Master Thesis

Cartan Connection in Sub-Riemannian Geometry

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Contents

Chapter 1

Introduction

A Riemannian geometry consists of a smooth manifold with a Riemannian metric. The fundamental theorem of Riemannian geometry states that there is a unique torsion free connection that preserves the metric [Lee97, Theorem 5.4]. This connection is called the Levi-Civita connection. The existence and uniqueness of the Levi-Civita connection gives a canonical way of defining the Riemannian curvature tensor

$$R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

where u, v and w are vector fields on a Riemannian manifold M with Levi-Civita connection ∇ [Lee97, Chapter 7]. The spaces $\mathbb{R}^n$, the sphere S^n and the hyperbolic space $\mathbb{H}^n$ are the model spaces of Riemannian geometry. Any complete, simply-conneted Riemannian manifold that has constant sectional curvature is isometric to one of these spaces [Lee97, Theorem 11.12]. This result is a consequence of the Cartan-Ambrose-Hicks theorem which in simple words state that if all the points in one manifold correspond to a point in another manifold with the same sectional curvatures, then the two manifolds are isometric. In this sense the Riemannian metric is determined by the Riemannian curvature tensor. This was first proven locally by Ellie Cartan. A global version was proven by Warren Ambrose in 1956 [Amb56], and the result was further generalized by Noel Hicks in 1959 [Hic59]. The general conception of curvature can be a measure how close your space is to some space you call flat, and the model spaces will be the natural choices of comparison.

The theory of curvature in Riemannian geometry is largely solved by the canonical Levi-Civita connection. In the case of sub-Riemannian geometry the question of finding a good definition of curvature is still being worked on. We can think of Riemannian manifolds as manifolds on which we can move freely in all directions, while a sub-Riemannian manifold is a manifold on which our movement is restricted to only directions within the distribution. An important question is when do we have enough freedom of movement that we could move from any point to any point, assuming our manifold is connected. I turns out that this is the case if we have a bracket generating distribution, and in this case we can define distance, see Theorem 3.1.8. This is a classical theorem that was proven independently by P.K. Rashevskiĭ in 1938 [Ras38] and W.L. Chow in 1939 [Cho39]. We will restrict ourselves to the case with bracket generating distributions. We want to generalize the notion of curvature in Riemannian geometry to include sub-Riemannian manifolds. Be aware there is no general notion of the Levi-Civita connection on sub-Riemannian manifolds. In regards to this Robert S. Strichartz, in his paper from 1986 defining sub-Riemannian geometry, wrote

”It appears that it would be barking up the wrong tree to try to distort the Riemannian definitions to make sense in this context. After all, curvature is a measurement of the higher order deviation of the manifold from the Euclidean model, and here there is no approximate Euclidean behavior.” [Str86]

Nevertheless it will be be the attempt of this thesis to give a canonical notion of curvature for the special case of sub-Riemannian manifolds that have the Heisenberg Lie algebra as its sub-Riemannian symbol, see Definition 3.3.2. For this to be possible we need a more general geometry than Riemannian geometry, which will be Cartan geometry. Historically Riemannian geometry was a generalization of Euclidean geometry put forth by Georg Riemann in which curvature became something of interest in the sense that it measured how far away our space was from the original Euclidean space. Soon thereafter Felix Klein made another generalization of Euclidean geometry when studying non-Euclidean spaces. These where classified as the quotients of Lie groups G/H (see 2.2.1) and is called homogeneous spaces of Klein geometries which are in general much more symmetrical than the Riemannian manifolds. Cartan geometry is a direct generalization of Klein geometries in the sense that any Klein geometry induce a principal bundle $H \to G \to G/H$ (see Example 2.3.3), while a Cartan geometry is built on a more general principal bundle where the bundles induced by a Klein geometry would be a highly symmetric case. In addition, a Cartan geometry has a Cartan connection that will be a 1-form on the principal bundle that takes values in a Lie algebra coming from a chosen model geometry. Note that from the Klein geometry we have a Maurer-Cartan form on the Lie group G taking values in the Lie algebra of G. This is a special case of a Cartan connection. Cartan geometry also generalize Riemannian geometry in the sense that we could consider the principal bundle over a Riemannian manifold M built by the Euclidean group $E(n)$ and the orthogonal group $O(n) \subset E(n)$. Specifying a Cartan connection on this principal bundle would give a Cartan geometry and if we chose a torsion free Cartan connection it will reflect the Levi-Civita connection. This indicates that the Cartan connection of a Cartan geometry is suitable for determining curvature of Cartan geometries in a way that generalize the notion of curvature in the Riemannian case.

The next question is if we can canonically associate a Cartan geometry with a sub-Riemannian manifold. When we have a bracket generating distribution on a manifold the distribution gives a filtration of the manifold and we can at each point associate a graded tangent space with an induced bracket that will make it isomorphic to a stratified Lie algebra, see Section 3.3. In this case we can use the method of Tanaka prolongation due to Noboru Tanaka from his paper [Tan70]. When we restrict to those that have constant sub-Riemannian symbol the Tanaka prolongation is trivial [Mor08]. Even with this method there is no unique choice of Cartan connection. While torsion freeness is a suitable normalizing condition for affine connections in Riemannian geometry, we need another normalizing condition to find a canonical Cartan connection. This normalizing condition was given by Tohru Morimoto in 2008 [Mor08]. When we use this method to determine the curvature our model space will be Carnot groups which can be thought of as the analogous of $\mathbb{R}^n$ in sub-Riemannian geometry.

The structure of the thesis is as follows: Chapter 2 will develop the language of Cartan geometry while we give some important results and examples to familiarize with the topic. This chapter will mainly use the language and theory presented in Sharpes book [Sha00]. In Chapter 3 we will introduce sub-Riemannian manifolds and examine how we can consturct a canonical Cartan geometry on these. In Chapter 4 we will examine the particular case of a sub-Riemannian manifold that has the Heisenberg Lie algebra as constant sub-Riemannian symbol. the following original results are included in this chapter as well: Theorem 4.2.2 which gives the Cartan connection for such a manifold, and Corollary 4.2.3 which gives the Cartan curvature function. The proof of these results will rely heavily on Theorem 3.5.5 and will largely be a specific computation of the sort that is more generally described in [AMS19]. The appendix is mostly reserved for results that has been used in Section 2.4. All these results can be found in [Sha00].

