Interest rate models in Solvency II

Master Thesis in Statistics

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Abstract

The best estimate of liabilities is important in the Solvency II framework. The best estimate of liabilities should be probability weighted average of future cash flows discounted to its present value. Life insurance companies need stochastic models to produce future paths for interest rates, bond returns and currency. These paths should be risk-neutral, meaning that interest rate models is important to consider in the Solvency II framework. In this thesis we have studied three different interest rate models, namely; the Hull-White extended Vasicek model, the CIR++ model and the G2++ model.

We calibrated our interest rate models to the same historical data and generated 10 000 simulations based on the yield curve and the parameter estimations. Based on the interest rates simulated we presented a synthetic example for calculating the best estimate of liabilities. In this example, the duration of the liabilities turned out to be an important factor.

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Introduction

1.1 Background

Insurance means protection of financial loss, used to hedge against the risk of an uncertainty. The concept behind an insurance company is that many are susceptible to the same risk, they equalize this risk by distributing losses. Insurance works like a contract between a policyholder and an insurance company. The insurance company commits to cover unexpected expenses that the policyholder may suffer, while the policyholder commits to pay a premium for this contract. This way the financial products sold by an insurance company often contain guarantees and options of numerous varieties.

We have two main types of insurance: life insurance and non-life insurance. Life insurance consist mainly in managing savings on behalf of the policyholder. This is policyholders funds and shall not be used to cover losses from e.g. non-life insurance. Non-life insurance is a broad category. It is a short term agreement that covers persons as well as things. When signing a contract with a non-life insurance company, it may be for your car, house, travel etc. In this thesis we will concentrate on life insurance. A life insurance company offers products which will pay out in case of death or disability as well as pension products e.g. occupational pension.

Occupational pension schemes are arrangements established by employers through e.g life insurance companies to provide pension and related benefits for their employees. We can divide occupational pension into two main categories: Group defined benefit pensions (DB) and Group defined contribution pensions (DC), where the employer can choose between DB and DC. DB involves a previously agreed pension benefit between the insurance company and the customer (employer). It guarantees pension payments from a specified age for as long as the insured person lives. Alternatively, it can be agreed that the pension will end at a specified age. Here, the the insurance company bears all the risk. In DC the premium is stated as a percentage of pay, while the pension payments are unknown. The customer bears all the financial risk during the saving period. In recent years, several companies have closed the defined benefit pension and switched to a defined contribution pension for new employees.

Pricing and managing contracts and return guarantees is important for the insurance companies, especially life insurance, and it is one of the most challenging problems faced by the companies today. The valuation of these policies sometimes requires complex analytical methods, as well as great deal of computer power and data. Moreover, life expectancy increase and interest rates are low, resulting in existing contracts from e.g. DB pensions, written in a period with higher interest rates, can be difficult to meet (Finanstilsynet, 2015a).

In light of current economic events and new legislation, insurance companies have realized the importance of properly managing their options and guarantees. For the company, the largest financial risk would be not to meet this guarantee as it affects the solvency of the company. A large part of a life insurance companies portfolios are invested in assets for which the returns depend on the interest rate level. Furthermore, insurance firms operating in the European Union started, from 2016, to use the Solvency II directive. This directive is a set of regulatory requirements based on economic principles for valuation of assets and liabilities. It is a risk-based system. The risk will be measured on consistent principles, and capital requirements will depend directly on this

1.2 Solvency II and Computing Liabilities

Solvency II is a new legislation that took effect 01.01.2016. The EU have collected all the central directives which covers the insurance and reinsurance businesses into one directive. Solvency II sets new demands on insurance companies regarding for example capital requirement, risk management processes and transparency.

The Solvency II framework, like the Basel II directive for credit and bonds, is based on three pillars.

- Pillar I covers the quantitative requirements, i.e. how assets and liabilities should be valued and the capital that a company is required to hold.
- Pillar II covers all the qualitative capital requirement i.e. how risks should be governed, managed and supervised. This covers the requirements for governance and risk management of insurers, as well as for the effective supervision of insurers. This pillar give supervisors greater power to challenge their firms on risk management issues.

- Pillar III sets out the requirements for supervisory reporting and disclosure of the information.

Pillar I defines two levels of capital requirements: Minimum Capital Requirement (MCR) and Solvency Capital Requirements (SCR). The MCR is the absolute minimum capital that an insurance company has to hold. If the capital falls below this level the supervisory authorities will intervene. The SCR is the required level of capital that an insurance company should hold. The SCR can be calculated either through a standard formula, through the use of an internal model or a combination of both. The internal model must be approved by the supervisory authorities.

In order to determine the capital requirements one have to calculate the technical provisions. Technical provisions is the amount that an insurance company must hold to ensure that it can meet its expected future obligations on insurance contracts. The technical provisions consist of risks that can be hedged and risks that cannot be hedged, and the value of the risks that can not be hedged should be the sum of a best estimate of the expected liabilities and risk margin.

1.2.1 Best Estimate of Liabilities

We want to calculate the best estimate of liabilities. This estimate should be probability weighted average of future cash-flows, discounted to its present value using a "risk-free" yield curve.

Methods for calculating the best estimate of the liabilities should be actuarial or statistical methods that take into account the risks that affect the future cash flow. There are different methods for calculating the best estimate including deterministic techniques, analytical techniques and simulation techniques. Simulation methods means using a stochastic model to produce future paths, while an analytical method means that the insurance company must be able to find a closed form solution for calculating the best estimate and in the deterministic approach the projection of the cash flows are based on a fixed set of assumptions.

For insurance contract with interest rate guarantee, a market consistent simulations or stochastic analysis is likely to be the most appropriate calculation approach, since changes in present values of liability are often a result of movements in economic variables.

1.3 Aim and scope

The Norwegian supervisory authority, Finanstilsynet, have said that the technical provisions for a life insurance company constitutes a dominant share of the balance sheet compared to a non-life insurance company (Finanstilsynet, 2015b). Even relatively small changes in the technical provisions can result in big changes in the capital and further affect the companies' solvency under the new framework. Therefore, a key aspect to the Solvency II framework is to compute the best estimate of liabilities.

The best estimate is the present value of expected future cash flows, discounting using a risk-free yield curve. For life insurance companies with financial guarantees and options, stochastic models help them produce future risk-neutral paths. Risk-neutral is when the expected return of all assets is equal to the risk-neutral rate used for discounting the cash-flows. Hence, interest rate models are a key component to consider within the Solvency II framework, in particular for life insurance products.

The aim with this thesis is to describe and calibrate interest rate models that can be used to simulate interest rates in order to calculate the best estimate of liabilities. We study three different models calibrated to data from the same historical period.

This thesis is outlined as follows. Chapter 2 will give us theoretical background, while Chapter 3 describes the interest rate models used in this thesis; the Hull-White model, the CIR++ model and the G2++ model. In Chapter 4 we calibrate and simulate from the three different models and in Chapter 5 we will present an example to calculate the best estimate of liabilities. Finally we will conclude and present a summary in Chapter 6.

Chapter 2

Theoretical Background

The concept of interest rates is used in our every-day life, if we lend money to the bank, we will expect this money to grow at some rate as time goes by. However expressing a such concept mathematically is more complex and we need to introduce many definitions.

The first part of this chapter covers basic definitions, focusing on different kinds of interest rates. We will use these definitions to look at a mathematical basis for interest rate modelling in Sections 2.2 and 2.3. Further, we describe two estimation tecnicques in Section 2.4 and correlation in Section 2.5. The approach is similar to Brigo and Mercurio (2007) and Björk (2004) with supplements from Rutkowski and Musiela (1998) and Øksendal (2003).

2.1 Definitions and notations

2.1.1 Short-Term Interest Rate

The first term that we need to introduce is the notation of a *bank account*. A bank account represents a risk-free security for which profit is accrued continuously at a risk-free rate.

Definition 2.1. Bank account We define $B(t)$ to be the value of a bank account at time $t \geq 0$. We assume $B(0) = 1$ and that the bank account evolves according to the following differential equation:

$$dB(t) = r_t B(t) dt, \quad B(0) = 1 \tag{2.1}$$

where r_t is the instantaneous rate, also referred to as instantaneous sport rate or short rate. Consequently,

$$B(t) = exp\left(\int_0^t r_s \; ds\right). \tag{2.2}$$

This definition tells us that investing a unit at time 0 yields, at time t, the value in Equation 2.2. In order to have one unit of currency at time T one has to invest the amount $1/B(T)$ at the beginning i.e. when $t = 0$. At time $t > 0$ the value of the initial investment is given by $B(t)(1/B(T))$, which leads to the next definition.

Definition 2.2. Stochastic discount factor The stochastic discount factor $D(t, T)$ is the value at time t of one unit of currency payable at time $T > t$. This gives us

$$D(t,T) = \frac{B(t)}{B(T)} = exp\left(\int_t^T r_s ds\right). \tag{2.3}$$

2.1.2 Zero-Coupon Bonds and Spot Interest Rates

Definition 2.3. Zero-Coupon Bond A zero-coupon bond with maturity T is a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. The contract valued at time $t < T$ is denoted $P(t, T)$.

Remark. The bond prices is assumed to follow a strictly positive and adapted process on a filtered probability space $(\Omega, \mathcal{F}, Q_0, (\mathcal{F}_t)_{0\leq t\leq T^*})$ where the filtration $\mathcal{F}_t$ is the Q_0-compounded version of the filtration generated by the underlying Browinan motion and T^* is the fixed horizon date for all market activities.

There is a close relationship between the discount factor $D(t, T)$ and $P(t, T)$. If r is deterministic, then D is deterministic and we have $D(t, T) = P(t, T)$ for each pair (t, T). However, in a more realistic case, the rates are stochastic, and it follows that $D(t, T)$ will be stochastic, while the zero-coupon bond price $P(t, T)$, has to be known.

Definition 2.4. Continuously-compounded spot interest rate The continuously-compounded spot interest rate, $R(t, T)$, also referred to as the yield on the zero-coupon bond $P(t, T)$, is the constant rate at which an investment of $P(t, T)$ units of currency at time t accrues continuously to yield one unit of currency at maturity T. In formulas:

$$R(t,T) = -\frac{\ln P(t,T)}{T-t}. \tag{2.4}$$

The continuously-compounded interest rate is a constant rate that is consistent with the the zero-coupon-bond price in that

$$e^{R(t,T)(T-t)}P(T,t) = 1, \tag{2.5}$$

from which we can express the bond price in terms of the continuously-compounded rate R

$$P(t,T) = e^{-R(t,T)(T-t)}. \tag{2.6}$$

Remark. The short term rate $r(t)$ is obtained as a limit of $R(t,T)$, that is

$$r(t) = \lim_{T \to t^+} R(t,T) = \lim_{T \to t^+} -\frac{\ln P(t,T)}{T-t}. \tag{2.7}$$

An alternative to continuous compounding is simple compounding, which applies when accruing occurs proportionally to time of the investment.

Definition 2.5. Simply-compounded spot interest rate The simply compounded spot interest rate, $L(t,T)$, is the constant rate at which an investment has to be made to produce an amount on one unit of currency at maturity T, starting from $P(t,T)$ units of currency at time t, when accruing is proportional to the investment time. In formulas:

$$L(t,T) = \frac{1-P(t,T)}{(T-t)P(t,T)}. \tag{2.8}$$

The market LIBOR rates are an example of simply-compounded rates. The LIBOR rates are typically linked to zero-coupon-bond prices by Actual/360 day-count[1]. The bond price can be expressed in terms of L as:

$$P(t,T) = \frac{1}{1+L(t,T)(T-t)}.$$

Definition 2.6. Zero-coupon curve The zero-coupon curve, also referred to as a yield-curve, at time t is the graph of the function

$$T \mapsto R(t,T), \quad T > t. \tag{2.9}$$

Such a zero-coupon curve shows the relation between the interest rate and the time to maturity. An example of such a curve is shown in Figure 2.1.

2.1.3 Forward rates

Forward rates are interest rates that can be locked today for an investment in a future time period. Their value can be derived directly from zero-coupon-bond prices. Forward rates are characterized by three time instants, the time t where the rate is considered, the expiry T, and the maturity S, $t \leq T \leq S$.

Definition 2.7. Continuously-compounded forward rate The continuously-compounded forward rate $f(t,T,S)$, at time t for the expiry T and maturity S is defined as

$$f(t,T,S) = \frac{1}{S-T} \ln \frac{P(t,T)}{P(t,S)}. \tag{2.10}$$

[1] Actual/360 day-count means that a year is assumed to be 360 days long, and the corresponding year fraction is the actual number of days between two dates divided by 360.

