University of Bergen

Department of Informatics
Algorithms

Maximum Number of Edges in Graphs Under Various Constraints

Student: Håvard Stuberg
Supervisor: Pinar Heggernes

Masters Thesis
June, 2019

Acknowledgements

I would like to thank my supervisor Pinar Heggernes for all the help, guidance and encouragement I have received while working on this thesis. I also want to thank Paloma Lima for her help and suggestions.

Contents

Chapter 1

Introduction

A graph is a mathematical structure used to model relationships between objects. A common example of how graphs can be used is the modelling of different transportation systems. Everything from the transportation of packets in computer networks to flights between airports can be modelled with graphs. In the latter example, the airports are represented by vertices, while a pair of vertices are connected by an edge if there is a flight between the corresponding airports.

Sometimes we know that an algorithm we are designing will be used on graphs that share some common property. Suppose that an airline advertises that they can take you from anywhere to anywhere on earth with at most two intermediate stops, and consider the graph modelling the direct flights of this airline. This graph has the property that for every pair of vertices, we can start in one vertex and end up at the other by going along at most three edges. This is an example of a property that, for some problems, may help us design a faster algorithm.

In this thesis, we will be looking at several variants of an extremal graph theory problem. In general, the objective for an extremal graph theory problem is to find the minimum or maximum of some property of a graph under some constraints. The property may, for example, be the number of vertices, edges or connected components. An example of an extremal graph theory problem could be the following. For graphs which have the property that the shortest path between every pair of vertices uses at most three edges, what is the minimum number of edges such a graph on n vertices can have?

1.1 Notation and Definitions

A graph G is defined as a pair $G = (V, E)$ where V is a finite set and $E \subseteq V \times V$. The elements in V are called vertices, and the elements in E are called edges. For a graph

G, $V(G)$ and $E(G)$ denote the vertex set and edge set of G. In directed graphs, an edge $uv \in E$ has start point u and endpoint v. In a simple undirected graph $uv = vu$ and $uu \notin E$. All graphs in this thesis are simple and undirected. In an undirected graph we say that both u and v are endpoints of an edge $uv \in E$. If $uv \in E$, then u and v are adjacent, and the edge uv is said to be incident to u and v. For a set of edges $E' \subseteq E$, $V(E')$ is the set of all vertices incident to an edge in E'. The degree of a vertex v is the number of edges incident to v and is denoted $deg(v)$. The maximum degree of a vertex in G is denoted $\Delta(G)$. The neighbourhood of a vertex v, is the set of all vertices adjacent to v and is denoted $N(v)$.

A path of length n is a sequence of vertices $v_0, v_1, \ldots, v_{n-1}$ where $v_i v_{i+1} \in E$, for $i = 0, 1, \ldots, n-2$. A cycle of length n is a sequence of vertices, $v_0, v_1, \ldots, v_{n-1}$ where $v_{(i \mod n)} v_{((i+1) \mod n)} \in E$, for $i = 0, 1, \ldots, n-1$. For a subset of vertices $S \subseteq V$, the induced subgraph $G[S]$ is the graph whose vertex set is S and whose edge set is all edges from E with both endpoints in S. For an integer k, C_k is a graph that consists of a single cycle on k vertices. P_k is a graph that consists of a single path on k vertices. A graph is C_k-free if it does not contain C_k as an induced subgraph, likewise a graph is P_k-free if it does not contain P_k as an induced subgraph.

A graph is connected if there is a path between every pair of its vertices. A connected component in a graph is a set of its vertices in which every pair of vertices is connected by a path. A tree is a graph that is both connected and acyclic. For a set of vertices $S \subset V(G)$, and two vertices $s, t \in V(G) \setminus S$, S is called an s,t-seperator if s and t are in different connected components in $G[V \setminus S]$. A minimal s,t-seperator is a set $S \in V(G)$ such that no subset of S is an s,t-seperator.

A complete graph is a graph where every pair of vertices are adjacent. A complete graph on k vertices is denoted K_k. A clique in a graph $G = (V, E)$, is a subset of vertices $K \subseteq V$ such that the induced subgraph $G[K]$ is a complete graph. K is a maximal clique if no superset of K is a clique. A clique of largest size is maximum. A complete bipartite graph is a graph whose vertex set can be partitioned into two sets V_1 and V_2, such that every vertex in V_1 is adjacent to every vertex in V_2 while no edge has both endpoints in the same set. The complete bipartite graph with vertex sets of size k_1 and k_2, is denoted K_{k_1,k_2}. If $k_1 = 1$, then K_{k_1,k_2} is called a k_2-star. $K_{1,3}$ is called a claw.

In a graph G, a vertex u is universal if for all $v \in V(G) \setminus \{u\}$ we have that $uv \in E(G)$. A vertex $u \in V(G)$ is simplicial if $G[N(u)]$ is a complete graph.

A matching M in a graph is a set of edges no two of which share a common endpoint. The matching number of a graph is the size of a maximum matching and is denoted $\nu(G)$. A graph $G = (V, E)$ has a perfect matching if $2\nu(G) = |V|$, and a near-perfect matching if $2\nu(G) = |V| - 1$. G is factor-critical if $G[V \setminus \{v\}]$ has a perfect matching for all $v \in V$. An induced matching M in a graph is a matching such that $G[V(M)] = M$.

The induced matching number of a graph is the size of a maximum induced matching and is denoted $\mu(G)$.

For a graph $G = (V, E)$, an independent set is a subset of vertices $I \subseteq V$ such that no pair of vertices in I are adjacent. The chromatic number of a graph is the fewest number of independent sets needed to cover the vertices $V(G)$ and is denoted $\chi(G)$. The clique number of a graph is the number of vertices in a maximum clique of G and is denoted $\omega(G)$.

1.2 Graph Classes

A graph class is an infinite set of graphs which share some common property. In the introduction, we described an example of a graph class, namely connected graphs that do not have induced paths containing more than 3 edges, thus 4 vertices. The longest induced path in such a graph is P_4, and therefore these graphs are connected and P_5-free. Studying a particular graph class might give us results relevant only for that graph class, but it is also insightful to see how properties of different graph classes vary. Results for a specific graph class may help us understand the problem better in general. Also, understanding the properties of a graph class might result in a more efficient algorithm for that graph class for some other problem.

There are some problems whose solutions require so much time or space that they can not be used in practice on large inputs. We say that these problems are intractable. Even though a graph problem is intractable on general graphs, it might have an efficient solution when restricted to particular graph classes. There are many examples of such problems, like computing the chromatic number, maximum induced matching, largest clique and largest independent set [20]. However, all of these problems have efficient solutions on, for example, chordal graphs [13].

1.2.1 Chordal Graphs

A chordal graph is a graph with no induced cycle of length four or more. Equivalently, a graph is chordal if there is an edge connecting non-consecutive vertices in every cycle on more than three vertices. Such an edge is called a chord. Chordal graphs have many practical applications; we find examples of such applications in sparse matrix multiplication [3], database management, knowledge-based systems and computer vision [14]. Chordal graphs are also referred to as triangulated, perfect elimination, monotone transitive and rigid circuit graphs, which reflects that chordal graphs have been studied in many different settings.

Equivalent definitions of chordal graphs are graphs that have a perfect elimination ordering and graphs where every minimal separator is a clique [9]. A perfect elimination ordering is an ordering of the vertices $v_1, v_2, \ldots, v_n$ such that for all i, we have that $G[N(v_i) \setminus \{v_1, v_2, \ldots, v_{i-1}\}]$ is a complete graph.

A clique tree $\mathcal{T}$ of a chordal graph G is a tree in which each node $T_i \in V(\mathcal{T})$ corresponds to a maximal clique K_i in G, such that for $T_i, T_j \in V(\mathcal{T})$ we have that if $K_i \cap K_j \neq \emptyset$ then $T_iT_j \in E(\mathcal{T})$. We will thus treat a tree node T_i as a set of vertices of G.

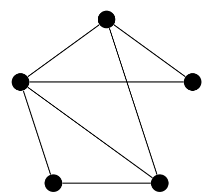

Figure 1.1: Example of a chordal graph.

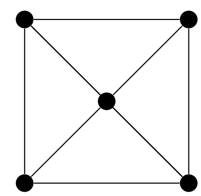

Figure 1.2: Example of a graph that is not chordal (because the outer cycle is induced and of length 4).

1.2.2 Interval Graphs

A graph $G = (V, E)$ is an interval graph if every vertex can be assigned an interval on the real line such that the intervals of a pair of vertices $v_1, v_2 \in V$ intersect if and only if $v_1v_2 \in E$. Interval graphs have many practical applications. In particular, they can be used to model problems that have a one-dimensional, or linear, component. This component could, for instance, be time or distance.

Every interval graph is chordal. To see this, let $\{v_1, v_2, \ldots, v_n\}$ be the vertices in a chordless cycle on more than three vertices in an interval graph G. Consider an interval representation of the graph $G[\{v_1, v_2, \ldots, v_n\}]$. Since $G[\{v_1, v_2, \ldots, v_n\}]$ is the chordless cycle C_n, the degree of every vertex in this graph is exactly two. If we consider the interval with start point furthest left on the real line, we see that if this vertex is adjacent to two vertices, the degree of its neighbours cannot both be two. We get that $G[\{v_1, v_2, \ldots, v_n\}]$ is not C_n, which means that every interval graph is chordal.

A unit interval graph is a graph that can be represented by a set of intervals of equal length. Unit interval graphs are exactly those interval graphs that do not contain $K_{1,3}$ as an induced subgraph [12].

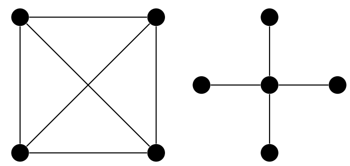

Figure 1.3: Example of an interval graph which is not unit interval.

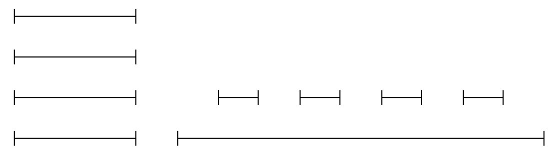

Figure 1.4: An interval representation of the graph in Figure 1.3.

1.2.3 Split Graphs

A split graph $G = (V, E)$ is a graph whose vertex set can be split into a clique and an independent set. Let $K \subseteq V$ and $I = V \setminus K$. If K is a clique and I is an independent set then (K, I) is a split partition of G.

Every split graph is chordal. This is easy to see since a chordless cycle on more than three vertices in a split graph with split partition (K, I) cannot contain more than two vertices from K. Since I is an independent set, there are no cycles in G that contain more than one vertex from I. Hence no split graph has an induced cycle on more than three vertices.

Neither the class of split graphs nor the class of interval graphs is a subset of the other. An example of a graph that is interval but not split is P_5. The graph in Figure 1.5 is an example of a graph that is split but not interval.

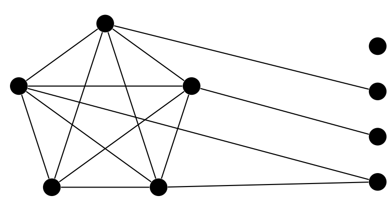

Figure 1.5: Example of a split graph.

1.2.4 Trivially Perfect Graphs

A graph is perfect if its chromatic number is equal to the size of the largest clique. A perfect graph is trivially perfect if $\omega(H) = \chi(H)$ for all induced subgraphs H of G. The class of trivially perfect graphs is exactly the class of graphs that are both P_4-free and C_4-free [4]. If a graph is trivially perfect, then every connected induced subgraph H has a universal vertex [23].

Since every connected trivially perfect graph has a universal vertex, and every subgraph of a trivially perfect graph is trivially perfect, it follows that a trivially perfect graph has an interval representation such that a pair of intervals are either disjoint or one is contained in the other. We can construct such an interval representation by repeatedly removing vertices that is universal in a connected component. We assign a subset of the interval of the universal vertex to each of the connected components that results when removing this vertex.

In the next section we are going to introduce cographs, and we will see that since trivially perfect graphs are P_4-free every trivially perfect graph is a cograph.

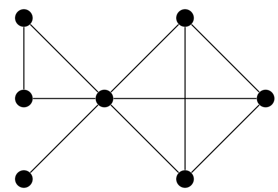

Figure 1.6: Example of a trivially perfect graph.

1.2.5 Cographs

The last graph class we will introduce is the class of cographs, it is the only graph class we will be studying that is not a subclass of chordal graphs. Before we define cographs, we need to define the complement of a graph and disjoint union of graphs. The complement $\overline{G}$ of a graph $G = (V, E)$, is a graph whose vertex set is V and whose edge set contains every pair of vertices not in E. The disjoint union of graphs $G_1, G_2, \ldots G_k$, is a graph $G = (V, E)$ where $V = V(G_1) \cup V(G_2) \cup \cdots \cup V(G_k)$ and $E = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_3)$.

Cographs are defined as graphs that can be generated by taking the complement or disjoint union of other cographs. With a graph that consists of a single vertex defined to be a cograph, these two rules create an infinite set of graphs. Interestingly, this turns out to be exactly the class of P_4-free graphs [6]. We see that in contrast to chordal graphs, which can be characterised by an infinite number of forbidden induced subgraphs, cographs can be characterised by a single forbidden induced subgraph.

A cotree is a structure to represent a cograph, and every cotree uniquely defines a cograph [6]. A cotree is a rooted tree where internal nodes are labelled 1 or 0 while the leaves represent vertices. A pair of vertices in a cograph are adjacent if their lowest common ancestor in the cotree of G is labelled 1. For every cotree, there is an equivalent cotree in which the parent of a node labelled 0 is labelled 1, and the parent of a node labelled 1 is labelled 0. With this added restriction on the structure of a cotree, there is a unique cotree representation for every cograph [6].

