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MULTISCALE SIMULATION OF NON-DARCY FLOWS

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Summary. In this work we present control volume multiscale methods which address problems on the interaction between pore scale and Darcy scale. For the case when linear equations govern the flow on the pore scale our solution converges to semi-analytical solution. Moreover, the method also extends to non-linear problems governed by Navier-Stokes equations. We show numerical results illustrating the applicability of the method.

1 INTRODUCTION

The standard way of modelling flow in porous media includes applying Darcy's law. Initially, this constitutive equation was phenomenologically derived in the middle of 19th century. In the 20th century it was also shown that under particular assumptions one can derive it from the Stokes equations on porous geometry [2]. Stokes equations, however, rely on several assumptions and are just a linear simplification of full physical problem on the fine scale. If non-linear Navier-Stokes terms are introduced on the pore scale, Darcy's law is amended with non-linear terms. The applications where we can come across such effects are flows that may occur near wells and in fractured regions in subsurface. Moreover, those flows are common for industrial and near-surface porous media.

To obtain the flux expression on Darcy (coarse) scale a multiscale modelling can be applied. The traditional derivations of Darcy type relations on the coarse scale achieve this by homogenization. From the above discussion, we understand that this approach relies on a priory assumptions of the nature of the fine scale flow. Furthermore, since homogenization yields a one-way upscaling, possible feedback from the coarse scale to the fine scale cannot be captured.

As an alternative, we present a multiscale method that for the coarse scale assumes only mass conservation on control volumes, that is, no phenomenological relationships are assumed a priory. The method is designed in the framework of the Heterogeneous Multiscale Methods (HMM) proposed by E and Engquist [4]. Additional information needed, such as a relationship between coarse scale pressure and fluxes, is obtained by locally solving continuum mechanics' equations on the pore scale. While the fine scale solver accounts for non-linear effects of the pore scale, the coarse solver takes care of slowly varying effective parameters.

The advantage of our HMM is that it converges to the correct upscaled solution derived by homogenization when the equations are essentially linear. Moreover, it gives flexibility of solving a wider range of problems, for which we cannot access the exact solution, without any need of additional user participation. As our method extends naturally to this variety of problems and, hence, should give a reasonable approximation that is better than a solution for a simplified model.

HMMs of such kind were considered previously; in particular, a control volume HMM formulation was considered by Abdulle and E[1]. Our method, however, is different not only in the fine scale problem but also in that it treats discontinuities more consistently as it is described in Sect. 3.2.

This paper presents an HMM for the problems of interaction between pore scale and Darcy scale. Our HMM is formulated on the coarse scale as a control volume method consistent with two-point flux approximation. The paper sketches the method algorithm and describes possible ways to optimize it.

Our numerical results test the algorithm for various aspects:

- convergence to semi-analytical homogenisation solution for linear flow regime;
- handling of heterogeneous porous media with discontinuities;
- difference of the presented method to linearised problems for non-linear regimes.

2 MODEL PROBLEM

The goal of our work is to investigate how the HMM framework can produce a solution to the coarse scale problem that accounts for the interplay between the coarse and the fine scale. As a model problems we choose an artificial pore structure that is periodic on macro subdomains. For simplicity only 2D porous media are considered, and the upscaled permeability is assumed to be aligned with the coarse grid.

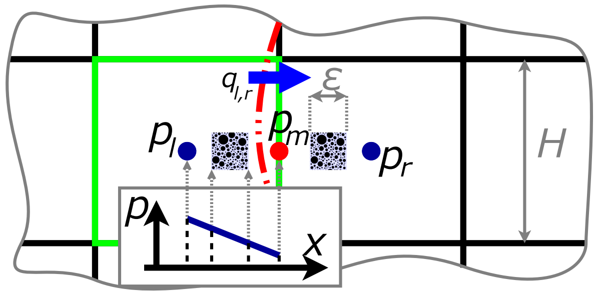
For the coarse scale we only use the (physically justified) continuity equation that provides mass conservation. We consider both problems with incompressible flow,

$$\nabla \cdot \vec{u} = f, \tag{1a}$$

and weakly compressible flow on the form

$$\begin{aligned} \phi p_t + \nabla \cdot \vec{u} &= f, \\ \rho_p &= c\rho. \end{aligned} \tag{1b}$$

In the equations above $\vec{u}$ is flux density vector, f is the mass source, p is the pressure, ρ is density, ϕ is the porosity, and c is compressibility.