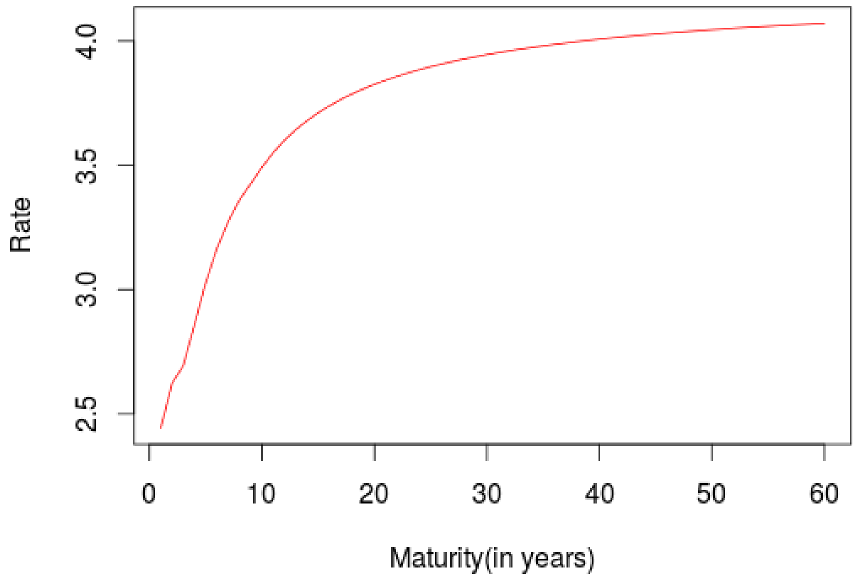

Figure 2.1: A Zero-coupon curve

Definition 2.8. Instantaneous forward rate Instantaneous forward rate, $f(t,T)$, at time t for maturity T is given by

$$f(t,T) = \lim_{S \to T^+} f(t,T,S) = -\frac{\partial \ln P(t,T)}{\partial T}, \tag{2.11}$$

so that

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right). \tag{2.12}$$

Definition 2.9. Simply-compounded forward interest rate The simply-compounded forward rate is denoted $F(t,T,S)$, and is defined by

$$F(t,T,S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right). \tag{2.13}$$

2.1.4 Interest Rate Swaps

An interest rate swap (IRS) is a contract that exchanges interest rate payments between two differently indexed legs. Here, one leg is fixed whereas the other one is floating. When the fixed leg is paid and the floating leg is received the IRS is termed Payer IRS, while in the other case we have Receiver IRS. Borrowing one unit at a fixed rate K with coupons paid at time T_i, $i = 1, ..., n$ and where $\tau_i = T_i - T_{i-1}$ gives the present value (PV):

$$PV_{fixed} = \sum_{i=1}^{n} P(t, T_i)\tau_i K + P(t, T_n), \tag{2.14}$$

where $t = T_0$[2]. The present value of a stream of floating rate cash flows is

$$PV_{floating} = \sum_{i=1}^{n} P(t, T_i)\tau_i L(T_{i-1}, T_i) + P(t, T_n). \tag{2.15}$$

The present value of a Payer IRS is thus given by

$$PV_{PayerIRS} = \sum_{i=1}^{n} P(t, T_i)\tau_i (L(T_{i-1}, T_i) - K). \tag{2.16}$$

A typical interest rate swap in the market has a fixed leg with annual payments and a floating leg with quarterly or semiannual payments, for simplicity we have assumed that the tenors of the floating and fixed legs are the same. The value of the swap at the initiation date will be zero to both parties. For this statement to be true, the values of the cash flow streams that the swap parties are going to exchange must be equal, i.e.

$$PV_{fixed} = PV_{floating}. \tag{2.17}$$

We can express the swap rate in terms of bond prices by first simplifying the value of the floating leg to 1 and then by using Equation 2.14. More specifically:

$$\begin{aligned} PV_{floating} &= \sum_{i=1}^{n} (P(t, T_i)\tau_i L(T_{i-1}, T_i)) + P(t, T_n) \\ &= \sum_{i=1}^{n} \left(P(t, T_i)\tau_i \frac{1 - P(T_{i-1}, T_i)}{\tau_i P(T_{i-1}, T_i)} \right) + P(t, T_n) \\ &= \sum_{i=1}^{n} (P(t, T_{i-1}) - P(t, T_i)) + P(t, T_n) = 1. \end{aligned} \tag{2.18}$$

[2] T_0 is the first reset date

Here, $L(T_{i-1}, T_i)$ is defined in 2.8. Further, we insert S for K in Equation 2.14 and we can express the swap rate $S(t, T_n)$ in terms of bond prices

$$S(t, T_n) = \frac{1 - P(t, T_n)}{\sum_{i-1}^{n} \tau_i P(t, T_i)}, \tag{2.19}$$

If $t < T_0$ cash flows are exchanged starting at a future time and thus we have *forward start swap rate*. The value of the floating leg must be discounted and is therefore $P(t, T_0)$. The forward start swap rate is when present value of the payer forward start swap is equal 0, i.e.

$$P(t, T_0) - \sum_{i=1}^{n} P(t, T_i)\tau_i K + P(t, T_n) = 0. \tag{2.20}$$

Replacing K with $S(t, T_0, T_n)$, we obtain

$$S(t, T_0, T_n) = \frac{P(t, T_0) - P(t, T_n)}{\sum_{i=1}^{n} P(t, T_i)}. \tag{2.21}$$

2.2 No-Arbitrage pricing

The absence of arbitrage opportunities in the market is a fundamental economical assumption. Absence of arbitrage is equivalent to the impossibility to invest zero today and receive a non negative amount tomorrow, with a positive probability. Future considerations are based on this assumption.

From Rutkowski and Musiela (1998) we have the following definition:

Definition 2.10. A family $P(t, T)$, $t \leq T \leq T^*$, of adopted processes is called an arbitrage-free family of bond prices relative to r if the following conditions hold:

- $P(T, T) = 1$ for all $T \in [0, T^*]$
- There exists a probability measure Q on $\Omega, \mathcal{F}_T^*$ equivalent to Q_0 [3], such that the discounted bond price, $\tilde{P}(t, T)$ for all $t \in [0, T]$ is given by:

$$\tilde{P}(t, T) = D(0, t)P(t, T) = \frac{B(0)}{B(t)} P(t, T) = \frac{P(t, T)}{B(t)}. \tag{2.22}$$

Here, $\tilde{P}(t, T)$ is a martingale under Q.

Any probability measure that satisfies these conditions are in fact a *martingale measure* for the family $P(t, T)$. As $\tilde{P}(t, T)$ follows a martingale under Q we have:

$$\tilde{P}(t, T) = E_Q(\tilde{P}(T, T)|\mathcal{F}_t), \quad t \leq T. \tag{2.23}$$

[3] Q^0 and Q are equivalent, that is $Q(A) = 0$ if and only if $Q^0(A) = 0$ for every $A \in \mathcal{F}_T^*$

Where E_Q denotes the expectation under the risk-neutral measure. We can use this to show that $P(t,T)$ is equal to the expectation of the stochastic discount factor $D(t,T)$ under Q. From Equations 2.22 and 2.23 we have:

$$D(0,t)P(t,T) = E_Q[D(0,T)P(T,T)|\mathcal{F}_t] = E_Q[D(0,t)|\mathcal{F}_t],$$

which leads to the following for the bond price

$$\begin{aligned} P(t,T) =& D(0,t)^{-1}E_Q[D(0,T)|\mathcal{F}_t] \\ =& \exp\left(\int_0^t r_s ds\right) E_Q\left[\exp\left(-\int_0^T r_s ds\right)\middle|\mathcal{F}_t\right] \\ =& E_Q\left[\exp\left(-\int_t^T r_s ds\right)\middle|\mathcal{F}_t\right] \\ =& E_Q[D(t,T)|\mathcal{F}_t]. \end{aligned} \tag{2.24}$$

This means that we can directly obtain the unique no-arbitrage price for bonds.

2.3 Factor models of the Term Structure

As we have seen, the zero-coupon bond price can be viewed as the expectation of the random variable $D(t,T)$ under probability measure Q. This means that whenever we are able to characterize the distribution of $\exp(-\int_t^T r_s ds)$, we are able to compute bond prices. Since the price $P(t,T)$ should in some sense depend on the behaviour of the rate over the interval $[t,T]$, a natural starting point is the dynamics of interest rates.

We model the evolution of interest rate as a stochastic differential equation(SDE). An SDE is a differential equation where one or more terms are stochastic, resulting in a stochastic solution. The general form is

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t, \quad X(0) = x_0, \tag{2.25}$$

where the function μ is called drift term, the function σ is the diffusion coefficient and W is a *Brownian motion*.

Remark. Equation 2.25 is a short form of the integrated representation

$$X_t = x_0 + \int_0^t \mu(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_t, \tag{2.26}$$

A Brownian motion is a stationary stochastic process with independent increments that follows a normal distribution, i.e

Definition 2.11. Brownian Motion. A Brownian motion is a stochastic process where the following conditions hold:

- $X(0) = 0$,
- $X(t_1) - X(t_0), .., X(t_k) - X(t_{k-1})$ are independent for $t_0 < t_1, .., < t_{k-1} < t_k$, and
- $X(t+s) - X(s) \sim N(0, t\sigma^2)$.

In order to guarantee the existence of a unique solution let $T > 0$ and $\mu : [0,T] \times \mathbb{R}^n \mapsto \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \mapsto \mathbb{R}^{n \times m}$ be measurable functions satisfying

$$\begin{aligned} |\mu(x,t)| + |\sigma(x,t)| &\leq C(1+|x|) \\ |\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| &\leq D|x-y| \end{aligned} \tag{2.27}$$

for some constants C and D, where $|\sigma^2| = \sum |\sigma_{ij}|^2$, (Øksendal, 2003).

If we consider a *time-homogenous* Itô process, the functions μ and σ will only depend on X and not t, and equation 2.25 will be on the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \tag{2.28}$$

where μ and σ satisfy the conditions in 2.27, which in this case can be simplified to

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq D|x-y| \tag{2.29}$$

The importance property of these processes is the *Markov property*:

Definition 2.12. Markov Process The stochastic process X_t is called Markov, if for every n and $t_1 < t_2 < .. < t_n$,

$$P(X_{t_n}|X_{t_{n-1}}, .., X_{t_1}) = P(X_{t_n}|X_{t_{n-1}}). \tag{2.30}$$

From Øksendal (2003) we have the following theorem:

Theorem 2.1. ***The Markov property for Itô processes*** *Let X_t^x be a time homogeneous Itô process of the form*

$$dX_t^x = \mu(X_t^x)dt + \sigma(X_t^x)dW_t, \tag{2.31}$$

and let f be a bounded Borel function from $\mathbb{R}^n$ to $\mathbb{R}$. Then, for $t \leq s$

$$E[f(X_{t+s}^x|\mathcal{F}_s)] = E[f(X_t^y)]|_{y=X_s^x} \tag{2.32}$$

However, interest rate models may depend on time. In fact, in a no-arbitrage model the drift is, in general, dependent on time. Øksendal (2003) describes how the general case from Equation 2.25 can be reduced to the time-homogeneous situation and thus we can use the Markov property to calculate the price of the zero-coupon bond $P(t,T)$ given in Equation 2.24.

2.3.1 Dynamics under risk-neutral measure Q

We are now interested in the dynamics of 2.25 under an equivalent martingale measure Q. The first approach for interest rate models, proposed by Vasicek (1977), was based on defining the instantaneous spot-rate dynamics under the real-world measure, Q^0, however construction of a locally-risk less portfolio leads to the existence of a stochastic process that only depend on the current time and instantaneous spot rate, not maturity of the claims. Such a process is commonly referred to as the *market price of risk*, when deriving such process we follow Björk (2004), and further we follow Brigo and Mercurio (2007) and use this process to define a Girsanov change of measure from the real-world to the risk-neutral world.

For a fixed maturity date T, the zero-coupon bond price $P(t,T)$ is a function of X_t and t, that is $P(t,T) = F(X_t,t) \quad t \leq T$. By applying the Itô formula we get the following dynamics

$$dP(t,T) = \alpha(X_t,t)dt + \beta(X_t,t)dW_t, \tag{2.33}$$

where

$$\begin{aligned}\alpha(X_t,t) =& \frac{\partial F(X_t,t)}{\partial t} + \frac{\partial F(X_t,t)}{\partial X}\mu(x_t,t)\\ &+\frac{1}{2}\frac{\partial^2 F(X_t,t)}{\partial X^2}\sigma(X_t,t)^2,\\ \beta(X_t,t) =& \frac{\partial F(X_t,t)}{\partial t}\sigma(X_t,t).,\end{aligned} \tag{2.34}$$

where $\frac{\partial F}{\partial X} = (\frac{\partial F}{\partial X_1},...,\frac{\partial F}{\partial X_n})$. We consider a portfolio consisting of u^T units of the zero-coupon bond P^T with maturity T, and u^S units of the zero-coupon bond P^S with maturity S. The value process of this portfolio is given by

$$V_t(u) = u_t^T P^T(t,T) + u_t^S P^S(t,S). \tag{2.35}$$

We assume a self-financing portfolio, hence

$$dV_t(u) = u_t^T dP^T(t,T) + u_t^S dP^S(t,S).$$

Inserting the differential for both maturity T and S, and after some reshuffling we obtain

$$dV_t(u) = (u_t^T\alpha(X_t,t) + u_t^S\alpha(X_t,t))dt + (u_t^T\beta(X_t,t) + u_t^S\beta(X_t,t))dW_t. \tag{2.36}$$

In order to guarantee a risk-less portfolio we have that

$$u_t^T\beta^T(X_t,t) + u_t^S\beta^S(X_t,t) = 0. \tag{2.37}$$

With this the dW-term in 2.36 will disappear so the value dynamics reduce to:

$$dV_t(u) = u_t^T \alpha(X_t, t) + u_t^S \alpha(X_t, t))dt \tag{2.38}$$

Further, in order to avoid arbitrage, the portfolio must have a rate of return equal to the short rate of interest. We have the condition:

$$dV_t(u) = r(t)V_t dt,$$

where $r(t)$ is the instantaneous spot rate, and V_t is defined in Equation 2.35. These two conditions leads us to the following linear system of equations:

$$u_t^T(\alpha^T(X_t, t) - r(t)P^T(t, T)) + u_t^S(\alpha^S(X_t, t) - r(t)P^S(t, T)) = 0$$
$$u_t^T \beta^T(X_t, t) + u_t^S \beta^S(X_t, t) = 0, \tag{2.39}$$

where the unknowns are u_t^T and u_t^S.

After some calculations the linear system solves as:

$$\frac{\alpha^T(X_t, t) - r(t)P^T(X_t, t)}{\beta^T(t, T)} = \frac{\alpha^S(X_t, t) - r(t)P^S(t, S)}{\beta^T(X_t, t)}. \tag{2.40}$$

From Equation 2.40 we see that we on the left-hand side have a stochastic process which does not depend on the choice of S, while on the right-hand side we have a process that does not depend on the choice of T. The common quotient will thus not depend on the choice of either T or S, which leads us to the stochastic process λ:

$$\lambda(t) = \frac{\alpha^T(X_t, t) - r(t)P^T(X_t, t)}{\beta^T(t, T)} \tag{2.41}$$

for each maturity T, thus λ that may depend on r but not T.

The process $\lambda(t)$ is known as *the market price of risk* and can be interpreted as follows: Equation 2.33 expresses how the bond price $P(t, T)$ evolves over time, and $r(t)$ is the instantaneous-return rate of a risk-free investment. The difference $\alpha^T - r(t)$ represents the difference in returns with respect to a risk-free case. The denominator, β^T, represents the volatility and when we divide by β^T, we are dividing the amount of risk we are subject to.

