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A random exchange model with constant decrements.

by

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Abstract.

Let d be a positive constant and let $\{U_n\}$ be a sequence of independent, identically distributed random variables. Define a new sequence of variables $\{X_n\}$ recursively by

$$X_n = \max(X_{n-1} - d, U_n).$$

A crude model of physical exchange processes, based on the sequence $\{X_n\}$, is analyzed. The stationary distribution of X_n, of X_n at an epoch of exchange $(X_n = U_n)$ and of the time between consecutive exchanges are determined. For the case where U_n is $N(\mu, \sigma^2)$, an asymptotic expansion for the expected number of years between exchanges as $d \to 0$ is sought.

1. Introduction.

The model to be considered here was proposed by Herman G. Gade (Gade 1973) in connection with his investigations of deep water exchanges in sill fjords. Though I want to discuss the model abstractly (it may be used to describe other physical phenomena), it is useful to have the following physical process in mind.

The deep water masses of a sill fjord are characterized by a relatively high degree of uniformity. As a first approximation the density of the water may be considered homogeneous throughout the basin. Various diffusion processes causes this density to decrease approximately linearly with time. In this paper I will follow Gade and assume a constant annual density decrement d .

It is an empirical fact that influxes of coastal water into the basin tend to take place at the same time of the year, thus establishing a recurring phenomenon with time intervals being essentially multiples of a full year. The deep water renewals are relatively rapid events, often completed within the course of a few weeks. The influx will take place when the coastal water at sill depth is heavier than the resident water in the fjord basin. I assume that in this case all the resident water is replaced by water with the same density as that of the coastal water present at sill depth. The density of the coastal water in adjacent years are assumed

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In a forthcoming paper a more general model will be considered in which also the annual density decrements are assumed to be i.i.d. random variables. This will prove to be a generalization of existing models in the theory of queues. My reason for discussing the special case of constant decrement here is twofold. As far as I know, it is only this specialized model which up to now has been used to describe a physical process. Furthermore, in this special case it is possible to proceed by relatively elementary mathematical methods. Thus it should be possible to follow the discussion without being a specialist in complex integral equations.

2. The model.

Let $\{U_n ; n = 1,2,3,\ldots\}$ be a sequence of i.i.d. random variables characterized by the distribution function

$$G(x) = P[U_n \leq x] \quad ; \quad n = 1,2,\ldots \quad .$$

(Here and in the following P denotes probability.) In the physical model, U_n is the density of coastal water at sill level in year number n , at the time of the year when exchanges tend to take place. We number the years consecutively. In most of the paper the U_n's are assumed to be absolute to be independent, identically distributed (i.i.d.) random variables.

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$$g(x) = \frac{d}{dx} G(x) \ .$$

Let X_o be independent of the sequence $\{U_n\}$ and define a new sequence $\{X_n\}$ of random variables by

$$(1) \qquad X_n = \max(U_n , X_{n-1} - d) \ ; \ n = 1,2,\ldots$$

where d is a positive constant (the annual density decrement). It follows from our assumptions that X_n may be interpreted as the density of the deep water masses of the sill fjord in year number n, assuming an initial density X_o. (X_o may be constant).

Equation (1) may be written

$$X_n = \begin{cases} U_n & \text{if} \quad U_n > X_{n-1} - d \\ X_{n-1} - d & \text{if} \quad X_{n-1} - d \geq U_n \ . \end{cases}$$

Thus we are led to study the events

$$A_n : \ U_n > X_{n-1} - d \quad \text{(exchange at time } n \text{)}$$

$$A_n^c : \ U_n \leq X_{n-1} - d \quad \text{(no exchange at time } n \text{)} \ .$$

Most of this paper is devoted to the study of the process $\{X_n \ ; \ n = 0,1,2,\ldots\}$. Some information may be drawn directly from equation (1), e.g.

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$$X_n \geq U_n$$

that is

$$E(X_n) \geq E(U_n)$$

provided both expectations exist. Here E denotes expectation, e.g.

$$E(U_n) = \int_{-\infty}^{\infty} x \cdot dG(x)$$

Thus the expected density of the resident water in the fjord is greater than or equal to the expected density of the coastal water.

Furthermore, $\{X_n\}$ is easily seen to be a Markov process. That is, the conditional distribution of X_n, given $X_k, X_{k+1}, \ldots, X_{n-1}$ $(k < n-1)$ is independent of $X_k, X_{k+1}, \ldots, X_{n-2}$. This is immediate from equation (1). It is well known that all the information about a Markov process can in principle be obtained from the initial distribution and the transition probability function

$$P(x;y) = P[X_n \leq y \mid X_{n-1} = x]$$

In general this function depends on n, but here we have a so-called homogeneous Markov process where $P(x;y)$ is independent of n. We find

$$P(x;y) = P[\max(U_n, X_{n-1}-d) \leq y \mid X_{n-1} = x]$$

(2)
$$= P[U_n \leq y \,,\, X_{n-1} \leq y + d \mid X_{n-1} = x]$$

$$= G(y)I(y + d - x)$$

where

(3)
$$I(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Here we have used the fact that X_{n-1} and U_n are independent, since X_{n-1} depends only on $X_o, U_1, U_2, \ldots, U_{n-1}$ through equation (1).

In a corresponding way we find the n step transition probability function

$$P^{(n)}(x;y) = P[X_{k+n} \leq y \mid X_k = x] \quad , \quad (n = 1,2,\ldots) ,$$

which is independent of k .

$$P^{(n)}(x;y) = P[\max(U_{k+n}, X_{k+n-1}-d) \leq y \mid X_k = x]$$

$$= P[\max(U_{k+n}, U_{k+n-1}-d, X_{k+n-2}-2d) \leq y \mid X_k = x]$$

$$= \ldots$$

(4)

$$= P[\max(U_{k+n}, U_{k+n-1}-d, \ldots, U_{k+1}-(n-1)d, X_k-nd) \leq y \mid X_k = x]$$

$$= P[X_k \leq y + nd, U_{k+n-j} \leq y + jd, (j=0,1,\ldots,n-1) \mid X_k = x]$$

$$= I(y + nd - x)\prod_{j=0}^{n-1} G(y + jd)$$

with $I(t)$ given by (3).

3. Stationary distribution.

When $n \to \infty$, $I(y + nd - x) \to 1$ for all x and y the sequence $\{\prod_{j=0}^{n-1} G(y + jd), n=1,2,\ldots\}$ decreases monotonically:

$$\prod_{j=0}^{n} G(y + jd) = G(y + nd) \prod_{j=0}^{n-1} G(y + jd) \leq \prod_{j=0}^{n-1} G(y + jd) .$$

Therefore the following limit always exists and is independent of x

$$(5) \qquad F(y) = \lim_{n \to \infty} P^{(n)}(x;y) = \prod_{j=0}^{\infty} G(y + jd) .$$

It is easily seen that $0 \leq F(y) \leq 1$ for all x and $F(x) \leq F(y)$ when $x < y$. Let X_o have the distribution function $F_o(x)$. Then by dominated convergence

$$(6) \qquad \lim_{n \to \infty} P[X_n \leq y] = \lim_{n \to \infty} \int_{x=-\infty}^{\infty} P^{(n)}(x;y) dF_o(x) = F(y) .$$

The following theorem is the main result of this section. It may be generalized to the case of random decrements (see Helland, 1973).

3. Stationary distribution.

When $n \to \infty$, $l(y + nd - x) \to 1$ for all x and y the sequence $\{\prod_{j=0}^{n-1} G(y + jd),\ n=1,2,\dots\}$ decreases monotonically:

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The following theorem is the main result of this section. It may be generalized to the case of random decrements (see Helland, 1975).

Theorem 1.

(i) If $E(U_1^+) < \infty$, then $F(y)$ is a distribution function and $\{X_n\}$ converges in law to the distribution F .

(ii) If $E(U_1^+) = \infty$, then

$$F(y) = \lim_{n \to \infty} P[X_n \leq y] = 0 \quad \text{for all } y .$$

Remark.

With U_n^+ is meant $\max(0, U_n)$. $E(U_n^+)$ is of course independent of n , since the distribution of U_n is. The term convergence in law is used in the usual probabilistic sense. It simply means that (6) holds (in general for all y where $F(y)$ is continuous), where F is a distribution function.

Proof.

We have to show that F is a distribution function when $E(U_1^+)$ is finite. Now we have already remarked that $F(y)$ is bounded by 0 and 1 and is monotonic in y . It is also easy to see that $F(y)$ is continuous from the right. When $G(y) = 0$ it is trivial, otherwise show by monotone convergence

$$\lim_{z \downarrow y} \log F(z) = \lim_{z \downarrow y} \sum_{j=0}^{\infty} \log G(z+jd)$$

$$= \sum_{j=0}^{\infty} \lim_{z \downarrow y} \log G(z+jd)$$

$$= \sum_{j=0}^{\infty} \log G(y+jd) = \log F(y)$$

Furthermore $F(y) \leq G(y)$ so that $F(y) \to 0$ as $y \to -\infty$.

Therefore F is a distribution function if and only if

$$\lim_{y \to +\infty} F(y) = \lim_{y \to +\infty} \prod_{j=0}^{\infty} G(y+jd) = 1 .$$

Now

$$F(y) = \prod_{j=0}^{\infty} G(y+jd)$$

$$= \prod_{j=0}^{\infty} (1 - P[U_1 > y + jd])$$

$$\geq 1 - \sum_{j=0}^{\infty} P[U_1 > y + jd]$$

and

$$\sum_{j=0}^{\infty} P[U_1 > y + jd] = \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} P[kd < U_1 - y \leq (k+1)d]$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} P[kd < U_1 - y \leq (k+1)d]$$

$$= \sum_{k=0}^{\infty} (k+1) P[kd < U_1 - y \leq (k+1)d]$$

$$= \frac{1}{d} \sum_{k=0}^{\infty} kd \; P[kd < U_1 - y \leq (k+1)d] + P[U_1 > y]$$

$$\leq \frac{1}{d} E((U_1 - y)^+) + P[U_1 > y] .$$

When $E(U_1^+) < \infty$, the last expression tends to 0 as $y \to +\infty$, thus proving (i) of the theorem.

