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A robust linearization scheme for finite volume based discretizations for simulation of two-phase flow in porous media

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ABSTRACT

In this work we consider a mathematical model for two-phase flow in porous media. The fluids are assumed immiscible and incompressible and the solid matrix non-deformable. The mathematical model for the two-phase flow is written in terms of the global pressure and a complementary pressure (obtained by using the Kirchhoff transformation) as primary unknowns. For the spatial discretization, finite volumes have been used (more precisely the multi-point flux approximation method) and in time the backward Euler method has been employed. We present here a new linearization scheme for the nonlinear system arising after the temporal and spatial discretization. We show that the scheme is linearly convergent. Numerical experiments are presented that sustain the theoretical results.

1. Introduction

Water and soil pollution, CO_2 storage, enhanced oil recovery and nuclear waste management are typical examples of two-phase porous media flows with obvious high societal relevance. A crucial role in understanding two-phase flow in porous media is played by numerical simulations, including mathematical modeling and numerical methods.

Mathematical models of two-phase flow in porous media consist of coupled, nonlinear and possibly degenerate partial differential equations. This makes the design and implementation of efficient numerical schemes for two-phase flow in porous media a challenging task. Locally conservative discretizations such as finite volume [1–5] and mixed finite element [6,7] methods are popular spatial discretization as they alleviate many stability issues. Furthermore, often long time-scales are of interest in applications, so fully implicit temporal discretizations are, in general, preferred.

At each time step, the spatial and temporal discretizations thus lead to a large system of nonlinear equations. This system is usually solved by either Picard's method [8] or Newton's method [9,8,10–13]. The former is linearly convergent while the latter is quadratically convergent. The quadratic convergence of Newton's method comes at the price of only local convergence in solution space, however it remains a very powerful tool when applied to systems arising from discretization of parabolic equations. This is because, in this case, the starting iteration is chosen as the solution at the last time step and the

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initial error can be controlled. In this way, by adjusting the time step size one ensures that the starting iteration is in the convergence region of the method. In order to apply Newton's method to degenerate problems one needs a further regularization step [11–13], which may alter the quality of the solution (in particular the mass balance). Moreover, the time-step restriction depends on the mesh diameter and on the size of the regularization step, which may be relatively restrictive in practice (see [12,13] for *a priori* derived convergence conditions for the Newton method when applied to transport equations). Thus, we identify two main concerns of Newton's method: the need of regularization for degenerate problems, as well as the time-step constraint implicitly imposed by the convergence region.

A possible improvement to Newton's method for degenerate problems is the semismooth Newton method [14]. This method is of a higher algorithmic complexity and requires additional reformulation of the model by adding so called complementary conditions. The semismooth Newton method can be applied to two-phase flow or multicomponent transport with much better results compared to Newton's method (see [14–16]).

In this work we propose and analyze a new linearization scheme for finite volume discretization schemes for two-phase flow. The proposed scheme is a monotone fixed point iteration [17,18]. We show the linear convergence and robustness of the scheme, subject to a relatively mild restriction on the time step size but independent of grid size. Moreover, the scheme does not involve the calculation of derivatives, which is an advantage over both Picard and Newton methods. Both analysis and numerical experiments indicate that the new scheme is a valuable alternative to Picard or Newton-type methods for solving two-phase flow in porous media.

The paper is structured as follows. In Section 2, we present the two-phase model considered here, describe the linearization scheme and show its convergence. The numerical results are given in Section 3, which show the applicability of the method. The paper ends with some concluding remarks in Section 4.

2. Mathematical model and discretization

We consider a simplified mathematical model for two-phase flow in porous media. The fluids are immiscible and incompressible and the solid matrix is non-deformable. The formulation adopted here uses the global pressure and a complementary pressure (obtained by using the Kirchhoff transformation) as primary unknowns (see [19–21]). For simplicity of exposition, we assume spatially homogeneous relative permeability and capillary pressure functions, and no gravity.

Throughout this paper we use common notations from functional analysis. The domain $\Omega \subset \mathbb{R}^d$, d being the dimension of the space, is open, bounded and with a Lipschitz continuous boundary. By C we mean a positive constant, not depending on the unknowns or the discretization parameters.