If $\lambda(t)$ satisfies the Novikov's condition, i.e.

$$E_{Q^0}\left[\exp\left(\frac{1}{2}\int_0^T \lambda^2(s)ds\right)\right] < \infty \tag{2.42}$$

we are able to define a probability measure Q equivalent to Q^0 by the Randon-Nikodym derivative

$$\left.\frac{\partial Q}{\partial Q^0}\right|_{\mathcal{F}_t} = \exp\left(\frac{1}{2}\int_0^t \lambda^2(s)ds - \int_0^t \lambda(s)dW^0(s)\right), \tag{2.43}$$

where $\mathcal{F}_t$ is generated by r up to t. As a consequence the process X evolves under Q according to

$$dX(t) = [\mu(t, X(t)) - \lambda(t)\sigma(t, X(t))]dt + \sigma(t, r(t))dW(t),$$

where the process

$$W(t) = W^0(t) + \int_0^t \lambda(s)ds \tag{2.44}$$

is a Brownian motion under Q[4].

Remark. Since we are using the no-arbitrage argument when constructing λ, Q satisfies the conditions in Definition 2.10 and is therefore an equivalent martingale measure.

2.3.2 Term-Structure Equation

By inserting the definitions of α and β from Equation 2.34 into $\lambda(t)$ we obtain:

$$\begin{aligned}\frac{\partial F(X_t,t)}{\partial t} &+ \frac{\partial F(X_t,t)}{\partial X}\left(\mu(X_t,t) - \lambda(t)\sigma(X_t,t)\right)\\ &+\frac{1}{2}\left(\frac{\partial^2 F(X_t,t)}{\partial X^2}\sigma^2(X_t,t)\right) - rF(X_t,t) = 0,\end{aligned} \tag{2.45}$$

with the terminal condition $P(T,T) = F(X_T,T) = 1$. This is, as Björk (2004) calls it, the *term structure equation*. It follows from Equations 2.34 and 2.41 that λ is of the form $\lambda = \lambda(r,t)$ so this equation is a standard PDE. The problem is that in order to solve the PDE we must specify λ, as it is not determined within the model. However if we are just concerned with the pricing of the interest derivatives, we can directly model the rate dynamics under Q, and λ will be implicit in our dynamics. In fact, the term $\mu - \lambda\sigma$ is precisely the drift term of the short rate under the martingale measure Q (Björk, 2004). We define the dynamics under the risk-neutral measure Q as

$$dX_t = \mu(X_t, t; \theta)dt + \sigma(X_t, t; \theta)dW_t, \quad X(0) = x_0$$

where μ and σ are given functions, and θ denotes the parameter vector. The term structure equation from 2.45 is now

$$\begin{aligned}\frac{\partial F(X_t,t)}{\partial t} + \frac{\partial F(X_t,t)}{\partial X}\left(\mu(X_t,t;\theta)\right) + \frac{1}{2}\left(\frac{\partial^2 F(X_t,t)}{\partial X^2}\sigma^2(X_t,t;\theta)\right) - rF(X_t,t) &=0,\\ F(X_T,T) &=1.\end{aligned} \tag{2.46}$$

It follows from the *Feynman-Kac*[5] formula that the risk neutral calculation formula for bond prices, $E_Q(\exp(-\int_t^T r(s)ds)|\mathcal{F}_t)$, solves the PDE in 2.46.

[4] See Øksendal (2003) for details about the Randon-Nikodyn derivative and the Girsanov change of measure

[5] See Björk (2004) for details

2.3.3 Affine Term Structure

Definition 2.13. Affine Term Structure If the term-structure has the form

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}. \tag{2.47}$$

where $A(t,T)$, and $B(t,T)$ are deterministic functions, then the model is said to possess an affine term structure.

We assume that the risk-neutral dynamics for the short rate are given in Equation 2.25. If the coefficients μ and σ are of the form

$$\mu(x,t) = \lambda(t)x + \eta(t), \quad \sigma(x,t) = \sqrt{\gamma(t)x + \delta}, \tag{2.48}$$

for suitable deterministic time functions λ, η, γ, and δ, then the model has an affine term structure (Björk, 2004). The functions A and B can be obtained from the coefficients λ, η, γ,and δ by solving the following differential equations:

$$\begin{cases} \frac{\partial}{\partial t}B(t,T) + \lambda(t)B(t,T) - \frac{1}{2}\gamma(t)B(t,T)^2 + 1 & = 0, \\ B(T,T) & = 0. \end{cases} \tag{2.49}$$

$$\begin{cases} \frac{\partial}{\partial t}[\ln A(t,T)] - \eta(t)B(t,T) - \frac{1}{2}\delta(t)B(t,T)^2 & = 0, \\ A(T,T) & = 1. \end{cases} \tag{2.50}$$

Note that the first equation is a *Ricatti equation* meaning that, in general, it needs to be solved numerically. However, as we shall see, in some cases it can be solved analytically.

2.4 Estimation

In order to estimate the parameters of the interest rate models, we need to introduce some methods.

2.4.1 Maximum likelihood estimator

The idea is is to find a parameter value for which the actual outcome has the maximum probability. Suppose we have a time series $r(t_i)$, $i = 1, .., n$, and that the transition density

$$p(t_{i+1}, r(t_i); t_i, r(t_i)|\theta) \tag{2.51}$$

is known.

Remark. In general, the transition density at time t is conditional on $\mathcal{F}_{t_i}$. However our interest rate models are based on Markov processes, which means that it is only conditional on values at time t_i.

The joint density of our observations is

$$p(r(t_1), ..., r(t_n)|\theta) = p_0(r(t_1)|\theta) \prod_{i=1}^{n} p(t_{i+1}, r(t_i); t_i, r(t_i)|\theta), \tag{2.52}$$

and the likelihood function is given by

$$L(\theta) = \prod_{i=1}^{n} p(t_{i+1}, r(t_i); t_i, r(t_i)|\theta). \tag{2.53}$$

The maximum likelihood estimator of θ is the value $\hat{\theta}$ which maximizes the likelihood function, $\hat{\theta} = \arg\max_\theta L(\theta)$. Equivalently $\hat{\theta}$ is the value which maximizes the log-likelihood function. Since the logarithmic function is monotonically increasing, it is often easier to work with.

2.4.2 Svensson Model

The market interest rate curve is only observed at some discrete time points. To obtain a finer resolution, we use the Svensson Model (Svensson, 1994):

$$f(0,\tau) = \beta_0 + \beta_1 \exp(-\frac{\tau}{\lambda_1}) + \beta_2 \frac{\tau}{\lambda_1} \exp(-\frac{\tau}{\lambda_1}) + \beta_3 \frac{\tau}{\lambda_2} \exp(-\frac{\tau}{\lambda_2}). \tag{2.54}$$

We use the following relationship between the yield curve and the forward rates when determining the parameters $\beta_0, \beta_1, \beta_2, \beta_3, \lambda_1$ and λ_2:

$$R(t,T) = \frac{\int_{\tau=t}^{T} f(t,\tau) d\tau}{T-t} \tag{2.55}$$

which means that the current yield curve may be represented by the same parameters as the forward rate:

$$\begin{aligned} R(0,\tau) = \beta_0 &+ \beta_1 \frac{1-\exp(-\frac{\tau}{\lambda_1})}{\frac{\tau}{\lambda_1}} + \beta_2 \left[\frac{1-\exp(-\frac{\tau}{\lambda_1})}{\frac{\tau}{\lambda_1}} - \exp(-\frac{\tau}{\lambda_1}) \right] \\ &+ \beta_3 \left[\frac{1-\exp(-\frac{\tau}{\lambda_2})}{\frac{\tau}{\lambda_2}} - \exp(-\frac{\tau}{\lambda_2}) \right] \end{aligned}$$

The parameters from Equation 2.54 can be estimated by minimizing the squared difference between observed data and the theoretical yield curve.

2.5 Correlation

In Section 3.2 we say that interest rates tend to exhibit non-perfect correlation, so in this section we will just give a quick review about correlation.

Correlation is a measure of the statistical relationship between two random variables or observed data values. The most common measure of correlation between two random variables, X and Y is the so-called Pearson's correlation

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}, \tag{2.56}$$

where

$$-1 \leq \rho \leq 1.$$

The interest rates are perfectly correlated when $\rho = 1$, uncorrelated when $\rho = 0$ and negatively correlated when $\rho = -1$.

Chapter 3

Interest Rate Models

Interest rate modelling have been developed during the last decades to estimate prices of interest rate derivatives. It is a branch of mathematical finance where no general model has yet been accepted. There are different models suited to different situations and products. Interest rate models can broadly be divided into short rate models, forward rate models, and LIBOR and swap market models (Björk, 2004). They range from simple one-factor models to more complex multi-factor models.

Short rate models describe the dynamics of the instantaneous spot rate, or short rate, while the forward rate models chooses the forward rate as a fundamental quantity to the model. The problem with both short rate models and forward rate models is that neither the instantaneous short rate, nor the instantaneous forward rate is observable in the market. The LIBOR and swap market models on the other hand are a class of models that describes the evolution of rates that are directly observable in the market. However, these models tend to be more complicated in their setup. When selecting a particular model, it is important to keep in mind the uncertainty principle of modeling: loosely speaking this principle asserts that *the more a model fits the given data, the less it explains* (Park, 2004).

With this principle in mind, and the fact that our context and purpose is limited to Solvency II, the following list contains the desired properties of the interest rate models:

- Available in the literature
- An intuitive model for decision makers
- Ability to calibrate to market prices and/or historical data
- Ability to simulate the model
- Numerical stability

- Reasonable results for use in Solvency II

We will concentrate on short rate models. They are very well documented in the literature, and widely used. Furthermore, our three models are quite intuitive and easy to calibrate and simulate. They all provide an exact fit to the current yield curve. When it comes to reasonable results for use in Solvency II, will we late see that the difference between the best estimate of the liabilities computed using the different models are small. Hopefully, this indicates that they all produce reasonable results. The following sections will explain the dynamics, estimation and simulation for the one-factor Hull-White, the CIR++ models, and the two factor G2++-model. Throughout this chapter we follow Brigo and Mercurio (2007) with supplements from Park (2004) and Dagıstan (2010).

3.1 One Factor Models

Short rate models can be classified as either equilibrium or no-arbitrage models. Equilibrium models are also referred to as *endogenous term structure models*. The reason for this is that the term-structure of interest rates is an output for these models. We start this section by introducing the first, simplest one-factor models, but our main focus in this thesis is on the no-arbitrage models; the Hull-White model and the CIR++-model.

3.1.1 Equilibrium models

The first short rate models that were introduced in the literature were time-homogeneous, which means that the short rate dynamics were only dependent on constant coefficients. Classical short rate models are the Vasicek model (Vasicek, 1977), and the Cox, Ingersoll and Ross (CIR) model (Cox et al., 1985). In these models the short term interest rate follows a mean-reverting process of the form

$$dr(t) = (b - ar(t))dt + \sigma r(t)^{\beta} dW(t). \tag{3.1}$$

Here, b, a, σ and β are positive constants and $W(t)$ is a standard Brownian motion.

The situation where $\beta = 0$ leads to the Vasicek model, i.e the dynamics, defined under risk-neutral measure Q, is

$$dr(t) = [b - ar(t)]dt + \sigma dW(t), \quad r(0) = r_0. \tag{3.2}$$

This dynamic makes the model attractive for several reasons. The equation is linear and can be solved explicitly. The distribution of $r(t)$ is Gaussian, and since the bond price $P(t,T) = E_t[e^{-\int_t^T r(s)ds}]$ can be calculated as an expression dependent on b, a, σ, and $r(t)$, we know the whole interest rate curve at time t.

Traditionally it has been regarded as a major drawback that the Vasicek model allows negative interest rate values, but this is no longer the case, since we now know we might face negative interest rate values.

The CIR model consider the situation where $\beta = 0.5$, introducing the square root in the diffusion coefficient, which together with a condition on the parameters ensure positive interest rates. In order to use the same notation for the CIR and CIR++ model we define $a = \beta$, and $b = \mu$. The CIR model has the following dynamics:

$$dr(t) = \beta(\mu - r(t))dt + \sigma\sqrt{x(t)}dW(t) \quad r(0) = r_0, \tag{3.3}$$

with $2\beta\mu > \sigma^2$. This model will always produce positive interest rates, which used to be a big advantage over the Vasicek model. The instantaneous rate is characterized by a non-central chi square distribution, which is an advantage over the Gaussian distribution as the chi-square distribution provides fatter tails. Furthermore, the CIR model is analytically tractable, but less so than the Vasicek model.

The main problem with both the Vasicek and the CIR model is their endogenous nature. To improve this situation, no-arbitrage models were introduced. These models are modified versions of the endogenous models, where the strategy is to include a time-varying parameter.

3.1.2 The Hull-White Extended Vasicek model

The need for an exact fit to the currently observed yield curve led John C. Hull and Alan White (Hull and White, 1990) to introduce a time-varying parameter in the Vasicek model. This ensures that the model can match the current term structure of interest rates exactly. Furthermore, the model implies a normal distribution for the short-rate process for any given time point. It is analytically tractable and it allows negative interest rates.

Hull and White (1990) assumed that the instantaneous short-rate process, r, under the risk-neutral measure Q has dynamics given by

$$dr(t) = [\theta(t) - \beta(t)r(t)] + \sigma(t)dW(t), \quad r(0) = r_0, \tag{3.4}$$

where r_0 is a positive constant and $\theta(t), a(t)$ and $\sigma(t)$ are deterministic functions of time. This model can be fitted to the term-structure of interest rates and the term structure of spot and forward rate volatility. However, future volatility structures implied by 3.4 are likely to be unrealistic in that they do not conform to typical market shapes (Brigo and Mercurio, 2002).

Therefore, one sets $\beta(t) = \beta$ and $\sigma(t) = \sigma$ and get the following extension of the Vasicek model:

$$dr(t) = [\theta(t) - \beta r(t)]dt + \sigma dW(t), \quad r(0) = r_0. \tag{3.5}$$

Here, β, σ and r_0 are positive constants and $\theta(t)$ is chosen so that the model exactly fit the current term structure of interest rates.

It can be shown that the following holds for θ[1]:

$$\theta(t) = \frac{\delta f^M(0,t)}{\delta T} + \beta f^M(0,t) + \frac{\sigma^2}{2\beta}(1 - e^{-2\beta t}), \tag{3.6}$$

where $f^M(0,t)$ is the market instantaneous forward rate at time 0 for maturity T, which according to Definition 2.5 is:

$$f^M(0,t) = -\frac{\delta \ln P^M(0,T)}{\delta T}. \tag{3.7}$$

Here, $P^M(0,T)$ is the market zero-coupon price for maturity T.