The proof of (ii) is similar.

$$0 \leq F(y) = \prod_{j=0}^{\infty} (1 - P[U_1 > y + jd])$$

$$\leq \exp \left\{ - \sum_{j=0}^{\infty} P[U_1 > y + jd] \right\}$$

where we have used the fact that $1 - t \leq \exp\{-t\}$ for all real t.

As in the first part of the proof we find

$$\sum_{j=0}^{\infty} P[U_1 > y + jd] = \frac{1}{d} \sum_{k=0}^{\infty} (k+1)d\ P[kd < U_1 - y \leq (k+1)d]$$

$$\geq \frac{1}{d} E((U_1 - y)^+) = +\infty$$

when $E(U_1^+) = +\infty$. Thus $F(y) \equiv 0$ and the proof is complete.

Assume from now on that $E(U_1^+)$ is finite. The following proposition shows that when we have an initial distribution $F_o(y) = F(y)$, all the variables X_n have the same distribution $F(y)$. This may be deduced from more general theorems on Markov process, but the direct verification in the present case is simple.

Proposition.

When $E(U_1^+) < +\infty$, $F(y)$ is a stationary distribution in the sense that

$$F(y) = \int_{-\infty}^{\infty} P(x;y)\ dF(x) \tag{7}$$

Proof.

The right-hand side of (7) is

$$\int_{-\infty}^{\infty} P(x;y)\, dF(x) = \int_{-\infty}^{\infty} G(y) I(y+d-x)\, dF(x)$$

$$= G(y) \int_{-\infty}^{y+d} dF(x)$$

$$= G(y)\, F(y+d)$$

$$= G(y) \prod_{j=0}^{\infty} G(y+d+jd)$$

$$= \prod_{j=0}^{\infty} G(y+jd)$$

$$= F(y)$$

Example.

Usually it is difficult to get an explicit expression for $F(y)$ from (5). However, the following example is easy to handle.

(8) $$G(x) = \exp\{-a\, e^{-bx}\}\ , \quad a,b > 0\ , \quad -\infty < x < \infty$$

The probability density for U_1 is

(9) $$g(x) = G'(x) = ab \exp\{-bx-a\, e^{-bx}\} \ .$$

For later reference we need the expectation and variance of U_1 .

$$E(U_1) = ab \int_{-\infty}^{\infty} x \exp\{-bx-a\, e^{-bx}\} dx$$

$$= -\frac{1}{b} \int_0^{\infty} \ln\left(\frac{z}{a}\right) e^{-z}\, dz$$

$$= \frac{1}{b}(\ln a + C)$$

where C is the Euler-Mascheroni constant

$$C = \lim_{m\to\infty}\left(\sum_{n=1}^{m}\frac{1}{n} - \ln m\right) = -\int_0^{\infty} \ln z\, e^{-z}\, dz \approx 0{,}5772 \ .$$

We have made the change of variable $x \to z = a\, e^{-bx}$.

Similarly we find

$$E(U_1^2) = ab \int_{-\infty}^{\infty} x^2 \exp\{-bx-a\, e^{-bx}\} dx$$

$$= \frac{1}{b^2}\int_0^{\infty} (\ln z - \ln a)^2 e^{-z}\, dz$$

$$= \frac{1}{b^2}(\beta + 2C \ln a + (\ln a)^2)$$

where

$$\beta = \int_0^{\infty} (\ln z)^2 e^{-z}\, dz = \frac{\pi^2}{6} + C^2 \ .$$

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The relevant integrals may be found in any large table, for istance Erdélyi et al.(1954). The variance of U_1 is

$$\sigma^2 = \mathrm{Var}(U_1) = \frac{1}{b^2}(\beta - C^2) = \frac{\pi^2}{6b^2}$$

which is independent of a .

The stationary distribution of $\{X_n\}$ is found from (5) and (8)

$$F(x) = \prod_{k=o}^{\infty} \exp\left\{-a\ e^{-b(x+kd)}\right\}$$

$$(10) \qquad = \exp\left\{-a\ e^{-bx} \sum_{k=o}^{\infty} (e^{-bd})^k\right\}$$

$$= \exp\left\{\frac{-a\ e^{-bx}}{1-e^{-bd}}\right\} = \exp\left\{-a_1\ e^{-bx}\right\}$$

where $a_1 = a(1-e^{-bd})^{-1}$.

Thus under stationary conditions

$$E(X_n) = \frac{1}{b}(\ln a_1 - C) = \frac{1}{b}\left(\ln a - C - \ln(1-e^{-bd})\right)$$

$$> E(U_n)$$

$$\mathrm{Var}(X_n) = \frac{\pi^2}{6b^2} = \mathrm{Var}(U_n)$$

The stationary probability density $f(x)$ for $\{X_n\}$ is given by an expression similar to (9). In fig.1 and

fig.2 f and g are drawn for the two cases $\frac{\sigma}{d} = 1$ and $\frac{\sigma}{d} = 2$, $a = 1$ in both cases.

4. The exchanges.

From the model (1) we see that the process $\{X_n\}$ (density of resident water at time n) developes as follows

$$
\begin{array}{lll}
X_o & & \\
X_1 = X_o - d & & (\geq U_1) \\
X_2 = X_o - 2d & & (\geq U_2) \\
\cdots & & \\
X_{N_1-1} = X_o - (N_1-1)d & & (\geq U_{N_1-1}) \\
X_{N_1} = U_{N_1} & & (> X_o - N_1 d) \\
X_{N_1+1} = U_{N_1} - d & & (\geq U_{N_1+1}) \\
\cdots & &
\end{array}
\tag{11}
$$

where N_1 is the year of the first exchange. Assume that the exchanges take place in the years N_1 , $N_1 + N_2$, $N_1 + N_2 + N_3$ and so on. We then have to study the associated process:

$\{N_k ; k = 1,2,\ldots\}$ (number of years between successive exchanges)

$\{S_k ; k = 1,2,\ldots\}$ with $S_k = U_{N_1+N_2+\ldots+N_k}$ (the density of water at the k'th exchange)

These two processes were also studied in Gade (1973). We shall attack the problem by different means, which give explicit formulae for the relevant stationary distributions.

First of all we want to find the probability of exchange in year n, given nothing but the initial density distribution (that is the distribution of X_o). This probability is

$$P[A_n] = P[U_n > X_{n-1} - d]$$

$$(12) \qquad = P[X_{n-1} < U_n + d]$$

$$= \int_{-\infty}^{\infty} F_{n-1}^{(-)}(u+d)\, dG(u)$$

where F_{n-1} is the distribution function of X_{n-1} and

$$F_{n-1}^{(-)}(x) = \lim_{h \downarrow o} F_{n-1}(x-h) .$$

We have utilized the independence of X_{n-1} and U_n.

From now on assume that $E(U_1^+) < \infty$ and that G is continuous. Then from (5) the stationary distribution function F is continuous and

$$\lim_{n \to \infty} F_{n-1}^{(-)}(x) = F(x) \quad \text{for all } x .$$

Therefore from (12)

$$(13) \qquad \pi_A = \lim_{n\to\infty} P[A_n] = \int_{-\infty}^{\infty} F(u+d)dG(u)$$

π_A is the stationary probability of exchange in a given year.

Now turn to the density of water at exchange. It is easy to see that $\{S_k ; k = 1,2,...\}$ is a Markov process, but the transition probability function is cumbersome to handle (see Gade (1973), formula (8) which gives the derivative of this function). Therefore we will find the stationary distribution function $T(x)$ of $\{S_k\}$ directly. Assume that stationary conditions have been reached, that is, all the variables X_n , $n = 1,2,...$ have the same distribution $F(x)$. Then

$$T(x) = P[S_k \leq x]$$

$$= P[X_n \leq x \mid A_n]$$

$$(14) \qquad = \frac{P[(X_n \leq x) \cap (U_n > X_{n-1}-d)]}{P[A_n]}$$

$$= \pi_A^{-1} P[(U_n \leq x) \cap (X_{n-1} < U_n+d)]$$

$$= \pi_A^{-1} \int_{-\infty}^{x} F(u+d)dG(u)$$

where again F is continuous and we have used the independence of X_{n-1} and U_n . By combining (5), (13) and (14)

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$$T(x) = P[S_k \leq x]$$

$$= P[X_n \leq x \mid A_n]$$

$$= \frac{P[(X_n \leq x) \cap (U_n \geq X_{n-1}-d)]}{P[A_n]} \tag{14}$$

$$= \pi_A^{-1}\, P[(U_n \leq x) \cap (X_{n-1} < U_n+d)]$$

$$= \pi_A^{-1} \int_{-\infty}^{x} F(u+d)\,dG(u)$$

where again F is continuous and we have used the independence of X_{n-1} and U_n. By combining (5), (13) and (14) we can give $T(x)$ directly in terms of the given distribution $G(x)$:

(15)
$$T(x) = \frac{\int_{-\infty}^{x} \left\{ \prod_{j=1}^{\infty} G(u+jd) \right\} dG(u)}{\int_{-\infty}^{\infty} \left\{ \prod_{j=1}^{\infty} G(u+jd) \right\} dG(u)} .$$

When $T(x)$ is known, we can also find the probability distribution of the number of years between two successive exchanges. First assume that $X_o = y$ is constant. Then we find by simple inspection of the scheme (11) :

$$P[N_1 = n | X_o = y] = Q(n;y)$$

(16)
$$= P\left[U_1 \leq y-d, U_2 \leq y-2d, \ldots, U_{n-1} \leq y-(n-1)d, U_n > y-nd\right]$$

$$= \{1 - G(y-nd)\} \prod_{j=1}^{n-1} G(y-jd)$$

This formula is also given by Gade (1973) (formula(6)), and it is valid whether or not an exchange takes place in the year 0 .