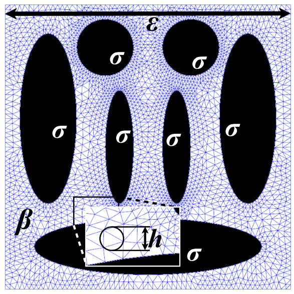


Figure 1: Building blocks of the multiscale method from the coarse grid perspective (a) and an example of a fine scale cell problem with triangulation (b).

As for the fine scale we study the Navier-Stokes equations on porous geometries like on Fig. 1b. The governing equations in this case are (1a) and

$$\rho\vec{u}\cdot\nabla\vec{u} = -\nabla p + \mu\Delta\vec{u} + \vec{g}, \tag{2}$$

where μ is viscosity, $\vec{u}$ is velocity and g represents body forces. Time dependent terms are omitted from the equations (1a,2) since we will only consider fine scale sub-problems that equilibrate much faster than the typical time scale of the coarse problem. For analysis purposes we will also consider Stokes equation, which is obtained by setting $\rho\vec{u}\cdot\nabla\vec{u} = 0$ in (2).

3 DESCRIPTION OF THE MULTISCALE METHOD

The method is formulated within the HMM framework[4]. To describe the method one should choose a fine scale method, a coarse method and operators to project information between the two. This section describes our choice of method's components.

3.1 The coarse scale control volume method

The coarse scale equation is discretized by a two-point-flux-style control volume method. The unknowns are pressures in the cell centers denoted p_l and p_r on Fig. 1a. Along with the cell center pressures, auxiliary variables $q_{l,r}$ denoting fluxes between neighbouring cells are considered. The method ensures local mass conservation, in other words for each cell it satisfies

$$\frac{d}{dt}\int\limits_{\tau_l}\rho_l\phi dx + \oint\limits_{\partial\tau_l}\rho\vec{u}\cdot\vec{n}ds = \int\limits_{\tau_l} f dx, \tag{3}$$

where $\partial\tau_l$ is the border of the cell τ_l and $\vec{n}$ is an outgoing unit normal. As we are using the simplified continuity equations (1) the method will ensure that the corresponding equation is satisfied locally in the integral sense instead; for (1b) it takes the form:

$$\frac{d}{dt}\int_{\tau_l} p_l \phi dx + \oint_{\partial \tau_l} \vec{u}\cdot\vec{n} ds = \int_{\tau_l} f dx. \tag{4}$$

The second term is nothing more than a sum of fluxes (such as $q_{l,r}$) over all edges of τ_l.

As previously mentioned the constitutive relationship between primary pressure variable and fluxes are not known a priori. The fluxes $q_{l,r}$ can be found from a formal relationship

$$q_{l,r} = F(\vec{x}_l, \vec{x}_r, p_l, p_r, \rho_l, \rho_r) \tag{5}$$

by interpolating information from fine scale problems. Here, $\vec{x}$ is the coordinate and subscripts denote the cells.

Before projecting coarse scale pressures to the fine scale, an auxiliary pressure p_m is introduced on all edges. Then we interpolate linearly between p_l and p_m and project the pressure values to form boundary conditions of the fine scale problem at point $(\vec{x}_m + \vec{x}_l)/2$ as shown on Fig. 1a. After solving the fine scale problem (see remark 1) we integrate the component of the velocity that is parallel to the flow direction $(\vec{x}_m - \vec{x}_l)$ across the cell and obtain $q_{l,m}$ by scaling the result (see also remark 2).