Solving for the instantaneous short rate, $r(t)$:

$$\begin{aligned} d(e^{\beta t}r) &= e^{\beta t}dr + \beta e^{\beta t}rdt = \theta(t)e^{\beta t}dt + \sigma e^{\beta}dW(t) \\ r(t)e^{\beta t} &= r(0) + \int_0^t \theta(u)e^{\beta u}du + \sigma \int_0^t e^{\beta u}dW(u) \\ r(t) &= r(0)e^{-\beta t} + \int_0^t \theta(u)e^{\beta(t-u)}du + \sigma \int_0^t e^{\beta(t-u)}dW(u). \end{aligned}$$

Since start time is arbitrary we get,

$$\begin{aligned} r(t) &= r(s)e^{-\beta(t-s)} + \int_s^t e^{-\beta(t-u)}\theta(u)du + \sigma \int_s^t e^{-\beta(t-u)}dW(u) \\ &= r(s)e^{-\beta(t-s)} + \alpha(t) - \alpha(s)e^{-\beta(t-s)} + \sigma \int_s^t e^{-\beta(t-u)}dW(u), \end{aligned} \tag{3.8}$$

as a solution for Equation 3.5, where

$$\alpha(t) = f^M(0,t) + \frac{\sigma^2}{2\beta^2}(1 - e^{-\beta t})^2. \tag{3.9}$$

The Hull-White model, conditional on $\mathcal{F}_t$, is Gaussian with mean and variance given by

$$E[r(t)|\mathcal{F}_s] = r(s)e^{-\beta(t-s)} + \alpha(t) - \alpha(s)e^{-\beta(t-s)}, \tag{3.10}$$

and

$$Var[r(t)|\mathcal{F}_s] = \frac{\sigma^2}{2\beta}[1 - e^{-2\beta(t-s)}]. \tag{3.11}$$

[1] see e.g. Björk (2004)

Defining the process x by

$$dx(t) = -\beta x(t)dt + \sigma dW(t), \quad x(0) = 0, \tag{3.12}$$

i.e an *Ornstein-Uhlenbeck process*[2], we see that for $s < t$

$$x(t) = x(s)e^{-\beta(t-s)} + \sigma \int_s^t e^{-\beta(t-u)} dW(u), \tag{3.13}$$

which yields $r(t) = x(t) + \alpha(t)$ for each t.

The possibility of negative interest rates have previously been regarded as a drawback for the Hull White model. The risk-neutral probability of negative rates at time t is given by

$$\mathcal{Q}(x(t) < 0) = \Phi\left(-\frac{\alpha(t)}{\sqrt{\frac{\sigma^2}{2\beta}[1 - e^{-2\beta t}]}}\right), \tag{3.14}$$

with Φ denoting the standard normal cumulative distribution function. Normally this probability has been very small, but in countries with low interest rates this probability is no longer negligible.

Parameter estimation

Equations 3.6 and 3.7 show how $\theta(t)$ may be completely determined by the current yield curve. We need to determine the remaining model parameters β and σ. We can estimate these parameters based on historical data by minimizing the sum of squares difference between theoretical and empirical volatilities of monthly absolute spot rate changes, or we can use maximum likelihood estimation. In the first approach we follow Park (2004). The stochastic dynamics for the spot rate implied by a one-factor Hull-White model can be derived by a application of the Itô's rule and is given by:

$$dR(t,T) = \frac{\frac{1}{2}\sigma^2 B^2(t,T) - r(t)}{T-t} dt + \frac{\sigma B(t,t)}{T-t} dw, \tag{3.15}$$

where

$$B(t,T) = \frac{1}{\beta}\left(1 - e^{-\beta(T-t)}\right). \tag{3.16}$$

Discretization of the spot rate $dR(t,T)$ leads to

$$R_{k+1} - R_k = \frac{\frac{1}{2}\sigma^2 B_k^2 - r_k}{T - t_k} \Delta t + \frac{\sigma B_k}{T - t_k} \sqrt{\Delta t} z_{k+1}, \tag{3.17}$$

[2] An Ornstein-Uhlenbeck process is a stationary stochastic process that satisfy the requirements for both a Gaussian process and a Markov process.

where the discrete times are denoted $t_1, t_2, .., $ and z_k is assumed to be independent standard zero-mean, unit standard deviation normal densities. We use simplified notation where,

$$\begin{aligned} R_k &= R(t_k, T), \\ B_k &= B(t_k, T), \\ &. \\ &. \\ &. \end{aligned}$$

and

$$r_k = r(t_k).$$

Equation 3.17 implies that, given r_k and R_k, the conditional density for $\Delta R_{k+1} = R_{k+1} - R_k$ is normal with mean $\frac{\frac{1}{2}\sigma^2 B_k^2 - r_k}{T - t_k}\Delta t$ and standard deviation

$$\sigma_{\Delta R} = \frac{\sigma B_k}{T - t_k}\sqrt{\Delta t}. \tag{3.18}$$

Substituting $T = t_k + \tau$ leads to

$$\sigma_{\Delta R}(\tau) = \frac{\sigma(1 - e^{-\beta\tau})}{\beta\tau}\sqrt{\Delta t}. \tag{3.19}$$

Estimates of β and σ can be obtained by minimizing the objective function

$$J(\beta, \sigma) = \sum_{k=1}^{p} \| \sigma_{\Delta R}(\tau_k) - \sigma_{\Delta R}^{obs}(\tau_k) \|^2 . \tag{3.20}$$

Denoting the time series of daily spot rate changes for a given maturity τ_k by $\Delta R_1, .., \Delta R_N$, the associated sample variance, $\sigma_{\Delta R}^{obs}(\tau_k)$ is determined from

$$E[\Delta R] = \frac{1}{N}\sum_{j=1}^{N} \Delta R_j, \tag{3.21}$$

$$\sigma_{\Delta R}^{obs}(\tau_k) = \frac{1}{N-1}\sum_{j=1}^{N} (\Delta R_j - E[\Delta R])^2. \tag{3.22}$$

We assume 12 trading months in a year, which gives $\Delta t = 1/12$. The objective function can then be expressed as

$$\min J(\beta, \sigma) = \sum_{k=1}^{p} \left|\left| \frac{\sigma(1 - e^{-\beta\tau_k})}{\beta\tau_k} - \sigma_{\Delta R}^{obs} \cdot \sqrt{12} \right|\right|^2 \tag{3.23}$$

Using the Maximum likelihood approach we utilize the fact that the Hull-White model has the same short rate dynamics as the Vasicek model, it is only the long-term mean parameter that is time-dependent in the Hull-White model. Therefore, for the mean-reversion and volatility parameters for the Hull-White model we will use the maximum likelihood estimates of these parameters in the Vasicek model. We follow Brigo and Mercurio (2007) and define the Vasicek model as in Equation 3.2. By integrating, between any instants s and t, we obtain

$$r(t) = r(s)e^{-a(t-s)} + \frac{b}{a}(1 - e^{-a(t-s)}) + \sigma \int_s^t e^{-a(t-su} dW(u). \tag{3.24}$$

The variable $r(t)$, conditional on $\mathcal{F}_s$, is normally distributed with mean $r(s)e^{-a(t-s)} + \frac{b}{a}(1 - e{-}a(t-s))$ and variance $\frac{\sigma^2}{2a}[1 - e^{-2a(t-s)}]$. Further, it is natural to estimate the following functions of the parameters; $\mu = \frac{b}{a}$, $\alpha = e^{-a\delta}$ and $V^2 = \frac{\sigma^2}{2a}(1-e^{-2a\delta})$, where δ is the time step $r_0, r_1, ..., r_n$ of r. The maximum likelihood for α, μ and V^2 are

$$\begin{aligned}
\hat{\alpha} &= \frac{n\sum_{i=1}^n r_i r_{i-1} - \sum_{i=1}^n r_i \sum_{i=1}^n r_{i-1}}{n\sum_{i=1}^n r_{i-1}^2 - (\sum_{i=1}^n r_{i-1})^2}, \\
\hat{\mu} &= \frac{\sum_{i=1}^n [r_i - \hat{\alpha} r_{i-1}]}{n(1-\hat{\alpha})}, \\
\hat{V^2} &= \frac{1}{n}\sum_{i=1}^n \left[r_i - \hat{\alpha} r_{i-1} - \hat{\beta}(1-\hat{\alpha}) \right],
\end{aligned}$$

meaning that

$$\begin{aligned}
\hat{\beta} &= \frac{-\log(\hat{\alpha})}{\delta}, \\
\hat{\sigma} &= \frac{2\beta \hat{V}^2}{(1 - e^{-2\hat{\beta}\delta})}.
\end{aligned}$$

Simulation

The first thing we need to do when we want to simulate the spot interest rate $R(t, T)$, is to generate $r(t)$. We start by generating forward paths for $x(t)$, from Equation 3.13, according to the discretization

$$x_{k+1} = e^{-\beta \Delta t} x_k + \sigma \sqrt{\frac{1 - e^{-2\beta \Delta t}}{2\beta}} z_{k+1}, \tag{3.25}$$

where $z_{k+1} \sim N(0, 1)$ i.e. a standard zero-mean, unit-variance Gaussian (Park, 2004). Once x has been generated, the corresponding value for $r(t)$ can be recovered from $x(t)$

$$r(x(t),t) = x(t) + f^M(0,t) + \frac{\sigma^2}{2\beta^2}(1-e^{-\beta t})^2, \tag{3.26}$$

where $f^M(0,t)$ is the market instantaneous forward rate given in Equation 3.7.

Given the short rate $r(t)$, the spot interest rate $R(t,T)$ is a deterministic function of $r(t)$. We simulate $R(t,T)$ by using the following relationship:

$$R(t,T) = -\frac{\ln P(t,T)}{T-t}, \tag{3.27}$$

where $P(t,T)$ is the price at time t of a zero-coupon bond with maturity T. For the Hull-White model, $P(t,T)$ can be derived by computing the expectation 2.24. Using the Markov property for Itô processes, defined in Theorem 2.1, this expression is equivalent to

$$E_Q\left(exp\left(-\int_0^{T-t} r(s)^y ds\right)\right)\Bigg|_{y=r(t)}, \tag{3.28}$$

where Q is the risk-neural measure. First, we calculate $\int_0^{T-t}(r(s)^y ds)|_{y=r(t)}$:

$$\begin{aligned}
&\left(\int_0^{T-t} r(s)^y ds\right)\Bigg|_{y=r(t)} = \\
&= \frac{\alpha}{\beta}(T-t) + \left(r(t)-\frac{\alpha}{\beta}\right)\left(\frac{1-e^{-\beta(T-t)}}{\beta}\right) + \sigma\int_0^{T-t}\int_0^s e^{\beta(u-s)}dW(u)ds \\
&= \frac{\alpha}{\beta}(T-t)\left(r(t)-\frac{\alpha}{\beta}\right)\left(\frac{1-e^{-\beta(T-t)}}{\beta}\right) + \sigma\int_0^{T-t}\left(\int_u^{T-t} e^{\beta(u-s)}ds\right)dW(u) \\
&= \frac{\alpha}{\beta}(T-t)\left(r(t)-\frac{\alpha}{\beta}\right)\left(\frac{1-e^{-\beta(T-t)}}{\beta}\right) + \sigma\int_0^{T-t}\frac{e^{\beta(u-(T-t))}}{\beta}dW(u).
\end{aligned} \tag{3.29}$$

This integral is a Gaussian variable, and hence we can calculate the expectation as

$$\begin{aligned}
P(t,T) &= E_Q\left[\exp\left(-\int_0^{T-t} r(s)^y ds\right)\right)\Bigg|_{y=r(t)} \\
&= \exp\left[E_Q\left(-\int_0^{T-t} r(s)^y ds\right)\Bigg|_{y=r(t)}\right. \\
&\quad \left. + \frac{1}{2}var_Q\left(-\int_0^{T-t} r(s)ds\right)\right],
\end{aligned}$$

with

$$E_Q\left(-\int_0^{T-t} r(s)^y ds\right)\Bigg|_{y=r(t)} = \left(\frac{\alpha}{\beta}(T-t) + (r(t) - \frac{\alpha}{\beta}\frac{1-e^{-\beta(T-t)}}{\beta}\right)$$

$$Var_Q\left(-\int_0^{T-t} r(s)^y ds\right)\Bigg|_{y=r(t)} = \frac{\sigma^2}{\beta^2}\int_0^{T-t}(1-e^{\beta(u-(T-t))})^2 du.$$

Hence, the bond price can be expressed in the form

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}, \tag{3.30}$$

i.e. affine term structure, where $B(t,T)$ and $A(t,T)$ are defined respectively as

$$B(t,T) = \frac{1}{\beta}\left(1 - e^{-\beta(T-t)}\right), \tag{3.31}$$

$$A(t,T) = \frac{P^M(0,T)}{P^M(0,t)}\exp\{B(t,T)f^M(0,t) - \frac{\sigma^2}{4\beta}(1-e^{-2\beta t})B(t,T)^2\}. \tag{3.32}$$

3.1.3 CIR++

Hull and White (1990) also proposed an extension of the CIR-model based on the same idea considering time dependent coefficients like for the Vasicek (1977) model:

$$dr(t) = [\theta(t) - \beta(t)r(t)]dt + \sigma(t)\sqrt{r(t)}dW(t), \tag{3.33}$$

where β, σ and θ are deterministic functions of time. However, this model is not analytically tractable. Indeed, no analytical solution exists for the zero-coupon bond prices. The same drawback holds for the simplified dynamics where β and σ are constant, and only θ is time dependent.

A different approach was illustrated by Brigo and Mercurio (2007) as a method to extend any time-homogenous short rate model, so as to exactly reproduce any observed term structure while preserving the possible analytical tractability of the original model. In this approach a deterministic shift function is added to the short rate process. In the case of the Vasicek model, the extension is equivalent to that of Hull and White (1990). However for the CIR model, this extension is more analytically tractable, and avoids the problem concerning numerical solutions.