Under stationary conditions the probability that two consecutive exchanges are separated by n years will be

$$Q(n) = \int_{-\infty}^{\infty} Q(n;y)dT(y)$$

$$= \pi_A^{-1} \int_{-\infty}^{\infty} \left\{1 - G(y-nd)\right\}\left\{\prod_{j=1}^{n-1} G(y-jd)\right\} F(y+d)dG(y)$$

(17)

$$= \pi_A^{-1} \int_{-\infty}^{\infty} \left\{\frac{1}{G(y)} \prod_{j=-n+1}^{\infty} G(y+jd) - \frac{1}{G(y)} \prod_{j=-n}^{\infty} G(y+jd)\right\} dG(y)$$

$$= \pi_A^{-1} \int_{-\infty}^{\infty} \left\{ F(y-nd+d) - F(y-nd)\right\} \frac{dG(y)}{G(y)}$$

from (16) and (5).

Alternatively

(18) $$P\left[N_k \geq n\right] = \sum_{m=n}^{\infty} Q(m) = \pi_A^{-1} \int_{-\infty}^{\infty} F(y-nd+d) \frac{dG(y)}{G(y)}$$

The (stationary) expectation of N_k is

$$\bar{n} = \sum_{n=1}^{\infty} n\, Q(n) = \sum_{n=1}^{\infty} P[N_k \geq n]$$

By a change of variable and partial integration in (18) we find

$$P[N_k \geq n] = \pi_A^{-1} \int_{-\infty}^{\infty} F(y+d)\, d\{\ln G(y+nd)\}$$

$$= - \pi_A^{-1} \int_{-\infty}^{\infty} \ln G(y+nd)\, dF(y+d) .$$

Therefore

$$\bar{n} = - \pi_A^{-1} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \ln G(y+nd)\, dF(y+d)$$

$$= - \pi_A^{-1} \int_{-\infty}^{\infty} \ln F(y+d)\, dF(y+d)$$

$$= - \pi_A^{-1} \int_{0}^{1} \ln x\, dx = \pi_A^{-1}$$

or by (13)

$$(19) \qquad \bar{n} = \left(\int_{-\infty}^{\infty} F(u+d)\, dG(u) \right)^{-1} .$$

The result $\bar{n}\, \pi_A = 1$ is also valid in the general model with random decrements. This was indicated in Helland (1973) and will be proved rigorously in a forthcoming paper.

By methods similar to those used above, we could find other quantities of interest, e.g. the (stationary) variance of N_k, the correlation between N_k and N_j $(k \neq j)$, the probability distribution of the increase of the density of water by exchange and so on.

Example (continued).

When $G(x)$ is given by (8) and $F(x)$ by (10) we find by straightforward integration

$$\int_{-\infty}^{x} F(u+d)\, dG(u) = \int_{-\infty}^{x} \exp\left\{ \frac{-a\, e^{-bu}}{1-e^{-bd}} \right\} .\, ab\, e^{-bu}\, du$$

(20)

$$= \left(1 - e^{-bd}\right) \exp\left\{ \frac{-a\, e^{-bx}}{1-e^{-bd}} \right\} .$$

Therefore from (13)

(21) $$\pi_A = 1 - e^{-bd} ,$$

and from (19)

(22) $$\bar{n} = \left(1 - e^{-bd}\right)^{-1} = \left(1 - \exp\left\{-\frac{\pi d}{\sigma\sqrt{6}}\right\}\right)^{-1}$$

where σ^2 is the variance of U_1 . In fig.3 $\bar{n}$ is drawn as a function of $\frac{\sigma}{d}$.

From (14), (20) and (21) we get

$$T(x) = \exp\left\{\frac{-a\, e^{-bx}}{1-e^{-bd}}\right\} = F(x)$$

by (10). That is, $\{X_n\}$ and $\{S_k\}$ have the same stationary distribution. In fact one can show that this property nearly characterizes the special distribution (8). More precisely,

if $G(x)$ is such that $T(x) = F(x)$, then either U_1 is concentrated on an interval of length less than d (in which case $\pi_A = 1$) or

$$G(x) = \exp\left\{ - a(x)\, e^{-bx} \right\}$$

where $a(x)$ is periodic with periode d and such that $G(x)$ is monotonic and continuous from the right. The proof is straightforward, but cumbersome, and will not be given.

The stationary distribution of $\{N_k\}$ is also easily found in this case. By (18), (8), (10) and (21) we find

$$P[N_k \geq n] = \pi_A^{-1} \int_{-\infty}^{\infty} F(y - (n-1)d)\, \frac{dG(y)}{G(y)}$$

$$= \pi_A^{-1} \int_{-\infty}^{\infty} \exp\left\{ \frac{-a\, e^{-by}\, e^{(n-1)bd}}{1 - e^{-bd}} \right\} .\ ab\, e^{-by} dy$$

$$= e^{-(n-1)bd}$$

Thus

(23) $$Q(n) = P[N_k = n] = (e^{bd} - 1)\, e^{-nbd} \quad , \; n = 1,2,\ldots$$

That is, under stationary conditions N_k is geometrically distributed. From this we find again (22) and

(24) $$\mathrm{Var}\,(N_k) = \frac{e^{-bd}}{(1 - e^{-bd})^2} \quad .$$

For other distributions than (8), the evaluation of $F(x)$, $T(x)$, π_A etc. is not so simple. Take for instance U_1 to be normally distributed. Without lack of generality we can take the mean of U_1 to be zero (translation off all the variables U_n , X_n , S_k will not alter the model). Then

$$(25) \qquad G(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} dt = \Phi(\frac{x}{\sigma}) ,$$

where σ^2 is the variance of U_1 , Φ is the standard normal distribution. From (5) we find

$$(26) \qquad F(x) = \prod_{k=0}^{\infty} \Phi(\frac{x}{\sigma} + k\frac{d}{\sigma}) .$$

(13) gives

$$(27) \qquad \pi_A = \int_{-\infty}^{\infty} F(y+d)g(y)dy = \int_{-\infty}^{\infty} \left\{ \prod_{k=1}^{\infty} \Phi(z + k\frac{d}{\sigma}) \right\} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz .$$

(14) gives

$$(28) \qquad T(x) = \pi_A^{-1} \int_{-\infty}^{x} F(u+d)g(y)dy = \pi_A^{-1} \int_{-\infty}^{\frac{x}{\sigma}} \left\{ \prod_{k=1}^{\infty} \Phi(z+k\frac{d}{\sigma}) \right\} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz .$$

Finally, the corresponding probability densities $f(x) = F'(x)$ and $t(x) = T'(x)$ are

$$(29) \qquad f(x) = \sum_{k=0}^{\infty} \frac{g(x+kd)}{G(x+kd)} F(x) = F(x) \sum_{k=0}^{\infty} e^{-\frac{(x+kd)^2}{2\sigma^2}} \left[\sqrt{2\pi}\sigma \, \Phi(\frac{x}{\sigma} + k\frac{d}{\sigma}) \right]^{-1}$$

For other distributions than (8), the evaluation of $F(x)$, $T(x)$, π_A etc. is not so simple. Take for instance U_1 to be normally distributed. Without lack of generality we can take the mean of U_1 to be zero (translation of all the variables U_n, X_n, S_k will not alter the model). Then

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$$(30) \qquad t(x) = \pi_A^{-1} F(x+d)g(x) = \frac{1}{\pi_A\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \prod_{k=1}^{\infty} \Phi(\frac{x}{\sigma} + k\frac{d}{\sigma})$$

The last two expressions have been evaluated numerically for $\frac{\sigma}{d} = 1,2$ and 4 and the corresponding curves are shown in fig.4,5 and 6.

5. Asymptotic expansions.

This section is matematically a little more involved than the previous ones. However, it is simple to pose the problem: Is it possible to find simpler, approximate expressions for quantities characterizing the model when d is small? Physically we should expect the following when $d \to 0$. As the annual decrease of the density of the water becomes small, the expected number of years between successive exchanges should increase. Hence the probability of exchange in a given year should decrease. Finally, exchanges take place only when the density of the coastal water is extremely high, therefore the expected density of the exchanged water will increase.

First of all we must make precise what we mean by small d. The model is in general determined by d and the distribution function $G(x)$ of U_1, hence small d should

$$(30) \qquad t(x) = \pi_A^{-1} F(x+d) g(x) = \frac{1}{\pi_A \sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} \prod_{k=1}^{\infty} \Phi\left(\frac{x}{\sigma} + k\frac{d}{\sigma}\right)$$

The last two expressions have been evaluated numerically for $\frac{\sigma}{d} = 1, 2$ and 4 and the corresponding curves are shown in fig. 4, 5 and 6.

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First of all we must make precise what we mean by small d. The model is in general determined by d and the distribution function G(x) of U_i, hence small d should mean that d is much less than some parameter characterizing the distribution $G(x)$. It is easy to see that $E(U_1)$ is irrelevant in this connection ($E(U_1)$ may be changed arbitrarily by translating all the variables U_n, X_n etc). Usually by small d we will mean

$$\frac{d}{\sigma} << 1 ,$$

where $\sigma^2 = \mathrm{Var}(U_1)$. Note , however, that the asymptotic expressions which we are going to develop are not simply functions of d/σ , but depend on the detailed form of $G(x)$.

Assume that $E(U_1^+)$ is finite and that $G(x)$ is absolute continuous with probability density $g(x)$. Then (see (5), (13) and (19))

$$(31) \qquad F(x) = \prod_{k=o}^{\infty} G(x+kd)$$

$$(32) \qquad \pi_A = (\bar{n})^{-1} = \int_{-\infty}^{\infty} \left\{ \prod_{k=1}^{\infty} G(x+kd) \right\} g(x)dx .$$

The infinite products occuring here are difficult to handle analytically. By taking logarithm these may be expressed by means of infinite sums, which in turn may be approximated by integrals when d is small. In this way we get the following

Theorem 2.

