Extended Supersymmetry and Superfields

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This thesis concerns the reconstruction of $N = 1$ *supersymmetry, starting with the Standard Model. Next, we partially construct an* $N = 2$ *superfield theory. The differential representation of the* $N = 2$ *supercharges is found. Issues regarding the necessity of mixing left and right chirality in extended supersymmetry is discussed, and possible ways to circumvent this problem.*

Contents

Chapter 1

Introduction

The Standard model in particle physics has been extremely successful in predicting the existence of particles and the fundamental forces that govern their behavior. The Standard Model is built on the basic assumption of symmetries in nature. A symmetry is an invariance of a physical law under some transformation of coordinates, which can be either internal or external. With an external symmetry we refer to transformations in space-time. Through Noether's theorem we know that every symmetry brings about a conserved quantity. Examples from classical physics involve the homogeneity and isotropy of space and the homogeneity of time. Because space is homogeneous, the laws of physics must be the same regardless of where an experiment takes place. This is a symmetry that leads to the conservation of momentum. Since the laws of physics are the same in any chosen direction (space is isotropic), then angular momentum is conserved. The homogeneity of time will lead to the conservation of energy.

In Quantum Field Theory, a free field has no interactions associated with it. This could be the free fermion field, not interacting with itself or with any electromagnetic field. This field possesses an internal symmetry related to a change in phase of the fields. This change of phase will not alter the equations of motion for the field, and thus the physics of the field is not altered. The phase change just mentioned must however be global in order to preserve the symmetry. This means that we cannot choose a different phase shift for each point in space. This does not seem appropriate for a field theory, which should be local in nature. If we require this symmetry of the phase change to be local, such that we may pick a different phase at each point in space and still preserve the symmetry, we require an extra field. The field that appears is the electromagnetic field. Thus, a requirement that an existing global symmetry be local leads to the necessary existence of the force field interacting with the matter field for which we required the symmetry to be local. In the Standard Model all the known forces of nature are constructed in this way.

Since symmetries have been so successful, modern physics strives to find other symmetries of nature. The Standard Model is not believed to be the end of the story. It is believed that it will break down at some energy level. A candidate for physics beyond the Standard Model is supersymmetry.

A supersymmetric theory will leave physics unchanged under a transformation relating fermionic degrees of freedom with bosonic degrees of freedom. All known matter particles are of fermionic nature. The yet undiscovered Higgs particle is of bosonic nature. Requiring supersymmetry in particle physics leads to the necessary existence of bosonic paticles for all existing fermions and vice versa.

In supersymmetry there exists a basic algebra that defines the supersymmetric group transformations. It turns out that this is not the most general algebra for supersymmetry. The basic algebra leads to what is called $N = 1$ supersymmetry. An extension of this algebra leads to what is called extended supersymmetry and the specific extensions are called $N = 2$, $N = 3$ etc.

$N = 1$ supersymmetry is believed to be the most promising for phenomenological reasons. It is however important to investigate the implications of $N > 1$ supersymmetry, since we as of yet do not have any experimental evidence for either model.

We will in this thesis reconstruct $N = 1$ supersymmetry, starting with the reconstruction of the Standard Model. Once we have seen how $N = 1$ supersymmetry is constructed we will move on to extended supersymmetry and find the implications of using an $N = 2$ algebra. We will also begin the construction of an $N = 2$ superfield theory.

Chapter 2

Background

2.1 Group theory in physics

Group theory is an important subject within physics. It is used to further analyze the symmetries found in nature. We will here go through a short summary of group theory which is based on information found in [9].

2.1.1 Groups

A mathematical group consists of a set of operations that have a common property. This property could for example be that all elements in the group must have determinant 1, or for example each element must be a transformation that under a unitary transform preserves some quantity or even both properties. It is however not certain that elements just having such a property will form a group. An operator having such a property, must also, when operating on another member of the group, give a new element in the group. The operators must also obey associativity and there must exist a unit element as well as an inverse within the group. We can state this as follows:

Let $G = \{g_1, \cdots, g_n\}$ be a set of operators and let $\circ$ define the group operation. Then if the following is satisfied

- Closure: if $g_i \in G$ and $g_j \in G$ then $g_i \circ g_j \in G$
- Associativity: for $\forall g_i, g_j, g_k \in G$ we have $(g_i \circ g_j) \circ g_k = g_i \circ (g_j \circ g_k)$
- Identity: there $\exists I \in G$ such that $g_i \circ I = I \circ g_i = g_i$
- Inverse: there $\exists g_i^{-1} \in G$ for $\forall g_i \in G$ such that $g_i^{-1} \circ g_i = g_i \circ g_i^{-1} = I$

A group can be either discrete, in which case there are a finite number of elements in the group, or it can be continuous. In a continuous group there are an infinite number of elements and each group operation can lead to an infinite number of possible elements, still in the group.

Although the group elements are in themselves abstract elements, where a group multiplication has been assigned, these elements can have a specific representation. This is done by assigning to each element in the group a map to a set of matrices having the same group properties. This map need not map to a set that describes the group completely. A representation of the croup could be the set of matrices only containing the identity element. This is still a valid map from the group elements to a set of matrices having the group properties, but it is not a faithful representation. In a faithful representation, the matrix elements will fully describe the properties and elements of the group they are mapped from.

2.1.2 Lie Groups

A Lie Group is a continuous group where an additional geometrical structure is developed on top of the group properties mentioned in section 2.1.1. For the elements in a continuous group we can select an appropriately sized matrix for the representation and assigning to each cell in the matrix, a free variable. By then imposing the constraints that must be present to preserve all the group properties (including the defining properties of the group), some of the degrees of freedom will be removed (i.e. some of the matrix cells are given by the variables in the rest of the matrix cells). This can be described as follows:

Let $M(x_1,\cdots,x_{n\times n})$ be a prospective representation of a group G. We must require that $M(x_1,\cdots,x_{n\times n})$ has the defining group properties (e.g. det M = 1). This will put a constraint on the matrix elements such that

$$M = M(x_1,\cdots,x_m, f_1(x_1,\cdots,x_m),\cdots,f_{n\times n-m}(x_1,\cdots,x_m))$$

This shows that every element in the group represented by M can be identified by a point $x = (x_1,\cdots,x_m)$ on the manifold generated by f.

Once the group manifold has been identified it is also possible to identify the inverse and identity group elements as coordinates x on the manifold. A map $\phi(x,y)$ can be found, that represents group multiplication. This is done by solving the equations

$$M(x, f(x))M(y, f(y)) = M(z,\phi(x,y))$$

We can now Taylor expand each of the elements of the representation of the Lie Group around the element $M = I$ where I is the identity. By only keeping terms up to the first order in the expansion coefficients we have

$$M \approx I + x_1T_1 + \cdots + x_mT_m$$

where T_1 to T_m are the resulting matrix coefficients in the Taylor expansion. T_1 to T_m are called the generators of the group G. The generators enable access to group elements close to the identity, but they also contain all the information needed to gain access to any other group element on the manifold. This can be seen as follows.

Let ε be an infinitesimal real number and let $T = \sum_{i=1}^{m} a_iT_i$ be a linear combination of the generators, then $I + \varepsilon T$ is still a group element infinitesimally close to the identity and belonging to the group G. By making repeated infinitesimal group operations we have

$$\lim_{N\to\infty}(I+\varepsilon T)^N = \lim_{N\to\infty}\left(I+\frac{\theta}{N}T\right)^N = e^{\theta T}$$

which shows that $\exp(\theta T)$ is a group element, in the representation of the group, far away from the identity. Thus, the generators contain the information needed to reconstruct the group.

The commutator of two group elements $g_1, g_2 \in G$ is defined as

$$g_1 \circ g_2 \circ (g_2 \circ g_1)^{-1} = g_1 \circ g_2 \circ g_1^{-1} \circ g_2^{-1}$$

Now, let M_1 and M_2 be the elements in the representation of G corresponding to g_1 and g_2, where we now take g_1 and g_2 to be elements close to the identity. Further, let εX and δY be linear combinations of the generators of the group, that correspond to the elements M_1 and M_2, respectively. ε and δ are infinitesimal numbers. The commutator to the lowest order expansion in linear combinations of the generators is then

$$M_1 M_2 M_1^{-1} M_2^{-1} = e^{\varepsilon X} e^{\delta Y} e^{-\varepsilon X} e^{-\delta Y} \approx (I + \varepsilon X)(I + \delta Y)(I - \varepsilon X)(I - \delta Y)$$

$$= I - [\delta Y]^2 - \varepsilon\delta YX + \varepsilon\delta^2 YXY - [\varepsilon X]^2 + \varepsilon^2\delta XXY + \varepsilon\delta XY - \varepsilon\delta^2 XYY - \varepsilon^2\delta XYX + [\varepsilon\delta XY]^2$$

Terms of second order in ε or δ will vanish, such that

$$M_1 M_2 M_1^{-1} M_2^{-1} \approx I - \varepsilon\delta YX + \varepsilon\delta XY = I + \varepsilon\delta [X,Y]$$

Since also, $M_1 M_2 M_1^{-1} M_2^{-1} \approx I + \kappa Z$ where Z is a linear combination of the generators of the group and κ is an infinitesimal real number, then we have an algebraic closure under the commutator $[X,Y]$ of any linear combinations X and Y of the generators. The algebra is called a Lie Algebra. This also implies that the generators T_i must satisfy the relation

$$[T_i, T_j] = \sum_k C_{ij}^k T_k$$

where C_{ij}^k are called structure constants and completely determine the structure of the Lie Algebra.

2.1.3 Lie Group $SO(n)$

The group $O(n)$ is the group consisting of all orthogonal $n \times n$ matrices A that satisfy the relation $AA^T = I$. As an example we let $n = 2$ and let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

By requiring orthogonality we get three equations

$$a^2 + b^2 = 1 \quad c^2 + d^2 = 1 \quad ac + bd = 0$$

The solutions to this set of equations give

$$A = \begin{pmatrix} \mp d & \pm c \\ c & d \end{pmatrix}$$

Now by using that $d = \pm\sqrt{1-c^2}$ we get that

$$A = \begin{pmatrix} \mp\sqrt{1-c^2} & \pm c \\ c & \sqrt{1-c^2} \end{pmatrix} \quad or \quad A = \begin{pmatrix} \pm\sqrt{1-c^2} & \pm c \\ c & -\sqrt{1-c^2} \end{pmatrix}$$

In this particular example we choose to look at the following solution of A

$$A = \begin{pmatrix} \sqrt{1-c^2} & -c \\ c & \sqrt{1-c^2} \end{pmatrix}$$

We Taylor expand to first order in c to get $A \approx I + \varepsilon M$ where

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Here M is the generator of $O(2)$.

2.1.4 Lie Group $U(1)$

The unitary group $U(n)$ is the group consisting of all $n \times n$ matrices A that satisfy the relation $A^\dagger A = I$. In the standard model, the transformation

$$T(x) = e^{iq\xi(x)}$$

is applied to fields in the theory. Here $\xi(x)$ is a complex valued function. T is a 1×1 matrix, and satisfies $T^\dagger(x)T(x) = 1$, as well as the axioms for a group. Thus T is an element in $U(1)$.

2.1.5 Lie Group $SU(n)$

The special unitary group $SU(n)$ is the group consisting of all $n \times n$ matrices A that satisfy $A^\dagger A = I$ as well as $\det A = 1$. In the standard model, the transformation

$$T(x) = e^{q\lambda_m(x)B^m}$$

is applied to multiplets of fields. λ_m are real parameters and summation over m is implied. $q \in \mathbb{R}$ represents a charge. If B^m is 2×2, T acts on vectors containing two fields. If B^m is 3×3, T acts on vectors containing three fields. Since

$$T^\dagger(x) = \left[e^{q\lambda_m(x)B^m}\right]^\dagger = \sum_{k=0}^{\infty}\left[\frac{(q\lambda_m(x)B^m)^k}{k!}\right]^\dagger = \sum_{k=0}^{\infty}\frac{(q\lambda_m(x)(B^m)^\dagger)^k}{k!} = e^{q\lambda_m(x)(B^m)^\dagger}$$

then $T^\dagger(x)T(x) = I$, assuming $B^\dagger = -B$. With these requirements, T is an element of $U(n)$. If B^m would be traceless, then

$$\log[\det T(x)] = \mathrm{Tr}[\log T(x)] = \mathrm{Tr}\left[\log e^{q\lambda_m(x)B^m}\right] = \mathrm{Tr}\left[q\lambda_m(x)B^m\right] = q\lambda_m(x)\mathrm{Tr}\left[B^m\right] = 0$$

$$\Leftrightarrow \det T(x) = e^0 = 1$$

Now, since $\det T(x) = 1$, then T is an element in $SU(n)$. We see that B^m constitute the generators of the group.

Generation	*Leptons*	*Quarks*
1	$e^- \ e^+ \ \bar{\nu}_e \ \nu_e$	u d
2	$\mu^- \ \mu^+ \ \bar{\nu}_\mu \ \nu_\mu$	s c
3	$\tau^- \ \tau^+ \ \bar{\nu}_\tau \ \nu_\tau$	b t

Table 2.1: The three families of quarks and leptons.

2.2 The Standard Model

The Standard Model is a description of all the known forces in physics, except gravity. In the following we will give a brief overview of the Standard Model based on information found in [5, 11, 13, 15, 16].

2.2.1 Introduction

The Standard Model fermions are listed in table 2.1. There are three families of leptons and three families of quarks. The electrons (e) in the first family are paired with their corresponding neutrinos (ν) in the same family. The three families of leptons are the electron family (e), the muon family (μ) and the tau family (τ). The six quarks are up (u), down (d), strange (s), charm (c), bottom (b) and top (t). Each quark carries electric charge, hypercharge, isospin and color charge. The color charge is represented by one of the colors red, green or blue. All the leptons are spin-1/2 fermions.

The forces between particles are mediated by the gauge bosons (i.e. integer spin particles). The electromagnetic force is mediated by the massless photon (γ), the weak force by the massive $W^\pm$ and Z^0 particles, and the strong force is mediated by the massless gluons (g). The Higgs mechanism gives particles their mass, and it carries its own particle, called the Higgs particle.

The Standard Model is built upon the principle of quantum fields and their symmetries. One can start with a massless Lagrangian density. By requiring an existing global symmetry to be local, then additional gauge fields must be added to the Lagrangian density to satisfy local gauge invariance, as opposed to a global invariance. In choosing the correct symmetries to make global, then the corresponding gauge field will correspond to the force mediators between the particles in the Lagrangian density. The corresponding theories are called gauge theories. In the case of QED, the massless photon field emerges as a consequence of making the existing global $U(1)$ symmetry local. For the weak interactions, massless gauge fields appear by requiring local $SU(2)$ invariance of the doublet containing the left chiral parts of a fermion and its associated neutrino. These gauge fields should according to experiment have mass. By realizing that the minima of the Lagrangian leads to a broken symmetry, one finds that these mediators have masses hidden by the broken symmetry. These are the $W^\pm$ and Z^0 bosons. For the quarks, one can arrange the three colors of the quarks into a multiplet of 3 quarks (red, green and blue), and insist on $SU(3)$ local gauge symmetry. This will lead to the gluon fields coupling to the Lagrangian. In short, the Standard Model is generated by requiring local gauge invariance under the combined group $U(1) \times SU(2)_L \times SU(3)$. The L stands for left, and refers to the fact that only the left chiral fermions are subject to the local gauge condition.

2.2.2 The QED sector

We start with a free Lagrangian $\mathcal{L}$, modeling a lepton $l \in \{e, \mu, \tau\}$ with mass m

$$\mathcal{L} = \sum_l \bar{\psi}_l(x) \left(i\gamma^\mu \partial_\mu - m\right) \psi_l(x) \tag{2.1}$$

where ψ_l is a 4-component Dirac spinor. $\mathcal{L}$ is invariant under the global $U(1)$ transformation $\psi(x) \rightarrow \psi^{'}(x) = \psi(x)e^{-iq\xi}$, where $\xi \in \mathbb{R}$ is a constant. These transformations represent rotations of the components of ψ in the complex plane (note that each component $\psi_a \in \mathbb{C}$ where a denotes the spinor index). If the invariance only holds when ξ is independent of x, then the field is not invariant separately at each space-time position x. It is then natural to believe that the invariance of $\mathcal{L}$ should hold for the local transformation $e^{-iq\xi(x)}$. Since ξ commutes with every object in the Lagrangian, this leads to an Abelian theory. Inserting the transformation into $\mathcal{L}$ gives

$$\mathcal{L} = \sum_l \left\{ \bar{\psi}_l(x)[i\gamma^\mu \partial_\mu - m]\psi_l(x) + q\bar{\psi}_l(x)\gamma^\mu \psi_l(x)\partial_\mu \xi(x) \right\}$$

where the derivative ∂_μ generated the additional term. By introducing the covariant derivative $\partial_\mu \rightarrow D_\mu$, satisfying

$$D^{'}_\mu \psi^{'}_l(x) = e^{-iq\xi(x)} D_\mu \psi_l(x) \tag{2.2}$$

then $\mathcal{L}$ will exhibit $U(1)$ symmetry. Note that

$$\partial_\mu(e^{-iq\xi(x)}\psi_l(x)) = [\partial_\mu - iq\partial_\mu \xi(x)]e^{-iq\xi(x)}\psi_l(x) \tag{2.3}$$

To construct the covariant derivative D_μ we need to start with the partial derivative ∂_μ, and then add a vectorial quantity $A_\mu(x)$ that transforms in such a way as to cancel the extra term emerging in equation (2.3). We then see that by defining $D_\mu \equiv [\partial_\mu + iqA_\mu]$, we have that

$$D^{'}_\mu \psi^{'}_l(x) = D^{'}_\mu[e^{-iq\xi(x)}\psi_l(x)] = [\partial_\mu + iqA^{'}_\mu(x) - iq\partial_\mu \xi(x)]\psi_l(x)e^{-iq\xi(x)}$$

By requiring the introduced field to transform as $A^{'}_\mu(x) = A_\mu(x) + \partial_\mu \xi(x)$, then equation (2.2) is satisfied. To make equation (2.1) invariant under the local $U(1)$ symmetry we make the substitution $\partial_\mu \rightarrow D_\mu$, leading to

$$\mathcal{L} = \sum_l \left\{ \bar{\psi}_l(x) \left(i\gamma^\mu \partial_\mu - m\right) \psi_l(x) - q\bar{\psi}_l(x)\gamma^\mu \psi_l(x)A_\mu(x) \right\}$$

This is not a complete Lagrangian, since it includes the field $A_\mu(x)$ with no kinetic term. A term that is invariant under the $U(1)$ symmetry and consisting of A_μ and its derivatives is needed. The Ricci identity

$$[D_\mu, D_\nu] = iq(\partial_\mu A_\nu - \partial_\nu A_\nu) = iqF_{\mu\nu}$$

shows that, since $[D_\mu, D_\nu]$ is manifestly invariant under $U(1)$ transformations, then $F_{\mu\nu}$ must retain $U(1)$ invariance. Restricting to a renormalizable theory, it is necessary to only include terms up to m^4, which leave two options

$$F^{\mu\nu}F_{\mu\nu} \qquad\qquad \varepsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}$$

where the second option breaks parity and time reversal symmetry. Using the familiar normalization from electrodynamics, we get the QED Lagrangian

$$\mathcal{L}_{QED} = \sum_l \left\{ \bar{\psi}_l(x)\left(i\gamma^\mu\partial_\mu - m\right)\psi_l(x) - q\bar{\psi}_l(x)\gamma^\mu\psi_l(x)A_\mu(x) \right\} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

where ψ_l describes the leptons $\{e^+, e^-, \mu^+, \mu^-, \tau^+, \tau^-\}$ and the gauge field A_μ describes the photon, which couples to the conserved current $J^\mu = q\bar{\psi}_l(x)\gamma^\mu\psi_l(x)$.

2.2.3 The electro-weak sector

There are three types of weak interactions. These are the purely leptonic, semi leptonic and purely hadronic interactions. An example of a semi leptonic process is $n \to p + e^- + \bar{\nu}_e$, and a pure hadronic processes is $\Lambda \to p + \pi^-$. To start with, only purely leptonic processes such as $\tau^- \to e^- + \bar{\nu}_e + \nu_\tau$ are considered.

We start with a system of three free massive Dirac fields for the fermions and three free massive Dirac fields for their corresponding neutrinos (we will in this section omit explicit x dependencies to simplify notation).

$$\mathcal{L} = \sum_l \left\{ \bar{\psi}_l(i\gamma^\mu\partial_\mu - m_l)\psi_l + \bar{\psi}_{\nu_l}(i\gamma^\mu\partial_\mu - m_{\nu_l})\psi_{\nu_l} \right\}$$

where $l \in e, \mu, \tau$ represents the three families of particles. Next we project the leptons into their left and right components using $\psi^L = \frac{1}{2}(1-\gamma^5)\psi$ and $\psi^R = \frac{1}{2}(1+\gamma^5)\psi$. Then $\psi = \psi^L + \psi^R$ and we get that

$$\begin{aligned}
\mathcal{L} = &\sum_l \left\{ \bar{\psi}_l^L(i\gamma^\mu\partial_\mu - m_l)\psi_l^L + \bar{\psi}_l^R(i\gamma^\mu\partial_\mu - m_l)\psi_l^R \right\} \\
+ &\sum_l \left\{ \bar{\psi}_{\nu_l}^L(i\gamma^\mu\partial_\mu - m_{\nu_l})\psi_{\nu_l}^L + \bar{\psi}_{\nu_l}^R(i\gamma^\mu\partial_\mu - m_{\nu_l})\psi_{\nu_l}^R \right\} \\
+ &\sum_l \left\{ \bar{\psi}_l^L(i\gamma^\mu\partial_\mu - m_l)\psi_l^R + \bar{\psi}_l^R(i\gamma^\mu\partial_\mu - m_l)\psi_l^L \right\} \\
+ &\sum_l \left\{ \bar{\psi}_{\nu_l}^L(i\gamma^\mu\partial_\mu - m_{\nu_l})\psi_{\nu_l}^R + \bar{\psi}_{\nu_l}^R(i\gamma^\mu\partial_\mu - m_{\nu_l})\psi_{\nu_l}^L \right\}
\end{aligned}$$

$\mathcal{L}$ has $U(1)$ global symmetry. Experiments show interactions between left handed leptons and their left handed neutrinos. Therefore we want to impose a gauge symmetry on the lepton - neutrino left doublets and leave the right handed components unchanged under the corresponding transformations. Since right and left parts of the spinors couple, $\mathcal{L}$ does not posses such a symmetry. We decouple the right and left handed spinors by setting $m_l = m_{\nu_l} = 0$ (this will be fixed by the Higgs mechanism). We introduce the doublet

$$\Psi_l \equiv \begin{pmatrix} \psi_{\nu_l}^L \\ \psi_l^L \end{pmatrix} \quad \bar{\Psi}_l = \begin{pmatrix} \bar{\psi}_{\nu_l}^L & \bar{\psi}_l^L \end{pmatrix}$$

Then $\mathcal{L}$ can be written as

$$\mathcal{L} = \sum_l \{ \bar{\Psi}_l (i\gamma^\mu \partial_\mu) \Psi_l + \bar{\psi}_l^R (i\gamma^\mu \partial_\mu) \psi_l^R + \bar{\psi}_{\nu_l}^R (i\gamma^\mu \partial_\mu) \psi_{\nu_l}^R \}$$

We now see that $\mathcal{L}$ still has a global symmetry under the $U(1)$ symmetry transformation

$$\psi_l^L \to \psi_l^{'L} = \psi_l^L e^{-ig^{'} Y \xi}$$

$$\psi_{\nu_l}^L \to \psi_{\nu_l}^{'L} = \psi_{\nu_l}^L e^{-ig^{'} Y \xi}$$

$$\psi_l^R \to \psi_l^{'R} = \psi_l^R e^{-ig' Y \xi}$$

$$\psi_{\nu_l}^R \to \psi_{\nu_l}^{'R} = \psi_{\nu_l}^R e^{-ig' Y \xi}$$

where $g^{'}, Y, \xi \in \mathbb{R}$. In addition, $\mathcal{L}$ now has a global $SU(2)$ symmetry through the transformation

$$\Psi_l \to \Psi_l^{'} = e^{\frac{i}{2} g \beta_n \sigma^n} \Psi_l$$

where $\beta_n \in \mathbb{R}$. We sum over $n = 1, 2, 3$ where σ^n are the Pauli matrices. We then have a combined global $U(1) \times SU(2)$ symmetry described by

$$\Psi_l \to \Psi_l^{'} = e^{\frac{i}{2} g \beta_n \sigma^n} \Psi_l e^{-ig' Y \xi}$$

$$\psi_l^R \to \psi_l^{'R} = \psi_l^R e^{-ig' Y \xi}$$

$$\psi_{\nu_l}^R \to \psi_{\nu_l}^{'R} = \psi_{\nu_l}^R e^{-ig' Y \xi}$$

The Pauli matrices satisfy the algebra

$$[\sigma^m, \sigma^n] = 2i\varepsilon_{mnk} \sigma^k \tag{2.4}$$

while Y is the weak hypercharge, which is related to the electric charge through $Y = Q/e - I_3^W$. I_3^W is the weak isocharge associated to the corresponding weak hypercharge current. Y is $-\frac{1}{2}$ when associated with Ψ_l, -1 when associated with ψ_l^R and 0 when associated with $\psi_{\nu_l}^R$. The transformations will hold locally by replacing the derivatives of $\mathcal{L}$ with the covariant derivatives D_μ. These transformations are then elements in the product group $U(1) \times SU(2)_L$. Going from global to local invariance, $\xi \to \xi(x)$ and $\lambda_n \to \lambda_n(x)$. There will be three variations of D_μ, depending on which field they act on (Ψ_l, ψ_l^R or $\psi_{\nu_l}^R$).