We will assume the reader is familiar with manifolds and Riemannian geometry. In addition it is useful to have some knowledge about Lie groups as our introduction to the topic is quite short with an emphasis on the Maurer-Cartan form.

Chapter 2

Cartan Geometry

In this chapter we will establish the theory that is needed to understand Cartan geometry, in particular we will see how the Cartan connection associated with a Cartan geometry gives an expression of the curvature. Cartan geometry is a generalization of Klein geometry in the sense that if we have a Klein geometry (G, H) as in Definition 2.2.1, then a principal bundle arise in a natural way. The Maurer-Cartan form of G will serve the role of the Cartan connection and the curvature will be the structural equation of G. The first section 2.1 is devoted to the Maurer-Cartan form which will be frequently used when working with Cartan geometries, as well as serving as a simple example of a Cartan connection. In section 2.2 we will define Klein geometries and give some examples. In section 2.3 we will define principal bundles which is the fundamental structure that any Cartan geometry will have. We will also define principal connections which will have many similarities to the Cartan connection, as well as some important distinctions. Section 2.4 is where we show how to constuct a Cartan geometry using an atlas of Cartan gauges on a manifold. Here we will define Cartan geometry and the Cartan connection as well as studying some intresting properties. We will look at the Cartan curvature in section 2.5.

Most of what is presented in this chapter can be found in Sharpe's book "Differential geometry: Cartans generalization of Kleins Erlangen program" [Sha00] as this has been the main source to the topic. A lot of the notation and definitions would therefore coincide with what is presented here, and in particular the construction of a Cartan geometry in section 2.4 is the exact construction presented by Sharpe in Chapter 5 of this book.

2.1 Lie Groups and The Maurer-Cartan Form

Definition 2.1.1. A Lie group G is a group that is also a smooth manifold with the properties that the map of the group multiplication and the inverse map are both smooth maps. ♠

Example 2.1.2. Let E be a vector space. The group of nondegenerate linear transformations $GL(E)$ is a Lie group, and so is the group of orthogonal transformations $O(E) \subset GL(E)$. In the case $E = \mathbb{R}^n$ we denote these groups simply by $GL(n)$ and $O(n)$ respectively. These Lie groups can be represented as $n \times n$ matrices such that matrix multiplication is the group operation. These examples can be found in [Hal15]. ♣

Let G be a Lie group. We let $e \in G$ be the group identity element and define $\mathfrak{g} = T_eG$ to be the Lie algebra of G. We will later give a more abstract definition of a Lie algebra independent of Lie groups.

$$\begin{aligned} L_g : G &\longrightarrow G \\ a &\longmapsto ga \end{aligned}$$

denote the left translation by $g \in G$. This is a diffeomorphism with inverse $L_{g^{-1}}$. The differential of a diffeomorphism gives an isomorphism [O'N83, Theorem 1.16]

$$L_{g^{-1}*} : T_g G \longrightarrow T_e G = \mathfrak{g}$$

for each $g \in G$. Technically the map presented above is $(L_{g^{-1}})_{*g} = (dL_{g^{-1}})_g$, but for a simpler notation the point at which we are differentiating will be omitted whenever it is the inverse of the the point of the left translation function. This gives a canonical trivialization of the tangent bundle $TG \to G \times \mathfrak{g}$.

Definition 2.1.3. Let G be a Lie group. The Maurer-Cartan form ω_G is a left-invariant $\mathfrak{g}$-valued 1-form on G defined by

$$\begin{aligned} \omega_G : TG &\longrightarrow \mathfrak{g} \\ v &\longmapsto L_{g^{-1}*}(v) \end{aligned}$$

for $v \in T_g G$. ♠

Left-invariant here refers to the fact that for any left translation L_h we have

$$\omega_G(L_{h*}(v)) = L_{(hg)^{-1}*}(L_{h*}(v)) = L_{g^{-1}*}(v) = \omega_G(v)$$

for $v \in T_g G$.

Lemma 2.1.4. *[Sha00, Lemma 3.2.2, p.101]*
Let G be a Lie group, and let V be a vector field on G. The following are equivalent:

(i) $\omega_G(V)$ is a constant (as a $\mathfrak{g}$-valued function on G),

(ii) $L_{g*}V_a = V_{ga}$ for all $a, g \in G$.

Proof. $\omega_G(V)$ is constant $\Leftrightarrow L_{a^{-1}*}(V_a) = L_{(ga)^{-1}*}(V_{ga})$ for all $a, g \in G \Leftrightarrow L_{g*}(V_a) = V_{ga}$ for all $a, g \in G$. □

Definition 2.1.5. Let G be a Lie group. Any vector field V satisfying the properties of the lemma above is called left-invariant. ♠

The space $\mathcal{L}(G)$ of left-invariant vector fields over a Lie group G is isomorphic to the Lie algebra $\mathfrak{g}$ of G. In fact the Maurer-Cartan form gives rise to an isomorphism between the spaces. This follows from the fact that ω_G is a 1-form, hence it is linear, together with the property that $\omega_G(V)$ is constant for any $V \in \mathcal{L}(G)$. This gives us a linear map between the vector space of left-invariant vector fields on G and the Lie algebra $\mathfrak{g}$ by $V \mapsto V_e$. This map is an isomorphism: If $V_e = 0$, then $V_g = L_{g*}V_e = 0$, which proves injectivity. If $X \in \mathfrak{g}$, then we can define $V_g = L_{g*}X$, and we get

$$L_{a*}V_g = L_{a*}L_{g*}X = L_{ag*}X = V_{ag},$$

so V is a left-invariant vector field with $V_e = X$, hence the map is surjective. Now we have a one to one correspondence between left invariant vector fields on a Lie group G and the elements of the Lie algebra $\mathfrak{g}$.