To see that there is no induced path on more than three vertices in a graph represented by a cotree, consider the cotree in Figure 1.8. There is a path $c \to d \to e$ on three vertices, to get a path on four vertices we need to add a leaf to the cotree such that the new vertex is adjacent to only c or e. It is easy to see that if the lowest common ancestor of this new leaf and c is 1 then the lowest common ancestor of d and the new vertex is also 1.

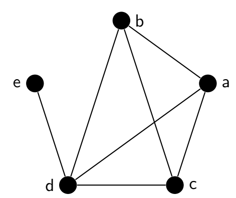

Figure 1.7: Example of a cograph.

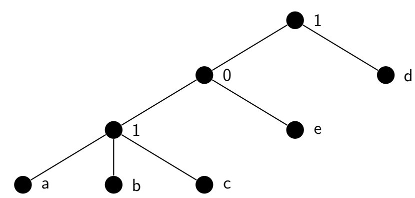

Figure 1.8: The cotree of the cograph in Figure 1.7.

1.2.6 Graph Class Hierarchy

We have introduced several graph classes in the preceding sections explaining how these relate to each other. Figure 1.9 gives us an overview of the graph classes we have introduced. In this figure, an arrow represents the subset relation, meaning that $A \to B$ indicates $A \subseteq B$. For example, the arrow from interval to chordal indicates that all interval graphs are chordal.

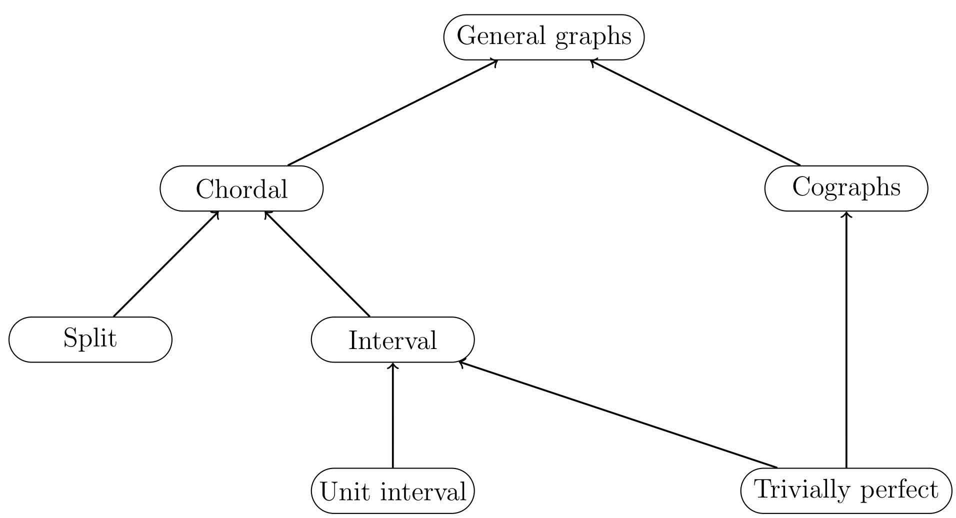

Figure 1.9: Relationship between graph classes. Note that the graph classes we have introduced are all perfect [13].

1.3 Overview of the Thesis

With the given definitions, we are now ready to present the main content of this thesis. This thesis is organized as follows. In the next chapter, we will give a more thorough introduction to the problems we are going to study in this thesis. We will also present the previously known results for these problems.

In Chapter 3, we are going to study the maximum number of edges in chordal graphs whose matching number and maximum degree are bounded. In this chapter, we are going to present the main result of this thesis, a tight bound on the number of edges in edge-extremal interval graphs.

In Chapter 4, we are going to initiate the study of the maximum number of edges in graphs whose induced matching number and maximum degree are bounded. We will solve the problem for interval graphs, split graphs and cographs.

We will use practical tests in both Chapter 3 and Chapter 4 to test some conjectures and hypotheses. In Chapter 5 we are going to describe the practical framework for this kind of testing. We will also take a quick look at a related extremal graph theory problem and give a summary of the work we have done on this thesis.

Chapter 2

Bounding the Number of Edges in a Graph

In this thesis, we will be studying problems that seek to determine the maximum number of edges in graphs under some constraints. We get a trivial example if the constraint is on the number of vertices. Let n be an upper bound on the number of vertices. This gives $\frac{n(n-1)}{2}$ as the upper bound on the number of edges. If a graph has as many edges as the upper bound, under the given constraints, then we call the graph edge-extremal.

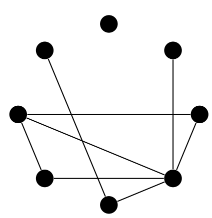

Figure 2.1: A graph on 8 vertices.

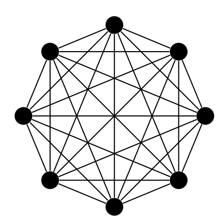

Figure 2.2: An edge-extremal graph on 8 vertices.

When we have such a general result for general graphs, we can then study the same question on graphs belonging to various graph classes. For example, we can observe that the extremal graph for the general case, a complete graph on n vertices, is C_k-free for $k > 3$. Therefore we can conclude that, for $k > 3$, an edge-extremal C_k-free graph on n vertices is a complete graph. What if we chose a graph class that cannot have a complete graph as an edge-extremal graph? Consider the class of C_3-free graphs. What is the maximum number of edges in C_3-free graphs on n vertices? As a warm up to our topic of study, let us resolve this question.

Lemma 2.1. A C_3-free graph on n vertices has at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.[1]

Proof. We first observe that the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ has $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges. It remains to show that this graph is edge-extremal. Assume for contradiction that there is a C_3-free graph G on n vertices such that $|E(G)| > \left\lfloor \frac{n^2}{4} \right\rfloor$.

We observe that for every edge $uv \in E(G)$, we have that $deg(u) + deg(v) \leq n$, since otherwise, there is a vertex $w \in N(u) \cap N(v)$ such that $\{u, v, w\}$ is an induced C_3 in G. It follows that $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ has the maximum number of vertices of degree $\lceil \frac{n}{2} \rceil$. Since the minimum degree of a vertex in this graph is $\lfloor \frac{n}{2} \rfloor$ and $|E(G)| > |E(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})|$, we have that $\Delta(G) > \lceil \frac{n}{2} \rceil$.

Let v be a vertex in V such that $deg(v) = \Delta(G)$. We have that $deg(v) = \lceil \frac{n}{2} \rceil + l$, for $1 \leq l < n$. For every vertex $u \in N(v)$, we have that $deg(u) \leq n - deg(v)$. It follows that there are at most $n - deg(v)$ vertices of degree at most $deg(v)$, the remaining $deg(v)$ vertices has degree at most $n - deg(v)$. We get the following upper bound on the number of edges in G.

$$\begin{aligned}
|E(G)| = \frac{1}{2}\sum_{u \in V} deg(u) &\leq \frac{1}{2}\Big((n - deg(v))deg(v) + deg(v)(n - deg(v))\Big) \\
&\leq \frac{1}{2}\big((n - deg(v))deg(v) + deg(v)(n - deg(v)\big) = deg(v)(n - deg(v)) \\
&= (\left\lceil \frac{n}{2} \right\rceil + l)(n - \left\lceil \frac{n}{2} \right\rceil - l)
\end{aligned}$$

First, let n be even. In this case we have that

$$|E(G)| = \frac{1}{2}\sum_{u \in V} deg(u) \leq (\frac{n}{2} + l)(n - \frac{n}{2} - l) = \frac{n^2}{4} - l^2 < |E(K_{\frac{n}{2}, \frac{n}{2}})| = |E(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})|$$

If n is odd, $n = 2k + 1$, we have that

$$\begin{aligned}
|E(G)| = \frac{1}{2}\sum_{u \in V} deg(u) \leq (\left\lceil \frac{n}{2} \right\rceil + l)(n - \left\lceil \frac{n}{2} \right\rceil - l) &= (k + 1 + l)(2k + 1 - (k + 1) - l) \\
&= (k + 1 + l)(k - l)
\end{aligned}$$

Substituting $\frac{n-1}{2}$ for k we get

$$|E(G)| \leq k(k + 1) - l(l + 1) = \frac{(n + 1)(n - 1)}{4} - l(l + 1) < |E(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})|$$

[1] After proving this result, we found out that it is a known result by Mantel [15]. It is also special case of a more general result on the number of edges in K_k-free graphs [22].

We get that $|E(G)| < |E(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})|$, which is a contradiction. $\square$

For C_3-free graphs on n vertices, we have that $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is edge-extremal, and $\left\lfloor \frac{n^2}{4} \right\rfloor$ is a tight upper bound on the number of edges. We see that by restricting the graph class we are looking at to C_3-free graphs, the maximum number of edges is around half the maximum number of edges for general graphs.

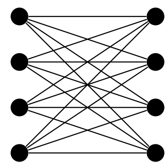

Figure 2.3: An edge-extremal C_3-free graph on 8 vertices.

Recall the question from Chapter 1, which asks for minimum instead of maximum: For graphs which have the property that the shortest path between every pair of vertices uses at most three edges, what is the minimum number of edges a graph on n vertices may have? Since this graph has to be connected, the minimum number of edges is at least $n-1$. For this problem, there are multiple edge-extremal graphs; every graph with $n-1$ edges that satisfies the property that the shortest path between every pair of vertices is of length at most three, is edge-extremal. The complete bipartite graph, $K_{1,n-1}$, is an example of an edge-extremal graph for this problem. $K_{1,n-1}$ is also called an $(n-1)$-star.

What if we put a constraint on the maximum degree instead of the number of vertices? It is easy to see that the number of edges is not bounded in this case. For graphs that satisfy $\Delta(G) < i$, we can get an arbitrary number of edges with disjoint cliques of size at most i or an arbitrarily long path.

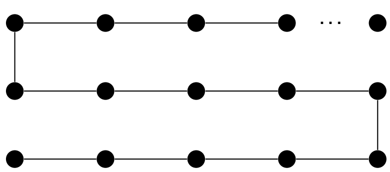

Figure 2.4: A long path.

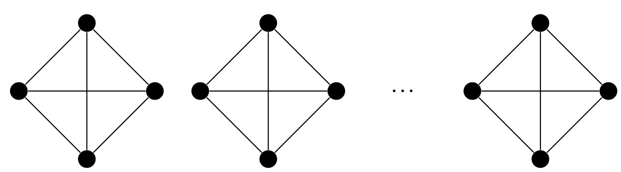

Figure 2.5: An arbitrary number of cliques.

Using maximum degree alone will not give a bound on the number of edges, but if we bound both the maximum degree and the matching number, we will see that the number of edges is bounded. First, observe that there is no upper bound on the number of edges in graphs bounded by just their matching number, either. A k-star has matching number 1, and k edges. Since the size of the star does not change the matching number, we can construct a graph with an arbitrary number of edges, although the matching number is 1.

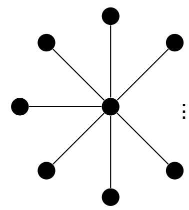

Figure 2.6: A star.

2.1 Maximum Degree and Matching Number

When we put a constraint on both the matching number and the maximum degree of a graph, then neither stars, cliques or paths can be used to construct a graph with an arbitrary number of edges. A single clique and a path both have a perfect or near-perfect matching, while a k-star has a vertex of degree k.

For general graphs, the solution to this problem is known, and we will be presenting the solution after introducing some notation. Let $M_{\mathcal{C}}(i,j)$ be all graphs G in a graph class $\mathcal{C}$ that satisfies the constraints $\Delta(G) < i$ and $\nu(G) < j$. Let $\mathcal{GEN}$ be the class of all general graphs. Balachandran and Khare [1] proved an exact bound on the number of edges in graphs belonging to $M_{\mathcal{GEN}}(i,j)$.

Theorem 2.2 (Balachandran and Khare [1]). For an edge-extremal general graph $G \in M_{\mathcal{GEN}}(i,j)$, we have that

$$|E(G)| = (i-1)(j-1) + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor .$$

How an edge-extremal graph is constructed depends on whether i is odd or even. For odd i, a graph consisting of $\left\lfloor \frac{j-1}{\frac{i-1}{2}} \right\rfloor$ disjoint cliques on i vertices and $j-1-\frac{i-1}{2}\left\lfloor \frac{j-1}{\frac{i-1}{2}} \right\rfloor$

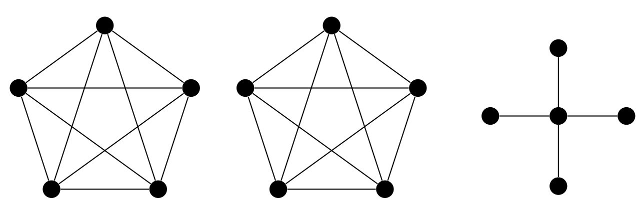

Figure 2.7: An edge-extremal graph in $M_{\mathcal{GEN}}(5,6)$.

disjoint $(i-1)$-stars is edge-extremal. An example of an edge-extremal graph is given in Figure 2.7.

For even i, a clique of size i has a perfect matching. By modifying a clique of size i, we can increase the number of edges by $i-1-\frac{i}{2}$ without increasing the matching number. We do this by removing a maximum matching and adding a new vertex adjacent to every vertex in the resulting subgraph but one. An example of an edge-extremal graph for even i is given in Figure 2.8.

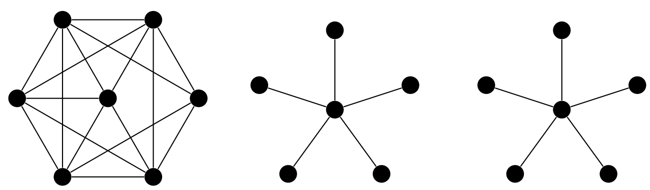

Figure 2.8: An edge-extremal graph in $M_{\mathcal{GEN}}(6, 6)$.

