^{\frac{1}{2}} \quad (0 < \theta < 2\pi)$. The surface Y cannot be displayed in 3d space without intersecting itself, so the self-intersecting surface in Figure 3.2 is our best representation.

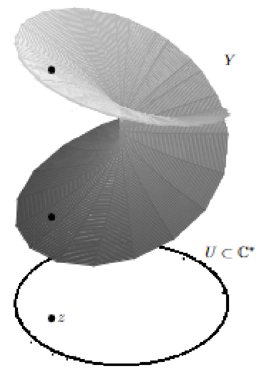

Figure 3.2: Riemann surface of $z^{\frac{1}{2}}$

Analytic continuations of $\log z$ and $z^{\frac{1}{2}}$ are considered at the end of Section 4.2.

3.2 Definition of Riemann surfaces

In this section, Riemann surfaces are defined. Section 2.1 introduced some properties of Riemann surfaces, namely that they are connected 2-manifolds, but to become a Riemann surface, the manifold also needs an added structure.
Before this structure can be introduced, some definitions are required.

Definition 3.2.1 (Chart)**.** *Let X be a 2-manifold. A complex chart on X is a homeomorphism $\psi : U \to V$ of an open subset $U \subset X$ onto an open subset $V \subset \mathbb{C}$.*

A chart can also be written as (ψ, U), where $U \subseteq X$ and $\psi : U \to \mathbb{C}$ is a homeomorphism from U to $\psi(U)$.
Two complex charts $\psi_1 : U_1 \to V_1$ and $\psi_2 : U_2 \to V_2$ are said to be *holomorphically compatible* if the map

$$\psi_2 \circ \psi_1^{-1} : \psi_1(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2)$$

is biholomorphic. We also consider the two charts to be holomorphically compatible if $U_1 \cap U_2 = \emptyset$.
Holomorphically compatible charts are important because we want charts that work well together and give the same result where they overlap.

Example 3.2.2 (Simple Chart). *Let* $U = \{p = (p_1, p_2, p_3) : \ (p_3 - 1)^2 + p_2^2 < 1, \ p_1 = 0\} \subset \mathbb{R}^3$ *be the disk in the* yz*- plane and let* $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\} \subset \mathbb{C}$ *be the unit disk in* $\mathbb{C}$*. Define a chart* $\psi : U \to \mathbb{D}$ *by* $\psi(p) = p_2 + i(p_3 - 1)$*, as shown in Figure 3.3.*

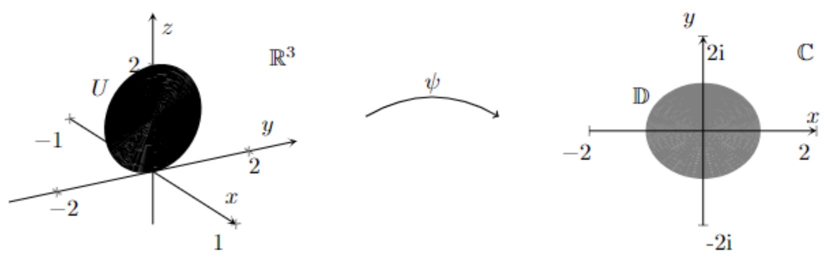

Figure 3.3: Example of a chart

Definition 3.2.3 (Atlas). *A complex atlas* $\mathscr{A}$ *on* X *is a system of holomorphically compatible charts which cover* X*.*

$$\mathscr{A} = \left\{ \psi_i : U_i \to V_i : \quad i, \ j \in I, \quad \psi_i, \psi_j \textit{ holomorphically compatible}, \quad \bigcup_{i \in I} U_i = X \right\}$$

We say that two atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ are analytically equivalent, denoted $\mathscr{A}_1 \sim \mathscr{A}_2$ if every chart of $\mathscr{A}_1$ is holomorphically compatible with every chart of $\mathscr{A}_2$. Note that $\mathscr{A}_1 \sim \mathscr{A}_2$ if and only if $\mathscr{A}_1 \cup \mathscr{A}_2$ is an atlas.

We now have what we need to define a structure on our manifold.

Definition 3.2.4 (Complex structure). *A complex structure on a manifold is an equivalence class of analytically equivalent atlases.*

A complex structure on X is given by choosing an atlas $\mathscr{A}$ on X. That is, if $\mathscr{A}$ is an atlas on X, then

$$\Sigma = \{\hat{\mathscr{A}} : \ \hat{\mathscr{A}} \sim \mathscr{A}\}$$

is a complex structure on X.

Atlases on X can be of different sizes, but there is one that is the largest.

Definition 3.2.5 (Maximal Atlas)**.** *Let $\mathscr{A}$ be an atlas. The maximal atlas $\hat{\mathscr{A}}$ is the collection of all charts holomorphically compatible with every other chart of $\mathscr{A}$.*

$$\hat{\mathscr{A}} = \{\varphi \text{ chart on } X : \quad \varphi \text{ and } \psi \text{ holomorphically compatible, for any } \psi \in \mathscr{A}\}$$

Each complex structure Σ has a unique maximal atlas given by

$$\hat{\mathscr{A}} = \bigcup_{\mathscr{A} \in \Sigma} \mathscr{A}$$

It is also possible to define a complex structure from a maximal atlas. If $\hat{\mathscr{A}}$ is a maximal atlas and $\mathscr{A}_1 \subset \hat{\mathscr{A}}$ and $\mathscr{A}_2 \subset \hat{\mathscr{A}}$, then $\mathscr{A}_1 \sim \mathscr{A}_2$. This enables us to write

$$\Sigma = \{\mathscr{A} : \mathscr{A} \subset \hat{\mathscr{A}}\}$$

This means that having a complex structure is equivalent to having a maximal atlas since we can construct one from the other.

This gives us all the pieces we need to define Riemann surfaces.

Definition 3.2.6 (Riemann Surface)**.** *A Riemann surface X is a connected 2-manifold with a complex structure Σ.*

As discussed above, we could write a maximal atlas instead of a complex structure. If X is a Riemann surface, then by a chart on X, we mean a complex chart in the maximal atlas of the complex structure on X.

3.3 Examples of Riemann Surfaces

Section 3.1 described how some Riemann surfaces could be constructed. This section looks at other well-known examples of Riemann surfaces and gives atlases to define their complex structure.

Example 3.3.1 (Complex plane)**.** *The complex plane $\mathbb{C}$ is a Riemann surface. Its complex structure is defined by the atlas, whose only chart is the identity map $\varphi : \mathbb{C} \to \mathbb{C}$.*

$$\mathscr{A} = \{\varphi : \mathbb{C} \to \mathbb{C} : \quad \varphi(z) = z \ for\ all\ z \in \mathbb{C}\}$$

Example 3.3.2 (Domain of a Riemann Surface). *Remember that a domain is a connected, open subset. We have that every domain of a Riemann surface is a Riemann surface. If X is a Riemann surface and $Y \subset X$ is a domain, then take as its atlas all those complex charts $\varphi : U \to V$ on X, where $U \subset Y$.*

$$\mathscr{A} = \{\varphi : U \to V : \ U \subset Y\}$$

Example 3.3.3 (The Riemann Sphere). *The Riemann Sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, shown in Figure 3.4 is a compact Riemann surface. An atlas on $\mathbb{P}^1$ is*

$$\mathscr{A} = \{\varphi_1, \varphi_2\}$$

where φ_1 is the identity map $\varphi_1 : \mathbb{C} \to \mathbb{C}$ and

$$\varphi_2 : \mathbb{P}^1 \setminus \{0\} \to \mathbb{C} = \begin{cases} \frac{1}{z} & for\ z \in \mathbb{C}^* \\ 0 & for\ z = \infty \end{cases} \qquad\qquad \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

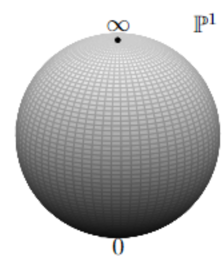

Figure 3.4: The Riemann Sphere

Example 3.3.4 (Torus). *In this example, we will look at the torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$, see Figure 3.5.*

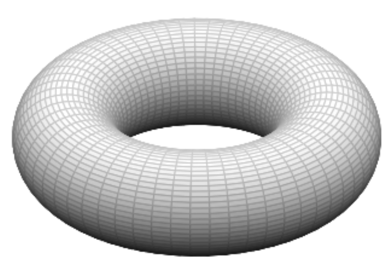

Figure 3.5: Torus

Before we can give the torus a complex structure, we first need to define a lattice. Let $\omega_1, \omega_2 \in \mathbb{C}$ *be linearly independent, that is* $Im(\frac{\omega_1}{\omega_2}) \neq 0$. *Then*

$$\Gamma = \{n_1\omega_1 + n_1\omega_2 : \quad n_1, n_2 \in \mathbb{Z}\}$$

shown in Figure 3.6, is called the lattice spanned by ω_1 *and* ω_2.

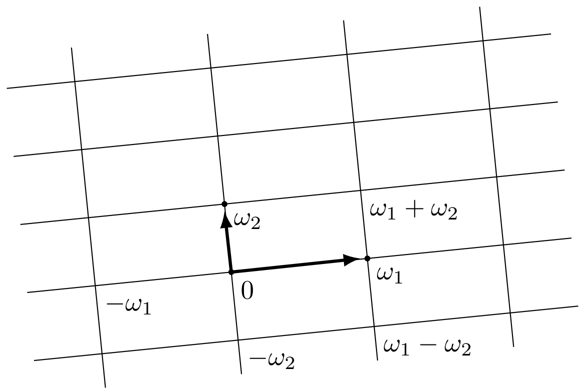

Figure 3.6: Lattice spanned by ω_1 and ω_2

Two complex numbers $z_1, z_2 \in \mathbb{C}$ *are called equivalent mod* Γ *if* $z_1 - z_2 \in \Gamma$. *We call the set of all the equivalence classes* $\mathbb{C}/\Gamma$.

We want to show that the torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ *is equivalent to* $\mathbb{T} = \mathbb{C}/\Gamma$. *The map*

$$F : \ \mathbb{C}/\Gamma \to \mathbb{S}^1 \times \mathbb{S}^1, \quad a_1\omega_1 + a_2\omega_2 \mapsto e^{i\pi a_1} \times e^{i\pi a_2}$$

is a homeomorphism of $\mathbb{C}/\Gamma$ *onto the torus* $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$.

