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On Acoustic Streaming in Magnetohydrodynamics

by

A. Kildal and S. Tjøtta

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UNIVERSITY OF BERGEN
Bergen, Norway

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Abstract.

Starting from a basic set of equations describing irreversible flow in magnetohydrodynamics, general equations governing the generation and transport of vorticity are obtained. These equations are solved by applying the method of successive approximations. Diffusion effects are accounted for in a first linearized approximation. To a second approximation it is then shown that the oscillatory motion within the boundary layers near a vibrating plate will generate steady circulations in the fluid. It is also shown that the absorption of magneto-acoustic wavebeam of finite amplitude leads to formation of a steady vortex motion.

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I. Introduction.

The many non-linear effects in hydrodynamics, arising from the non-linearity of the governing equations of motion, will also appear in magneto-hydrodynamics. However, they may be more or less modified by the electromagnetic fields, and in addition some new, purely hydromagnetic effects may occur.

In this work we shall study a non-linear effect, which in a non-conducting fluid is known as acoustic streaming. In acoustics there are two kinds of such streaming. One occurs near solid boundaries oscillating in a viscous fluid or near a boundary with which a standing sound wave is allowed to interact; the other kind of streaming is produced by the interaction of a free progressing soundbeam with the surrounding fluid. In the latter case, the linearized motion will be irrotational, whereas the first case is characterized by having a rotational first order solution in the motion of the viscous fluid near the solid boundaries. The acoustic streaming cannot be explained from the linearized equations of motion, as the oscillatory solutions of these are equal to zero when averaged in time. The explanations are found in the higher approximations of these equations.

The present work is a theoretical study of a similar effect in magneto-hydrodynamics. First we derive generell equations governing the generation and transport of vorticity, and give a short discussion of these equations in general. Thereafter two examples are worked out in detail: (i) the fluid motion near a flexible membrane vibrating, with finite amplitude in a uniform, external magnetic field, (ii) the flow generated

by the absorption of a magneto-acoustic wave of finite amplitude propagating perpendicularly to a uniform, external magnetic field. In the last case the linearized motion is irrotational while in the first case the linearized motion is rotational near the membrane. For more general cases in magneto-hydrodynamics the situation is more complicated as the propagation of $\nabla \times \underline{v}$ and $\nabla \cdot \underline{v}$ in the linearized approximation are coupled to each other.

II. Basic equations.

The starting point will be the following set of equations, describing irreversible flow in magnetohydrodynamics:

$$(1) \qquad \frac{D\rho}{Dt} + \rho\, \nabla\cdot\underline{v} = 0 \quad ,$$

$$(2) \quad \rho\frac{D\underline{v}}{Dt} = -\nabla(p-\kappa \text{ div } \underline{v}-\tfrac{1}{3}\eta \text{ div } \underline{v}-\underline{v}\nabla\eta)+\mu\nabla^2\underline{v}+\nabla\times(\underline{v}\times\nabla\mu)-\underline{v}\nabla^2\mu+\underline{j}\times \underline{B}$$

$$(3) \qquad \nabla\times\underline{B} = \mu\underline{j} \quad , \quad \nabla\times\underline{E} = -\frac{\partial \underline{B}}{\partial t} \quad , \quad \nabla\cdot\underline{B} = 0 \quad ,$$

$$(4) \qquad \underline{j} = \sigma(\underline{E} + \underline{v}\times\underline{B}) \quad .$$

Here $\underline{v}$ denotes the fluid velocity, ρ the density, p the pressure, $\underline{E}$ and $\underline{B}$ the electric and magnetic field respectively, and $\underline{j}$ the current density. Further, η and κ are the coefficients of shear and bulk viscosity respectively, and σ is the electrical conductivity. The magnetic permeability is taken to be that pertaining to free space, i.e. $\mu = \mu_o = 4.10^{-7}\,\frac{\text{henry}}{\text{meter}}$. The rationalized MKS units are used in all equations.

This set of equations is, of course, not complete before we have specified the heat-exchange equation, an equation of state, and the coefficients η and σ, but for the moment

these equations are not needed.

III. Vorticity equations.

Let us put

(5) $$\underline{B} = \underline{B}_o + \underline{b}$$

where $\underline{B}_o$ denotes a uniform, time-independent external field.

By applying well known vector identities, we can now readily derive differential equations governing the fluid vorticity and the magnetic vorticity.

From Eqs. (2) and (3) we obtain:

(6)
$$\left(\frac{\eta}{\rho}\nabla^2-\frac{\partial}{\partial t}\right)\nabla\times\underline{v}+\frac{\underline{B}_o}{\mu\rho}\cdot\nabla\nabla\times\underline{b} = \frac{\nabla\rho}{\rho}\times\frac{D\underline{v}}{Dt}-\nabla\times(\underline{v}\times\nabla\times\underline{v})+\frac{1}{\mu\rho}\nabla\times(\underline{b}\times(\nabla\times\underline{b}))$$
$$+\frac{1}{\rho}[\nabla^2\underline{v}\times\nabla\eta-\underline{v}\times\nabla^2\nabla\eta+\nabla^2(\underline{v}\times\nabla\eta)-\nabla\nabla\cdot(\underline{v}\times\nabla\eta)+\nabla^2\eta\ \nabla\times\underline{v}]$$

From Eqs. (3) and (4) we obtain:

(7)
$$\left(\frac{1}{\mu\sigma}\nabla^2-\frac{\partial}{\partial t}\right)\nabla\times\underline{b}+\underline{B}_o\cdot\nabla\nabla\times\underline{v}+\underline{B}_o\times\nabla\nabla\cdot\underline{v} = \nabla\times\nabla\times(\underline{b}\times\underline{v}) +$$
$$+\frac{\nabla\sigma}{\sigma}\times\frac{\partial\underline{b}}{\partial t}+\frac{\nabla\sigma}{\sigma}\times[-\underline{B}_o\cdot\nabla\underline{v}+(\nabla\cdot\underline{v})\underline{B}_o]+\frac{\nabla\sigma}{\sigma}\times\nabla\times(\underline{b}\times\underline{v})-\frac{1}{\mu\sigma}\nabla\times[\frac{\nabla\sigma}{\sigma}\times(\nabla\times\underline{b})]$$

Some conclutions can already be drawn from these equations. All terms on the right hand side of the equations (6) and (7) are of second or higher order in $\underline{v}$, $\nabla\rho$, $\nabla\eta$ and $\nabla\sigma$. Let us neglect these terms and take η/ρ and σ to be constants on the left-hand side of the equations, and assuming for a moment that all magnetic field is produced by the motion of the fluid, i.e., we put $\underline{B}_o = 0$. Then we obtain linearized diffusion equations, and the well known result that the propagation of linearized vorticity in $\underline{v}$ and $\underline{b}$, for the

~~case of no external field, depends only on the kinematic~~ shear viscosity and the electric conductivity, respectively. Further, no generation of linearized vorticity can take place in an unbounded space where there is no external magnetic field.