Definition 2.1.6. If $X \in \mathfrak{g}$, the Lie algebra of G, then $X^\sharp$ denotes the corresponding left-invariant vector field on G defined by

$$(X^\sharp)_g = L_{g*}(X).$$

♠

Definition 2.1.7. A Lie algebra is a vector space $\mathfrak{g}$ over a field $\mathbb{R}$ or $\mathbb{C}$ with a binary operation $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket that satisfy the following properties

(i) $[ax + by, z] = a[x, z] + b[y, z]$ (Bilinearity),

(ii) $[x, y] = -[y, x]$ (Skew symmetry),

(iii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity),

for all $x, y, z \in \mathfrak{g}$ and a, b scalars.

♠

We can define a bracket operation $[.,.]$ on the tangent space at the identity $\mathfrak{g} = T_eG$ such that is satisfies all the requirements above to be a Lie algebra.

Lemma 2.1.8. *Let $V, W \in \mathcal{L}(G)$ be two left invariant vector fields on a Lie group G. Then $[V, W] \in \mathcal{L}(G)$.*

Proof. A vector field V is left-invariant if and only if $V(f \circ L_g) = (V(f))$ for any $f \in \mathcal{C}^\infty(G)$ and any $g \in G$. To see this, let V be left-invariant, then

$$V(f \circ L_g)(a) = ((L_{g*}V)(f))(ga) = V(f)(ga) = V(f)L_g(a),$$

and on the contrary if V is any vector field satisfying $V(f \circ L_g) = (V(f))L_g$, then $((L_{g*}V)(f))(a) = (V(f))(ga)$ for all $g, a \in G$ and all f, hence $L_{g*}V = V$. This is the same as saying V is L_g-related to itself [ONe83, Definition 1.20].

Assume V, W are left-invariant vector fields on a Lie group G. Then we have

$$\begin{aligned}
([V, W]f)L_g &= (VWf)L_g - (WVf)L_g \\
&= V((Wf)L_g) - W((Vf)L_g) \\
&= V(W(f \circ L_g) - W(V(f \circ L_g) \\
&= [V, W](f \circ L_g)
\end{aligned}$$

using the relation established above repeatedly. □

Corollary 2.1.9. *Let G be a Lie group with identity element e. For any $X, Y \in T_eG$, define $[X, Y] = \omega_G([X^\sharp, Y^\sharp])$. This makes the tangent space of the Lie group at the identity T_eG into a Lie algebra.*

Proof. Since any element X of the Lie algebra correspond to a left-invariant vector field $X^\sharp$ and by Lemma 2.1.8 the bracket of two left-invariant vector fields is left-invariant. This means that the bracket we defined on T_eG is well-defined. Bilinearity, skew symmetry and the Jacobi identity all follow from the Lie bracket on vector fields [O'N83, Lemma 1.18]. $\square$

There is one representation of a Lie group that is particularly important, so we will introduce it here. Let G be a Lie group, then define a map

$$\begin{aligned} \psi_g : G &\longrightarrow G \\ a &\longmapsto gag^{-1}. \end{aligned}$$

This is an automorphism on G, and the differential of this map at the identity will be an automorphism of the Lie algebra:

$$Ad_g = (\psi_{g*})_e : \mathfrak{g} \longrightarrow \mathfrak{g}$$

Definition 2.1.10. For any Lie group G, the adjoint representation is defined as

$$\begin{aligned} Ad : G &\longrightarrow Aut(\mathfrak{g}) \\ g &\longmapsto Ad_g, \end{aligned}$$

where Ad_g is as defined above. ♠

If we take the differential of the adjoint representation, we get

$$ad : \mathfrak{g} \to End(\mathfrak{g}),$$

and in fact we could define the Lie bracket on $\mathfrak{g}$ by $[X, Y] = (ad(X))(Y)$, and this would give an isomorphic Lie algebra [Kna05, prop.1.74].

If ω_G is the Maurer-Cartan form of a Lie group G we compute the exterior derivative of ω_G

$$d\omega_G(V, W) = V(\omega_G(W)) + W(\omega_G(V)) - \omega_G([V, W])$$

where V, W are left-invariant vector fields on G. We can also look at the wedge product of $\mathfrak{g}$-valued forms, but to obtain a new $\mathfrak{g}$-valued form we must compose it with the Lie bracket. Let ω_1 be a $\mathfrak{g}$-valued p-form and ω_2 a $\mathfrak{g}$-valued q-form, and we define

$$[\omega_1, \omega_2](v_1, \dots, v_{p+q}) = \sum_\sigma \operatorname{sign}(\sigma)[\omega_1(v_{\sigma(1)}, \dots, v_{\sigma(p)}), \omega_2(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)})]$$

where σ is the permutation of the indexes with $\operatorname{sign}(\sigma)$ being negative for odd permutations and positive for even permutations. In particular, if $p = q = 1$ we get

$$\begin{aligned} [\omega_1, \omega_2](v_1, v_2) &= [\omega_1(v_1), \omega_2(v_2)] - [\omega_1(v_2), \omega_2(v_1)] \\ &= [\omega_1(v_1), \omega_2(v_2)] + [\omega_2(v_1), \omega_1(v_2)] \end{aligned}$$

and for the Maurer-Cartan form we get

$$[\omega_G, \omega_G](V, W) = 2[\omega_G(V), \omega_G(W)].$$

Proposition 2.1.11. *Let G be a Lie group with Maurer-Cartan form ω_G. Then we have*

$$d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0.$$

Proof. For the exterior derivative we have

$$d\omega_G(V, W) = V(\omega_G(W)) + W(\omega_G(V)) - \omega_G([V, W])$$

where $V, W \in \mathcal{L}(G)$. Since the Maurer-Cartan form is constant on left-invariant vector fields, we have

$$V(\omega_G(W)) = W(\omega_G(V)) = 0.$$

This gives

$$\begin{aligned} d\omega_G(V, W) &= -\omega_G([V, W]) \\ &= -[V, W]_e \\ &= -[V_e, W_e] \\ &= -[\omega_G(V), \omega_G(W)] \\ &= -\frac{1}{2}[\omega_G, \omega_G](V, W), \end{aligned}$$

which proves that the equation in the proposition holds when applied to left-invariant vector fields. Since the equation is a linear combination of 2-forms it must be true for any vectors $v, w \in T_g G$ that is the restriction of a left-invariant vector field. But any tangent vectors of G can be extended to a left-invariant vector field, hence the equation holds for all vector fields of G. □

This equation is sometimes referred to as the Maurer-Cartan equation.