Let

$$\pi : \ \mathbb{C} \to \mathbb{C}/\Gamma, \quad z \mapsto [z]$$

be the canonical projection. That is, a map from the point $z \in \mathbb{C}$ *to its equivalence class modulo* Γ. *Since the map* π *is open and surjective, it induces a quotient topology on* $\mathbb{C}/\Gamma$. $U \subset \mathbb{C}/\Gamma$ *is open if and only if* $\pi^{-1}(U) \in \mathbb{C}$ *is open. With this topology,* $\mathbb{C}/\Gamma$ *is a compact connected Hausdorff topological space.*

We now define a complex structure on $\mathbb{C}/\Gamma$. *Let* $V \subset \mathbb{C}$ *such that if* $z_1 \in V$ *then* $z_2 \notin V$ *for all* $z_2 \in [z_1]$. *That is, no two points in* V *are equivalent under* Γ. $U = \pi(V)$ *is open and* $\pi|_V$ *is a homeomorphism. Then the inverse of* $\pi|_V$, $\varphi : U \to V$, *is a complex chart on* $\mathbb{C}/\Gamma$. *We can repeat this process for all* $V \in \mathbb{C}$. *Let now* $\mathscr{A}$ *be the atlas of all such* φ.

$$\mathscr{A} = \{\varphi : \ U \to V : \ V \subset \mathbb{C}, \ U = \pi(V), \ \varphi = (\pi|_V)^{-1}\}$$

Explicitly if $q = e^{2\pi i a_1} \times e^{2\pi i a_2}$ *with* $a_1, a_2 \neq 0 \bmod \mathbb{Z}$*. We can choose* $0 < a_1, a_2 < 1$ *(or* $-\frac{1}{2} < a_1, a_2 < \frac{1}{2}$ *if* q *is on the center line) and use the chart*

$$\varphi(e^{2\pi i a_1} \times e^{2\pi i a_2}) = a_1\omega_1 + a_2\omega_2$$

This gives a complex structure $\mathscr{A}_{\omega_1,\omega_2}$ *on* $\mathbb{T}$*.*

The following theorem states the equivalence of complex structures on the torus.

Theorem 3.3.5. *Let* $\Gamma = n_1\omega_1 + n_2\omega_2$*. All complex tori* $(\mathbb{T}, \mathscr{A}_{\omega_1,\omega_2})$ *are isomorphic to a torus* $(\mathbb{T}, \mathscr{A}_{1,\tau})$*, with* $Im(\tau) \neq 0$

In Section 4.3, we will look at more results on the torus.

3.4 Holomorphic Functions

This section defines holomorphic functions and mappings on Riemann surfaces and considers some of their properties.

Definition 3.4.1 (Holomorphic Function)**.** *Let* X *be a Riemann surface and* $Y \subset X$ *an open subset. A function* $f : Y \to \mathbb{C}$ *is called holomorphic, if for every chart* $\psi : U \to V$ *on* X *the function*

$$f \circ \psi^{-1} : \psi(U \cap Y) \to \mathbb{C}$$

is holomorphic in the usual sense on the open set $\psi(U \cap Y) \subset \mathbb{C}$*.*

The set of all functions that are holomorphic on Y is denoted by $\mathcal{O}(Y)$. That is, for $Y \subset X$ open

$$\mathcal{O}(Y) = \{f : \ Y \to \mathbb{C} \ \textit{holomorphic}\}$$

Definition 3.4.2 (Holomorphic Mapping)**.** *Suppose* X *and* Y *Riemann surfaces. A continuous mapping* $f : X \to Y$ *is called holomorphic, if for every pair of charts* $\psi_1 : U_1 \to V_1$ *on* X *and* $\psi_2 : U_2 \to V_2$ *on* Y *with* $f(U_1) = U_2$*, the mapping*

$$\psi_2 \circ f \circ \psi_1^{-1} : V_1 \to V_2$$

is holomorphic in the usual sense. A mapping is called biholomorphic if it is bijective and both f *and* f^{-1} *are holomorphic. Two Riemann surfaces* X *and* Y*, are called isomorphic or biholomorphic if there exists a biholomorphic mapping* $f : X \to Y$*.*

It is enough to check f for a single atlas. If we choose a chart $\hat{\psi}_1$ on X and $\hat{\psi}_2$ on Y, such that $\hat{\psi}_2 \circ f \circ \hat{\psi}_1^{-1}$ is holomorphic, then $\psi_2 \circ f \circ \psi_1^{-1}$ is holomorphic for arbitrary charts ψ_1 and ψ_2 on

X and Y respectively. For simplicity let ψ_1 and ψ_2 be defined on the same domains as $\hat{\psi}_1$ and $\hat{\psi}_2$. Then

$$\psi_2 \circ f \circ \psi_1^{-1} = (\psi_2 \circ \hat{\psi}_2^{-1}) \circ \ \hat{\psi}_2 \circ f \circ \hat{\psi}_1^{-1} \circ \ (\hat{\psi}_1 \circ \psi_1^{-1})$$

which is holomorphic because each of its composed parts is holomorphic. By our initial assumption $\hat{\psi}_2 \circ f \circ \hat{\psi}_1^{-1}$ is holomorphic. The function $(\psi_2 \circ \hat{\psi}_2^{-1})$ consists of two charts on the same Riemann surface, which by definition are holomorphically compatible, and thus $(\psi_2 \circ \hat{\psi}_2^{-1})$ is holomorphic. By the same argument $(\hat{\psi}_1 \circ \psi_1^{-1})$ is also holomorphic.

The next two theorems consider the equivalence of Riemann surfaces and are regarded as some of the most important theorems in complex analysis and the theory of holomorphic functions.

Theorem 3.4.3 (Riemann Mapping Theorem)**.** *[Con78, Theorem 7.4.2] Let $G \subsetneq \mathbb{C}$ be a simply connected region and let $a \in G$. Then there is a unique holomorphic function $f : G \to \mathbb{C}$ having the properties:*

(i) $f(a) = 0$ *and* $f'(a) > 0$;

(ii) f *is one-one*

(iii) $f(G) = \{z : \ |z| < 1\}$

This theorem states that every simply connected proper region of $\mathbb{C}$ is biholomorphic to $\mathbb{D}$. Since $\mathbb{C}$ is biholomorphic to itself, any simply connected region $G \subseteq \mathbb{C}$ is biholomorphic to either $\mathbb{D}$ or $\mathbb{C}$. The following theorem is a more general version of the Riemann mapping theorem that gives us a statement about all simply connected Riemann surfaces and not just regions of $\mathbb{C}$.

Theorem 3.4.4 (Uniformization Theorem)**.** *Suppose X is a simply connected Riemann surface. Then X can be mapped biholomorphically onto either the Riemann sphere $\mathbb{P}^1$, the complex plane $\mathbb{C}$, or else the unit disk $\mathbb{D}$.*

This means there are really only three simply connected Riemann surfaces, as every other simply connected Riemann surface is biholomorphic to either $\mathbb{P}^1$, $\mathbb{C}$, or $\mathbb{D}$. It is important to note that this only works for simply connected Riemann surfaces, so the theorem does not apply to surfaces like the torus and the cylinder. The Riemann surface of $\log z$ is also not simply connected. Most of the Riemann surfaces we have looked at in this thesis, except those listed in the theorem, are not simply connected, but it is too nice of a result to be left out.

The two final theorems of this section are similar to the classic versions of the Identity Theorem [Theorem 2.2.7] and Riemann's removable singularities Theorem [Theorem 2.2.6], but they are now generalized to Riemann surfaces.

Theorem 3.4.5 (Riemann's Removable Singularities Theorem)**.** *Let V be an open subset of a Riemann surface and let $a \in U$. Suppose the function $f \in \mathcal{O}(U \setminus \{a\})$ is bounded in some neighborhood of a. Then f can be extended uniquely to a function $\tilde{f} \in \mathcal{O}(U)$.*

Theorem 3.4.6 (Identity Theorem)**.** *Suppose X and Y are Riemann surfaces and $f_1,\ f_2 : X \to Y$ are two holomorphic mappings which coincide on a set $A \subset X$ having a limit point $a \in X$. Then f_1 and f_2 are identically equal.*

We will include a proof of the latter result.

Proof. Let $G \subset X$ be the set of all points $x \in X$ having an open neighborhood W such that $f_1|_W = f_2|_W$. By definition, G is open. We want to show that G is also closed. Suppose b is a boundary point of G. Then $f_1(b) = f_2(b)$ since f_1 and f_2 are continuous. Choose charts $\psi_1 : U_1 \to V_1$ on X and $\psi_2 : U_2 \to V_2$ on Y with $b \in U_1$ and $f_i(U_1) \subset U_2$. We may also assume that U_1 is connected. The mappings

$$g_i := \psi_2 \circ f_i \circ \psi_1^{-1} : V_1 \to V_2 \subset \mathbb{C}$$

are holomorphic since f_1 and f_2 are holomorphic. Since $H = U_1 \cap G \neq \emptyset$ there exists a domain $V' = \psi_1(H) \subset V_1$ such that $g_1|_{V'} = g_2|_{V'}$, with a limit point in $\psi_1(b)$ in V_1. By the classic Identity Theorem [Theorem 2.2.7], g_1 and g_2 are identically equal on V_1. Thus $f_1|_{U_1} = f_2|_{U_1}$ and since $b \in U_1$ we have that $b \in G$ and thus G is closed.
Now since X is connected, and both open and closed, either $G = \emptyset$ or $G = X$ by the definition of connected sets. We can show that the first case is excluded by repeating a similar process to earlier around the set A. Let $A \subset X$ be a set with a limit point $a \in X$ such that $f_1|_A = f_2|_A$. Since f_1 and f_2 are continuous, and a is a limit point of A, we get that $f_1(a) = f_2(a)$. Choose charts $\varphi_1 : U_1 \to V_1$ on X and $\varphi_2 : U_2 \to V_2$ on Y with $a \in U_1$, $A \subset U_1 \subset X$ and $f_i(U_1) \subset U_2$. We may also assume that U_1 is connected. The mappings

$$h_i := \varphi_2 \circ f_i \circ \varphi_1^{-1} : V_1 \to V_2 \subset \mathbb{C}$$

are holomorphic. Since $g_1|_{\varphi(A)} = g_2|_{\varphi(A)}$ and $\varphi(A)$ has a limit point $\varphi(a)$, we can use the Identity Theorem in the plane to show that g_1 and g_2 are equal on V_1. Thus f_1 and f_2 are equal on U_1 and since $a \in U_1$, there exists an open neighborhood W around a such that $f_1|_W = f_2|_W$. Thus $a \in G$, which shows that G is nonempty. Hence f_1 and f_2 coincide on all of X.