Observe that for most i and j, the edge-extremal graphs have at least one connected component which is a star. A natural question to ask might be what the maximum number of edges is in graphs that do not contain $K_{1,3}$, the smallest star, as an induced subgraph. Dibek, Ekim and Heggernes [8] have studied this problem. Let $\mathcal{CF}$ denote the class of claw-free graphs.

Theorem 2.3 (Dibek, Ekim and Heggernes [8]). For an edge-extremal graph $G \in M_{\mathcal{CF}}(i, j)$ we have that

1. if $i \geq 2j$, then $|E(G)| = (2j-1)(j-1)$,
2. if $i < 2j$, then $|E(G)| = (i-1)(j-1) + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{j-1}{\left\lceil \frac{i-1}{2} \right\rceil} \right\rfloor$.

For $i \geq 2j$ K_{2j-1} is a unique edge-extremal graph, which means that for bounding the number of edges in claw-free graphs, in contrast to general graphs, it is sufficient to bound the matching number. For $i < 2j$, the authors gave two types of claw-free graphs which can be used to construct edge-extremal graphs in $M_{\mathcal{CF}}(i, j)$.

Theorem 2.4 (Dibek, Ekim and Heggernes [8]). If $i < 2j$, then an edge-extremal graph in $M_{\mathcal{CF}}(i, j)$ can be constructed by taking

$$\begin{cases} q-1 \textit{ copies of } K_i & \textit{and } 1 \textit{ copy of } R_{i+2r,i-1} & , \textit{if } i \textit{ is odd} \\ q-1 \textit{ copies of } R'_{i+1,i-1} & \textit{and } 1 \textit{ copy of } R'_{i+2r+1,i-1} & , \textit{if } i \textit{ is even} \end{cases}$$

where q and r are the quotient and remainder of the division of $j-1$ by $\left\lceil \frac{i-1}{2} \right\rceil$. So, we have that $(j-1) = q \left\lfloor \frac{i-1}{2} \right\rfloor + r$.

The graph $R_{a,b}$ has vertex set $\{v_0, v_1, v_2, \ldots, v_{a-1}\}$ and a vertex v_r is adjacent to $v_{r-\frac{b}{2}}$, $v_{r-\frac{b}{2}+1}$, ..., $v_{r+\frac{b}{2}-1}$, $v_{r+\frac{b}{2}}$ where indices are of modulo a.

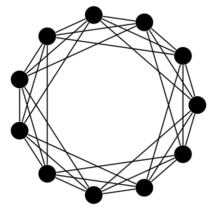

Figure 2.9: $R_{11,6}$

Note that when a is even, $R_{a,b}$ has a perfect matching. Similar to the situation for edge-extremal graphs in $M_{\mathcal{GEN}}(i,j)$, there is a graph with a higher number of edges that is similar to $R_{a,b}$ for even i. Let $R_{a,b-1} = (V, E)$, the vertex set of this graph $R'_{a,b}$ is V and its edge set is $E \cup E'$. Without going into any details, E' is defined by the following sequence.

$$\begin{gathered}\{1, 1+\frac{b+1}{2}, 1+2\frac{b+1}{2}, \ldots, 1+(\frac{a}{m}-1)\frac{b+1}{2},\\ 2, 2+\frac{b+1}{2}, 2+2\frac{b+1}{2}, \ldots, 2+(\frac{a}{m}-1)\frac{b+1}{2},\\ \ldots,\\ m, m+\frac{b+1}{2}, m+2\frac{b+1}{2}, \ldots, m+(\frac{a}{m}-1)\frac{b+1}{2}\},\end{gathered}$$

where m is the greatest common divisor of a and $b+1$, and the vertices are labelled from 1 to a in the order they appear around the circle. There is an edge connecting the vertices corresponding to the first and second element of the sequence, third and fourth element, fifth and sixth etc.

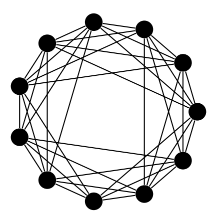

Figure 2.10: $R'_{11,7}$

We see that for even i, the edge-extremal graphs in $M_{\mathcal{GEN}}(i,j)$ or $M_{\mathcal{CF}}(i,j)$ are not chordal. Therefore, along the lines of the previous discussion, it might be interesting to study the maximum number of edges for chordal graphs. The solution for chordal graphs is so far not known, but Måland [17] studied two subclasses of chordal graphs, namely split graphs and unit interval graphs. The results for those graphs give us a lower bound on the number of edges in edge-extremal chordal graphs. Let $\mathcal{SPLIT}$ be the class of split graphs, and $\mathcal{UNIT}$ the class of unit interval graphs.

Theorem 2.5 (Måland [17]). For an edge-extremal graph $G \in M_{\mathcal{SPLIT}}(i,j)$, we have that

1. if $i-1 \leq 2(j-1)+1$, then $|E(G)| = \frac{i(i-1)}{2}$,

2. if $i-1 > 2(j-1)+1$, then $|E(G)| = max\{\frac{(2(j-1)+1(2(j-1))}{2}, (i-1)(j-1) - \frac{(j-1)((j-1)-1)}{2}\}$.

In the first case, K_i is an edge-extremal graph. In the second case, depending on which term is largest, either $K_{2(j-1)+1}$ or a split graph on i vertices where $|C| = j-1$ and every vertex in C is adjacent to every vertex in I, is edge-extremal. If we instead look at the disjoint union of split graphs, the edge-extremal graphs have exactly $(i-1)(j-1)$ edges for even i and $(i-1)(j-1) + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor$ edges for odd i.

Unit interval graphs are both chordal and claw-free. For most i and j, the edge-extremal graphs found for general graphs, claw-free graphs and split graphs are not unit interval graphs. Måland found that the edge-extremal unit interval graphs consist of disjoint cliques of varying size.

Theorem 2.6 (Måland [17]). For an edge-extremal graph $G \in M_{\mathcal{UNIT}}(i,j)$ we have that

1. if i is odd, then $|E(G)| = i(j-1) + (2r+1-i)r$, $r = (j-1) \mod \frac{i-1}{2}$,

2. if i is even, then $|E(G)| = (i-1)(j-1) - \mathcal{R}$,
   where $\mathcal{R} = min\{(i-2(p+1))p, (i-2(s+1))s\}$,
   $a = \frac{i}{2}$, $b = \frac{i-2}{2}$, $j-1 = q_1 a + r_1 = q_2 b + r_2$, $p = max\{0, r_2 - q_2\}$, $s = min\{q_1 + r_2, b-1\}$.

Note that for even i we have $(i-1)(j-1)$ as an upper bound on the number of edges for both split graphs and unit interval graphs. This study by Måland left open the case for interval graphs. In fact, the problem for interval graphs has remained open until now. In Chapter 3 we will be looking further at this problem for chordal graphs and we will resolve the problem on interval graphs.

2.2 Maximum Degree and Induced Matching Number

What if we, instead of bounding the matching number, bound the induced matching number? Since the induced matching number of a graph is at most the same as the matching number, bounding the induced matching number does not bound the number of edges. Similar to the problem introduced in the previous section, we will bound the number of edges by setting a constraint on maximum degree in addition to the induced matching number. This problem does not seem to have been studied before, and in Chapter 4 we will initiate the study of this problem.

So far in this chapter, $M_{\mathcal{C}}(i,j)$ has denoted the set of all graphs in a graph class $\mathcal{C}$ that satisfies $\Delta(G) < i$ and $\nu(G) < j$. To differentiate if the constraint is on the matching number or induced matching number, $M_{\nu,\mathcal{C}}(i,j)$ will from now on be the set of all graphs in a graph class $\mathcal{C}$ that satisfies $\Delta(G) < i$ and $\nu(G) < i$, while $M_{\mu,\mathcal{C}}(i,j)$ will denote the set of all graphs G in a graph class $\mathcal{C}$ that satisfies $\Delta(G) < i$ and $\mu(G) < j$.

To see that the number of edges in edge-extremal graphs in $M_{\nu,\mathcal{C}}(i,j)$ and $M_{\mu,\mathcal{C}}(i,j)$ are not necessarily equal, consider the examples of edge-extremal graphs in $M_{\nu,\mathcal{C}}(i,j)$. The induced matching number of each connected component in these graphs is one. Consequently, no edge-extremal graph in $M_{\mu,\mathcal{C}}(i,j)$ contains a star since the other connected components have a higher number of edges with the same induced matching number.

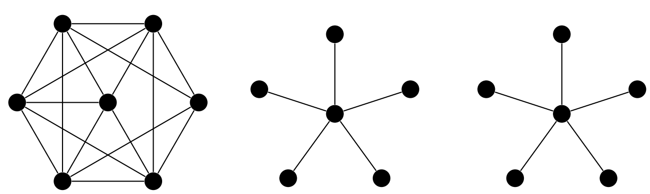

Figure 2.11: Edge-extremal graph in $M_{\nu,\mathcal{GEN}}(6,6)$.

Consider a clique with an even number of vertices. The procedure of deleting a perfect matching and adding a vertex adjacent to all but one of the vertices in the clique does not increase the matching number or the induced matching number. This means that a graph that consists of disjoint cliques do not give the maximum number of edges for graphs in $M_{\mu,\mathcal{GEN}}(i,j)$. Repeating this procedure for another perfect matching in the clique does increase the matching number, but the induced matching number still does not change. In fact, the procedure can be repeated for every perfect matching of the clique without increasing the induced matching number.

We see that by repeating this procedure, we get a complete bipartite graph. A complete bipartite graph has maximum induced matching 1 and $(i-1)^2$ edges. This means that a graph that consists of $j-1$ disjoint copies of $K_{i-1,i-1}$ is in $M_{\mu,\mathcal{GEN}}(i,j)$. This graph

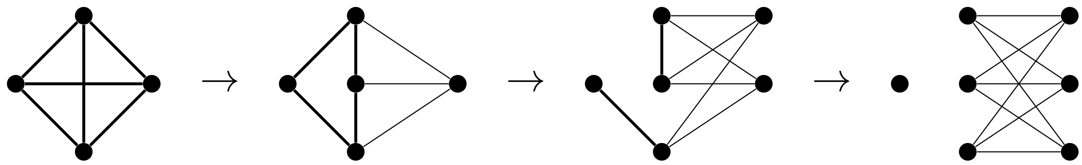

Figure 2.12: By repeating the procedure for every perfect matching in a clique we get a disjoint union of complete bipartite graph and an isolated vertex.

has $(i-1)^2(j-1)$ edges, in comparison to $\frac{i(i-1)}{2}(j-1)$ edges for disjoint cliques. In Chapter 4 we will see that the edge-extremal cographs are disjoint union of complete bipartite graphs. However, we will see that there are general graphs with a higher number of edges.

Chapter 3

Chordal Graphs Whose Matching Number and Maximum Degree are Bounded

In this chapter, we aim to get an insight into how edge-extremal chordal graphs may be constructed. The examples we gave in the previous chapter of edge-extremal graphs in $M_{\nu,\mathcal{GEN}}(i,j)$ for odd i, belong to the graph classes we will consider in this chapter. Our objective for this chapter is finding the maximum number of edges when i is even. In the previous chapter, we saw that $(i-1)(j-1)$ is an upper bound on the number of edges for two subclasses of chordal graphs when i is even. In this chapter we will add interval graphs to that list. In fact, we will see that, for all i and j, an edge-extremal interval graph has $(i-1)(j-1)$ edges.

The graph classes we will consider in this chapter have in common that they contain stars. At first, this might not seem that significant, but for graph classes which include stars, we may assume that there is no vertex whose removal reduces the matching number, by the following observation.

Observation 3.1. For a graph G with a vertex v such that $\nu(G \setminus \{v\}) < \nu(G)$, the disjoint union of $G \setminus \{v\}$ and an $(i-1)$-star, has at least as many edges as G without exceeding the constraints set for the matching number and maximum degree of the graph.

This tells us that there is an edge-extremal component that consists of stars and components with no vertices whose removal reduces the matching number. Gallai's lemma tells us that a graph that has this property is factor-critical.

Theorem 3.2 (Gallai's lemma)**.** If a graph G is connected and $\nu(G \setminus \{v\}) = \nu(G)$ for all $v \in V(G)$, then G is factor-critical.

Recall that the definition of a factor-critical graph is a graph in which every subgraph of $n-1$ vertices has a perfect matching, which implies that $|V(G)| = 2\nu(G)+1$. It follows that, for graph classes $\mathcal{C}$ that contain stars, if there is a graph in $M_{\nu,\mathcal{C}}(i,j)$ with more than $(i-1)(j-1)$ edges, there is a connected graph $G \in M_{\nu,\mathcal{C}}(i,j)$ where $|V(G)| = 2(j-1)+1$ and $|E(G)| > (i-1)(j-1)$ for some i and j.

3.1 Chordal Graphs

In this section, we will investigate a hypothesis on how edge-extremal chordal graphs can be constructed. We will also be running tests on chordal graphs to see how many edges the graphs we are able to generate have, for given i and j. Let $\mathcal{CHO}$ be the class of chordal graphs. In the previous chapter, we saw that for even i, $(i-1)(j-1)$ is an upper bound on the number of edges for graphs in both $M_{\nu,\mathcal{SPLIT}}(i,j)$ and $M_{\nu,\mathcal{UNIT}}(i,j)$. Since a graph of disjoint stars is chordal, we have that for all i and j the edge-extremal chordal graphs have at least $(i-1)(j-1)$ edges.

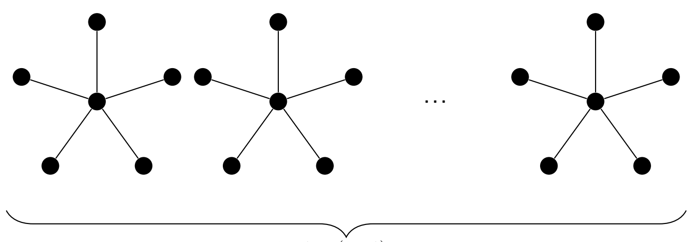

Figure 3.1: Chordal graph with $(i-1)(j-1)$ edges.