2.2 Klein Geometry

Definition 2.2.1. A Klein geometry is a pair (G, H) where G is a Lie group and $H \subset G$ is a closed subgroup such that G/H is connected. The kernel of a Klein geometry is the largest subgroup K of H that is normal in G. A Klein geometry (G, H) is called effective if the kernel is trivial, and it is called locally effective if the kernel is discrete. ♠

When we write G/H, we use the fact that H acts on G from the right by

$$\begin{aligned} \mu : G \times H &\longrightarrow G \\ g \times h &\longmapsto gh. \end{aligned}$$

Now we can make a quotient space by the equivalence relation

$$g_1 \sim g_2 \quad \Leftrightarrow \quad g_1 = \mu(g_2, h) \text{ for some } h \in H.$$

In fact, for any Klein geometry the quotient space $M = G/H$ is a smooth manifold [Sha00, Theorem 4.2.4, p.145]. The kernel of a Klein geometry as defined above is well-defined, and whenever we have a Klein geometry (G, H) with kernel K there is an associated effective Klein geometry $(G/K, H/K)$ that give the same smooth manifold $(G/K)/(H/K) \cong G/H$ [Sha00, Prop 4.3.1, p.150]. An effective Klein geometry is often called a homogeneous manifold [War83, Chapter 3, p 120].

Example 2.2.2. (i) Let $G = E(n)$ be the group of Euclidean transformations of $\mathbb{R}^n$. We can give a matrix representation of this group

$$E(n) = \left\{ \begin{pmatrix} 1 & 0 \\ t & A \end{pmatrix} \mid t \in \mathbb{R}^n,\, A \in O(n) \right\}$$

where we let this work on an element $v \in \mathbb{R}^n$ by

$$v \longmapsto Av + t.$$

It is clear form this representation that $E(n)$ is a Lie group with matrix multiplication as group operation. Let $H = O(n)$ be the orthogonal group, that is the group of rotations and reflections of $\mathbb{R}^n$. For convenience we use a slightly unusual representation of $O(n)$:

$$O(n) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \mid A \in GL(n),\; AA^T = A^TA = I \right\}$$

where A would be an element of $O(n)$ in the standard representation. This gives $E(n)/O(n) = T(n)$, the group of translations, here represented as

$$T(n) = \left\{ \begin{pmatrix} 1 & 0 \\ t & I \end{pmatrix} \mid t \in \mathbb{R}^n,\, I \text{ is the } n \times n \text{ identity matrix} \right\}.$$

With this representation, we see that $T(n)$ is clearly isomorphic to $\mathbb{R}^n$ as a smooth manifold:

$$E(n)/O(n) \cong \mathbb{R}^n.$$

(ii) Let $G = O(n+1)$, the group of orthogonal transformations of $\mathbb{R}^{n+1}$, and let $H = O(n) \subset O(n+1)$, where we use the same representation of $O(n)$ as above. Write an element $R \in O(n+1)$ as a matrix

$$R = \begin{pmatrix} r_1 & a_1 \\ r_2 & a_2 \\ \vdots & \vdots \\ r_{n+1} & a_{n+1} \end{pmatrix}$$

where $r_i \in \mathbb{R}$ and $a_i \in \mathbb{R}^n$. The scalar product of any column or row with itself must be 1 since $RR^T = R^TR = I$, in particular we have $|(r_i, a_i)| = 1$. The elements of $O(n)$ acts on R from the right by

$$R \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} r_1 & Aa_1 \\ r_2 & Aa_2 \\ \vdots & \vdots \\ r_{n+1} & Aa_{n+1} \end{pmatrix}$$

Here Aa_i is any element satisfying the equation

$$|r_i|^2 + |Aa_i|^2 = 1,$$

in fact, any vector satisfying the equation above for a fixed r_i can be written on the form $|Aa_i|$ for some orthogonal matrix A. This means that we can represent any element of the quotient space $O(n+1)/O(n)$ uniquely by the r_i, and the only restriction we have is

$$|r_1|^2 + \ldots + |r_{n+1}|^2 = 1.$$

This is exactly the defining equation of the n-dimensional sphere S^n embedded in $\mathbb{R}^{n+1}$. We conclude that

$$O(n+1)/O(n) \cong S^n.$$

♣

Whenever we have a Klein geometry (G, H), we also have a pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{h}$ is the Lie algebra of H. Of course it follows from the definition of a Klein geometry that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. This motivates the definition of a Klein pair.

Definition 2.2.3. A Klein pair $(\mathfrak{g}, \mathfrak{h})$ is a Lie algebra $\mathfrak{g}$ with a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The kernel $\mathfrak{k}$ of $(\mathfrak{g}, \mathfrak{h})$ is the largest ideal of $\mathfrak{g}$ such that $\mathfrak{k} \subset \mathfrak{h}$. If $\mathfrak{k} = \{0\}$ we say that the Klein pair $(\mathfrak{g}, \mathfrak{h})$ is effective.