$\square$

3.5 Meromorphic Functions

As with the classical case, meromorphic functions are holomorphic except for poles. This short section defines meromorphic functions on Riemann surfaces and covers an important theorem for relating them to holomorphic functions.

Definition 3.5.1 (Meromorphic function)**.** *Let X be a Riemann surface, and Y be an open subset of X. By a meromorphic function on Y, we mean a holomorphic function $f : Y' \to \mathbb{C}$, where $Y' \subset Y$ is an open subset, such that the following hold:*

(i) *$P = Y \setminus Y'$ contains only isolated points.*

(ii) *For every point $p \in P$ one has*

$$\lim_{x \to p} |f(x)| = \infty$$

The points of P are called the poles of f. The set of all meromorphic functions on Y is denoted by $\mathscr{M}(Y)$.

Theorem 3.5.2. *Suppose X is a Riemann surface and $f \in \mathscr{M}(X)$ For each pole p of f, define $f(p) := \infty$. Then $f : X \to \mathbb{P}^1$ is a holomorphic mapping. Conversely, if $f : X \to \mathbb{P}^1$ is a holomorphic mapping, then f is either identically equal to ∞ or else $f^{-1}(\infty)$ consists of isolated points and $f : X \setminus f^{-1}(\infty) \to \mathbb{C}$ is a meromorphic function on X.*

From now on, we will identify a meromorphic function $f \in \mathscr{M}(X)$ with the corresponding holomorphic mapping $f : X \to \mathbb{P}^1$. For this reason the holomorphic mapping $f : X \to \mathbb{P}^1$, given by the theorem, is often used to define a meromorphic function instead of the definition.

Proof. We will split the proof into two parts, the main statement and its converse.

(i) Let $f \in \mathscr{M}(X)$ and let P be the set of poles of f. Then f induces a mapping $f : X \to \mathbb{P}^1$, which is continuous. Suppose $\psi_1 : U_1 \to V_1$ and $\psi_2 : U_2 \to V_2$ are charts on X and $\mathbb{P}^1$ respectively, such that $f(U_1) \subset U_2$. Choose the chart

$$\psi_2 = \begin{cases} \frac{1}{z} & for\ z \in \mathbb{C}^* \\ 0 & for\ z = \infty \end{cases}$$

introduced in Example 3.3.3. To show that f is holomorphic, we need to show that

$$g := \psi_2 \circ f \circ \psi_1^{-1} : V_1 \to V_2 \subset \mathbb{C}$$

is holomorphic. Since f is holomorphic on $X \setminus P$, it follows that g is holomorphic on $V_1 \setminus \psi_1(P)$. Since $g(\psi_1(P)) = \psi_2(\infty) = 0$, there exist neighborhoods around the points $\psi_1(P)$ where g

is bounded and thus by Riemann's Removable Singularities Theorem [Theorem 3.4.5], g is holomorphic on all of V_1.

(ii) Conversely if f is a holomorphic mapping, define $P = f^{-1}(\infty)$. P either consists of isolated points or has a limit point p. If $f|_P = \infty$, and P has a limit point, then by the Identity Theorem [Theorem 3.4.6], we have that $f \equiv \infty$.

□

3.6 Properties of Holomorphic Maps

Returning to look at holomorphic maps, this section considers some more of their properties.

Theorem 3.6.1 (Local Behavior of Holomorphic Mappings)**.** *Suppose X and Y are Riemann surfaces and $f : X \to Y$ is a non-constant holomorphic mapping. Suppose $a \in X$ and $b := f(a)$. Then there exists an integer $k \geq 1$, and charts $\varphi : U \to V$ on X and $\psi : U' \to V'$ on Y with the following properties:*

(i) $a \in U, \quad \varphi(a) = 0; \quad b \in U', \quad \psi(b) = 0.$

(ii) $f(U) \subset U'$.

(iii) *The map $F := \psi \circ f \circ \varphi^{-1} : V \to V'$ is given by*

$$F(z) = z^k \text{ for all } z \in V.$$

We say that f has multiplicity k at $a \in X$ if there exist charts $\varphi : U \to V$ and $\psi : U' \to V'$, with $a \in U \subset X$ and $f(a) \in U' \subset Y$, such that

$$F(z) = \psi \circ f \circ \varphi^{-1}(z) = z^k$$

independent of the choice of charts φ *and* ψ.

Proof. From the definition of Riemann surfaces, we have that there exist charts $\varphi_0 : U_0 \to V_0$ and $\psi_0 : U' \to V'$ such that $a \in U_0$ and $b \in U'$. Using these charts, we can construct charts φ and ψ that have the properties we want by taking

$$\hat{\varphi}(\zeta) = \varphi_0(\zeta) - \varphi_0(a) \quad \text{and} \quad \psi(\zeta) = \psi_0(\zeta) - \psi_0(b)$$

Define $U = U_0 \cap f^{-1}(U')$. Remark that $a \in U$ since $a \in U_0$ and $a \in f^{-1}(b) \subset f^{-1}(U')$. Hence $f(U) = f(U_0 \cap f^{-1}(U')) \subset U'$

We have that $\hat{F} := \psi \circ f \circ \hat{\varphi}^{-1} : V_0 \to V'$ is a holomorphic function since f is holomorphic. We also have that $\hat{F}(0) = 0$ because $\hat{\varphi}^{-1}(0) = a, \quad f(a) = b \quad \text{and} \quad \psi(b) = 0$. From Theorem 2.2.9, we have that $\hat{F}(\hat{z}) = \hat{z}^k g(\hat{z})$ where $g(\hat{z})$ is a holomorphic function with $g(0) \neq 0$. Then a neighborhood exists where we can define $h = \sqrt[k]{g}$. Define a biholomorphic mapping $\alpha : V_1 \to V$, of an open neighborhood $V_1 \subset V_0$ of zero to an open neighborhood V of zero, such that $\alpha(\hat{z}) = \hat{z}h(\hat{z})$ and let $\varphi = \alpha \circ \hat{\varphi}$. Let $z = \hat{z}h(\hat{z}) \in V$ then $F := \psi \circ f \circ \varphi^{-1} : V \to V'$ and $F(z) = z^k$ by construction.

$$F(z) = \hat{F}(\alpha^{-1}(z)) = \hat{F}(\alpha^{-1}(\hat{z}h(\hat{z}))) = \hat{F}(\hat{z}) = \hat{z}^k g(\hat{z}) = (\hat{z}h(\hat{z}))^k = z^k$$

□

Corollary 3.6.2. *Let X and Y be Riemann surfaces and let $f : X \to Y$ be a non-constant holomorphic mapping. Then the following results follow from the theorem.*

(i) *f is open. That is, the image of every open set under f is open.*

(ii) *If f is injective, then f is a biholomorphic mapping of X onto $f(X)$.*

(iii) *If $Y = \mathbb{C}$, then the absolute value of f does not attain its maximum.*

Theorem 3.6.3. *Suppose X and Y are Riemann surfaces. Suppose X is compact and $f : X \to Y$ is a non-constant holomorphic mapping. Then Y is compact, and f is surjective.*

Proof. Since f is a non-constant holomorphic mapping, it is open by Corollary 3.6.2. Since X is compact, $f(X)$ is compact and thus closed. Since the only subsets of a connected topological space that are both open and closed are the empty set and the whole space, it follows that $f(X) = Y$. Thus f is surjective, and Y is compact. □

We have the following corollaries.

Corollary 3.6.4. *Every holomorphic function on a compact Riemann surface is constant.*

Corollary 3.6.5 (Liouville's Theorem)**.** *Every bounded holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant.*

Corollary 3.6.6. *Every meromorphic function f on $\mathbb{P}^1$ is rational; that is, it can be written as the quotient of two polynomials.*

Note that a meromorphic function f on $\mathbb{P}^1$ is a holomorphic function $f : \mathbb{P}^1 \to \mathbb{P}^1$.

Proof. The set, $P = f^{-1}(\infty)$, of the poles of f, must be finite because otherwise, P would have a limit point, and then by the Identity Theorem [Theorem 3.4.6] $f \equiv \infty$. Assume that $f(\infty) \neq \infty$. If not, consider the function $\frac{1}{f}$ instead. Suppose $p_1, ..., p_n \in \mathbb{C}$ are the poles of f, the Laurent series of f around the point p_m is given by

$$f(z) = \sum_{i=-k_m}^{\infty} a_{m_i}(z - p_m)^i = h_m(z) + \sum_{i=0}^{\infty} a_{m_i}(z - p_m)^i$$

where

$$h_m(z) = \sum_{i=-k_m}^{-1} a_{m_i}(z - p_m)^i$$

is the principal part of f at the pole p_m. Note that the principal part of f is a finite sum because it is at a pole. Since $h_m(z)$ is finite on $\mathbb{P}^1 \setminus \{p_m\}$ and $h_m(p_m) = \infty$, it is well defined and the function $g = f - (h_1 + ... + h_n)$ is holomorphic on $\mathbb{P}^1$. Since $\mathbb{P}^1$ is compact, then by Corollary 3.6.4, g is constant. Thus $f = c + h_1 + ... + h_n$, for some constant c, which is a rational function. $\square$

3.7 Coverings

This section and the next are on the topic of coverings. Here liftings of maps and curves will be introduced, and coverings will be defined to ensure their existence. First, some definitions.

Definition 3.7.1 (Fiber)**.** *Suppose X and Y are topological spaces and $p : Y \to X$ is a continuous map. For $x \in X$, the set $p^{-1}(x)$ is called the fiber of p over x. If $y \in p^{-1}(x)$, one says that the point y lies over x.*

Definition 3.7.2 (Fiber preserving)**.** *If $p : Y \to X$ and $q : Z \to X$ are continuous maps, then a map $f : Y \to Z$ is called fiber-preserving if $p = q \circ f$. This means that any point $y \in Y$, lying over the point $x \in X$, is mapped to a point that also lies over x.*

Definition 3.7.3 (Branch point)**.** *Suppose X and Y are Riemann surfaces and $p : Y \to X$ is a non-constant holomorphic map. A point $y \in Y$ is called a branch point of p if there is no neighborhood V of y such that $p|_V$ is injective.*

If y is a branch point, then the map p has multiplicity $k > 1$ at the point y, and any local definition of the map p^{-1} is multi-valued.