We start with the covariant derivative acting on Ψ_l. We require that

$$D_\mu^{'} \Psi_l^{'} = e^{\frac{i}{2} g \lambda_n \sigma^n} (D_\mu \Psi_l) e^{-ig' Y \xi}$$

Since the transformations $e^{\frac{i}{2} g \lambda_n \sigma^n}$ constitute group elements in a Lie group on a smooth manifold, then it suffices to treat the transformations for λ_n infinitesimal. Thus

$$D_\mu^{'} \Psi_l^{'} = [1 + \frac{i}{2} g \lambda_n \sigma^n] (D_\mu \Psi_l) e^{-ig' Y \xi} \tag{2.5}$$

There are four degrees of freedom in the transformation (three in λ_n and one in ξ). We may choose an ansatz for D_μ, using three gauge fields $W_{n,\mu}$ and one gauge field B_μ,

$$D_\mu = \partial_\mu + ig'YB_\mu + \frac{i}{2}gW_{n,\mu}\sigma^n \tag{2.6}$$

Inserting the ansatz of D'_μ into equation (2.5), we get

$$\begin{aligned}
D'_\mu \Psi'^L_l &= [1+\frac{i}{2}g\lambda_n\sigma^n]\left\{[\partial_\mu + ig'YB''_\mu + \frac{i}{2}gW''_{n,\mu}\sigma^n]\Psi^L_l\right\}e^{-ig'Y\xi} \\
&- [1+\frac{i}{2}g\lambda_{n\sigma^n}]ig'Y(\partial_\mu\xi)\Psi^L_l e^{-ig'Y\xi} + \frac{i}{2}g\sigma^n(\partial_\mu\lambda_n)\Psi^L_l e^{-ig'Y\xi} \\
&- \frac{i}{4}g^2\lambda_n W''_{n,\mu}[\sigma^m,\sigma^n]\Psi^L_l e^{-ig'Y\xi}
\end{aligned} \tag{2.7}$$

Requiring $B''_\mu = B_\mu + \partial_\mu\xi$, will make the second term in equation (2.7) cancel, while $B''_\mu \to B_\mu$ in the first term. A further requirement of $W''_{n,\mu} = W'_{n,\mu} - \partial_\mu\lambda_n$ gives

$$\begin{aligned}
D'_\mu \Psi'^L_l &= [1+\frac{i}{2}g\lambda_n\sigma^n]\left\{[\partial_\mu + ig'YB_\mu + \frac{i}{2}gW'_{n,\mu}\sigma^n]\Psi^L_l\right\}e^{-ig'Y\xi} \\
&- \frac{i}{2}g^2\lambda_n W'_{n,\mu}\varepsilon_{mnk}\sigma^k\Psi^L_l e^{-ig'Y\xi}
\end{aligned}$$

where $\mathcal{O}(\lambda_n^2)$ terms are left out and the algebra of equation (2.4) has been used to write the result in terms of the structure constants. Finally we may cancel the last term, by requiring the transformation $W'_{n,\mu} = W_{n,\mu} - g\lambda_n W_{k,\mu}\varepsilon_{nkm}$ and disregarding $\mathcal{O}(\lambda_n^2)$ terms. We have then found that equation (2.6) satisfies the requirements of the covariant derivative with the gauge field transformations being

$$B_\mu \to B_\mu + \partial_\mu\xi$$

and

$$W_{n,\mu} \to W_{n,\mu} - \partial_\mu\lambda_n - g\lambda_n W_{k,\mu}\varepsilon_{nkm}$$

Using the same calculations we find that

$$D'_\mu \psi'^R_l = (D_\mu \psi^R_l)e^{-ig'Y\xi}$$

leaves $\mathcal{L}$ invariant, where $D_\mu = \partial_\mu + ig'YB_\mu$, and is satisfied if the gauge field transforms as $B_\mu \to B_\mu + \partial_\mu\xi$. Inserting the values of the hypercharges we have

$$D_\mu\Psi_l = [\partial_\mu - \frac{i}{2}g'B_\mu + \frac{i}{2}gW_{n,\mu}\sigma^n]\Psi_l$$

$$D_\mu\psi^R_l = [\partial_\mu - ig'B_\mu]\psi^R_{\nu_l}$$

$$D_\mu\psi^R_{\nu_l} = \partial_\mu\psi^R_{\nu_l}$$

$$B_\mu \to B_\mu + \partial_\mu\xi$$

$$W_{n,\mu} \to W_{n,\mu} - \partial_\mu \lambda_n - g\lambda_n W_{k,\mu}\varepsilon_{nkm}$$

By making the replacement $\partial_\mu \to D_\mu$ in $\mathcal{L}$ we get

$$\mathcal{L} = \sum_l \{\bar{\Psi}_l(i\gamma^\mu\partial_\mu)\Psi_l + \bar{\psi}_l^R(i\gamma^\mu\partial_\mu)\psi_l^R + \bar{\psi}_{\nu_l}^R(i\gamma^\mu\partial_\mu)\psi_{\nu_l}^R\}$$

$$+\sum_l \left\{-g\left[\frac{1}{2}\bar{\Psi}_l\gamma^\mu\sigma^n\Psi_l\right]W_{n,\mu} - g'\left[-\frac{1}{2}\bar{\Psi}_l\gamma^\mu\Psi_l - \bar{\psi}_l^R\gamma^\mu\psi_l^R\right]B_\mu\right\}$$

There are no free fields for the four gauge fields $W_{n,\mu}$ and B_μ in this Lagrangian. In analogy with QED, the $U(1)\times SU(2)_L$ invariant free field for B_μ will be $-\frac{1}{4}B^{\mu\nu}B_{\mu\nu}$ with $B^{\mu\nu} \equiv \partial^\nu B^\mu - \partial^\mu B^\nu$ (where B_μ is defined to be $SU(2)_L$ invariant). As in section 2.2.2 we may use the covariant derivative to find a $U(1)\times SU(2)_L$ invariant term consisting of $W_{n,\mu}$ and its derivatives. We have that

$$[D_\mu, D_\nu]\Psi_l = \frac{i}{2}[g'B_{\mu\nu} - gW_{n,\mu\nu}\sigma^n - g^2 W_{n,\mu}W_{m,\nu}\varepsilon_{nmk}\sigma^k]\Psi_l$$

where $W_{n,\mu\nu} \equiv \partial_\nu W_{n,\mu} - \partial_\mu W_{n,\nu}$. Since $B_{\mu\nu}$ is $U(1)$ invariant and by definition $SU(2)_L$ invariant and $W_{n,\mu}$ by definition is $U(1)$ invariant, then

$$G_{n,\mu\nu} \equiv W_{n,\mu\nu} - gW_{k,\mu}W_{m,\nu}\varepsilon_{kmn}$$

must be a $U(1)\times SU(2)_L$ invariant. With the normalization conditions we employ, we get the following massless electro-weak Lagrangian $\mathcal{L}_{EW}$

$$\mathcal{L}_{EW} = \sum_l \{\bar{\Psi}_l(i\gamma^\mu\partial_\mu)\Psi_l + \bar{\psi}_l^R(i\gamma^\mu\partial_\mu)\psi_l^R + \bar{\psi}_{\nu_l}^R(i\gamma^\mu\partial_\mu)\psi_{\nu_l}^R\} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}G^{n,\mu\nu}G_{n,\mu\nu}$$

$$+\sum_l \left\{-g\left[\frac{1}{2}\bar{\Psi}_l\gamma^\mu\sigma^n\Psi_l\right]W_{n,\mu} - g'\left[-\frac{1}{2}\bar{\Psi}_l\gamma^\mu\Psi_l - \bar{\psi}_l^R\gamma^\mu\psi_l^R\right]B_\mu\right\}$$

Next, we need to attach two Higgs fields to $\mathcal{L}_{EW}$ and require symmetry breaking to regain the lepton masses and to acquire masses for the $W_{n,\mu}$ gauge fields. The Higgs field that will generate masses for the $W_{n,\mu}$ gauge fields is a scalar doublet, with a degenerate ground state. The corresponding Lagrangian density has global $U(1)\times SU(2)$ symmetry.

$$\mathcal{L}_{HW} = [\partial^\mu\Phi]^\dagger[\partial_\mu\Phi] - \mu^2\Phi^\dagger\Phi - \lambda[\Phi^\dagger\Phi]^2$$

where $\Phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$. The Hamiltonian density is $\mathcal{H} = \sum_i \pi_i\phi_i - \mathcal{L}_{HW}$. By stating this in terms of $|\phi_1|^2 = \phi_1^\dagger\phi_1$ and $|\phi_2|^2 = \phi_2^\dagger\phi_2$ we get

$$\mathcal{H} = |\dot{\phi}_1|^2 - \partial^K\phi_1^\dagger\partial_K\phi_1 + |\dot{\phi}_2|^2 - \partial^K\phi_2^\dagger\partial_K\phi_2 + \mu^2|\phi_1|^2 + \mu^2|\phi_2|^2 + \lambda[|\phi_1|^2 + |\phi_2|^2]^2$$

where we define the Higgs potential

$$V(|\phi_1|, |\phi_2|) \equiv \mu^2|\phi_1|^2 + \mu^2|\phi_2|^2 + \lambda[|\phi_1|^2 + |\phi_2|^2]^2$$

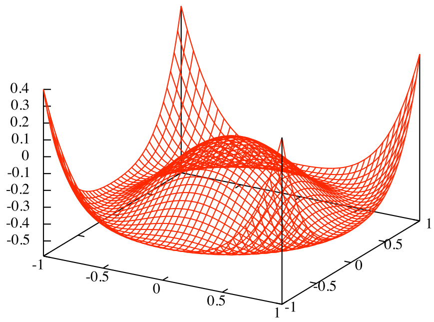

Figure 2.1: Higgs potential $V(|\phi_1|,|\phi_2|)$ illustrated with $\mu^2=-1$ and $\lambda=0.6$.

From figure 2.1 we see that with $\mu^2<0$ and $\lambda>0$, the classical ground state of $\mathcal{H}$ as well as the potential V are negative. This degeneracy of the ground state will break the $U(1)\times SU(2)$ symmetry upon a choice of ground state. When demanding that

$$\frac{\partial\mathcal{H}}{\partial|\phi_1|}=0 \qquad \text{and} \qquad \frac{\partial\mathcal{H}}{\partial|\phi_2|}=0$$

we find the minimum of the classical fields to be given by

$$|\phi_1|^2+|\phi_2|^2=\frac{-\mu^2}{2\lambda}\equiv\frac{v^2}{2} \tag{2.8}$$

where we are free to choose a ground state in the $|\phi_1|$, $|\phi_2|$ plane, that satisfy this relation. We make the $U(1)\times SU(2)$ symmetry local in the same manner as for the spinors, using the covariant derivative used for Ψ_l (this is also a doublet). Then, by making the substitution $\partial_\mu\rightarrow D_\mu$ in $\mathcal{L}_{EW}$, setting the hypercharge $Y=+\frac{1}{2}$ in equation (2.6), we have that

$$\begin{aligned}\mathcal{L}_{HW}=&[\partial^\mu\Phi^\dagger][\partial_\mu\Phi]+\frac{i}{2}g'B_\mu[\partial^\mu\Phi^\dagger]\Phi+\frac{i}{2}gW_{n,\mu}[\partial^\mu\Phi^\dagger]\sigma^n\Phi\\&-\frac{i}{2}g'B^{\mu\dagger}\Phi^\dagger\partial_\mu\Phi+\frac{1}{4}(g')^2B^{\mu\dagger}B_\mu\Phi^\dagger\Phi+\frac{1}{4}g'gW_{n,\mu}B^{\mu\dagger}\Phi^\dagger\sigma^n\Phi\\&-\frac{i}{2}gW^\dagger_{m,\mu}\Phi^\dagger(\sigma^m)^\dagger\partial_\mu\Phi+\frac{1}{4}gg'W^{\mu\dagger}_mB_\mu\Phi^\dagger(\sigma^m)^\dagger\Phi+\frac{1}{4}g^2W^{\mu\dagger}_mW_{n,\mu}\Phi^\dagger(\sigma^m)^\dagger\sigma^n\Phi\\&-\mu^2\Phi^\dagger\Phi-\lambda[\Phi^\dagger\Phi]^2\end{aligned}$$

From equation (2.8) we see that we are free to choose

$$\Phi_0=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\v\end{pmatrix}$$

as the ground state of the system. By doing this, the $SU(2)$ symmetry is broken. We may now vary the fields around Φ_0 by defining the field variable as

$$\Phi=\Phi_0+\frac{1}{\sqrt{2}}\begin{pmatrix}a(x)+ib(x)\\\beta(x)+ic(x)\end{pmatrix} \tag{2.9}$$

where a, b, c and β are real fields. We are allowed to choose the unitary gauge, $a = b = c = 0$ (we will see that the gauge fields that are now mass-less will acquire masses, which leads to three additional degrees of freedom used to choose a gauge for a, b and c). We may now insert equation (2.9) into $\mathcal{L}_{HW}$ while applying the unitary gauge. We will then be able to identify the following terms

$$\mathcal{L}_1 = \frac{1}{2}\partial_\mu \beta \partial^\mu \beta - \frac{1}{2}\left(\mu^2 + 3\lambda v^2\right)\beta^2$$

$$\mathcal{L}_2 = \sum_{n=1}^{3} \frac{1}{2}\frac{1}{4} g^2 v^2 W_n^{\mu\dagger} W_{n,\mu}$$

$$\mathcal{L}_3 = \frac{1}{2}\frac{1}{4}(g')^2 v^2 B^{\mu\dagger} B_\mu$$

$$\mathcal{L}_4 = -\frac{1}{4} g g' v^2 B^{\mu\dagger} W_{3,\mu}$$

Here, $\mathcal{L}_1$ is the free real scalar Higgs field with mass $m_H^2 = (\mu^2 + 3\lambda v^2) = -2\mu^2$. $\mathcal{L}_2$ represents the mass terms for the three $W_{n,\mu}$ bosons with mass $m_W^2 = \frac{1}{4}v^2 g^2$. We notice from $\mathcal{L}_3$ that the $U(1)$ gauge field B_μ also acquires a mass term. $\mathcal{L}_4$ consists of those additional terms that contain only B_μ and $W_{3,\mu}$. The remaining terms can be identified as interaction terms. By substituting into $\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$,

$$W_{3,\mu} = \cos\theta_W Z_\mu + \sin\theta_W A_\mu$$

$$B_\mu = -\sin\theta_W Z_\mu + \cos\theta_W A_\mu$$

where θ_W is the weak mixing angle and demanding that $g\sin\theta_W = g'\cos\theta_W$, we will be left with no terms quadratic in A_μ. Then, the field A_μ has no mass and is taken to be the photon field. Isolating terms quadratic in Z_μ, we find

$$\mathcal{L}_{ZM} = \frac{1}{2}\frac{1}{4} g^2 v^2 \frac{1}{\cos^2\theta_W} Z^{\mu\dagger} Z_\mu$$

where we find the mass of the Z boson,

$$m_Z^2 = \frac{g^2 v^2}{4\cos^2\theta_W} = \frac{m_W^2}{\cos^2\theta_W}$$

.

Next, we introduce masses for the leptons, by coupling to the Higgs field through the Yukawa interaction term,

$$\mathcal{L}_Y = -g_l[\bar{\Psi}_l \psi_l^R \Phi + \Phi^\dagger \bar{\psi}_l^R \Psi_l] - g_{\nu_l}[\bar{\Psi}_l \psi_{\nu_l}^R \tilde{\Phi} + \tilde{\Phi}^\dagger \bar{\psi}_{\nu_l}^R \Psi_l]$$

where g_l and g_{ν_l} are the Yukawa coupling constants and $\tilde{\Phi} = -i[\Phi^\dagger \sigma^2]^T$. $\mathcal{L}_Y$ is invariant under $U(1)\times SU(2)$ transformations. We now substitute Φ with equation (2.9) and employ the unitary gauge. We can then identify the following terms

$$\mathcal{L}_5 = -\frac{1}{\sqrt{2}} g_l v \bar{\psi}_l^L \psi_l^R - \frac{1}{\sqrt{2}} g_l v \bar{\psi}^R \psi_l^L$$

$$\mathcal{L}_6 = -\frac{1}{\sqrt{2}} g_{\nu_l} v \bar{\psi}^L_{\nu_l} \psi^R_{\nu_l} - \frac{1}{\sqrt{2}} g_{\nu_l} v \bar{\psi}^R_{\nu_l} \psi^L_{\nu_l}$$

These are the mass terms of the leptons, with masses $m_l = \frac{1}{\sqrt{2}} g_l v$ and $m_{\nu_l} = \frac{1}{\sqrt{2}} g_{\nu_l} v$. The remaining terms are interaction terms between leptons and the Higgs field β. Putting together all the terms from $\mathcal{L}_{EW}$, the Higgs term and the Yukawa interaction terms, we arrive at the full massive electro-weak Lagrangian, unifying the electromagnetic and the weak interactions. To include quarks in this model it is possible to consider left quark $U(1) \times SU(2)$ doublets, eg. $(u^c, d^c)_L$ and u^c_R, d^c_R, where c is the color of the quarks. This would enable, not only purely leptonic processes, but also hadronic and semi-leptonic processes.

2.2.4 The QCD sector

The development of QCD, as a gauge theory, is very similar to the electro-weak gauge theory described in section 2.2.3. We start out with a free, massive quark color triplet field of leptons.

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$$

where Ψ represents a color triplet for one specific family of quarks, and is defined by

$$\Psi \equiv \begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix}$$

where r, g and b denotes red, green and blue, respectively. The triplet is invariant under $U(1) \times SU(3)$ global transformations

$$e^{-iq\xi} e^{iq' t_n M^n}$$

where q is the photon coupling constant and q' is the gluon coupling constant. $t_n \in \mathbb{R}$ while $n = 1, \ldots, 8$. M^n are the generators and, since the above transformations form a group, they satisfy the algebra[1]

$$[M^n, M^m] = C_{nmk} M^k$$

where C_{mnk} are the structure constants. We require $\mathcal{L}$ to be invariant under the corresponding local $U(1) \times SU(3)$ transformations. The covariant derivative is

$$D_\mu = \partial_\mu + iqA_\mu + iq' K_{n,\mu} M^n$$

with the corresponding transformations of the 9 gauge fields A_μ, $K_{n,\mu}$

$$A_\mu \to A_\mu + \partial_\mu \xi$$

$$K_{m,\mu} \to K_{m,\mu} - \partial_\mu t_m - iq' t_n K_{k,\mu} C_{knm}$$

where summations over k and n are implied. Making the replacement $\partial_\mu \to D_\mu$ in $\mathcal{L}$ yields

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi - q\bar{\Psi}\gamma^\mu A_\mu \Psi - q' \bar{\Psi}\gamma^\mu M^n \Psi K_{n,\mu}$$

[1] Some authors pull out an i from C_{nmk} as convention

We now need free fields for the gauge fields. For A_μ this is $F^{\mu\nu}F_{\mu\nu}$. For $K_{n,\mu}$ we can use the commutator of the covariant derivatives, only leaving terms involving $K_{n,\mu}$. Then we get

$$[D_\mu, D_\nu] = iq'M^k[\partial_\mu K_{k,\nu} - \partial_\nu K_{k,\mu}] - (q')^2 C_{nmk} M^k K_{n,\mu} K_{m,\nu}$$

We may now define the $SU(3)$ invariant quantity

$$K_{k,\mu\nu} \equiv \partial_\mu K_{k,\nu} - \partial_\nu K_{k,\mu} - (q')^2 C_{nmk} K_{n,\mu} K_{m,\nu}$$

Using the standard normalization constants, we arrive at the QCD Lagrangian

$$\mathcal{L}_{QCD} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}K^{n,\mu\nu}K_{n,\mu\nu} - q\bar{\Psi}\gamma^\mu A_\mu \Psi - q'\bar{\Psi}\gamma^\mu M^n \Psi K_{n,\mu}$$

A_μ is the photon field, while $K_{n,\mu}$ are the 8 gluon fields binding the quarks.

2.3 Dirac spinors

Four-component Dirac spinors are used throughout the standard model. They are defined through its transformation properties, as are contravariant and covariant vectors. In the following we derive the transformation properties of the Dirac spinor based on the presentation found in [2, 8, 13].

2.3.1 Transformations in general

The difference between a type of spinor and a vector is how they transform under Lorentz transformations. As an introduction we look at how the components of a vector and the components of a covector transform. For a vector we have the following (see [8]):

Let $x'^\mu = x'^\mu(x)$ be a map between a primed and an unprimed coordinate system, and let $x = x(t)$, where $t \in \mathbb{R}$ is an arbitrary parameter and $x', x \in \mathbb{R}^n$, then

$$\frac{dx'^\mu}{dt} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{dx^\nu}{dt}$$

$$v'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}v^\nu \tag{2.10}$$

Since $\frac{dx'^\mu}{dt} \equiv v'^\mu$ and $\frac{dx^\nu}{dt} \equiv v^\mu$ are any vectors in the primed and unprimed systems respectively, then the transformation (2.10) is an intrinsic property of vectors. Any function failing (2.10) is therefore, by definition, not a vector.

There exist other objects that do not transform as a vector. To show this, we let $\sigma^\mu : TM \to \mathbb{R}$ be a linear map, from a linear space TM of vectors v to $\mathbb{R}$. Let $\{\hat{e}_\nu\}$ be a basis for TM, such that $\sigma^\mu(\hat{e}_\nu) = \delta^\mu_\nu$. Then, because of linearity of σ we have

$$\sigma^\mu(v) = \sigma^\mu(v^\nu \hat{e}_\nu) = v^\nu \sigma^\mu(\hat{e}_\nu) = v^\nu \delta^\mu_\nu = v^\mu$$

Next, create a linear functional $\alpha : TM \to \mathbb{R}$ using σ as basis. Then $\alpha = a_\mu \sigma^\mu$. Now, interpret a_μ as the components of an n-tuple, similar to how v^μ are the components of the n-tuple which transforms as a vector. Since v transforms as a vector and $\sigma^\mu(v) = v^\mu$, then σ also transforms as a vector. Using (2.10) on σ we find

$$\alpha = a_\mu \sigma^\mu = a_\mu \frac{\partial x'^\mu}{\partial x^\nu} \sigma^\nu = a'_\nu \sigma^\nu$$

where the transformation property of a emerges as

$$a'_\nu = \frac{\partial x'^\mu}{\partial x^\nu} a_\mu \Rightarrow a'_\mu = \frac{\partial x'^\nu}{\partial x^\mu} a_\nu \tag{2.11}$$

Comparing equations (2.10) and (2.11) it is evident that the summation indices have been exchanged and thus a is not a vector. It is in fact a covariant vector (distinguished from a vector which is often refered to as a contravariant vector). The familiar connection between the two is seen by Riesz Representation Theorem [3], which says that any linear functional α can be represented by an inner product $\langle v, w \rangle$. Then

$$\alpha(\hat{e}_\nu) = a_\mu \sigma^\mu(\hat{e}_\nu) = a_\mu \delta^\mu_\nu = a_\nu = \langle \hat{e}_\nu, v \rangle = \langle \hat{e}_\nu, v^\mu \hat{e}_\mu \rangle = v^\mu \langle \hat{e}_\nu, \hat{e}_\mu \rangle = v^\mu g_{\mu\nu}$$

where $g_{\mu\nu}$ is the familiar metric and $a_\nu \equiv v_\nu$.

Let $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ be a Lorentz transformation from a non-primed to a primed system. Then

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu$$

and, according to (2.10), any x^ν transforming according to the above Lorentz transformation is a contravariant vector. Now, multiply both sides of $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ with $g_{\alpha\mu}$, then $x'_\alpha = \Lambda_{\alpha\nu} x^\nu = \Lambda_{\alpha\nu} g^{\nu\beta} x_\beta = \Lambda_\alpha{}^\beta x_\beta$. x_β does not transform according to (2.10) because of the exchange of summation indices. x_β transforms according to (2.11), which confirms that x_β is a covariant vector as the notation suggests.

A two-component spinor χ is defined by its transformation property [7]

$$\chi \to e^{-\frac{i}{2}\theta \cdot \sigma} \chi$$

where σ are the Pauli matrices, and θ, denotes the free parameter of the transformation. This transformation forms the group $SU(2)$, as seen in section 2.1. We can also mention that the group $SO(5)$ has a four dimensional spinor representation [9].

2.3.2 Definition of a Dirac spinor

The Dirac equation is $(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$. Lorentz covariance requires that this equation of motion has the same form in any inertial system. The matrix γ^μ is equivalent up to a unitary transform and no distintions need to be made on γ^μ between different inertial systems [2]. Since x transforms as $x'^\nu = \Lambda^\nu{}_\mu x^\mu$, then ψ must also transform. The transformation will enter in the following way (see [2]):

Let $S(\Lambda)$ be a linear transformation such that

$$\psi'(x') = S(\Lambda)\psi(x) \tag{2.12}$$

We find a relation between $S(\Lambda)$ and γ^μ by inserting $S^{-1}(\Lambda)S(\Lambda) = 1$ and left multiplying with $S(\Lambda)$ on the equation of motion, and then assume Lorentz covariance:

$$S(\Lambda)(i\gamma^\mu \partial_\mu - m)S^{-1}(\Lambda)S(\Lambda)\psi(x) = 0$$

$$(iS(\Lambda)\gamma^\mu S^{-1}(\Lambda)\partial_\mu - m)(\Lambda)\psi'(x') = 0$$

Since

$$\partial_\mu \psi'(x') = \frac{\partial \psi'(x')}{\partial x^\mu} = \frac{\partial \psi'(x')}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\mu} = \frac{\partial \psi'(x')}{\partial x'^\nu}\Lambda^\nu{}_\mu = \Lambda^\nu{}_\mu \partial'_\nu \psi'(x')$$

we have the replacement $\partial_\mu \to \Lambda^\nu{}_\mu \partial'_\nu$ and

$$(iS(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu{}_\mu \partial'_\nu - m)(\Lambda)\psi'(x') = 0$$

Lorentz covariance requires

$$\gamma^\nu \equiv S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu{}_\mu \tag{2.13}$$

such that $(i\gamma^\nu \partial'_\nu - m)(\Lambda)\psi'(x') = 0$ in the primed inertial system. Equation (2.13) can be solved and has the solution

$$S(\Lambda) = e^{-\frac{i}{4}\omega\sigma_{\mu\nu}I^{\mu\nu}} \tag{2.14}$$

where

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \tag{2.15}$$

$I^{\mu\nu}$ together with ω specifies the transformation. A 4-component Dirac spinor is defined by its transformation (2.12) together with (2.14).