♠

While it is clear that any Klein geometry gives a Klein pair, it is not clear that there can be associated a Klein geometry to any Klein pair. In fact the later is not true, see [Sha00, Remark 3.8.10 and Definition 4.3.16]. It is however the case that effective Klein geometries provide effective Klein pairs. This follows from Lemma A.0.8 in the appendix.

2.3 Principle Bundles and Principal Connections

Definition 2.3.1. Let $\xi = (E, M, \pi, F)$ be a smooth fiber bundle. A group G makes ξ together with the right action $E \times G \to E$ into a principal bundle if the right action is fiber preserving and acts simply transitively on each fiber, i.e. for each $x, y \in F$ there exist a unique $g \in G$ such that $xg = y$.

♠

Notice that a simply transitive action is a transitive action with the additional property that the element $g \in G$ such that $xg = y$ is unique for any $x, y \in F$. One could also make the equivalent demand that the action should be free and transitive, where free means that for any $x \in F$, if $xg = x$ then $g = e$, the identity element of G.

Lemma 2.3.2. *A group action is simply transitive if and only if it is free and transitive.*

Proof. If the action $F \times G \to F$ is simply transitive, then for any $x \in F$ there is a unique $g \in G$ with $xg = x$, but $xe = x$ for all $x \in F$, hence $g = e$ and the group action is free.

If the action is free and transitive, assume $xg_1 = xg_2$ for $x \in F$, $g_1, g_2 \in G$. Then $xg_1g_2^{-1} = x$, and since the action is free this means $g_1g_2^{-1} = e$, hence $g_1 = g_2$ which means the group action is simply transitive. □

Example 2.3.3. Any Klein geometry (G, H) gives a natural principle bundle $H \to G \to G/H$. We already have a right action defined, that is the restricted group action of H acting from the right on G. This action becomes transitive by construction. Also this action is free, since if we have $gh = g$ for some $g \in G$ and $h \in H$, then $g^{-1}gh = g^{-1}g = e$. In particular the cases from Example 2.2.2 are principle bundles:

(i) $O(n) \longrightarrow E(n) \xrightarrow{\pi} \mathbb{R}^n$,

(ii) $O(n) \to O(n+1) \xrightarrow{\pi} S^n$.

♣

Definition 2.3.4. Let P be a smooth manifold. Let H be a Lie group and let $\mu : P \times H \to P$ be a smooth right action. Let $\mu_p : H \to P$ be defined by $\mu_p(h) = \mu(p, h)$, and then

$$\mu_{p*} : T_e H \longrightarrow T_p P.$$

If $X \in \mathfrak{h}$, the Lie algebra of H, then we define the vector field $X^\sharp \in \mathcal{T}(P)$ on P by

$$(X^\sharp)_p = \mu_{p*}(X)$$

♠

Notice that any principal bundle by definition has a smooth manifold P with a Lie group H acting from the right, hence for any principal bundle we have a vector field $X^\sharp$ for each $X \in \mathfrak{h}$ where $\mathfrak{h}$ is the Lie algebra of H. This is analogous to the left-invariant vector fields on a Lie group in some sense, as the example bellow will make clear.

Example 2.3.5. If G is a Lie group and $H \subset G$ is a subgroup with $\mu : G \times H \to G$ defined by $\mu(g, h) = gh$, then we have a right action on a smooth manifold by a Lie group. In this case, notice that $\mu_g = L_g$ is just the left translation. Then, for any $X \in \mathfrak{h}$ we have $(X^\sharp)_g = \mu_{g*}(X) = L_{g*}(X)$. In the special case were $H = G$ we see that this definition agrees with Definition 2.1.6. Thus we see that using the right action of a Lie group G on itself we can construct the left-invariant vector fields on G. ♣

Definition 2.3.6. Let $P \to M$ be a principal bundle with fiber G. A principal connection on P is a differential 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ with values in the Lie algebra $\mathfrak{g}$ of G such that

(i) $Ad_g(R_g^*\omega) = \omega$ for all $g \in G$, where R_g is the right multiplication by g;

(ii) If $X \in \mathfrak{g}$ and $X^\sharp$ is the vector field on P associated to X, then $\omega(X^\sharp) = X$.

♠

Example 2.3.7. In the trivial case where $M = \{e\}$ is a single point manifold, we get $P = G$ a Lie group. Then the Maurer-Cartan form ω_G on G would be a principal connection on $G \to \{e\}$. It is clear that the Maurer-Cartan form satisfy the second property, since in this case $X^\sharp$ is exactly the left-invariant vector fields on G, see Example 2.3.5. That the first property is satisfied is a known result. To prove it we see that it is equivalent to

$$R_g^*\omega_G(V) = Ad_{g^{-1}}\omega(V).$$

for $V \in TP$. Look at $V_p \in T_pP$ and let $X^\sharp$ be a left invariant vector field associated with $X \in \mathfrak{g}$ such that $X_p^\sharp = V_p$. On the left side we get

$$\begin{aligned} R_g^*\omega_G(V_p) &= R_g^*\omega_G(X^\sharp) \\ &= \omega_G(R_{g*}X^\sharp) \\ &= (R_{g*}X^\sharp)_e \end{aligned}$$

and on the right side we get

$$\begin{aligned} Ad_{g^{-1}}\omega_G(V_p) &= R_{g*}(L_{g^{-1}*}(\omega_G(X^\sharp))) \\ &= R_{g*}(X^\sharp_{g^{-1}}) \\ &= (R_{g*}X^\sharp)_e. \end{aligned}$$

Since V was an arbitrary vector field on P and p an arbitrary point in P we can conclude that the Maurer-Cartan form is a principal connection on $G \to \{e\}$. ♣