Let $E(U_1^+) < \infty$ and let $G(x)$ have continuous second derivative in $(-\infty,\infty)$. Assume that the density $g(y) \to 0$ as $y \to \infty$. Let $a \in [-\infty,\infty)$ be such that $G(x) = 0$ when $x \leq a$, $G(x) > 0$ when $x > a$ (eventually $a = -\infty$). Then as $d \to 0$

$$(33) \qquad F(x) = \sqrt{G(x)}\, \exp\left\{d^{-1} \int_x^{\infty} [\ln G(y)]\, dy\right\} (1 + O(d))$$

uniformly in x when $x \geq c$ for all $c > a$, and

$$(34) \qquad \pi_A = \int_a^{\infty} (G(x))^{-\frac{1}{2}} \exp\left\{d^{-1} \int_x^{\infty} [\ln G(y)]\, dy\right\} g(x)dx\, (1+O(d))$$

Proof.

From (31) $F(x) = 0$ when $x \leq a$. When $x > a$

$$(35) \qquad F(x) = \exp\left\{ \sum_{k=o}^{\infty} \ln G(x+kd)\right\}$$

and the sum converges because $E(U_1^+) < \infty$ (cf. Theorem 1). First we have to show

$$(36) \qquad \sum_{k=o}^{\infty} \ln G(x+kd) = d^{-1} \int_x^{\infty} [\ln G(y)]\, dy + \tfrac{1}{2}\ln G(x) + O(d)$$

uniformly in x $(x \geq c)$ as $d \to 0$. To this end use the

trapezoidal formula

$$\int_{x+kd}^{x+(k+1)d} \left[\ln G(y)\right] dy = \frac{d}{2}\left[\ln G(x+kd) + \ln G(x + (k+1)d)\right] - \frac{d^3}{12}\left[\frac{d^2}{dy^2} \ln G(y)\right]_{y=\eta_k}$$

where $x + kd < \eta_k < x + (k+1)d$. This may be shown by approximating the area corresponding to the integral by the area of a trapezoid and estimating the error. (Cf. e.g. Conte (1965), p.122). Now put

$$\varphi(y) = \frac{d}{dy} \ln G(y) = \frac{g(y)}{G(y)} \qquad (y \geq c) .$$

Then the trapezoidal formula may be written

$$(37) \quad \tfrac{1}{2} \ln G(x+kd) + \tfrac{1}{2} \ln G(x + (k+1)d) = d^{-1} \int_{x+kd}^{x+(k+1)d} \left[\ln G(y)\right] dy + \frac{d^2}{12} \varphi'(\eta_k)$$

Since $g(y) \to 0$ as $y \to \infty$, also $\varphi(y) \to 0$ as $y \to \infty$ and $\varphi(y)$ will be uniformly bounded for $y \in [c,\infty)$. Therefore

$$d \sum_{k=o}^{\infty} \varphi'(\eta_k) = \int_x^{\infty} \varphi'(y) dy + o(1) = - \varphi(x) + o(1) = O(1)$$

uniformly in x $(x \geq c)$. By adding the equations (37)

for $k = 0,1,2,\ldots$, (36) follows.

Put

$$r(x) = - \sum_{k=0}^{\infty} \ln G(x+kd)$$

$$s(x) = - d^{-1} \int_{x}^{\infty} [\ln G(y)]dy - \tfrac{1}{2} \ln G(x) .$$

Then r and s are nonnegative and from (36)

$$\sup_{x \geq c} |r(x) - s(x)| = O(d).$$

Now

$$\sup_{x \geq c} \left| \frac{F(x) - \exp(-s(x))}{F(x)} \right|$$

$$= \sup_{x \geq c} \left| \frac{\exp(-r(x)) - \exp(-s(x))}{\exp(-r(x))} \right|$$

$$= \sup_{x \geq c} \left| 1 - \exp(r(x) - s(x)) \right|$$

$$\leq \max\left\{ 1 - \exp\left(-\sup_{x \geq c} |r(x) - s(x)| \right), \exp\left(\sup_{x \geq c} |r(x) - s(x)| \right) - 1 \right\}$$

$$= O(d)$$

and this proves (33).

Finally (34) is proved by noting that

$$\pi_A = \int_a^\infty \frac{F(x)}{G(x)} g(x)\,dx = \int_a^c \frac{F(x)}{G(x)} g(x)\,dx + \int_c^\infty \frac{F(x)}{G(x)} g(x)\,dx$$

Since from (31) $0 \le \frac{F(x)}{G(x)} \le 1$, we can make the first integral as small we wish by taking c small enough. In the second integral we can replace $F(x)$ by the uniform asymptotic expression (33) . The proof of the theorem is completed by letting $c \to a$.

To analyse (34) further, we want to change the integration variable from x to

(38) $$z = - \int_x^\infty [\ln G(y)]dy \quad , \quad (x > a)$$

When x grows from a to ∞, z will decrease monotonically from

(39) $$h = - \int_a^\infty [\ln G(y)]dy$$

to zero. Let $b \in (a,\infty]$ be such that $G(x) = 1$ when $x \ge b$, $G(x) < 1$ when $x < b$ (eventually $b = \infty$). Then the upper integration limits ∞ in (33), (34), (38) and (39) may be replaced by b . When $a < x < b$, $\ln G(y) < 0$ and the transformation (38) is one-to-one. The inverted transformation is

(40) $$x = \psi(z) \quad , \quad 0 < z < h \quad ,$$

where ψ is differentiable. Straightforward change of variable

in (34) gives

$$\pi_A = - \int_0^h \frac{g(\psi(z))}{\sqrt{G(\psi(z))}\,\ln G(\psi(z))} e^{-d^{-1}z}\,dz(1+O(d)) \tag{41}$$

By combining this with a well known result from the theory of asymptotic expansions we get the following

Theorem 3.

Let $G(x)$ and $g(x)$ satisfy the conditions in theorem 2. Let ψ be given by (38) and (40). Suppose that we can find an asymptotic expansion

$$- \frac{g(\psi(z))}{G(\psi(z))\ \ln G(\psi(z))} \sim \sum_n a_n \Phi_n(z) \quad , \quad z \to 0 \tag{42}$$

and that each Φ_n has a Laplace transform

$$r_n(t) = \int_0^\infty \Phi_n(z) e^{-tz} dz \quad , \quad t \geq t_o$$

for some $t_o \geq 0$. Suppose that

$$e^{\varepsilon t} r_n(t) \to \infty \quad \text{as} \quad t \to \infty$$

for each $\varepsilon > 0$ and each n. Then

$$\pi_A \sim \sum_n a_n r_n(d^{-1}) \quad (1 + O(d)) \quad , \quad d \to 0 .$$

Proof.

The integral in (41) is the Laplace transform of

$$k(z) = \begin{cases} \dfrac{g(\psi(z))}{G(\psi(z)) \ln G(\psi(z))} & 0 < z < h \\ 0 & z \geq h \end{cases}$$

as a function of d^{-1}. Theorem 3 is therefore a simple consequence of (41) and the theorem on pp. 31-32 in Erdélyi (1956).

Example.

The last two theorems may easily be applied to the special distribution (8) discussed in section 3 and 4. However, here the results are of little value, as we already have closed expressions for π_A and $F(x)$. Instead we will consider the case where U_1 is normally distributed. As before we can take the mean to be zero. Furthermore, from (27), π_A is a function of $d\sigma^{-1}$ only. Therefore, without lack of generality, we can take the variance σ^2 of U_1 to be unity. Then

$$G(x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

and these functions satisfy the conditions of theorem 2 with $a = -\infty$.

From this theorem we have

$$(43) \qquad \pi_A \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{\Phi(x)}} e^{-d^{-1}\theta(x)} dx \, (1+O(d))$$

where

$$(44) \qquad \theta(x) = -\int_x^{\infty} \ln \Phi(y)\, dy = x \ln \Phi(x) + \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \frac{y\, e^{-\frac{y^2}{2}}}{\Phi(y)} dy$$

by partial integration.

Now $\theta(x) > 0$ for all x and $\theta(x) \to 0$ as $x \to \infty$. Therefore, as $d \to 0$, the important contribution from the integrand in (43) comes for large positive values of x. We know that $\Phi(x) \to 1$ very fast as $x \to \infty$. Therefore we may safely neglect the factors $\Phi(x)$ and $\Phi(y)$ in the denominators in (43) and (44). Also we may neglect the integral from $-\infty$ to 0 in (43) This gives

$$(45) \qquad \pi_A \sim \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{x^2}{2}} e^{-d^{-1}\theta_1(x)} dx$$

where

$$(46) \qquad \theta_1(x) = x \ln \Phi(x) + \frac{1}{\sqrt{2\pi}} \int_x^{\infty} y\, e^{-\frac{y^2}{2}} dy = x \ln \Phi(x) + \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

To simplify (45) further, it is natural to utilize the well-known asymptotic expansion

$$(47) \qquad \Phi(x) \sim 1 - \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{15}{x^7} \pm \cdots \right) \qquad (x \to \infty)$$

(Cf. e.g. Dettman (1965) p.451-52). This gives

$$(48) \qquad \theta_1(x) \sim \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(\frac{1}{x^2} - \frac{3}{x^4} + \frac{15}{x^6} \mp \cdots \right) \qquad (x \to \infty)$$

Note that the approximation obtained by only taking a finite number of terms in (48) is coarser than the one used in going from (43) to (45).