In order to omit p_m and hence obtain $q_{l,r}$ we solve the equation $q_{l,m} = q_{l,r} = q_{m,r}$, that is coupled naturally to the two fine scale problems associated with the coarse edge; see Fig. 1a. This equation is true under the assumption that for this local coarse problem the flow is only happening between the cells l and r driven by the corresponding pressures and ρ is constant, i.e. (1a) is valid.

Equipped with flux expressions on each edge, one can perform coarse scale time stepping, discretizing equation in time by backward Euler or solve the system by the Newton method for the stationary case.

Remark 1 : Fine scale. *We define our fine scale problem as a PDE on a domain, that for simplicity is taken to be a square. The size of this square is equal to the period of the porous media ε, and it is positioned around the coarse sampling point. In homogenization, this fine scale problem is referred to as a cell problem.*

The boundary conditions for velocity are: no flow in the solid grains σ and on their boundaries, and periodicity on the square edges; see Fig. 1b. Pressure is also periodically transversal to the flow and has a jump of a given Δp in the direction of the flow. This boundary value problem is consistent with the homogenization cell problem for Stokes flow (see Sect. 4.1 for more details).

To solve the fine scale problem we use the lowest order Taylor-Hood finite element method on a triangular grid (see Fig. 1b) as implemented in FEniCs[5].

Remark 2 : Optimization of the algorithm. *In order to reduce the total computational time we reuse fine scale computations if possible. If for a given cell problem* α *and pressure drop* p *we have computed* $F_\alpha(p_0)$ *and* $F_\alpha(p_1)$*, such that* $p_0 \le p \le p_1$ *and* $|F_\alpha(p_1) - F_\alpha(p_0)| < \delta$ *then we approximate* $F_\alpha(p)$ *by linear interpolation between the two values. In the case where no information is available an on-demand solver is invoked for the fine scale problem and the resulting flux is stored for future use.*

We can also reduce the computational cost by reformulating half of the original problem (5) $F(\vec{x}_l, p_l, \rho_l)$ *to a one parametric dimensionless problem* $F_\alpha(\tilde{p})$ *where* α *corresponds to geometry in point* $\vec{x}_l$*. The resulting problem can be solved by techniques described in this remark.*

3.2 Handling discontinuities

The method proposed in this paper differs from the control volume HMM proposed earlier, e.g. by Abdulle and E[1], in how the coarse scale sampling is performed. In[1], the sampling domains are located in the middle of the border between coarse cells. In contrast, we use two domains, one in each cell, between the center of the cell and the center of the considered border. The old approach resembles control volume finite elements, whereas our approach coincides with two-point flux approximation.

For problems with continuous parameter fields both methods perform similarly, and our method tends to be slower due to more sampling points. The extra sampling points are however necessary to get a proper coarse flux expression for porous media with discontinuities.

It is natural to assume that the discontinuities can be resolved by the coarse grid in average (see the red line on Fig. 1a). A perturbation of order $\varepsilon << H$ in the interface can cause the old method to give an arbitrary wrong sample. For the problems where the scale of discontinuities is small (of order l) or where the grid is non-uniform the old approach will give an error of order H/l. Our approach however will give an error that is only proportional to the ε perturbation of discontinuity; thus this sampling strategy is superior to using a single point only.

4 NUMERICAL EXPERIMENTS

Numerical experiments presented below show applicability of the discussed method. In Sect. 4.1 it is shown that our method converges to homogenization solution for Stokes flow. In Sect. 4.2 we show full flexibility of the proposed method on a heterogeneous domain with discontinuity using realistic parameters. The final example demonstrates the importance of using a non-linear approximation on the fine scale and relates it to various applications.