Definition 3.7.4 (Unbranched Holomorphic map)**.** *Suppose X and Y are Riemann surfaces. The map $p : Y \to X$ is called an unbranched holomorphic map if it has no branch points.*

An unbranched holomorphic map p is injective, and p has multiplicity $k = 1$ at every point $y \in Y$. Thus, around every point y, there exist charts such that

$$F(z) = \psi_2 \circ p \circ \psi_1^{-1}(z) = z$$

We continue with some theorems on holomorphic maps.

Theorem 3.7.5. *Suppose X and Y are Riemann surfaces. A non-constant holomorphic map $p : Y \to X$ has no branch points if and only if p is a local homeomorphism; that is, every point $y \in Y$ has an open neighborhood V which is mapped homeomorphically by p onto an open set U in X.*

Theorem 3.7.6. *Suppose X is a Riemann surface, Y is a Hausdorff topological space, and $p : Y \to X$ is a local homeomorphism. Then there is a unique complex structure on Y such that p is holomorphic.*

Proof. Let $\mathscr{A}$ be an atlas on X and $\mathscr{B}$ an atlas on

$$Y = \{(p^{-1}(U) \cap V,\ \zeta \circ p) : V \subset Y \textit{ open},\ \textit{such that } p|_V \textit{ is invertible},\ (U, \zeta) \in \mathscr{A}\}$$

We have that $\zeta \circ p$ is invertible on $p^{-1}(U) \cap V$
Let (ζ_1, U_1) and (ζ_2, U_2) be charts on X, then $(\zeta_1 \circ p)$ is a chart on Y and we get

$$\zeta_2 \circ p \circ (\zeta_1 \circ p)^{-1}\big|_{\zeta_1(U_1 \cap p(V))} = \zeta_2 \circ p \circ (p|_V)^{-1} \circ \zeta_1^{-1}\big|_{\zeta_1(U_1 \cap p(V))} = \zeta_2 \circ \zeta_1^{-1}\big|_{\zeta_1(U_1 \cap p(V))}$$

which is holomorphic from the definition of $\mathscr{A}$. Thus p is holomorphic.
Now for uniqueness. Assume $\mathscr{C}$ is another complex atlas such that $p : (Y, \mathscr{C}) \to X$ is holomorphic and thus locally biholomorphic. Hence the identity map from $(Y, \mathscr{B}) \to (Y, \mathscr{C})$ is locally biholomorphic and thus biholomorphic. This means $\mathscr{B}$ and $\mathscr{C}$ define the same complex structure. □

Next, we introduce the concept of lifting.

Definition 3.7.7 (Lifting)**.** *Suppose X, Y, and Z are topological spaces and $p : Y \to X$ and $f : Z \to X$ are continuous maps. Then by a lifting of f with respect to p we mean a continuous mapping $g : Z \to Y$ such that $f = p \circ g$, that is, the diagram in Figure 3.7 commutes.*

$$\begin{tikzcd}
 & Y \arrow[d, "p"] \\
Z \arrow[ur, "g"] \arrow[r, "f"] & X
\end{tikzcd}$$

Figure 3.7: Lifting

Theorem 3.7.8 (Uniqueness of Lifting)**.** *Suppose X and Y are Hausdorff spaces and $p : Y \to X$ is a local homeomorphism. Suppose Z is a connected topological space and $f : Z \to X$ is a continuous mapping. If $g_1,\ g_2 : Z \to Y$ are two liftings of f and $g_1(z_0) = g_2(z_0)$ for some point $z_0 \in Z$ then $g_1 = g_2$.*

This means that a single point uniquely determines a lifting of f.

Theorem 3.7.9. *Suppose X, Y, and Z are Riemann surfaces, $p : Y \to X$ is an unbranched holomorphic map, and $f : Z \to X$ is any holomorphic map. Then every lifting $g : Z \to Y$ of f is holomorphic.*

We are especially interested in lifting curves, but before looking closer at lifting curves, we define an equivalence relation between curves.

Definition 3.7.10 (Homotopic curves)**.** *Suppose X is a topological space and $a, b \in X$. Two curves $u, v : I \to X$ ($I = [0, 1]$) from a to b are called homotopic, denoted $u \sim v$, if there exists a continuous mapping $A : I \times I \to X$ with the following properties:*

(i) *$A(t,\ 0) = u(t)$ for every $t \in I$,*

(ii) *$A\ (t,\ 1) = v(t)$ for every $t \in I$,*

(iii) *$A(0,\ s) = a$ and $A(1,\ s) = b$ for every $s \in I$.*

If we let $u_s(t) = A(t, s)$, then every u_s is a curve from a to b as shown in Figure 3.8.

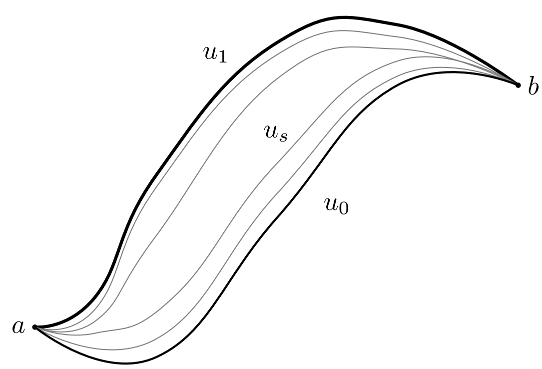

Figure 3.8: Homotopic Curves

Theorem 3.7.11 (Lifting of Homotopic Curves)**.** *Suppose X and Y are Hausdorff spaces and $p : Y \to X$ is a local homeomorphism. Suppose $a,\ b \in X$ and $\hat{a} \in Y$ is a point such that $p(\hat{a}) = a$. Further, suppose a continuous mapping $A : I \times I \to X$ is given such that $A(0, s) = a$ and $A(1, s) = b$ for every $s \in I$. Set*

$$u_s(t) := A(t, s)$$

If every curve u_s can be lifted to a curve $\hat{u}_s$ with initial point $\hat{a}$, then $\hat{u_0}$ and $\hat{u_1}$ have the same endpoint and are homotopic.

If the liftings exist, this theorem tells us about the properties of the lifted curves, but it does not give us insight into when the liftings exist.

The following definition will give us a condition that ensures the existence of the lifting of a curve.

Definition 3.7.12 (Covering Map)**.** *Suppose X and Y are topological spaces. A mapping $p : Y \to X$ is called a covering map if the following holds. Every point $x \in X$ has an open neighborhood U such that its preimage $p^{-1}(U)$ can be represented as*

$$p^{-1}(U) = \bigcup_{j \in J} V_j$$

where the V_j, $j \in J$, are disjoint open subsets of Y, and all the mappings $p|_{V_j} \to U$ are homeomorphisms. In particular, p is a local homeomorphism.

The covering map $p : Y \to X$ is sometimes called the covering of X, and Y is called the covering space of X. From Theorem 3.7.5, we have that p has no branch points, but the converse is not true. If p has no branch points, it is a local homeomorphism, but that does not necessarily make p a covering map.

Example 3.7.13. *The map $p : \mathbb{D} \to \mathbb{C}$, $z \mapsto z$ is a local homeomorphism but not a covering map. For any $a \in \mathbb{C}$ with $|a| \geq 1$, there does not exist an open neighborhood U such that*

$$p^{-1}(U) = \bigcup_{j \in J} V_j$$

where the V_j, $j \in J$, are disjoint open subsets of $\mathbb{D}$.

We have said that we want to be able to lift every curve, but let us state it more rigorously.

Definition 3.7.14 (Curve Lifting Property)**.** *A continuous map $p : Y \to X$ is said to have the curve lifting property if the following condition holds. For every curve, $u : [0,1] \to X$ and every point $y_0 \in Y$ with $p(y_0) = u(0)$ there exists a lifting $\hat{u} : [0,1] \to Y$ of u such that $\hat{u}(0) = y_0$, as shown in Figure 3.9.*

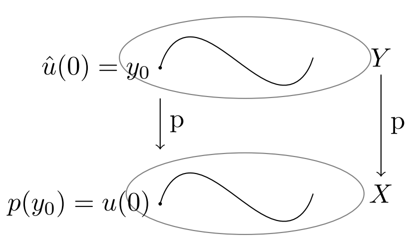

Figure 3.9: Curve lifting

If p has the curve lifting property, Theorem 3.7.11 states that if $u \sim v$, then $\hat{u} \sim \hat{v}$.

Theorem 3.7.15. *Every covering map $p : Y \to X$ of topological spaces X and Y has the curve lifting property.*

Thus, as we stated earlier, having a covering map ensures that the lifting of a curve is always possible.

We finish the section with an example of a covering.

Example 3.7.16 (Covering of $\mathbb{C}^*$)**.** *We will show that the map $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering map. Suppose $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$ with $\exp(b) = a$. Since exp is a local homeomorphism, there exist open neighborhoods V_0 and U of a and b, respectively, such that $\exp|_{V_0} \to U$ is a homeomorphism. Then*

$$\exp^{-1}(U) = \bigcup_{n\in\mathbb{Z}} V_n \quad V_n := 2\pi in + V_0$$

The V_n are pairwise disjoint, that is, $V_i \cap V_j = \emptyset$ for any $i, j \in \mathbb{Z}$, and each map $\exp|_{V_n} \to U$ is a homeomorphism. Hence $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering map, and thus the covering of $\mathbb{C}^*$.

3.8 Universal Covering

A space X can have multiple coverings, but there will always be a choice of covering space, which can also be considered as a covering of the other covering spaces. This universal covering is the topic of this section.

Definition 3.8.1 (Universal Covering)**.** *Suppose X, Y, and Z are connected topological spaces and $p : Y \to X$ is a covering map. $p : Y \to X$ is called the universal covering of X if, for every covering map, $q : Z \to X$, and every choice of points $y_0 \in Y$, $z_0 \in Z$ with $p(y_0) = q(z_0)$ there exists precisely one continuous fiber-preserving mapping $f : Y \to Z$ such that $f(y_0) = z_0$. See the diagram in Figure 3.10.*

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

Figure 3.10: Universal covering

Note that

(i) The universal covering is unique up to isomorphism.

(ii) All other covering spaces can be obtained as quotients of the universal covering.

Theorem 3.8.2. *Suppose X and Y are connected manifolds, Y is simply connected, and $p : Y \to X$ is a covering map. Then p is the universal covering of X.*

We will have a universal covering for any connected manifold, as stated by the following theorem.