2.4 Weyl spinors

In supersymmetric theories we wish to relate bosonic to fermionic degrees of freedom. It is then convenient to have a spinorial object like the Dirac spinor, but with two degrees of freedom. The Weyl spinor is such an object. In the following we look at the transformation properties of these objects including the dotted index notation (van der Waerden notation) used to construct invariants out of the Weyl spinors. The presentation below is based on [1] and [12].

2.4.1 Definition of Weyl spinors

It is possible to separate the Dirac equation into two equations coupled by the mass, m, each represented by using 2-component spinors ψ and χ. The two resulting equations are Lorentz covariant, provided the spinors ψ and χ transform in the correct manner. These transformations lead to the definition of 2-component Weyl spinors. The calculations are shown below (see [1]).

Let Ψ be a 4-component Dirac spinor and represent the components in this spinor as

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

where ψ and χ each have two components. Next, use the following representation of the Dirac equation:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix} \rightarrow \gamma^0 \cdot \gamma = \alpha = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}$$

By using Ψ and the above representation we get

$$(i\gamma^\mu \partial_\mu - m)\Psi = i\gamma^0 \partial_0 \Psi + i\gamma^K \partial_K \Psi - m\Psi = 0$$

Multiply from left with γ^0

$$i\partial_0 \Psi + i\gamma^0 \gamma^K \partial_K \Psi - \gamma^0 m\Psi = 0$$

to get

$$E\begin{pmatrix} \psi \\ \chi \end{pmatrix} - \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}\begin{pmatrix} \psi \\ \chi \end{pmatrix} \cdot p - m\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \chi \end{pmatrix} = 0$$

which leads to the following form of the Dirac equation

$$(E + \sigma \cdot p)\chi = m\psi \tag{2.16}$$

$$(E - \sigma \cdot p)\psi = m\chi \tag{2.17}$$

where $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ and σ^K is the K'th Pauli matrix. We see that a nonzero mass term will cause a mixing between the two spinors ψ and χ. Next, perform a Lorentz transformation on equations (2.16) and (2.17) insisting on Lorentz covariance. The energy and momentum transform as

$$E' = E - \eta \cdot p$$

$$p' = p - \eta E - \varepsilon \times p$$

Here ε is an infinitesimal rotation and η is an infinitesimal boost. Lorentz covariance then gives a transformation on ψ and χ

$$\psi'(x') = \left(1 + i\frac{1}{2}\varepsilon \cdot \sigma - \frac{1}{2}\eta \cdot \sigma\right)\psi(x) = V(\Lambda)\psi(x) \tag{2.18}$$

$$\chi'(x') = \left(1 + i\frac{1}{2}\varepsilon\cdot\sigma + \frac{1}{2}\eta\cdot\sigma\right)\chi(x) = V^{\dagger^{-1}}(\Lambda)\chi(x) \tag{2.19}$$

which are the transformations defining 2-component Weyl spinors.

ψ is called a right-chiral spinor while χ is called a left-chiral spinor. The purpose of the 2-component spinors is to reduce the number of degrees of freedom that are present in the 4-component spinors to two. This further poses a requirement of representing a 4-component spinor using only χ (or alternatively ψ). There must then be a way of retrieving ψ from χ (or vice versa). Lorentz-invariant constructions using 2-component spinors are of great importance. These constructions will consist of products of ψ and χ. Since we require using only χ in our constructions, then a ψ construction that transforms like χ must be found. Converting between spinor types and creating objects that transform as χ from ψ can be done as follows (see [1])

Let $\sigma^\mu = (1,\sigma)$ and $\bar{\sigma}^\mu = (1,-\sigma)$, then

$$\sigma^\mu p_\mu \psi = m\chi$$

$$\bar{\sigma}^\mu p_\mu \chi = m\psi$$

which shows that from the two components of χ it is possible to find the components of ψ and vice-versa. Now, let σ_2 be the second Pauli matrix, then from equations (2.18) and (2.19)

$$\sigma_2\chi'^* = V(\Lambda)\sigma_2\chi^*$$

which shows that $\sigma_2\chi^*$ transforms as ψ through $V(\Lambda)$. Denote

$$\psi_\chi \equiv i\sigma_2\chi^*$$

Similarly, $\chi_\psi \equiv -i\sigma_2\psi^*$ is a ψ construction transforming like χ.

2.4.2 Dotted index notation

We wish to have a notational tool to help constructing Lorentz invariants. Specifically it will be a useful tool in constructing invariants only containing left-chiral spinors. From a 4-component Dirac spinor we can construct the Lorentz invariants

$$\bar{\Psi}\Psi = \Psi^\dagger\gamma^0\Psi = \psi^\dagger\chi + \chi^\dagger\psi$$

It turns out that, through the transformations V and $V^{\dagger^{-1}}$ for the right-chiral and left-chiral spinors respectively, $\psi^\dagger\chi$ and $\chi^\dagger\psi$ are separately Lorentz invariant. We now define

$$\chi_a \equiv \text{ The components of } \chi$$

and

$$\psi^a \equiv \text{ The components of } \psi^\dagger$$

where we always will denote with a lower index, quantities that transform as left-chiral spinors. Then we may write

$$\psi^\dagger\chi = \psi^a\chi_a$$

where a is summed over. It is important to have the summation indices from top left to bottom right. Having the indices from bottom left to top right will give a sign reversal since the quantities ψ^a and χ_a are Grassmanian:

$$\psi^a\chi_a = \psi^1\chi_1 + \psi^2\chi_2 = -\chi_1\psi^1 - \chi_2\psi^2 = -\chi_a\psi^a$$

We may further look at the second invariant quantity $\chi^\dagger\psi$. To describe this invariant, we define

$$\bar{\psi}^{\dot{a}} \equiv \text{ The components of } \psi$$

and

$$\bar{\chi}_{\dot{a}} \equiv \text{ The components of } \chi^\dagger$$

such that

$$\chi^\dagger\psi = \bar{\chi}_{\dot{a}}\bar{\psi}^{\dot{a}}$$

We have then made sure that components transforming as right chiral spinors have an upper index. Notice that the dotted indices must be summed from the bottom left to the top right to get the correct value. Also notice the bar that always accompanies the dotted index. This allows us, in some instances, to drop the indices and use a dot product notation instead. We have seen that $-i\sigma^2\psi*$ transforms as a left chiral spinor. We would like to use a lower un-dotted index for such a component.

$$\psi_a \equiv \text{ The components of } -i\sigma^2\psi*$$

We are then left with $i\sigma^2\chi*$ that transforms as a right chiral quantity. We may define

$$\bar{\chi}^{\dot{a}} \equiv \text{ The components of } i\sigma^2\chi*$$

From the above definitions we see that

$$(\chi_a)^\dagger = \bar{\chi}_{\dot{a}}$$

and

$$(\psi^{\dot{a}})^\dagger = \psi^a$$

such that, taking the hermitian conjugate will switch between a dotted and un-dotted index. We may write $\chi* = (\chi^\dagger)^T$ where T denotes the transpose. $i\sigma^2(\chi^\dagger)^T$ transforms as a right chiral quantity and is denoted $\bar{\chi}^{\dot{a}}$ and $\chi^\dagger$ transforms as a left chiral quantity denoted $\bar{\chi}_{\dot{b}}$, thus

$$\bar{\chi}^{\dot{a}} = i[\sigma^2]^{\dot{a}\dot{b}}\bar{\chi}_{\dot{b}} \equiv \varepsilon^{\dot{a}\dot{b}}\bar{\chi}_{\dot{b}}$$

Similarly we get that

$$\psi_a = -i[\sigma^2]_{ab}\psi^b \equiv \varepsilon_{ab}\psi^b$$

$$\bar{\chi}_{\dot{a}} = -i[\sigma^2]_{\dot{a}\dot{b}}\bar{\chi}^{\dot{b}} \equiv \varepsilon_{\dot{a}\dot{b}}\bar{\chi}^{\dot{b}}$$

$$\bar{\psi}^{\dot{a}} = i[\sigma^2]^{\dot{a}\dot{b}}\bar{\psi}_{\dot{b}} \equiv \varepsilon^{\dot{a}\dot{b}}\bar{\psi}_{\dot{b}}$$

We see that the σ^2 Pauli matrix is used to raise and lower Weyl spinor indices. Here, ε^{ab} is the totally antisymmetric tensor which has the property that $\varepsilon_{ab}\varepsilon^{bc} = \delta_a^c$ [1].

We have now got a notation where invariants are made by combining either dotted or un-dotted quantities (i.e. in a summation dotted and un-dotted indices are treated as different indices even though they have the same letter).

Chapter 3

$N = 1$ Supersymmetry

In the following, an overview of $N = 1$ SUSY will be provided, based on information found in [1, 6, 10, 12–14, 16].

3.1 Introduction

There are unwanted infinities in the Standard Model when trying to calculate the vacuum expectation values of bosonic and fermionic fields. This can be seen as follows.

The Lagrangian $\mathcal{L}_B$ for a bosonic complex scalar field and the Lagrangian $\mathcal{L}_F$ for a fermionic field and their corresponding field mode expansions are [13]

$$\mathcal{L}_B = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi$$

$$\phi(x) = \sum_k \frac{1}{\sqrt{2V\omega_k}} [a(k)e^{-ikx} + b^\dagger(k)e^{ikx}]$$

and

$$\mathcal{L}_F = \bar{\psi}[i\gamma^\mu \partial_\mu - m]\psi$$

$$\psi(x) = \sum_{p,s} \sqrt{\frac{m}{2VE_p}} [c(s,p)u_s(p)e^{-ipx} + d^\dagger(s,p)v_s(p)e^{ipx}]$$

where the Einstein summing convention is employed. The Hamiltonian densities of the two Lagrangians are calculated as $\mathcal{H}_B = \pi\dot{\phi} + \pi^\dagger\dot{\phi}^\dagger - \mathcal{L}_B$ and $\mathcal{H}_F = p\dot{\psi} + p^\dagger\dot{\phi}^\dagger - \mathcal{L}_F$, where π and p are the respective conjugate momenta. To calculate the energy of the vacuum we ignore all normal ordering. Then we get that

$$\langle 0|H_B|0\rangle = \langle 0|\int d^3x \mathcal{H}_B|0\rangle = \langle 0|\sum_k \omega_k [a^\dagger(k)a(k) + b^\dagger(k)b(k) + 1]|0\rangle = 2\sum_k \frac{1}{2}\omega_k$$

and

$$\langle 0|H_F|0\rangle = \langle 0|\int d^3x \mathcal{H}_F|0\rangle = \langle 0|\sum_p \frac{E_p}{2}[c^\dagger(s,p)c(s,p) + d^\dagger(s,p)d(s,p) - 1]|0\rangle = -2\sum_p \frac{1}{2}E_p$$

where the relations

$$u^\dagger_{s,\alpha}(p)u^\alpha_{s'}(p) = \frac{E_p}{m}\delta_{ss'} = v^\dagger_{s,\alpha}(p)v^\alpha_{s'}(p)$$

and

$$u^{\dagger}_{s,\alpha}(p)v^{\alpha}_{s'}(-p)=0$$

have been employed. Both energies of the vacuum are infinite, but for the fermion field, the energy is negative. Thus, by combining fermionic and bosonic degrees of freedom we may extend the theory such that these vacuum energies cancel each other. This is the case in a supersymmetric theory where each fermionic degree of freedom is partnered with a bosonic degree of freedom and vice versa, through an invariance under transformations between the fermionic and bosonic fields. Then, when calculating the vacuum expectation value of a bosonic field, this will automatically entail a corresponding negative expectation value of the partnered fermionic field. Note, however, that because of SUSY breaking, the vacuum expectation value will still be non-zero.

In a similar manner, supersymmetry will cancel quantum corrections that occur when any field f couples to the higgs field H [14]

$$\Delta m_H^2 = -\frac{|\lambda_f|^2}{8\pi^2}\Lambda_{UV}^2 + \cdots$$

where Λ_{UV} is the cutoff and λ_f is the coupling constant between f and H. The fact that this correction is very large is referred to as the Hierarchy problem in particle physics. Being able to fix the Hierarchy problem makes supersymmetry an attractive extension to the Standard Model. The resulting extension is called the Minimal Supersymmetric Standard Model (MSSM).

Bosonic string theory is described through the Polyakov action [10]

$$S_p = -\frac{T}{2}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

where $h_{\alpha\beta}$ is the world sheet metric, $h=\det h_{\alpha\beta}$ and $\eta_{\mu\nu}$ is the D-dimensional Minkowski metric. This action must experience Poincaré symmetry which leads to conserved currents that in turn should satisfy the Poincaré algebra. This is only satisfied in 26 dimensions. By pairing the bosonic $h_{\alpha\beta}$ and X^μ with fermionic, anticommuting partners, the same requirements will lead to 11 dimensions. Supersymmetric particles will automatically emerge from this partnering of bosonic and fermionic degrees of freedom. The theory emerging from the action S_p gives rise to tachyon particles in the theory. The tachyons are removed from the theory by making it supersymmetric. This leads to superstring theory. We will in the following not focus on supersymmetry in string theory, but supersymmetry in particle physics.

3.2 SUSY transformations

In section 2.2, global symmetries such as invariance under $\phi \to \phi e^{-iq\xi}$, were encountered. This is an invariance on rotation of the phase in the complex plane and is therefore an internal symmetry. An invariance on rotations and boosts in Minkowski space constitutes a space-time symmetry. A SUSY transformation is an invariance under a change of the bosonic degrees of freedom due to a small change in the fermionic degrees of freedom and vice versa.

Let $\mathcal{L}_f$ be a free fermion field, and let $\Psi = \begin{pmatrix}\psi \\ \chi\end{pmatrix}$. We then have that

$$\mathcal{L}_f = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi = \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + \psi^\dagger i\sigma^\mu\partial_\mu\psi$$

where χ represents the left-chiral fermion. We only wish to consider SUSY transformations related to the left-chiral part of the lagrangian density. We want to have a lagrangian consisting of both fermionic and bosonic degrees of freedom and corresponding transformations between these degrees of freedom that leave it invariant up to a surface term. We may consider

$$\mathcal{L} = \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + \partial_\mu\phi^\dagger\partial^\mu\phi$$

where ϕ is a complex scalar field. Let ξ be a left-chiral spinor. Then the transformations

$$\delta_\xi\chi = -i\sigma^\mu\bar{\xi}\partial_\mu\phi \tag{3.1}$$

$$\delta_\xi\bar{\chi} = -i\partial_\mu\phi^\dagger\bar{\sigma}^\mu\xi \tag{3.2}$$

$$\delta_\xi\phi = \xi\cdot\chi \tag{3.3}$$

$$\delta_\xi\phi^\dagger = \bar{\xi}\cdot\bar{\chi} \tag{3.4}$$

leave the lagrangian $\mathcal{L}$ invariant up to a surface term. These transformations can be found by requiring that the left and right-hand sides transform in the same manner under Lorentz transformations, the dimensions on the left and right hand sides match, Lorentz contractions must be consistent from one side of the equation to the other, and that the degrees of freedom match on both sides. Note that ξ is a left-chiral Weyl spinor since we want $\xi\cdot\chi$ to form an invariant. We see that the transformations on the fermionic degrees of freedom of χ are due to changes in the bosonic degrees of freedom of ϕ and vice versa.

If these transformations constitute a group, then we should have an associated algebra. We will now try to find the associated algebra. Let U_α be the SUSY transformation on the complex scalar field ϕ, due to the infinitesimal real parameters α_i, where i runs from 1 to n. Let U_α be some member of a group, then

$$U_\alpha = e^{i\alpha_i G^i} = e^{i\alpha\cdot G} = 1 + \alpha\cdot G + \mathcal{O}(\alpha^2)$$

where i is the sum over group generators $G_1, \ldots, G_n$. We have collected α_i and G_i into

$$\alpha = \begin{pmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{pmatrix} \qquad G = \begin{pmatrix}G_1 \\ \vdots \\ G_n\end{pmatrix}$$

We start by calculating [1, 12]

$$\delta_\alpha\delta_\beta\phi = U_\alpha^\dagger U_\beta^\dagger\phi U_\beta U_\alpha$$

where β_i are real quantities analogous to α_i. By only keeping terms of $\mathcal{O}(\alpha)$ and $\mathcal{O}(\beta)$, we get that

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\phi = [[\beta\cdot G, \alpha\cdot G], \phi] \tag{3.5}$$

By promoting α and $\beta = \bar{\alpha}$ to left-chiral spinors, we may then form the spinors

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \equiv \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$$

and

$$Q^\dagger = \begin{pmatrix} Q_1^\dagger \\ Q_2^\dagger \end{pmatrix} \equiv \begin{pmatrix} G_3 \\ G_4 \end{pmatrix}$$

We then have the invariant quantities, $\alpha\cdot Q$, $\beta\cdot Q$, $\bar{\alpha}\cdot\bar{Q}$, $\bar{\beta}\cdot\bar{Q}$, which can be inserted into equation (3.5), to get

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\phi = [[\beta\cdot Q, \alpha\cdot Q] + [\beta\cdot Q, \bar{\alpha}\cdot\bar{Q}] + [\bar{\beta}\cdot\bar{Q}, \alpha\cdot Q] + [\bar{\beta}\cdot\bar{Q}, \bar{\alpha}\cdot\bar{Q}], \phi]$$

By utilizing the fact that α and β are Grassmann numbers, and that Q and $Q^\dagger$ are Grassmann operators, we can move α and β outside the commutators. We then get that

$$\begin{aligned}
[\beta\cdot Q, \alpha\cdot Q] &= \sigma^{2,ac}\sigma^{2,bd}\beta_c\alpha_d\{Q_a, Q_b\} \\
[\beta\cdot Q, \bar{\alpha}\cdot\bar{Q}] &= -\sigma^{2,ac}\sigma^{2,bd}\beta_c\alpha_d^*\{Q_a, Q_b^\dagger\} \\
[\bar{\beta}\cdot\bar{Q}, \alpha\cdot Q] &= -\sigma^{2,ac}\sigma^{2,bd}\beta_c^*\alpha_d\{Q_a^\dagger, Q_b\} \\
[\bar{\beta}\cdot\bar{Q}, \bar{\alpha}\cdot\bar{Q}] &= \sigma^{2,ac}\sigma^{2,bd}\beta_c^*\alpha_d^*\{Q_a^\dagger, Q_b^\dagger\}
\end{aligned}$$

From this we see that

$$\begin{gathered}
(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\phi = \\
\sigma^{2,ac}\sigma^{2,bd}[\beta_c\alpha_d\{Q_a, Q_b\} - \beta_c\alpha_d^*\{Q_a, Q_b^\dagger\} - \beta_c^*\alpha_d\{Q_a^\dagger, Q_b\} + \beta_c^*\alpha_d^*\{Q_a^\dagger, Q_b^\dagger\}, \phi]
\end{gathered} \tag{3.6}$$

We now calculate the differential $(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\phi$ using a different approach, such that a comparison of results leads to an algebra. In this approach we use equations (3.1) and (3.3) to calculate the differential. We get that

$$\delta_\alpha\delta_\beta\phi = \delta_\alpha(\beta^a\chi_a) = \beta^a\delta_\alpha\chi_a = -i\beta_b[\bar{\sigma}^{\mu T}]^{bd}\alpha_d^*\partial_\mu\phi \tag{3.7}$$

where we in the last step inserted equation (3.1) and used the equality

$$\bar{\sigma}^{\mu T} = -\sigma^2\sigma^\mu\sigma^2 \tag{3.8}$$

Using equation (3.7) we find that

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\phi = \sigma^{2,ac}\sigma^{2,bd}[\beta_c^*\alpha_d\sigma^\mu_{ba} + \beta_c\alpha_d^*\sigma^\mu_{ab}]\partial_\mu\phi \tag{3.9}$$

Further we have that symmetries under translation lead to the conserved momentum P_μ. Such a translation is described by $\delta\phi = \varepsilon^\mu\partial_\mu\phi$. The generator of the corresponding group is P_μ such that

$$T = \exp(i\varepsilon^\mu P_\mu) = 1 + i\varepsilon^\mu P_\mu + \mathcal{O}(\varepsilon^2)$$

We may perform a unitary transformation, using the expansion of the transformation to get

$$\phi' = T\phi T^\dagger = \phi + \delta\phi + \mathcal{O}(\delta\phi^2) = \phi + i\varepsilon^\mu [P_\mu, \phi] + \mathcal{O}(\varepsilon^2)$$

ε is an infinitesimal parameter and we need only keep terms to order $\mathcal{O}(\varepsilon)$. Having an expression for $\delta\phi$, we obtain

$$\partial_\mu \phi = i[P_\mu, \phi]$$

Inserting this into equation (3.9) we ge

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\phi = \sigma^{2,ac}\sigma^{2,bd}[-\beta_c^*\alpha_d\sigma^\mu_{ba}P_\mu - \beta_c\alpha_d^*\sigma^\mu_{ab}P_\mu, \phi] \tag{3.10}$$

By comparing equation (3.10) with equation (3.6), we find the following algebras

$$\{Q_a, Q_b\} = \{Q_a^\dagger, Q_b^\dagger\} = 0 \tag{3.11}$$

$$\{Q_a, Q_b^\dagger\} = \sigma^\mu_{ab}P_\mu \tag{3.12}$$

This should also hold for $(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\chi$ in the case that these were true algebras associated with the transformations (3.1) to (3.4). This is not the case. This means that in order for these transformations to form a closed group, the transformations must be altered. This can be done by adding the term $FF^\dagger$ to $\mathcal{L}$, where F is referred to as an auxiliary field. The SUSY transformations, still satisfying the above algebra, will then become

$$\delta_\xi\chi = -i\sigma^\mu\bar{\xi}\partial_\mu\phi + \xi F \tag{3.13}$$

$$\delta_\xi\bar{\chi} = -i\partial_\mu\phi^\dagger\bar{\sigma}^\mu\xi + F^\dagger\bar{\xi} \tag{3.14}$$

$$\delta_\xi\phi = \xi\cdot\chi \tag{3.15}$$

$$\delta_\xi\phi^\dagger = \bar{\xi}\cdot\bar{\chi} \tag{3.16}$$

$$\delta_\xi F = -i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\chi \tag{3.17}$$

$$\delta_\xi F^\dagger = i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\xi \tag{3.18}$$

All $N = 1$ SUSY transformations are characterized by the algebras in equations (3.11) and (3.12) and it is possible to find other lagrangian densities that satisfy these. All such lagrangians are said to possess supersymmetry.

3.3 Supermultiplets

In section 3.2 we found the $N = 1$ SUSY algebra, described in equations (3.11) and (3.12). The charges, Q_a and $Q_a^\dagger$, represent infinitesimal SUSY transformations on the fields that are contained in the action which is invariant under these. The algebra tells us that the SUSY transformations contained in the group generated by the algebra, does not generally commute. We are interested in finding eigenstates that can be used to describe an irreducible representation of the fields. We may therefore not use the SUSY charges directly. We will need operators that commute with all the generators of the group generated by the SUSY algebra (Casimir operators). The Casimir operators for the SUSY algebra are $P^\mu P_\mu$ and $W^\mu W_\mu$ (W^μ is the Pauli-Lubanski operator, and is given below). Since these commute with all of the generators, then they will be independent of which SUSY transformations have been applied to the fields, and can therefore be used to classify the irreducible representations.