From now on the rest is only technicalities and is defered to the appendix. By change of variable from x to $u = \theta_1(x)$ in (45) and by means similar to those used in proving theorem 3, we find an asymptotic expansion for π_A as $d \to 0$. I choose to state the result in terms of

$$\bar{n} = \pi_A^{-1} = E(N_k) ,$$

the expected number of years between consecutive exchanges under stationary conditions. Also, we allow U_1 to have a general $N(\mu,\sigma^2)$-distribution, that is, we replace d by $d\,\sigma^{-1}$. To three terms the expansion is

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the expected number of years between consecutive exchanges under stationary conditions. Also, we allow U_1 to have a general $N(\mu, \sigma^2)$-distribution, that is, we replace d by $d\sigma^{-1}$. To three term's the expansion is

$$(49) \qquad \bar{n} = \frac{\sigma}{d\sqrt{\rho}}\left(1 + \frac{\ln\rho + a - 1}{\rho} + \frac{2(3\ln\rho + 3a - 5)^2 + \pi^2 + 22}{12\rho^2} + O\left(\frac{\ln\rho}{\rho}\right)^3\right),$$

where

$$\rho = \ln\left(\frac{\sigma}{d}\right)^2$$

$$a = \ln\sqrt{2\pi} - C \approx 0{,}3417$$

$$C \approx 0{,}5772 \qquad \text{(Euler-Mascheroni constant).}$$

In fig.7 and fig.8 the right-hand side of (49) with 1, 2 and 3 terms is drawn as a function of σd^{-1}. This is compared with a few exact values of $\bar{n}$ calculated directly from (27) and with Gades "semiempirical" formula.

$$(50) \qquad \bar{n} \approx 1 + 0{,}729 \left(\frac{\sigma}{d}\right)^{\frac{\sqrt{3}}{2}} .$$

6. Conclusions

The purpose of this paper was to analyze a concrete stochastical model. In general the model was determined by a constant d (the annual density decrement) and a distribution function $G(x)$ (the distribution of the density of coastal water in the physical interpretation). We succeeded in finding explicit expressions for many quantities characterizing the model : the stationary distribution of the density of the resident water (5), the stationary density-distribution of the

water entering the basin (15), the expected number of years between exchanges (19) etc. Mostly these results depended on the detailed form of $G(x)$ and were rather difficult to analyze. Some results were remarkably simple, however. For instance, when $G(x)$ is absolutely continuous with probability density $g(x)$, from (14) the probability density of the water entering the basin is

$$(51) \qquad t(x) = \pi_A^{-1}\, g(x) \prod_{j=1}^{\infty} G(x+jd)$$

The normalizing factor $\bar{n} = \pi_A^{-1}$ may in principle be found by requiring the area under $t(x)$ to be unity. This is to be compared to Gades procedure, finding $t(x)$ by solving numerically a complicated integral equation.

Also, it is interesting to note how thing simplify when $G(x)$ is the special distribution (8). This distribution resembles somewhat the normal one (see figs. 1 and 2), and may be useful in applications.

When $G(x)$ is the normal distribution, we do not find as simple answers. One main result here is the asymptotic expansion (49). To three terms it seems to give a better approximation to $\bar{n}$ than (50) when $\sigma d^{-1} \gtrsim 5$. Still more important, however, is that it indicates that the behavior of $\bar{n}$ as $s = \sigma d^{-1} \to \infty$ is not like s^{α} $(\alpha < 1)$, but rather like $s(\ln s)^{-\frac{1}{2}}$, that is, only a slight deflect from linearity.

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Appendix.

Here I want to fill in the gaps in the derivation of (49) from (45) and (48).

Put

$$u = \theta_1(x) \tag{52}$$

where $\theta_1(x)$ is given by (46). Then

$$\frac{du}{dx} = \ln \Phi(x) + \frac{x\, e^{-\frac{x^2}{2}}}{\sqrt{2\pi}\,\Phi(x)} - \frac{x\, e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

$$\sim -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} \mp \dots\right) + x\, e^{-\frac{x^2}{2}} . O\left(e^{-\frac{x^2}{2}}\right)$$

as $x \to \infty$ from (47). Therefore

$$\frac{dx}{du} \sim -\sqrt{2\pi}\, e^{\frac{x^2}{2}}\left(x + \frac{1}{x} - \frac{2}{x^3} + O(x^{-5})\right) \qquad (x \to \infty) . \tag{53}$$

Note that $u \downarrow 0$ as $x \to \infty$ and $\frac{du}{dx}$ is negative when $x \geqq 0$. When $x = 0$, $u = (2\pi)^{-\frac{1}{2}}$. Therefore from (45) and (53)

$$\pi_A \sim \int_0^{(2\pi)^{-\frac{1}{2}}} \left[x(u) + \frac{1}{x(u)} - \frac{2}{x(u)^3}\right] e^{-d^{-1}u}\, du . \tag{54}$$

The problem is to invert (52). From (48)

$$\ln u \sim -\frac{x^2}{2} - \ln x^2 - \ln\sqrt{2\pi} - \frac{3}{x^2} + \frac{6}{x^4} \qquad \begin{matrix}(x \to \infty)\\ (u \to 0)\end{matrix}$$

or

(55) $$x^2 \sim -2\ln u - 2\ln\sqrt{2\pi} - 2\ln x^2 - \frac{6}{x^2} + \frac{12}{x^4}$$

Put $w = -2\ln u$ and take the logarithm in (55). This gives

$$\ln x^2 \sim \ln w + \ln\left(1 - \frac{2\ln x^2}{w} - \frac{2\ln\sqrt{2\pi}}{w}\right)$$

$$\ln x^2 \sim \ln w - \frac{2\ln(\sqrt{2\pi}\,w)}{w}$$

Insert this in (55) to get

$$x \sim \sqrt{w} - \frac{\ln(\sqrt{2\pi}\,w)}{\sqrt{w}} - \frac{(\ln(\sqrt{2\pi}\,w))^2 - 4\ln(\sqrt{2\pi}\,w) + 6}{2\,w^{\frac{3}{2}}}$$

Inserted in (54) this gives

(56) $$\pi_A \sim \int_0^{(2\pi)^{-\frac{1}{2}}} \left\{ \sqrt{w} - \frac{\ln(\sqrt{2\pi}\,w) - 1}{\sqrt{w}} - \frac{(\ln(\sqrt{2\pi}\,w))^2 - 6\ln(\sqrt{2\pi}\,w) + 10}{2\,w^{\frac{3}{2}}} \right\} e^{-d^{-1}u}\,du$$

where

$$w = -2\ln u\,.$$

To perform the integrations in (56), we change variable from u to $t = d^{-1}u$. Then

$$w = -2\ln t - 2\ln d = \rho - 2\ln t$$

where

$$\rho = -2\ln d \to \infty \quad \text{as} \quad d \to 0 .$$

The factor in curly brackets in (56) has the following asymptotic expansion as $\rho \to \infty$

$$(57) \qquad \sqrt{\rho} - \frac{\ln(\sqrt{2\pi}\,\rho) + \ln t - 1}{\sqrt{\rho}} - \frac{(\ln(\sqrt{2\pi}\,\rho))^2 - 6\ln(\sqrt{2\pi}\,\rho) + 10 + (\ln t)^2 - 6\ln t + 2(\ln t)\ln(\sqrt{2\pi}\rho)}{2\rho^{\frac{3}{2}}}$$

This expansion is valid when

$$|\ln t| < \tfrac{1}{2}\rho \quad ,$$

that is

$$d < t < d^{-1} \qquad): \quad d^2 < u < 1$$

Now for each $\varepsilon > 0$ the factor in curly brackets in (56) is bounded in absolute value by $u^{-\varepsilon}$ as $u \to 0$. Therefore the part of the integral from 0 to d^2 is bounded by

$$\int_0^{d^2} u^{-\varepsilon} e^{-d^{-1}u} du \leq \int_0^{d^2} u^{-\varepsilon} du = \frac{d^{2-2\varepsilon}}{1-\varepsilon} = o\left(d^{\frac{3}{2}}\right)$$

when $\varepsilon < \frac{1}{4}$. Neglecting corrections of this order, we may multiply (57) by e^{-t}, integrate from 0 to ∞ to get

$$(58)\quad \pi_A \sim d\sqrt{\rho}\left(1 - \frac{\ln(\sqrt{2\pi\rho}) - C - 1}{\rho} - \frac{(\ln(\sqrt{2\pi\rho}))^2 - 6\ln(\sqrt{2\pi\rho}) + 10 + C^2 + \frac{\pi^2}{6} + 6C - 2C\ln(\sqrt{2\pi\rho})}{2\rho^2}\right)$$

Here we have again utilized the definite integrals which we used to find $E(U_1)$ and $E(U_1^2)$ in the first example in section 3. The order of the first neglected term may easily be checked. From (58) we find $\bar{n} = \pi_A^{-1}$ and replace d by $d\sigma^{-1}$ to obtain (49).