However, the non-linear equations (6) and (7) show that higher order vorticity can be produced in an unbounded space, and an external magnetic field will nearly always lead to generation of linearized vorticity. For an incompressible, electric conducting fluid, however, the vorticity in both $\underline{v}$ and $\underline{b}$ will, even with $\underline{B}_o = 0$, remain equal to zero in the linearized approximation, if both were zero at a certain time.

In the following we shall see examples of non-linear vorticity generation. The method of successive approximation will be applied in solving one basic set of non-linear equations and the derived vorticity equations. We set

$$\begin{aligned} \underline{v} &= \underline{0} + \underline{v}_1 + \underline{v}_2 + \ldots \\ \rho &= \rho_o + \rho_1 + \rho_2 + \ldots \\ \underline{B} &= \underline{B}_o + \underline{b}_1 + \underline{b}_2 + \ldots \end{aligned} \tag{8}$$

where the subscript indicates the orders of magnitude, the first order being the linearized solution. This we insert into our equation and study the solutions to second order. For all first order variables we assume a harmonic time-dependence with frequency ω. The square terms in the equations will then generally be terms with double frequency 2ω, as well as terms independent of time. The former ones will modify the main oscillation with frequency ω, while the latter ones generate steady vortex motion. We obtain general vorticity equations governing this steady second order motion

by substituting Eqs. (8) in Eqs. (6) and (7) and taking the time-average. Furthermore, we note that

(9) $$\nabla \cdot \langle \underline{b}_2 \rangle = 0 \quad ,$$

and from the equation of continuity we find

(10) $$\nabla \cdot \langle \underline{v}_2 \rangle = - \frac{1}{\rho_o} \nabla \cdot \langle \rho_1 \underline{v}_1 \rangle \quad ,$$

where $\langle \ \rangle$ indicates that the average in time is taken. When adjusted so as to satisfy appropriate boundary conditions, the solution of our time-averaged equations completely specifies the steady second order velocity, $\langle \underline{v}_2 \rangle$. It is of interest to note that this steady velocity field is not, in general, divergence-free when as here specified in Eulerian coordinates. In a Lagrangian description, however, it turns out that the generated steady velocity of second order will always be divergence-free. (See Westervelt 1953.)[1] Therefore, we obtain a more convenient set of equations by using the latter description. In experiments the steady velocity field is observed by tracing small particles following the flow, and the velocity observed will therefore be the Lagrangian one.

If we have computed $\langle \underline{v}_2 \rangle$ we must therefore make a transformation to a Lagrangian reference system before we compare our result with experiments. The transformation is given by

(11) $$\langle \underline{v}_2 \rangle_L = \langle \underline{v}_2 \rangle + \langle (\int \underline{v}_1 \, dt) \cdot \nabla \underline{v}_1 \rangle \quad .$$

In fluid mechanics a transformation of this kind has shown to bring the theory into good agreement with experiments.[2,3]

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1) Westervelt, P.J.: .... J. Acoust.Soc.Am. 25,60 (1953).
2) Skavlem, S., and S. Tjøtta: J. Acoust.Soc.Am. 27,26 (1955).
3) Raney, W.P., J.C. Corelli and P.J. Westervelt: J. Acoust.Soc. Am. 26,6(1954).

IV. Two dimensional streaming.

We now proceed to discuss in more detail the two dimensional streaming generated near an insulating plate vibrating with finite amplitude in an incrompessible MHD fluid. The motion is assumed to take place in the (x,y) plane, and for the magnetic field we put

(12) $$\underline{B} = B_o\, \underline{e}_2 + \underline{b} \quad ,$$

where $\underline{e}_2$ is the unit vector in the y-direction. The functions $\Psi = \Psi(x,y)$ and $\varphi = \varphi(x,y)$ are introduced by

(13) $$\underline{v} = \nabla \times (\Psi\, \underline{e}_3) \quad ,$$
$$\underline{b} = \nabla \times (\varphi\, \underline{e}_3) \quad ,$$

and thus

(14) $$\nabla \times \underline{v} = -\, \underline{e}_3 \nabla^2 \Psi \quad ,$$
$$\nabla \times \underline{b} = -\, \underline{e}_3 \nabla^2 \varphi \quad .$$

The vorticity equations (6) and (7) now take the forms

(15) $$\left(\frac{\eta}{\rho}\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\psi + \frac{B_o}{\mu\rho}\frac{\partial}{\partial y}\nabla^2\varphi = \underline{v}\cdot\nabla\nabla^2\Psi - \frac{1}{\mu\rho}\,\underline{b}\cdot\nabla\nabla^2\varphi \quad ,$$

(16) $$\left(\frac{1}{\mu\sigma}\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2\varphi + \underline{B}_o\frac{\partial}{\partial y}\nabla^2\Psi = [\nabla \times \nabla \times (\underline{v} \times \underline{b})]\; \underline{e}_3 \; ,$$

where we also have assumed all diffusion coefficients to be constants. Furthermore, we also need the following relation between $\underline{v}$ and $\underline{b}$ obtained from Eqs. (3) and (4):

(16b) $$\left(\frac{1}{\mu\sigma}\nabla^2 - \frac{\partial}{\partial t}\right)\underline{b} + \underline{B}_o\frac{\partial}{\partial y}\underline{v} = -\,\nabla \times (\underline{v} \times \underline{b}) \quad .$$

Eq. (16) is obtained by taking the curl of this equation.

1. First approximation.

As a first approximation, we take the solution of the linearized equations, which we obtain by neglecting the right-hand side of Eqs. (15) and (16). Furthermore, we assume for the first order solutions,

$$\Psi_1 = \sum_j C_j \cos kx\, e^{-q_j y + i\omega t} ,$$

(17)

$$\varphi_1 = \sum_j K_j \cos kx\, e^{-q_j y + i\omega t} ,$$

where $q_j (j = 1, 2, \ldots, 6)$ are solutions of the following characteristic equation in q:

(18) $$\left[(\gamma_1^2 - q^2)(\gamma_2^2 - q^2) - \gamma_3^2 q^2\right](q^2 - k^2) = 0 ,$$

Here

$$\gamma_1^2 = k^2 + i\omega/\nu , \quad \gamma_2^2 = k^2 + i\omega/\nu^* , \quad \gamma_3^2 = \frac{\alpha}{\beta}\frac{\omega}{\nu} , \quad \nu = \frac{\eta}{\rho}, \nu^* = \frac{1}{\mu\sigma}$$

(19)

$$\alpha = \frac{B_o^2}{\rho\mu\ \nu\ \omega} , \quad \beta = \frac{1}{\mu\sigma\nu} , \quad \varepsilon = \frac{k^2\nu}{\omega} , \quad \varepsilon^* = \frac{k^2\nu^*}{\omega}$$

k = wavenumber and ω = frequency with which the plate vibrates.

The coefficients K_j are related to C_j by Eq. (16b):

(20) $$K_j = -\frac{B_o\, q_j\, C_j}{\alpha_j} , \quad \alpha_j = i\omega - \frac{1}{\mu\sigma}(q_j^2 - k^2) .$$

Boundary conditions for the first approximation.