3.1.1 Cliques and Stars

One hypothesis is that the edge-extremal graphs for even i, as for odd i, consist of disjoint cliques and stars. Assuming that this hypothesis is true, Observation 3.2 gives us the number of edges in edge-extremal chordal graphs.

Observation 3.3. Let G be a graph that consists of disjoint cliques and stars, and that satisfies $\Delta(G) < i$ and $\nu(G) < j$. If i is even, then the maximum number of edges G can have is $(i-1)(j-1)$.

Proof. An $(i-1)$-star has $i-1$ edges and matching number 1, which means that a graph that is a disjoint union of $j-1$ copies of an $(i-1)$-star has $(i-1)(j-1)$ edges. If there is a graph with a higher number of edges, there has to be a complete graph K such that $\frac{|E(K)|}{\nu(K)} > i-1$. Since $\Delta(G) < i$, we may have cliques on up to i vertices. We will consider cliques on even and odd number of vertices separately. First consider a clique on $i-2k$ vertices, $k \geq 0$. Since $i-2k$ is even, the clique has matching number $\frac{i-2k}{2}$. We get that

$$\frac{|E(K)|}{\nu(K)} = \frac{1}{\frac{i-2k}{2}} \frac{(i-2k)(i-1-2k)}{2} = \frac{2}{i-2k} \frac{(i-2k)(i-1-2k)}{2} = i-1-2k \leq i-1.$$

Consider a clique on $i-1-2k$ vertices. Since $i-1-2k$ is odd the clique has matching number $\frac{i-2-2k}{2}$. In this case, we get that

$$\frac{|E(K)|}{\nu(K)} = \frac{1}{\frac{i-2-2k}{2}} \frac{(i-1-2k)(i-2-2k)}{2} = i-1-2k \leq i-1.$$

Which implies that for graphs that consist of disjoint union of cliques and stars, the maximum number of edges is $(i-1)(j-1)$. □

We also observe that for an edge-extremal graph in $M_{\nu,\mathcal{CHO}}(i,j)$, the number of maximal cliques in G cannot be two. Since such a graph has a perfect or near-perfect matching and there is a vertex adjacent to every other vertex in the graph, we may add an edge between every non-adjacent pair of vertices. In the next section, we will run practical tests to see if it enables us to make additional observations on the structure of edge-extremal chordal graphs.

3.1.2 Test Results

Since we have a hypothesis about edge-extremal chordal graphs, but no proof, we generated chordal graphs for various i and j, to see how these look like, and what kind of bounds we could deduce from them. If we were to find a chordal graph with more than $(i-1)(j-1)$ edges we could immediately discard our hypothesis. In this section, we will be presenting the results we got after testing every chordal graph on up to 13 vertices.

We did not find a chordal graph with more than $(i-1)(j-1)$ edges, we did, however, find chordal graphs with $(i-1)(j-1)$ edges which did not consist of just cliques and stars. We present these results in Tables 3.1 and 3.2. In Table 3.2 the entries correspond to the maximum number of edges found for given i and j. The entries in Table 3.1 are corresponding graphs we found with $(i-1)(j-1)$ edges.

We see that the graphs on the first row are all stars. Apart from stars and complete graphs, there are eight entries in Table 3.1 for graphs with $(i-1)(j-1)$ edges, these are $G_1, G_2, \ldots, G_8$. These graphs are depicted below. Although these are not disjoint union of cliques and stars, they can be turned into disjoint unions of cliques and

		i					
		2	4	6	8	10	12
	2	K_2	$K_{1,3}$	$K_{1,5}$	$K_{1,7}$	$K_{1,9}$	$K_{1,11}$
	3	-	K_4	G_5	-	-	-
j	4	-	G_1	K_6	-	-	-
	5	-	G_2	G_6	K_8	-	-
	6	-	G_3	G_7	-	K_{10}	-
	7	-	G_4	G_8	-	-	K_{12}

Table 3.1: Chordal graphs with $(i-1)(j-1)$ edges.

		i					
		2	4	6	8	10	12
	2	1	3	5	7	9	11
	3	-	6	10	13	17	21
j	4	-	9	15	18	24	30
	5	-	12	20	28	30	38
	6	-	15	25	31	45	47
	7	-	18	30	34	46	66

Table 3.2: Maximum number of edges in connected chordal graphs for various i and j.

stars with a few modifications. For G_1, G_2 and G_3 this can be done by changing one endpoint of at most two edges.

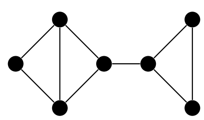

Figure 3.2: G_1

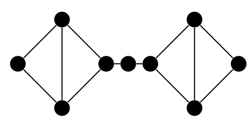

Figure 3.3: G_2

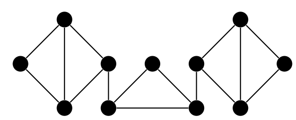

Figure 3.4: G_3

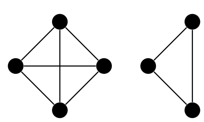

Figure 3.5: G'_1

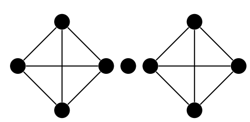

Figure 3.6: G'_2

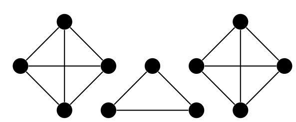

Figure 3.7: G'_3

G_4 is similar to G_2, and it can be turned into a graph that consists of cliques of size four in a similar manner to G_2. G_5 is two stars that share a common vertex. For G_6 we get the disjoint union of clique and a star by changing one endpoint of two edges.

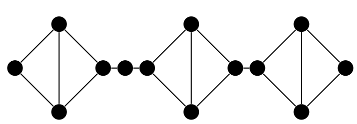

Figure 3.8: G_4

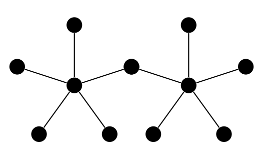

Figure 3.9: G_5

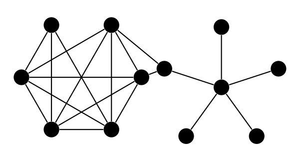

Figure 3.10: G_6

From Observation 3.3, we have that for even i, we get $(i-1)(j-1)$ edges with cliques of size i and $i-1$. G_7 can be seen as two cliques of size $i-1$ connected by an $(i-1)$-star. G_8 is similar to G_2, instead of cliques on four vertices we have cliques on six vertices.

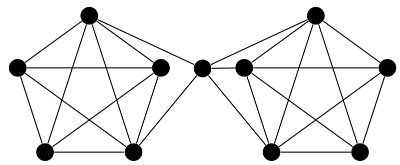

Figure 3.11: G_7

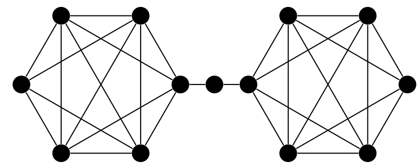

Figure 3.12: G_8

We have not listed every chordal graph we found with $(i-1)(j-1)$ edges here, but all of them can be said to be similar to graphs that consist of just cliques and stars. In fact, all of them can be turned into graphs that only consists of disjoint cliques and stars by making modifications similar to the ones we have made for the graphs presented in this section.

The graphs we have presented in this section share some common properties. We found that, apart from cliques on $i-1$ vertices, there is no chordal graph on up to 13 vertices with $(i-1)(j-1)$ edges that is factor-critical. Also, every graph we have presented in this section is interval. There are, however, chordal graphs with $(i-1)(j-1)$ edges that are not interval graphs. An example of such a graph is given in Figure 3.13, this graph has the same matching number, maximum degree and number of edges as G_4.

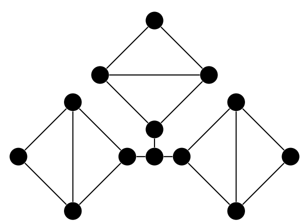

Figure 3.13: A chordal graph with $(i-1)(j-1)$ edges that is not an interval graph.

In the next sections we will prove that for even i, no graph in $M_{\nu,\mathcal{INT}}(i,j)$ has more than $(i-1)(j-1)$ edges. In particular, we will see that every graph we have presented in this section, apart from the graph in Figure 3.13, is edge-extremal in $M_{\nu,\mathcal{INT}}(i,j)$. But first, we will start with a subclass of interval graphs, namely trivially perfect graphs.

3.2 Trivially Perfect Graphs

In this section, we will solve the edge-extremal problem for trivially perfect graphs. Let $\mathcal{TP}$ be the class of trivially perfect graphs. Since a graph that consists of disjoint stars is trivially perfect, Figure 3.1 gives us a trivially perfect graph with $(i-1)(j-1)$ edges.

Theorem 3.4. For an edge-extremal graph $G \in M_{\nu,\mathcal{TP}}(i,j)$ we have that

1. if i is odd, then $|E(G)| = (i-1)(j-1) + \lfloor \frac{i-1}{2} \rfloor \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor$,
2. if i is even, then $|E(G)| = (i-1)(j-1)$.

Proof. The case when i is odd follows from that the edge-extremal graphs in $M_{\nu,\mathcal{GEN}}(i,j)$ we described in Chapter 2 are trivially perfect.

The proof for the case when i is even is by contradiction. Suppose we have a trivially perfect graph G such that $|E(G)| > (i-1)(j-1)$. For a connected component C in G let $i_C = \Delta(C)+1$ and $j_C = \nu(C)+1$. Since $|E(G)| > (i-1)(j-1)$, G has a connected component C, where $|E(C)| > (i_C-1)(j_C-i)$. Consider a vertex v that is universal in C. Similar to the argument in the introduction to this chapter, if $\nu(C-\{v\}) < \nu(C)$, then the disjoint union of an (i_C-1)-star and $C-\{v\}$ has the same matching number, maximum degree and number of edges as C. We can remove such vertices until C has no universal vertex whose removal reduces the matching number. If removing a universal vertex does not decrease the matching number, then the connected component has a near-perfect matching. If C has a near-perfect matching, $|C| \leq i$ and C has the maximum number of edges given i_C and j_C, then we have that C is a clique. Since i is even we have that $|C| \leq i-1$. Observation 3.3 tells us that $|E(C)| \leq (i_C-1)(j_C-i)$. This is a contradiction. □

3.3 Interval Graphs

In this section, we will solve the edge-extremal problem for interval graphs. Let $\mathcal{INT}$ be the class of interval graphs. For even i, recall that $(i-1)(j-1)$ is an upper bound on the number of edges for graphs in $M_{\nu,\mathcal{UNIT}}(i,j)$, a proper subset of $M_{\nu,\mathcal{INT}}(i,j)$. Depending on i and j this number could not always be reached. Since a disjoint union of stars is interval, the edge-extremal interval graphs has at least $(i-1)(j-1)$ edges.

To prove that no interval graph exceeds $(i-1)(j-1)$ edges, we will first prove that there is an edge-extremal interval graph that has no connected component with more than i vertices.

Lemma 3.5. For all i and j, there is an edge-extremal graph in $M_{\nu,\mathcal{INT}}(i,j)$ with no connected component that has more than i vertices.

Proof. Assume there is a graph $G \in M_{\nu,\mathcal{INT}}(i,j)$ that has a connected component with more than i vertices. We will present a procedure to construct a new graph $G' \in M_{\nu,\mathcal{INT}}(i,j)$ with no connected component with more than i vertices with at least as many edges. Let C be a connected component in G with more than i vertices. If C has a vertex whose removal reduces the matching number, we may by Observation 3.1 then remove this vertex and add an $(i-1)$-star to the graph. If we have that $|V(C)| \leq i$ after iteratively removing such vertices then we are done.

By Gallai's lemma we now have that $|V(C)| = 2\nu(C) + 1$. Consider an interval representation of C, and let $v_1, v_2, \dots, v_i, \dots, v_{|V(C)|}$ be the vertices of C ordered by the start point of the intervals representing the vertices. Let $V_i = \{v_1, v_2, \dots, v_i\}$.

We construct a new graph G' by taking the disjoint union of K_i and $G[V \setminus V_i]$. We claim that if G is an edge-extremal graph, then so is G'. To do this we need to verify that (1) $\Delta(G') < i$, (2) $\nu(G') < j$ and (3) $|E(G')| \geq |E(G)|$.

(1) Since $\Delta(K_i) = i - 1$ and $\Delta(G) < i$, we have that $\Delta(G') < i$.

(2) The number of vertices in G' is

$$|V(G)| - i + |V(K_i)| = 2\nu(G) + 1 - i + i = 2\nu(G) + 1.$$

Which means that we have the following upper bound on the matching number of G'

$$\nu(G') \leq \left\lfloor \frac{|V(G')|}{2} \right\rfloor = \left\lfloor \frac{2\nu(G)+1}{2} \right\rfloor = \nu(G) < j.$$

(3) We need to verify that $|\{v_a v_b \in E(G) \mid v_a \in V_i \wedge v_b \in V \setminus V_i\}| \leq |E(K_i)| - |E(G[V_i])|$. Since the vertices are sorted by the start point of their interval in the interval representation of G, for vertices $v_a \in V_i \setminus \{v_i\}$ and $v_b \in V \setminus V_i$, if the interval of v_a intersects the interval of v_b, the interval of v_a also intersects the interval of v_i. In other words, for $a < i$ and $b > i$, if $v_a v_b \in E(G)$ then $v_a v_i \in E(G)$, which means that v_i is adjacent to every vertex in $V_i \setminus \{v_i\}$ adjacent to some vertex in $V \setminus V_i$. Since $|V_i \setminus \{v_i\}| = i - 1$ and $d(v_i) < i$, we have that if v_i is adjacent to p vertices in $V \setminus V_i$, then there are at least p vertices in V_i which v_i is not adjacent to. For a vertex $u \in V_i$ that v_i is not adjacent to, we have that u is not adjacent to any vertex in $V \setminus V_i$ and thereby we have that $d(u) \leq i - 2$. Since v_i is adjacent to every vertex of degree $i - 1$ in V_i, we can change the interval of v_i such that v_i is first in the sorted list and adjacent

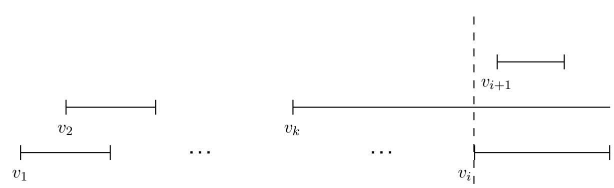

Figure 3.14: Any interval crossing the dashed line intersects the interval of v_i.

to every vertex in $V_i \setminus \{v_i\}$. By doing this we get a graph that belongs to $M_{\nu,\mathcal{INT}}(i,j)$ with at least as many edges. We can repeat this procedure until there are no edges incident to vertices in both V_i and $V \setminus V_i$. It follows that $|\{v_a v_b \in E(G) \mid v_a \in V_i \wedge v_b \in V \setminus V_i\}| \leq |E(K_i)| - |E(G[V_i])|$.