From Brigo and Mercurio (2007), this extension of the CIR model yields the unique short-rate model featuring properties such as:

- Exact fit of any observed term structure
- Analytical formula for e.g bond prices
- The distribution of the instantaneous spot rates has fatter tails than in the Gaussian case, and through restrictions it is possible to guarantee positive interest rates

The extension of the CIR models is referred to as the CIR++-model. The short rate dynamics, under the risk neutral measure, are given by

$$dx(t) = \beta(\mu - x(t))dt + \sigma\sqrt{x(t)}dW(t), \quad x(0) = x_0 \tag{3.34}$$
$$r(t) = x(t) + \varphi(t), \tag{3.35}$$

where x_0, β, μ and σ are positive constants such that $2\beta\mu > \sigma^2$, thus ensuring that the process x remains positive. $W(t)$ denotes a standard Brownian motion, and $\varphi(t)$ is the deterministic shift function chosen to fit the initial term structure. The CIR++ model provides an exact fit to the current term structure of the interest rates by setting

$$\varphi(t) = \varphi^{CIR}(t) = f^M(0,t) - f^{CIR}(0,t), \tag{3.36}$$

where

$$f^{CIR}(0,t) = \frac{2\beta\mu(e^{th}-1)}{2h + (\beta + h)(e^{th}-1)} + x_0 \frac{4h^2 e^{th}}{[2h + (\beta + h)(e^{th}-1)]^2}. \tag{3.37}$$

Here $h = \sqrt{\beta^2 + 2\sigma^2}$ and $f^M(0,t)$ is the instantaneous forward rate.

Parameter Estimation

The parameters β, μ and σ of the CIR++-model may be estimated using Maximum likelihood estimation. One then uses the fact that the increments of the short rate follows a non-central chi-square distribution (Brigo and Mercurio, 2007).

To obtain the likelihood function for CIR++ we need the probability density to the short rate in this model, which is given by

$$x(t) = \frac{\sigma^2(1 - e^{-\beta(t-u)})}{4\beta} \chi_d^2 \left(\frac{4\beta e^{-\beta(t-u)}}{\sigma^2(1 - e^{-\beta(t-u)})} x(u) \right), \quad t > u. \tag{3.38}$$

Here

$$d = \frac{4\mu\beta}{\sigma^2}, \tag{3.39}$$

This shows that, given $r(u)$, $r(t)$ is distributed as $(\sigma^2(1-e^{-\beta(t-u)}))/(4\beta)$ times a non-central chi-square random variable with d degrees of freedom and the non-centrality parameter is given by

$$\lambda = \frac{4\beta e^{-k(t-u)}}{\sigma^2(1-e^{-\beta(t-u)})} r(u) \tag{3.40}$$

(Dagıstan, 2010). The likelihood function for an interest rate series with N observations is:

$$L(\beta, \mu, \sigma) = \prod_{t=1}^{N-1} p(r_{t+\Delta t} | r_t; \beta, \mu, \sigma). \tag{3.41}$$

It can be shown that for the CIR++-model

$$p(r_{t+\Delta t}|r_t) = 2 \cdot c \cdot g(2 \cdot r_{t+\Delta t} | 2 \cdot c \cdot r_t), \tag{3.42}$$

where

$$c = \frac{2\beta}{\sigma^2(1-e^{-\beta\Delta t})}, \tag{3.43}$$

and $g(\cdot)$ is the non-central χ^2 distribution with the degrees of freedom is defined in Equation 3.39 and the non-central parameter is λ, defined in Equation 3.40.

Since the CIR++ model is non-gaussian, and follows a chi-square distribution we need to use numerical optimization to find the MLEs. Furthermore, to determine the instantaneous forward rate we use the Svensson model described in Section 2.4.

Simulation

Simulations of the spot rate $R(t, T)$ are obtained by first simulating the short rate, which again is generated by first simulating x using

$$x(t_{k+1}) = \frac{Y}{2c}. \tag{3.44}$$

Here Y is a variate from a non-central chi-squared distribution defined in the "Parameter Estimation" section, and c is defined in Equation 3.43. The short rate is then computed as

$$r(t) = x(t) + \varphi(t), \tag{3.45}$$

where $\varphi(t)$ is given in Equation 3.36.

Once we have simulated the short rate, we obtain the spot rate by using the following relationship:

$$R(t,T) = -\frac{\ln P(t,T)}{T-t}, \tag{3.46}$$

where $P(t,T)$ is the price at time t of a zero-coupon bond with maturity T.

The CIR++ model is in the class of affine term structure models which will help us to find $P(t,T)$. In Section 2.3.3 we described the affine term structure and that the differential equation solving B is a Riccati equation, meaning that it, in general, needs to be solved numerically. However in case of CIR we have that the equations are explicitly solvable for A and B since $\lambda(t) = -\theta$, $\eta(t) = \theta\beta$, $\gamma(t) = \sigma^2$ and $\delta(t) = 0$. (Brigo and Mercurio, 2007). The Equations 2.50 and 2.49 become

$$\begin{cases} \frac{\partial}{\partial t}B(t,T) + \theta B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) + 1 & = 0, \\ B(T,T) & = 0. \end{cases} \tag{3.47}$$

$$\begin{cases} \frac{\partial}{\partial t}A(t,T) - \theta\beta B(t,T) & = 0 \\ A(T,T) & = 0. \end{cases} \tag{3.48}$$

Since dynamics of CIR++ is the dynamics of CIR model plus the deterministic function φ, we can calculate the bond price from 3.47 and 3.48. From Brigo and Mercurio (2007), the price at time t, maturing at T is given by

$$P(t,T) = \overline{A}(t,T)e^{-B(t,T)r(t)}, \tag{3.49}$$

where

$$\overline{A}(t,T) = \frac{P^M(0,T)A(0,t)e^{B(0,t)x_0}}{P^M(0,t)A(0,T)e^{B(0,T)x_0}} A(t,T)e^{B(t,T)\varphi^{CIR}(t;\alpha)}, \tag{3.50}$$

$$A(t,T) = \left[\frac{2he^{(\beta+h)(T-t)/2}}{2h + (\beta+h)(e^{(T-t)h}-1)}\right]^{2\beta\mu/\sigma^2} \tag{3.51}$$

and

$$B(t,T) = \frac{2(e^{(T-t)h}-1)}{2h + (\beta+h)(e^{(T-t)h}-1)}. \tag{3.52}$$

$P^M(0,t)$ is the market discount factor for the maturity T and $\varphi^{CIR}(t;\alpha)$ is defined as in Equation 3.36.

3.2 Two Factor Models

The Hull-White- and CIR++ models are one-factor models, which means that at every time step, interest rates for all maturities in the yield curve are perfectly correlated. However, interest rates tend to exhibit non-perfect correlation. Therefore, when modelling interest rates we also want to study models with a more realistic correlation pattern. This can be achieved using multi-factor models.

The choice of the number of factors involves a compromise between numerically efficient implementation and the capability of the model to represent a realistic correlation pattern, and to give satisfactory fit market data. Historical analysis of the yield curve suggests that one-factor models explain from 68% to 76% of the total variation, and two factor models 85% to 90% of variations in the yield curve (Brigo and Mercurio, 2007). In this thesis the focus will be on the two-factor Gaussian(G2++)-model.

3.2.1 The G2++-model

Gaussian models like the G2++ model is attractive since they are analytically tractable, and the Gaussian distribution allows the derivation of explicit expression for the zero-coupon curve. Further, the G2++ model allows negative interest rate which, according to Brigo and Mercurio (2007), is an *unpleasant feature*. However, negative interest rates is no longer regarded as a drawback.

In the G2++ model, the short rate process is given by the sum of two correlated Gaussian factors plus a deterministic function. The G2++-model bears many similarities with the two-factor Hull-White model, however the G2++ model leads to less complicated formula and is easier to implement. The dynamics of the instantaneous-short rate process is given as

$$\begin{aligned} dx(t) &= -\alpha x(t)dt + \gamma dW_1(t), \quad x(0) = 0, \\ dy(t) &= -\beta y(t)dt + \eta dW_2(t), \quad y(0) = 0, \\ r(t) &= x(t) + y(t) + \varphi(t), \end{aligned} \tag{3.53}$$

where α and β are constants reflecting the rate of mean reversion, γ and η are volatility constants, and W_1 and W_2 are standard Brownian motions with correlation κ.

From Brigo and Mercurio (2007), integration of Equation 3.53 implies that for each $s < t$

$$\begin{aligned} r(t) =& x(s)e^{-\alpha(t-s)} + y(s)e^{\beta(t-s)} \\ &+\gamma \int_s^t e^{\alpha(t-u)} dW_1(u) + \eta \int_s^t e^{\beta(t-u)} dW_2(u) + \varphi. \end{aligned} \tag{3.54}$$

This means that $r(t)$, conditional on $\mathcal{F}_s$, is normally distributed with mean and variance given respectively by

$$E[r(t)|\mathcal{F}_s] = x(s)e^{\alpha(t-s)} + y(s)e^{\beta(t-s)} + \varphi, \tag{3.55}$$

$$\begin{aligned} Var[r(t)|\mathcal{F}_s] =& \frac{\gamma^2}{2\alpha}\left[1 - e^{-2\alpha(t-s)}\right] + \frac{\eta}{2\beta}\left[1 - e^{-2\beta(t-s)}\right] \\ &+ 2\kappa\frac{\gamma\eta}{\alpha+\beta}\left[1 - e^{-(\alpha+\beta)(t-s)}\right]. \end{aligned} \tag{3.56}$$

In particular

$$r(t) = \sigma\int_0^t e^{-\alpha(t-u)}dW_1(u) + \eta\int_0^t e^{-\beta(t-u)}dW_2(u) + \varphi(t). \tag{3.57}$$

Parameter Estimation

The G2++-model have five parameters that need to be estimated; $\alpha, \beta, \gamma, \eta$ and κ. We estimate these parameters based on historical data, by minimizing the sum of squared differences between theoretical and empirical volatilities of monthly absolute spot rate changes. The approach is from Park (2004), and is similar to the one for the Hull-White model, described in Section 3.1.2.

The spot rates can be expressed in terms of the state variables $x(t)$ and $y(t)$ as

$$R(t, t+\tau) = c_1(\tau)x(t) + c_2(\tau)y(t) + v(t), \tag{3.58}$$

where

$$c_1(\tau) = \frac{1 - e^{-\alpha\tau}}{\alpha\tau}, \tag{3.59}$$

$$c_2(\tau) = \frac{1 - e^{-\beta\tau}}{\beta\tau}, \tag{3.60}$$

and

$$\begin{aligned} v(t) =& -\frac{1}{2\tau}[V(t, t+\tau) - V(0, t-\tau) + V(0, t)] \\ &- \frac{1}{\tau}\log\frac{P(0, t+\tau)}{0, t}. \end{aligned} \tag{3.61}$$

In Equation 3.61 $V(t, T)$ is given by:

$$\begin{aligned} V(t,T) =& \frac{\gamma^2}{\alpha^2}\left[T - t + \frac{2}{\alpha}e^{-\alpha(T-t)} - \frac{1}{2\alpha}e^{-2\alpha(T-t)} - \frac{3}{2\alpha}\right] \\ &+ \frac{\eta^2}{\beta^2}\left[T - t + \frac{2}{\beta}e^{-\beta(T-t)} - \frac{1}{2\beta}e^{-2\beta(T-t)} - \frac{3}{2\beta}\right] \\ &+ 2\kappa\frac{\gamma\eta}{\alpha\beta}\left[T - t + \frac{e^{-\alpha(T-t)} - 1}{\alpha} + \frac{e^{-\beta(T-t)} - 1}{\beta} - \frac{e^{-(\alpha+\beta)(T-t)} - 1}{\alpha+\beta}\right]. \end{aligned} \tag{3.62}$$

Applying Itô's rule, one obtains the following stochastic dynamics of $R(t,T)$

$$dR(t,t+\tau) = \left(\frac{\delta v(t)}{\delta t} - c_1(\tau)\alpha x(t) - c_2(\tau)\beta y(t)\right) dt \\ + c_1(\tau)\gamma dW_1(t) + c_2(\tau)\eta dW_2(t), \tag{3.63}$$

where $dW_1 \cdot dW_2 = \kappa dt$. Discretization leads to

$$\Delta R_{k+1} = \left(\frac{\delta v(t)}{\delta t} - c_1(\tau)\alpha x_k - c_2(\tau)\beta y_k\right) \Delta t \\ + c_1(\tau)\gamma\sqrt{\Delta t}\epsilon^1_{k+1} + c_2(\tau)\eta\sqrt{\Delta t}\epsilon^2_{k+1}, \tag{3.64}$$

where $\Delta R_{k+1} = R_{k+1} - R_k$, and

$$\begin{bmatrix} \epsilon^1_k \\ \epsilon^2_k \end{bmatrix} \sim N(0, \begin{bmatrix} \kappa & 1 \\ 1 & \kappa \end{bmatrix} \tag{3.65}$$

are independent and identically distributed(iid) over k. Given R_k, x_k and y_k, it follows that ΔR_{k+1} is normally distributed with variance

$$\sigma^2_{\Delta R}(\tau) = \Delta t[(c_1(\tau)\gamma)^2 + (c_2(\tau)\eta)^2 + 2c_1(\tau)c_2(\tau)\gamma\eta\kappa]. \tag{3.66}$$

Inserting $c_1(\tau)$ and $c_2(\tau)$ from Equations 3.59 and 3.60, and preforming a little bit of calculations, we see that standard deviation is

$$\sigma_{\Delta R}(\tau) = \frac{\sqrt{\Delta t}}{\alpha\beta\tau}\left[[\beta\gamma(1-e^{-\alpha\tau})]^2[\alpha\eta(1-e^{-\beta\tau})]^2 + 2\alpha\beta\gamma\eta\kappa(1-e^{-\alpha\tau})(1-e^{-\beta\tau})\right]^{\frac{1}{2}}. \tag{3.67}$$

The spot rate volatility calibration for G2++ is identical to that for the Hull-White model. The objective function to be minimized can be expressed as

$$\min J(\alpha,\beta,\gamma,\eta,\kappa) = \sum_{k=1}^{p} \left\| \frac{\sigma_{\Delta R}(\tau_k)}{\sqrt{\Delta t}} - \sigma^{obs}_{\Delta R}(\tau_k)\cdot\sqrt{12} \right\|^2. \tag{3.68}$$