The boundary conditions for this first approximation are taken to be

(21) $$v_{1x} = 0 , \quad v_{1y} = v_o \sin kx\, e^{i\omega t} \quad \text{for } y = 0 ,$$

v_o being the velocity amplitude of the plate, and further that all effects from the boundary shall vanish for $y \to \infty$. For the electromagnetic field we would in general have the conditions that $\underline{E}$ and $\frac{B}{\mu}$ should both have continuous tangential components at the boundary, and in addition that the normal component of B also should be continuous. We shall, however, here assume $\mu = \mu_o$ in the plate as well, a consequence of which is that B must be continuous at the boundary. Extension to other cases can readily be made. For $y \leq 0$ we then have

$$(22) \quad \underline{B} = B_o \underline{e}_2 + b_o \underline{e}_1 \cos kx \, e^{ky+i\omega t} + b_o \underline{e}_2 \sin kx \, e^{ky+i\omega t} \quad ,$$

which satisfies $\nabla \cdot \underline{B} = 0$ and $\nabla \times \underline{B} = 0$, and gives $\underline{b}_1 \to 0$ for $y \to -\infty$.

Not all solutions in q are relevant, as we require the effect from the boundary to vanish for $y \to \infty$. From Eq. (18) we can besides the root $q_3 = k$ only use the two roots, say q_1 and q_2, having a positive real part.

The boundary conditions thus impose the following relations between the constants C_j and K_j:

$$(23) \quad \begin{aligned} \sum_{j=1}^{3} q_j C_j &= 0 \\ \Sigma C_j &= -\frac{v_o}{k} \\ \Sigma q_j K_j &= b_o \\ \Sigma K_j &= -\frac{b_o}{k} \end{aligned}$$

Assuming v_o, B_o and k fixed, Eqs. (20) and (23) determine b_o, C_j and K_j.

Before we proceed to discuss some limiting cases, we note the following expressions for the fluid and magnetic vorticity of first order

$$(24)\quad \nabla^2\psi_1 = \left[C_2(q_2{}^2 - k^2)e^{-q_1 y} + C_3(q_3{}^2 - k^2)e^{-q_2 y}\right]\cos kx\, e^{i\omega t} \quad ,$$

$$(25)\quad \nabla^2\varphi_1 = \left[K_2(q_2{}^2 - k^2)e^{-q_1 y} + K_2(q_3{}^2 - k^2)e^{-q_2 y}\right]\cos kx\, e^{i\omega t} \quad ,$$

Far outside the boundary layers with thicknesses $(\mathrm{Re}(q_1))^{-1}$ and $(\mathrm{Re}(q_2))^{-1}$ we have

$$\nabla \times \underline{v}_1 = 0 \quad ,$$
$$\nabla \times \underline{b}_1 = 0 \quad ,$$

and therefore no coupling between the fluid motion and the electromagnetic field. Near the boundary, however, the motion is greatly changed due to viscosity, and here the motion will also be affected by magnetic field. Requirements which must be satisfied in order to have typical boundary layer effects are that $\mathrm{Re}(q_1) \gg k$ and $\mathrm{Re}(q_2) \gg k$.

<u>Limiting cases.</u>

Since the boundary layers play a dominant role in the discussion of acoustic streaming, we shall discuss the roots q_1, q_2 versus k for different limiting cases of the parmeters α, β and ε. Other limiting cases have already been discussed by Kildal in a previous paper(4). For cases with $k = 0$, Hide and Roberts(5) have given a throughout discussion.

We commence our discussion with the case

1) $\beta \gg 1$, $\alpha \ll \beta$, $\varepsilon^* \ll 1$.

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4) Kildal, A.: Z. Physik 172, 49 (1963).

5) Hide, R. and P. H. Roberts: Rev. Mod. Phys. 32, 799 (1960).

$$q_2 = \sqrt{\frac{\omega}{2\nu}}\left[1 + i + \frac{1}{2}(1 - i)(\varepsilon + \frac{\alpha}{\beta})\right]$$

$$q_3 = \sqrt{\frac{\omega}{2\nu^*}}\left[1 + i + \frac{1}{2}(1 - i)(\varepsilon^* - \frac{\alpha}{\beta})\right]$$

This is a case with low conductivity and weak magnetic field. Two waves are generated near the plate, one with the well known penetration depth $\delta_1 = \sqrt{\frac{2\nu}{\omega}}$ in hydrodynamics and one with the analogue magnetic penetration depth $\delta_2 = \delta_1\sqrt{\beta}$. Both skin depths are small compared to the wavelength. It is only inside these non-steady boundary layers that the first order fluid motion is influenced by viscous and magnetic forces.

2. $\beta << 1, \quad \alpha << 1, \quad \varepsilon\beta << 1.$

$$q_1 = \sqrt{\frac{\omega}{2\nu^*}}\ (1 + i + \text{terms of higher orders}) \quad ,$$

$$q_2 = \sqrt{\frac{\omega}{2\nu}}\ (1 + i + \quad " \quad " \quad " \quad) \quad .$$

This case of high conductivity and weak magnetic field leads to results in quantitative agreement with 1.

3. $\beta >> 1, \quad \frac{\alpha}{\beta} >> 1. \quad \varepsilon\beta << 1.$

$$q_1 = \sqrt{\frac{\omega}{\nu}\frac{\alpha}{\beta}}\ (1 + i\ \frac{\beta}{2\alpha}) \quad ,$$

(26)

$$q_2 = \sqrt{\frac{\omega}{2\alpha}}\ (i + \frac{1}{2}\beta\varepsilon + \frac{1}{2}\frac{\beta}{\alpha}) \quad .$$

This is a case with a strong magnetic field. One Alfvén wave is generated at the plate. The penetration depth is

$$\delta_2 = \left(\frac{\nu\alpha}{\omega}\right)^{1/2} (\beta\varepsilon + \frac{1}{2}\frac{\beta}{\alpha})^{-1}$$

To calculate the second approximation in this case we need to know q_1 and q_2 to a rather high approximation in ε, α and β. Numerical results show one circulation for the secondary flow.

2. approximation.