We construct the new graph by repeating this procedure until G does not contain any connected components with more than i vertices. □

We will use the preceding lemma to prove the upper bound, but first, as an example of how the procedure works, we will be running the procedure on G_6 from the previous section. Notice that none of the graphs with $(i-1)(j-1)$ edges from the preceding section are factor-critical. In the proof, it was important that we could assume that the edge-extremal graphs are all factor-critical since then we know that $|V(G)| = 2\nu(G)+1$. However, this is still the case for G_6, even though G_6 is not factor-critical. To explain the procedure in the proof, we will use G_6 as an example without removing the vertex whose removal reduces the matching number.

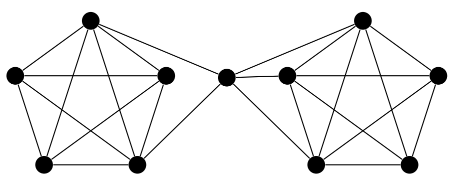

Figure 3.15: G_6

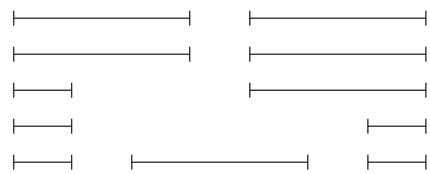

Figure 3.16: Interval representation of G_6.

For this graph $\Delta(G_6) = 5$ and $\nu(G_6) = 5$, so we have that $i = 6$ and $j = 6$. Figure 3.16 is an interval representation of G_6. By removing the i first vertices in this interval representation of G_6, a clique on five vertices remain. In this graph there is only one vertex adjacent to vertices in both V_i and $V \setminus V_i$, we see that the number of adjacent vertices of this vertex in $V \setminus V_i$ is exactly the number of vertices not adjacent to this vertex in V_i. We see that by running this procedure one time, the new graph has no connected component with more than i vertices, while the matching number, maximum degree and the number of edges remain the same.

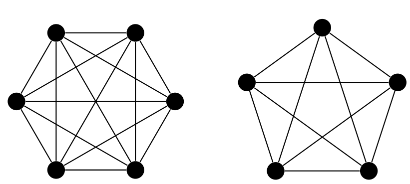

Figure 3.17: G'_6

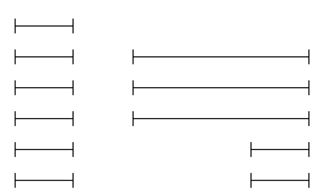

Figure 3.18: Interval representation of G'_6.

We will now prove the upper bound for interval graphs.

Theorem 3.6. For an edge-extremal graph $G \in M_{\nu,\mathcal{INT}}(i,j)$ we have that

1. if i is odd, then $|E(G)| = (i-1)(j-1) + \lfloor \frac{i-1}{2} \rfloor \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor$,
2. if i is even, then $|E(G)| = (i-1)(j-1)$.

Proof. The case when i is odd follows from that the edge-extremal graphs in $M_{\nu,\mathcal{GEN}}(i,j)$ we described in Chapter 2 are interval graphs.

Now consider the case when i is even. For all i and j there is a graph that consist of disjoint stars in $M_{\nu,\mathcal{INT}}(i,j)$ that has $(i-1)(j-1)$ edges. It remains to show that no graph in $M_{\nu,\mathcal{INT}}(i,j)$ has more than $(i-1)(j-1)$ edges. Assume for contradiction that there is a graph $G \in M_{\nu,\mathcal{INT}}(i,j)$ such that $|E(G)| > (i-1)(j-1)$. Since $|E(G)| > (i-1)(j-1)$, G has a connected component C such that $|E(C)| > (i_C - 1)(j_C - 1)$, where $i_C = \Delta(C) + 1$, and $j_C = \nu(C) + 1$. By Lemma 3.5 we can assume that $|V(C)| \leq i$.

It follows from Observation 3.1 and Gallai's lemma that we can assume that C is factor-critical and thereby $|V(C)| = 2\nu(C) + 1$. We have that C has a near-perfect matching, $|C| \leq i_C$ and G is edge-extremal. Hence C has to be a complete graph. From Observation 3.3 we have that $|E(C)| \leq (i-1)(j-1)$ which is a contradiction and the proof is complete. $\square$

As we have seen in this chapter, the edge-extremal interval graphs are not unique. A graph G that is a disjoint union of cliques on i and $i-1$ vertices, and $(i-1)$-stars is edge-extremal if $\nu(G) = j-1$. G_4 from the previous section is also an edge-extremal graph in $M_{\nu,\mathcal{INT}}(6,3)$. Some examples of edge-extremal graphs in $M_{\nu,\mathcal{INT}}(6,3)$ are given below.

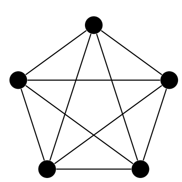

Figure 3.19: K_5

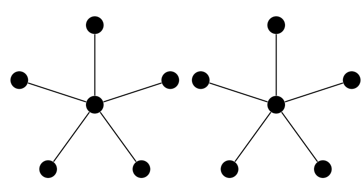

Figure 3.20: Two 5-stars

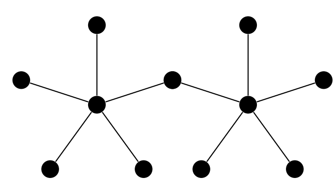

Figure 3.21: G_4

Chapter 4

Graphs Whose Induced Matching Number and Maximum Degree are Bounded

In this chapter, we will study a slightly different problem compared to the previous chapter. We will initiate the study on the problem of finding the maximum number of edges in graphs whose maximum induced matching number and maximum degree are bounded. We will solve this problem for split graphs, interval graphs and cographs.

Let us first have a look at general graphs. The size of a maximum induced matching in a complete graph is one. The same is true for a complete bipartite graph. Since we can have more vertices in a complete bipartite graph than in a complete graph, with the same degree bound, complete bipartite graphs seem like natural candidates for edge-extremal general graphs.

To get an idea of how edge-extremal graphs in $M_{\mu,\mathcal{GEN}}(i, j)$ might look like, we generated and tested every connected graph on up to 11 vertices. We found that there are graphs with a higher number of edges than a disjoint union of complete bipartite graphs. An example of this is the graph that results when a single edge is removed from a complete bipartite graph and both endpoints of this edge are connected to a

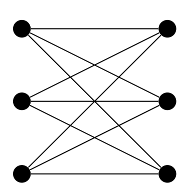

Figure 4.1: $K_{3,3}$

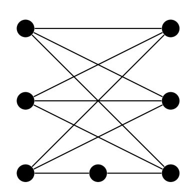

Figure 4.2: $K'_{3,3}$

new vertex. Let $K'_{i-1,i-1}$ be the graph we obtain when we start from $K_{i-1,i-1}$. The induced matching number of both of these graphs is 1. The number of edges in $j-1$ disjoint copies of $K'_{i-1,i-1}$ is $((i-1)^2+1)(j-1)$. This is thus a lower bound on the number of edges in edge-extremal graphs in $M_{\mu,\mathcal{GEN}}(i,j)$.

4.1 Chordal Graphs

In this section, we will look at the problem for chordal graphs. In Chapter 3 we conjectured that, for even i, edge-extremal graphs in $M_{\nu,\mathcal{CHO}}(i,j)$ have fewer edges than edge-extremal graphs in $M_{\nu,\mathcal{GEN}}(i,j)$. In this section, we will be able to prove that edge-extremal graphs in $M_{\mu,\mathcal{CHO}}(i,j)$ have fewer edges than edge-extremal graphs in $M_{\mu,\mathcal{GEN}}(i,j)$, for all $i>2$ and all j.

We start by making an observation on the structure of some edge-extremal chordal graphs and its clique trees. Note, first of all, that $\frac{i(i-1)}{2}(j-1)$ is a trivial lower bound on the number of edges in edge-extremal chordal graphs. This corresponds to $(j-1)$ cliques of size i.

Observation 4.1. There is an edge-extremal chordal graph G, and a clique tree $\mathcal{T}$ of G, such that for a leaf clique L with parent clique P, we have that $|L\setminus P|=1$.

Proof. Suppose we have a chordal graph G, and a clique tree $\mathcal{T}$ of G with a leaf clique L that has parent clique P such that $|L\setminus P|\geq 2$. An induced matching that contains no vertices from $P\cup L$ is not maximal since a pair of vertices from $L\setminus P$ could be added to the induced matching. It follows that $\mu(G)>\mu(G[V\setminus(L\cup P)])$. Since the vertices in P form a clique, we have that

$$\begin{aligned}|E(G)|-|E(G[V\setminus(L\cup P)])| &\leq i-1+i-2+\cdots+i-|L\cup P|\\ &\leq \sum_{k=1}^{|L\cup P|} i-k \leq \sum_{k=1}^{i} i-k = |E(K_i)|\end{aligned}$$

This means that the disjoint union of K_i and $G[V\setminus(L\cup P)]$ has at least as many edges as G. Since $\mu(K_i)=1$ the induced matching number has not changed and the proof is complete. □

With this observation, we can give an upper bound on the number of edges in graphs in $M_{\mu,\mathcal{CHO}}(i,j)$. This upper bound proves that no graph is edge-extremal in both $M_{\mu,\mathcal{CHO}}(i,j)$ and $M_{\mu,\mathcal{GEN}}(i,j)$ for $i>2$.

Observation 4.2. For an edge-extremal graph $G\in M_{\mu,\mathcal{CHO}}(i,j)$, $i>2$, we have that $|E(G)|<(i-1)^2(j-1)$.

Proof. Let G be an edge-extremal chordal graph in $M_{\mu,\mathcal{CHO}}(i,j)$ and $\mathcal{T}$ be a clique tree of G. By Observation 4.1 we can assume that for a leaf clique L and its parent P we have that $|L \setminus P| = 1$. Let v be this vertex. We want to show that there is a set of vertices whose removal from the graph reduces the number of edges by at most $(i-1)^2$ while reducing the induced matching number by at least one. We will remove $N(u)$ for an arbitrary vertex $u \in N(v)$ from the graph.

First consider every maximum induced matching that contains v. For a vertex $u \in N(v)$, we have that $\mu(G \setminus N(u)) < \mu(G)$ since $N(v) \subseteq N(u)$.

Let us now consider every maximum induced matching M that does not contain v. In this case we have that for all vertices $u \in N(v)$, there is a vertex $w \in N(u)$ that is in the induced matching. Since otherwise vu could be added to M, which would imply that the induced matching is not maximum. This implies that $\mu(G[V \setminus N(u)]) < \mu(G)$.

The procedure of removing $N(u)$ from the graph can be repeated until $\mathcal{T}$ contains one maximal clique. Since $|E(G)| - |E(G[V \setminus N(u)]| \leq (i-1)^2$, we have that $|E(G)| \leq (i-1)^2(j-2) + \frac{i(i-1)}{2} < (i-1)^2(j-1)$. $\square$

In Chapter 3, we conjectured that there are edge-extremal graphs in $M_{\nu,\mathcal{CHO}}(i,j)$ that consist of disjoint cliques and stars. Note that the induced matching number of a star and a complete graph are both one. Thus, there is no advantage of using stars when trying to construct edge-extremal chordal graphs. Could it be that edge-extremal chordal graphs are disjoint unions of cliques?

By running tests, we found that there is no chordal graph on up to 13 vertices with more than $\frac{i(i-1)}{2}(j-1)$ edges. The results for graphs on up to 13 vertices are given in Table 4.1.

		i						
		2	3	4	5	6	7	8
	2	1	3	6	10	15	21	28
	3	-	6	12	20	30	36	38
j	4	-	9	17	23	29	34	38
	5	-	12	15	22	26	30	32
	6	-	-	13	15	19	22	24

Table 4.1: Maximum number of edges in connected chordal graphs on up to 13 vertices for various i and j.

Of course, this does not prove that disjoint union of K_i cliques is edge-extremal. Even if the disjoint copies of K_i were edge-extremal, we found that the edge-extremal graphs are not unique. G_2 from the previous chapter is an example, G_2 has the same induced matching number, maximum degree and number of edges as two disjoint copies of K_4.

These graphs, depicted in Figure 4.2 and Figure 4.3, have maximum induced matching of size 2, maximum degree of 3 and 12 edges.

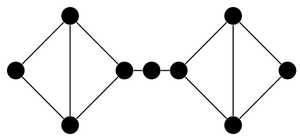

Figure 4.3: G_2

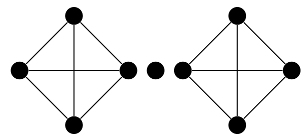

Figure 4.4: G_2'

4.2 Interval Graphs

Although we did not succeed to solve the problem on chordal graphs, we will see in this section, we are able to solve it for interval graphs. Since a graph that consist of a disjoint union of cliques is interval, we get an interval graph with $\frac{i(i-1)}{2}(j-1)$ edges by taking disjoint union of $j-1$ cliques on i vertices.