Simulation

Similar to the Hull-White model we want to simulate the spot interest rate $R(t,T)$ by first generating $r(t)$. To generate the forward paths for x and y, we use the fact that $(x(t), y(t))$ is a two-dimensional Ornstein-Uhlenbeck process. From Park (2004) we have the following iterations:

$$x_{k+1} = e^{-\alpha\Delta t}x_k + \gamma\sqrt{\frac{1-e^{-2\alpha\Delta t}}{2\alpha}}z^1_{k+1}, \tag{3.69}$$

$$y_{k+1} = e^{-\beta\Delta t}y_k + \eta\sqrt{\frac{1-e^{-2\beta\Delta t}}{2\beta}}z^2_{k+1}. \tag{3.70}$$

$$\tag{3.71}$$

Once x and y have been generated, the corresponding value for r can be recovered from Equation 3.53, which we repeat here for convenience:

$$r(t_k) = x(t_k) + y(t_k) + \varphi(t_k). \tag{3.72}$$

Here $\varphi(t_k)$ is given by

$$\varphi(t_k) = f^M(0,t_k) + \frac{\gamma^2}{2\alpha^2}(1-e^{-\alpha t_k})^2 + \frac{\eta^2}{2\beta^2}(1-e^{-\beta t_k})^2 + \kappa\frac{\gamma\eta}{\alpha\beta}(1-e^{-\alpha t_k})(1-e^{-\beta t_k}).$$

Furthermore, $t_k = k\Delta t$, $f^M(0,t_k)$ is the instantaneous market forward rate, and each (z_k^1, z_k^2) is a two-dimensional zero-mean Gaussian distribution with covariance matrix equal to

$$Q = \begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}. \tag{3.73}$$

Simulations of spot rates $R(t,T)$ are obtained by using the following relationships:

$$R(t,T) = -\frac{\ln P(t,T)}{T-t}, \tag{3.74}$$

where P(t,T) is the price at time t of a zero-coupon bond maturing at T so that

$$P(t,T) = E\left(e^{-\int_t^T r(s)ds} \mid \mathcal{F}_t\right). \tag{3.75}$$

Here E denotes the expectation under the risk-adjusted measure Q. We follow Brigo and Mercurio (2007) when deriving φ and $P(t,T)$. First, in order to compute this expectation, we need the following lemma:

Lemma 3.1. *For each t, T the random variable*

$$I(t,T) := \int_t^T [x(u) + y(u)]du$$

conditional to the sigma-field $\mathcal{F}_t$ is normally distributed with mean $M(t,T)$ and variance $V(t,T)$, respectively given by

$$M(t,T) = \frac{1-e^{-\alpha(T-t)}}{\alpha}x(t) + \frac{1-e^{-\beta(T-t)}}{\beta}y(t), \tag{3.76}$$

and

$$\begin{aligned} V(t,T) =& \frac{\gamma}{\alpha^2}\left[T - t + \frac{2}{\alpha}e^{-\alpha(T-t)} - \frac{1}{2\alpha}e^{-2\alpha(T-t)} - \frac{3}{2\alpha}\right] \\ &+ \frac{\eta}{\beta^2}\left[T - t + \frac{2}{\beta}e^{-\beta(T-t)} - \frac{1}{2\beta}e^{-2\beta(T-t)} - \frac{3}{2\beta}\right] \\ &+ 2\kappa\frac{\gamma\eta}{\alpha\beta}\left[T - t + \frac{e^{-\alpha(T-t)}-1}{\alpha} - \frac{e^{-\beta(T-t)}-1}{\beta} - \frac{e^{-(\alpha+\beta)(T-t)}}{\alpha+\beta}\right]. \end{aligned} \tag{3.77}$$

See Brigo and Mercurio (2007) for proof.
Applying Lemma 3.1 and the fact that $r(t)$ is normally distributed, we have

Theorem 3.1. *The price at time t of a zero-coupon bond maturing at time T and with unit face value is*

$$P(t,T) = \exp\left[-\int_t^T \varphi(u)du - \frac{1-e^{\alpha(T-t)}}{\alpha}\, x(t) - \frac{1-e^{\beta(T-t)}}{\beta} y(t) + \frac{1}{2}V(t,T)\right]. \tag{3.78}$$

Further assume that the term structure of discount factors that is currently observed in the market is given by the smooth function $T \mapsto P^M(0,T)$. Denote the instantaneous forward rate at time 0 for a maturity by $f^M(0,T)$, i.e

$$f^M(0,T) = -\frac{d\ln P^M(0,T)}{dT}. \tag{3.79}$$

The model 3.53 fits the currently-observed term structure of discount factors if and only if, for each maturity $T \le T^*$ the discount factor $P(0,T)$ produced by 3.53 agree with the one in the market, i.e if and only if

$$P^M(0,T) = exp\left(-\int_0^T \varphi(u)du + \frac{1}{2}V(0,T)\right). \tag{3.80}$$

To derive the φ so that the model fits the currently observed term structure we take the logarithms of both sides and differentiate with respect to T, i.e.

$$\frac{d\log P^M(0,T)}{dT} = -\varphi(T) + \frac{\gamma^2}{2\alpha^2}(1-e^{-\alpha t})^2 + \frac{\eta^2}{2\beta^2}(1-e^{-\beta t})^2 \frac{\kappa\gamma\eta}{\alpha\beta}(1-e^{-\alpha t})(1-e^{-\beta t}). \tag{3.81}$$

We solve for φ and let $f^M(0,T)$ defined as in Equation 3.79:

$$\varphi(t) = f^M(0,t) + \frac{\gamma^2}{2\alpha^2}(1-e^{-\alpha t})^2 + \frac{\eta^2}{2\beta^2}(1-e^{-\beta t})^2 \frac{\kappa\gamma\eta}{\alpha\beta}(1-e^{-\alpha t})(1-e^{-\beta t}). \tag{3.82}$$

Note that

$$V(t,T) = \frac{\gamma}{\alpha}\int_t^T [1-e^{-\alpha(T-u)}]^2 du + \frac{\eta^2}{\beta^2}\int_t^T [1-e^{-\beta(T-t)}]^2 du + 2\kappa\frac{\gamma\eta}{\alpha\beta}\int_t^T [1-e^{-\alpha(T-u)}][1-e^{-\beta(T-u)}]du. \tag{3.83}$$

Finally, in order to get an expression for the zero-coupon prices, we have

$$exp\left[-\int_t^T \varphi(u)du\right] = exp\left[-\int_0^T \varphi(u)du\right] exp\left[-\int_0^t \varphi(u)du\right] = \frac{P^M(0,T)\exp[-\frac{1}{2}V(0,T)]}{P^M(0,t)\exp[-\frac{1}{2}V(0,t)]}. \tag{3.84}$$

The corresponding zero-coupon bond price at time t follows from Equations 3.80 and 3.84:

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)} e^{A(t,T)}, \tag{3.85}$$

where

$$A(t,T) = \frac{1}{2}[V(t,T) - V(0,T) + V(0,t)] - \frac{1 - e^{-\alpha(T-t)}}{\alpha} x(t) - \frac{1 - e^{-\beta(T-t)}}{\beta} y(t).$$

Here $V(t,T)$ is given by Equation 3.62.

Chapter 4

Estimation and Simulation

4.1 Parameter Estimation

When using the interest rate models in Solvency II framework one has to use a risk free term structure specified by European Insurance an Occupational Pension Authority (EIOPA). The term structure of interest rate models is the relationship between the interest rate or bond and different terms or maturity, also known as a zero-coupon curve, or yield curve. This risk free yield curve is published for different currencies, including NOK, for the current reference date. See Figure 4.1.

The interest rate models should further be calibrated to an appropriate volatility measure. One alternative is calibration to market prices of different interest rate derivatives. A problem with this approach however, is that derivatives with long maturities are often not available. We can also use historical interest rate data to determine the model parameters. Calibration based on historical data are often more stable than those implied by derivative prices.

If liquid market prices exist, they should be used, since the purpose of the use of the interest rate model is pricing. The model parameters are typically fitted to match the prices of liquid market instruments such as swaps, caps, or liquid bonds seen in the market, Park (2004). However, for the Norwegian market the availability of option prices across different maturities is limited. When calibrating interest rate models, Brigo and Mercurio (2007) suggest that illiquid market data should be left out. We are left with three options:

- Using interest rate derivative prices from a more liquid market as proxies.
- Using fictitious prices.
- Using historical data.

The method of calibration depends on the particular application. For our purpose calibration using historical data is reasonable.

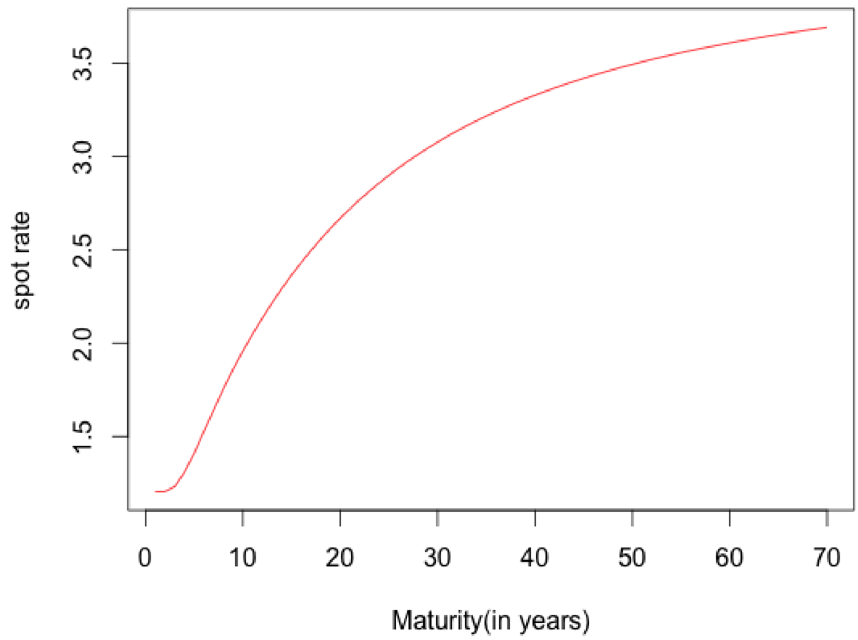

Figure 4.1: The yield curve specified by EIOPA in January 2016.

The short rate, $r(t)$, is a key ingredient in all our models. Since the short rate in reality is stochastic, and we assume a risk-neutral world, the natural way of evaluating $P(t,T)$ is to take the expectation, as shown in 2.24. Knowledge of $P(t,T)$ for arbitrary values of t and T completely define the interest rate term-structure, hence knowledge of the short rate process completely determines the term-structure. However, the theoretical definition of the short rate does not exist in reality. In other words, the short rate can not be directly observed. The overnight rate, or the daily rate is not considered to be a good proxy for the short rate because they are driven by a different set of economic factors and do not accurately reflect the principal factors driving the term-structure. Common practice is to use the 1-month or 3-month rates as a proxy. We will use the 3-month rate in this thesis.

The same period is used to estimate the parameters of all three interest rate models. Our data set consists of monthly data from the 3-month rate and swap rates with maturities 1 to 10 years, from the period December 2003 to December 2013. Figure 4.2 shows the swap rates and the 3-month rate. Here, the dark green line corresponds to the 1-year rate, the dark blue line to the 2-year rate and so on. The purple line with the smallest volatility corresponds to the 10-year maturity rate. Finely, the dotted blue line is the 3-month rate.

Figure 4.2: Norwegian swap rates from December 31, 2003 to December 31, 2013 and the 3-month rate (dotted line).

When estimating the parameters we have used maximum likelihood for the Hull-White and CIR++ model, while for G2++ we have minimized the sum of squared differences between theoretical and empirical volatilities. The values of the estimated parameters are given in Table 4.1.

Hull White-parameters					
Parameter	$\hat{\mu}$	$\hat{\beta}$	$\hat{\sigma}$		
Value	0.0268	0.2026	0.0099		
CIR++-parameters					
Parameter	μ	β	σ		
Value	0.0467	0.0471	0.0468		
G2++-parameters					
Parameter	α	β	γ	η	κ
Value	0.6838	0.0931	1e-6	0.0099	-0.8809

Table 4.1: Estimated parameters for our three models.

4.2 Simulation

Our goal is to predict future interest rates, which we can use in the computation of the best estimate of liabilities under Solvency II. For this we want the expected future values of the interest rates to correspond to the yield curve prescribed by EIOPA. We have used the yield curve from January 2016, which is shown in Figure 4.1. From the yield curve we can derive $P^M(0,T)$ which is needed in all of our models.

With the yield curve described and the parameter estimates shown in Section 4.1 we generated 10 000 simulations with a time horizon of 60 years for maturities 1 to 3 years for each of the three models. The short rate is shown in Figure 4.3. Here we can see that both the Hull-White model and the G2++ model produce negative interest rates, while for the CIR++ model all rates are above zero. We can also notice that the CIR++ model leads to slightly larger short rates in the long run.

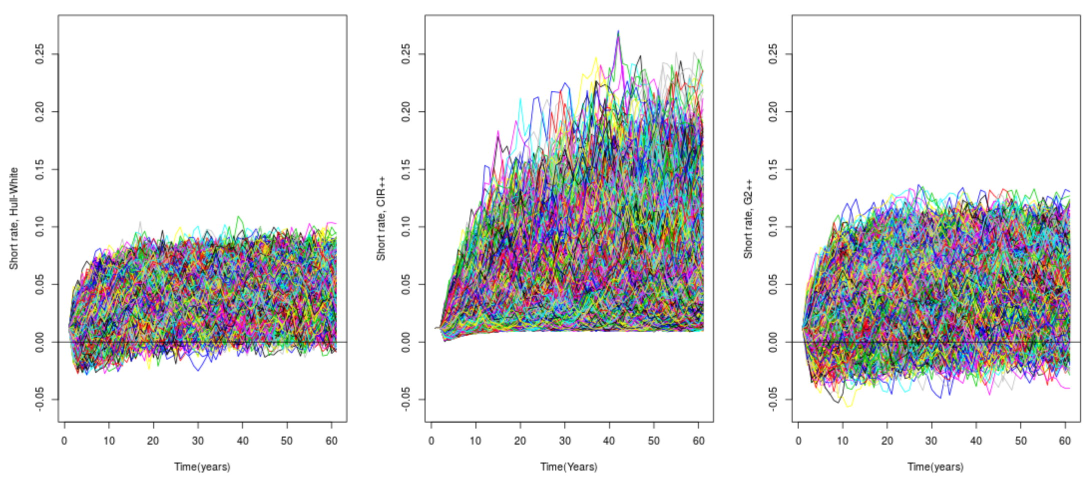

Figure 4.3: Simulated future paths for the instantaneous short rate for the Hull-White model (left), the CIR++ model (middle) and the G2++ model (right)
.