We now pass to the second approximation and find that the steady flow field is governed by the following set of, equations:

$$\nabla p = \underline{P} + \frac{1}{\mu}(\nabla \times \langle \underline{b}_2 \rangle) \times \underline{B}_o + \eta \, \nabla^2 \langle \underline{v}_2 \rangle ,$$

$$0 = \underline{Q} + B_o \frac{\partial}{\partial y} \langle \underline{v}_2 \rangle + \frac{1}{\mu\sigma} \nabla^2 \langle \underline{b}_2 \rangle , \tag{27}$$

$$\nabla \cdot \langle \underline{v}_2 \rangle = 0 \quad ,$$

$$\nabla \cdot \langle \underline{b}_2 \rangle = 0 \quad ,$$

where now all variables refer to the time-averaged second order quantities, and P and Q are the time-average of the square terms in first order quantities, being determined when ψ_1 and φ_1 of Eq. (17) are known. Thus P and Q are the force terms that drive the streaming. By timeaveraging of the vorticity equations (15) and (16) we find that the source terms in steady vorticity equations are given by $\nabla \times \underline{P}$ and $\nabla \times \underline{Q}$, which vanish outside the two non-steady boundary layers. The source terms are proportional to $\sin 2k\, x$. Having assumed $\nabla \cdot \langle \underline{v}_2 \rangle = 0$, we therefore obtain as the result of the second approximation, a steady motion consisting of a series of vortices periodic with respect to x with period π/k. We have also computed the transform given in Eq. (11). Near the plate the difference between the Lagrangian velocity $\langle \underline{v}_2 \rangle_L$ and the Eulerian one $\langle \underline{v}_2 \rangle$ is considerable.

In the numerical calculations the velocity is made non-

To calculate the second approximation in this case we need to know q_1 and q_2 to a rather high approximation in ϵ, α and δ. Numerical results show one circulation for the secondary flow.

2. approximation.

We now pass to the second approximation and find that the steady flow field is governed by the following set of equations:

$$\nabla p = \underline{P} + \frac{1}{\mu}(\nabla \times \langle \underline{h}_2 \rangle) \times \underline{H}_0 + \eta \nabla^2 \langle \underline{v}_2 \rangle ,$$

$$0 = \underline{Q} + H_0 \frac{\partial}{\partial y} \langle \underline{v}_2 \rangle + \frac{1}{\mu\sigma} \nabla^2 \langle \underline{h}_2 \rangle ,$$

$$\nabla \cdot \langle \underline{v}_2 \rangle = 0 ,$$

$$\nabla \cdot \langle \underline{h}_2 \rangle = 0 , \qquad (27)$$

where now all variables refer to the time-averaged second order quantities, and P and Q are the time-average of the square terms in first order quantities, being determined when Ψ_1 and Φ_1 of Eq. (17) are known. Thus P and Q are the force terms that drive the streaming. By timeaveraging of the vorticity equations (15) and (16) we find that the source terms in steady vorticity equations are given by $\nabla \times \underline{P}$ and $\nabla \times \underline{Q}$, which vanish outside the two non-steady boundary layers. The source terms are proportional to $\sin 2k\,x$. Having assumed $\nabla \cdot \langle \underline{v}_2 \rangle = 0$, we therefore obtain as the result of the second approximation, a steady motion consisting of a series of vortices periodic with respect to x with period π/k. We have also computed the translation given in Eq. (11). Near the plate the difference between the Lagrangian velocity $\langle \underline{v} \rangle_L$ and the Eulerian one $\langle \underline{v}_2 \rangle$ is considerable.

In the numerical calculations the velocity is made non-dimensional by division by $\frac{v_o^2}{8\omega}\sqrt{\frac{\omega}{\nu}}$ and the x-dependence is supressed. The physical distance y is made non-dimensional by division by the non-steady boundary layer thickness $\delta_{AC} = \sqrt{\frac{\nu}{\omega}}$.

In Fig. 1 we show $V_y = \langle v_{2y}\rangle_L \Big/ \frac{v_o^2}{8\omega}\sqrt{\frac{\omega}{\nu}}$ versus $z = {}^{y}/\delta_{AC}$ for different values of α defined in Eq. (19).

We see from the curves that the steady boundary layer thickness δ_{DC} defined as the value y where $\langle v_{2y}\rangle_L$ changes direction varies with α. To see the variation in detail, a curve (Fig. 2) showing ${}^{\delta_{DC}}/\delta_{AC}$ versus α has been drawn.

We also give (in Fig. 3) a set of curves showing V_y vs. z for different values of ε from which we learn that δ_{DC} depends on ε. A curve giving ${}^{\delta_{DC}}/\delta_{AC}$ vs. $\sqrt{\varepsilon}$ has been given. For cylinders oscillating in a non-magnetic fluid, universal curves showing ${}^{\delta_{DC}}/\delta_{AC}$ vs. cylinder radius are well known.[2,3,7,8]

If V_y does not change direction, only one circulation is formed. This always takes place when ε or ${}^{\alpha}/\beta$ becomes sufficiently large.

Other cases are discussed in a paper by one of the authors (Kildal[9]).

5. Effect of compressibility, expanding waves.

In the case of a vibrating plate, we find that the non-steady boundary layers decrease in thickness with increasing frequency as $\omega^{-\frac{1}{2}}$. Furthermore, to a second approximation we

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7) Holtsmark, J., I. Johnsen, T. Sikkeland and S. Skavlem: J. Acoust. Soc. Am. 26,26 (1954).
8) Olsen, T.: J. Acoust. Soc. Am. 28, L (1956).
9) Kildal, A.: Årbok for Universitetet i Bergen. Mat.-Naturv. Serie 1964, No. 8.

find that the generating steady volume forces $\underline{P}$ and $\underline{Q}$ are non-zero only within these non-steady boundary layers. In the limit of very high frequency we therefore find that the steady flow of second order is generated in a vanishing thin layer near the plate. the inner flow system becomes very thin and the flow velocity decreases with increasing ω, the other parameters being constants.

So far we have assumed an incompressible MHD fluid. We have also studied the effect of compressibility on the generation of second order streaming in general. In the case of a vibrating plate discussed in the preceding section, this effect will only slightly modify the general flow picture for low frequencies. Inclusion of compressibility, however, leads to generation of expanding waves, and at frequencies in the ultrasonic range, it turns out that absorption of these waves may lead to a steady flow of second order.

We shall here study an example of a MHD wave propagating perpendicularly to a uniform external magnetic field. Linearized, we then have a longitudinal magneto-acoustic wave with phase velocity

$$a = (a_1{}^2 + a_s{}^2)^{1/2}$$

where $a_1 = \left(\frac{B_o{}^2}{\mu\rho}\right)^{1/2}$ is the Alfven velocity and $a_s = (\gamma\frac{p}{\rho})^{1/2}$ is the sound velocity. Since we take into account diffusion effects, the wave will be attenuated, and we shall see how this leads to generation of higher order vorticity.

In the first, linearized approximation we put

$$\nabla \times \underline{v}_1 = 0$$

since we limit ourselves to the case of propagation perpendicularly to the magnetic field. In this case $\nabla \times \underline{v}_1$ satisfies the ordinary diffusion equation and therefore it cannot propagate out of the thin boundary layer near the wavesource. On the other hand $\nabla \times \underline{b}_1 \neq 0$, as the propagation of $\nabla \times b_1$ is coupled to that of $\nabla \cdot \underline{v}_1$; only in an incompressible MHD fluid will $\nabla \times b_1$ diffuse in direction perpendicularly to B_o.