4.1 Verification of convergence for the Stokes flow

For method validation, it is of interest to verify that the multiscale method gives correct result to problems with a well known solution. In the paper Alyaev et al[3], an a priori error

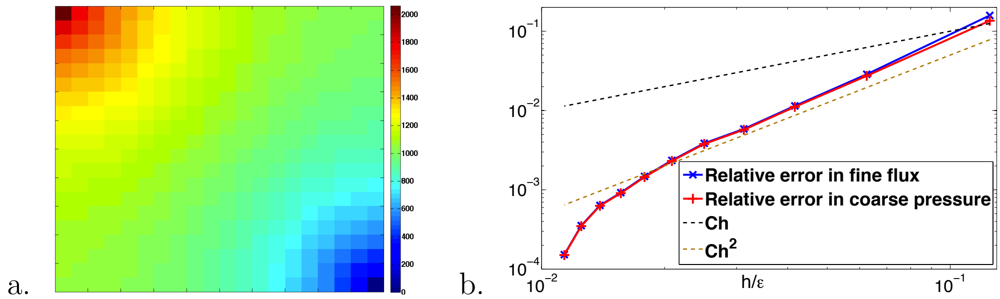


Figure 2: A coarse scale pressure solution to a linear problem (a) and convergence of solution to the reference (b).

estimate for the HMM described here is proved for the case of incompressible Stokes-type flow. The error between the homogenization solution[2] and the fully discrete solution has the form

$$\left\| p - p_{MS}^{H,h} \right\|_{L^2} \leq C \quad \left(\left(\frac{h}{\varepsilon} \right)^{\alpha} + H \right), \tag{6}$$

where the terms represent the propagation of the fine scale error and the coarse scale error respectively. The sizes involved in (6) are shown on Figures 1a and 1b: H is the coarse cell size, ε is the period of the porous medium and h is the fine grid size. The parameter $1 \leq \alpha < 2$ comes from the estimate for Taylor-Hood elements for Stokes problem and depends on the cell geometry[6]. The analysis[3] shows that the coarse scale error in the pressure is proportional to the fine scale error in velocity that is $C\,(h/\varepsilon)^{\alpha}$. The coarse scale error is the error of the two-point flux approximation, which is well analysed and verified in the literature.

Here we present a verification of the propagation of the fine scale error specific for this method. We consider a quarter of a 5-well problem and look for a coarse scale solution in a square domain with no-flow Neumann boundary conditions except for the right-bottom cell that has zero Dirichlet condition (Fig. 2a). In the top-left corner cell the forcing term is introduced. The fine scale structure is taken from Fig. 1b. For the comparison H and ε are fixed. We consider the relative error between solution on h grid and a reference grid with very small cells. Figure 2b shows that the L^2 error in coarse scale pressure is indeed proportional to the error in the coarse flux and the rate of convergence is not lower when expected ($\alpha \approx 2$ for our geometry Fig. 1b).

4.2 Non-linear compressible flow in heterogeneous media

This numerical example shows the full potential of the HMM described in this paper. We consider weakly compressible non-Darcy flow of water in the domain consisting of 2 regions: a low permeable region with anisotropic porous structure as on Fig. 1b in the middle and an isotropic high permeable region made of circular grains in the square

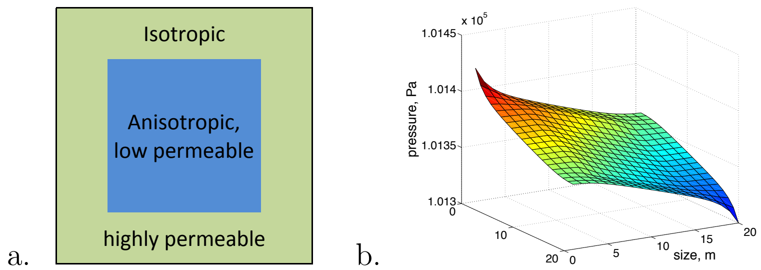


Figure 3: Permeability field structure (a) and a solution after a number of time steps (b) for the heterogeneous problem.