Further, is it interesting to have a look at the mean and standard deviation, corresponding to the 10 000 simulations shown in Figures 4.4 and 4.5 respectively. Notice even when we used the same historical data for estimation, the simulations from the three models have quite different mean and volatility characteristics, especially far into the future.

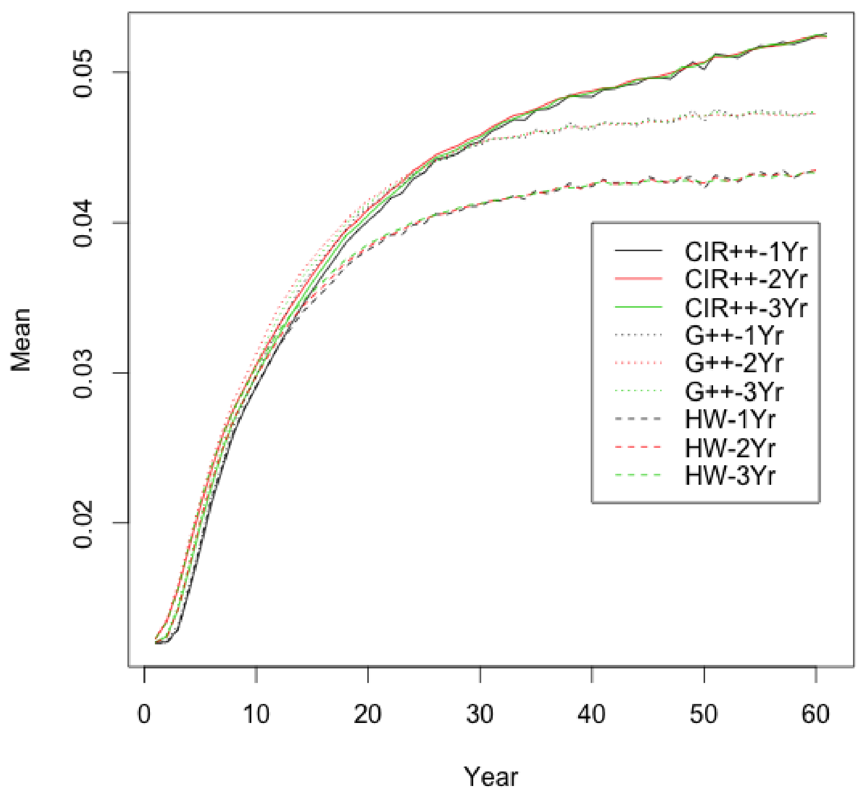

Figure 4.4: Mean over 10 000 simulations of spot rates with 1, 2 and 3 years maturity using Hull White, CIR++ and G2++.

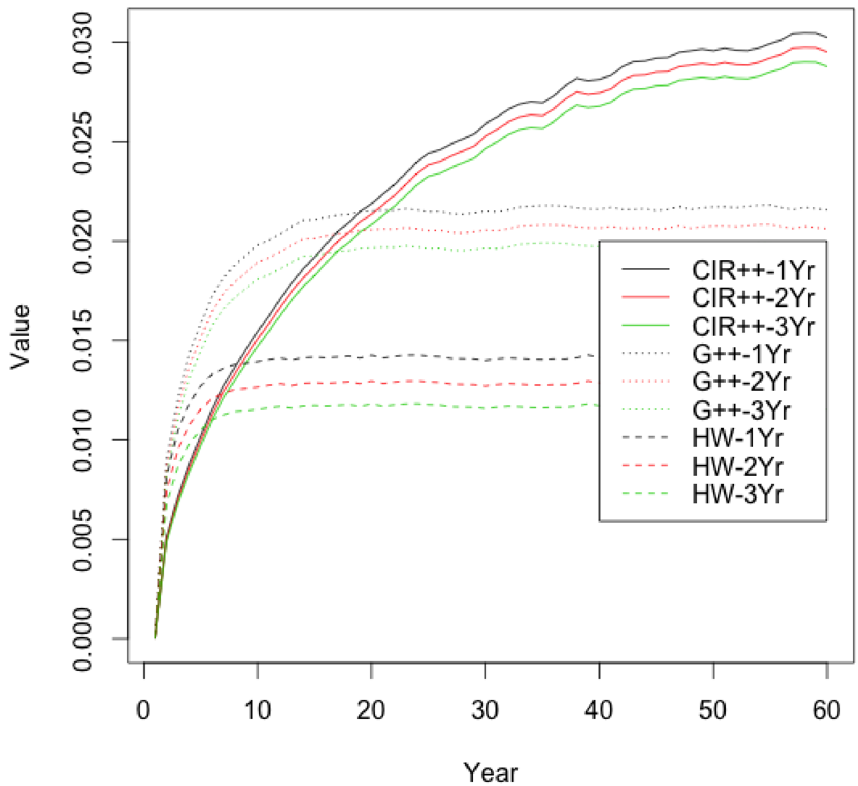

Figure 4.5: Standard deviation of 10 000 simulations of spot rate with 1, 2 and 3 years maturity generated using Hull White, CIR++ and G++.

We can show that the difference far into the future does not mean that not all three models fit the initial term structure. Let the discount factor in simulation s and year r be $d_{t,s}$ and define it as

$$d_{t,s} = \frac{1}{\prod_{u=1}^{t}(1 + R(u, u+1, s))}, \tag{4.1}$$

where $R(u, u+1, s)$ is the value of the risk-free 1-year interest rate in year u and simulation s. For each year t, we compute the average $\bar{d}_t$ of $d_{t,s}$ over all simulations, and finally we derive the model-implied yield curve from these average values as

$$y_t = (\bar{d}_t)^{-1/t} - 1. \tag{4.2}$$

The difference between the input yield curve from Figure 4.1 and the yield curve defined by $y_t, \quad t = 1, ..., T$ should be small if our simulations are appropriate. Figure 4.6 show us these differences for all our models in basis points. As we can see from the figure the difference is minor.

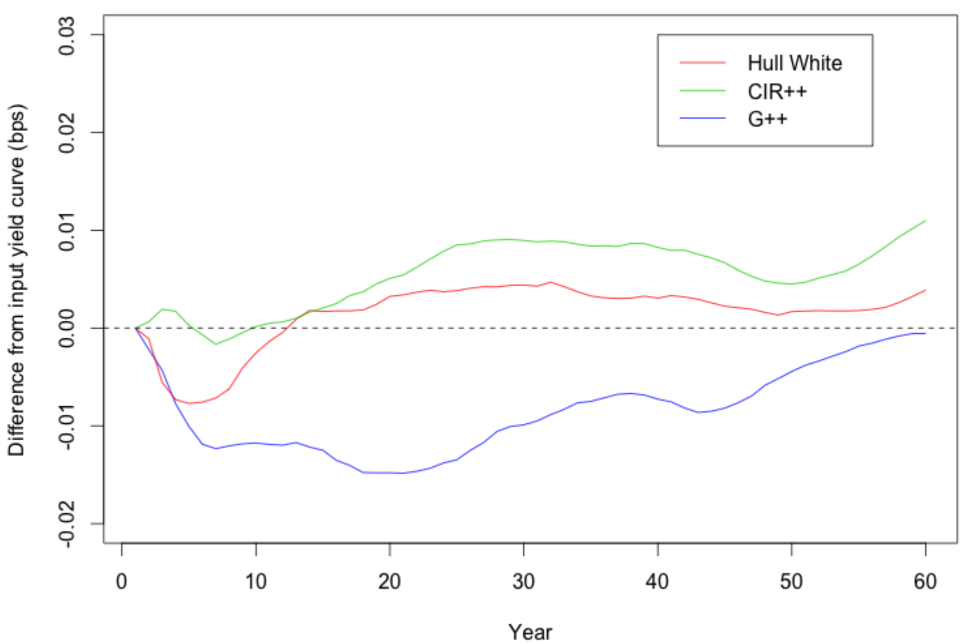

Figure 4.6: Difference between the implied yield curve and the market yield curve when calibrating the three models

Chapter 5

Computing Liabilities

A key aspect to the Solvency II framework is the Solvency Capital Requirement (SCR). It represents required level of capital that an insurance company must hold to ensure that it can meet its expected future obligations on insurance contracts, in other words it is the required risk capital for a one-year time horizon. The SCR is given as the 99.5% quantile of the distribution of the one-year loss. It can be calculated either trough standard formula, through the use of an internal mode, or a combination of both. The internal model must be approved by the supervisory authorities.

In Solvency II standard approach the SCR is first calculated for each module listed in 5.1. For market and insurance risk module, each individual shock is performed according to detailed rules, e.g. stock prices fall with $x\%$ or mortality goes down by $y\%$

The SCR for each individual risk is then determined as the difference between the net asset value (assets minus the best estimate of liabilities) in the unstressed balance sheet and the net asset value in the stressed balance sheet. More specifically, one determines the value of the asset A_0^C under different shocks C and computes the corresponding values of liability, L_0^C. The SCR corresponding to the shock C is determined as

$$SCR_C = (A_0 - L_0) - (A_0^C - L_0^C) = E_0 - E_0^C, \tag{5.1}$$

where E_0 is the net asset value today. The market value of the assets today, A_0 is readily available. For the liabilities we use the best estimate of liabilities from the technical provisions. A method of calculate the best estimate of liabilities is described in Equation 5.5.

These individual risk capital amounts are then combined across the risks within the module using a specified correlation matrix.

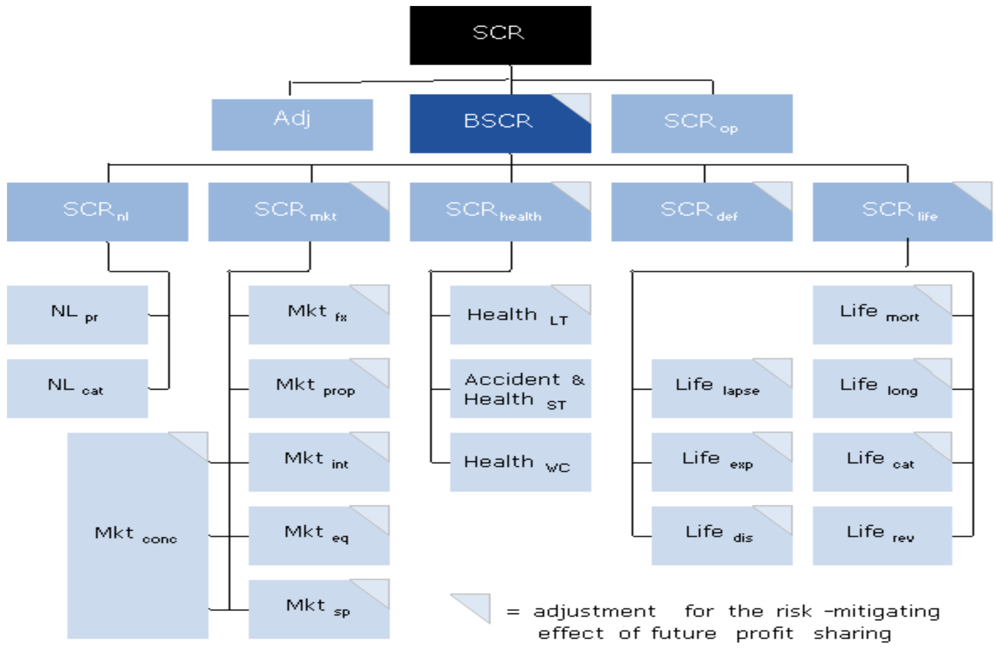

Figure 5.1: The structure of SCR calculation

5.1 Liability Models and Parameters

In our example we will study three different profiles of the guaranteed benefits of an insurance portfolio of which two are based on real world portfolios from the life insurance company Spare-Bank 1 Forsikring. Our profiles corresponds to the products; *old-age pension for individuals with profit sharing (old-age pension)* and *paid-up defined benefit pension policies with pension sharing (paid-up policies)*.

The product *old-age pension* consists of an old-age pension with the possibility of a further payment (lump sum or annuity) if the policyholder dies. The old-age pension is either paid out for a defined number of years or as a lifelong benefit, usually starting at the age 67. In Spare Bank 1's portfolio the guaranteed interest rate is between 2.5% and 4%, depending on a policy's inception date, and the average duration of the liabilities is approximately 5.5 years.

The product *paid-up policies* are fully paid contracts from a defined benefit plan. The benefits are old-age pension, spouse pension and disability pension. The old-age pension, with benefits usually starting at age 67, and spouse pension, are either paid out in a defined number of years, or as a lifelong benefit. A disability pension may be paid out until age 67. In Spare Bank 1s portfolio the guaranteed interest rate is between 2.5% and 4%, depending on when the premiums were paid, and the average duration of the liabilities is approximately 16 years.

Further, according to Norwegian legislation one must split these products' profit into three main elements when allocating profit between insurance company and policyholder:

- *Risk result*: Pure risk premium income minus benefits paid to policyholders and changes in premium.
- *Administration result*: Administration fees minus expenses and commissions.
- *Financial market result*: Financial market income minus guaranteed interest and changes in market risk related reserves.

For the old-age pension if the sum of the risk result, administration result and financial market result is positive, the policyholder will receive a minimum of 65% of the profit. For paid-up policies however, the profit sharing rules are as follows; if the risk result is positive, the policyholder will receive the profit from this element. Further, the company will receive the administration result in any case, and finally, if the sum of financial market result and risk result is positive, the policyholder gets a minimum 80% of the profit.

5.2 Synthetic Example.

The synthetic example presented in this section has the same structure as the one presented in Aas et al. (2015). We study three different profiles of the guaranteed benefits:

- The average duration of the liabilities is 5.5 years. The development of the guaranteed benefits over the time horizon are shown in the upper panel of Figure 5.3.
- The average duration of the liabilities is 16 years. The development of the guaranteed benefits over the time horizon are shown in the middle panel of Figure 5.3.
- The average duration of the liabilities is 50 years, The development of the guaranteed benefits over the time horizon are shown in the lower panel of Figure 5.3.