In other directions of propagation the situation is more complex as the propagation of $\nabla \times \underline{v}_1$ is coupled to that of $\nabla \cdot \underline{v}_1$ Therefore both the effect of $\nabla \times \underline{v}_1$ and $\nabla \cdot \underline{v}$, have to be accounted for in the square terms determining the steady forces driving the steady flow. Only the component of $\nabla \times \underline{v}_1$ along the magnetic lines of forces propagates - along the lines of force - uncoupled to $\nabla \cdot \underline{v}_1$. This can be seen from the general vorticity equations, in which $\nabla \cdot \underline{v}_1$ enters only through the term $\underline{B}_o \times \nabla\nabla \cdot \underline{v}_1$, having no component along $\underline{B}_o$.

For an incompressible MHD fluid, on the other hand, this term disappears, and therefore both $\nabla \times \underline{v}_1$ and $\nabla \times \underline{b}_1$ then propagate in one dimension along the lines of force, being attenuated only because of finite electric conductivity, and non-zero viscosity. In this case all non-linear terms in the general vorticity equations are zero, and thus there will be no generation of second order vorticity, either in $\underline{v}$ or $\underline{b}$ for this case of $\nabla \cdot \underline{v} = 0$. In an incompressible fluid higher order steady vorticity can only be generated in the boundary layers near solid bodies.

We now return to our example with a longitudinal wave. For the moment we neglect the effect of a gradient in the since we limit ourselves to the case of propagation perpendicularly to the magnetic field. In this case $\nabla \times \underline{v}_1$ satisfies the ordinary diffusion equation and therefore it cannot propagate out of the thin boundary layer near the wavesource. On the other hand $\nabla \times \underline{h}_1 \neq 0$, as the propagation of $\nabla \times h_1$ is coupled to that of $\nabla \cdot \underline{v}_1$; only in an incompressible MHD fluid will $\nabla \times h_1$ diffuse in direction perpendicularly to B_0.

In other directions of propagation the situation is more complex as the propagation of $\nabla \times \underline{v}_1$ is coupled to that of $\nabla \cdot \underline{v}_1$. Therefore both the effect of $\nabla \times \underline{v}_1$ and $\nabla \cdot \underline{v}_1$ have to be accounted for in the square terms determining the steady forces driving the steady flow. Only the component of $\nabla \times v_1$ along the magnetic lines of forces propagates - along the lines of force - uncoupled to $\nabla \cdot \underline{v}_1$. This can be seen from the general vorticity equations, in which $\nabla \cdot \underline{v}_1$ enters only through the term $\underline{B}_0 \times \nabla \nabla \cdot \underline{v}_1$, having no component along $\underline{B}_0$.

For an incompressible MHD fluid, on the other hand, this term disappears, and therefore both $\nabla \times \underline{v}_1$ and $\nabla \times \underline{h}_1$ then propagate in one dimension along the lines of force, being attenuated only because of finite electric conductivity, and non-zero viscosity. In this case all non-linear terms in the general vorticity equations are zero, and thus there will be no generation of second order vorticity, either in $\underline{v}$ or $\underline{h}$ for this case of $\nabla \cdot \underline{v} = 0$. In an incompressible fluid higher order steady vorticity can only be generated in the boundary layers near solid bodies.

We now return to our example with a longitudinal wave. For the moment we neglect the effect of a gradient in the coefficient of shear-viscosity and shall return to the validity of this approximation later. The vorticity equation in $\underline{v}$ now takes the form

$$(28) \qquad \frac{\eta}{\rho_o} \nabla^2 \nabla \times \underline{v}_2 - \frac{\partial}{\partial t} \nabla \times \underline{v}_2 = \frac{\nabla \rho_1}{\rho_o} \times \frac{\partial \underline{v}_1}{\partial t}$$

to the second order of approximation.

If we substitute the isentropic values for ρ_1 and $\underline{v}_1$ in equation (1), the right-hand side becomes zero, and there will be no generation of second order vorticity. Since we here, however, have taken into account diffusion effects in the linearized approximation, the wave becomes attenuated and it turns out that the right-hand side of Eq. (28) is proportional to the attenuation coefficient. Furthermore, this side of the equation becomes independent of time, and therefore only steady vorticity can be generated to this order of approximation.

We get

$$(29) \qquad \frac{\eta}{\rho_o} \nabla^2 \nabla \times \underline{v}_2 - \frac{\partial}{\partial t} \nabla \times \underline{v}_2 = - \frac{2D}{\rho_o} \nabla \rho_1 \times \nabla \frac{\partial \rho_1}{\partial t}$$

where D is the diffusivity and ρ_1 the isentropic density fluctuation.

If we introduce the intensity vector

$I = \langle p \, \underline{v} \rangle = \frac{a^4}{\rho_o \omega} \, \mathrm{Im} \, (\tfrac{1}{2} \rho_1 \nabla \rho_1)$ we find:

$$(30) \qquad \nabla^2 \nabla \times \underline{v}_2 = - \frac{2\alpha}{\eta a} \nabla \times \underline{I} \quad ,$$

where $\alpha = \frac{D\omega^2}{a^3}$ is the absorption coefficient. Thus the effect increases strongly with an increase in the frequency, whereas the boundary layer effect discussed previously decreased with increasing values of the frequency. Furthermore, we find this effect only if $\nabla \times \underline{I} \neq 0$ somewhere. For a plane wavebeam this will be the case at the edge of the beam. The volume force

driving the flow can for this case be traced back to the gradient in the radiation pressure. This we shall not discuss further here. Instead we refer the reader to a work by Piercy and Lamb (1954)[10], in which the concept of radiation pressure was introduced to explain the steady streaming observed in ultrasonics.

So far we have neglected the effect from a gradient in the coefficient of shear-viscosity. This coefficient will in general depend on the temperature and therefore oscillate with the excess temperature in the wavefield. For a well defined wavebeam, however, it turns out that the contribution to the source term in the vorticity equation in $\underline{v}$ is negligible small. For further discussion of this and a general discussion of Eq. (30) in fluid mechanics, we refer to previous works by one of the authors.[11,12]. Here one also finds a bibliography on streaming problems in general.

We further have from the equation of continuity

$$\nabla \cdot \langle \underline{v}_2 \rangle = - \frac{1}{\rho_o} \langle \rho_1 \, \underline{v}_1 \rangle = \frac{2D}{\rho_o^2} \langle \nabla \rho_1 \rangle^2$$

This is very small and to our order of approximation we can put

(31) $$\nabla \cdot \langle \underline{v}_2 \rangle = 0 \quad ,$$

which together with Eq. (3) and given boundary conditions determine the flow field. A transformation into a Lagrangian reference system is not needed in the case of ultrasonic frequencies as the transforms become very small for high frequencies.