Now, suppose (P, M, π, G) is a principal bundle over a smooth manifold M. Let $p \in P$, and consider

$$d\pi_p : T_pP \longrightarrow T_{\pi(p)}M.$$

Define $\mathcal{V}_p = \ker(d\pi_p) \subset T_pP$. The subspaces $\mathcal{V}_p$ form a subbundle $\mathcal{V} \subset TP$ called the vertical subbundle. Notice that $\mathcal{V}_p = T_p(\pi^{-1}(m))$, the tangent space to the fibre of $\pi : P \to M$ over $m = \pi(p)$. But the fibers of the principal bundle is the Lie group G, hence $\mathcal{V}_p \cong T_gG \cong \mathfrak{g}$ for some $g \in G$, where $\mathfrak{g}$ is the Lie algebra of G. Notice that for any $X \in \mathfrak{g}$ we have that $X^\sharp \in \mathcal{V}$, hence any principal connection on P will look like the Maurer-Cartan form ω_G on $\mathcal{V}$ in the sense of property (ii) of Definition 2.3.6 being fulfilled. This means that any principal connection $\omega_p : T_pP \to \mathfrak{g}$ must be a linear isomorphism when restricted to $\mathcal{V}_p$. As a consequence, we could write $T_pP = \mathcal{V}_p \oplus \ker(\omega_p)$. Now we might think of defining a connection on a principal bundle by choosing a subbundle of the tangent bundle of the principal bundle $\mathcal{H} \subset TP$ such that $TP = \mathcal{H} \oplus \mathcal{V}$. Some literature use this approach when defining a principal bundle. We give a definition similar to [Joy09, Def.2.1.6].

Definition 2.3.8. Let $P \to M$ be a principal bundle with fiber G. A prncipal Ehresmann connection on P is a vector subspace $\mathcal{H} \subset TP$ called the horizontal subbundle, that is invariant under the G-action on P, and which satisfies $T_pP = \mathcal{V}_p \oplus \mathcal{H}_p$ for each $p \in P$.

♠

The statement that $\mathcal{H}$ is invariant under the action of G on P means that $R_{g*}(\mathcal{H}_p) = \mathcal{H}_{pg}$ for all $p \in P$, $g \in G$. Notice that $d\pi_p$ is a linear map that maps T_pP onto $T_{\pi(p)}M$, and since $\mathcal{V}_p = \ker(d\pi_p)$ we have an isomorphism $T_{\pi(p)}M \cong \mathcal{H}_p$. It is worth noting that a principal Ehresmann connection is a special case of the more general Ehresmann connection that could be defined on any fiber bundle. We shall see bellow that there is a one-to-one correspondence between principal connections and principal Ehresmann connections. This means that the two definitions are equivalent in some sense, and we might choose to work with one or the other.

Lemma 2.3.9. *There is a one to one correspondence between principle connections and principle Ehresmann connections.*

Proof. We start by showing that a principal Ehresmann connection $\mathcal{H} \subset TP$ induces a principal connection. Let $\phi : TP \to \mathcal{V}$ be a bundle morphism, i.e. a fiber preserving continuous map considering both $\mathcal{V}$ and TP as fiber bundles over M, such that

- $\phi(\phi(v)) = \phi(v)$ for all $v \in TP$,

- $\phi|_{\mathcal{V}} = id_{\mathcal{V}}$, the identity map on $\mathcal{V}_p$.

Such a projection is uniquely determined by its kernel, hence any principal Ehresmann connection $\mathcal{H} \subset TP$ induces a projection $\phi : TP \to \mathcal{V}$ by $\ker(\phi) = \mathcal{H}$. Let $\imath : \mathcal{V} \longrightarrow \mathfrak{g}$ be a map such that $\imath_p(X_p^\sharp) = X \in \mathfrak{g}$. Define the $\mathfrak{g}$-valued 1-form

$$\eta = \imath \circ \phi : TP \longrightarrow \mathfrak{g}.$$

We need to show that this 1-form satisfies the two properties that makes it a principal connection:

(i) Notice that the adjoint map $Ad_g = d(R_{g^{-1}} \circ L_g)$, so for any $X \in \mathfrak{g}$ we have

$$\begin{aligned} Ad_g(X) &= d(R_{g^{-1}} \circ L_g)(X) \\ &= R_{g^{-1}*}(L_{g*}(X)) \\ &= R_{g^{-1}*}(X) \end{aligned}$$

since the Lie algebra can be represented as the left-invariant vector fields on G. It remains to be shown that $R_{g^{-1}*}(\eta(R_{g*}(v))) = \eta(v)$ for any $v \in TP$. We can write $v = v_{\mathcal{H}} + v_{\mathcal{V}}$ where $v \in T_pP$, $v_{\mathcal{H}} \in \mathcal{H}_p$ and $v_{\mathcal{V}} \in \mathcal{V}_p$. Then $\eta(v) = \imath_p(\phi(v_{\mathcal{H}} + v_{\mathcal{V}})) = \imath_p(v_{\mathcal{V}})$. What we end up with is

$$\begin{aligned} R_{g^{-1}*}(\eta(R_{g*}(v))) &= R_{g^{-1}*}(\eta(R_{g*}(v_{\mathcal{H}} + v_{\mathcal{V}}))) \\ &= R_{g^{-1}*}(\eta(\tilde{v_{\mathcal{H}}} + R_{g*}v_{\mathcal{V}})) \\ &= R_{g^{-1}*}(\imath(R_{g*}(v_{\mathcal{V}}))) \\ &= \imath(v_{\mathcal{V}}) \\ &= \eta(v) \end{aligned}$$

(ii) Let the vector fields $X^\sharp$ be constructed by using the right action $\mu : P \times G \to P$. Recall that this right action is fiber preserving. That means that, using the local trivialization of P, the map

$$\mu_{p*} : \mathfrak{g} \longrightarrow T_pP \cong T_mM \oplus T_aG$$

can be evaluated as $\tau(\mu_{p*}(X)) = (O_{T_mM}, R_{a*}X) \in T_mM \oplus T_aG$, where $p = (m, a)$ with $m = \pi(p) \in M$ and $a \in G$, and τ is the map of the local trivialization. Going back to T_pP, recall that $\mathcal{V}_p \cong \mathfrak{g} \cong T_aG$ and $\mathcal{H}_p \cong T_mM$, hence $X_p^\sharp = \mu_{p*}(X) \in \mathcal{V}_p$ for each $p \in P$, $X \in \mathfrak{g}$.