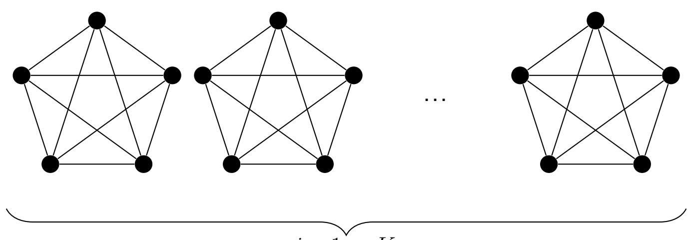

Figure 4.5: Interval graph in $M_{\mu,\mathcal{INT}}(i,j)$ with $\frac{i(i-1)}{2}(j-1)$ edges.

To prove that this bound is also an upper bound, we first need to prove that there is an edge-extremal interval graph with no connected component with more than i vertices. We do this in a similar manner as we did in the proof of Lemma 3.5 which states that there are edge-extremal graphs in $M_{\nu,\mathcal{INT}}(i,j)$ with no connected component with more than i vertices.

Lemma 4.3. For all i and j, there is an edge-extremal graph in $M_{\mu,\mathcal{INT}}(i,j)$ with no connected component with more than i vertices.

Proof. Assume that there is a graph $G \in M_{\mu,\mathcal{INT}}(i,j)$ that has a connected component with more than i vertices. We will present a procedure to construct a new graph G' with no connected component of more than i vertices while $\Delta(G') < i$, $\nu(G') < j$ and $|E(G')| \geq |E(G)|$. Let C be a connected component in G with more than i vertices. Consider an interval representation of C, and let $v_1, v_2, \ldots, v_i, \ldots, v_{|V(C)|}$ be the vertices of C ordered by the start point of the interval representing the vertices. Let $V_i = \{v_1, v_2, \ldots, v_i\}$.

We construct a new graph G' by taking the disjoint union of K_i and $G[V \setminus V_i]$. We claim that if G is an edge-extremal graph, then so is G'. By the same argument as in the proof of Lemma 3.5, we have that $\Delta(G) < i$ and $|E(G')| \geq |E(G)|$.

It remains to show that $\mu(G') < j$. Assume for contradiction that $\mu(G') \geq j$. We have that $\mu(G') = \mu(K_i) + \mu(G[V \setminus V_i])$. Since $\mu(G[V \setminus V_i]) \leq \mu(G)$ we have that, by assumption, $\mu(G[V \setminus V_i]) = \mu(G)$. It follows that there has to be a maximum induced matching M such that $M \cap V_i = \emptyset$. Since the vertices are ordered by the start point of the intervals representing the vertices and C is connected, we have that v_1 and v_2 are adjacent. Since $\Delta(C) < i$ we have that neither v_1 nor v_2 are adjacent to any vertex in $V \setminus V_i$. We arrive at a contradiction since if $M \cap V_i = \emptyset$, then the induced matching would not be maximum since $v_1 v_2$ could then be added to the induced matching. □

Theorem 4.4. For an edge-extremal graph $G \in M_{\mu,\mathcal{INT}}(i, j)$, we have that $|E(G)| = \frac{i(i-1)}{2}(j-1)$.

Proof. By Lemma 4.3 we may assume that there is an edge-extremal graph $G \in M_{\mu,\mathcal{INT}}(i, j)$ that has no connected component with more than i vertices. Since no graph on up to i vertices has more edges than K_i and $\mu(K_i) = 1$, we have that G consist of $j-1$ disjoint cliques on i vertices. The number of edges in this graph is $\frac{i(i-1)}{2}(j-1)$. □

Now that we have this upper bound, we see that both the graph in Figure 4.3 and in Figure 4.4 is edge-extremal in $M_{\mu,\mathcal{INT}}(4, 3)$. So we have that there are edge-extremal interval graphs that has connected components with more than i vertices, and the edge-extremal interval graphs are not unique. Note that this result also gives a tight bound on the number of edges in edge-extremal graphs in $M_{\mu,\mathcal{UNIT}}(i, j)$, since a disjoint union of cliques is unit interval.

4.3 Split Graphs

In this section, we will solve the problem for split graphs. Note that a disjoint union of cliques is not a split graph. Therefore above result does not apply to split graphs. First, we observe that for every split graph G we have that $\mu(G) = 1$.

Lemma 4.5. For every split graph G and an induced matching M in G, $|M| = 1$.

Proof. Let (C, I) be a split partition of G. Since I is an independent set, every edge in M has at least one endpoint in C. Since C is a clique, the induced subgraph $G[V(M)] \neq M$ if M contains more than one edge incident to a vertex in C. □

Since split graphs cannot be disconnected, apart from isolated vertices, we have the following corollary:

Corollary 4.5.1. For positive integers i_1, i_2 and j, and for two edge-extremal split graphs $G \in M_{\mu,\mathcal{SPLIT}}(i_1, j)$ and $G' \in M_{\mu,\mathcal{SPLIT}}(i_2, j)$, we have that $|E(G)| = |E(G')|$.

Therefore the problem of finding edge-extremal graphs in $M_{\mu,\mathcal{SPLIT}}(i, j)$ is the same as finding edge-extremal split graphs bounded just by their maximum degree. The solution to this problem is given in the following theorem.

Theorem 4.6. For an edge-extremal split graph G bounded by $\Delta(G) < i$ we have that $|E(G)| = \frac{i(i-1)}{2}$.

Proof. Assume we have an edge-extremal split graph G and let (C, I) be a split partition of G. If $I = \emptyset$, then C contains i vertices. A complete graph on i vertices has $\frac{i(i-1)}{2}$ edges. Suppose G has a non-empty independent set. We may assume that for every vertex $v \in C$ we have $d(v) = i-1$, since otherwise adding a vertex to I which is adjacent to only v increases the number of edges without exceeding the boundary on the maximum degree. Since every edge of G has at least one endpoint in C we may replace I with a set I' consisting of $i-|C|$ vertices where every vertex in I' is connected to every vertex in C. For every vertex $v \in C$ we have $d(v) = i-1$ and v is adjacent to every vertex in the graph. Every vertex in I' could also be adjacent to every other vertex in I'. We have that as long as $|I'| > 1$ the graph is not edge-extremal, if $|I'| = 1$ then G is a complete graph. $\square$

The solution for the more general problem of finding the maximum number of edges in graphs in $M_{\mu,\mathcal{SPLIT}}(i, j)$ follows immediately.

Corollary 4.6.1. For an edge-extremal graph $G \in M_{\mu,\mathcal{SPLIT}}(i, j)$ we have that $|E(G)| = \frac{i(i-1)}{2}$.

Comparing the maximum number of edges in graphs in $M_{\mu,\mathcal{SPLIT}}(i, j)$ to the maximum number of edges in graphs in $M_{\nu,\mathcal{SPLIT}}(i, j)$, presented in Chapter 2, we see that there are graphs that are edge-extremal in both $M_{\mu,\mathcal{SPLIT}}(i, j)$ and $M_{\nu,\mathcal{SPLIT}}(i, j)$. This is the only graph class we are studying for which this is the case.

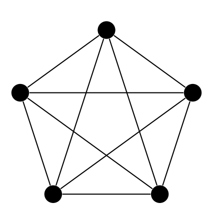

Figure 4.6: A graph that is edge-extremal in both $M_{\nu,\mathcal{SPLIT}}(5, 3)$ and $M_{\mu,\mathcal{SPLIT}}(5, 3)$.

4.4 Cographs

I this section we will solve the problem for cographs. Let $\mathcal{CO}$ denote the class of cographs. The edge-extremal graphs in $M_{\nu,\mathcal{GEN}}(i,j)$ we presented in Chapter 2 are cographs, but we see that the graph $K'_{i-1,i-1}$ we presented in the introduction to this chapter is not. We will see that, in contrast to the situation for edge-extremal graphs in $M_{\nu,\mathcal{CO}}(i,j)$ and $M_{\nu,\mathcal{GEN}}(i,j)$, an edge-extremal graph in $M_{\mu,\mathcal{CO}}(i,j)$ has strictly fewer edges than an edge-extremal graph in $M_{\mu,\mathcal{GEN}}(i,j)$.

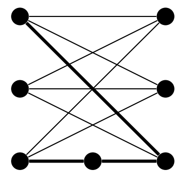

Figure 4.7: An induced P_4 in $K'_{3,3}$.

Since the class of cographs is exactly the class of P_4-free graphs, for every path on three vertices two nonconsecutive vertices have to be adjacent. This gives us the impression that bounding the maximum degree of a cograph, also bounds the number of vertices. The following lemma tells us that no matter what the induced matching number of the graph is, no cograph has more than $2\Delta(G)$ vertices.

Lemma 4.7. For a connected cograph $G = (V, E)$, we have that $|V| \leq 2\Delta(G)$, and if $|V| = 2\Delta(G)$ then G is a complete bipartite graph.

Proof. Let G be a connected cograph, with cotree T. Since G is connected the root of T is labelled 1. Let $k \geq 2$ be the number of children of the root node of T. For an arbitrary order of the children of the root node of T, let T_i be the subtree of T rooted at child i and n_i the number of leaf nodes in T_i. Since $k \geq 2$ we have that $n_i \leq \Delta(G)$ for all i. For a vertex v corresponding to some leaf node in T_1 we have that

$$\Delta(G) \geq d(v) \geq \sum_{i=2}^{k} n_i$$

For a vertex u corresponding to some leaf node in T_2 we have that

$$\Delta(G) \geq d(u) \geq n_1 + \sum_{i=3}^{k} n_i$$

Since $n_i \geq 1$, this implies that $n_1 \leq \Delta(G) - (k-2)$. We get the following upper bound on the number of vertices in G:

$$|V| = \sum_{i=1}^{k} n_i = n_1 + \sum_{i=2}^{k} n_i \leq \Delta(G) - (k-2) + \Delta(G)$$

For $k > 2$ we have that $|V| < 2\Delta(G)$. Consider the maximum number of vertices when $k = 2$, since $n_i \leq \Delta(G)$ we have $|V| = 2\Delta(G)$ when $n_1 = n_2 = \Delta(G)$. Since every vertex in T_1 is adjacent to every vertex in T_2 and $n_2 = \Delta(G)$ no pair of vertices in T_1 are adjacent, likewise no pair of vertices in T_2 are adjacent. G is, therefore, a complete bipartite graph. $\square$

Since the maximum induced matching number of a complete bipartite graph is one, this immediately solves the problem for cographs.

Theorem 4.8. For an edge-extremal graph $G \in M_{\mu,\mathcal{CO}}(i,j)$, G is the disjoint union of $j-1$ complete bipartite graphs $K_{i-1,i-1}$ and $|E(G)| = (i-1)^2(j-1)$

Proof. Let G be an edge-extremal in $M_{\mu,\mathcal{CO}}(i,j)$. Lemma 4.7 tells us that no connected graph in $M_{\mu,\mathcal{CO}}(i,j)$ has as many vertices as $K_{i-1,i-1}$. Since $\mu(K_{i-1,i-1}) = 1$ and every vertex in $K_{i-1,i-1}$ has degree $i-1$, we have that no cograph in $M_{\mu,\mathcal{CO}}(i,j)$ has as many edges as a disjoint union of $K_{i-1,i-1}$. We have that G is the disjoint union of $j-1$ complete bipartite graphs $K_{i-1,i-1}$. The number of edges in $K_{i-1,i-1}$ is $(i-1)^2$, the total number of edges in G is $(i-1)^2(j-1)$. $\square$

It follows that to bound the number of edges in connected cographs, it is sufficient to bound the maximum degree. The same is true for split graphs, for interval graphs however, bounding the maximum degree does not bound the number of edges since a long path is an interval graph. Another interesting observation is that, for these graph classes, there are edge-extremal graphs, for all i and j, that consist of disjoint union of connected components whose maximum induced matching number is one. For split graphs and cographs the edge-extremal graphs are unique, for interval graphs however, Figure 4.3 is an example of a connected graph that is edge-extremal and has induced matching number two.

Chapter 5

Conclusion

In this thesis, we have studied two extremal graph theory problems for various graph classes. One of the ways we have tried to get an insight into the solution to these problems is by running tests on graphs. In the first section of this chapter, we will discuss our approach to testing. In the proceeding section, we will look at another extremal graph theory problem. For this problem, we can use the practical framework we have developed for testing to see if we find graphs that are of interest. Finally, we will conclude the chapter by giving a summary of this thesis and stating some open problems.

5.1 The Practical Framework for Testing Hypotheses

In this section, we will describe our approach to generating and testing graphs. In Chapter 3 we made the hypothesis that the edge-extremal chordal graphs for even i consisted of disjoint cliques and stars and had $(i-1)(j-1)$ edges. If we were to find a counter-example of this hypothesis, we would immediately be able to discard it. We, therefore, wanted to generate chordal graphs and test those to see if we could find such a counter-example. Our aim in this thesis has been to gain an insight into the problem of finding the maximum number of edges in graphs satisfying the constraints. Therefore, our focus has not been on developing or implementing procedures that are as fast as possible, but rather creating a framework that works for our purposes.

5.1.1 Generating Graphs

To generate graphs efficiently, it is important only to generate the graphs we need. If we consider two graphs G and H with labelled vertices. We say that G and H are isomorphic if there is a mapping $M : V(G) \to V(H)$ such that for every pair of vertices $v_1, v_2 \in V(G)$, we have that $v_1v_2 \in E(G)$ if and only if $M(v_1)M(v_2) \in E(H)$. If G and H are isomorphic, they are structurally identical, meaning that for our purposes we do not want to generate both G and H. Below is an example of three isomorphic graphs, generating one of these graphs is sufficient.