Theorem 3.8.3. *Suppose X is a connected manifold. Then there exists a connected, simply connected manifold $\tilde{X}$ and a covering map $p : \tilde{X} \to X$.*

Here p is the universal covering of X. If X is a Riemann surface, then we can find the universal covering of X, and by Theorem 3.7.6, the covering space $\tilde{X}$ is a Riemann surface.

We finish with some examples of universal coverings.

Example 3.8.4 (Universal covering of $\mathbb{C}^*$)**.** *In Example 3.7.16 we showed that* $\exp : \mathbb{C} \to \mathbb{C}^*$ *is a covering of* $\mathbb{C}^*$*. We now want to show that* $\exp : \mathbb{C} \to \mathbb{C}^*$ *is the universal covering of* $\mathbb{C}^*$*. Since C^* is connected,* $\mathbb{C}$ *is simply connected, and* $\exp : \mathbb{C} \to \mathbb{C}^*$ *is a covering map, then by Theorem 3.8.2* $\exp : \mathbb{C} \to \mathbb{C}^*$ *is the universal covering of* $\mathbb{C}^*$*.*

Example 3.8.5 (Universal covering of Torus, Cylinder and Plane)**.** *If we have the complex plane* $\mathbb{C}$*, the complex cylinder* $\mathbb{C} \times \mathbb{S}^1$*, and the complex torus* $\mathbb{T}$*,* $\mathbb{C}$ *will be the universal covering. This is because both the torus and the cylinder can be obtained as quotients of the plane, but* $\mathbb{C}$ *is not a quotient of either* $\mathbb{T}$ *or* $\mathbb{C} \times \mathbb{S}^1$*. Figure 3.11 shows how the spaces are quotients of each other.*

$$\begin{tikzcd}
& \mathbb{T} & \\
\mathbb{C} \arrow[ur, "/\{\mathbb{Z} \times \mathbb{Z}\}"] \arrow[rr, "/\mathbb{Z}"] & & \mathbb{C} \times \mathbb{S}^1 \arrow[ul, "/\mathbb{Z}"']
\end{tikzcd}$$

Figure 3.11: Universal covering maps

Chapter 4

Analytic Continuation

This chapter considers analytic continuation. Analytic continuation is important for Riemann surfaces because, among other things, the analytic continuation can be viewed as a Riemann surface with some functions and a covering map. The structure of this chapter is as follows. The first section introduces some necessary terminology for analytic continuation. Section 4.2 covers the subject of analytic continuation, and the final section, Section 4.3, looks at some results on the torus.

4.1 Sheaves

Sheaves are essential to define analytic continuation. This section, covering sheaves, mostly follows Forster's [For99, Chapter 1.6] definitions but starts with some definitions from Conway changed to be defined on a Riemann surface X instead of just $\mathbb{C}$.

Definition 4.1.1 (Function element). *[Con78, Definition 9.2.1] Let X be a Riemann surface, G a non-empty connected open subset of X, and let $f : G \to X$ be a holomorphic mapping on G. Then we call the pair (f, G) a function element.*

Definition 4.1.2 (Germ). *[Con78, Definition 9.2.1] For a function element (f, G) and an inner point $z \in G$, we define the germ of f as $[f]_z = \{(g, G_g) : f|_V = g|_V$ for some neighborhood V of $z\}$*

We will later introduce Forster's definition of a germ, which relies on first having defined sheaves, but we will continue to use this notation for the germ. The following proposition follows from the definition.

Proposition 4.1.3.

(i) $g \in [f]_z \iff f \in [g]_z$

(ii) *If $(g_1, G_1),\ (g_2, G_2) \in [f]_z$ then $g_1 \equiv g_2$ on the connected part of $G_1 \cap G_2$ that contains the point z.*

We now introduce a sheaf, but first, the more general case.

Definition 4.1.4 (Presheaf)**.** *Suppose X is a topological space and $\mathcal{I}$ is the system of open sets in X. A presheaf of abelian groups on X is a pair $(\mathscr{F}, p)$ consisting of*

(i) *a family. $\mathscr{F} = (\mathscr{F}(V))_{U \in \mathcal{I}}$ of abelian groups,*

(ii) *a family $p = \left(p_V^U\right)_{U,\ V \in \mathcal{I},\ V \subset U}$ of group homeomorphisms*

$$p_V^U : \mathscr{F}(U) \to \mathscr{F}(V)\ , \text{ where } V \text{ is open in } U,$$

with the following properties:

$$p_U^U = id_{\mathscr{F}(U)} \text{ for every } U \in \mathcal{I}$$

$$p_W^V \circ p_V^U = p_W^U \text{ for } W \subset V \subset U$$

Generally one just writes $\mathscr{F}$ instead of $(\mathscr{F}, p)$. The homeomorphism p_V^U are called restriction homeomorphisms. Instead of $p_V^U(f)$ for $f \in \mathscr{F}(V)$ one writes just $f|_V$. Analogous to presheaves of abelian groups, one can also define presheaves of vector spaces, rings, and sets, among other things.

Definition 4.1.5 (Sheaf)**.** *A presheaf $\mathscr{F}$ on a topological space X is called a sheaf if, for every open set $V \subset X$ and every family of open subsets $U_i \subset U, i \in I$, such that $U = \bigcup_{i \in I} U_i$ the following conditions, called the Sheaf Axioms, are satisfied:*

(i) *If $f,\ g \in \mathscr{F}(V)$ are elements such that $f|_{U_i} = g|_{U_i}$ for every $i \in I$, then f= g.*

(ii) *Given elements $f_i \in \mathscr{F}(U_i),\ i \in I$, such that*

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \text{ for all } i,\ j \in I,$$

then there exists an $f \in \mathscr{F}(V)$ such that $f|_{U_i} = f_i$ for every $i \in I$.

Example 4.1.6 (Sheaf of Holomorphic Functions)**.** *Let X be a Riemann surface and $U \subset X$ be open. The sheaf of holomorphic functions $\mathcal{O}$ is a sheaf defined by*

$$\mathcal{O}(U) = \{f : U \to \mathbb{C} : f \text{ is holomorphic}\}$$

and the restriction map

$$p_V^U : \mathcal{O}(U) \to \mathcal{O}(V), \quad V \subset U \text{ open}$$

which is the usual restriction map of the domain of definition. Similarly, the sheaf of meromorphic functions is defined by

$$\mathscr{M}(U) = \{g : U \to \mathbb{C} : g \text{ is meromorphic}\}$$

and the restriction maps

$$p_V^U : \mathscr{M}(U) \to \mathscr{M}(V), \quad V \subset U \text{ open}$$

are again the usual restriction map of the domain of definition.

Now we only need one more definition before defining a germ again.

Definition 4.1.7 (Stalk of a Presheaf)**.** *Define an equivalence relation $\sim_a$ in the following way: Two elements $f \in \mathscr{F}(U)$ and $g \in \mathscr{F}(V)$ are related $f \sim_a g$ precisely if there exists an open set W with $a \in W \subset U \cap V$ such that $f|_W = g|_W$, which is equivalent to $[f]_a = [g]_a$. The set of all equivalence classes is given by*

$$\mathscr{F}_a := \big(\bigsqcup_{a \in U} \mathscr{F}(U) \big) / \sim_a$$

and is called the stalk of $\mathscr{F}$ at the point a.

As commented on in the definition, we recognize Conway's condition for being in the germ of f at a as the equivalence relation $\sim_a$. This makes the germ of f at a the equivalence class of f with respect to $\sim_a$, as stated in the following definition.

Definition 4.1.8 (Germ)**.** *For any open neighborhood U of a, let*

$$\rho_a : \mathscr{F}(U) \to \mathscr{F}_a$$

be the mapping which assigns to each element $f \in \mathscr{F}(U)$ its equivalence class modulo $\sim_a$. One calls $[f]_a = \rho_a(f)$ the germ of f at a.

We now write the stalk as:

$$\mathscr{F}_a := \big(\bigsqcup_{a \in U} \mathscr{F}(U) \big) / \sim_a = \{[f]_a : f \in \mathscr{F}(U)\}$$

Example 4.1.9 (The Stalk of $\mathcal{O}$)**.** *The sheaf of holomorphic functions $\mathcal{O}$ on X has the stalk $\mathcal{O}_a$ for $a \in X$.*

$$\mathcal{O}_a := \big(\bigsqcup_{a \in U} \mathcal{O}(U) \big) / \sim_a = \{[f]_a : f \in \mathcal{O}(U)\}$$

A germ $\varphi \in \mathcal{O}_a$ can be represented as a Taylor series. This is because if $f \in \varphi$, then f is a holomorphic function in an open neighborhood around a, which means that f has a Taylor series around a. If f and g both are holomorphic in a neighborhood around a, then $f \sim_a g$ if and only if they have the same Taylor series around a. Hence there is an isomorphism between the stalk $\mathcal{O}_a$ and the ring of all convergent Taylor series. Similarly, $\mathscr{M}_a$ is isomorphic to the ring of all convergent Laurent series.

We want to work with topological spaces, so we construct a topological space associated to a presheaf. Suppose X is a topological space and $\mathscr{F}$ is a presheaf on X. Let

$$|\mathscr{F}| := \bigsqcup_{x \in X} \mathscr{F}_x$$

be the disjoint union of all the stalks. Denote by $p : |\mathscr{F}| \to X$ the mapping which assigns to each element $\varphi \in \mathscr{F}_x$ the point x.

$$p : |\mathscr{F}| \to X, \quad [f]_z \mapsto z$$

Now introduce a topology on $|\mathscr{F}|$ as follows: For any open subset $U \subset X$ and an element $f \in \mathscr{F}(U)$, let

$$[U, f] := \{[f]_x : x \in U\} \subset |\mathscr{F}|$$

Hence $|\mathscr{F}|$ is a topological space associated to the presheaf $\mathscr{F}$. The following theorem gives nice results on $|\mathscr{F}|$.

Theorem 4.1.10. *The system $\mathscr{B}$ of all sets $[U, f]$, where U is open in X and $f \in \mathscr{F}(U)$, is a basis for a topology on $|\mathscr{F}|$. The projection*

$$p : |\mathscr{F}| \to X, \quad [f]_z \mapsto z$$

is a local homeomorphism.

This is important because it gives us a topology on $|\mathscr{F}|$, and from the definition of a covering map, this also makes p a covering map and $|\mathscr{F}|$ a covering space of X. In particular, it gives us a topology on $|\mathcal{O}|$ and makes it a covering space.