2.1. Governing equations

Mass (volume) balance:

$$\frac{\partial s}{\partial t} + \nabla \cdot \vec{q}_w = f_1(s). \tag{1}$$

Fractional phase flux

$$\vec{q}_w = -k\nabla\theta + f_w(s)\vec{q}. \tag{2}$$

Conservation of total phase volumes

$$\nabla \cdot \vec{q} = f_2(s). \tag{3}$$

Darcy's law for total flow

$$\vec{q} = -\lambda(s)k\nabla p. \tag{4}$$

The equations hold true in $\Omega \times [0, T]$, with T denoting the final time. The system is closed by an invertible relationship between the saturation s and the complementary pressure θ, i.e. $s = s(\theta)$ and constitutive laws for the various functions f_α, $\alpha \in \{1, 2, w\}$. The model is a reformulation of the two-phase system

$$\frac{\partial(\phi\rho_\alpha s_\alpha)}{\partial t} + \nabla \cdot (\rho_\alpha \vec{q}_\alpha) = \Psi_\alpha, \quad \alpha = w, n, \tag{5}$$

$$\vec{q}_\alpha = -\frac{k_{r,\alpha}}{\mu_\alpha} k\nabla p_\alpha, \quad \alpha = w, n, \tag{6}$$

$$s_w + s_n = 1, \tag{7}$$

$$p_n - p_w = p^{cap}(s_w). \tag{8}$$

We denoted by w and n the wetting and the non-wetting phase, respectively. Here: the porosity ϕ, the densities ρ_w, ρ_n and the viscosities μ_w, μ_n are constants. Furthermore, there are no exchange terms between the phases, and the capillary pressure p^{cap} and the relative permeabilities $k_{r,w}$, $k_{r,n}$ are assumed known, monotone and Lipschitz continuous functions of the wetting phase saturation s_w. To transform the system (5)–(8) into the system (1)–(4) one introduces the global pressure

and a complementary pressure, defined by

$$p(\mathbf{x}, s) := p_n(\mathbf{x}) - \int_0^s f_w(\mathbf{x}, \xi)\frac{\partial p^{cap}}{\partial \xi}(\mathbf{x}, \xi)d\xi, \tag{9}$$

$$\theta(\mathbf{x}, s) := -\int_0^s f_w(\mathbf{x}, \xi)\lambda_n(\mathbf{x}, \xi)\frac{\partial p^{cap}}{\partial \xi}(\mathbf{x}, \xi)d\xi, \tag{10}$$

where $s := s_w$. The phase mobilities $\lambda_\alpha := \frac{k_{r,\alpha}}{\mu_\alpha}$, $\alpha = w, n$ and the fractional flow function $f_w := \frac{\lambda_w}{\lambda_w+\lambda_n}$ are given, Lipschitz continuous functions of s. For the details of the transformation please see [19–21].

The system (1)–(4) consists of two coupled nonlinear partial differential equations, one degenerate elliptic–parabolic and the other elliptic. Existence and uniqueness of a solution for the system (1)–(4) has been proved in [21]. Initial and boundary conditions complete the model.

2.2. Assumptions on the data

Throughout this paper we consider the following physically reasonable assumptions on the data.

(A1) $s(\cdot)$ is monotone increasing and Lipschitz continuous.

(A2) The permeability k is strictly positive, the total mobility $\lambda(\cdot)$ is Lipschitz continuous and there exists $a_\star, a^\star \in \mathbb{R}$ such that for all $y \in \mathbb{R}$

$$0 < a_\star \le \lambda(y) \le a^\star < \infty. \tag{11}$$

(A3) $f_1(\cdot), f_2(\cdot)$ and $f_w(\cdot)$ are Lipschitz continuous and bounded functions.

2.3. Temporal discretization

Let $N \ge 1$ be an integer giving the time step $\Delta t = T/N$, and let $t_n = n\Delta t$, for $n = 1, \ldots, N$ the discrete time points. For the time discretization of (1) we use the backward Euler method. This leads to a nonlinear system of equations. The variables in (2)–(4) are evaluated at the current time t^n, while (1) becomes

$$s^n + \Delta t(\nabla \cdot \vec{q}_w^n - f_1(s^n)) = s^{n-1}. \tag{12}$$

2.4. Iterative approach

The standard approach to solving the coupled equations resulting after applying the backward Euler time discretization to (1)–(4) is to apply Newton's method. As mentioned in the introduction, this has several drawbacks. First, the Jacobian matrix of the system needs to be assembled, and secondly, the convergence of the algorithm is not guaranteed when the initial guess is not *close enough* to the solution, which implies a restriction on the time step size. We consider therefore the applicability of a fixed point linearization method. Let $L > 0$ be a constant such that

$$L \ge \sup_\theta \frac{d}{d\theta}s(\theta). \tag{13}$$

At time t_n and for given $\theta^{n,i-1}$, θ^{n-1}, the iterate i is obtained by solving the following system of equations

$$L(\theta^{n,i} - \theta^{n,i-1}) + s^{n,i-1} + \Delta t(\nabla \cdot \vec{q}_w^{n,i} - f_1(s^{n,i-1})) = s^{n-1}, \tag{14}$$