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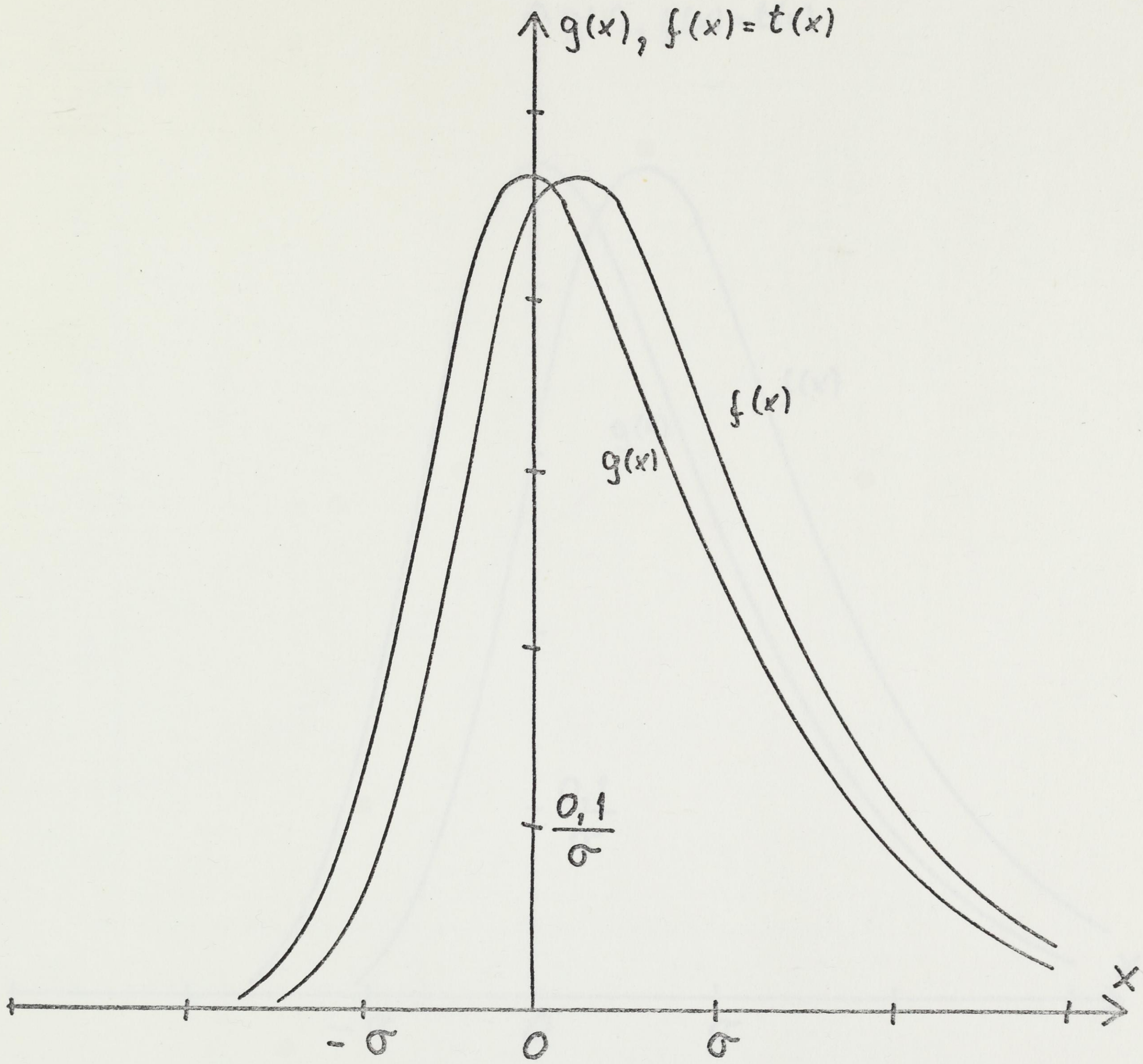

Fig. 1.

The stationary distribution of the density of coastal water (g) and resident water (f) when

$$G(x) = \exp(-e^{-bx})$$

$$\frac{\sigma}{d} = \frac{\pi}{bd\sqrt{6}} = 1$$

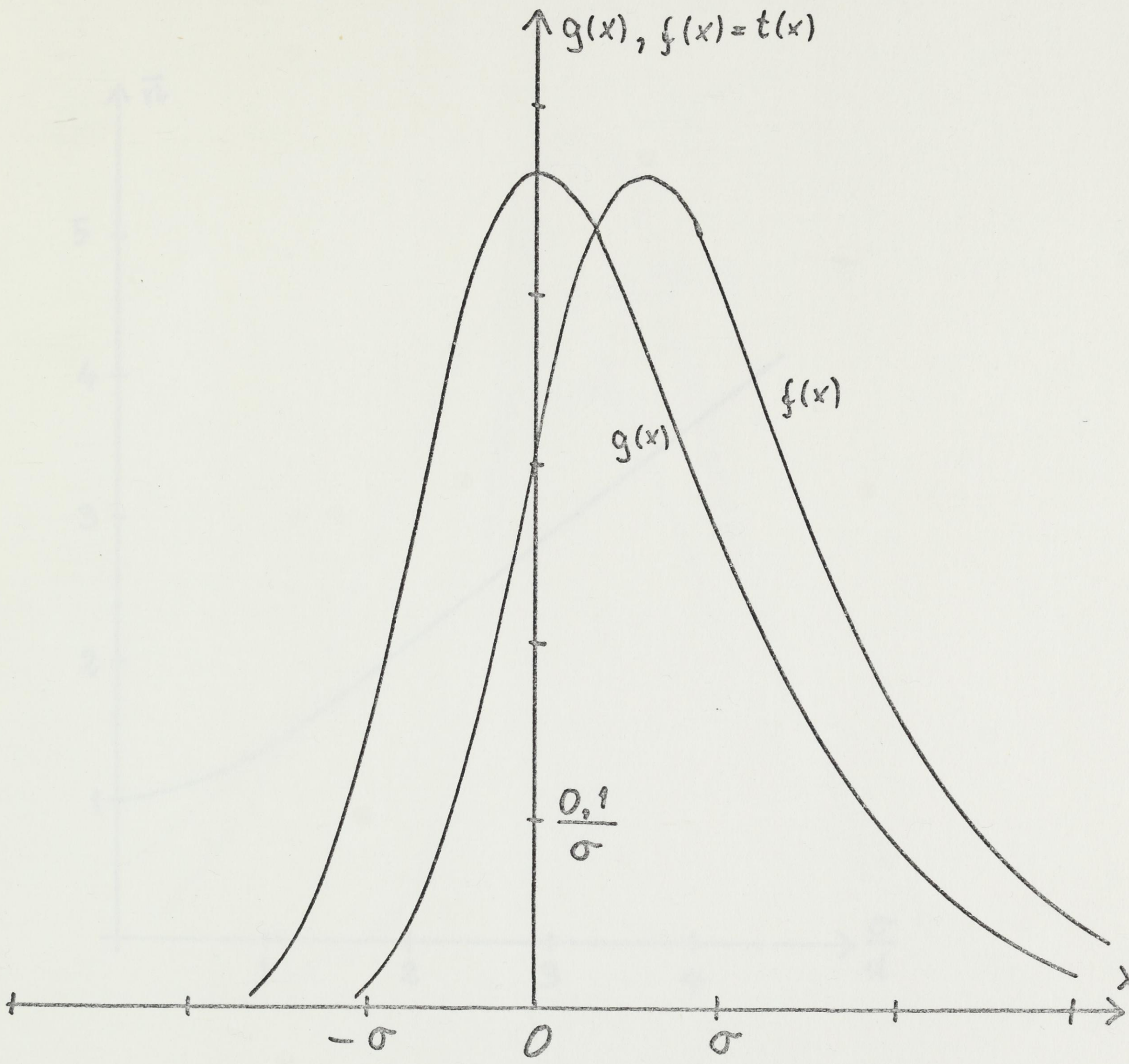

Fig. 2.

The stationary distribution of the density of coastal water (g) and resident water (f) when

$$G(x) = \exp(-e^{-bx})$$

$$\frac{\sigma}{d} = \frac{\pi}{bd\sqrt{6}} = 2$$

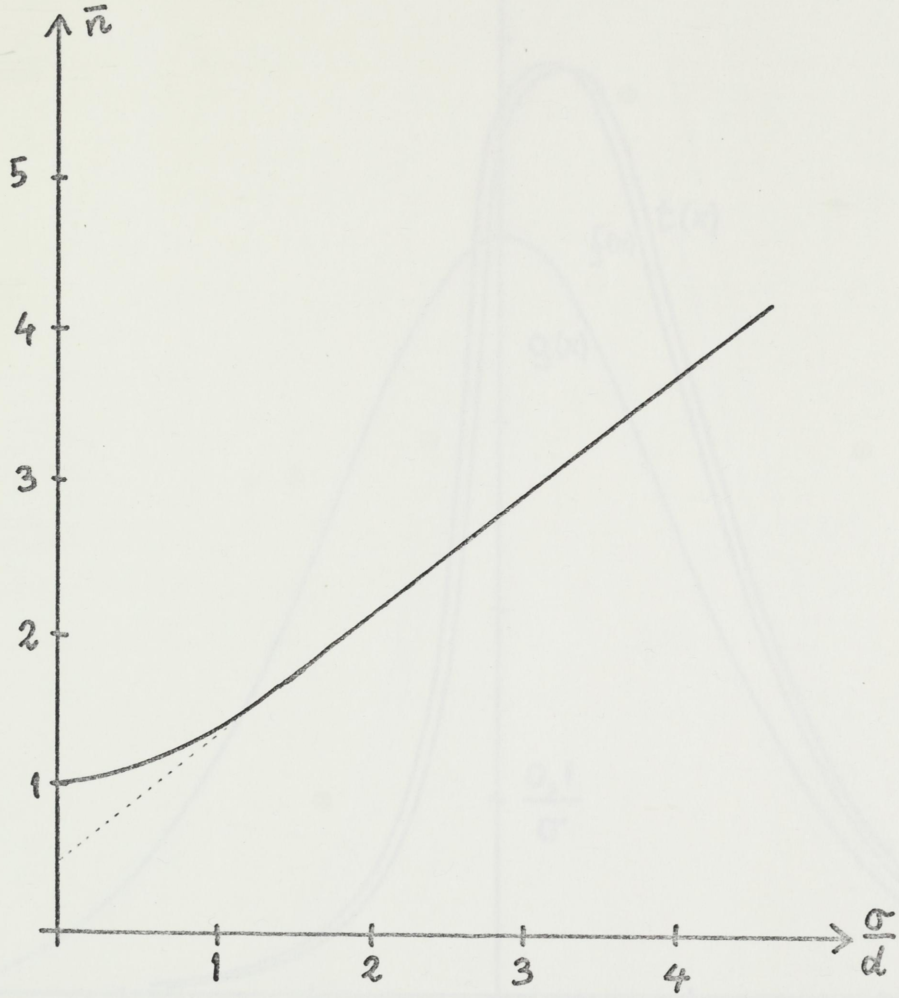

Fig. 3.