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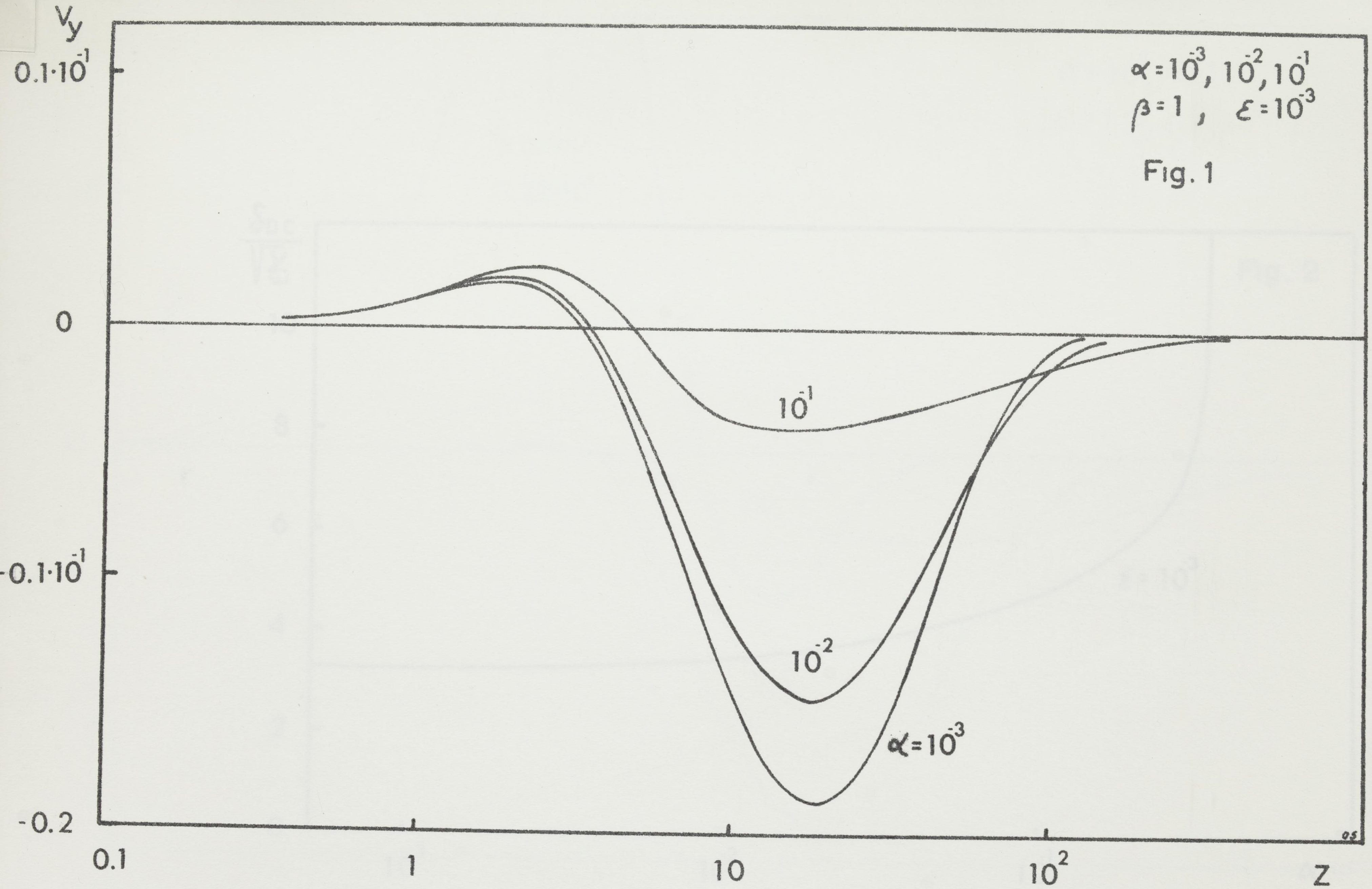
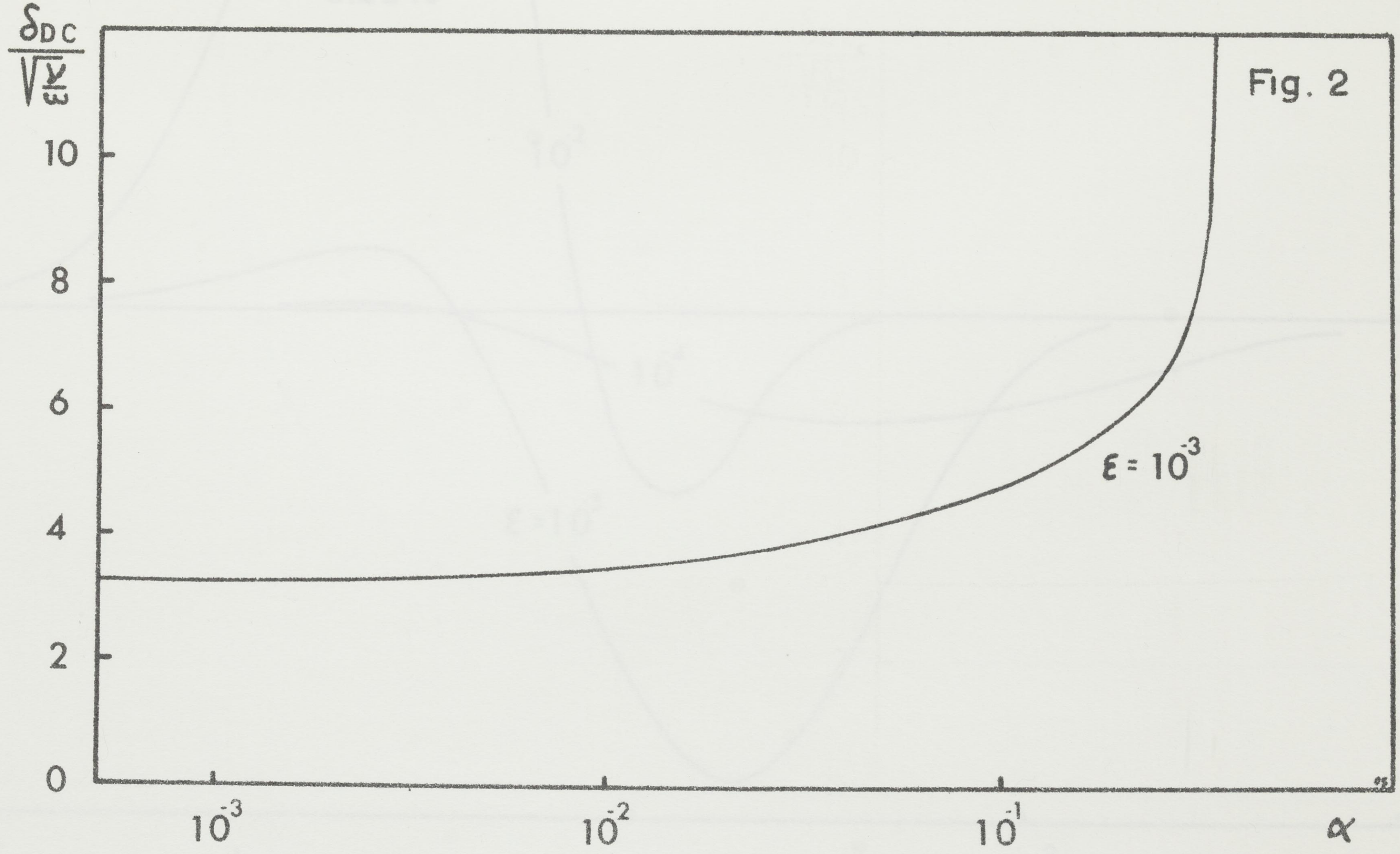
10) Piercy, J.E. and Lamb, J.: Proc. Roy. Soc. A. 226, 43 (1954).