$$\vec{q}_w^{n,i} = -k\nabla\theta^{n,i} + f_w(s^{n,i-1})\vec{q}^{n,i}, \tag{15}$$

$$\nabla \cdot \vec{q}^{n,i} = f_2(s^{n,i-1}), \tag{16}$$

$$\vec{q}^{n,i} = -\lambda(s^{n,i-1})k\nabla p^{n,i}, \tag{17}$$

where $s^{n,i-1} = s(\theta^{n,i-1})$ and $s^{n-1} = s(\theta^{n-1})$. Note that this system is decoupled, in the sense that Eqs. (16)–(17) and (14)–(15) can be solved sequentially. Thus, only one global (elliptic or parabolic) system needs to be solved at a time. We point out that the addition of the term $L(\theta^{n,i} - \theta^{n,i-1})$ is necessary to overcome the degeneracy of the problem. This term is essential for obtaining a robust, linear convergent numerical scheme.

2.5. Spatial discretization

Eqs. (14)–(17) have previously been solved in the context of mixed finite element approximations to the spatial derivatives. Here, in contrast, we consider a finite volume setting. We introduce a finite volume duplex $\mathcal{D} = (\mathcal{T}, \mathcal{F})$, representing the mesh Tessellation and Faces [22]:

- $\mathcal{T}$ is a non-overlapping partition of the domain Ω. Furthermore, let m_K denote the d-dimensional measure of $K \in \mathcal{T}$.
- $\mathcal{F}$ is a set of faces of the partitioning $\mathcal{T}$. Naturally, the faces must be compatible with the mesh, such that for all $K \in \mathcal{T}$ there exists a subset $\mathcal{F}_K \subset \mathcal{F}$ such that $\partial K = \cup_{\sigma\in\mathcal{F}_K}\sigma$.

Additionally, we define the following useful subset of the mesh duplex:

- For each face $\sigma \in \mathcal{F}$, we denote the neighboring cells $\mathcal{T}_\sigma$. Note that for all internal faces $\mathcal{T}_\sigma$ will contain exactly two elements, while it contains a single element when $\sigma \subset \partial\Omega$.

This is sufficient to provide an abstract definition of a finite volume method for Eqs. (14)–(17). With respect to Eqs. (14)–(17), we identify p and θ as cell variables and $q^n_{w\sigma} = \vec{q}_w \cdot \vec{\nu}$, $q_\sigma = \vec{q} \cdot \vec{\nu}$, as face variables, and denote the corresponding discrete spaces as $\mathcal{H}_\mathcal{T}$ and $\mathcal{H}_\mathcal{F}$. We denoted by $\vec{\nu}$ the outer unit normal.

A finite volume method is characterized by the existence of a discrete operator $\tilde{\nabla}\cdot : \mathcal{H}_\mathcal{F} \to \mathcal{H}_\mathcal{T}$. For $\vec{q} \in \mathcal{H}_\mathcal{F}$ and for each $K \in \mathcal{T}$, it is defined as

$$(\tilde{\nabla} \cdot \vec{q})_K = \frac{1}{m_K} \sum_{\sigma \in \mathcal{F}_K} \pm m_\sigma q_\sigma . \tag{18}$$

The choice of sign in Eq. (18) is determined by the convention on normal vectors for faces.

Various finite volume methods differ in their definition and construction of discrete operators representing Eq. (4), which we denote by $F(p, s) : (\mathcal{H}_\mathcal{T} \times \mathcal{H}_\mathcal{T}) \to \mathcal{H}_\mathcal{F}$. It is typical to consider this mapping as a product $F = F_1(p)F_2(s; F_1(p))$ where $F_1 : \mathcal{H}_\mathcal{T} \to \mathcal{H}_\mathcal{F}$ and $F_2 : (\mathcal{H}_\mathcal{T} \times \mathcal{H}_\mathcal{F}) \to \mathcal{H}_\mathcal{T}$. Classically, F_1 is a discrete representation of a Darcy flux (in this work we consider only linear finite volume methods, therefore F_1 is a linear function), while F_2 is a (potentially smoothed) upstream value of the function $\lambda(s)$. Standard formulations for upstream weighting used in porous media follow those for hyperbolic conservation laws [23], or their multi-dimensional generalizations [24].