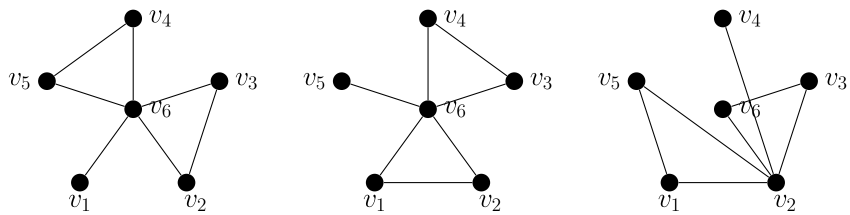

Figure 5.1: Isomorphic graphs.

We get a simple procedure that generates every graph on n vertices by branching on every pair of vertices, either add an edge between the pair or not. The problem with this procedure is that it does not take advantage of isomorphism. The procedure would generate $2^{\frac{n(n-1)}{2}}$ graphs, while the number of non-isomorphic graphs is much lower. For $n = 8$, the procedure would generate 268435456 graphs. However, there are only 12346 non-isomorphic graphs on 8 vertices [16].

For generating general graphs, we used a program called geng from Nauty [16], which considers isomorphism and generates small graphs efficiently. Using this program, we were able to generate every connected graph on up two 13 vertices. On a computer running 12 threads at 3.5 GHz, this can be done in about a month. On the same computer, generating every connected graph on 14 vertices would take more than a decade.

For generating chordal graphs, we chose to use geng, proceeded by a self-implemented procedure that recognises which of the generated graphs are chordal. A more natural approach would be to generate chordal graphs directly. However, in [19] and [21] the authors found that geng is very efficient and that they were able to generate larger graphs using geng than a self-implemented method. Based on these previous experiences, the limited time for a master thesis and the fact that implementation is not our main focus, we decided to use geng for generating chordal graphs.

The proportion of chordal graphs relative to the number of general graphs decreases as the graphs get bigger, which means that this approach gets less efficient as the number of vertices increases. For graphs on up to 12 vertices, there are about 9000 times as many non-chordal as there are chordal graphs. For graphs on up to 13 vertices, there are more than 160000 times as many non-chordal as there are chordal graphs.

n	**Chordal graphs**	**General graphs**
3	2	2
4	5	6
5	15	21
6	58	112
7	272	853
8	1614	11117
9	11911	261080
10	109539	11716571
11	1247691	1006700565
12	17566431	164059830476
13	305310547	50335907869219

Table 5.1: The number of connected non-isomorphic graphs.

From Chapter 1 we have that a graph is chordal if and only if it has a perfect elimination ordering, and any simplicial vertex can start this ordering. There are algorithms, for example Maximum Cardinality Search [18], that finds a perfect elimination ordering in time $O(n+m)$ if the graph is chordal. However, for our purposes, we found that a straight forward procedure that finds a perfect elimination ordering if the graph is chordal, in time $O(n^4)$ was faster in practice.

The straight forward procedure works by repeatedly iterating through the vertices, deleting every simplicial vertex. If the procedure does not find any simplicial vertices while the graph is not empty, then it returns that the graph is not chordal. Since every graph on 3 vertices is chordal, we stop the procedure when it has deleted $n-3$ vertices. Since every chordal graph has at least two simplicial vertices, the algorithm can return that the graph is not chordal if the procedure does not find more than one simplicial vertex while iterating through the vertices. The outer loop of the algorithm then runs at most $\frac{n}{2}$ times.

Since we wanted a procedure that was fast for small graphs, in practice the constant factor associated with the running time of an algorithm may be more important than the asymptotic running time, since the size of the input is bounded. One reason that the constant factor is smaller for our straight forward procedure is that there are no data structures which need to be initialised. Also, geng outputs a string which gives us the upper triangular adjacency matrix which means that we could run our procedure directly on the output from geng. We found that it was more efficient to use the adjacency matrix, using entry (i,i) to indicate whether vertex i has been deleted, than creating the adjacency list representation of the graph.

Another reason the constant factor is smaller may be that Maximum Cardinality

Search always finds an ordering of the vertices regardless of whether the graph is chordal or not. We then need to verify whether the ordering is a perfect elimination ordering to determine if the graph is chordal. In contrast, the straight forward procedure returns that the graph is not chordal as soon as it does not find any simplicial vertices. 70% of all connected graphs on 12 vertices have no simplicial vertex, meaning that for graphs on 12 vertices the procedure is effectively cubic for 70% of the graphs. The time spent recognising which graphs were chordal took approximately $\frac{1}{4}$ of the time it took to generate the graphs.

5.1.2 Testing the Graphs

The objective with the tests we have done has been to find which sets the graphs belong to. In particular we wanted to find, for a chordal graph G, for which i and j does G belong to $M_{\nu,\mathcal{CHO}}(i,j)$ and $M_{\mu,\mathcal{CHO}}(i,j)$, and for a general graph G for which i and j does G belong to $M_{\mu,\mathcal{GEN}}(i,j)$. This way, we could compare the number of edges in G to the number of edges in the graph with the highest number of edges that had previously been determined to belong to the same set.

For all of these tasks we need to determine the maximum degree of the graph. This is done by simply counting the number of edges incident to each vertex. Since we used geng to generate graphs with a given number of edges, we did not need to count the total number of edges ourselves. We also need to determine the matching number and the induced matching number.

The problem of finding the matching number of a graph can be solved by Edmonds' Blossom algorithm in time $O(mn^2)$ [10]. We used a Java implementation [2] of this algorithm as a foundation for creating an algorithm that works for testing graphs of the format used by geng.

The problem of finding a maximum induced matching in general graphs is intractable, but for chordal graphs the problem is efficiently solvable [5]. However, since the graphs we tested were small, and since the number of chordal graphs is small relative to the number of general graphs, we found that an exponential time algorithm was sufficiently fast. In fact, since the graphs we are dealing with have at most 13 vertices, we can consider our algorithm to be of constant time. The bottleneck for our approach is on generating the chordal graphs, not finding the maximum induced matching. We tested every chordal graph on up to 13 vertices and every general graph on up to 11 vertices.

We get an exhaustive search algorithm for finding the size of a maximum induced matching by checking every subset of vertices and then returning the size of the largest set that is an induced matching in the graph. However, we found that it was more efficient to check if there is a set of 4 vertices that induces a matching in the graph, and if there such a set of vertices we search for a set on 6 vertices that induces a matching and so forth. Most graphs of the size we tested do not have an induced matching of size more than two, meaning that the procedure would stop after iterating through every subset of 6 vertices and output that the size of a maximum induced matching in the graph is 2.

In the next section, we will take a look at another extremal graph theory problem. We will also see that the framework for testing hypotheses that we described in this section, can without much work be used on this problem as well.

5.2 A Related Problem: Minimal Feedback Vertex Sets in Chordal Graphs

The problem of edge-maximality under some constraints that we studied in this thesis is just one example of extremal problems on graphs. Another extremal graph problem asks for the maximum number of minimal feedback vertex sets in graphs on n vertices. A feedback vertex set in a graph $G = (V, E)$ is a set of vertices F such that $G[V \setminus F]$ is acyclic. A minimal feedback vertex set is a set of vertices F such that no proper subset $F' \subset F$ is a feedback vertex set.

An induced forest is a set of vertices F such that $G[F]$ is acyclic. F is a maximal induced forest if no set F' is an induced forest and $F \subset F'$. The problem of finding the maximum number of maximal induced forests in graphs on n vertices is the same problem as finding the maximum number of minimum feedback vertex sets. To see this, we observe that for every minimal feedback vertex set F, $V \setminus F$ is a maximal induced forest. It follows that the number of minimal feedback vertex sets in a graph is equal to the number of maximal induced forests.

5.2.1 Maximum Number of Minimal Feedback Vertex Sets in Chordal Graphs

For general graphs, Fomin, Gaspers, Pyatkin and Razgon showed that 1.8638^n is an upper bound on the maximum number of minimal feedback vertex sets [11]. The authors also gave an infinite family of graphs with $105^{\frac{n}{10}} \approx 1.5926^n$ minimal feedback vertex sets. This family consists of disjoint unions of the graph given in Figure 5.3.

Since the graph in Figure 5.3 is not chordal, Couturier, Heggernes, van't Hof and Villanger studied the same question on chordal graphs [7]. For chordal graphs they proposed $10^{\frac{n}{5}} = 1.585^n$ as a tight bound on the number of minimal feedback vertex sets. In a complete graph on k vertices, any set of $k - 2$ vertices intersects every cycle in the graph. A complete graph on k vertices has therefore $\binom{k}{k-2}$ minimal feedback

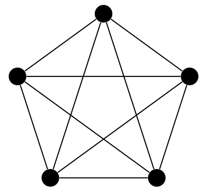

Figure 5.2: Chordal graph with 10 minimal feedback vertex sets.

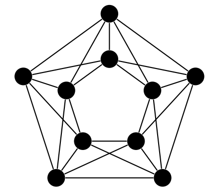

Figure 5.3: Non-chordal graph with 105 minimal feedback vertex sets.

vertex sets. For $k = 5$, $\binom{k}{k-2}$ is maximized, and $\binom{5}{3} = 10$. A graph on n vertices that is a disjoint union of $\frac{n}{5}$ cliques of size 5 has $10^{\frac{n}{5}}$ minimal feedback vertex sets.

We see that infinite families of graphs with extremal properties are usually generated by taking disjoint union of copies of a single graph. The known examples of graphs with maximum number of minimal feedback vertex sets are all disconnected. The following observation tells us that there are also connected graphs with the maximum number of minimal feedback vertex sets.

Observation 5.1. For all n, there is a connected graph on n vertices that has the maximum number of minimal feedback vertex sets.

Proof. The proof is by contradiction. Let $\mathcal{F}_n$ be the set of graphs that have the maximum number of minimal feedback vertex sets on n vertices. $\mathcal{F}_n$ is also the set of graphs that have the maximum number of maximal induced forests on n vertices. Let G be the graph that has the least number of connected components in $\mathcal{F}_n$. Let $\mathcal{K}$ be the set of the connected components of G. If $|\mathcal{K}| = 1$ then G is a connected graph. For two connected components $K_1, K_2 \in \mathcal{K}$, construct G' by adding the edge v_1v_2 for an arbitrary vertex v_1 in K_1 and an arbitrary vertex v_2 in K_2. We now claim that G' and G have the same number of minimal feedback vertex sets, which contradicts the assumption that G was the graph with the minimum number of connected components in $\mathcal{F}_n$. Recall that for every minimal feedback vertex set F, we have that $V \setminus F$ is a maximal induced forest.

If the number of minimal feedback vertex sets is lower in G' than in G, there is a set of vertices $F \subset V(G)$ that is a maximal induced forest in G, but not G'. We have that either (1) $G[F]$ has a cycle, or (2) F induces a forest, but the set is not maximal.

(1) If $G[F]$ has a cycle, the same cycle would be in $G'[F]$ since $E(G) = E(G') \setminus \{v_1v_2\}$.

(2) Observe that since v_1v_2 is not part of any cycle, we have that if F is maximal in G, it is also maximal in G'.

This contradicts the assumption that the number of minimal feedback vertex sets is lower in G' than G.

If the number of minimal feedback vertex sets is higher in G' than in G, there is a set of vertices $F \subset V(G)$ that is a maximal induced forest in G', but not G. We have that (1) $G'[F]$ has a cycle or (2) $G'[F]$ is acyclic but F is not maximal.

(1) If there is a cycle in $G'[F]$ but not $G[F]$, v_1v_2 has to be part of this cycle since $E(G) = E(G) \setminus \{v_1v_2\}$. But v_1v_2 cannot be part of the cycle since its removal increases the number of connected components.

(2) If F is not maximal, there exists a vertex which when included creates a cycle in G' but not G. v_1v_2 has to be part of that cycle, but v_1v_2 is not part of any cycle.

This contradicts the assumption that the number of minimal feedback vertex sets is higher in G' than G.

The number of minimal feedback vertex sets is neither higher or lower in G' than G. We have arrived at a contradiction since $G' \in \mathcal{F}_n$ and has fewer connected components than G. □

As a consequence of this observation, we see that disjoint union of cliques on 5 vertices are not the only graphs with the maximum number of minimal feedback vertex sets. For a graph that consists of more than one connected component, we may add an edge between any pair of vertices in different connected components.

In the proof of the upper bound presented by the authors in [7], there was a problem verifying one of the cases of the proof. Using the framework we have described in the previous section, we can easily test chordal graphs of up to 13 vertices, to see if we find a graph that contradicts the hypothesis that 1.585^n is a upper bound. It follows from Observation 5.1 that we get the maximum number of minimal feedback sets for all n by testing connected graphs as we have done for the other problems. As it was the case for induced matchings, a trivial algorithm that iterates through every subset of vertices is sufficiently fast to test every connected graph on up to 13 vertices. In Table 5.2 $G + H$ is the disjoint union of G and H.

We see that for chordal graphs on up to 13 vertices, there is no graph that has more than $10^{\frac{n}{5}}$ minimal feedback vertex sets. For up to 7 vertices there is no other graph with as many minimal feedback vertex sets as the graph listed in the table. For $8 \leq n \leq 12$ there is one other graph that has the same number of minimal feedback vertex sets, we get this graph by adding an edge with one endpoint in each of the cliques. For $n = 13$ there are 6 ways of connecting 2 or all 3 cliques. In total there are 7 graphs with 360 minimal feedback vertex sets.

n	**Graph**	**#minFVS**	$\mathbf{10^{n/5}}$
1	K_1	1	1.58
2	K_1	1	2.51
3	K_3	3	3.98
4	K_4	6	6.31
5	K_5	10	10.00
6	K_6	15	15.84
7	K_7	21	25.12
8	$K_4 + K_4$	36	39.81
9	$K_4 + K_5$	60	63.10
10	$K_5 + K_5$	100	100.00
11	$K_5 + K_6$	150	158.49
12	$K_6 + K_6$	225	251.19
13	$K_4 + K_4 + K_5$	360	398.11

Table 5.2: Graphs with maximum number of minimal feedback vertex sets.