The (stationary) expected number of years between consecutive exchanges when

$$G(x) = \exp(-e^{-bx})$$

as a function of

$$\frac{\sigma}{d} = \frac{\pi}{bd\sqrt{6}}$$

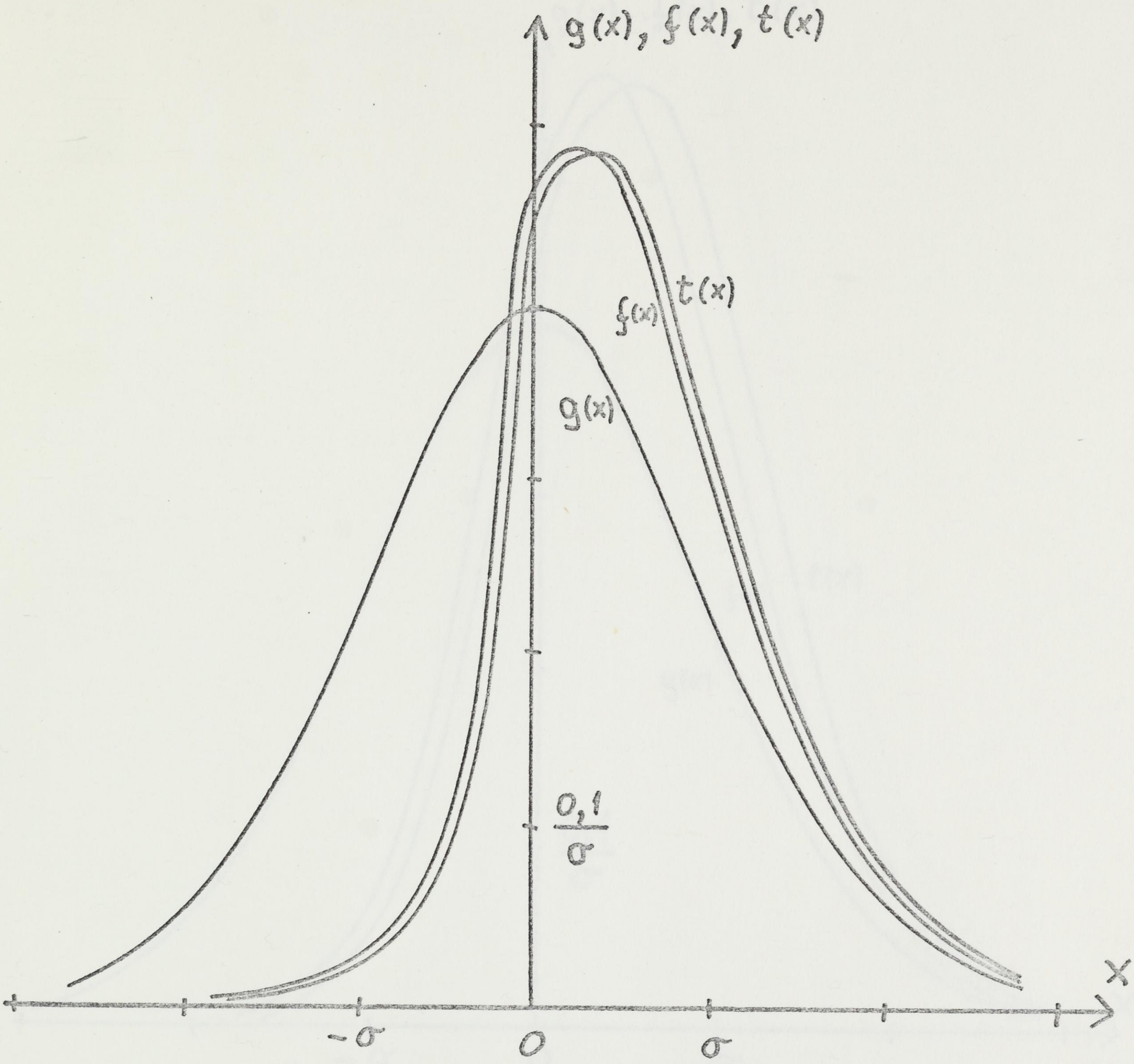

Fig. 4.

The stationary distribution of the density of coastal water (g), resident water (f) and exchanged water (t) when

$$U_1 \sim N(0,\sigma^2) \quad , \quad \frac{\sigma}{d} = 1$$

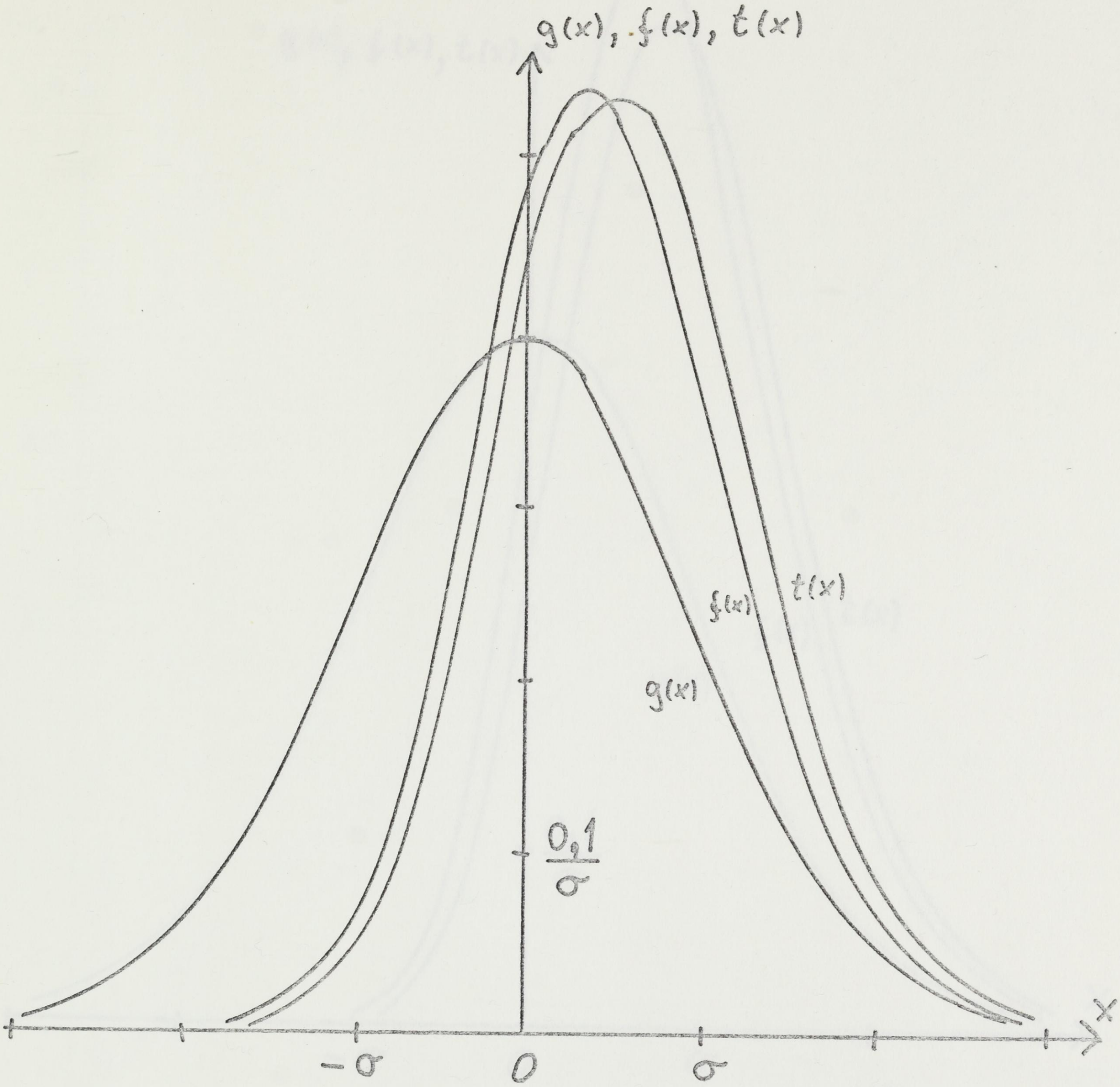

Fig. 5.

The stationary distribution of the density of coastal water (g), resident water (f) and exchanged water (t) when