We denote by $\|\cdot\|$ the discrete L_2 norm

$$\|f\|^2 := \sum_{K\in\mathcal{T}} m_K |f_K|^2, \tag{19}$$

and by $\langle\cdot,\cdot\rangle$ the associated scalar product

$$\langle f, g\rangle := \sum_{K\in\mathcal{T}} m_K f_K g_K . \tag{20}$$

We further assume:

(A4) The finite volume method is stable: if u is the solution vector of the Laplace equation, and $\vec{q}$ denotes its discrete gradient, there exist two constants $C, C' > 0$ such that

$$\langle F_1 u, u\rangle \geq C\|\vec{q}\|^2 \quad \text{and} \quad \langle F_1 u, u\rangle \geq C'\|u\|^2. \tag{21}$$

Remark 2.1. The assumption (A4) is satisfied for most finite volume methods. One usually has $\langle F_1 u, u\rangle = C\|\vec{q}\|^2$, while $\langle F_1 u, u\rangle \geq C'\|u\|^2$ follows from the discrete Poincaré inequality.

Let now $n \in \mathbb{N}$, $n \geq 1$ denote the time index, t_n being the actual time step. The fully discrete (nonlinear) scheme reads:
Having computed θ^{n-1}, find $(\theta^n, p^n, \vec{q}^n_w, \vec{q}^n) \in \mathcal{H}_\mathcal{T} \times \mathcal{H}_\mathcal{T} \times \mathcal{H}_\mathcal{F} \times \mathcal{H}_\mathcal{F}$ such that

$$s^n + \Delta t(\tilde{\nabla} \cdot \vec{q}^n_w - f_1(s^n)) = s^{n-1}, \tag{22}$$

$$\vec{q}^n_w = F_1(\theta^n) + f_w(F_2(s^n; F_1(\theta^n)))\vec{q}^n, \tag{23}$$

$$\tilde{\nabla} \cdot \vec{q}^n = f_2(s^n), \tag{24}$$

$$\vec{q}^n = F_1(p^n)F_2(s^n; F_1(p^n)), \tag{25}$$

where $s^n = s(\theta^n)$ and $s^{n-1} = s(\theta^{n-1})$. Eqs. (22) and (24) are understood elementwise for each $K \in \mathcal{T}$, while Eqs. (23) and (25) hold for each face $\sigma \in \mathcal{F}$. Furthermore, note that Eqs. (24)–(25) are a nonlinear elliptic system, which is decoupled from (22)–(23) (itself also a nonlinear system). The unknowns θ^n, p^n are cellwise constant (implicitly the same holds for $s^n = s(\theta^n)$).

In line with (14)–(17), the linearization of (22)–(25) reads ($i \geq 1$ being the iteration step):
Given $\theta^{n,i-1}$, find $(\theta^{n,i}, p^{n,i}, \vec{q}^{n,i}_w, \vec{q}^{n,i}) \in \mathcal{H}_\mathcal{T} \times \mathcal{H}_\mathcal{T} \times \mathcal{H}_\mathcal{F} \times \mathcal{H}_\mathcal{F}$ such that

$$L(\theta^{n,i} - \theta^{n,i-1}) + s^{n,i-1} + \Delta t(\tilde{\nabla} \cdot \vec{q}^{n,i}_w - f_1(s^{n,i-1})) = s^{n-1}, \tag{26}$$

$$\vec{q}^{n,i}_w = F_1(\theta^{n,i}) + f_w(F_2(s^{n,i-1}; F_1(\theta^{n,i-1})))\vec{q}^{n,i}, \tag{27}$$

$$\tilde{\nabla} \cdot \vec{q}^{n,i} = f_2(s^{n,i-1}), \tag{28}$$

$$\vec{q}^{n,i} = F_1(p^{n,i})F_2(s^{n,i-1}; F_1(p^{n,i-1})), \tag{29}$$

where $s^{n,i-1} = s(\theta^{n,i-1})$ and $s^{n-1} = s(\theta^{n-1})$. We start the iterations by taking $\theta^{n,0} = \theta^{n-1}$. Eqs. (28)–(29) now form a linear elliptic system, which is decoupled from (26)–(27) (itself also a linear system).

2.6. Convergence of the iterative method

We introduce now the errors at the iteration step i:

$$e_\theta^{n,i} = \theta^{n,i} - \theta^n, \qquad e_p^{n,i} = p^{n,i} - p^n, \qquad e_s^{n,i} = s^{n,i} - s^n = s(\theta^{n,i}) - s(\theta^n).$$

In order to show the convergence of the scheme (26)–(29) we will prove that

$$\|e_\theta^{n,i}\|, \|e_p^{n,i}\| \to 0 \quad \text{when } i \to \infty,$$

where $\|\cdot\|$ is the L_2 discrete norm (introduced in (19)). From $\|e_\theta^{n,i}\| \to 0$, obviously it follows that also $\|e_s^{n,i}\| \to 0$.

The convergence of the proposed scheme is proved in the theorem below.