To show the other direction, let ω be a principal connection on P, and define $\mathcal{H} \subset TP$ as

$$\mathcal{H} = \ker(\omega).$$

We need to show that

(i) $T_pP = \mathcal{H}_p \oplus \mathcal{V}_p$ for all $p \in P$;

(ii) $R_{g*}(\mathcal{H}_p) = \mathcal{H}_{pg}$ for all $p \in P$ and $g \in G$.

To see the first one, let $v \in T_pP$ and let $\omega(v) = X$. Let $v_{\mathcal{V}} = \mu_{p*}(X) \in \mathcal{V}_p \subset T_pP$, and let $v_{\mathcal{H}} = v - v_{\mathcal{V}}$. By definition $v_{\mathcal{V}} = X_p^\sharp$, and $\omega(X_p^\sharp) = X$, then

$$\omega(v_{\mathcal{H}}) = \omega(v - v_{\mathcal{V}}) = \omega(v) - \omega(X_p^\sharp) = X - X = 0,$$

hence $v_{\mathcal{H}} \in ker(\omega) = \mathcal{H}_p$, which proves that $T_pP = \mathcal{H}_p \oplus \mathcal{V}_p$.

Let $v = v_{\mathcal{H}} + v_{\mathcal{V}}$, then $R_{g*}(v) = v_{\mathcal{H}} + R_{g*}(v_{\mathcal{V}})$. If we let $v \in \mathcal{H}_p$ we see that $\omega(R_{g*}(v)) = \omega(v) = 0$, hence

$$R_{g*}(\mathcal{H}_p) \subset \mathcal{H}_{pg}.$$

Since this is true for any $p \in T_pP$ and $g \in G$, we also have

$$R_{g^{-1}*}(\mathcal{H}_{pg}) \subset \mathcal{H}_p$$

which means that $R_{g*}(\mathcal{H}_p) = \mathcal{H}_{pg}$ for all $p \in P$ and $g \in G$.

□

2.4 Cartan Geometry

In this section we will go through the details in constructing a Cartan geometry on a smooth manifold by the means of Cartan gauges. This construction is in many ways similar to how one can construct a manifold from a topological space by the means of charts. This can be recognized in the terminology; just like a sufficient collection of charts is called an atlas, we will give a definition of a Cartan atlas, which in words can be called a sufficient collection of Cartan gauges. In the construction we will need a model geometry that will resemble a Klein geometry; or more specifically we will use a Klein pair as in Definition 2.2.3. In this chapter when we build a canonical principal bundle on our Cartan geometry we will give four claims. Even though these claims are not trivial, the proof will not be given here, but the interested reader is encouraged to go to the appendix which is mostly reserved to verify these claims. Everything presented here can be found in [Sha00].

Definition 2.4.1. A model geometry for a Cartan geometry consists of

(i) an effective Klein pair $(\mathfrak{g}, \mathfrak{h})$,

(ii) a Lie group H such that $\mathfrak{h}$ is the Lie algebra of H,

(iii) a representation

$$\begin{aligned} Ad : H &\longrightarrow \mathrm{Aut}(\mathfrak{g}) \\ h &\longrightarrow Ad_h. \end{aligned}$$

♠

Let $(\mathfrak{g}, \mathfrak{h})$ be a Klein pair, let M be a smooth manifold with an open subset $U \subset M$ and a $\mathfrak{g}$-valued 1-form θ defined on U. If

$$\bar{\theta_u} : T_uU \xrightarrow{\theta_u} \mathfrak{g} \xrightarrow{\rho} \mathfrak{g}/\mathfrak{h}$$

is an isomorphism for each $u \in U$, we call (U, θ) a Cartan gauge on M. Here $\bar{\theta}_u$ is the composition of θ_u and the projection $\rho : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. Notice that for such a Cartan gauge to exist we need $T_xM \cong \mathfrak{g}/\mathfrak{h}$ for all $x \in U$. This means that the model geometry we use on a manifold to construct a Cartan geometry must in some sense be compatible with the manifold itself. Also notice that for any Klein geometry this will follow automatically since $M = G/H$.

Definition 2.4.2. A Cartan atlas on M is a collection $\mathcal{A} = \{(U_\alpha, \theta_\alpha)\}$ of Cartan gauges with model geometry consisting of the Klein pair $(\mathfrak{g}, \mathfrak{h})$ and group H such that

(i) $\bigcup_\alpha U_\alpha = M$;

(ii) if (U, θ_U), $(V, \theta_V) \in \mathcal{A}$, then there exist a smooth map $k : U \cap V \to H$ such that

$$(\theta_V)_x = Ad(k(x)^{-1})(\theta_U)_x + (k^*)_x\omega_H$$

for each $x \in U \cap V$.

$\spadesuit$

We will say that θ_U and θ_V are k-related and write $\theta_U \Rightarrow_k \theta_V$ whenever the second condition is fulfilled. Examining this condition, we see that

$$(\theta_V)_x : T_xV \cong T_xM \longrightarrow \mathfrak{g}, \qquad (\theta_U)_x : T_xU \cong T_xM \longrightarrow \mathfrak{g},$$

and if $k(x) = h \in H$, then

$$Ad(k(x)^{-1}) = Ad_{h^{-1}} : \mathfrak{g} \longrightarrow \mathfrak{g},$$

as defined earlier in relation to Definition 2.1.10. The last term is the pullback of the Maurer-Cartan form Definition 2.1.3. By the definition of pullbacks we get

$$(k^*)_x\omega_H : T_x(U \cap V) \cong T_xM \longrightarrow \mathfrak{h} \subset \mathfrak{g},$$

given by

$$(k^*)_x\omega_H(W_x) = \omega_H(dk_x(W_x))$$

where W is a vector field on $U \cap V$. Notice that Ad_h is a linear automorphism on $\mathfrak{g}$ that sends $\mathfrak{h}$ to itself whenever $h \in H$. This is clear since if we look at the Lie group H by itself, then $Ad_h \in Aut(\mathfrak{h})$ for all $h \in H$. That means that the diagram bellow commutes:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "Ad_h"] \arrow[dr, "\rho"] & \mathfrak{g} \arrow[d, "\rho"] \\
& \mathfrak{g}/\mathfrak{h}
\end{tikzcd}$$