The Lorentz transformations of a space-time independent left-chiral spinor χ is given by

$$\chi \to \chi' = e^{\frac{i}{2}\omega^{\mu\nu}\sigma_{\mu\nu}}\chi$$

which is analogous to the 4-component transformation $S(\Lambda)$ of equation (2.14). We also have an algebra similar to equation (2.15), which is given by

$$\sigma_{\mu\nu} = \frac{i}{4}[\sigma_\mu, \bar{\sigma}_\nu]$$

where now the 2×2 matrices σ^μ and $\bar{\sigma}^\mu$ replace the 4×4 gamma matrices. For an infinitesimal Lorentz transformation we have that $x^{\mu'} = x^\mu + \omega^{\mu\nu}x_\nu$, such that for a space-time dependent spinor $\chi(x^\mu)$ we have that

$$\begin{aligned}
\chi(x^\mu) \to \chi'(x^{\mu'}) &= e^{\frac{i}{2}\omega^{\alpha\beta}\sigma_{\alpha\beta}}\chi(x^\mu + \omega^{\mu\nu}x_\nu) \\
&= e^{\frac{i}{2}\omega^{\alpha\beta}\sigma_{\alpha\beta}}[\chi(x^\mu) + x_\beta\omega^{\alpha\beta}\partial_\alpha\chi(x^\mu) + \mathcal{O}(\omega^2)] \\
&= \left[1 + \frac{i}{2}\omega^{\alpha\beta}\sigma_{\alpha\beta}\right][\chi(x^\mu) + x_\beta\omega^{\alpha\beta}\partial_\alpha\chi(x^\mu) + \mathcal{O}(\omega^2)] \\
&= \chi(x^\mu) + \frac{1}{2}\omega^{\alpha\beta}[i\sigma_{\alpha\beta} + x_\beta\partial_\alpha - x_\alpha\partial_\beta]\chi(x^\mu) + \mathcal{O}(\omega^2)
\end{aligned}$$

We may however also describe the Lorentz transformation above as

$$\begin{aligned}
\chi(x^\mu) \to \chi'(x^{\mu'}) &= e^{\frac{i}{2}\omega^{\alpha\beta}M_{\alpha\beta}}\chi(x^\mu) \\
&= \chi(x^\mu) + \frac{i}{2}\omega^{\alpha\beta}M_{\alpha\beta}\chi(x^\mu) + \mathcal{O}(\omega^2)
\end{aligned}$$

By comparing, we see that the generators of Lorentz transformations for space-time dependent spinors are

$$M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + \sigma_{\mu\nu}$$

The Pauli-Lubanski operator is now defined as

$$W^\mu \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}M_{\rho\sigma}P_\nu$$

We already know that P^μ has the momentum p^μ as its eigenstate. Then this is one of the parameters characterizing the irreducible representations. We still need to know what the eigenvalues of W^μ are and what they represent. In the Standard Model as well as for the MSSM (Minimal Supersymetric Standard Model), all masses are generated by symmetry breaking. We are therefore entitled to only consider massless states. We may choose a frame where the four-momentum is $p^\mu = (p^0, 0, 0, p^3) = (E, 0, 0, E)$, and label this state as $|p\rangle_0$. We then get that

$$W^\mu P_\mu |p\rangle_0 = W^\mu P^\nu \eta_{\mu\nu}|p\rangle_0 = (W^0P^0 - W^3P^3)|p\rangle_0 = E(W^0 - W^3)|p\rangle_0$$

and, since $\varepsilon^{\mu\nu\rho\sigma}$ is antisymmetric while $P_\mu P_\nu$ is symmetric under $\mu \leftrightarrow \nu$, we have

$$W^\mu P_\mu |p\rangle_0 = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} M_{\rho\sigma} P_\nu P_\mu |p\rangle_0 = 0$$

such that $W^0 = W^3$. Further we can use

$$\begin{aligned} W^\mu |p\rangle_0 &= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} M_{\rho\sigma} P_\nu |p\rangle_0 \\ &= \frac{1}{2}\varepsilon^{\mu 0\rho\sigma} M_{\rho\sigma} p_0 |p\rangle_0 + \frac{1}{2}\varepsilon^{\mu 3\rho\sigma} M_{\rho\sigma} p_3 |p\rangle_0 \end{aligned}$$

to calculate $W^3 = W^0$:

$$\begin{aligned} W^3 |p\rangle_0 &= \frac{1}{2}(-\varepsilon^{0123} M_{12} p_0 - \varepsilon^{0213} M_{21} p_0)|p\rangle_0 \\ &= \frac{1}{2}(M_{21} - M_{12}) E |p\rangle_0 \\ &= \frac{1}{2}(x_2 P_1 - x_1 P_2 + \sigma_{21} - x_1 P_2 + x_2 P_1 - \sigma_{12}) E |p\rangle_0 \\ &= \left[(x_2 P_1 - x_1 P_2) + \frac{1}{2}(\sigma_{21} - \sigma_{12})\right] E |p\rangle_0 \\ &= \left[L^3 + S^3\right] E |p\rangle_0 \end{aligned}$$

where L^3 is the angular momentum and S^3 is the spin. We will assume that the angular momentum for the massless particle is zero. Thus

$$W^0 |p\rangle_0 = W^3 |p\rangle_0 = s_z E |p\rangle_0 = s \cdot \hat{p} E |p\rangle_0$$

where $h = s \cdot \hat{p}$ is the helicity of the massless particle. We also see that

$$\begin{aligned} 0 = W^\mu W_\mu |p\rangle_0 &= (W^0 W^0 - W^1 W^1 - W^2 W^2 - W^3 W^3)|p\rangle_0 \\ &= -(W^1 W^1 + W^2 W^2)|p\rangle_0 \end{aligned}$$

which means that $W^1 |p\rangle_0 = 0$ and $W^2 |p\rangle_0 = 0$.

We now know that the momentum operator P_μ gives momentum eigenstates while the Pauli-Lubanski opertator W^μ gives helicity eigenstates h combined with the particle energy E. From the corresponding Casimir operators $P^\mu P_\mu$ and $W^\mu W_\mu$ we then know that the irreducible representations for the massless particles must be classified through the helicity h and the energy E (E represents the four momentum p), and we will denote the corresponding states as $|p, h\rangle$. To look at how the supercharges affect the momentum and helicity part of the state, we must know how the momentum operator P^μ and the Pauli-Lubanski operator W^μ commute with the supercharges Q_a and $Q_a^\dagger$. Since we may write

$$Q_a \sim \int d^4x J_a^0$$

for some conserved current J^μ, related to the symmetry through Noether's theorem, then Q_a has no x dependence. Also, $P_\mu = i\partial_\mu$ and we see that

$$[Q_a, P^\mu] = 0$$

$$[Q_a^\dagger, P^\mu] = 0$$

Using that [12]

$$[Q_a, M_{\mu\nu}] = (\sigma_{\mu\nu})_a{}^b Q_b \tag{3.19}$$

and

$$[Q_a^\dagger, M_{\mu\nu}] = -Q_b^\dagger (\bar{\sigma}_{\mu\nu})^b{}_a \tag{3.20}$$

we have for W^0 that

$$\begin{aligned}
[Q_a, W^0]|p\rangle_0 &= \frac{1}{2}\varepsilon^{0\nu\rho\sigma}[Q_a, M_{\rho\sigma}]P_\nu|p\rangle_0 \\
&= \frac{1}{2}\varepsilon^{03\rho\sigma}[Q_a, M_{\rho\sigma}]P_0|p\rangle_0 \\
&= \frac{1}{2}\varepsilon^{0\rho\sigma 3}(\sigma_{\rho\sigma}))_a{}^b Q_b P_3|p\rangle_0 \\
&= -\frac{1}{2}(\sigma_3)_a{}^b Q_b P_3|p\rangle_0
\end{aligned}$$

and similarly

$$[Q_a^\dagger, W^0]|p\rangle_0 = \frac{1}{2}Q_b^\dagger(\sigma_3)^b{}_a P_3|p\rangle_0$$

More specifically we get that

$$[Q_1, W^0] = -\frac{1}{2}Q_1 E$$

$$[Q_1^\dagger, W^0] = \frac{1}{2}Q_1^\dagger E$$

$$[Q_2, W^0] = \frac{1}{2}Q_2 E$$

$$[Q_2^\dagger, W^0] = -\frac{1}{2}Q_2^\dagger E$$

We may now calculate the effect of the supercharges Q_a and $Q_a^\dagger$ on the states $|p,h\rangle$. First we look at how the supercharges affect the momentum state:

$$P^\mu(Q_a|p,h\rangle) = Q_a P^\mu|p,h\rangle = p^\mu(Q_a|p,h\rangle) \tag{3.21}$$

$$P^\mu(Q_a^\dagger|p,h\rangle) = Q_a^\dagger P^\mu|p,h\rangle = p^\mu(Q_a^\dagger|p,h\rangle) \tag{3.22}$$

We see that Q_a and $Q_a^\dagger$ do not affect the momentum state p. For the helicity we get

$$\begin{aligned}
W^0(Q_1|p,h\rangle) &= \left[Q_1 W^0 + \frac{1}{2}Q_1 E\right]|p,h\rangle \\
&= \left[Q_1 hE + \frac{1}{2}Q_1 E\right]|p,h\rangle \\
&= \left[h + \frac{1}{2}\right]E(Q_1|p,h\rangle)
\end{aligned}$$

and similarly

$$W^0(Q_1^\dagger|p,h\rangle) = \left[h - \frac{1}{2}\right] E(Q_1^\dagger|p,h\rangle)$$

From this we conclude that Q_1 raises the helicity by $1/2$, $Q_1^\dagger$ lowers the helicity by $1/2$, Q_2 lowers the helicity by $1/2$ and $Q_2^\dagger$ raises the helicity by $1/2$. Mathematically we have that

$$\begin{aligned}
Q_1|p,h\rangle &= |p,h+\frac{1}{2}\rangle \\
Q_1^\dagger|p,h\rangle &= |p,h-\frac{1}{2}\rangle \\
Q_2^\dagger|p,h\rangle &= |p,h+\frac{1}{2}\rangle \\
Q_2|p,h\rangle &= |p,h-\frac{1}{2}\rangle
\end{aligned}$$

With these relations we are able to calculate how many states are in an $N = 1$ supermultiplet. Also note that every state within any given supermultiplet must transform in the same manner. If this was not the case, then the state that did not transform equivalently, would not be in the group that forms the supermultiplet and thus not in the supermultiplet itself. From equations (3.21) and (3.22) we see that every member of the supermultiplet will have the same momentum. We may assume a lowest helicity state and denote it $|p,-j\rangle$. Then we have by definition that

$$Q_1^\dagger|p,-j\rangle = Q_2|p,-j\rangle = 0$$

We may now try to raise the helicity by using either Q_1 or $Q_2^\dagger$. Starting with Q_1 we have from the SUSY algebra in equation (3.12) that

$$\begin{aligned}
\{Q_1,Q_1^\dagger\}|p,-j\rangle &= (\sigma^\mu)_{11}P^\nu\eta_{\mu\nu}|p,-j\rangle \\
&= [(\sigma^0)_{11}P^0 - (\sigma^3)_{11}P^3]|p,-j\rangle \\
&= [E-E]|p,-j\rangle = 0
\end{aligned}$$

Then we also have that

$$\langle p,-j|Q_1^\dagger Q_1|p,-j\rangle = -\langle p,-j|Q_1 Q_1^\dagger|p,-j\rangle = 0$$

Therefore, although Q_1 is a raising operator on the helicity state, it is not equipped to raise the helicity from its minimum. For $Q_2^\dagger$ the result differs

$$\begin{aligned}
\{Q_2,Q_2^\dagger\}|p,-j\rangle &= (\sigma^\mu)_{22}P^\nu\eta_{\mu\nu}|p,-j\rangle \\
&= [(\sigma^0)_{22}P^0 - (\sigma^3)_{22}P^3]|p,-j\rangle \\
&= [E+E]|p,-j\rangle = 2E|p,-j\rangle
\end{aligned}$$

Thus, we may only use $Q_2^\dagger$ to raise the helicity state from the minimum $-j$. We then get another state in the supermultiplet in addition to the existing state $|p,-j\rangle$:

$$Q_2^\dagger|p,-j\rangle = |p,\frac{1}{2}-j\rangle$$

Since $Q_2^\dagger$ anticommutes with itself (it is Grassmannian) then $(Q_2^\dagger)^2 = 0$ and it cannot raise the helicity any further. Also, $\{Q_1, Q_2^\dagger\} = 0$, such that $Q_1 Q_2^\dagger|p,-j\rangle = 0$. Thus Q_1 will not raise the helicity any further. We then have only two possible states in the irreducible representation of the $N=1$ SUSY algebra. These are $|p,-j\rangle$ and $|p,\frac{1}{2}-j\rangle$ where $-j$ is the lowest helicity value. Also note that due to demanding TCP invariance, we must insist on the existence of a an anti-particle version of the irreducible representation, with opposite signs on the helicities.

3.4 Superfield formalism

The superfield formalism is used to systematically build supersymmetric Lagrangians. The idea is to write the SUSY generators Q_a and $Q_a^\dagger$ as differential operators, working on superspace. Superspace is a space spanned by the space-time coordinate x and the two spinor parameters θ and $\bar{\theta}$. We will first find how the superspace coordinates $(x,\theta,\bar{\theta})$ change under a SUSY transformation.

Let $\Phi(x,\theta,\bar{\theta})$ be a superfield and let $U(a,\xi,\bar{\xi})$ be a unitary SUSY transformation generated by the charges Q_a and $Q_a^\dagger$. Here, a is a space-time transformation, and ξ, $\bar{\xi}$ are SUSY variations. Then

$$U(a,\xi,\bar{\xi}) = e^{ia\cdot P + i\xi\cdot Q + i\bar{\xi}\cdot\bar{Q}} \tag{3.23}$$

We may use this transformation to represent $\Phi(0) \to \Phi(x,\theta,\bar{\theta})$ by

$$\Phi(x,\theta,\bar{\theta}) = U(x,\theta,\bar{\theta})\Phi(0)U^\dagger(x,\theta,\bar{\theta})$$

The transformation $\Phi(x,\theta,\bar{\theta}) \to \Phi(x',\theta',\bar{\theta}')$, where $x' = x + a$, $\theta' = \theta + \xi$ and $\bar{\theta}' = \bar{\theta} + \bar{\xi}$ is represented by

$$\Phi(x',\theta',\bar{\theta}') = U(a,\xi,\bar{\xi})U(x,\theta,\bar{\theta})\Phi(0)U^\dagger(x,\theta,\bar{\theta})U^\dagger(a,\xi,\bar{\xi})$$

Here a, ξ and $\bar{\xi}$ are infinitesimal transformations. We may also write

$$U(x',\theta',\bar{\theta}')\Phi(0)U^\dagger(x',\theta',\bar{\theta}') = U(a,\xi,\bar{\xi})U(x,\theta,\bar{\theta})\Phi(0)U^\dagger(x,\theta,\bar{\theta})U^\dagger(a,\xi,\bar{\xi})$$

such that

$$U(x',\theta',\bar{\theta}') = U(a,\xi,\bar{\xi})U(x,\theta,\bar{\theta})$$

By using equation (3.23), we get that

$$e^{ix'^\mu P_\mu + i\theta'^a Q_a + i\bar{\theta}'_{\dot{a}}\bar{Q}^{\dot{a}}} = e^{ia^\mu P_\mu + i\xi^a Q_a + i\bar{\xi}_{\dot{a}}\bar{Q}^{\dot{a}}} e^{ix^\mu P_\mu + i\theta^a Q_a + i\bar{\theta}_{\dot{a}}\bar{Q}^{\dot{a}}} \tag{3.24}$$

We may use the BCH (Baker-Campbell-Hausdorff) identity

$$e^A e^B = e^{(A+B+\frac{1}{2}[A,B]+\cdots)}$$

to calculate the right hand side of (3.24). Note that x^μ is not an operator, in the same way as a^μ is not an operator. Therefore, since P_μ commutes with the SUSY generators, every commutator with P_μ in the BCH expansion will vanish. Since the components of

ξ and θ are Grassmannian, we are left with only one non-vanishing commutator in the BCH expansion. The right hand side of (3.24) now becomes

$$e^{i(a^\mu+x^\mu)P_\mu+i(\xi^a+\theta^a)Q_a+i(\bar{\xi}_{\dot{a}}+\bar{\theta}_{\dot{a}})\bar{Q}^{\dot{a}}+\frac{1}{2}\{[i\xi^a Q_a,i\theta^b Q_b]+[i\xi^a Q_a,i\bar{\theta}_{\dot{b}}\bar{Q}^{\dot{b}}]+[i\bar{\xi}_{\dot{a}}\bar{Q}^{\dot{a}},i\theta^b Q_b]+[i\bar{\xi}_{\dot{a}}\bar{Q}^{\dot{a}},i\bar{\theta}_{\dot{b}}\bar{Q}^{\dot{b}}]\}}$$

For the commutators in the exponent we have

$$\begin{aligned}[i\xi^a Q_a, i\theta^b Q_b] &= -[\xi^a Q_a, \theta^b Q_b] = \\ &= -\xi^a Q_a\theta^b Q_b + \theta^b Q_b \xi^a Q_a = \xi^a\theta^b Q_a Q_b + \xi^a\theta^b Q_b Q_a \\ &= \xi^a\theta^b\{Q_a, Q_b\} = 0\end{aligned}$$

where the sign changes are due to ξ and θ being Grassmann variables. Further we have

$$\begin{aligned}[i\xi^a Q_a, i\bar{\theta}_{\dot{b}}\bar{Q}^{\dot{b}}] &= \xi^a\bar{\theta}_{\dot{b}}\{Q_a,\bar{Q}^{\dot{b}}\} = \xi^a\bar{\theta}_{\dot{b}}\{Q_a,(i\sigma^2)^{\dot{b}\dot{c}}\bar{Q}_{\dot{c}}\} \\ &= \xi^a\bar{\theta}_{\dot{b}}(i\sigma^2)^{\dot{b}\dot{c}}\{Q_a,\bar{Q}_{\dot{c}}\} = -\xi^a(i\sigma^2)^{\dot{c}\dot{b}}\bar{\theta}_{\dot{b}}[\sigma^\mu]_{a\dot{c}}P_\mu \\ &= -\xi^a[\sigma^\mu]_{a\dot{c}}\bar{\theta}^{\dot{c}}P_\mu = -\xi\sigma^\mu\bar{\theta}P_\mu\end{aligned}$$

$$[i\bar{\xi}_{\dot{a}}\bar{Q}^{\dot{a}}, i\theta^b Q_b] = -[i\theta^b Q_b, i\bar{\xi}_{\dot{a}}\bar{Q}^{\dot{a}}] = \theta^a[\sigma^\mu]_{a\dot{c}}\bar{\xi}^{\dot{c}}P_\mu = \theta\sigma^\mu\bar{\xi}P_\mu$$

$$[i\bar{\xi}_{\dot{a}}\bar{Q}^{\dot{a}}, i\bar{\theta}_{\dot{b}}\bar{Q}^{\dot{b}}] = -\bar{\theta}_{\dot{b}}\bar{\xi}_{\dot{a}}\{\bar{Q}^{\dot{a}},\bar{Q}^{\dot{b}}\} = 0$$

The right-hand side of equation (3.24) becomes

$$\begin{aligned}&e^{i(a^\mu+x^\mu)P_\mu+i(\xi^a+\theta^a)Q_a+i(\bar{\xi}_{\dot{a}}+\bar{\theta}_{\dot{a}})\bar{Q}^{\dot{a}}+\frac{1}{2}\{-\xi^a[\sigma^\mu]_{a\dot{c}}\bar{\theta}^{\dot{c}}P_\mu+\theta^a[\sigma^\mu]_{a\dot{c}}\bar{\xi}^{\dot{c}}P_\mu\}} \\ =&e^{(ia^\mu+ix^\mu-\frac{1}{2}\xi^a[\sigma^\mu]_{a\dot{c}}\bar{\theta}^{\dot{c}}+\frac{1}{2}\theta^a[\sigma^\mu]_{a\dot{c}}\bar{\xi}^{\dot{c}})P_\mu+i(\xi^a+\theta^a)Q_a+i(\bar{\xi}_{\dot{a}}+\bar{\theta}_{\dot{a}})\bar{Q}^{\dot{a}}}\end{aligned}$$

Comparing this result with the left hand side of equation (3.24) we read off the superspace coordinate transformations

$$x'^\mu = x^\mu + a^\mu + \frac{i}{2}\xi^a[\sigma^\mu]_{a\dot{c}}\bar{\theta}^{\dot{c}} - \frac{i}{2}\theta^a[\sigma^\mu]_{a\dot{c}}\bar{\xi}^{\dot{c}} \tag{3.25}$$

$$\theta'^a = \theta^a + \xi^a \tag{3.26}$$

$$\bar{\theta}'_{\dot{a}} = \bar{\theta}_{\dot{a}} + \bar{\xi}_{\dot{a}} \tag{3.27}$$

Next we will find the SUSY generators as differential operators by first expanding the SUSY transformation $U(a,\xi,\bar{\xi})$ in terms of the charges and comparing with the expansion of $\Phi(x',\theta',\bar{\theta}')$ in terms of a, ξ and $\bar{\xi}$ in the transformations (3.25), (3.26) and (3.27).

Expansion in terms of the charges gives

$$\Phi(x',\theta',\bar{\theta}') = (1 - ia^\mu P_\mu - i\xi^a Q_a - i\bar{\xi}_{\dot{a}}\bar{Q}^{\dot{a}})\Phi(x,\theta,\bar{\theta}) \tag{3.28}$$

while a Taylor expansion in terms of a, $\xi$and $\bar{\xi}$ used in the transformations on the superspace coordinates gives

$$\begin{aligned}\Phi' &= \Phi(x^\mu + a^\mu + \frac{i}{2}\xi^a[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{\dot{b}} - \frac{i}{2}\theta^a[\sigma^\mu]_{a\dot{b}}(i\sigma^2)^{\dot{b}\dot{d}}\bar{\xi}_{\dot{d}}, \theta^a + \xi^a, \bar{\theta}_{\dot{a}} + \bar{\xi}_{\dot{a}}) \\ &= \Phi + \delta^\mu_\nu a^\nu[\partial_\mu\Phi] + \frac{i}{2}\xi^c[\sigma^\mu]_{c\dot{b}}\bar{\theta}^{\dot{b}}[\partial_\mu\Phi] \\ &\quad + \delta^a_c\xi^c[\partial_a\Phi] + \frac{i}{2}\theta^a[\sigma^\mu]_{a\dot{b}}(i\sigma^2)^{\dot{b}\dot{c}}\bar{\xi}_{\dot{c}}[\partial_\mu\Phi] + \delta^{\dot{c}}_{\dot{a}}\bar{\xi}_{\dot{c}}[\bar{\partial}^{\dot{a}}\Phi]\end{aligned}$$

The fourth term in the Taylor expansion was calculated by

$$\frac{\partial}{\partial\xi^c}(\theta^a + \xi^a)\,\xi^c\frac{\partial}{\partial(\theta^a + \xi^a)}\Phi(\ldots, \theta^a + \xi^a, \ldots)\Big|_{a^\mu, \xi, \bar{\xi}=0}$$

where we needed to be careful to put the expansion variable ξ^c at the correct place (to avoid sign changes, due to the Grassmann nature of ξ^c). The sixth term in the Taylor expansion is calculated in a similar way. We will continue by reorganizing the terms to a more suggestive form:

$$\begin{aligned}\Phi' &= \left\{1 + a^\mu\partial_\mu + \frac{i}{2}\xi^a[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{\dot{b}}\partial_\mu + \xi^a\partial_a + \frac{i}{2}\theta^a[\sigma^\mu]_{a\dot{b}}(i\sigma^2)^{\dot{b}\dot{c}}\bar{\xi}_{\dot{c}}\partial_\mu + \bar{\xi}_{\dot{c}}\bar{\partial}^{\dot{c}}\right\}\Phi \\ &= \left\{1 + a^\mu\partial_\mu + \xi^a\left(\partial_a + \frac{i}{2}\bar{\theta}^{\dot{b}}[\sigma^\mu]_{a\dot{b}}\partial_\mu\right) + \bar{\xi}_{\dot{c}}\left(\bar{\partial}^{\dot{c}} - \frac{i}{2}(i\sigma^2)^{ad}\theta_d[\sigma^\mu]_{a\dot{b}}(i\sigma^2)^{\dot{b}\dot{c}}\partial_\mu\right)\right\}\Phi \\ &= \left\{1 + a^\mu\partial_\mu + \xi^a\left(\partial_a + \frac{i}{2}\bar{\theta}^{\dot{b}}[\sigma^\mu]_{a\dot{b}}\partial_\mu\right) + \bar{\xi}_{\dot{c}}\left(\bar{\partial}^{\dot{c}} + \frac{i}{2}(i\sigma^2)^{da}\theta_d[\sigma^\mu]_{a\dot{b}}(i\sigma^2)^{\dot{b}\dot{c}}\partial_\mu\right)\right\}\Phi \\ &= \left\{1 + a^\mu\partial_\mu + \xi^a\left(\partial_a + \frac{i}{2}\bar{\theta}^{\dot{b}}[\sigma^\mu]_{a\dot{b}}\partial_\mu\right) + \bar{\xi}_{\dot{c}}\left(\bar{\partial}^{\dot{c}} - \frac{i}{2}\theta_d(\sigma^2)^{da}[\sigma^\mu]_{a\dot{b}}(\sigma^2)^{\dot{b}\dot{c}}\partial_\mu\right)\right\}\Phi \\ &= \left\{1 + a^\mu\partial_\mu + \xi^a\left(\partial_a + \frac{i}{2}[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{\dot{b}}\partial_\mu\right) + \bar{\xi}_{\dot{c}}\left(\bar{\partial}^{\dot{c}} + \frac{i}{2}\theta_d[\bar{\sigma}^\mu]^{d\dot{c}}\partial_\mu\right)\right\}\Phi \qquad (3.29)\end{aligned}$$

where we have used the notations $\partial^a \equiv \frac{\partial}{\partial\theta_a}$, $\bar{\partial}_{\dot{a}} \equiv \frac{\partial}{\partial\bar{\theta}^{\dot{a}}}$, $\Phi \equiv \Phi(x, \theta, \bar{\theta})$ and $\Phi' \equiv \Phi(x', \theta', \bar{\theta}')$. In the last step we also applied equation (3.8). Comparing the result in (3.29) with equation (3.28), we can read off the differential representation of the SUSY generators

$$P_\mu = i\partial_\mu \qquad (3.30)$$

$$Q_a = i\partial_a - \frac{1}{2}[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{\dot{b}}\partial_\mu \qquad (3.31)$$

$$\bar{Q}^{\dot{a}} = i\bar{\partial}^{\dot{a}} - \frac{1}{2}[\bar{\sigma}^\mu]^{b\dot{a}}\theta_b\partial_\mu \qquad (3.32)$$

Next we need to impose a constraint on the superfield Φ. This is because the most general, Lorentz invariant, expansion of Φ

$$\begin{aligned}\Phi(x, \theta, \bar{\theta}) &= A(x) + \theta\cdot\alpha(x) + \bar{\theta}\cdot\bar{\beta}(x) + (\theta\cdot\theta)B(x) + (\bar{\theta}\cdot\bar{\theta})H(x) \\ &\quad + \theta\sigma^\mu\bar{\theta}V_\mu(x) + (\theta\cdot\theta)\bar{\theta}\cdot\bar{\gamma}(x) + (\bar{\theta}\cdot\bar{\theta})\theta\cdot\eta(x) + (\theta\cdot\theta)(\bar{\theta}\cdot\bar{\theta})P(x)\end{aligned} \qquad (3.33)$$

where A, B, H, P are scalar fields, α, β, γ, η are left chiral fermion fields and V_μ is a vector field, contains too many fields to match the Wess-Zumino Lagrangian (see [12]), which is based on symmetry arguments alone, and not superfields. We only wish to keep terms containing A, α and B. This can be done by making Φ holomorphic in $\bar{\theta}$, i.e. $\bar{\partial}_{\dot{a}}\Phi = 0$. Removing the unwanted terms, would not pose this as a covariant condition under SUSY transformations, since a SUSY transformation would reinstate the unwanted terms. This is also seen from equations (3.31) and (3.32). We will need to find a covariant derivative $\bar{D}_{\dot{a}}$ such that

$$\bar{D}_{\dot{a}}\Phi' = \bar{D}_{\dot{a}}(1 - ia^\mu P_\mu - i\xi^a Q_a - i\bar{\xi}_{\dot{a}}\bar{Q}^{\dot{a}})\Phi = 0$$