Theorem 2.1. *Assuming* (A1)–(A4)*, the linearization scheme* (26)–(29) *is (at least) linearly convergent if the time step* Δt *is small enough.*

Proof. We give the main idea of the proof, pointing out only what is peculiar for the convergence of the new scheme and assuming that the standard estimates can be obtained for the considered finite volume method. By subtracting (26)–(29) from (22)–(25) one gets

$$L(e_\theta^{n,i} - e_\theta^{n,i-1}) + e_s^{n,i-1} + \Delta t G_1(\theta^{n,i}, \theta^n, p^{n,i}, p^n, s^{n,i-1}, s^n) = 0, \tag{30}$$

and

$$\tilde{\nabla} \cdot G_2(p^{n,i}, p^n, s^{n,i-1}, s^n) = 0, \tag{31}$$

with the expressions G_1 and G_2 depending on the considered finite volume scheme. By standard techniques and using (A2)–(A4), (31) implies

$$\|e_p^{n,i}\| + \|\vec{q}^{n,i} - \vec{q}^n\| \le C_0 \|e_s^{n,i-1}\|, \tag{32}$$

with C_0 not depending on the discretization parameters. Multiplying (30) with $e_\theta^{n,i}$ (the multiplication is done element wise), weighing by the cell volume m_K and then summing up the resulting gives

$$L\langle e_\theta^{n,i} - e_\theta^{n,i-1}, e_\theta^{n,i}\rangle + \langle e_s^{n,i-1}, e_\theta^{n,i}\rangle + \Delta t \langle G_1(\theta^{n,i}, \theta^n, p^{n,i}, p^n, s^{n,i-1}, s^n), e_\theta^{n,i}\rangle = 0. \tag{33}$$

Using Young's inequality, i.e. $|ab| \le \frac{\epsilon}{2}|a|^2 + \frac{1}{2\epsilon}|b|^2$ for all $\epsilon > 0$ and (A4) leads to

$$\langle G_1(\theta^{n,i}, \theta^n, p^{n,i}, p^n, s^{n,i-1}, s^n), e_\theta^{n,i}\rangle \ge C_1\|e_\theta^{n,i}\|^2 - C_2\|e_s^{n,i-1}\|^2 - C_3\|e_p^{n,i}\|^2 \tag{34}$$

where the constants $C_1 > 0, C_2 \ge 0, C_3 \ge 0$ are not depending on the time step size Δt or mesh diameter h. Putting together now (32)–(34), after some algebraic manipulations we obtain

$$\left(\frac{L}{2} + C_1\Delta t\right)\|e_\theta^{n,i}\|^2 + \frac{L}{2}\|e_\theta^{n,i} - e_\theta^{n,i-1}\|^2 + \langle e_s^{n,i-1}, e_\theta^{n,i}\rangle \le \frac{L}{2}\|e_\theta^{n,i-1}\|^2 + \Delta t(C_2 + C_3C_0^2)\|e_s^{n,i-1}\|^2, \tag{35}$$

which can be further written as

$$\left(\frac{L}{2} + C_1\Delta t\right)\|e_\theta^{n,i}\|^2 + \frac{L}{2}\|e_\theta^{n,i} - e_\theta^{n,i-1}\|^2 + \langle e_s^{n,i-1}, e_\theta^{n,i-1}\rangle \le \frac{L}{2}\|e_\theta^{n,i-1}\|^2 + \Delta t(C_2 + C_3C_0^2)\|e_s^{n,i-1}\|^2 + \langle e_s^{n,i-1}, e_\theta^{n,i-1} - e_\theta^{n,i}\rangle. \tag{36}$$

By using (13), i.e. that the constant L is bigger than or equal to the Lipschitz constant of $s(\cdot)$, the monotonicity of $s(\cdot)$ and Young's inequality, the above implies

$$\left(\frac{L}{2} + C_1\Delta t\right)\|e_\theta^{n,i}\|^2 + \frac{1}{L}\|e_s^{n,i-1}\|^2 \le \frac{L}{2}\|e_\theta^{n,i-1}\|^2 + \Delta t(C_2 + C_3C_0^2)\|e_s^{n,i-1}\|^2 + \frac{1}{2L}\|e_s^{n,i-1}\|^2. \tag{37}$$

From (37) we immediately obtain

$$\left(\frac{L}{2} + C_1\Delta t\right)\|e_\theta^{n,i}\|^2 + \left(\frac{1}{2L} - \Delta t(C_2 + C_3C_0^2)\right)\|e_s^{n,i-1}\|^2 \le \frac{L}{2}\|e_\theta^{n,i-1}\|^2. \tag{38}$$