11) Tjøtta, S.: On some non-linear effects in sound fields with special emphasis on the generation of vorticity and the formation of streaming patterns I and II. Archiv. for Mathem. og Naturv. B LIV Nr. 102, Oslo.

12) Tjøtta, S.: On Quartz-Wind, Universitetet i Bergen, Årbok 1962, No. 7.

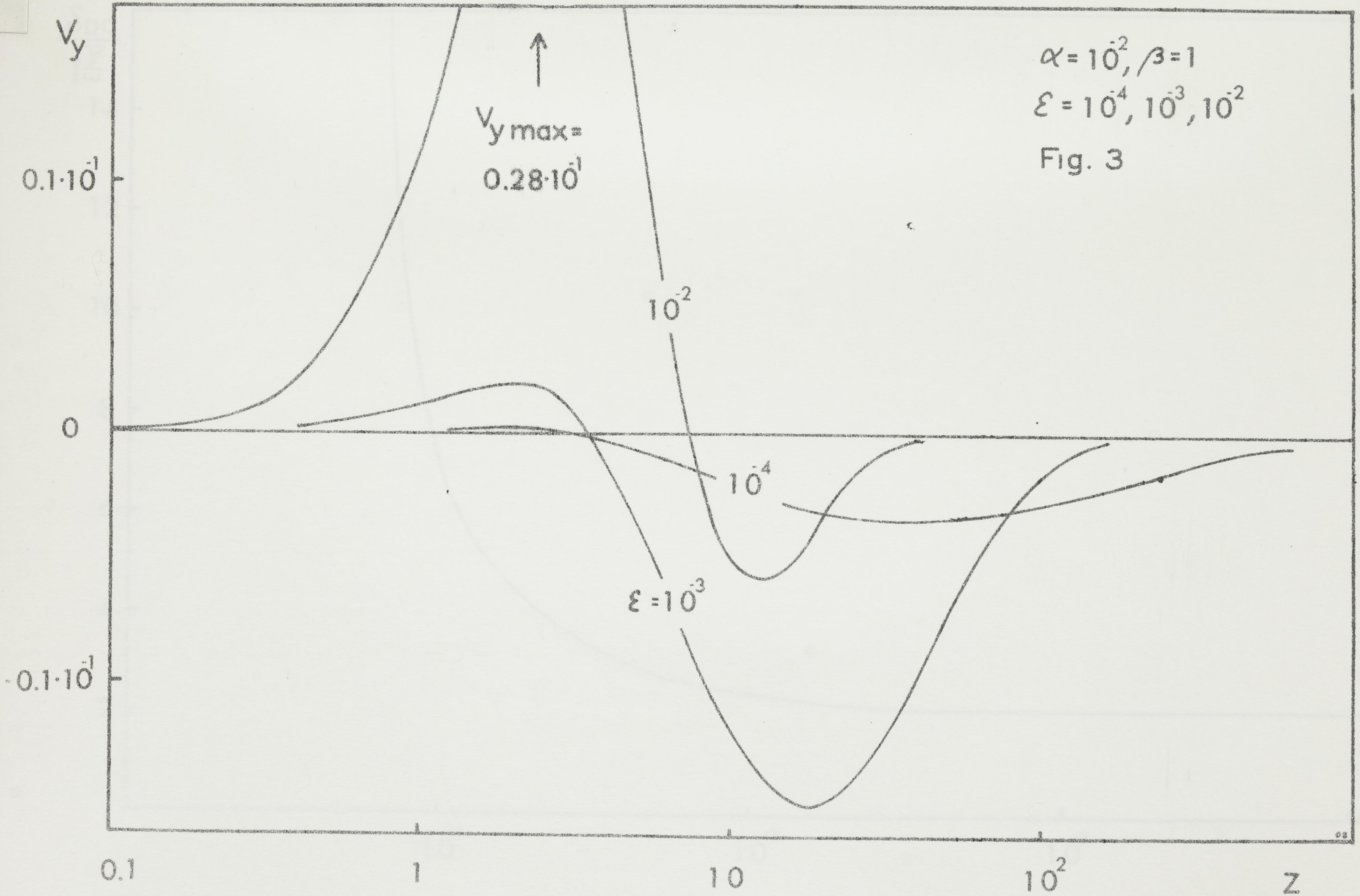


V_y
$0.1 \cdot 10^{-1}$
0
$-0.1 \cdot 10^{-1}$
-0.2
0.1
1
10
10^2
Z
$\alpha = 10^{-3}, 10^{-2}, 10^{-1}$
$\beta = 1$, $\varepsilon = 10^{-3}$
Fig. 1
10^{-1}
10^{-2}
$\alpha = 10^{-3}$