We have by definition that $\bar{D}_{\dot{a}}\Phi = 0$. Therefore, we can find $\bar{D}_{\dot{a}}$ by requiring that it commutes with 1, $a^\mu P_\mu$, $\xi^a Q_a$ and $\bar{\xi}_{\dot{a}}\bar{Q}^{\dot{a}}$. It will commute trivially with 1. We require that an ansatz will not contain any x dependencies such that a commutation with $a^\mu P_\mu$ is also trivial. We require $\bar{D}_{\dot{a}}$ to contain the derivative $\bar{\partial}_{\dot{a}}$, and therefore $\bar{D}_{\dot{a}}$ is a Grassmann derivative. Based on this we may try the ansatz

$$\bar{D}_{\dot{a}} = \bar{\partial}_{\dot{a}} + C_{\dot{a}b}\theta^b$$

The remaining conditions are

$$[\bar{D}_{\dot{a}}, i\xi^b Q_b] = i\bar{D}_{\dot{a}}\xi^b Q_b - i\xi^b Q_b \bar{D}_{\dot{a}} = -i\xi^b\{\bar{D}_{\dot{a}}, Q_b\} = 0 \tag{3.34}$$

$$[\bar{D}_{\dot{a}}, i\bar{\xi}_{\dot{b}}\bar{Q}^{\dot{b}}] = i\bar{D}_{\dot{a}}\bar{\xi}_{\dot{b}}\bar{Q}^{\dot{b}} - i\bar{\xi}_{\dot{b}}\bar{Q}^{\dot{b}}\bar{D}_{\dot{a}} = i\bar{\xi}_{\dot{b}}\{\bar{D}_{\dot{a}}, \bar{Q}^{\dot{b}}\} = 0 \tag{3.35}$$

Note that

$$\begin{aligned}
\{\bar{\partial}_{\dot{a}}, \bar{Q}^{\dot{b}}\} &= i\{\bar{\partial}_{\dot{a}}, \bar{\partial}^{\dot{b}}\} - \frac{1}{2}[\bar{\sigma}^\mu]^{c\dot{b}}\{\bar{\partial}_{\dot{a}}, \theta_c\partial_\mu\} \\
&= i(\bar{\partial}_{\dot{a}}\bar{\partial}^{\dot{b}} + \bar{\partial}^{\dot{b}}\bar{\partial}_{\dot{a}}) - \frac{1}{2}[\bar{\sigma}^\mu]^{c\dot{b}}(\bar{\partial}_{\dot{a}}\theta_c\partial_\mu + \theta_c\partial_\mu\bar{\partial}_{\dot{a}}) \\
&= i(\bar{\partial}_{\dot{a}}\bar{\partial}^{\dot{b}} - \bar{\partial}_{\dot{a}}\bar{\partial}^{\dot{b}}) - \frac{1}{2}[\bar{\sigma}^\mu]^{c\dot{b}}(-\theta_c\partial_\mu\bar{\partial}_{\dot{a}} + \theta_c\partial_\mu\bar{\partial}_{\dot{a}}) = 0
\end{aligned}$$

We may now insert the ansatz into equation (3.35)

$$\begin{aligned}
\{\bar{D}_{\dot{a}}, \bar{Q}^{\dot{b}}\} = \{\bar{\partial}_{\dot{a}} + C_{\dot{a}c}\theta^c, \bar{Q}^{\dot{b}}\} &= \{\bar{\partial}_{\dot{a}}, \bar{Q}^{\dot{b}}\} + C_{\dot{a}c}\{\theta^c, \bar{Q}^{\dot{b}}\} \\
&= C_{\dot{a}c}\{\theta^c, i\bar{\partial}^{\dot{b}} - \frac{1}{2}[\bar{\sigma}^\mu]^{d\dot{b}}\theta_d\partial_\mu\} \\
&= C_{\dot{a}c}i(\theta^c\bar{\partial}^{\dot{b}} + \bar{\partial}^{\dot{b}}\theta^c) - \frac{1}{2}[\bar{\sigma}^\mu]^{d\dot{b}}(\theta^c\theta_d\partial_\mu + \theta_d\partial_\mu\theta^c) \\
&= C_{\dot{a}c}i(\theta^c\bar{\partial}^{\dot{b}} - \theta^c\bar{\partial}^{\dot{b}}) - \frac{1}{2}[\bar{\sigma}^\mu]^{d\dot{b}}(\theta^c\theta_d\partial_\mu - \theta^c\theta_d\partial_\mu) = 0
\end{aligned}$$

Thus, equation (3.35) is satisfied for all $C_{\dot{a}b}$. We now insert the ansatz into equation (3.34), to get

$$\begin{aligned}
\{\bar{D}_{\dot{a}}, Q_b\} &= \{\bar{\partial}_{\dot{a}} + C_{\dot{a}c}\theta^c, i\partial_b - \frac{1}{2}[\sigma^\mu]_{b\dot{d}}\bar{\theta}^{\dot{d}}\partial_\mu\} \\
&= i\{\bar{\partial}_{\dot{a}}, \partial_b\} - \frac{1}{2}[\sigma^\mu]_{b\dot{d}}\{\bar{\partial}_{\dot{a}}, \bar{\theta}^{\dot{d}}\partial_\mu\} + iC_{\dot{a}c}\{\theta^c, \partial_b\} - \frac{1}{2}C_{\dot{a}c}[\sigma^\mu]_{b\dot{d}}\{\theta^c, \bar{\theta}^{\dot{d}}\partial_\mu\} \\
&= -\frac{1}{2}[\sigma^\mu]_{b\dot{d}}\{\bar{\partial}_{\dot{a}}, \bar{\theta}^{\dot{d}}\partial_\mu\} + iC_{\dot{a}c}\{\theta^c, \partial_b\} \\
&= -\frac{1}{2}[\sigma^\mu]_{b\dot{d}}(\bar{\partial}_{\dot{a}}\bar{\theta}^{\dot{d}}\partial_\mu + \bar{\theta}^{\dot{d}}\partial_\mu\bar{\partial}_{\dot{a}}) + iC_{\dot{a}c}(\theta^c\partial_b + \partial_b\theta^c) \\
&= -\frac{1}{2}[\sigma^\mu]_{b\dot{d}}([\bar{\partial}_{\dot{a}}\bar{\theta}^{\dot{d}}]\partial_\mu - \bar{\theta}^{\dot{d}}\bar{\partial}_{\dot{a}}\partial_\mu + \bar{\theta}^{\dot{d}}\bar{\partial}_{\dot{a}}\partial_\mu) + iC_{\dot{a}c}(\theta^c\partial_b + [\partial_b\theta^c] - \theta^c\partial_b) \\
&= -\frac{1}{2}[\sigma^\mu]_{b\dot{d}}([\bar{\partial}_{\dot{a}}\bar{\theta}^{\dot{d}}]\partial_\mu) + iC_{\dot{a}c}([\partial_b\theta^c]) = -\frac{1}{2}[\sigma^\mu]_{b\dot{a}}\partial_\mu + iC_{\dot{a}b}
\end{aligned}$$

For the above expression to be zero, we need $C_{\dot{a}b}$ to be

$$C_{\dot{a}b} = -\frac{i}{2}[\sigma^\mu]_{b\dot{a}}\partial_\mu = -\frac{1}{2}[\sigma^\mu]_{b\dot{a}}P_\mu$$

Thus, the covariant derivative is

$$\bar{D}_{\dot{a}} = \bar{\partial}_{\dot{a}} - \frac{i}{2}\theta^b[\sigma^\mu]_{b\dot{a}}\partial_\mu$$

All superfields that have the property

$$\bar{D}_{\dot{a}}\Phi(x, \theta, \bar{\theta}) = 0$$

are called left-chiral superfields. All fields that have the property

$$D_a\Phi(x, \theta, \bar{\theta}) = 0$$

are called right-chiral superfields [12].

3.5 Left chiral superfields

We will next try to find the most general left chiral superfield. This field might still have a dependence on $\bar{\theta}$. To satisfy this possibility, let $y(x, \bar{\theta})$ be the most general function of both x and $\bar{\theta}$, that satisfies

$$\bar{D}_{\dot{a}}y = 0$$

then we automatically have that (note that $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$)

$$\begin{aligned}
\bar{D}_{\dot{a}}\Phi(y, \theta) &= \left(\bar{\partial}_{\dot{a}} - \frac{i}{2}\theta^b[\sigma^\mu]_{b\dot{a}}\partial_\mu\right)\Phi(y, \theta) \\
&= \frac{\partial\Phi}{\partial y^\nu}\frac{\partial y^\nu}{\partial\bar{\theta}^{\dot{a}}} - \frac{i}{2}\theta^b[\sigma^\mu]_{b\dot{a}}\frac{\partial\Phi}{\partial y^\nu}\frac{\partial y^\nu}{\partial x^\mu} \\
&= \frac{\partial\Phi}{\partial y^\nu}\left(\bar{\partial}_{\dot{a}} - \frac{i}{2}\theta^b[\sigma^\mu]_{b\dot{a}}\partial_\mu\right)y^\nu = \frac{\partial\Phi}{\partial y^\nu}\bar{D}_{\dot{a}}y^\nu = 0
\end{aligned}$$

regardless of the contents of Φ. We make an ansatz for y

$$y^\nu = x^\nu + \bar{\theta}^{\dot{a}} K^\nu_{\dot{a}}$$

where K is independent of x and $\bar{\theta}$. We then get that

$$\begin{aligned}
0 = \bar{D}_{\dot{b}} y^\nu &= \left(\bar{\partial}_{\dot{b}} - \frac{i}{2}\theta^c [\sigma^\mu]_{c\dot{b}} \partial_\mu\right)\left(x^\nu + \bar{\theta}^{\dot{a}} K^\nu_{\dot{a}}\right) \\
&= \bar{\partial}_{\dot{b}} x^\nu + \bar{\partial}_{\dot{b}} \bar{\theta}^{\dot{a}} K^\nu_{\dot{a}} - \frac{i}{2}\theta^c [\sigma^\mu]_{c\dot{b}} \partial_\mu x^\nu - \frac{i}{2}\theta^c [\sigma^\mu]_{c\dot{b}} \partial_\mu \bar{\theta}^{\dot{a}} K^\nu_{\dot{a}} \\
&= [\delta^{\dot{a}}_{\dot{b}}] K^\nu_{\dot{a}} - \frac{i}{2}\theta^c [\sigma^\mu]_{c\dot{b}} \delta^\nu_\mu \\
&= K^\nu_{\dot{b}} - \frac{i}{2}\theta^c [\sigma^\nu]_{c\dot{b}}
\end{aligned}$$

such that

$$K^\nu_{\dot{a}} = \frac{i}{2}\theta^c [\sigma^\nu]_{c\dot{a}}$$

Inserting this into the original ansatz we get that

$$y^\nu = x^\nu - \frac{i}{2}\theta^c [\sigma^\nu]_{c\dot{a}} \bar{\theta}^{\dot{a}}$$

We may now make a Lorentz invariant expansion based on the new variable y, and θ, to retrieve the most general left-chiral superfield.

$$\Phi(y,\theta) = \phi(y) + \theta\cdot\chi(y) + \frac{1}{2}\theta\cdot\theta F(y)$$

We now proceed by Taylor expanding each component field in terms of $g^\mu \equiv \frac{i}{2}\theta\sigma^\mu\bar{\theta}$ around $g^\mu = 0$ (note that by the previous definition, $y^\mu = x^\mu - g^\mu$).

$$\begin{aligned}
\Phi(y^\mu,\theta) &= \phi(x^\mu - g^\mu) + \theta\cdot\chi(x^\mu - g^\mu) + \theta\cdot\theta F(x^\mu - g^\mu) \\
&= \phi(x^\mu) + \frac{\partial\phi}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial g^\beta} g^\beta + \frac{1}{2}\frac{\partial}{\partial g^\gamma}\left(\frac{\partial\phi}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial g^\beta}\right) g^\beta g^\gamma + \cdots \\
&+ \theta\cdot\chi(x^\mu) + \theta\cdot\frac{\partial\chi}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial g^\beta} g^\beta + \theta\cdot\frac{1}{2}\frac{\partial}{\partial g^\gamma}\left(\frac{\partial\chi}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial g^\beta}\right) g^\beta g^\gamma + \cdots \\
&+ \frac{1}{2}\theta\cdot\theta F(x^\mu) + \frac{1}{2}\theta\cdot\theta\frac{\partial F}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial g^\beta} g^\beta + \theta\cdot\theta\frac{1}{4}\frac{\partial}{\partial g^\gamma}\left(\frac{\partial F}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial g^\beta}\right) g^\beta g^\gamma + \cdots
\end{aligned}$$

Next we may calculate

$$\begin{aligned}
\frac{\partial}{\partial g^\gamma}\left(\frac{\partial\phi}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial g^\beta}\right) &= \frac{\partial^2\phi}{\partial g^\gamma\partial y^\alpha}\frac{\partial y^\alpha}{\partial g^\beta} + \frac{\partial\phi}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial g^\gamma\partial g^\beta} = \frac{\partial^2\phi}{\partial g^\gamma\partial y^\alpha}\frac{\partial y^\alpha}{\partial g^\beta} \\
&= \left(\frac{\partial}{\partial y^\alpha}\frac{\partial\phi}{\partial g^\gamma}\right)(-\delta^\alpha_\beta) = \left(\frac{\partial}{\partial y^\alpha}\frac{\partial\phi}{\partial y^\rho}\frac{\partial y^\rho}{\partial g^\gamma}\right)(-\delta^\alpha_\beta) \\
&= \left(\frac{\partial}{\partial y^\alpha}\frac{\partial\phi}{\partial y^\rho}\right)(-\delta^\rho_\gamma)(-\delta^\alpha_\beta) \\
&= \partial_\beta\partial_\gamma\phi(x)
\end{aligned}$$

By using this result and only considering terms that do not vanish due to the Grassmann nature of the variables we get the left chiral superfield

$$\begin{aligned}\Phi(y^\mu,\theta) &= \phi(x^\mu) - g^\beta\partial_\beta\phi(x^\mu) + \frac{1}{2}g^\beta g^\gamma\partial_\beta\partial_\gamma\phi(x^\mu)\\ &+ \theta\cdot\chi(x^\mu) - g^\beta\theta\cdot\partial_\beta\chi(x^\mu) + \frac{1}{2}\theta\cdot\theta F(x^\mu)\\ &= \phi(x^\mu) - \frac{i}{2}\theta\sigma^\beta\bar\theta\partial_\beta\phi(x^\mu) - \frac{1}{8}\theta\sigma^\beta\bar\theta\theta\sigma^\gamma\bar\theta\partial_\beta\partial_\gamma\phi(x^\mu)\\ &+ \theta\cdot\chi(x^\mu) - \frac{i}{2}\theta\sigma^\beta\bar\theta\theta\cdot\partial_\beta\chi(x^\mu) + \frac{1}{2}\theta\cdot\theta F(x^\mu)\end{aligned} \tag{3.36}$$

The transformation properties of the component fields ϕ, χ and F are in accordance with the transformations in equations (3.13) to (3.18) [12]. This can be seen by first realizing that Φ' is calculated by replacing all component fields by their primed (i.e. transformed) quantities ϕ', χ' and F'. On the other hand, it is possible to find Φ' in terms of the infinitesimal changes a, ξ and $\bar\xi$ by using the explicit differential representations of the SUSY charges.

Note that in equations (3.13) to (3.18), the only two that transform, by themselves, as a total derivative of the space-time coordinates are the F and $F^\dagger$ field transformations. The other transformations must be combined to transform as a total derivative (i.e. the Lagrangian that is built out of the free left-chiral fermion field χ, the free complex scalar field ϕ and the auxiliary field F is of course an invariant under SUSY up to a total derivative). If we combine several left chiral-superfields Ψ_i, we get the following result

$$\bar D_{\dot a}[\Phi_i\Phi_j] = \left[\bar\partial_{\dot a} - \frac{i}{2}\theta^b[\sigma^\mu]_{b\dot a}\partial_\mu\right][\Phi_i\Phi_j] = [\bar D_{\dot a}\Phi_i]\Phi_j + \Phi_i[\bar D_{\dot a}\Phi_j] = 0$$

We have then reached two important facts

1. Any generic left-chiral superfield Φ contains an F-term. The F-term, i.e. the coefficient of $\frac{1}{2}\theta\cdot\theta$, transforms as a total derivative under SUSY.
2. Any combination $\Phi_i\ldots\Phi_j$ of left-chiral superfields is also a superfield.

This implies that, by projecting out the F-term of any combination of left-chiral superfields one obtains a surface term under SUSY transformations. We may use this fact to construct SUSY invariant Lagrangians. In general we have, for a potential, $\mathcal{W}(\Phi_1,\ldots,\Phi_n)$, of left chiral superfields Φ_i that the SUSY invariant Lagrangian is

$$\sim \mathcal{W}(\Phi_1,\ldots,\Phi_n)\Big|_F = \int d\theta_1 d\theta_2\mathcal{W}(\Phi_i,\ldots,\Phi_k)$$

where F denotes the F-term and the indices on $d\theta_1$ and $d\theta_2$ are spinor indices. $\mathcal{W}(\Phi_i,\ldots,\Phi_k)$ is called the superpotential. The constant in this expression can be found by dimensional analysis. We will look at the constant of $\mathcal{W}=\Phi_i\Phi_j$. Notice, that since

$$\int d\theta_1 d\theta_2\theta\cdot\theta = 2$$

and $[\theta] = -1/2$, we must have that $[d\theta_1] = 1/2$ and $[d\theta_2] = 1/2$, where the brackets denote the dimension of the object that they enclose. The first part of the expansion of the superfield $\Phi_i\Phi_j$ is the complex scalar field combination $\phi_i(x)\phi_j(x)$, as shown earlier. Since $[\phi] = 1$, then $[\Phi_i\Phi_j] = 2$. Therefore, the F-term of $\Phi_i\Phi_j$ has dimension 3. Since a renormalizable Lagrangian needs dimension 4 we must multiply with constants of dimension 1. The only choices are the masses m^{ij}. m^{ij} is taken to be symmetric, since $\Phi_i\Phi_j$ is symmetric. Therefore, we also need a factor of $1/2!$ in front. The SUSY invariant term appropriate for a Lagrangian is then

$$\frac{1}{2}m^{ij}\Phi_i\Phi_j\Big|_F = \frac{1}{2}m^{ij}\int d^2\theta\Phi_i\Phi_j$$

where $d^2\theta \equiv d\theta_1 d\theta_2$. We see that

$$\int d^2\theta\Phi_i\Phi_j\Phi_k$$

has dimension 4, and thus requires dimensionless constants y^{ijk}. In addition, we may choose y^{ijk} to be symmetric under interchange of indices, which gives a pre-factor of $1/3! = 1/6$. With the combination of three left-chiral superfields, and dimension 4 of the F-term, we have reached a limit for what is allowed in a renormalizable Lagrangian.

We will see later, that the F-terms contain interactions between the component fields (no gauge fields are included yet). The free parts of the Lagrangians can not be extracted from the F-terms. This can be seen by realizing that the free parts of a Lagrangian contain hermitian conjugates of the component fields. These are not present in the left-chiral superfields. We must then make the combination $\Phi^\dagger\Phi$, in the hope of extracting the SUSY invariant free fields. This combination does not constitute a left-chiral superfield, since $\bar{D}_{\dot{a}}[\Phi^\dagger\Phi] \neq 0$.

Looking at the component fields that are available for any superfield we see that the component field with largest dimension is the one connected to the most factors of θ and least factors of ∂_μ. This is because the dimension of θ is negative ($[\theta] = -1/2$) and the dimension of ∂_μ is positive ($[\partial_\mu] = +1$). Any SUSY transformation of this component field $C(x)$ must be constructed from a left chiral variation ξ, which has dimension $[\xi] = -1/2$, and the available component fields (that all have a lower dimension).

$$\delta C \sim \xi \times \text{ other component fields}$$

A SUSY transformation will by definition, not multiply more than one field to ξ (i.e. we want to transform between bosonic and fermionic degrees of freedom and vice versa). We must therefore only pick one field to multiply for each term contained in δC.

For a left-chiral superfield, the component field with the largest dimension is F, with dimension $[F] = 2$ linked to $(1/2)\theta\cdot\theta$. Since F is a complex scalar field we need to combine ξ with a left chiral field. The only one at our disposal is χ. We then have the following dimensional relations

$$2 = [\delta F] = [\xi] + [\chi] + [u] = -\frac{1}{2} + \frac{3}{2} + [u] = 1 + [u]$$

We see that $[u] = 1$. The only available structure (except for a scalar field) that can be used is the derivative ∂_μ which has dimension 1. Therefore, the transformation of the F-term has the structure of a total derivative and will vanish under a space-time integration. This is why the largest-dimension component field F is a SUSY invariant.

We repeat this analysis for $\Phi^\dagger\Phi$ which is not a left chiral superfield, but a general superfield. In a general superfield, the component field with largest dimension is the one linked to the terms with $(\theta\cdot\theta)(\bar{\theta}\cdot\bar{\theta})$, as shown in equation (3.33). Also here, we are forced to include a derivative, to increase the dimension from the lower-dimensional field. Therefore, this component field must also transform as a total derivative. The term with $(\theta\cdot\theta)(\bar{\theta}\cdot\bar{\theta})$ is referred to as the D-term. The projection of the D-term is

$$\Phi^\dagger\Phi\Big|_D \sim \int d^2\theta d^2\bar{\theta}\Phi^\dagger\Phi$$

The D-terms contain free fields that can be used in the construction of SUSY invariant Lagrangians because of the adjoint component fields that mix with non-adjoint fields.

3.6 Gauge interactions

In order to include electromagnetic, weak and strong interactions and find the corresponding superpartners of the photon, gluon, $W^\pm$ and the Z^0, we need to implement $U(1)$, $SU(2)$ and $SU(3)$ gauge invariance into the superfield formalism. Starting with $U(1)$, we require an invariance of the superpotential $\mathcal{W}(\Phi_i,\cdots\Phi_j)$ under

$$\Phi' = e^{2iq_k\Lambda}\Phi_k$$

where Λ must be a left-chiral superfield (such that Φ' is still a left-chiral superfield). The only requirement that needs to be imposed on the transformations working on the superpotential $\mathcal{W}$ is that the charges q_k of all the transformations add to zero (Λ does not impose any differentiation on Φ). Λ is a left-chiral superfield, and has an expansion according to equation (3.36)

$$\begin{aligned}\Lambda = \phi_\Lambda(x^\mu) - \frac{i}{2}\theta\sigma^\beta\bar{\theta}\partial_\beta\phi_\Lambda(x^\mu) - \frac{1}{8}\theta\sigma^\beta\bar{\theta}\theta\sigma^\gamma\bar{\theta}\partial_\beta\partial_\gamma\phi_\Lambda(x^\mu)\\ + \theta\cdot\chi_\Lambda(x^\mu) - \frac{i}{2}\theta\sigma^\beta\bar{\theta}\theta\cdot\partial_\beta\chi_\Lambda(x^\mu) + \frac{1}{2}\theta\cdot\theta F_\Lambda(x^\mu)\end{aligned} \tag{3.37}$$

In contrast to the left superfields Φ which have dimension 1, Λ is dimensionless, and we get that

$$[\phi_\Lambda] = 0$$

$$[\chi_\Lambda] = -[\theta] = \frac{1}{2}$$

$$[F_\Lambda] = -2[\theta] = 1$$

The above requirement on the sum of all the charges q_k is not sufficient to make sure that the transformation

$$\Phi'^\dagger\Phi' = \Phi^\dagger e^{-2iq\Lambda^\dagger}e^{2iq\Lambda}\Phi$$

is gauge invariant. This is because $\Lambda \neq \Lambda^\dagger$. To make this gauge invariant, we need to introduce a gauge field $\mathcal{V}$, such that

$$\Phi'^\dagger e^{2q\mathcal{V}'}\Phi' = \Phi^\dagger e^{-2iq\Lambda^\dagger} e^{2q\mathcal{V}'} e^{i2q\Lambda}\Phi = \Phi^\dagger e^{2q\mathcal{V}}\Phi$$

by imposing an appropriate transformation on $\mathcal{V}$. We then have that

$$\mathcal{V}' = \mathcal{V} - i(\Lambda - \Lambda^\dagger) \tag{3.38}$$

Notice that the real components of Λ and $\Lambda^\dagger$ are the same. Therefore $\Lambda - \Lambda^\dagger$ is purely imaginary. By multiplying with i, this becomes real, which implies that $\mathcal{V}$ is a real quantity, i.e.

$$\mathcal{V}^\dagger = \mathcal{V}$$

From equation (3.38) we see that $\mathcal{V}$ must also be a superfield, but not a left-chiral superfield. The most general expansion of a superfield consists of nine terms and is given by equation (3.33):

$$\begin{aligned}\mathcal{V} = {} & A + \theta\cdot\alpha + \bar\theta\cdot\bar\beta + (\theta\cdot\theta)B + (\bar\theta\cdot\bar\theta)H \\ & + \theta\sigma^\mu\bar\theta V_\mu + (\theta\cdot\theta)\bar\theta\cdot\bar\gamma + (\bar\theta\cdot\bar\theta)\theta\cdot\eta + (\theta\cdot\theta)(\bar\theta\cdot\bar\theta)P\end{aligned} \tag{3.39}$$

where we have that $\mathcal{V}$ is dimensionless, as opposed to Φ which has dimension 1. In the expansion of $\mathcal{V}$, the component fields therefore have the same dimensions as in the expansion of Λ and

$$\begin{aligned}[A] &= 0 \\ [\alpha] = [\beta] &= \frac{1}{2} \\ [B] = [H] = [V_\mu] &= 1 \\ [\gamma] = [\eta] &= \frac{3}{2} \\ [P] &= 2\end{aligned}$$