$$U_1 \sim N(0,\sigma^2) \quad , \quad \frac{\sigma}{d} = 2$$

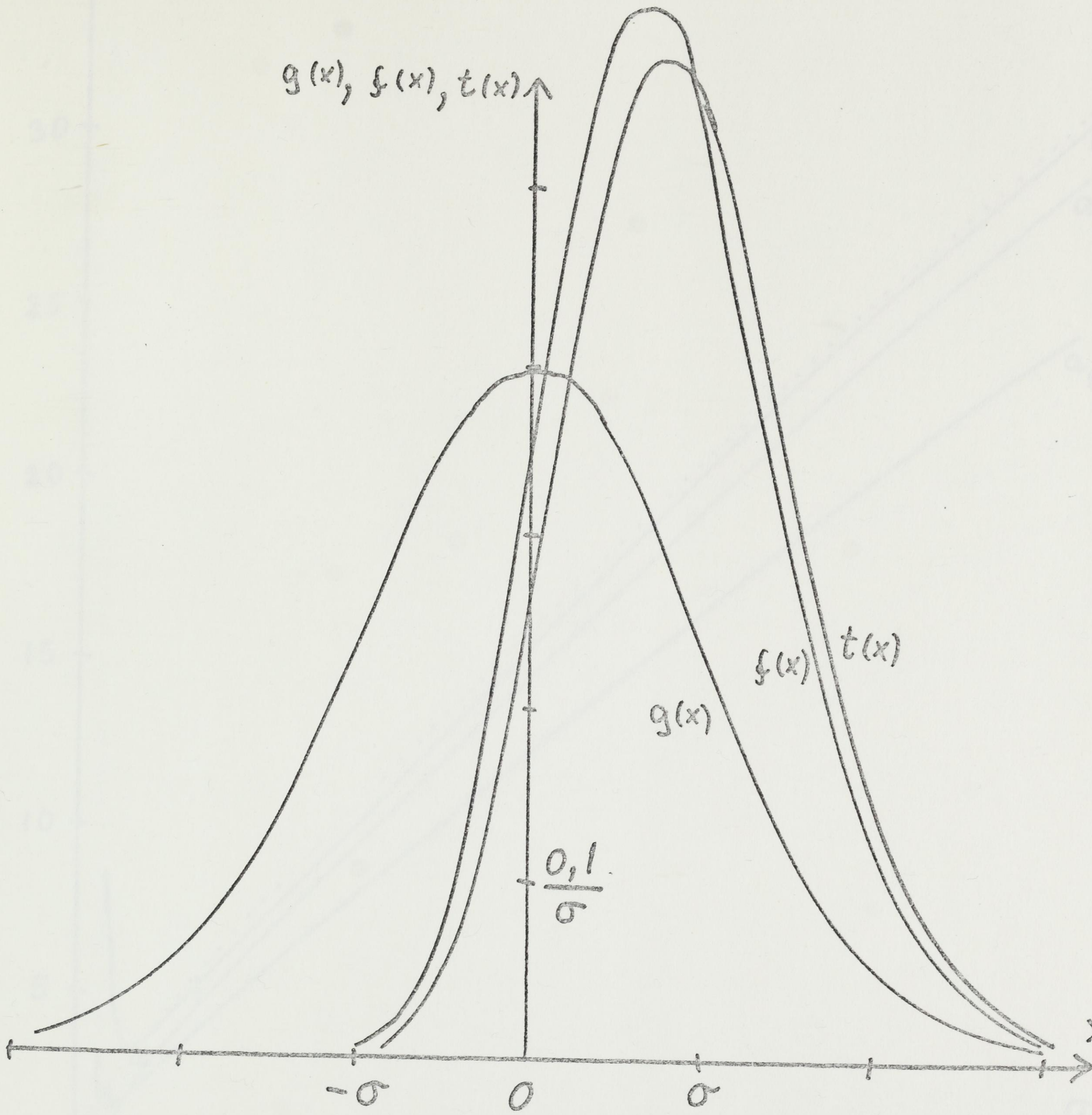

Fig. 6

The stationary distribution of the density of coastal water (g), resident water (f) and exchanged water (t) when

$$U_1 \sim N(0,\sigma^2) \quad , \quad \frac{\sigma}{d} = 4$$

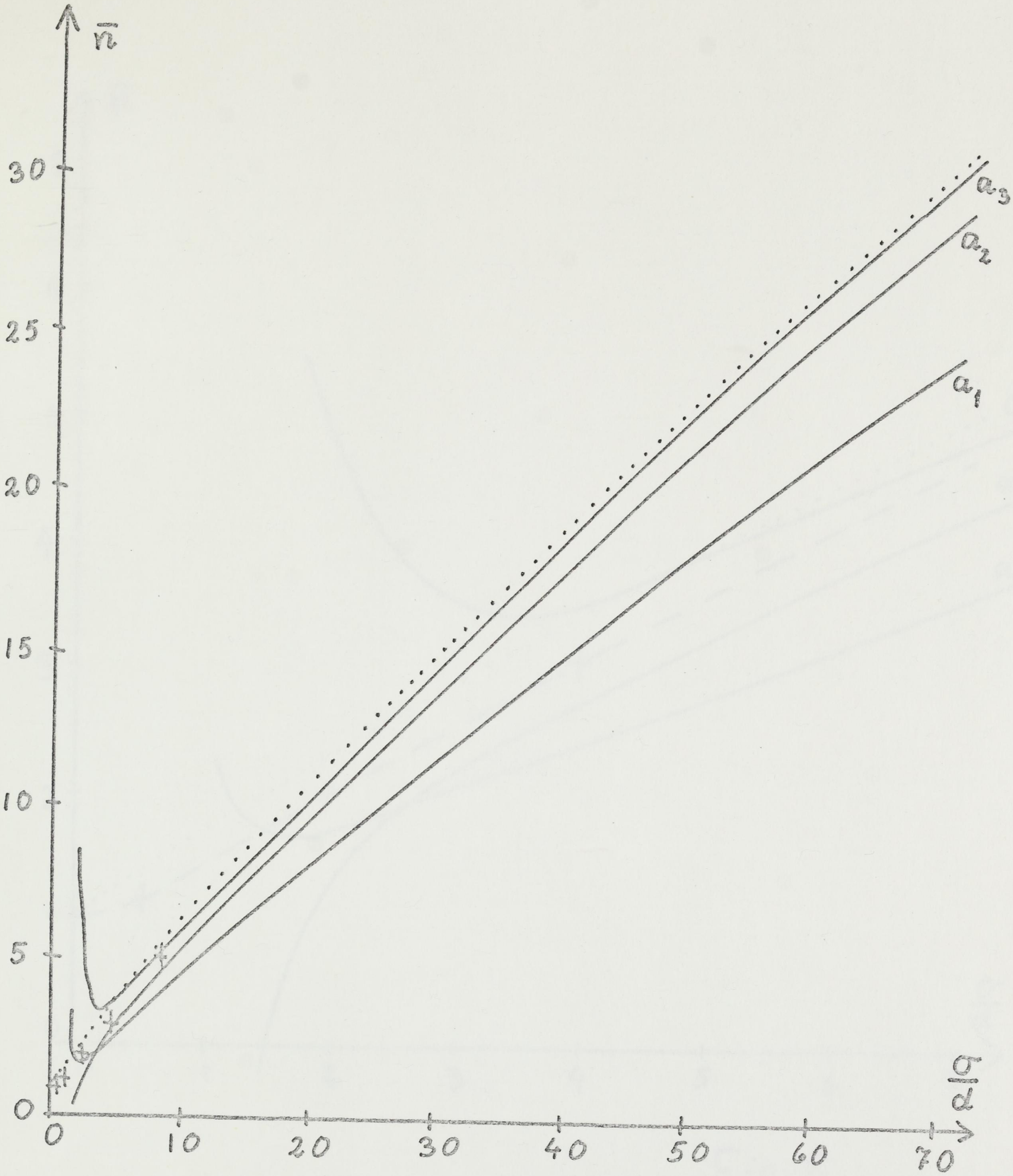

Fig. 7.

The (stationary) expected number of years between consecutive exchanges when $U_1 \sim N(0,\sigma^2)$ as a function of σ/d.

Crosses: Exact values, calculated from (27).

a_1, a_2, a_3: The asymptotic expression (49) with 1, 2 and 3 terms.

Dotted line: Gades approximation (50).

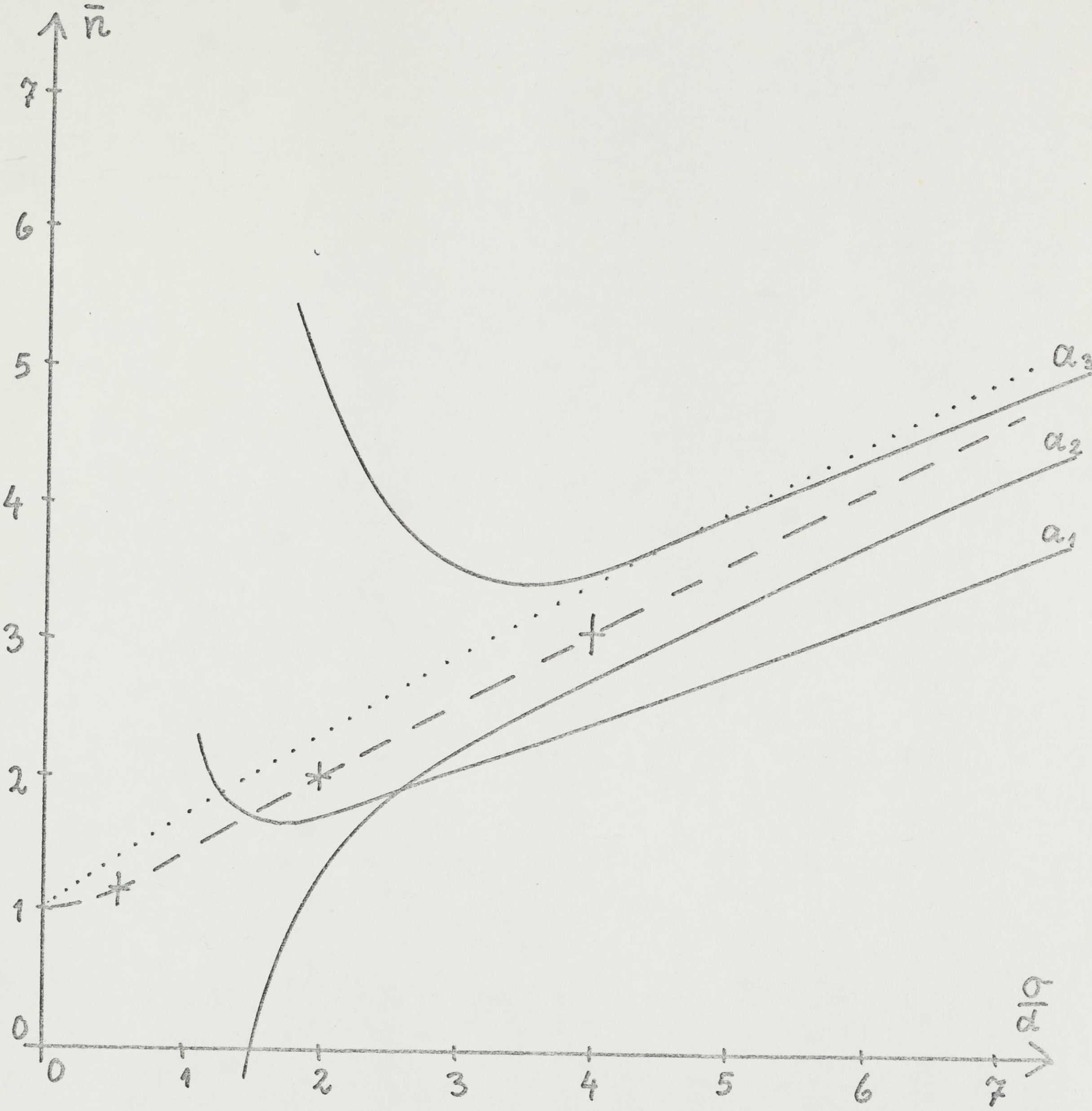

Fig. 8.

Enlarged portion of fig.7.

Dashed line: Drawn through exact, calculated values.