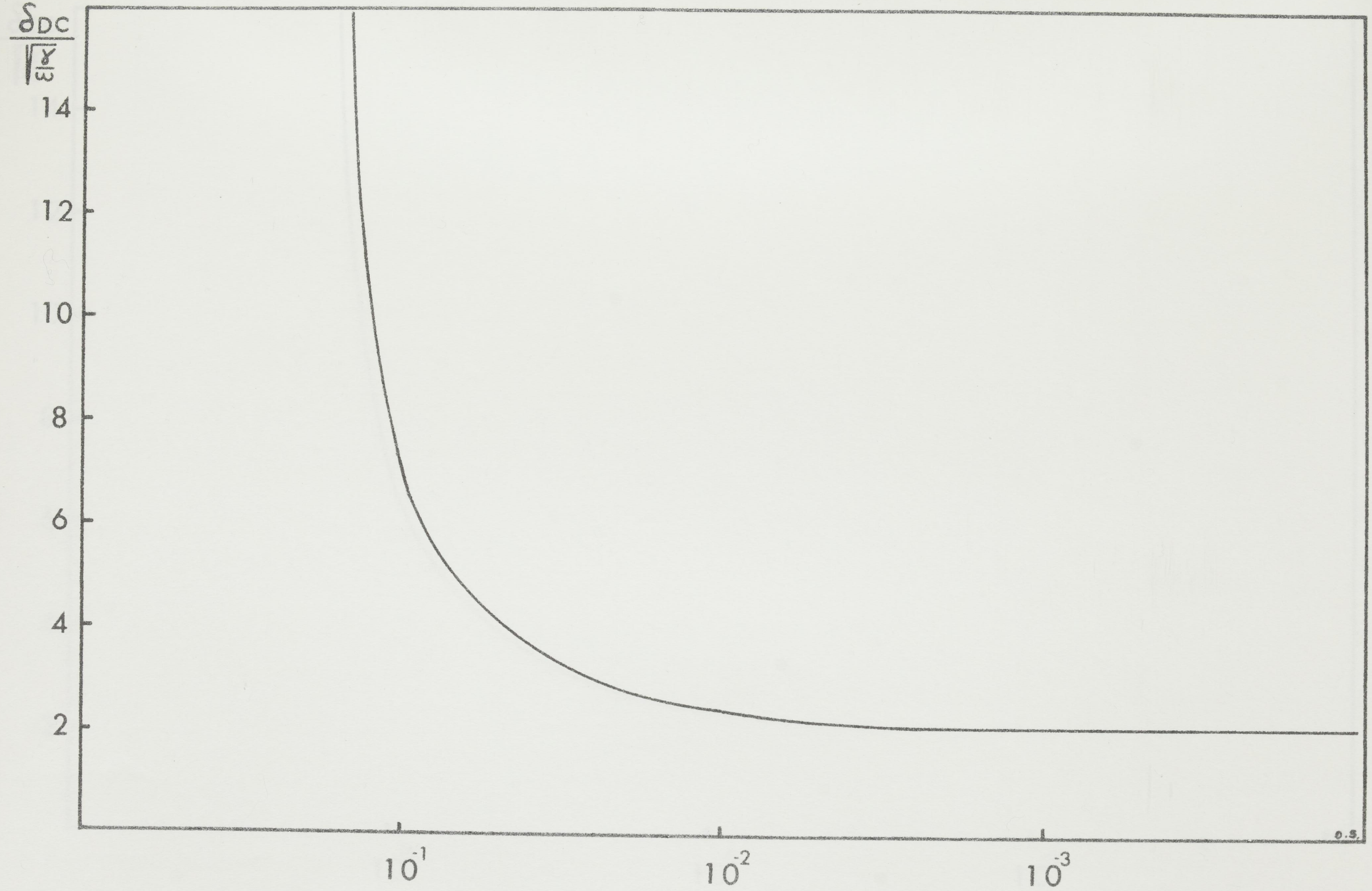
$\frac{\delta_{DC}}{\sqrt{\nu/\omega}}$
10
8
6
4
2
0
10^{-3}
10^{-2}
10^{-1}
α
$\varepsilon = 10^{-3}$

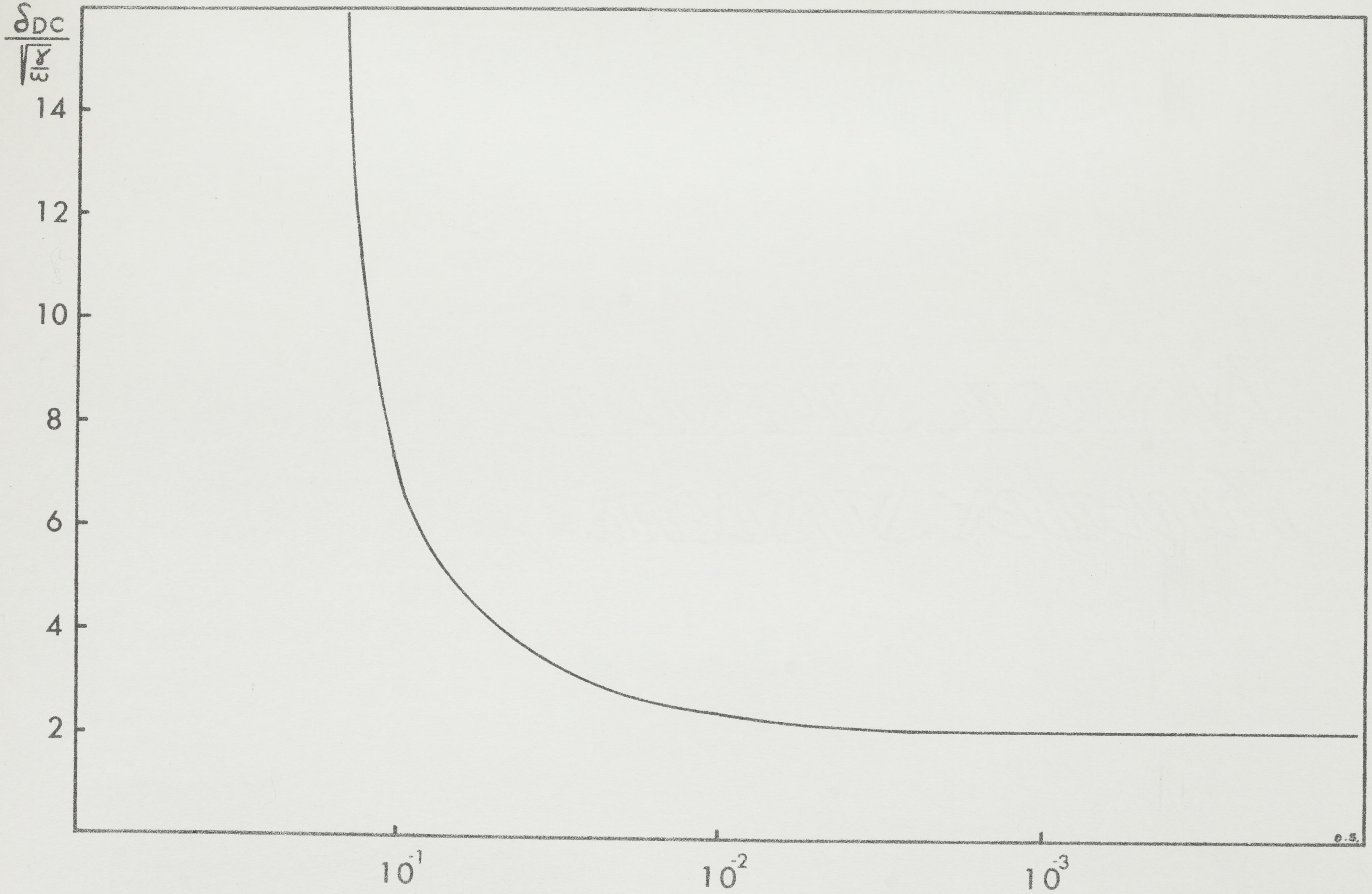
Fig. 2



V_y
$V_{y\,max} = 0.28 \cdot 10^{-1}$
$\alpha = 10^{-2}, \beta = 1$
$\varepsilon = 10^{-4}, 10^{-3}, 10^{-2}$
$0.1 \cdot 10^{-1}$
0
$-0.1 \cdot 10^{-1}$
10^{-2}
10^{-4}
$\varepsilon = 10^{-3}$
0.1
1
10
10^{2}
Z

Fig. 3


$\frac{\delta_{DC}}{\sqrt{\frac{\gamma}{\omega}}}$
14
12
10
8
6
4
2
10^{-1}
10^{-2}
10^{-3}
o.s.



δDC / √(γ/ω)
14
12
10
8
6
4
2
10^-1
10^-2
10^-3
o.s.