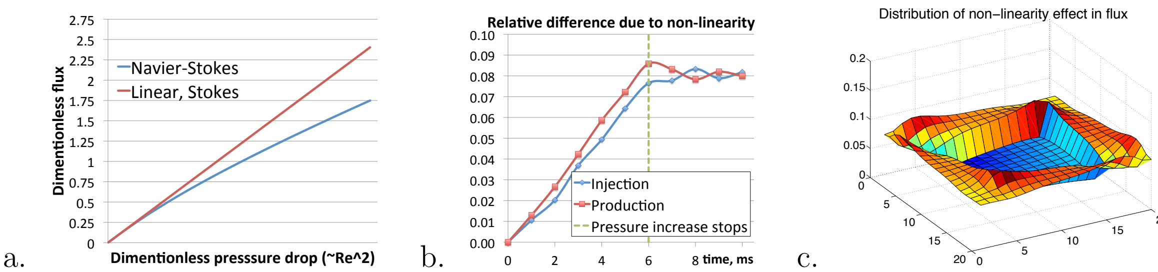


Figure 4: Comparison of linear and non-linear fluxes for the cell problem on Fig. 1b (a), dynamics of injection and production error in time for the linear approximation (b) and spatial distribution of relative non-linear effects in flux for the heterogeneous example (c).

arrangement forming a frame; see Fig. 3a. In analogy to the previous example we place an injector in one corner and a producer in the opposite corner. The pressure in the producer increases linearly in time for a number of steps and then remains constant.

Figure 3b shows a solution before injection stops. There are 3 noticeable features of the solution: fast decrease of pressure around the injector before the low permeable region (forming a peak) with a little build-up on the border of it; almost linear variation within the low-permeable region; and a smaller (due to compressibility) negative peak in the production point.

To investigate the importance of non-linearities on the fine scale, we first consider the coarse flux as a function of the pressure drop over a single edge when the pore geometry is as shown in Fig. 1b. Figure 4a shows the dependency of the non-dimensional mean flux to the non-dimensional pressure drop, that can be interpreted as a scaled Reynolds number squared. As seen from the figure the difference between the linear approximation and the true flux reaches 35% even in the non-turbulent stationary regime.

Next, let us consider again the problem from Fig. 3 and compare it to the same problem but with linear flux approximation. Fig. 4b shows how the deviation in the injection and production will evolve in time. As expected higher pressure drop results in higher influence of the non-linear flow on the fine scale, and which persists after the solution reaches steady-state. For this problem the linear constitutive law overestimates the production by almost

10% which can be crucial for applications such as oil recovery.

Finally, we consider the spatial distribution of the error when using a linear flux relation for the heterogeneous example. Fig. 4c shows the relative magnitude of the difference between a linear approximation and the true flux. As expected, the non-linear effects are largest near the injection and production wells, where the pressure drop is highest. Predictably, error is also significant throughout high permeable channels forming the outer part of the domain, in particular there are large errors close to the boundary between the two permeability regions. Comparing with Fig. 3, we see that non-linear flux effects can be important not only in regions with large pressure drops, but also close to material discontinuities, where flow focusing may occur. The small error in the low permeable region was to be expected, since the pressure drop is much smaller there.

5 CONCLUSIONS

We have presented a new control volume multiscale method for handling effects on the pore and Darcy scale. The method utilizes either Stokes or Navier-Stokes formulations of the fine scale problem, coupled with an incompressible or weakly compressible conservation law on the coarse scale. The coarse-scale fluxes are treated by a multi-scale extension of the standard two-point flux approximation, and this makes the scheme capable of handling discontinuities that are resolved by the coarse grid.

Our numerical results verify analytically obtained convergence estimates of the method for problems where a homogenized coarse-scale formulation is known. Furthermore, we illustrate the applicability of the method to non-linear flows on heterogeneous domains, observing that non-linear flow formulations may be of importance not only near wells, but also in regions where high flow rates are induced by material heterogeneities.

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