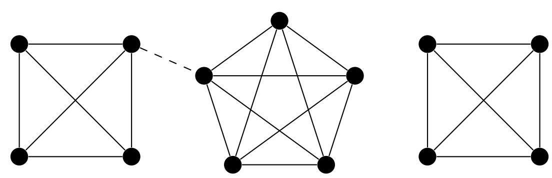

Figure 5.4: There are 6 non-isomorphic graphs that can be constructed by adding edges connecting the cliques without changing the number of minimal feedback vertex sets. We get one of these by adding an edge along the dashed line.

Testing did not give us a counter-example for the claim that $10^{\frac{n}{5}}$ is the maximum number of minimal feedback vertex sets for a graph on n vertices. The problem in [7] arose when the difference between a leaf clique and its parent is exactly one vertex. In a sun-graph (to be defined in the next section), this is the case for every leaf clique in a clique-tree representation of the graph. This makes sun-graphs an interesting graph class to study.

5.2.2 Number of Minimal Feedback Vertex Sets in Sun-Graphs

In a sun-graph $G = (V, E)$ there is a set of vertices $K \subset V$ of size $\frac{n}{2}$ that is a clique in G. Every vertex $v \in V \setminus K$ is adjacent to 2 vertices in K such that the degree of every vertex in K is $\frac{n}{2} + 1$. No pair of vertices in $V \setminus K$ are adjacent.

Observation 5.2. Let G be a sun-graph on n vertices. G has $\frac{n^2-2n}{8}$ minimal feedback vertex sets.

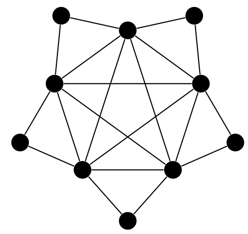

Figure 5.5: A sun graph on 10 vertices.

Proof. We prove this observation by proving that a graph G on n vertices has $\frac{n^2-2n}{8}$ maximal induced forests. Let K denote the clique consisting of half the vertices in G and I denote the other vertices. First, we observe that no induced forest contains more than two vertices from K, since this would result in a cycle. Every maximal induced forest has to contain exactly two vertices from K. To see this, let S denote a maximal induced forest. If S contains only vertices from I, S will not be maximal, since there are always two vertices in K which do not have a common neighbour in S. This holds even when S contains every vertex in I. G also has a maximal induced forest for every pair of vertices in K. To see this, let u and v denote two vertices from K. Any vertex x from I will be part of a maximal induced forest as longs as not both ux and vx are edges in the graph. It follows that the set of vertices from I included in the maximal induced forest is uniquely determined by which two vertices are chosen from K. We then conclude that the number of maximal induced forests of G is equal to the number of pairs of vertices from K. This number is $\frac{(n/2)(n/2-1)}{2} = \frac{n^2-2n}{8}$. □

Since $\frac{n^2-2n}{8}$ is less than $10^{\frac{n}{5}}$ for all positive n, no sun-graph has more minimal feedback vertex sets than the disjoint union of K_5. This observation concludes this section. We have not found any graph that contradicts the hypothesis that $10^{\frac{n}{5}}$ is the maximum number of minimal feedback vertex sets in chordal graphs on n vertices.

5.3 Summary

In this thesis, we have studied two similar problems which both ask how many edges a graph under certain constraints can have. In one of the problems the constraints are on the matching number and maximum degree of the graph, in the other problem the constraints are on the induced matching number and maximum degree.

In Chapter 3 we investigated the first problem. Our aim was to find the maximum number of edges in graphs that belong to $M_{\nu,\mathcal{CHO}}(i,j)$. We conjectured that a graph that consists of $j-1$ disjoint $(i-1)$-stars has the maximum number of edges. We already believed that this conjecture was true, but by testing every chordal graph of up to 13 vertices, we reinforced our belief that $(i-1)(j-1)$ is the maximum number of edges under these constraints. Since we were not able to prove this conjecture, we turned our attention to interval graphs, a subclass of chordal graphs for which the solution to the problem was not known. Surprisingly, we found that proving that $(i-1)(j-1)$ is the maximum number of edges in interval graphs was an easier task, and we were able to do this.

In Chapter 4 we initiated the study on the maximum number of edges in graphs whose induced matching number and maximum degree is bounded. Since this problem did not seem to have been studied before, we first wanted to look at the problem for general graphs. We were not able to solve this problem, but by generating and testing graphs we were able to give a lower bound on the number of edges in edge-extremal graphs in $M_{\mu,\mathcal{GEN}}(i,j)$. We then turned our attention to other graph classes.

First, we looked at the problem for chordal graphs were we found that the edge-extremal general graphs are not chordal. We then looked at two subclasses of chordal graphs.

The first subclass of chordal graphs we looked at was split graphs. The solution for split graphs turned out to be easy to prove. Split graphs is the only graph class we have studied were all graphs are connected, this results in that there are split graphs that are edge-extremal for both problems for some i and j.

For interval graphs, we found that cliques could be used to construct edge-extremal graphs. Since the induced matching number of a clique is one, the edge-extremal graphs in $M_{\mu,\mathcal{INT}}(i,j)$ is just graphs with a higher number of disjoint cliques than the edge-extremal graphs in $M_{\mu,\mathcal{SPLIT}}(i,j)$ and $M_{\nu,\mathcal{INT}}(i,j)$.

We also solved the problem for cographs. In a cograph, we may have an induced cycle on more than three vertices, it was interesting to find that this resulted in that graphs of disjoint cliques were no longer edge-extremal. Instead, the edge-extremal cographs consist of disjoint copies of complete bipartite graphs.

The results are summarised in Table 5.3. Note that the edge-extremal graphs in $M_{\nu,\mathcal{GEN}}(i,j)$ we gave in Chapter 2 are cographs. Since the class of trivially perfect graphs is a subclass of interval graphs, and a graph that consists of $j-1$ disjoint cliques on i vertices is trivially perfect and edge-extremal in $M_{\nu,\mathcal{INT}}(i,j)$, we have that the edge-extremal graphs in $M_{\mu,\mathcal{TP}}(i,j)$ have $\frac{i(i-1)}{2}(j-1)$ edges.

We see that split graphs is the only graph class for which edge-extremal graphs for both problems have the same number of edges. We also see that for graphs in $M_{\nu,\mathcal{C}}(i,j)$, every graph class we have studied, apart from split graphs, has edge-extremal graphs with as many edges as the maximum for general graphs for odd i. While we see that for the other problem, we have that for $i > 2$, the edge-extremal general graphs have a higher number of edges than every other graph class.

Graph class $\mathcal{C}$	$\boldsymbol{M_{\nu,\mathcal{C}}(i,j)}$	$\boldsymbol{M_{\mu,\mathcal{C}}(i,j)}$
General	$\|E(G)\| = c$, from [1]	$\|E(G)\| \geq ((i-1)^2+1)(j-1)$
Chordal	If i is even, $\|E(G)\| \geq (i-1)(j-1)$ If i is odd, $\|E(G)\| = c$, from [1]	$\|E(G)\| \geq \frac{i(i-1)}{2}(j-1)$
Interval	If i is even, $\|E(G)\| = (i-1)(j-1)$ If i is odd, $\|E(G)\| = c$, from [1]	$\|E(G)\| = \frac{i(i-1)}{2}(j-1)$
Trivially perfect	If i is even, $\|E(G)\| = (i-1)(j-1)$ If i is odd, $\|E(G)\| = c$, from [1]	$\|E(G)\| = \frac{i(i-1)}{2}(j-1)$
Split	$\|E(G)\| \leq \frac{i(i-1)}{2}$, from [17]	$\|E(G)\| = \frac{i(i-1)}{2}$
Cographs	$\|E(G)\| = c$, from [1]	$\|E(G)\| = (i-1)^2(j-i)$

Table 5.3: The number of edges in edge-extremal graphs in $M_{\nu,\mathcal{C}}(i,j)$ and $M_{\mu,\mathcal{C}}(i,j)$. $c = (i-1)(j-1) + \lfloor \frac{i-1}{2} \rfloor \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor$

5.4 Open Problems

There are two main problems that remain open. From Chapter 3 we have an open problem:

1. How many edges do the edge-extremal graphs in $M_{\nu,\mathcal{CHO}}(i,j)$ have?

If our hypothesis that $(i-1)(j-1)$ is the maximum number of edges fails, a perhaps simpler problem is to prove that there are edge-extremal graphs in $M_{\nu,\mathcal{INT}}(i,j)$ that are not edge-extremal in $M_{\nu,\mathcal{CHO}}(i,j)$. This would be a conterexample of our hypothesis, although it would not necessarily give us the correct bound.

The second open problem is from Chapter 4:

2. How many edges do the edge-extremal graphs in $M_{\mu,\mathcal{GEN}}(i,j)$ have?

Depending on the solution for $M_{\mu,\mathcal{GEN}}(i,j)$, some graph classes might be interesting to study. By Observation 4.2 we have that every graph in $M_{\mu,\mathcal{CHO}}(i,j)$ has less than $(i-1)^2(j-1)$ edges, thus the edge-extremal graphs in $M_{\mu,\mathcal{CHO}}(i,j)$ have fewer edges than the edge-extremal graphs in $M_{\mu,\mathcal{GEN}}(i,j)$. Which means that solving the problem for general graphs, does not solve the problem for chordal graphs. One question may be if the edge-extremal graphs in $M_{\mu,\mathcal{INT}}(i,j)$ are edge-extremal in $M_{\mu,\mathcal{CHO}}(i,j)$ also. More generally, the problem is:

- How many edges do the edge-extremal graphs in $M_{\mu,\mathcal{CHO}}(i,j)$ have?

Bibliography

[1] Niranjan Balachandran and Niraj Khare. Graphs with restricted valency and matching number. *Discrete Mathematics*, 309(12):4176 – 4180, 2009.

[2] Manish Bhojasia. Java program to implement the edmond's algorithm for maximum cardinality matching. *https://www.sanfoundry.com/java-program-implement-edmonds-algorithm-maximum-cardinality-matching/*, 2014.

[3] Jean R. S. Blair and Barry Peyton. An introduction to chordal graphs and clique trees. In Alan George, John R. Gilbert, and Joseph W. H. Liu, editors, *Graph Theory and Sparse Matrix Computation*, pages 1–29, New York, NY, 1993. Springer New York.

[4] Andreas Brandstädt, Van Bang Le, and Jeremy P. Spinrad. Graph classes: A survey. *Society for Industrial and Applied Mathematics*, 1999.

[5] Kathie Cameron. Induced matchings. *Discrete Applied Mathematics*, 24(1):97 – 102, 1989.

[6] D.G. Corneil, H. Lerchs, and L.Stewart Burlingham. Complement reducible graphs. *Discrete Applied Mathematics*, 3(3):163 – 174, 1981.

[7] Jean-François Couturier, Pinar Heggernes, Pim van 't Hof, and Yngve Villanger. Maximum number of minimal feedback vertex sets in chordal graphs and cographs. 08 2012.

[8] Cemil Dibek, Tınaz Ekim, and Pinar Heggernes. Maximum number of edges in claw-free graphs whose maximum degree and matching number are bounded. *Discrete Mathematics*, 340(5):927 – 934, 2017.

[9] G. A. Dirac. On rigid circuit graphs. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 25(1):71–76, Apr 1961.

[10] Jack Edmonds. *Paths, Trees, and Flowers*, pages 361–379. Birkhäuser Boston, Boston, MA, 1987.

[11] Fedor V. Fomin, Serge Gaspers, Artem V. Pyatkin, and Igor Razgon. On the minimum feedback vertex set problem: Exact and enumeration algorithms. *Algorithmica*, 52(2):293–307, Oct 2008.

[12] Frédéric Gardi. The roberts characterization of proper and unit interval graphs. *Discrete Mathematics*, 307(22):2906 – 2908, 2007.

[13] Martin Charles Golumbic. *Algorithmic Graph Theory and Perfect Graphs*, volume 57 of *Annals of Discrete Mathematics*. Elsevier, 2004.

[14] Pinar Heggernes. Minimal triangulations of graphs: A survey. *Discrete Math.*, 306(3):297–317, February 2006.

[15] W Mantel. Problem 28 (solution by h. gouwentak, w. mantel, j. teixeira de mattes, f. schuh and w. a. wythoff). *Wiskundige Oppgaven*, 10:61–62, 1907.

[16] Brendan D. McKay and Adolfo Piperno. Practical graph isomorphism, {II}. *Journal of Symbolic Computation*, 60(0):94 – 112, 2014.

[17] Erik Måland. Maximum number of edges in graph classes under degree and matching constraints. Master's thesis, University of Bergen, Norway, 2005.

[18] Donald J Rose, R Endre Tarjan, and George S Lueker. Algorithmic aspects of vertex elimination on graphs. *SIAM Journal on computing*, 5(2):266–283, 1976.

[19] Ida Ryland. New lower bounds on the maximum number of minimal connected vertex covers. Master's thesis, University of Bergen, Norway, 2017.

[20] M. Sipser. *Introduction to the Theory of Computation*. Cengage Learning, 2012.

[21] Ida B. Skjørten. Faster enumeration of minimal connected dominating sets in split graphs. Master's thesis, University of Bergen, Norway, 2017.

[22] Paul Turán. On an extremal problem in graph theory. *Matematikai és Fizikai Lapok*, 48:436–452, 1941.

[23] E. S. Wolk. The comparability graph of a tree. *Proceedings of the American Mathematical Society*, 13:789–795, 1962.