We have that

$$\begin{aligned}\mathcal{V}^\dagger = {} & A^\dagger + \bar\theta\cdot\bar\alpha + \theta\cdot\beta + (\bar\theta\cdot\bar\theta)B^\dagger + (\theta\cdot\theta)H^\dagger \\ & + \theta\sigma^\mu\bar\theta V_\mu^\dagger + (\bar\theta\cdot\bar\theta)\theta\cdot\gamma + (\theta\cdot\theta)\bar\theta\cdot\bar\eta + (\theta\cdot\theta)(\bar\theta\cdot\bar\theta)P^\dagger\end{aligned}$$

from which it follows, by comparison to $\mathcal{V}$, that

$$\begin{aligned} A &= A^\dagger & \alpha &= \beta \\ B &= H^\dagger & V_\mu &= V_\mu^\dagger \\ \eta &= \gamma & P &= P^\dagger \end{aligned}$$

We may then re-state the expansion of $\mathcal{V}$ as

$$\begin{aligned}\mathcal{V} = {} & A + \theta\cdot\alpha + \bar\theta\cdot\bar\alpha + (\theta\cdot\theta)B + (\bar\theta\cdot\bar\theta)B^\dagger \\ & + \theta\sigma^\mu\bar\theta V_\mu + (\theta\cdot\theta)\bar\theta\cdot\bar\gamma + (\bar\theta\cdot\bar\theta)\theta\cdot\gamma + (\theta\cdot\theta)(\bar\theta\cdot\bar\theta)P\end{aligned}$$

in addition to noticing the hermiticity conditions on A, P and V_μ. We may now look at how the individual component fields of $\mathcal{V}$ transform (keep in mind that even though the notation is similar, these component fields are not the same as in the expansion of Φ, but they are of the same type as in the expansion of Λ, due to their dimensions). Before doing that, we will for later convenience restate $\mathcal{V}$ in a different representation [12]:

$$
\begin{aligned}
\mathcal{V} = {} & C + \frac{i}{\sqrt{2}}\theta\cdot\rho - \frac{i}{\sqrt{2}}\bar{\theta}\cdot\bar{\rho} + \frac{i}{4}\theta\cdot\theta(M+iN) - \frac{i}{4}\bar{\theta}\cdot\bar{\theta}(M^\dagger - iN^\dagger) \\
& + \frac{1}{2}\theta\sigma^\mu\bar{\theta}A_\mu + \frac{1}{2\sqrt{2}}\theta\cdot\theta\left(\bar{\theta}\cdot\bar{\lambda} + \frac{1}{2}\bar{\theta}\sigma^\mu\partial_\mu\rho\right) + \frac{1}{2\sqrt{2}}\bar{\theta}\cdot\bar{\theta}\left(\theta\cdot\lambda - \frac{1}{2}\theta\sigma^\mu\partial_\mu\bar{\rho}\right) \\
& - \frac{1}{8}\theta\cdot\theta\bar{\theta}\cdot\bar{\theta}\left(D + \frac{1}{2}\eta^{\mu\nu}\partial_\nu\partial_\mu C\right)
\end{aligned}
$$

where $\eta^{\mu\nu}$ denotes the Minkowski metric. We have divided α into its real and imaginary parts in addition to pulling out a factor of $i/4$. The field B is divided into to fields λ and ρ and the field P is divided into C and D. The pre-factors that are pulled out are adjusted such that the fields D and λ transform into themselves under a gauge transformation. Using this representation we may calculate $\mathcal{V}'$ from (3.38) and (3.37)

$$
\begin{aligned}
\mathcal{V}' = {} & C + \frac{i}{\sqrt{2}}\theta\cdot\rho - \frac{i}{\sqrt{2}}\bar{\theta}\cdot\bar{\rho} + \frac{i}{4}\theta\cdot\theta(M+iN) - \frac{i}{4}\bar{\theta}\cdot\bar{\theta}(M^\dagger - iN^\dagger) \\
& + \frac{1}{2}\theta\sigma^\mu\bar{\theta}A_\mu + \frac{1}{2\sqrt{2}}\theta\cdot\theta\left(\bar{\theta}\cdot\bar{\lambda} + \frac{1}{2}\bar{\theta}\sigma^\mu\partial_\mu\rho\right) + \frac{1}{2\sqrt{2}}\bar{\theta}\cdot\bar{\theta}\left(\theta\cdot\lambda - \frac{1}{2}\theta\sigma^\mu\partial_\mu\bar{\rho}\right) \\
& - \theta\cdot\theta\bar{\theta}\cdot\bar{\theta}\left(D + \frac{1}{2}\eta^{\mu\nu}\partial_\nu\partial_\mu C\right) \\
& - i\phi_\Lambda - \frac{1}{2}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi_\Lambda + \frac{i}{8}\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta}\partial_\mu\partial_\nu\phi_\Lambda - i\theta\cdot\chi_\Lambda - \frac{1}{2}\theta\sigma^\mu\bar{\theta}\theta\cdot\partial_\mu\chi_\Lambda - \frac{i}{2}\theta\cdot\theta F_\Lambda \\
& + i\phi_\Lambda^\dagger - \frac{1}{2}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi_\Lambda^\dagger - \frac{i}{8}\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta}\partial_\mu\partial_\nu\phi_\Lambda^\dagger + i\bar{\theta}\cdot\bar{\chi}_\Lambda - \frac{1}{2}\theta\sigma^\mu\bar{\theta}\bar{\theta}\cdot\partial_\mu\bar{\chi}_\Lambda + \frac{i}{2}\bar{\theta}\cdot\bar{\theta}F_\Lambda^\dagger
\end{aligned}
$$

We will use the identity [12]

$$(\lambda\sigma^\mu\bar{\chi})(\lambda\sigma^\nu\bar{\chi}) = \frac{1}{2}\eta^{\mu\nu}\lambda\cdot\lambda\bar{\chi}\cdot\bar{\chi}$$

on terms number 12 and 18, and the identity [12]

$$\theta\sigma^\mu\bar{\lambda}\theta\cdot\partial_\mu\chi = -\frac{1}{2}\theta\cdot\theta(\partial_\mu\chi)\sigma^\mu\bar{\lambda}$$

on term number 14. By taking the adjoint of the last identity we get

$$\lambda\sigma^\mu\bar{\theta}\partial_\mu\bar{\chi}\cdot\bar{\theta} = -\frac{1}{2}\bar{\theta}\cdot\bar{\theta}\lambda\sigma^\mu(\partial_\mu\bar{\chi})$$

which will be used on term number 20. We then get that

$$\begin{aligned}
\mathcal{V}' = & C - i(\phi_\Lambda - \phi_\Lambda^\dagger) + \frac{i}{\sqrt{2}}\theta\cdot[\rho - \sqrt{2}\chi_\Lambda] - \frac{i}{\sqrt{2}}\bar{\theta}\cdot[\bar{\rho} - \sqrt{2}\bar{\chi}_\Lambda] \\
& + \frac{i}{4}\theta\cdot\theta[M + iN - 2F_\Lambda] - \frac{i}{4}\bar{\theta}\cdot\bar{\theta}[M^\dagger - iN^\dagger - 2F_\Lambda^\dagger] + \frac{1}{2}\theta\sigma^\mu\bar{\theta}[A_\mu - \partial_\mu(\phi_\Lambda + \phi_\Lambda^\dagger)] \\
& + \frac{1}{2\sqrt{2}}\theta\cdot\theta\left[\bar{\theta}\cdot\bar{\lambda} + \frac{1}{\sqrt{2}}(\partial_\mu\chi_\Lambda)\sigma^\mu\bar{\theta} + \frac{1}{2}\bar{\theta}\sigma^\mu\partial_\mu\rho\right] \\
& + \frac{1}{2\sqrt{2}}\bar{\theta}\cdot\bar{\theta}\left[\theta\cdot\lambda + \frac{1}{\sqrt{2}}\theta\sigma^\mu(\partial_\mu\bar{\chi}_\Lambda) - \frac{1}{2}\theta\sigma^\mu\partial_\mu\bar{\rho}\right] \\
& - \frac{1}{8}\theta\cdot\theta\bar{\theta}\cdot\bar{\theta}\left[D + \frac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu\{C - i(\phi_\Lambda - \phi_\Lambda^\dagger)\}\right]
\end{aligned}$$

By comparing this result with $\mathcal{V}$ we see that the following field transformation have taken place

$$\begin{aligned}
C &\to C - i(\phi_\Lambda - \phi_\Lambda^\dagger) \\
\rho &\to \rho - \sqrt{2}\chi_\Lambda \\
M + iN &\to M + iN - 2F_\Lambda \\
A_\mu &\to A_\mu - \partial_\mu(\phi_\Lambda + \phi_\Lambda^\dagger) \\
D &\to D
\end{aligned}$$

The transformation of λ can be seen by first explicitly using the transformation property of ρ

$$\theta\cdot\lambda' - \frac{1}{2}\theta\sigma^\mu\partial_\mu\bar{\rho}' = \theta\cdot\lambda' - \frac{1}{2}\theta\sigma^\mu\partial_\mu\bar{\rho} + \frac{1}{\sqrt{2}}\theta\sigma^\mu\partial_\mu\bar{\chi}_\Lambda$$

and then noticing that the calculation of $\mathcal{V}'$ gives the transformation

$$\theta\cdot\lambda - \frac{1}{2}\theta\sigma^\mu\partial_\mu\bar{\rho} \to \theta\cdot\lambda + \frac{1}{\sqrt{2}}\theta\sigma^\mu(\partial_\mu\bar{\chi}_\Lambda) - \frac{1}{2}\theta\sigma^\mu\partial_\mu\bar{\rho}$$

which implies that

$$\lambda \to \lambda$$

By using appropriate gauge conditions on ϕ_Λ, χ_Λ and F_Λ we are entitled to set ρ, M, N and C to zero [12]. This is the Wess-Zumino gauge, and leads to

$$\mathcal{V} = \frac{1}{2}\theta\sigma^\mu\bar{\theta}A_\mu + \frac{1}{2\sqrt{2}}\theta\cdot\theta\bar{\theta}\cdot\bar{\lambda} + \frac{1}{2\sqrt{2}}\bar{\theta}\cdot\bar{\theta}\theta\cdot\lambda - \frac{1}{8}\theta\cdot\theta\bar{\theta}\cdot\bar{\theta}D \tag{3.40}$$

Since $\Phi^\dagger e^{2q\mathcal{V}}\Phi$ is a superfield, it is still the component field with the highest dimension that will transform as a total derivative in space-time. Therefore, we can use the D-term of this expression as a SUSY term in a Lagrangian density

$$\Phi^\dagger e^{2q\mathcal{V}}\Phi\Big|_D = \int d^2\theta d^2\bar{\theta}\Phi^\dagger e^{2q\mathcal{V}}\Phi$$

where we must make sure that we have the correct pre-factor such that the Lagrangian density has dimension 4. This term will now contain the gauge interaction terms in addition to the free fields. The potential $\mathcal{W}$, consisting only of combinations of left-chiral superfields will not be altered by the gauge transformation, and thus the F-term will still contain the particle-particle interaction terms as before. The extension to non-abelian gauge superfield theory entails $\mathcal{V}_i$ and Λ_i, where i runs over all the generators of the transformation.

3.7 MSSM interactions and R-parity

We now want to retrieve possible SUSY interactions using the superfield approach. We first define the superfields corresponding to the Standard Model particles and thus defining their superpartners. We will only specify the first generation of particles. Extending to the second and third families will be trivial.

In order to keep the contents of the MSSM to be left-chiral states, the right-chiral states of the fermions are replaced by the left-chiral antiparticle states. This is possible since the operation of charge conjugation of a Weyl spinor is $i\sigma^2\psi^{\dagger T}$. But this is also how we find the left-chiral component of a left-chiral spinor ψ [1]. We will therefore need to have a superfield that contains the left-chiral anti-particle states of the positron (which is the same as the right-chiral state of the electron). The positron superfield is denoted by $\mathcal{E}_1$, where 1 denotes the first generation of particles, and must contain a spin-$1/2$ left-chiral particle. We note that in any combination of left-chiral superfields $\mathcal{W}(\Phi_i,\ldots,\Phi_j)$ the terms in the superfield expansion (3.36) containing

$$\theta\sigma^\mu\bar{\theta}\partial_\mu$$

$$\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta}\partial_\mu\partial_\nu$$

and

$$\theta\sigma^\mu\bar{\theta}\theta\cdot\partial_\mu$$

can never combine with any other term to create an F-term. We therefore only need to keep the other terms of the expansion of the superfield. The exact expansion of the superfield $\mathcal{E}_1$ is then

$$\mathcal{E}_1 = \tilde{\phi}_{\bar{e}} + \theta\cdot\chi_{\bar{e}} + \frac{1}{2}\theta\cdot\theta F_{\bar{e}} \tag{3.41}$$

where the tilde denotes the superpartner of the particle and $\bar{e}$ denotes the positron (e is the electron). Thus, the positron field is represented by $\chi_{\bar{e}}$ while the spositron field is represented by $\tilde{\phi}_{\bar{e}}$. This constitutes the doublet left-chiral supermultiplet containing the right-chiral electron state and its superpartner.

We will also need to have a superfield containing the left-chiral doublets of the electron and the neutrino. We denote this by $\mathcal{L}_1$ (which stands for leptons). This superfield must contain two left-chiral spin-$1/2$ particles that represent the electron and the neutrino. The corresponding SUSY partners in the supermultiplet must then be a spin-0 selectron and a spin-0 sneutrino. We then have

$$\mathcal{L}_1 = \begin{pmatrix} \tilde{\phi}_{\nu_e} \\ \tilde{\phi}_e \end{pmatrix} + \theta \cdot \begin{pmatrix} \chi_{\nu_e} \\ \chi_e \end{pmatrix} + \frac{1}{2}\theta \cdot \theta \begin{pmatrix} F_{\nu_e} \\ F_e \end{pmatrix} \tag{3.42}$$

The gauge fields corresponding to the weak interactions are the $U(1)_Y$ gauge field of the hypercharge and the $SU(2)_L$ gauge fields of the $W^\pm$ particles. The field B_μ, related to the hypercharge, is a spin-1 vector gauge field and should be related to the dimension 1 field A_μ of $\mathcal{V}$ in section 3.6. Thus, the $U(1)_Y$ superfield is represented by equation (3.40), and we will denote it by $\mathcal{B}$

$$\mathcal{B} = \frac{1}{2}\theta\sigma^\mu\bar{\theta}B_\mu + \frac{1}{2\sqrt{2}}\theta\cdot\theta\bar{\theta}\cdot\bar{\lambda}_Y + \frac{1}{2\sqrt{2}}\bar{\theta}\cdot\bar{\theta}\theta\cdot\lambda_Y - \frac{1}{8}\theta\cdot\theta\bar{\theta}\cdot\bar{\theta}D_Y$$

where λ_Y is the spin-3/2 superpartner of B_μ. The gauge superfield linked to $\mathcal{E}_1$ is represented in a similar way, and we denote it with $\mathcal{A}$. It is

$$\mathcal{A} = \frac{1}{2}\theta\sigma^\mu\bar{\theta}A_\mu + \frac{1}{2\sqrt{2}}\theta\cdot\theta\bar{\theta}\cdot\bar{\lambda}_\gamma + \frac{1}{2\sqrt{2}}\bar{\theta}\cdot\bar{\theta}\theta\cdot\lambda_\gamma - \frac{1}{8}\theta\cdot\theta\bar{\theta}\cdot\bar{\theta}D_\gamma$$

The superfield representing the $SU(2)_L$ gauge fields is denoted $\mathcal{W}^i$ where i refers to the particular generator that the superfield is connected to:

$$\mathcal{W}^i = \frac{1}{2}\theta\sigma^\mu\bar{\theta}W^i_\mu + \frac{1}{2\sqrt{2}}\theta\cdot\theta\bar{\theta}\cdot\bar{\lambda}^i_L + \frac{1}{2\sqrt{2}}\bar{\theta}\cdot\bar{\theta}\theta\cdot\lambda^i_L - \frac{1}{8}\theta\cdot\theta\bar{\theta}\cdot\bar{\theta}D^i_L \tag{3.43}$$

We have covered the $U(1)_Y$ and the $SU(2)_L$ sector. We now need to define the superfields that represent the $SU(3)_c$ sector. First we want to have a superfield that contains the spin-1/2 right-chiral quarks of the Standard Model. Again, we will rather use the left-chiral antiquarks. We denote the anti-up-quark superfield by $\mathcal{U}_1$, where again 1 refers to the generation of quarks. The anti-down-quark superfield is denoted $\mathcal{D}_1$. Similarly, the anti-strange-quark superfield would be denoted by $\mathcal{U}_2$ and the anti-charm-quark superfield by $\mathcal{D}_2$. We then have, in analogy with (3.41)

$$\mathcal{U}_1 = \tilde{\phi}_{\bar{u}} + \theta\cdot\chi_{\bar{u}} + \frac{1}{2}\theta\cdot\theta F_{\bar{u}}$$

$$\mathcal{D}_1 = \tilde{\phi}_{\bar{d}} + \theta\cdot\chi_{\bar{d}} + \frac{1}{2}\theta\cdot\theta F_{\bar{d}}$$

where $\chi_{\bar{u}}$ and $\chi_{\bar{d}}$ are the spin-1/2 anti-up and down-quark fields. Here, $\tilde{\phi}_{\bar{u}}$ and $\tilde{\phi}_{\bar{d}}$ are the corresponding spin-0 superpartners. The $SU(2)_L$ left-chiral quarks are represented by $\mathcal{Q}_1$ where 1 refers to the generation of quarks. We then have, in analogy with (3.42)

$$\mathcal{Q}_1 = \begin{pmatrix} \tilde{\phi}_u \\ \tilde{\phi}_d \end{pmatrix} + \theta \cdot \begin{pmatrix} \chi_u \\ \chi_d \end{pmatrix} + \frac{1}{2}\theta \cdot \theta \begin{pmatrix} F_u \\ F_d \end{pmatrix}$$

The gluon gauge field is a spin-1 gauge field. We denote it by $\mathcal{G}^a$ where a takes eight values, depending on which of the generators the superfield couples to. We have, in analogy with (3.43)

$$\mathcal{G}^a = \frac{1}{2}\theta\sigma^\mu\bar{\theta}G^a_{c\mu} + \frac{1}{2\sqrt{2}}\theta\cdot\theta\bar{\theta}\cdot\bar{\lambda}^a_c + \frac{1}{2\sqrt{2}}\bar{\theta}\cdot\bar{\theta}\theta\cdot\lambda^a_c - \frac{1}{8}\theta\cdot\theta\bar{\theta}\cdot\bar{\theta}D^a_c$$

where c denotes the color.

Finally we need to add a Higgs superfield. The Higgs field in the standard model is a spin-0 scalar field. We will therefore have a spin-$1/2$ left-chiral spinor field representing the superpartner of the Higgs particle. We denote the Higgs superfield generating the mass of the up quark by $\mathcal{H}_u$ and the Higgs superfield generating the down quark mass, by $\mathcal{H}_d$ [12]. We have

$$\mathcal{H}_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix} + \theta \cdot \begin{pmatrix} \tilde{\chi}_u^+ \\ \tilde{\chi}_u^0 \end{pmatrix} + \frac{1}{2}\theta \cdot \theta \begin{pmatrix} F_u^+ \\ F_u^0 \end{pmatrix}$$

and

$$\mathcal{H}_d = \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix} + \theta \cdot \begin{pmatrix} \tilde{\chi}_d^0 \\ \tilde{\chi}_d^- \end{pmatrix} + \frac{1}{2}\theta \cdot \theta \begin{pmatrix} F_d^0 \\ F_d^- \end{pmatrix}$$

where $+$, $-$ and 0 refer to the charges of the particles. H are the Higgs fields and $\tilde{\chi}$ are the corresponding Higgsinos.

We are now going to find which allowed combinations the above superfields $\mathcal{E}_i$, $\mathcal{L}_i$, $\mathcal{U}_i$, $\mathcal{D}_i$, $\mathcal{Q}_i$, $\mathcal{H}_u$ and $\mathcal{H}_d$ can make, in such a way that the charges add up to zero and that the combination $\Phi^\dagger e^{2i\mathcal{V}}\Phi$ is gauge invariant. We have that

$$\Phi \in \{\mathcal{E}_i, \mathcal{L}_i, \mathcal{U}_i, \mathcal{D}_i, \mathcal{Q}_i, \mathcal{H}_u, \mathcal{H}_d\}$$

and

$$\mathcal{V} \in \{\mathcal{B}, \mathcal{W}^i, \mathcal{G}^a\}$$

To make the correct combinations we need to know the $U(1)_Y$ hypercharge, $SU(2)_L$ and $SU(3)_c$ quantum numbers of the fields we want to combine. These are shown in table 3.1 [12]. Note that **1** assigned to $SU(2)_L$ means that the field with this quantum number does not transform at all under $SU(2)_L$. We saw an example of this in section 2.2, where the right-chiral leptons do not transform under the $SU(2)_L$ symmetry, and thus must have the quantum number **1**. The left-chiral doublet of the electron and the neutrino will however transform under the $SU(2)_L$ symmetry and will therefore have the quantum number **2** assigned to it. A similar argumentation holds for $SU(3)_c$ symmetries. For $SU(3)_c$ both the left and right-chiral components transform. The different ways in which the left and the right components transform, is denoted with a bar over the quantum number. Two superfields, one with quantum number **3** and another with quantum number $\bar{\mathbf{3}}$, will combined have the quantum number **1**. Such a combination of superfields will therefore be gauge invariant.

Note that it is possible to combine two $SU(2)_L$ doublets, with quantum number **2**, in a gauge invariant way without having access to fields of quantum numbers $\bar{\mathbf{2}}$. Let Φ_A and Φ_B be two left-chiral $SU(2)_L$ superfields. These then both consist of two left-chiral superfields each. By making the combination

$$\Phi_A^\dagger (i\sigma^2)\Phi_B \equiv \Phi_A \circ \Phi_B$$

we create a SUSY gauge invariant out of superfields with $SU(2)_L$ quantum numbers **2** [1, 12]. Note also that we must make sure that the dimension of the combined superfields will not exceed 3, such that we still have a renormalizable theory when picking

Superfield	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
$\mathcal{E}_i$	**1**	**1**	1
$\mathcal{L}_i$	**1**	**2**	-1/2
$\mathcal{U}_i$	$\bar{\mathbf{3}}$	**1**	-2/3
$\mathcal{D}_i$	$\bar{\mathbf{3}}$	**1**	1/3
$\mathcal{Q}_i$	**3**	**2**	1/6
$\mathcal{H}_u$	**1**	**2**	1/2
$\mathcal{H}_d$	**1**	**2**	-1/2

Table 3.1: Quantum numbers connected to the superfields (Note that we will follow the convention in section 2.2, which uses half of the hypercharge listed in [12]).

out the F-term (which adds 1 to the dimension).

We start by trying to find the possible combinations of $\mathcal{E}_i$ with other superfields. We start looking at the hypercharge. We will need to combine two superfields with hypercharge $-1/2$ to cancel the hypercharge from $\mathcal{E}_i$ ($\mathcal{E}_i$ has hypercharge 1). We start with $\mathcal{L}_i$, which is an $SU(2)_L$ superfield, that necessarily must be combined with another $SU(2)_L$ superfield with quantum numbers $\{\mathbf{1},\mathbf{2},-1/2\}$. We find the possibilities

$$\mathcal{E}_i\mathcal{L}_i \circ \mathcal{L}_i \qquad \mathcal{E}_i\mathcal{L}_i \circ \mathcal{H}_d \qquad \mathcal{E}_i\mathcal{H}_d \circ \mathcal{H}_d$$

However, if we look at $\mathcal{E}_i\mathcal{H}_d \circ \mathcal{H}_d$, it has the form

$$A^T i\sigma^2 A = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

such that it will vanish identically.

Next we will look at gauge invariant superfield combinations containing $\mathcal{L}_i$. $\mathcal{L}_i$ has the quantum numbers $\{\mathbf{1},\mathbf{2},-1/2\}$ and can only be combined with an $SU(2)_L$ superfield with the associated $SU(2)_L$ quantum number **2**. In addition the hypercharge and $SU(3)_c$ quantum numbers must add to 0 and **1** respectively. The only possible options are (not counting $\mathcal{E}_i$ treated previously)

$$\mathcal{L}_i \circ \mathcal{H}_u \qquad \mathcal{L}_i \circ \mathcal{Q}_i\mathcal{D}_i$$

We move on to find expressions involving $\mathcal{U}_i$. This superfield has quantum numbers $\{\bar{\mathbf{3}},\mathbf{1},-2/3\}$, and we must therefore combine this with a superfield having the $SU(3)_c$ quantum number **3**. Only $\mathcal{Q}_i$ offers such a possibility. The combination $\mathcal{U}_i\mathcal{Q}_i$ has quantum numbers $\{\mathbf{1},\mathbf{2},-1/2\}$. There is only one possibility to cancel these quantum numbers. We thus have the gauge invariant combination

$$\mathcal{U}_i\mathcal{Q}_i \circ \mathcal{H}_u$$

There is also a way to combine three quantum numbers $\bar{\mathbf{3}}$ into a gauge invariant quantity [12]. This leads to the following invariant

$$f_c^{abc}\mathcal{U}_i^a\mathcal{D}_i^b \circ \mathcal{D}_i^c$$

where f_c^{abc} are the structure constants of QCD.

The superfield $\mathcal{D}_i$ has quantum numbers $\{\bar{\mathbf{3}}, \mathbf{1}, 1/3\}$. Of the remaining fields not covered previously, this must be combined with a field containing the $SU(3)_c$ quantum number $\mathbf{3}$. The only possibility is $\mathcal{Q}_i$. The quantum number of this combination is $\{\mathbf{1}, \mathbf{2}, 1/2\}$. We therefore have only one option to cancel the quantum numbers

$$\mathcal{D}_i \mathcal{Q}_i \circ \mathcal{H}_d$$

$\mathcal{Q}_i$ has quantum numbers $\{\mathbf{3}, \mathbf{2}, 1/6\}$ and must, from the superfields not already previously covered, be combined with a superfield containing the $SU(3)_c$ quantum number $\bar{\mathbf{3}}$. There are no such options.