Under the mild restriction on the time step size

$$\Delta t \le \frac{1}{2L(C_2 + C_3C_0^2)},$$

the convergence of the scheme follows. The time step restriction does not depend on the grid size, which is by far not that restrictive as a stability condition of an explicit scheme. ■

2.7. An appropriate finite volume method

For concreteness, we consider MPFA-type discretizations to obtain the linear operator F_1 (see [1] for an introduction). These are constructed such that the support of q_σ is local (in terms of connectivity of $\mathcal{D}$). Furthermore, the MPFA discretizations have been shown to be robust and convergent for a range of relevant parameters and grids [25–29]. While any of the established MPFA methods are applicable, our particular interest is the MPFA method recently proposed in [30], which is the scalar counter-part of the finite volume method for elasticity given in [31]. This MPFA method retains the finite volume structure of Eq. (18) exactly, and uses a stabilization of the so-called O-method [1] to approximate the discrete flux F_1. The convergence of the method is established in [30]. In particular, this implies also that (A4) holds true.

3. Numerical results

Using the scheme as described above, we consider the 2D and 3D domains as given respectively by the unit square and unit cube with Dirichlet boundary conditions. Our main aim is to provide a numerical verification of the results of Theorem 2.1. Motivated by coarse-scale models for CO_2 storage [32], linear relative permeability functions are considered. More exactly, we take $\lambda_w(s) = \lambda_b s$ and $\lambda_n(s) = \lambda_c(1-s)$ (with $\lambda_b < \lambda_c$). The fractional flow function f_w thus becomes

$$f_w(s) = \frac{s}{s + \frac{\lambda_c}{\lambda_b}(1-s)}.$$

Furthermore, we consider a capillary pressure function of the form

$$p^{cap}(s) = c(s + (\lambda_b/\lambda_c - 1)^{-1}\ln(s)).$$

This choice ensures that the complementary pressure function is also nonlinear, yet still analytically tractable (cf. (10)):

$$s(\theta) = 1 - \left(1 + \frac{2(\lambda_b - \lambda_c)\theta}{c\lambda_c\lambda_b}\right)^{1/2}.$$

The derivative of $s(\cdot)$ is unbounded, as $s(\cdot)$ is only Hölder continuous at the endpoint $s = 1$ (i.e. $\theta = \frac{c\lambda_c\lambda_b}{2(\lambda_c - \lambda_b)}$). Nevertheless, we take $s = 0$ as initial and boundary data, and a (time limited) source term at the center of the domain such that the saturation remains uniformly bounded below $s = 0.2$. In this case $s(\cdot)$ becomes Lipschitz continuous, with the Lipschitz constant $L = \frac{5}{4}\frac{\lambda_c - \lambda_b}{c\lambda_c\lambda_b}$. A typical simulation on various unstructured and structured grids is depicted in Fig. 1. All simulations were conducted on a standard laptop using the Matlab implementation environment, and the total run-time for the test suite amounts to less than 10 min.

We consider grids varying from 400 to 10,000 cells in 2D and 3D. In all cases, we use a constant time-step of 0.1 (dimensionless units). For all simulations, we give a tolerance in absolute residual error of 10^{-3} for the nonlinear system. Throughout all numerical experiments we observe that the linear error reduction factor is stable between 3 and 8.5, independent of grid type, grid resolution, or dimensionality of the problem, as illustrated by the results shown in Fig. 2. This implies that in no instance more than a maximum of 7 iterations was needed. Furthermore, the experiments support the assertion that no grid-dependent time-step restriction is introduced by the solution of the nonlinear system. This verifies the robustness and generality of the iterative scheme, and the suitability of the finite volume discretization for complex geometries.

4. Conclusions

We presented a robust linearization scheme for FV discretizations of two-phase flow in porous media. The scheme is linearly convergent with a mild mesh independent restriction on the time step size. The restriction is very mild compared to the stability condition for explicit temporal discretizations or with typical restrictions for Newton's method. In these cases, the time step restriction depends on the mesh diameter, this not being the case for the new scheme. Another advantage of the presented scheme is that it does not involve the calculations of derivatives. Numerical examples (both 2D and 3D) sustain the theoretical results. We especially remark that there is almost no difference in terms of number of iterations between the 2D and 3D computations, which is an argument for the efficiency of the scheme. The new numerical scheme is relatively simple to implement and is a valuable alternative to Picard or Newton methods.