The following desired property of a presheaf is closely related to the Identity Theorem [Theorem 3.4.6].

Definition 4.1.11. *A presheaf $\mathscr{F}$ on a topological space X is said to satisfy the Identity Theorem if the following holds. If $U \subset X$ is a domain and $f,\ g \in \mathscr{F}(U)$ are elements whose germs $[f]_a$ and $[g]_a$ coincide at a point $a \in U$, then f= g. That is*

$$[f]_a = [g]_a\,,\ a \in U \implies f|_U = g|_U$$

The converse is always true, as per the definition of a germ. That is

$$f|_U = g|_U \implies [f]_a = [g]_a\,,\ for\ all\ a \in U$$

The sheaves $\mathcal{O}$ and $\mathscr{M}$ of holomorphic and meromorphic functions satisfy the Identity Theorem. This is one of the reasons that we want to work with holomorphic and meromorphic functions.

We finish this section with the following theorem concerning Hausdorff topological spaces.

Theorem 4.1.12. *Suppose X is a locally connected Hausdorff space and $\mathscr{F}$ is a presheaf on X, which satisfies the Identity Theorem. Then the topological space $|\mathscr{F}|$ is Hausdorff.*

Proof. Suppose $\varphi_1,\ \varphi_2 \in |\mathscr{F}| = \{[f]_z : z \in X\}$ and $\varphi_1 \neq \varphi_2$. Let

$$p : |\mathscr{F}| \to X, \qquad [f]_z \mapsto z$$

be a mapping. We then have two cases.

(i) $p(\varphi_1) = z_1 \neq z_2 = p(\varphi_2)$. Since X is Hausdorff, there exist disjoint neighborhoods V_1 and V_2 of z_1 and z_2, respectively. Then $p^{-1}(V_1)$ and $p^{-1}(V_2)$ are disjoint neighborhoods of φ_1 and φ_2 respectively.

(ii) $p(\varphi_1) = p(\varphi_2) = z$. Let $f_1 \in \mathscr{F}(U_1)$ and $f_2 \in \mathscr{F}(U_2)$ such that $[f_1]_z = \varphi_1$ and $[f_2]_z = \varphi_2$ respectively, where U_1 and U_2 are open neighborhoods of z. Let U be a connected set such that $z \in U \subset U_1 \cap U_2$. Then $\tilde{V}_1 = [U, f_1]$ and $\tilde{V}_2 = [U, f_2]$ are open neighborhoods of φ_1 and φ_2 respectively. We will now show that $\tilde{V}_1 \cap \tilde{V}_2 = \emptyset$ by contradiction.
Assume there exists a $\psi \in \tilde{V}_1 \cap \tilde{V}_2$. If $p(\psi) = w \in U$, we have $\psi = [f_1]_w = [f_2]_w$. Then, since $\mathscr{F}$ satisfies the Identity theorem, we get that $f_1|_U = f_2|_U$, which implies $[f_1]_z = [f_2]_z$. This contradicts $[f_1]_z = \varphi_1 \neq \varphi_2 = [f_2]_z$

□

This theorem can be used to show that the topological space $|\mathcal{O}|$ is Hausdorff. Let X be a Riemann surface; it is then a connected Hausdorff space, and since the sheaf $\mathcal{O}$ on X satisfies the Identity Theorem, $|\mathcal{O}|$ is Hausdorff.

4.2 Analytic Continuation

Now analytic continuations on Riemann surfaces are considered. As in the last section, Forster [For99, Chapter 1.7] is mainly followed, and it starts with some definitions from Conway, changed to be defined on a Riemann surface X.

Definition 4.2.1 (Analytic continuation along a path)**.** *[Con78, Definition 9.2.2] Let X be a Riemann surface and $\gamma : [0,1] \to X$ be a path. Suppose that for each $t \in [0,1]$ there is a function element (f_t, D_t) such that:*

(i) $\gamma(t) \in D_t$

(ii) *For each $t \in [0,1]$ there is a $\delta > 0$ such that $|s - t| < \delta$ implies $\gamma(s) \in D_t$ and*

$$[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$$

Then (f_1, D_1) is the analytic continuation of (f_0, D_0) along the path γ. We also say (f_1, D_1) is obtained from (f_0, D_0) by analytic continuation along γ.

In the second condition, there will always be a δ such that $|s-t| < \delta$ implies $\gamma(s) \in D_t$, since γ is continuous and D_t is open. We want the rest of the statement to hold because that gives us a function element (f_s, D_s) such that $f_s = f_t$ on $D_t \cap D_s$. We recognize this as the analytic continuation of f_t into D_s. Since t is arbitrary, we can repeat this process along the whole path γ by letting our new t be our previous s.

The following proposition states that the analytic continuation of a function element along a path is unique.

Proposition 4.2.2. *[Con78, Proposition 9.2.4] Let X be a Riemann surface and $\gamma : [0,1] \to X$ be a path from a to b. Let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ and $\{(g_t, B_t) : 0 \leq t \leq 1\}$ be analytic continuations along γ such that $[f_0]_a = [g_0]_a$. Then $[f_1]_b = [g_1]_b$.*

Definition 4.2.3 (Analytic Continuation of a Germ)**.** *[Con78, Definition 9.2.6] Let X be a Riemann surface and $\gamma : [0,1] \to X$ be a path from a to b. If $\{(f_t,\ G_t) : t \in [0,\ 1]\}$ is an analytic continuation along γ then the germ $[f_1]_b$ is the analytic continuation of $[f_0]_a$ along γ.*

Definition 4.2.4 (Complete Analytic Function)**.** *[Con78, Definition 9.2.7] If (f, G) is a function element then the collection of germs*

$$\mathcal{F} = \{[g]_b : [g]_b \ \textit{is the analytic continuation of}\ [f]_a \ \textit{along some path}\ \gamma\}$$

is called the complete analytic function obtained from (f, G). $\mathcal{F}$ is called a complete analytic function if there is a function element (f, G) such that $\mathcal{F}$ is the complete analytic function obtained from (f, G).

If $\mathcal{F}$ is the complete analytic function associated with (f, G), then $[f]_z \in \mathcal{F}$ for all $z \in G$.

Consider the set $\mathcal{R} = \{(z, [g]_z) : [g]_z \in \mathcal{F}\}$. It lets us define $\mathcal{F}$ as a map

$$\mathcal{F} : \mathcal{R} \to X, \quad (z, [f]_z) \mapsto f(z)$$

The pair $(\mathcal{R}, \rho)$, with $\rho : \mathcal{F} \to X, \quad (z, [f]_z) \mapsto z$, which Conway calls the Riemann surface of $\mathcal{F}$, is biholomorphic to the maximal analytic continuation (Y, p, f, b), which will be defined later in this section.

We will now look at how Forster defines analytic continuation. Suppose X is a Riemann surface. $u : [0,1] \to X$ is a curve and $a := u(0),\ b := u(1)$. The holomorphic function germ $\psi \in \mathcal{O}_b$ is said to result from the analytic continuation along the curve u of the holomorphic function germ $\varphi \in \mathcal{O}_a$ if the following holds. There exists a family $\varphi_t \in \mathcal{O}_{u(t)},\ t \in [0,1]$ of function germs with $\varphi_0 = \varphi$ and $\varphi_1 = \psi$ with the property that for every $\tau \in [0,1]$ there exists a neighborhood $T \subset [0,1]$ of τ, an open set $U \subset X$ with $u(T) \subset U$ and a function $f \in \mathcal{O}(U)$ such that

$$[f]_{u(t)} = \varphi_t, \text{ for every } t \in T$$

Here $[f]_{u(t)}$ is the germ of f at the point $u(t)$. Because of the compactness of $[0,1]$, this condition is equivalent to the following. There exists a partition $0 = t_0 < t_1 < ... < t_n - 1 < t_n = 1$ of the interval $[0,1]$, domains $U_i \subset X$ with $u([t_{i-1}, t_i]) \subset U_i$ and holomorphic functions $f_i \in \mathcal{O}(U_i)$ for $i = 1, ..., n$ such that:

(i) φ is the germ of f_1 at the point a and ψ is the germ of f_n at the point b.

(ii) $f_i|_{V_i} = f_{i+1}|_{V_i}$ for $i = 1, ..., n-1$, where V_i denotes the connected component of $U_i \cap U_{i+1}$ containing the point $u(t_i)$

Figure 4.1 illustrates the analytic continuation of φ along the curve u.

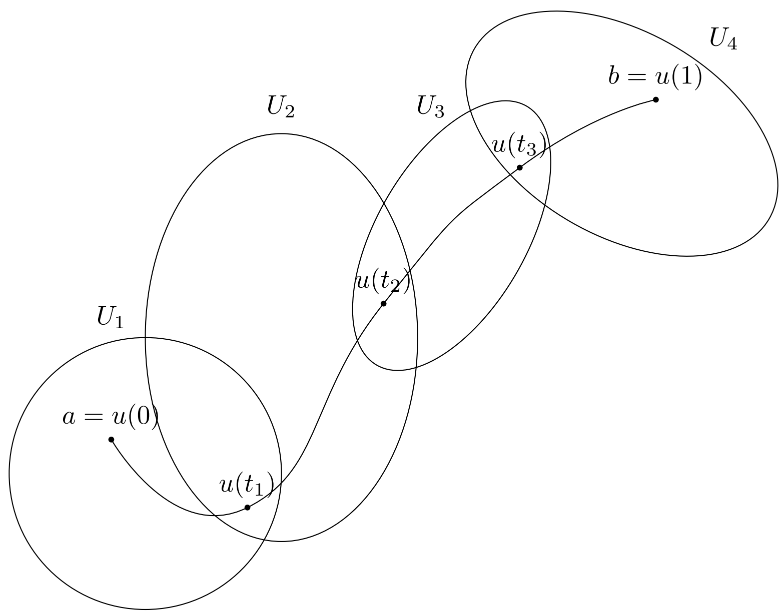

Figure 4.1: Analytic continuation along a curve

The following theorem states the analytic continuation along homotopic curves is unique.