$\mathcal{H}_u$ has quantum numbers $\{\mathbf{1}, \mathbf{2}, 1/2\}$. The only field left that has not been covered already is $\mathcal{H}_d$, which gives the gauge-invariant combination

$$\mathcal{H}_u \circ \mathcal{H}_d$$

We may project out the F-term of any of the above combinations to retain a gauge and SUSY invariant Lagrangian density which is also Lorentz invariant and renormalizable.

We also want to determine the possible gauge interactions we can have. We then have to make combinations of the form $\Phi^\dagger e^{2i\mathcal{V}} \Phi$ and project out the D-term. We first consider $\mathcal{E}_i$. The quantum numbers in table 3.1 show that $\mathcal{E}_i$ only has a non-trivial $U(1)_Y$ quantum number. Thus, the gauge invariance is only linked to the field $\mathcal{A}$ and the possible gauge interactions are retrieved from the D-term of

$$\mathcal{E}_i^\dagger e^{2g\mathcal{A}} \mathcal{E}_i$$

$\mathcal{L}_i$ has both a $U(1)_Y$ quantum number and an $SU(2)_L$ quantum number. Therefore, both $\mathcal{B}$ and $\mathcal{W}^|$ must be associated with the gauge invariance. We may pick out the possible gauge interactions from the D-term of

$$\mathcal{L}_i^\dagger e^{-2\cdot\frac{1}{2}g'\mathcal{B}+g\mathcal{W}^j\sigma_j} \mathcal{L}_i$$

Similarly we get for the other superfields

$$\mathcal{U}_i^\dagger e^{-2\cdot\frac{2}{3}g\mathcal{B}-g_s\mathcal{G}^a\lambda_a} \mathcal{E}_i$$

$$\mathcal{D}_i^\dagger e^{2\cdot\frac{1}{3}g\mathcal{B}-g_s\mathcal{G}^a\lambda_a} \mathcal{E}_i$$

$$\mathcal{Q}_i^\dagger e^{2\cdot\frac{1}{6}g\mathcal{B}+g_s\mathcal{G}^a\lambda_a+g\mathcal{W}^j\sigma_j} \mathcal{E}_i$$

$$\mathcal{H}_u^\dagger e^{2\cdot\frac{1}{2}g'\mathcal{B}+g\mathcal{W}^j\sigma_j} \mathcal{L}_i$$

$$\mathcal{H}_d^\dagger e^{-2\cdot\frac{1}{2}g'\mathcal{B}+g\mathcal{W}^j\sigma_j} \mathcal{L}_i$$

where g, g' and g_s are coupling constants, and λ_a are the Gell-Mann matrices.

As shown above, we have in total eight gauge invariant left-chiral superfield combinations. The F-terms of these will give possible interaction vertices between the particles. Looking at the following left-chiral combinations

$$\mathcal{L}_i \circ \mathcal{H}_u \qquad \mathcal{L}_i \circ \mathcal{Q}_i \mathcal{D}_i \qquad \mathcal{E}_i \mathcal{L}_i \circ \mathcal{L}_i$$

we see that they all consist of an uneven number of lepton and anti-lepton fields. This results in vertices which violate lepton-number conservation. In a similar manner the left-chiral superfield

$$f_c^{abc} \mathcal{U}_i^a \mathcal{D}_i^b \circ \mathcal{D}_i^c$$

will contain terms which violate baryon number conservation [12]. This can lead to non-phenemenological events such as the rapid decay of the proton. The way to remove the unwanted terms is to insist on a global symmetry of the Grassmann variables θ, of the following form [12]

$$\begin{pmatrix} \theta \\ \bar{\theta} \end{pmatrix} \rightarrow e^{i\gamma_5 \alpha} \begin{pmatrix} \theta \\ \bar{\theta} \end{pmatrix}$$

This is called R-symmetry. Setting $\alpha = \pi$ it is called R-parity. The R-parity of component fields can be calculated by

$$R = (-1)^{3B+L+2s}$$

where B is the baryon number, L the lepton number and s the spin of the component field.

Chapter 4

$N > 1$ Supersymmetry

In the following we discuss some aspects of $N > 1$ SUSY. The calculations are based on the $N > 1$ SUSY algebra found in [12] and $N > 1$ SUSY results from [1, 6].

4.1 Supermultiplets

$N > 1$ SUSY (also called Extended SUSY) is a generalization of the algebra in equations (3.11) and (3.12)

$$\{Q_a^I, Q_b^J\} = (-i\sigma_2)_{ab} Z^{IJ} \tag{4.1}$$

$$\{Q_a^I, Q_b^{J\dagger}\} = (\sigma^\mu)_{ab} \delta^{IJ} P_\mu \tag{4.2}$$

$$\{Q_a^{I\dagger}, Q_b^{J\dagger}\} = (-i\sigma_2)_{ab} (Z^{IJ})^* \tag{4.3}$$

where Z^{IJ} is antisymmetric in its indices I and J and $I, J = 1, \ldots, N$. We may try to find the supermultiplet associated with this kind of algebra by going through an identical set of calculations as in section 3.3. The SUSY charges were not used in finding the Casimir operators or the irreducible representation related to massless states. Therefore we must still categorize the states by the helicity and energy through $|p, h\rangle$. Q_a^I and $Q_a^{I\dagger}$ are not dependent on the space-time coordinates. As before we get that

$$[Q_a^I, P^\mu] = 0$$

$$[Q_a^{I\dagger}, P^\mu] = 0$$

Equations (3.19) and (3.20) can be derived by using equation (3.5) to find that

$$(\delta_\omega \delta_\beta - \delta_\beta \delta_\omega)\phi = -\frac{1}{2}[[\beta \cdot Q, \omega^{\mu\nu} M_{\mu\nu}], \phi]$$

The left-hand side is calculated by inserting the appropriate infinitesimal SUSY transformations $\delta_\xi \phi$ and $\delta_\xi \chi$. A comparison will then show the results of equations (3.19) and (3.20). Since Z^{IJ} is antisymmetric, then $Z^{KK} = 0$, and equations (4.1) and (4.3) will read

$$\{Q_a^K, Q_b^K\} = 0$$

and

$$\{Q_a^{K\dagger}, Q_b^{K\dagger}\} = 0$$

This means that each charge pair Q_a^K, Q_b^K separately will satisfy the $N = 1$ algebra. It is only for $I \neq J$ that the extended algebra differs. This implies that the infinitesimal SUSY transformations $\delta_\xi \phi$ and $\delta_\xi \chi$ are of the same form. Thus, the derivation of equations (3.19) and (3.20) are identical in the $N > 1$ case, which in turn gives a trivial extension to these equations, which for $N > 1$ SUSY become

$$[Q_a^I, M_{\mu\nu}] = (\sigma_{\mu\nu})_a{}^b Q_b^I$$

and

$$[Q_a^{I\dagger}, M_{\mu\nu}] = -Q_b^{I\dagger}(\bar{\sigma}_{\mu\nu})^b{}_a$$

It now follows that the momentum p in $|p,h\rangle$, as in the $N = 1$ case, will not be affected by any of the charges Q_a^I or $Q_a^{I\dagger}$. By applying the Pauli-Lubanski operator W^μ to $Q_a^I|p,h\rangle$ and $Q_a^{I\dagger}|p,h\rangle$ in order to calculate the helicity eigenvalue of a state, as for the $N = 1$ case, we find that

$$Q_2^{I\dagger}|p,h\rangle = Q_1^I|p,h\rangle = |p,h+\frac{1}{2}\rangle$$

and

$$Q_2^I|p,h\rangle = Q_1^{I\dagger}|p,h\rangle = |p,h-\frac{1}{2}\rangle$$

The difference between the $N = 1$ and $N > 1$ theories becomes apparent when we next try to find all the particles belonging to one supermultiplet. We will start with $N = 2$. As before, we start with the lowest helicity state, which we label $|p,-j\rangle$. We have that

$$Q_2^K|p,-j\rangle = 0$$

We may apply any of the two operators $Q_2^{I\dagger}$. We may choose to start with $I = 1$ (although $I = 2$ will provide a separate state with same helicity), such that

$$Q_2^{1\dagger}|p,-j\rangle = |p,\frac{1}{2}-j\rangle$$

Since $Q_2^{1\dagger}$ does not anti-commute with $Q_2^{2\dagger}$, then using $Q_2^{2\dagger}$ to the above result gives

$$Q_2^{2\dagger}Q_2^{1\dagger}|p,-j\rangle = Q_2^{2\dagger}|p,\frac{1}{2}-j\rangle = |p,1-j\rangle$$

Any other attempt to use $Q_2^{1\dagger}$ or $Q_2^{2\dagger}$ will give zero because of their Grassmanian nature. Therefore, for $N = 2$ SUSY we will have four states in one supermultiplet, as opposed to $N = 1$ where we have two. The supermultiplet for $N = 2$ SUSY with $j = \frac{1}{2}$ then consists of $|p,-1/2\rangle$, two $|p,0\rangle$ states and $|p,1/2\rangle$. We recognize the $h = -1/2$ state as a left-chiral spinor state and the $h = 0$ states as bosonic states, while the $h = 1/2$ state represents a right-chiral spinor. In $N = 1$ SUSY, the right chiral spinor was not part of the supermultiplet. Since all these states are in the same group they must also transform in the same manner. From section 2.2 we saw that the weak sector utilized an $U(1) \times SU(2)$ gauge transformation on the left-chiral part of the Dirac spinor, while only a $U(1)$ gauge transformation was utilized on the right-chiral part. The only viable SUSY theory for the weak interactions would be one containing a left-chiral spinor and its SUSY partner. Since $N > 1$ SUSY theories also must contain the corresponding right-chiral spinor in the supermultiplet, that must transform the same way as the left-chiral spinor, they are considered not phenomenologically viable. It is however possible to make a hybrid $N = 1$ / $N = 2$ model by only considering $N = 2$ in the QCD sector, in order to avoid the chirality problem [4].

Chapter 5

$N = 2$ Superfields

5.1 The superfield coordinates

In this section we will use the methods of section 3.4 and section 4.1, to start building a superfield formalism for $N = 2$ SUSY. This formalism can then be used to pick out the possible $N = 2$ SUSY interaction vertices in a consistent manner.

In the $N = 2$ case the algebra of equations (4.1), (4.2) and (4.3) can have the values $(I,J) \in \{(u,u),(u,v),(v,u),(v,v)\}$. The transformations are naturaly extended to

$$U(a,\xi^u,\bar{\xi}^u,\xi^v,\bar{\xi}^v)$$

where the indices refer to the specific charge, Q^u_a or Q^v_a, that the infinitesimal parameters are associated with (not to be confused with spinor indices). However, there are other operators, Z^{IJ}, associated with the algebra of $N = 2$ SUSY, as equations (4.1), (4.2) and (4.3) shows. This additional operator will need to be transformed under a SUSY transformation. The SUSY charges in $N = 1$ have the associated variable θ, that it operates on, and the infinitesimal parameter ξ representing the change under a transformation. The momentum operator P_μ has an associated variable x_μ, and an infinitesimal change a_μ. We need to assign similar variables to to the operators Z^{IJ}. Before we do this we need to make some assumptions on Z^{IJ}. We already know from the definition of the algebra that Z^{IJ} is antisymmetric in its indices I and J. We will assume that

$$[Z^{IJ},Z^{KL}] = 0 \qquad\qquad [Z^{IJ},(Z^{KL})^*] = 0 \tag{5.1}$$

We will also assume, as for P_μ, that Z^{IJ} commutes with Q^u_a, Q^v_a, $\bar{Q}^u_a$ and $\bar{Q}^v_a$. We still want P_μ to commute with all the generators, thus we will assume that it commutes with Z^{IJ} and $(Z^{IJ})^*$.

Because of equations (5.1) we see that Z^{IJ} is not of a Grassmann nature. Therefore, the associated variable and infinitesimal displacements should not be of a Grassmann nature. We assign the variables A_{IJ} and A^*_{IJ} to denote the current positions in superspace of Z^{IJ} and $(Z^{IJ})^*$ respectively. For the infinitesimal displacements in superspace we assign the variables α_{IJ} and α^*_{IJ} to Z^{IJ} and $(Z^{IJ})^*$ respectively.

Note that since Z^{IJ} is antisymmetric, then the symmetric part s_{IJ} of A_{IJ} will lead to

$$\sum_{IJ} s_{IJ} Z^{IJ} = -\sum_{IJ} s_{IJ} Z^{IJ}$$

such that we must have

$$\sum_{IJ} s_{IJ} Z^{IJ} = 0$$

Since we have to include this sum in the transformation U, we may ignore the symmetric part of A_{IJ} and rather take it to be antisymmetric. The same reasoning holds for α_{IJ}. Therefore we also take α_{IJ} to be antisymmetric in its indices. Since any antisymmetric matrix must have zero diagonal, we see that $A_{uu} = A_{vv} = 0$. We also have that

$$A_{uv} Z^{uv} + A_{vu} Z^{vu} = 2A_{uv} Z^{uv}$$

We now make the following definition for the $N = 2$ superspace coordinate:

$$C_s \equiv (x^\mu, \theta^u_a, \theta^v_a, \bar{\theta}^{u,\dot{a}}, \bar{\theta}^{v\dot{a}}, A_{uv}, A_{vu}, A^*_{uv}, A^*_{vu})$$

We also define the infinitesimal superspace displacement to be

$$c_s \equiv (a^\mu, \xi^u_a, \xi^v_a, \bar{\xi}^{u,\dot{a}}, \bar{\xi}^{v\dot{a}}, \alpha_{uv}, \alpha_{vu}, \alpha^*_{uv}, \alpha^*_{vu})$$

As was done in section 3.4, we use the relation

$$U(C'_s) = U(c_s)U(C_s) \tag{5.2}$$

to find the transformation of the superspace coordinates.

The transformation $U(c_s)$ is now defined as

$$U(c_s) = e^{ia^\mu P_\mu + i\xi^{u,a} Q^u_a + i\xi^{v,a} Q^v_a + i\bar{\xi}^u_{\dot{a}} \bar{Q}^{u,\dot{a}} + i\bar{\xi}^v_{\dot{a}} \bar{Q}^{v,\dot{a}} + 2i\alpha_{uv} Z^{uv} + 2i\alpha^*_{uv} (Z^{uv})^*}$$

and $U(C_s)$ as

$$U(C_s) = e^{ix^\mu P_\mu + i\theta^{u,a} Q^u_a + i\theta^{v,a} Q^v_a + i\bar{\theta}^u_{\dot{a}} \bar{Q}^{u,\dot{a}} + i\bar{\theta}^v_{\dot{a}} \bar{Q}^{v,\dot{a}} + 2iA_{uv} Z^{uv} + 2iA^*_{uv} (Z^{uv})^*}$$

By using the BCH (Baker-Campbell-Hausdorff) identity

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$$

for non-commutative quantities, we can calculate the resulting exponent of the right-hand side in equation (5.2):

$$\begin{aligned}
&ia^\mu P_\mu + i\xi^{u,a} Q^u_a + i\xi^{v,a} Q^v_a + i\bar{\xi}^u_{\dot{a}} \bar{Q}^{u,\dot{a}} + i\bar{\xi}^v_{\dot{a}} \bar{Q}^{v,\dot{a}} + 2i\alpha_{uv} Z^{uv} + 2i\alpha^*_{uv} (Z^{uv})^* \\
&+ix^\mu P_\mu + i\theta^{u,a} Q^u_a + i\theta^{v,a} Q^v_a + i\bar{\theta}^u_{\dot{a}} \bar{Q}^{u,\dot{a}} + i\bar{\theta}^v_{\dot{a}} \bar{Q}^{v,\dot{a}} + 2iA_{uv} Z^{uv} + 2iA^*_{uv} (Z^{uv})^* \\
&+\frac{1}{2}\big[ia^\mu P_\mu + i\xi^{u,a} Q^u_a + i\xi^{v,a} Q^v_a + i\bar{\xi}^u_{\dot{a}} \bar{Q}^{u,\dot{a}} + i\bar{\xi}^v_{\dot{a}} \bar{Q}^{v,\dot{a}} + 2i\alpha_{uv} Z^{uv} + 2i\alpha^*_{uv} (Z^{uv})^*, \\
&\quad ix^\nu P_\nu + i\theta^{u,b} Q^u_b + i\theta^{v,b} Q^v_b + i\bar{\theta}^u_{\dot{b}} \bar{Q}^{u,\dot{b}} + i\bar{\theta}^v_{\dot{b}} \bar{Q}^{v,\dot{b}} + 2iA_{uv} Z^{uv} + 2iA^*_{uv} (Z^{uv})^*\big]
\end{aligned} \tag{5.3}$$

Since P_μ, Z^{IJ} and $(Z^{IJ})^*$ commute with all generators, all commutators containing P_μ, Z^{IJ} or $(Z^{IJ})^*$ will vanish. Because of this and the nature of the Grassmann variables, the remaining commutators in the BCH identity will vanish. We now calculate the commutators in equation (5.3) that do not contain P_μ, Z^{IJ} or $(Z^{IJ})^*$. The first one is

$$\begin{aligned}[i\xi^{u,a}Q^u_a, i\theta^{u,b}Q^u_b] &= -\xi^{u,a}Q^u_a\theta^{u,b}Q^u_b + \theta^{u,b}Q^u_b\xi^{u,a}Q^u_a \\ &= \xi^{u,a}\theta^{u,b}Q^u_aQ^u_b + \xi^{u,a}\theta^{u,b}Q^u_bQ^u_a = \xi^{u,a}\theta^{u,b}\{Q^u_a, Q^u_b\} \\ &= \xi^{u,a}\theta^{u,b}(-i\sigma^2)_{ab}Z^{uu} = 0\end{aligned}$$

where we used the anti-commutation between the Grassmann variables. In the last steps we replaced the anti-commutator, using the $N = 2$ algebra. Then we could use the anti-symmetry of Z^{IJ} to get zero.

We continue evaluating the commutators resulting from equation (5.3):

$$[i\xi^{u,a}Q^u_a, i\theta^{v,b}Q^v_b] = \xi^{u,a}\theta^{v,b}\{Q^u_a, Q^v_b\} = \xi^{u,a}\theta^{v,b}(-i\sigma^2)_{ab}Z^{uv} = \xi^{u,a}\theta^v_aZ^{uv}$$

$$[i\xi^{u,a}Q^u_a, i\bar\theta^u_{\dot b}\bar Q^{u,\dot b}] = \xi^{u,a}\bar\theta^u_{\dot b}\{Q^u_a, \bar Q^{u,\dot b}\} = \xi^{u,a}\bar\theta^u_{\dot b}[\sigma^\mu]_a{}^{\dot b}P_\mu$$

$$[i\xi^{u,a}Q^u_a, i\bar\theta^v_{\dot b}\bar Q^{v,\dot b}] = \xi^{u,a}\bar\theta^v_{\dot b}\{Q^u_a, \bar Q^{v,\dot b}\} = \xi^{u,a}\bar\theta^v_{\dot b}[\sigma^\mu]_a{}^{\dot b}\delta^{uv}P_\mu = 0$$

$$[i\xi^{v,a}Q^v_a, i\theta^{u,b}Q^u_b] = \xi^{v,a}\theta^{u,b}\{Q^v_a, Q^u_b\} = \xi^{v,a}\theta^{u,b}(-i\sigma^2)_{ab}Z^{vu} = \xi^{v,a}\theta^u_aZ^{vu}$$

$$[i\xi^{v,a}Q^v_a, i\theta^{v,b}Q^v_b] = \xi^{v,a}\theta^{v,b}\{Q^v_a, Q^v_b\} = \xi^{v,a}\theta^{v,b}(-i\sigma^2)_{ab}Z^{vv} = 0$$

$$[i\xi^{v,a}Q^v_a, i\bar\theta^u_{\dot b}\bar Q^{u,\dot b}] = \xi^{v,a}\bar\theta^u_{\dot b}\{Q^v_a, \bar Q^{u,\dot b}\} = \xi^{v,a}\bar\theta^u_{\dot b}[\sigma^\mu]_a{}^{\dot b}\delta^{vu}P_\mu = 0$$

$$[i\xi^{v,a}Q^v_a, i\bar\theta^v_{\dot b}\bar Q^{v,\dot b}] = \xi^{v,a}\bar\theta^v_{\dot b}\{Q^v_a, \bar Q^{v,\dot b}\} = \xi^{v,a}\bar\theta^v_{\dot b}[\sigma^\mu]_a{}^{\dot b}P_\mu$$

$$[i\bar\xi^u_{\dot a}\bar Q^{u,\dot a}, i\theta^{u,b}Q^u_b] = \bar\xi^u_{\dot a}\theta^{u,b}\{\bar Q^{u,\dot a}, Q^u_b\} = \bar\xi^u_{\dot a}\theta^{u,b}[\sigma^\mu]_b{}^{\dot a}P_\mu$$

$$[i\bar\xi^u_{\dot a}\bar Q^{u,\dot a}, i\theta^{v,b}Q^v_b] = \bar\xi^u_{\dot a}\theta^{v,b}\{\bar Q^{u,\dot a}, Q^v_b\} = \bar\xi^u_{\dot a}\theta^{v,b}[\sigma^\mu]_b{}^{\dot a}\delta^{uv}P_\mu = 0$$

$$[i\bar\xi^u_{\dot a}\bar Q^{u,\dot a}, i\bar\theta^u_{\dot b}\bar Q^{u,\dot b}] = \bar\xi^u_{\dot a}\bar\theta^u_{\dot b}\{\bar Q^{u,\dot a}, \bar Q^{u,\dot b}\} = \bar\xi^u_{\dot a}\bar\theta^u_{\dot b}(-i\sigma^2)^{\dot a\dot b}(Z^{uu})^* = 0$$

$$[i\bar\xi^u_{\dot a}\bar Q^{u,\dot a}, i\bar\theta^v_{\dot b}\bar Q^{v,\dot b}] = \bar\xi^u_{\dot a}\bar\theta^v_{\dot b}\{\bar Q^{u,\dot a}, \bar Q^{v,\dot b}\} = \bar\xi^u_{\dot a}\bar\theta^v_{\dot b}(-i\sigma^2)^{\dot a\dot b}(Z^{uv})^* = -\bar\xi^u_{\dot a}\bar\theta^{v,\dot a}(Z^{uv})^*$$

$$[i\bar\xi^v_{\dot a}\bar Q^{v,\dot a}, i\theta^{u,b}Q^u_b] = \bar\xi^v_{\dot a}\theta^{u,b}\{\bar Q^{v,\dot a}, Q^u_b\} = \bar\xi^v_{\dot a}\theta^{u,b}[\sigma^\mu]_b{}^{\dot a}\delta^{vu}P_\mu = 0$$

$$[i\bar{\xi}^v_{\dot{a}}\bar{Q}^{v,\dot{a}}, i\theta^{v,b}Q^v_b] = \bar{\xi}^v_{\dot{a}}\theta^{v,b}\{\bar{Q}^{v,\dot{a}}, Q^v_b\} = \bar{\xi}^v_{\dot{a}}\theta^{v,b}[\sigma^\mu]_b{}^{\dot{a}}P_\mu$$

$$[i\bar{\xi}^v_{\dot{a}}\bar{Q}^{v,\dot{a}}, i\bar{\theta}^u_{\dot{b}}\bar{Q}^{u,\dot{b}}] = \bar{\xi}^v_{\dot{a}}\bar{\theta}^u_{\dot{b}}\{\bar{Q}^{v,\dot{a}}, \bar{Q}^{u,\dot{b}}\} = \bar{\xi}^v_{\dot{a}}\bar{\theta}^u_{\dot{b}}(-i\sigma^2)^{\dot{a}\dot{b}}(Z^{vu})^* = -\bar{\xi}^v_{\dot{a}}\bar{\theta}^{u,\dot{a}}(Z^{vu})^*$$

$$[i\bar{\xi}^v_{\dot{a}}\bar{Q}^{v,\dot{a}}, i\bar{\theta}^v_{\dot{b}}\bar{Q}^{v,\dot{b}}] = \bar{\xi}^v_{\dot{a}}\bar{\theta}^v_{\dot{b}}\{\bar{Q}^{v,\dot{a}}, \bar{Q}^{v,\dot{b}}\} = \bar{\xi}^v_{\dot{a}}\bar{\theta}^v_{\dot{b}}(-i\sigma^2)^{\dot{a}\dot{b}}(Z^{vv})^* = 0$$

We insert the results of the commutators into equation (5.3) and collect terms linear in the generators to get

$$\begin{aligned}
&i(x^\mu + a^\mu)P_\mu + i(\theta^{u,a} + \xi^{u,a})Q^u_a + i(\theta^{v,a} + \xi^{v,a})Q^v_a + i(\bar{\theta}^u_{\dot{a}} + \bar{\xi}^u_{\dot{a}})\bar{Q}^{u,\dot{a}} + i(\bar{\theta}^v_{\dot{a}} + \bar{\xi}^v_{\dot{a}})\bar{Q}^{v,\dot{a}} \\
&+2i(A_{uv} + \alpha_{uv})Z^{uv} + 2i(A^*_{uv} + \alpha^*_{uv})(Z^{uv})^* \\
&+\frac{1}{2}\xi^{u,a}\bar{\theta}^u_{\dot{b}}[\sigma^\mu]_a{}^{\dot{b}}P_\mu + \frac{1}{2}\xi^{v,a}\bar{\theta}^v_{\dot{b}}[\sigma^\mu]_a{}^{\dot{b}}P_\mu + \frac{1}{2}\bar{\xi}^u_{\dot{a}}\theta^{u,b}[\sigma^\mu]_b{}^{\dot{a}}P_\mu + \frac{1}{2}\bar{\xi}^v_{\dot{a}}\theta^{v,b}[\sigma^\mu]_b{}^{\dot{a}}P_\mu \\
&+\frac{1}{2}\xi^{u,a}\theta^v_a Z^{uv} - \frac{1}{2}\xi^{v,a}\theta^u_a Z^{uv} + \frac{1}{2}\bar{\xi}^v_{\dot{a}}\bar{\theta}^{u,\dot{a}}(Z^{uv})^* - \frac{1}{2}\bar{\xi}^u_{\dot{a}}\bar{\theta}^{v,\dot{a}}(Z^{uv})^*
\end{aligned}$$