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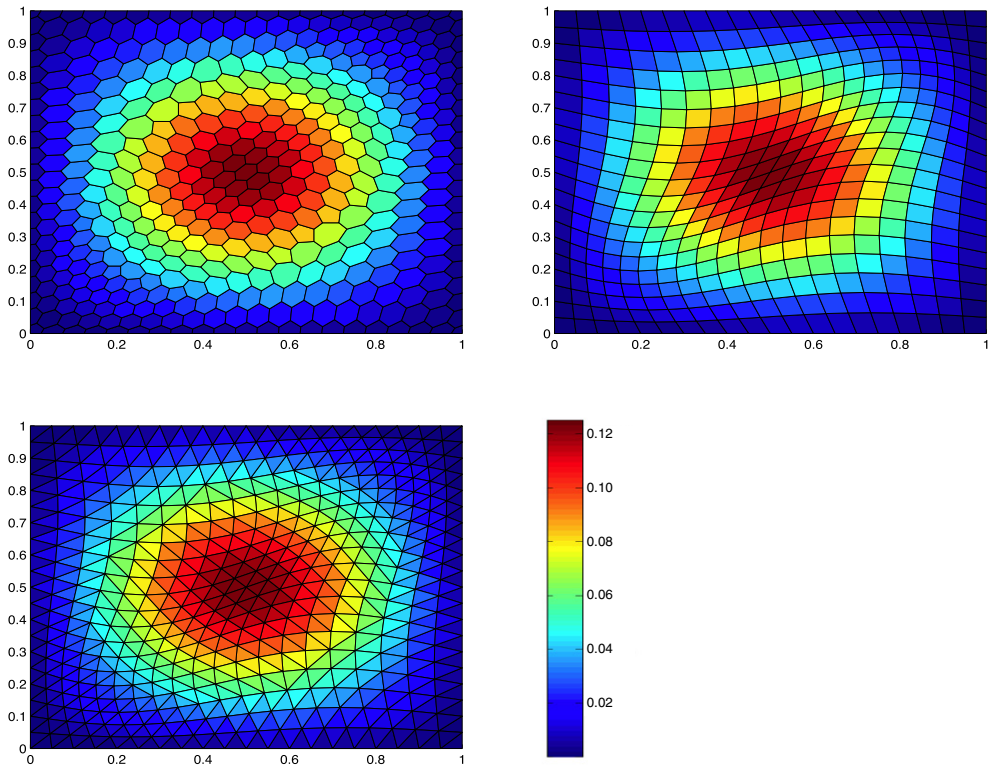

Fig. 1. Saturation s from example simulation results after 4 time steps on various grids. The use of MPFA spatial discretizations yields very little grid orientation effects, while the combination of implicit time integration allows for long time steps.

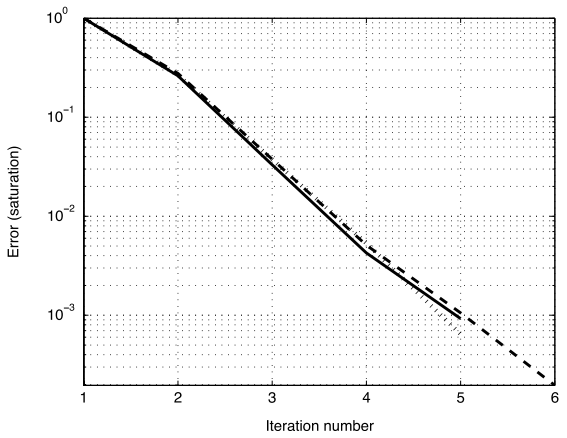

Fig. 2. Convergence history of relative saturation error for the first time-step for various grids. Illustrated are 100 × 100 quadrilateral grid (solid line), triangular grid with 800 cells (dashed line), and a 3D unstructured grid with 3840 cells (dotted lines).

References

[1] I. Aavatsmark, An introduction to multipoint flux approximations for quadrilateral grids, Comput. Geosci. 6 (2002) 404–432.
[2] M.G. Edwards, Unstructured, control-volume distributed, full-tensor finite-volume schemes with flow based grids, Comput. Geosci. 6 (2002) 433–452.
[3] M.G. Edwards, C.F. Rogers, Finite volume discretization with imposed flux continuity for the general tensor pressure equation, Comput. Geosci. 2 (1998) 259–290.
[4] R.A. Klausen, F.A. Radu, G.T. Eigestad, Convergence of MPFA on triangulations and for Richards' equation, Internat. J. Numer. Methods Fluids 58 (2008) 1327–1351.
[5] I.S. Pop, M. Sepulveda, F.A. Radu, O.P. Vera Villagran, Error estimates for the finite volume discretization for the porous medium equation, J. Comput. Appl. Math. 234 (2010) 2135–2142.
[6] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, New York, 1991.
[7] T. Arbogast, M.F. Wheeler, I. Yotov, Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences, SIAM J. Numer. Anal. 34 (2) (1997) 828–852.
[8] P. Knabner, L. Angermann, Numerical Methods for Elliptic and Parabolic Partial Differential Equations, Springer Verlag, 2003.