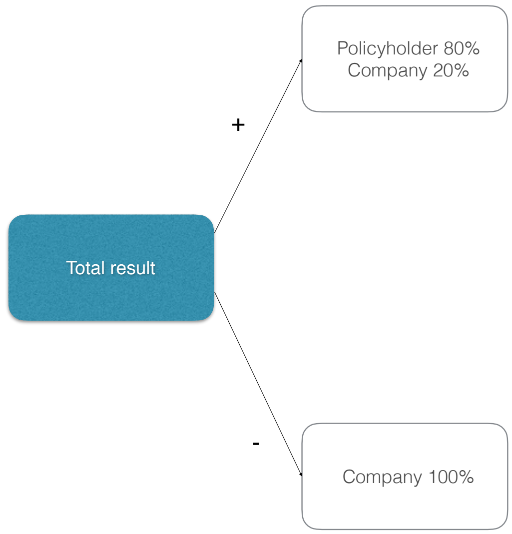

Figure 5.2: Profit sharing rules for **Profit sharing rules example**

The first two cases correspond to the average duration of the liabilities for the products *old-age pension* and *paid-up pension*, respectively. The third case is extreme, here all the liabilities are paid out at the end of the time horizon. We assume we have a product where the holder of the policy gets a fixed guaranteed interest rate. The annual guaranteed interest rate is assumed to be 3.5%. In addition, we assume a bonus which is annually added to the policyholder's account. That is, any surplus is divided 20/80 between the company and the policyholders, while any deficits must be covered by the insurance company. See Figure 5.2. It should be noted that our synthetic example is a simplified version of a real world case.

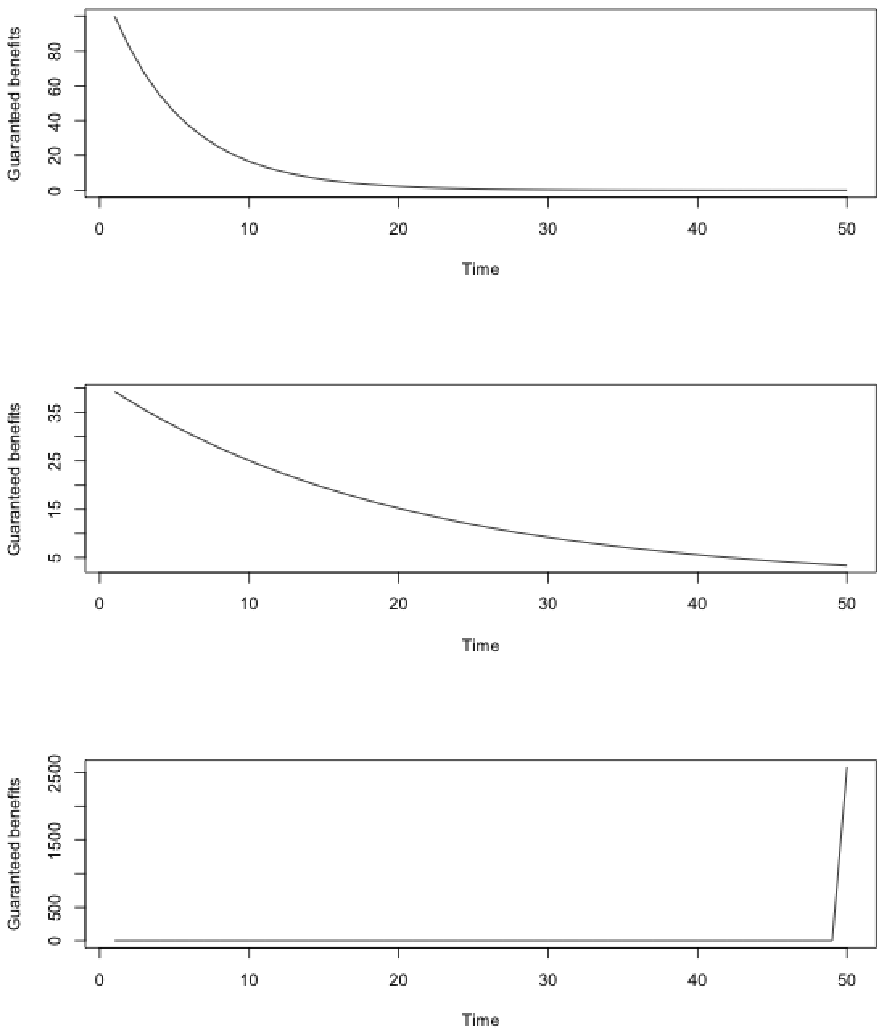

Figure 5.3: The development of guaranteed benefits over the time horizon for the three different profiles. Top panel: Average duration of the liabilities is 5.5 years, *old-age pension*. Middle panel: Average duration og the liabilities is 16 years, *paid-up policies*. Lower panel: Average duration of the liabilities is 50 years, *extreme case*

The initial premium reserve $V_0 = 462.39$ MNOK and the time horizon is set to 50 years. Let V_t be the premium reserve at the beginning of year t. We then have

$$V_t = V_{t-1}(1 + g_t) - b_t, \tag{5.2}$$

where b_t and g_t are the guaranteed benefits and the guaranteed interest rate in year t, respectively.

Let z_t be the achieved financial return in year t. Then, the company's financial market related profit, e_t in year t is given by:

$$e_t = \begin{cases} V_t(z_t - g_t) & z_t \leq g, \\ 0.2V_t(z_t - g_t) & z_t > g. \end{cases}$$

The costumers financial market related profit, c_t, in year t is given by:

$$c_t = \begin{cases} 0 & z_t \leq g \\ 0.8V_t(z_t - g_t) & z_t > g. \end{cases}$$

To generate the probability distribution of c_t we simulate the financial returns z_t in two different ways. The first model assumes that the returns are generated from a geometric random walk with volatility σ. We generate the returns as follows:

$$z_t = \exp(\log(1 - R(t, t + 1)) + \sigma\epsilon_t - 0.5\sigma^2) - 1, \tag{5.3}$$

where ϵ_t follows a standard normal distribution, $\epsilon_t \sim N(0, 1)$, and $R(t, t+1)$ are the 1-year interest rates simulated in Section 4.2.

The second model represents a bond portfolio that at the end of each year t is sold and replaced by a new portfolio for which the duration is D.

$$z_t = R(t, t + D) + \frac{\exp(-(D - 1)R(t + 1, t + D - 1))}{\exp(-(D - 1)R(t, t + D))} - 1. \tag{5.4}$$

The numerator of the fraction is the value of a zero coupon bond with duration $D - 1$ issued at the beginning of year $t + 1$, while the denominator is the value at the end of year t of a zero-coupon bond with duration D issued at the beginning of year t. In this example we assume that the credit bond portfolio has a fixed duration D of 3 years. This means we only need to simulate three interest rates; the 1-year interest rate, which is used for discounting, and the 2-year and 3-year rates, that are used to determine the yearly changes in the market value of the bond portfolio. The interest rates in both Model 1 and

Model 2 are simulated as described in Section 4.2.

When we have generated the simulations of $c_{t,s}$, the best estimate of the liabilities may be computed as

$$\hat{L} = \frac{1}{10000} \sum_{s=1}^{10000} \sum_{t=1}^{50} (b_t + c_{t,s}) d_{t,s}, \tag{5.5}$$

where $d_{t,s}$ is the discount factor in year t and simulation s, defined in Section 4.2. In this example the best estimate of the liabilities is given as the sum of the guaranteed benefits and the future discretionary benefits.

To ensure that the simulation error is negligible, we have for all experiments also computed the distance.

$$D = \left[V_0 - \frac{1}{10000} \sum_{s=1}^{10000} \sum_{t=1}^{50} (b_t + c_{t,s} + e_{t,s}) d_{t,s} \right] / V_0. \tag{5.6}$$

If there is no leakage in our cash-flow model and we have enough simulations, this distance should be small.

5.3 Results

We have presented all the results in Tables 5.1 to 5.4. In Table 5.1 we have studied the effect of different durations when we use Model 1 to simulate the financial return, z_t. Here the volatility, σ is set to 5%. Columns 2 to 4 show us the computed value of the best estimate of the liabilities for the three different interest rate models, while the last column contains the difference between value obtained by the G2++ model and that obtained by the Hull-White model. As we can see from the table, the difference between the two models increases for longer duration.

In Table 5.2 we have studied the effect of different values for the portfolio volatility σ, when the average duration of the benefits is fixed to 5.5 years. The difference between the models becomes smaller when the volatility increase. For large values of σ, the portfolio volatility is dominating the interest rate volatility. We can also see that the larger portfolio volatility, the larger is the value of the best estimate, as expected. A large volatility means increased risk of poor returns, which makes the interest rate guarantee more valuable.

Duration of benefits	Hull-White	CIR++	G2++	Diff.
5.5 years	527.4807	527.0704	527.9334	0.086%
16 years	597.3782	598.3376	599.6215	0.374%
50 years	889.7522	902.9500	906.3713	1.834%

Table 5.1: **Effect of duration of benefits:** The computed value of the best estimate of the liabilities for different durations of the benefits when Model 1 is used to generate the financial returns and the portfolio volatility is fixed to 5% ($\sigma = 0.05$).

Port. vol.	Hull-White	CIR++	G2++	Diff.
2%	508.6433	508.1434	509.5693	0.182%
5%	527.4807	527.0704	527.9334	0.086%
10%	564.1759	563.4995	564.5638	0.068%

Table 5.2: **Effect of portfolio volatility:** The computed values of the best estimate liabilities for different portfolio volatilities when Model 1 is used to to generate the financial returns and the average duration of the benefits is fixed to 5.5 years.

Table 5.3 shows the values of the best estimate liabilities for different durations of the benefits, using Model 2 with $D = 3$ for generating financial returns. Like Table 5.1, the difference between the two models increase for longer duration. If we further compare the two tables, we can see that the differences are larger in Table 5.3. This is due to the fact that the volatility of the financial returns in Model 2 is totally determined by the combination of duration and interest rate volatility, while in Model 1 σ is the far largest component of financial return volatility.

Duration of benefits	Hull-White	CIR++	G2++	Diff.
5.5 years	506.2328	504.5735	508.0189	0.352%
16 years	540.5984	539.1161	548.2993	1.404%
50 years	637.0246	657.2196	685.9579	7.134%

Table 5.3: **Effect of duration of benefits:** The computed value of the best estimate liabilities for different durations of the benefits when Model 2, with $D = 3$ is used to generate the financial returns.

Duration of benefits	Hull-White	CIR++	G2++	Diff.
5.5 years	516.6535	516.1918	517.1612	0.098 %
16 years	569.1382	569.6572	572.4405	0.577 %
50 years	764.6523	783.0286	789.5600	3.155 %

Table 5.4: **Effect of duration of benefits:** The computed value of the best estimate liabilities for different durations of the benefits when both Model 1 and Model 2 are used to generate the financial returns. Here, $\sigma = 0.05$ and $D = 3$.

In our last experiment we use a combination of Model 1 and 2. From Aas et al. (2015), the asset portfolio of the life insurance company may be divide into 5 main assets classes; *Norwegian stocks* (2%), *International stocks* (10%), *Real estate*(20%), *Credit bonds* (33%) and *Government bonds*(35%), where approximate portfolio wights are given in parenthesis. Since we are in a risk neutral world for Solvency II purposes, all kinds of assets will earn the risk free return on average. For the Norwegian stocks, International stocks, real estate and government bonds we will use Model 1. For credit bonds, however, we will use Model 2. More specifically, we assume financial returns are generated by

$$z_t = 0.67 z_{1,t} + 0.33 z_{2,t}, \tag{5.7}$$

where $z_{1,t}$ and $z_{2,t}$ are simulated using Model 1 and Model 2, respectively. In this combination model we assume that $\sigma = 5\%$ and $D = 3$. The results are shown in Table 5.4. As we can see from the table, the model differences are in this case slightly larger than those in Table 5.1. They are however smaller than those in Table 5.3.

Chapter 6

Summary and Conclusion

The best estimate of liabilities is important in the Solvency II framework. It is used to calculate e.g the solvency capital requirement, which is one of two capital requirements in Pillar I. The best estimate of liabilities should be probability weighted average of future cash flows discounted to its present value. Life insurance companies need stochastic models to produce future paths for interest rates, bond returns and currency. These paths should be risk-neutral, meaning that interest rate models is important to consider in the Solvency II framework. In this thesis we have studied three different interest rate models, namely; the Hull-White extended Vasicek model, the CIR++ model and the G2++ model.

Interest rate modelling has been developed during the last decades to estimate prices of interest rate derivatives. It is a branch of mathematical finance where no general model has yet been accepted. There are different models suited to different situations and products. Our choice of models were based on reasons as; well documented in the literature, easy to calibrate and simulate, and that they all provided an exact fit to the yield curve.

Further, interest rate models range from simple one-factor models to more complex multi-factor models. Both the Hull-White and CIR++ models are one-factor, which means that they, at every time point, are perfectly correlated. However, interest rates tend to exhibit non-perfect correlation. We therefore also studied the two-factor G2++ model. From Brigo and Mercurio (2007) two-factor models explain from 85% to 90% of the variation in the yield curve. Our experiments can be used for different models, as they have done in Aas et al. (2015). Here, they have studied the LIBOR Market model as well as the CIR++ and G2++ models.

When using interest rate models in Solvency II framework, one has to use the risk free term structure specified by EIOPA. Furthermore the interest rate models should be calibrated to an appropriate volatility measure. We calibrated our interest rate models to the same historical data and generated 10 000 simulations based on the yield curve and the parameter estimations. Alternatively to using historical data in an illiqued market is to use interest rate derivatives from a more liquid market as proxies or use fictitious prices. It would be interesting to compare the three alternatives.

Based on the interest rates simulated we have presented a synthetic example for calculating the best estimate of liabilities. Here, we saw that the duration of the liabilities was an important factor. If the duration and the proportion of bonds in the asset portfolio are both low, the three interest rate models produce quite similar values for the best estimate. The largest difference between the Hull-White and the G2++ model occurred when we used a bond portfolio to generate the best estimate, and the duration of benefits were 50 years. This was as large as 7.134%.

The experiments performed here could further be used for real insurance products, different interest rate models, different yield curves and different data. It would be interesting to see if our results are valid also when using market prices from a more liquid market.

6.1 Related work

This thesis is based on the work of Aas et al. (2015). They have studied the CIR++, G2++ and the LIBOR market model to compute the best estimate of liabilities. Their historical data is from the period March, 2001 to March, 2011. Overall, the results presented in this thesis and the results presented in Aas et al. (2015) are similar and the conclusion is the same. In addition to the synthetic example they have compared the interest rate models in terms of using them for two real-world products.

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