Theorem 4.2.5 (Monodromy Theorem)**.** *Suppose X is a Riemann surface and $u_0,\ u_1 : [0,1] \to X$ are homotopic curves from a to b. Suppose $u_s, 0 \leq s \leq 1$, is a deformation of u_0 into u_1 and $\varphi \in \mathcal{O}_a$ a is a function germ which admits an analytic continuation along every curve u_s Then the analytic continuations of φ along u_0 and u_1 yield the same function germ $\psi \in \mathcal{O}_b$.*

Corollary 4.2.6. *Suppose X is a simply connected Riemann surface, $a \in X$ and $\varphi \in \mathcal{O}_a$ is a function germ which admits an analytic continuation along every curve starting at a. Then there exists a globally defined holomorphic function $f \in \mathcal{O}(X)$ such that $[f]_a = \varphi$*

Another way of looking at analytic continuation is in terms of a Riemann surface with some functions and a covering map. To do this, we first need to define some maps.

Definition 4.2.7 (Pullback and Pushforward)**.** *Suppose X and Y are Riemann surfaces, and suppose $F : Y \to X$ is an unbranched holomorphic map. We call the map*

$$F^* : \mathcal{O}(X) \to \mathcal{O}(Y), \quad f \mapsto f \circ F$$

the pullback.

The pullback is generally not invertible, but suppose $p : Y \to X$ is an unbranched holomorphic map. Since p is unbranched, it is locally biholomorphic. For each $y \in Y$ introduce an isomorphism

$$p^* : \mathcal{O}_{p(y)} \to \mathcal{O}_y, \quad [f]_{p(y)} \mapsto [f \circ p]_y$$

Since p is locally biholomorphic, it is locally invertible, and thus p^* is invertible. Let

$$p_* : \mathcal{O}_y \to \mathcal{O}_{p(y)}, \quad [f]_y \mapsto \left[f \circ p^{-1}\right]_{p(y)}$$

be the inverse, which we call the *pushforward*.

The following definition of analytic continuation is the one we will be using for the rest of the chapter.

Definition 4.2.8 (Analytic Continuation)**.** *Suppose X is a Riemann surface, $a \in X$ is a point, and $\varphi \in \mathcal{O}_a$ is a function germ. A quadruple (Y, p, f, b) is called an analytic continuation of φ if:*

(i) *Y is a Riemann surface and $p : Y \to X$ is an unbranched holomorphic map.*
(ii) *f is a holomorphic function on Y, that is, $f \in \mathcal{O}(Y)$.*
(iii) *b is a point of Y such that $p(b) = a$ and $p_*([f]_b) = \varphi$.*

The holomorphic map p does not have to be surjective, as shown by the following example.

Example 4.2.9 (Analytic Continuation on Riemann Sphere)**.** *Let* $X = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ *be the Riemann sphere, shown earlier in Figure 3.4.*

$$U = \mathbb{C} \subset \mathbb{P}^1, \qquad g_0 : U \to \mathbb{C}, \qquad y_0(z) = z, \qquad \varphi = [g_0]_0$$

We can make an analytic continuation (Y, p, f, b) with

$$Y = \mathbb{C}, \qquad f(z) = z, \qquad b = 0$$

$$p : \mathbb{C} \to \mathbb{P}^1, \quad z \mapsto z$$

which is not surjective, since $\infty \notin p(\mathbb{C})$*.*

Definition 4.2.10 (Maximal Analytic Continuation)**.** *An analytic continuation* (Y, p, f, b) *of* φ *is said to be maximal if it has the following universal property. If* (Z, q, g, c) *is any other analytic continuation of* φ*, then there exists a fiber-preserving holomorphic mapping* $F : Z \to Y$ *such that* $F(c) = b$ *and* $F^*(f) = g$*. See the diagram in Figure 4.2.*

A maximal analytic continuation is unique up to isomorphism.

$$\begin{tikzcd}
& \mathbb{C} & \\
Z \arrow[ur, "g"] \arrow[rr, "F"] \arrow[dr, "q"'] & & Y \arrow[ul, "f"'] \arrow[dl, "p"] \\
& X &
\end{tikzcd}$$

Figure 4.2: Maximal analytic covering

Although the diagram looks similar to the diagram of the universal covering, Figure 3.10, they are not the same and should not be confused. The Riemann surface in the maximal analytic continuation differs from the Riemann surface, which is the universal covering space. We will see this in Example 4.3.5.

Lemma 4.2.11. *Suppose* X *is a Riemann surface,* $a \in X$*,* $\varphi \in \mathcal{O}_a$*, and* (Y, p, f, b) *is an analytic continuation of* φ*. If* $v : [0, 1] \to Y$ *is a curve with* $v(0) = b$ *and* $v(1) =: y$*, then the function germ* $\psi := p_*([f]_y) \in \mathcal{O}_{p(y)}$ *is an analytic continuation of* φ *along the curve* $u := p \circ v$*.*

We can also write the analytic continuation of φ along $u(t) = p \circ v(t)$ as

$$\varphi_t = p_*([f]_{v(t)})$$

The following theorem gives the existence of a maximal analytic continuation.

Theorem 4.2.12. *Suppose X is a Riemann surface, $a \in X$ and $\varphi \in \mathcal{O}_a$ is a holomorphic function germ at the point a. Then there exists a maximal analytic continuation (Y, p, f, b) of φ.*

Proof. First, we want to show that (Y, p, f, b) is an analytic continuation of φ. We must show the following three points from the definition to do this.

(i) *Y is a Riemann surface and p is an unbranched holomorphic map.*
Let Y be the connected component of $|\mathcal{O}|$ containing φ, where $|\mathcal{O}| = \bigsqcup_{a \in X} \mathcal{O}_a$.
By Theorem 4.1.10, the open sets $[U, f] = \{[f]_z : z \in U \textit{ open}\}$ make up the basis of the topology on $|\mathcal{O}|$ and the map $p : |\mathcal{O}| \to X$ is a local homeomorphism. Thus the restriction

$$p : Y \to X, \quad [f]_z \mapsto z$$

is also a local homeomorphism. Since $|\mathcal{O}|$ is Hausdorff, we have by Theorem 3.7.6 that there is a unique complex structure on Y, such that p is holomorphic. The complex structure also makes Y a Riemann surface.

(ii) *f is a holomorphic function on Y.*
For all $\eta = [g]_{p(\eta)} \in Y,\ g \in \mathcal{O}(X)$ define

$$f : Y \to \mathbb{C}, \quad \eta \mapsto g(p(\eta))$$

f is holomorphic, because locally around η we have

$$f \circ (\zeta \circ p)^{-1} = g \circ \zeta^{-1}$$

which is holomorphic, since both g and ζ^{-1} are holomorphic.

(iii) *b is a point of Y such that $p(b) = a$ and $p_*([f]_b) = \varphi$.*
Let $b = \varphi = [g]_a \in \mathcal{O}_a = Y$, then locally around b, f is given by $f(\tilde{b}) = g(p(\tilde{b}))$, which is equivalent to $f([g]_{\tilde{a}}) = g(\tilde{a})$, a

$$p_*([f]_b) = \left[f \circ p^{-1}\right]_a = [g]_a = \varphi$$

This shows that (Y, p, f, b) is an analytic continuation of φ. Now we want to show that it is a maximal analytic continuation.

Assume we have another analytic continuation (Z, q, g, c) and let F be a mapping defined in the following way. For all $\zeta \in Z$ let

$$F(\zeta) = q_*([g]_\zeta) = \left[g \circ q^{-1}\right]_{q(\zeta)} \in \mathcal{O}_{q(\zeta)}$$

We want to show that F satisfies the following conditions from the definition of a maximal analytic continuation.

(iv) $F : Z \to Y$.
Let $\zeta \in Z$. From the definition of an analytic continuation, we can get $q_*([g]_\zeta)$ as an analytic continuation of φ. This implies $q_*([g]_\zeta) \in Y$, thus F is a mapping from Z to Y.

(v) *F is holomorphic.*
Choose a chart $\psi : U \subset X \to \mathbb{C}$ around the point $q(\zeta)$. Then we locally have charts $\psi_1 = \psi \circ q$ around $\zeta \in Z$ and $\psi_2 = \psi \circ p$ around $F(\zeta) \in Y$. This gives us

$$\psi_2 \circ F \circ \psi_1^{-1}(z) = \psi(p(\left[g \circ q^{-1}\right]_{q(q^{-1}(\psi^{-1}(z)))})) = \psi(\psi^{-1}(z)) = z$$

which is holomorphic. Thus F is holomorphic.

(vi) *F is fiber-preserving, that is $p(F(\zeta)) = q(\zeta)$.*

$$p(F(\zeta)) = p(q_*([g]_\zeta)) = p(\left[(q^{-1})^*(g)\right]_{q(\zeta)}) = q(\zeta)$$

by the definitions of q_* and p.

(vii) $F(c) = b$.
$q(c) = a$ and from the definition of analytic continuation $q_*([g]_c) = \varphi \in \mathcal{O}_a$. This implies that $F(c) = q_*([g]_c) = \varphi = b$.

(viii) $F^*(f) = f \circ F = g$.

$$F^*(f) = f(F(\zeta) = f(q_*([g]_\zeta)) = f(\left[g \circ q^{-1}\right]_{q(\zeta)}) = g(q^{-1}(q(\zeta))) = g(\zeta) = g$$

since $f([g]_{\tilde{a}}) = g(\tilde{a})$

$\square$

We finish this section by looking at how the graph of a function relates to the maximal analytic continuation.

Proposition 4.2.13. *[Con78, Proposition 9.6.19] Suppose $a \in X$, $\varphi = [f]_a \in \mathcal{O}_a$ and let $(Y, p, \hat{f}, b)$ be a maximal analytic continuation of φ. If g is an entire function, such that $g(f(z)) = z$, then Y is biholomorphic to $graph(g) = \{(z, g(z)) \in \mathbb{C}^2\}$. The map is given by*

$$F : Y \to graph(g), \quad [h]_c \mapsto (h(c), c) = (c, g(c))$$

Here Y is the connected component in $|\mathcal{O}| = \bigsqcup_{a \in X} \mathcal{O}_a$.

This proposition has been rewritten using Forster's notation.

In Section 3.1, we looked at the functions $\log z$ and $z^{\frac{1}{2}}$. There our focus was on constructing their Riemann surfaces. Here we use the preceding theorem to study their maximal analytic continuations.