We can again collect terms linear in the generators to get the following exponent of the right-hand side of equation (5.2)

$$\begin{aligned}
&i(x^\mu + a^\mu - \frac{i}{2}\xi^{u,a}\bar{\theta}^u_{\dot{b}}[\sigma^\mu]_a{}^{\dot{b}} - \frac{i}{2}\xi^{v,a}\bar{\theta}^v_{\dot{b}}[\sigma^\mu]_a{}^{\dot{b}} - \frac{i}{2}\bar{\xi}^u_{\dot{a}}\theta^{u,b}[\sigma^\mu]_b{}^{\dot{a}} - \frac{i}{2}\bar{\xi}^v_{\dot{a}}\theta^{v,b}[\sigma^\mu]_b{}^{\dot{a}})P_\mu \\
&+i(\theta^{u,a} + \xi^{u,a})Q^u_a + i(\theta^{v,a} + \xi^{v,a})Q^v_a + i(\bar{\theta}^u_{\dot{a}} + \bar{\xi}^u_{\dot{a}})\bar{Q}^{u,\dot{a}} + i(\bar{\theta}^v_{\dot{a}} + \bar{\xi}^v_{\dot{a}})\bar{Q}^{v,\dot{a}} \\
&+2i(A_{uv} + \alpha_{uv} - \frac{i}{4}\xi^{u,a}\theta^v_a + \frac{i}{4}\xi^{v,a}\theta^u_a)Z^{uv} + 2i(A^*_{uv} + \alpha^*_{uv} - \frac{i}{4}\bar{\xi}^v_{\dot{a}}\bar{\theta}^{u,\dot{a}} + \frac{i}{4}\bar{\xi}^u_{\dot{a}}\bar{\theta}^{v,\dot{a}})(Z^{uv})^* \\
&= i(x^\mu + a^\mu + \frac{i}{2}\xi^{u,a}[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{u,\dot{b}} + \frac{i}{2}\xi^{v,a}[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{v,\dot{b}} - \frac{i}{2}\theta^{u,b}[\sigma^\mu]_{b\dot{a}}\bar{\xi}^{u,\dot{a}} - \frac{i}{2}\theta^{v,b}[\sigma^\mu]_{b\dot{a}}\bar{\xi}^{v,\dot{a}})P_\mu \\
&+i(\theta^{u,a} + \xi^{u,a})Q^u_a + i(\theta^{v,a} + \xi^{v,a})Q^v_a + i(\bar{\theta}^u_{\dot{a}} + \bar{\xi}^u_{\dot{a}})\bar{Q}^{u,\dot{a}} + i(\bar{\theta}^v_{\dot{a}} + \bar{\xi}^v_{\dot{a}})\bar{Q}^{v,\dot{a}} \\
&+2i(A_{uv} + \alpha_{uv} - \frac{i}{4}\xi^{u,a}\theta^v_a + \frac{i}{4}\xi^{v,a}\theta^u_a)Z^{uv} + 2i(A^*_{uv} + \alpha^*_{uv} - \frac{i}{4}\bar{\xi}^v_{\dot{a}}\bar{\theta}^{u,\dot{a}} + \frac{i}{4}\bar{\xi}^u_{\dot{a}}\bar{\theta}^{v,\dot{a}})(Z^{uv})^*
\end{aligned} \tag{5.4}$$

The left hand exponent of equation (5.2) is

$$ix'^\mu P_\mu + i\theta'^{u,a}Q^u_a + i\theta'^{v,a}Q^v_a + i\bar{\theta}'^u_{\dot{a}}\bar{Q}^{u,\dot{a}} + i\bar{\theta}'^v_{\dot{a}}\bar{Q}^{v,\dot{a}} + 2iA'_{uv}Z^{uv} + 2iA'^*_{uv}(Z^{uv})^* \tag{5.5}$$

A comparison of (5.4) and (5.5) shows that the superspace coordinates in $N=2$ SUSY transform as

$$x^\mu \to x^\mu + a^\mu + \frac{i}{2}\xi^{u,a}[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{u,\dot{b}} + \frac{i}{2}\xi^{v,a}[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{v,\dot{b}} - \frac{i}{2}\theta^{u,b}[\sigma^\mu]_{b\dot{a}}\bar{\xi}^{u,\dot{a}} - \frac{i}{2}\theta^{v,b}[\sigma^\mu]_{b\dot{a}}\bar{\xi}^{v,\dot{a}}$$

$$\theta^{u,a} \to \theta^{u,a} + \xi^{u,a}$$

$$\theta^{v,a} \to \theta^{v,a} + \xi^{v,a}$$

$$\bar{\theta}^u_{\dot{a}} \to \bar{\theta}^u_{\dot{a}} + \bar{\xi}^u_{\dot{a}}$$

$$\bar{\theta}^v_{\dot{a}} \to \bar{\theta}^v_{\dot{a}} + \bar{\xi}^v_{\dot{a}}$$

$$A_{uv} \to A_{uv} + \alpha_{uv} - \frac{i}{4}\xi^{u,a}\theta^v_a + \frac{i}{4}\xi^{v,a}\theta^u_a$$

$$A^*_{uv} \to A^*_{uv} + \alpha^*_{uv} - \frac{i}{4}\bar{\xi}^v_{\dot{a}}\bar{\theta}^{u,\dot{a}} + \frac{i}{4}\bar{\xi}^u_{\dot{a}}\bar{\theta}^{v,\dot{a}}$$

5.2 Differential representation of charges

We will now expand the $N = 2$ SUSY charges in two different ways. First we will expand the transformation $U(c_s)$, where c_s was defined in section 5.1, in terms of the charges. Next we will Taylor expand $\Phi(C'_s)$ to obtain the necessary differentials. By comparing these expansions we will get the $N = 2$ generators in the differential representation.

We will start with the expansion of $U(c_s)$. Note that we use $U^\dagger(c_s)$ in the expansion to follow the conventions used in [12]. Using $U(c_{s)}$ directly will give a sign shift on all the charges which will cancel in any commutation or anti-commutation relation containing the charges. The resulting algebra will therefore be the same, regardless of the convention used. We have

$$\begin{aligned}\Phi(C'_s) = [1 - ia^\mu P_\mu - i\xi^{u,a} Q^u_a - i\xi^{v,a} Q^v_a \\ - i\bar{\xi}^u_{\dot{a}} \bar{Q}^{u,\dot{a}} - i\bar{\xi}^v_{\dot{a}} \bar{Q}^{v,\dot{a}} - 2i\alpha_{uv} Z^{uv} - 2i\alpha^*_{uv} (Z^{uv})^*]\Phi(C_s)\end{aligned} \tag{5.6}$$

The Taylor expansion of $\Phi(C'_s)$ is calculated from

$$\begin{aligned}\Phi(C'_s) = \Phi\big(&x^\mu + a^\mu + \frac{i}{2}\xi^{u,a}[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{u,\dot{b}} + \frac{i}{2}\xi^{v,a}[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{v,\dot{b}} - \frac{i}{2}\theta^{u,b}[\sigma^\mu]_{b\dot{a}}\bar{\xi}^{u,\dot{a}} - \frac{i}{2}\theta^{v,b}[\sigma^\mu]_{b\dot{a}}\bar{\xi}^{v,\dot{a}}, \\ &\theta^{u,a} + \xi^{u,a}, \theta^{v,a} + \xi^{v,a}, \\ &\bar{\theta}^u_{\dot{a}} + \bar{\xi}^u_{\dot{a}}, \bar{\theta}^v_{\dot{a}} + \bar{\xi}^v_{\dot{a}}, \\ &A_{uv} + \alpha_{uv} - \frac{i}{4}\xi^{u,a}\theta^v_a + \frac{i}{4}\xi^{v,a}\theta^u_a, A^*_{uv} + \alpha^*_{uv} - \frac{i}{4}\bar{\xi}^v_{\dot{a}}\bar{\theta}^{u,\dot{a}} + \frac{i}{4}\bar{\xi}^u_{\dot{a}}\bar{\theta}^{v,\dot{a}}\big)\end{aligned}$$

We define

$$\Phi \equiv \Phi|_{c_s=0} = \Phi(x^\mu)$$

Note that the expansion terminates to the first order due to the Grassmann nature of the expansion variables. We also define

$$\partial^u_a \equiv \frac{\partial}{\partial\theta^{u,a}}$$

and

$$\bar{\partial}^{u,\dot{a}} \equiv \frac{\partial}{\partial\bar{\theta}^u_{\dot{a}}}$$

We introduce the notation C'^k, which means, $C'^0 = x'^\mu$, $C'^1 = \theta'^u_a$, $C'^5 = A'_{uv}$ and $C'^6 = A'^*_{uv}$. And the notation c'^k, which means, $c'^0 = a'^\mu$, $C'^1 = \xi'^u_a$, $c'^5 = \alpha'_{uv}$ and $c'^6 = \alpha'^*_{uv}$. It is also important to keep the original position of the Grassmann variables in the Taylor expansion (i.e. one must not place the Grassmann variables outside to the right in the expansion terms, as is usual in a Taylor expansion, without performing the correct sign alterations). In the Taylor expansion sums occurring below, we will place the expansion variables outside the sum, keeping in mind that these must be placed at the proper position when the sums are expanded. To be able to differentiate upper index Weyl spinors with respect to lower index Weyl spinors, and vice versa, we use $(i\sigma^2)$ to raise and lower indices. We now perform the Taylor expansion:

$$\begin{aligned}
\Phi' =& \Phi + \sum_{k=0}^{6}\left[\left(\frac{\partial}{\partial c_s^k}\Phi\right)\frac{\partial}{\partial a^\nu}C_s'^k|_{c_s=0}\right]a^\nu + \sum_{k=0}^{6}\left[\left(\frac{\partial}{\partial c_s^k}\Phi\right)\frac{\partial}{\partial \xi^{u,a}}C_s'^k|_{c_s=0}\right]\xi^{u,a} \\
&+\cdots+\sum_{k=0}^{6}\left[\left(\frac{\partial}{\partial c_s^k}\Phi\right)\frac{\partial}{\partial A^*_{uv}}C_s'^k|_{c_s=0}\right]A^*_{uv} \\
=& \Phi + \delta_\nu^\mu\partial_\mu\Phi a^\nu + \frac{i}{2}\xi^{u,a}[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{u,\dot{b}}\partial_\mu\Phi + \delta_c^a\xi^{u,c}\partial_a^u\Phi \\
&-\frac{i}{4}\xi^{u,a}\theta_a^v\frac{\partial}{\partial A_{uv}}\Phi + \frac{i}{2}\theta^{u,a}[\sigma^\mu]_{a\dot{b}}(i\sigma^2)^{\dot{b}\dot{c}}\bar{\xi}^u_{\dot{c}}\partial_\mu\Phi + \delta_{\dot{a}}^{\dot{c}}\bar{\xi}^u_{\dot{c}}\bar{\partial}^{u,\dot{a}}\Phi + \frac{i}{4}\bar{\xi}^u_{\dot{a}}\bar{\theta}^{v,\dot{a}}\frac{\partial}{\partial A^*_{uv}}\Phi \\
&+\frac{i}{2}\xi^{v,a}[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{v,\dot{b}}\partial_\mu\Phi + \delta_c^a\xi^{v,c}\partial_a^v\Phi + \frac{i}{4}\xi^{v,a}\theta_a^u\frac{\partial}{\partial A_{uv}}\Phi + \frac{i}{2}\theta^{v,a}[\sigma^\mu]_{a\dot{b}}(i\sigma^2)^{\dot{b}\dot{c}}\bar{\xi}^v_{\dot{c}}\partial_\mu\Phi \\
&+\delta_{\dot{a}}^{\dot{c}}\bar{\xi}^v_{\dot{c}}\bar{\partial}^{v,\dot{a}}\Phi - \frac{i}{4}\bar{\xi}^v_{\dot{a}}\bar{\theta}^{u,\dot{a}}\frac{\partial}{\partial A^*_{uv}}\Phi + \alpha_{uv}\frac{\partial}{\partial A_{uv}}\Phi + \alpha^*_{uv}\frac{\partial}{\partial A^*_{uv}}\Phi
\end{aligned} \tag{5.7}$$

Rewriting equation (5.7) we have

$$\begin{aligned}
\Phi(C_s') = \{&1 + a^\mu\partial_\mu + \xi^{u,a}\left(\partial_a^u + \frac{i}{2}[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{u,\dot{b}}\partial_\mu - \frac{i}{4}\theta_a^v\frac{\partial}{\partial A_{uv}}\right) \\
&+\left(-\bar{\partial}^{u,\dot{c}} + \frac{i}{2}\theta^{u,a}[\sigma^\mu]_{a\dot{b}}(i\sigma^2)^{\dot{b}\dot{c}}\partial_\mu - \frac{i}{4}\bar{\theta}^{v,\dot{c}}\frac{\partial}{\partial A^*_{uv}}\right)\bar{\xi}^u_{\dot{c}} \\
&+\xi^{v,a}\left(\partial_a^v + \frac{i}{2}[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{v,\dot{b}}\partial_\mu + \frac{i}{4}\theta_a^u\frac{\partial}{\partial A_{uv}}\right) \\
&+\left(-\bar{\partial}^{v,\dot{c}} + \frac{i}{2}\theta^{v,a}[\sigma^\mu]_{a\dot{b}}(i\sigma^2)^{\dot{b}\dot{c}}\partial_\mu + \frac{i}{4}\bar{\theta}^{u,\dot{c}}\frac{\partial}{\partial A^*_{uv}}\right)\bar{\xi}^v_{\dot{c}} \\
&+\alpha_{uv}\frac{\partial}{\partial A_{uv}} + \alpha^*_{uv}\frac{\partial}{\partial A^*_{uv}}\}\Phi
\end{aligned} \tag{5.8}$$

By comparing (5.6) with (5.8) we see that the differential representation of the $N=2$ SUSY charges are

$$\begin{aligned}
P_\mu &= i\partial_\mu \\
Q_a^u &= i\partial_a^u - \frac{1}{2}[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{u,\dot{b}}\partial_\mu + \frac{1}{4}\theta_a^v\frac{\partial}{\partial A_{uv}} \\
Q_a^v &= i\partial_a^v - \frac{1}{2}[\sigma^\mu]_{a\dot{b}}\bar{\theta}^{v,\dot{b}}\partial_\mu - \frac{1}{4}\theta_a^u\frac{\partial}{\partial A_{uv}} \\
\bar{Q}^{u,\dot{c}} &= i\bar{\partial}^{u,\dot{c}} - \frac{1}{2}\theta_a^u[\bar{\sigma}]^{a\dot{c}}\partial_\mu - \frac{1}{4}\bar{\theta}^{v,\dot{c}}\frac{\partial}{\partial A^*_{uv}} \\
\bar{Q}^{v,\dot{c}} &= i\bar{\partial}^{v,\dot{c}} - \frac{1}{2}\theta_a^v[\bar{\sigma}^\mu]^{a\dot{c}}\partial_\mu + \frac{1}{4}\bar{\theta}^{u,\dot{c}}\frac{\partial}{\partial A^*_{uv}} \\
Z^{uv} &= \frac{i}{2}\frac{\partial}{\partial A_{uv}} \\
(Z^{uv})^* &= \frac{i}{2}\frac{\partial}{\partial A^*_{uv}}
\end{aligned}$$

5.3 Chirality

In $N = 1$ superfield theory, we want the superfields to have the same component fields as the fields in the Wess-Zumino Lagrangian. To accomplish this a covariant derivative $\bar{D}_{\dot{a}}$ is constructed. To remove unwanted terms in the superfield the constraint $\bar{D}_{\dot{a}}\Phi = 0$ is required on the superfield Φ. The superfield Φ is then said to be a left-chiral superfield, since it only contains the left-chiral fermions of the original superfield. In the $N = 2$ case we have more non-vanishing terms in the superfield than in the $N = 1$ case. This is because terms might contain combinations of θ^u and θ^v that does not vanish when coupled. We could try to impose two constraints

$$\bar{D}^u_{\dot{a}}\Phi(x, \theta^u, \theta^v, \bar{\theta}^u, \bar{\theta}^v, A_{uv}, A^*_{uv}) = 0$$

$$\bar{D}^v_{\dot{a}}\Phi(x, \theta^u, \theta^v, \bar{\theta}^u, \bar{\theta}^v, A_{uv}, A^*_{uv}) = 0$$

where u and v refers to the explicit charges Q^I for $I \in \{u, v\}$ of the $N = 2$ algebra. However, the purpose would then be to impose a similar condition as for $N = 1$ in order to reduce the number of terms in the expansion of Φ, not to retrieve the Wess-Zumino Lagrangian. The phenomenological reasons for doing this is unclear, but it would still be an $N = 2$ model. Since we have combinations of θ^u and θ^v and not only θ as in $N = 1$ SUSY, we will still have a combination of left-chiral and right-chiral component fields in the $N = 2$ superfields after applying the above constraints. This is due to the fact that $[\theta^u] = [\theta^v] = [\theta]$.This is in accordance with what was discovered using the $N = 2$ algebra to find possible helicity states of an $N = 2$ supermultiplet.

Chapter 6

$N = 2$ Decay chains

6.1 Quantum numbers

As seen in section 3.7, we may exclude many of the possible interaction vertices by considering Lorentz invariance and the conservation of quantum numbers. We also saw how to build SUSY invariant combinations in section 2.4. Table 6.1 lists the quantum numbers of the $N = 1$ particles available in the MSSM, and table 6.2 shows the MSSM gauge fields and their quantum numbers [1]. R-parity will also prevent some important decays such as the rapid decay of the proton, but it will also prevent the decay of the neutralino. The baryon number is calculated through

$$B = \frac{1}{3}(n_q - n_{\bar{q}})$$

where n_q is the number of quarks and $n_{\bar{q}}$ is the number of anti-quarks. The baryon number must also be conserved in a decay.

6.2 Interaction terms

We saw in section 4, that for any $N = 2$ supermultiplet containing a fermion (with helicity $-1/2$), we will need to include two particles of helicity 0 and one particle with helicity $1/2$. This is in contrast to the $N = 1$ case where we only need the helicities $-1/2$ and 0. The electro-weak sector is left-chiral by nature and we cannot extend this sector to $N = 2$ supersymmetry. We will therefore look at a hybrid model, where we only allow the QCD sector to be extended to $N = 2$ supersymmetry. Therefore, we will in the electro-weak sector only have vertices containing left-chiral particles. We will allow for both left and right-chiral particles in the vertices of QCD.

We will now look at how the topology changes when $N = 2$ SUSY is allowed in the QCD-sector. The act of multiplying a left-chiral quantity with $(i\sigma^2)$ and taking the transpose is equivalent to both charge conjugation, and to changing the transformation properties to right-chiral. We no longer have $SU(2)_L$ singlets in the QCD sector since we have right-chiral fermions in the supermultiplet. Therefore we must collect the anti-up-quark, anti-down-quark, anti-up-squark and the anti-down-squark into two $SU(2)_R$ doublets, and no longer allow these as singlets under $SU(2)$. We then have the additional particle fields shown in table 6.3. One could possibly imagine an $N = 2$ decay

Particle	*Name*	*Spin*	*Lepton #*	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
$Q = \begin{pmatrix} \chi_u \\ \chi_d \end{pmatrix}$	quark	1/2	+1	$\mathbf{3}$	$\mathbf{2}$	1/6
$\tilde{Q} = \begin{pmatrix} \tilde{\phi}_u \\ \tilde{\phi}_d \end{pmatrix}$	squark	0	0	$\mathbf{3}$	$\mathbf{2}$	1/6
$\chi_{\bar{u}}$	anti-up-quark	1/2	-1	$\bar{\mathbf{3}}$	$\mathbf{1}$	-2/3
$\chi_{\bar{d}}$	anti-down-quark	1/2	-1	$\bar{\mathbf{3}}$	$\mathbf{1}$	1/3
$\tilde{\phi}_{\bar{u}}$	anti-up-squark	0	0	$\bar{\mathbf{3}}$	$\mathbf{1}$	-2/3
$\tilde{\phi}_{\bar{d}}$	anti-down-squark	0	0	$\bar{\mathbf{3}}$	$\mathbf{1}$	1/3
$L = \begin{pmatrix} \chi_e \\ \chi_{\nu_e} \end{pmatrix}$	electron / neutrino	1/2	+1	$\mathbf{1}$	$\mathbf{2}$	-1/2
$\tilde{L} = \begin{pmatrix} \tilde{\phi}_e \\ \tilde{\phi}_{\nu_e} \end{pmatrix}$	selectron / sneutrino	0	0	$\mathbf{1}$	$\mathbf{2}$	-1/2
$\chi_{\bar{e}}$	Positron	1/2	-1	$\mathbf{1}$	$\mathbf{1}$	1
$\tilde{\phi}_{\bar{e}}$	Spositron	0	0	$\mathbf{1}$	$\mathbf{1}$	1
$H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix}$	Higgs	0	0	$\mathbf{1}$	$\mathbf{2}$	1/2
$\tilde{H}_u = \begin{pmatrix} \tilde{H}_u^+ \\ \tilde{H}_u^0 \end{pmatrix}$	Higgsino	1/2	+1	$\mathbf{1}$	$\mathbf{2}$	1/2
$H_d = \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix}$	Higgs	0	0	$\mathbf{1}$	$\mathbf{2}$	-1/2
$\tilde{H}_d = \begin{pmatrix} \tilde{H}_d^0 \\ \tilde{H}_d^- \end{pmatrix}$	Higgsino	1/2	+1	$\mathbf{1}$	$\mathbf{2}$	-1/2

Table 6.1: Quantum numbers connected to the MSSM particles [1] (Note that we will follow the convention in section 2.2, which uses half of the hypercharge listed in [1]).

Particle	*Name*	*Spin*	*Lepton #*	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
$G^a_{c\mu}$ (g)	Gluon	1	0	$\mathbf{8}$	$\mathbf{1}$	0
λ^a_c $(\tilde{g})$	Gluino	1/2	0	$\mathbf{8}$	$\mathbf{1}$	0
W^i_μ $(W^\pm, W^0)$	W boson	1	0	$\mathbf{1}$	$\mathbf{3}$	0
λ^i_L $(\tilde{W}^\pm, \tilde{W}^0)$	Wino	1/2	0	$\mathbf{1}$	$\mathbf{3}$	0
B_μ (B)	B boson	1	0	$\mathbf{1}$	$\mathbf{1}$	0
λ_Y $(\tilde{B})$	Bino	1/2	0	$\mathbf{1}$	$\mathbf{1}$	0

Table 6.2: Quantum numbers connected to the MSSM gauge fields [1] (Note that we will follow the convention in section 2.2, which uses half of the hypercharge listed in [1]).

Particle	*Name*	*Spin*	*Lepton #*	$SU(3)_c$	$SU(2)_R$	$U(1)_Y$
$Q_R = \begin{pmatrix} \chi_{\bar{u}} \\ \chi_{\bar{d}} \end{pmatrix}$	quark	-1/2	-1	$\bar{\mathbf{3}}$	$\mathbf{2}$	-1/6
$\tilde{Q}_R = \begin{pmatrix} \tilde{\phi}_{\bar{u}} \\ \tilde{\phi}_{\bar{d}} \end{pmatrix}$	squark	0	0	$\bar{\mathbf{3}}$	$\mathbf{2}$	-1/6

Table 6.3: Quantum numbers connected to the extended SUSY particles (Note that we will follow the convention in section 2.2, which uses half of the hypercharge listed in [1]).

where an $N = 2$ spin-$1/2$ particle may decay via generalized R-parity violation (i.e. going from $N = 2$ to $N = 1$) to quark(s) and gluon(s).

6.3 The Neutralino

The neutralino is a candidate for dark matter, and can only be detected by particle accelerators through missing energy, due to its lack of interactions with the electromagnetic forces. It consists of a mixture of $\tilde{B}$ (bino), $\tilde{W}^0$ (neutral wino) and $\tilde{H}_u^0$ / $\tilde{H}_d^0$ (neutral Higgsinos) [1]. The neutralino cannot decay any further due to the assumed R-parity.

Chapter 7

Conclusion

We have been through the construction of the Standard Model and $N = 1$ supersymmetry using the superfield formalism. From the construction of the $N = 1$ superfield formalism we found a way to construct the $N = 2$ supercharges in a differential representation. We found that in order to construct $N = 2$ superfield theory it is not enough to extend the existing superspace by the trivial extension of the superspace coordinates by the number of added charges. We must also add coordinates for the new operators that appear in the extended supersymmetry algebra.

The $N = 1$ general superfield, not restricted to left chirality, has more component fields than are phenomenologically sound. This is of course also true for the general unrestricted $N = 2$ superfield. It is, however, not so clear which guidelines to use for the restriction of these superfields. We could try two constraints similar to the condition of left chirality in $N = 1$ SUSY, but it would not lead to a reconstruction of the Wess-Zumino Lagrangian as was the purpose in $N = 1$. We also note that even with the analogous restriction to $N = 1$ SUSY, the $N = 2$ superfield will not be left-chiral.

We found that in $N = 1$ supersymmetry, chiral fermions occur naturally as a consequence of the the $N = 1$ supersymmetry algebra. Chiral fermions are a necessity in the Standard Model, and must be possible as a viable extension of it. Using the $N > 1$ supersymmetry algebra we were able to calculate all the helicity states that must be present in one supermultiplet. It is evident that for any supermultiplet, we must couple a right-chiral field to a left-chiral field. This implies that a full $N > 1$ supersymmetric model will not be possible in an extension of the Standard Model. The chirality problem is however only present in the electro-weak sector of the Standard Model. A hybrid $N = 1/N > 1$ model could be an alternative. Here, only the QCD sector would be implemented using $N > 1$ supersymmetry,

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