[9] L. Bergamaschi, M. Putti, Mixed finite elements and Newton-type linearizations for the solution of Richards' equation, Internat. J. Numer. Methods Engrg. 45 (1999) 1025–1046.
[10] E.J. Park, Mixed finite elements for nonlinear second-order elliptic problems, SIAM J. Numer. Anal. 32 (1995) 865–885.
[11] F.A. Radu, I.S. Pop, P. Knabner, On the convergence of the Newton method for the mixed finite element discretization of a class of degenerate parabolic equation, in: A. Bermudez de Castro, et al. (Eds.), Numerical Mathematics and Advanced Applications, Springer, 2006, pp. 1194–1200.
[12] F.A. Radu, I.S. Pop, Newton method for reactive solute transport with equilibrium sorption in porous media, J. Comput. Appl. Math. 234 (2010) 2118–2127.
[13] F.A. Radu, I.S. Pop, Mixed finite element discretization and Newton iteration for a reactive contaminant transport model with nonequilibrium sorption: convergence analysis and error estimates, Comput. Geosci. 15 (2011) 431–450.
[14] S. Kräutle, The semismooth Newton method for multicomponent reactive transport with minerals, Adv. Water Resour. 34 (2011) 137–151.
[15] E. Marchand, T. Müller, P. Knabner, Fully coupled generalized hybrid-mixed finite element approximation of two-phase two-component flow in porous media. Part I: formulation and properties of the mathematical model, Comput. Geosci. 17 (2013) 431–442.
[16] E. Marchand, T. Müller, P. Knabner, Fully coupled generalized hybrid-mixed finite element approximation of two-phase two-component flow in porous media. Part II: numerical scheme and numerical results, Comput. Geosci. 16 (2012) 691–708.
[17] I.S. Pop, F.A. Radu, P. Knabner, Mixed finite elements for the Richards' equation: linearization procedure, J. Comput. Appl. Math. 168 (2004) 365–373.
[18] M. Slodicka, A robust and efficient linearization scheme for doubly nonlinear and degenerate parabolic problems arising in flow in porous media, SIAM J. Sci. Comput. 23 (2002) 1593–1614.
[19] T. Arbogast, The existence of weak solutions to single porosity and simple dual-porosity models of two-phase incompressible flow, J. Nonlinear Anal.: TMA 19 (1992) 1009–1031.
[20] G. Chavent, J. Jaffre, Mathematical Models and Finite Elements for Reservoir Simulation, Elsevier, 1991.
[21] Z. Chen, Degenerate two-phase incompressible flow. Existence, uniqueness and regularity of a weak solution, J. Differential Equations 171 (2001) 203–232.
[22] P. Ciarlet, J.L. Lions (Eds.), Handbook for Numerical Analysis, North Holland, 2000, pp. 715–1022.
[23] R.J. LeVeque, Finite Volume Methods for Hyperbolic Problems, Cambridge University Press, 2002.
[24] E. Keilegavlen, J.E. Kozdon, B.T. Mallisonand, Multidimensional upstream weighting for multiphase transport on general grids, Comput. Geosci. 16 (2012) 1021–1042.
[25] R. Klausen, R. Winther, Robust convergence of multi point flux approximation on rough grids, Numer. Math. 104 (2006) 317–337.
[26] L. Agelas, C. Guichard, R. Masson, Convergence of finite volume MPFA O type schemes for heterogeneous anisotropic diffusion problems on general meshes, Int. J. Finite Vol. 7 (2010) 1–34.
[27] E. Keilegavlen, J. Nordbotten, I. Aavatsmark, Sufficient criteria are necessary for monotone control volume methods, Appl. Math. Lett. 22 (2009) 1178–1180.
[28] E. Keilegavlen, I. Aavatsmark, Monotonicity for MPFA methods on triangular grids, Comput. Geosci. 15 (2011) 3–16.
[29] J. Nordbotten, I. Aavatsmark, G. Eigestad, Monotonicity of control volume methods, Numer. Math. 106 (2007) 255–288.
[30] J.M. Nordbotten, Convergence of a cell-centered finite volume discretization for linear elasticity, arXiv:1503.05040 (2015), http://arxiv.org/abs/1503.05040.
[31] J.M. Nordbotten, Cell-centered finite volume discretizations for deformable porous media, Int. J. Numer. Methods Eng. 100 (2014) 399–418.
[32] M.A. Celia, J.M. Nordbotten, Practical modeling approaches for geological storage of carbon dioxide, Ground Water 47 (2009) 627–638.