Example 4.2.14 (An Unambiguous Square root)**.** *Let $f(z) = \sqrt{z}$. Then by the map F from the preceding proposition, we get*

$$\begin{array}{lcl} Y & \xrightarrow{F} & graph(g) \\ p([h]_c) = c & & \underline{p}(z, z^2) = z^2 \\ \hat{f}([h]_c) = h(c) & & \underline{\hat{f}}(z, z^2) = z \\ b = \varphi = [f]_a & & \underline{b} = (a, a^2) \end{array}$$

The diagram in Figure 4.3 shows the relation between the functions $\underline{p}$, $\underline{\hat{f}}$, and f for the square root function.

$$\begin{tikzcd} (\zeta, \zeta^2) \arrow[d, "\underline{p}"'] \arrow[dr, "\underline{\hat{f}}"] & \\ z = \zeta^2 \arrow[r, "f"'] & \zeta \end{tikzcd}$$

Figure 4.3: Square root

We want $f(z) = \sqrt{z}$ to be the germ of a function defined independent of the continuation. Let

$z = \zeta^2$ then

$$z^{\frac{1}{2}} = \{\underline{\hat{f}}(\zeta, \zeta^2) : \ \underline{p}(\zeta, \zeta^2) = z\} = \{\zeta : \ \zeta^2 = z\}$$

To see why this formulation of the square root is desirable, let us look at the function at the point $z = i$. $f(i) = \pm\frac{1}{\sqrt{2}}(1+i)$ is multi-valued and thus

$$\underline{p}^{-1}(i) = \{(\pm i^{\frac{1}{2}}, i)\} = \{(\frac{1}{\sqrt{2}}(1+i), i), (-\frac{1}{\sqrt{2}}(1+i), i)\}$$

gives us two points on graph(g). By using $\underline{\hat{f}}$, those two points now give different values.

$$\underline{\hat{f}}(\frac{1}{\sqrt{2}}(1+i), i) = \frac{1}{\sqrt{2}}(1+i)$$

$$\underline{\hat{f}}(-\frac{1}{\sqrt{2}}(1+i), i) = -\frac{1}{\sqrt{2}}(1+i)$$

Hence $\underline{\hat{f}}(\zeta, \zeta^2)$, $(\zeta, \zeta^2) \in \underline{p}^{-1}(i)$ expresses the multi-value of $i^{\frac{1}{2}}$, while it itself being single valued.

Example 4.2.15 (An Unambiguous Logarithm)**.** *Let $f(z) = Log\ z$. Then similarly to the last example, we get*

$$\begin{array}{lcl}
Y & \xrightarrow{F} & graph(g) \\
p([h]_c) = c & & \underline{p}(z, e^z) = e^z \\
\hat{f}([h]_c) = h(c) & & \underline{\hat{f}}(z, e^z) = z \\
b = \varphi = [f]_a & & \underline{b} = (a, e^a)
\end{array}$$

The diagram in Figure 4.4 shows the relation between the functions $\underline{p}$, $\underline{\hat{f}}$, and f for the log function.

$$\begin{tikzcd}
(\zeta, e^\zeta) \arrow[d, "\underline{p}"'] \arrow[dr, "\underline{\hat{f}}"] & \\
z = e^\zeta \arrow[r, "f"'] & \zeta
\end{tikzcd}$$

Figure 4.4: Logarithm

The function $f(z) = Log\ z$ does not have an analytic continuation independent of the path, but

$g(z) = e^z$ *does. If we let* $z = e^\zeta$*, we can define an unambiguous logarithm function*

$$\log z = \{\underline{\hat{f}}(\zeta, e^\zeta) :\ \underline{p}(\zeta, e^\zeta) = z\} = \{\zeta :\ e^\zeta = z\}$$

Let us look at $z = i$*. Then* $f(i) = \{\frac{\pi}{2} + 2\pi n :\ n \in \mathbb{Z}\}$ *has infinitely many values and*

$$\underline{p}^{-1}(i) = \{(\frac{\pi}{2} + 2\pi n, i) :\ n \in \mathbb{Z}\}$$

gives us infinite points on graph(g). Those points now give different values using $\underline{\hat{f}}$*.*

$$\underline{\hat{f}}(\frac{\pi}{2}, i) = \frac{\pi}{2}$$

$$\underline{\hat{f}}(\frac{\pi}{2} + 2\pi n, i) = \frac{\pi}{2} + 2\pi n$$

We now write $\log i = \{\underline{\hat{f}}(\zeta, e^\zeta) :\ (\zeta, e^\zeta) \in \underline{p}^{-1}(i)\} = \{\frac{\pi}{2} + 2\pi n :\ n \in \mathbb{Z}\}$*.*

4.3 The Complex Torus

In Example 3.3.4, the torus $\mathbb{T}$ was defined, and a complex structure on it was introduced to make it a Riemann surface. In this final section, some more results on the torus are considered. The results are taken from Ahlfors *Complex Analysis* [Ahl79] unless stated otherwise.

We start by looking at periodic functions on $\mathbb{C}$ before we show how they relate to the torus.

Definition 4.3.1 (Periodic Function)**.** *A function f(z) is said to be periodic with period* $\omega \neq 0$ *if* $f(z + \omega) = f(z)$ *for all z.*

Definition 4.3.2 (Doubly Periodic Function)**.** *A function f(z) is said to be doubly periodic with periods* ω_1 *and* ω_2 *if* $f(z + \omega_1) = f(z + \omega_2) = f(z)$ *for all z. Where* $\omega_2 \neq 0$ *and* $Im(\frac{\omega_1}{\omega_2}) \neq 0$*, that is* ω_1*,* ω_2 *are linearly independent.*

Doubly periodic meromorphic functions are called elliptic, but the terms are equivalent for our purposes. Elliptic functions are an important topic in mathematics but will only be covered briefly here.

Since ω_1 and ω_2 are linearly dependent, they make a lattice

$$\Gamma = \{n_1\omega_1 + n_1\omega_2 :\quad n_1, n_2 \in \mathbb{Z}\}$$

The statement $f(z + \omega_1) = f(z + \omega_2) = f(z)$ is then equivalent to saying $z \in \mathbb{C}/\Gamma$. Thus any function on the torus, $f : \mathbb{C}/\Gamma \to \mathbb{C}$, is doubly periodic.

The following theorem shows why, when it comes to doubly periodic functions, we are mainly interested in meromorphic functions.

Theorem 4.3.3. *[For99, Theorem 1.2.13]*
Every doubly periodic holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant. Every non-constant doubly periodic meromorphic function $f : \mathbb{C} \to \mathbb{P}^1$ attains every value $c \in \mathbb{P}^1$.

A classic example of an elliptic function is the Weierstrass $\wp$-function made by Weierstrass. In the expression of this function, we sum over the lattice

$$\Gamma = \{n_1\omega_1 + n_1\omega_2 : \quad n_1, n_2 \in \mathbb{Z}\}$$

Definition 4.3.4 (Weierstrass $\wp$-Function)**.**

$$\wp(z) = \frac{1}{z} + \sum_{\omega \neq 0} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big) \qquad (\omega \in \Gamma)$$

We finish with an example.

Example 4.3.5 (Analytic Continuation on Torus)**.** *Let $X = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, isomorphic to the torus, be a Riemann surface and let $U = \{z + \mathbb{Z} + i\mathbb{Z} : |z| < \frac{1}{4}\}$ be a subset of X. Define a function $g(z + \mathbb{Z} + i\mathbb{Z}) = \sin(2\pi z)$ on U. The germ $\varphi = [g]_{0+\mathbb{Z}+i\mathbb{Z}}$ is represented by the series*

$$\varphi = \sum_{n=0}^{\infty} \frac{(2\pi)^{2n+1}}{(2n+1)!} z^{2n+1}$$

We look at the part of X which surrounds U. Let $\psi_1,\ \psi_2$ and ψ_3 be the result of analytic continuation along the curves $u_1,\ u_2$ and u_3 shown in Figure 4.5. Then $\psi_1 = \psi_2 = \varphi$, since U can be expanded to include u_1 and g is periodic along the real axis.

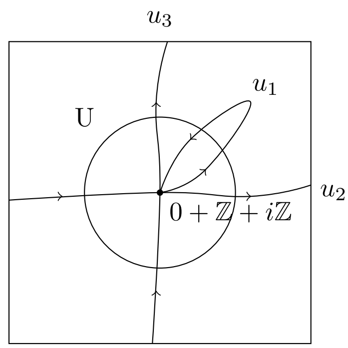

Figure 4.5: Curves u_1, u_2 and u_3

The same is not true for ψ_3 since g is not periodic along the imaginary axis. Here we have $\psi_3 = [g_3(z)]_0$ where $g_3(z+\mathbb{Z}+i\mathbb{Z}) = \sin(2\pi(z+i)))$. This gives the Taylor expansion

$$\psi_3 = \sum_{n=0}^{\infty} \frac{\cosh(2\pi)(-1)^n(2\pi)^{2n+1}}{(2n+1)!} z^{2n+1} + i \sum_{n=0}^{\infty} \frac{\sinh(2\pi)(-1)^n(2\pi)^{2n}}{(2n)!} z^{2n}$$

We can take an analytic continuation along u_3, but then we need a different function than for the analytic continuation along the other curves.

We now look at two analytic continuations. The first analytic continuation $(Z, q, \hat{g}, c)$ is given by
$Z = \mathbb{C}/\mathbb{Z}$*, which is isomorphic to the cylinder.*
$q(z+\mathbb{Z}) = z+\mathbb{Z}+i\mathbb{Z}$
$\hat{g} = (z+\mathbb{Z}) = \sin(2\pi z)$
$c = 0+\mathbb{Z}$
$q(c) = 0+\mathbb{Z}+i\mathbb{Z}$
The second analytic continuation (Y, p, f, b) is given by
$Y = \mathbb{C}$
$p(z) = z+\mathbb{Z}+i\mathbb{Z}$
$f(z) = \sin(2\pi z)$
$b = 0$

The analytic continuation $(Z, q, \hat{g}, c)$ is a maximal analytic continuation. Let $F : Y \to Z$, then $F^*(\hat{g}) = \hat{g} \circ F = f$ and $F(b) = F(0) = 0+\mathbb{Z} = c$. This shows that only $(Z, q, \hat{g}, c)$ can be a maximal analytic continuation of the two we have looked at. We will not be checking every other analytic continuation here.

In Example 3.8.5, we also considered the complex plane, cylinder, and torus. If we compare the examples, we see that in Example 3.8.5, the universal covering space of the three is the complex plane. This contrasts this example, where we have stated that the cylinder is the Riemann surface in the maximal analytic continuation. Thus as we stated in the previous section, the Riemann surface in the maximal analytic continuation differs from the Riemann surface, which is the universal covering space.

That concludes this introduction to Riemann surfaces.

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