Here ρ is the projection of $\mathfrak{g}$ to $\mathfrak{g}/\mathfrak{h}$ as a projection of vector spaces. Since $(k^*)_x\omega_H$ takes values in $\mathfrak{h}$, we have the following corollary:

Corollary 2.4.3. *If (U, θ_U) and (V, θ_V) are Cartan gauges in the same Cartan atlas, then the following diagram commutes:*

$$\begin{array}{ccc} T_xM & \xrightarrow{(\theta_V)_x} & \mathfrak{g} \\ \big\downarrow{\scriptstyle (\theta_U)_x} & & \big\downarrow{\scriptstyle \rho} \\ \mathfrak{g} & \xrightarrow{\rho} & \mathfrak{g}/\mathfrak{h} \end{array}$$

Definition 2.4.4. We define a Cartan stucture on a smooth manifold M as an equivalent class of Cartan atlases on M. A Cartan geometry is a smooth manifold M with a Cartan structure. The Cartan geometry is called effective if the model geometry is effective.

♠

Here two Cartan atlases is equivalent if the union of the two is a Cartan atlas. Note that for any Cartan atlas there is a unique maximal Cartan atlas such that the two are equivalent. We will later give another definition in which the constructions bellow will be apparent.

From any effective Cartan geometry a principal bundle arises in a natural way. Let M be a smooth manifold with a Cartan atlas $\mathcal{A} = \{(U_\alpha, \theta_\alpha)\}$ on the model geometry consisting of an effective Klein pair $(\mathfrak{g}, \mathfrak{h})$ and the Lie group H associated with $\mathfrak{h}$. Let $\mathcal{W} = \{W_\beta\}$ be an open cover of M such that:

(i) For any i, W_i is connected,

(ii) for any i there is an l such that $W_i \subset U_l$,

(iii) for any i and j the intersection $W_i \cap W_j$ is connected.

The idea is to glue together the product spaces $W \times H$ for all $W \in \mathcal{W}$. For each W_i we have by condition (ii) above a gauge (U_l, θ_l) such that $W_i \subset U_l$. Let Choose $\phi_i = \theta_l|_{W_i}$ as a representative 1-form on W_i. If $W_1, W_2 \in \mathcal{W}$ have the corresponding 1-forms ϕ_1, ϕ_2 respectively, then we have a gauge equivalence $\phi_1 \Rightarrow_k \phi_2$ along $W_1 \cap W_2$, where k is a smooth map $k : W_1 \cap W_2 \to H$.

Claim 1: This k is unique, see Proposition A.0.9.

Now, we can glue together the product spaces $W_1 \times H$ and $W_2 \times H$ along $(W_1 \cap W_2) \times H$ by the equivalence relation $(w, h) \sim (w, k(w)^{-1}h)$ for all $w \in (W_1 \cap W_2)$ and $h \in H$.

Claim 2: If $\theta_1 \Rightarrow_k \theta_2$ and $\theta_2 \Rightarrow_r \theta_3$, then $\theta_1 \Rightarrow_{kr} \theta_3$, see Lemma A.0.4 (iii).

By the claim, if we have three open sets $W_1, W_2, W_3 \in \mathcal{W}$ such that $w \in W_1 \cap W_2 \cap W_3 \neq \emptyset$, then $(w, h) \sim (w, (k(w)r(w))^{-1}h) = (w, r(w)^{-1}k(w)^{-1}h)$ as we would want. By this construction we have a fiber bundle over M, $\xi = (P, M, \pi, H)$ where P is the quotient space $P = (\bigcup_\beta W_\beta \times H)/\sim$ and π is the projection $[(w, h)] \mapsto w$. Moreover ξ is a smooth fiber bundle since the transition functions are smooth:

$$\begin{tikzcd} & \pi^{-1}(W) \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ W_1 \times H \arrow[rr, "id\times f"] & & W_2 \times H \end{tikzcd}$$

Here you see the natural trivializations where ρ_i sends the equivalence class $[(w, h)]$ to its corresponding element $(w, h_i) \in W_i \times H$, i.e. $(w, h_i) \sim (w, h)$. The transition function

$$\begin{aligned} id \times f : W_1 \times H &\longrightarrow W_2 \times H \\ (w,h) &\longmapsto (w, k(w)^{-1}h), \end{aligned}$$

and this function is smooth since k is smooth by definition and multiplication of elements in a Lie group is smooth.

Since the fibers of ξ is the Lie group H, we have a principle bundle: the Lie group H acts on P from the right by $[(w,h)]g = [(w,hg)]$. This is well defined since the diagram bellow commutes:

$$\begin{tikzcd} (w,h) \arrow[r, mapsto, "R_g"] \arrow[d, mapsto, "\sim"] & (w,hg) \arrow[d, mapsto, "\sim"] \\ (w,k(w)^{-1}h) \arrow[r, mapsto, "R_g"] & (w,k(w)^{-1}hg). \end{tikzcd}$$

At this point we will omit the brackets and simply to refer to an element of P as (w,h), implicitly understanding that this is an equivalence class of elements. Clearly H acts simply transitively on each fiber. This makes $\xi = (P, M, \pi, H)$ into a principle bundle by Definition 2.3.1. Note that this principle bundle was uniquely determined from an effective Cartan geometry.

From such a principal bundle we get a $\mathfrak{g}$-valued 1-form ω on P called the Cartan connection. Given a gauge (U,θ), we have a linear isomorphism

$$\begin{aligned} \omega : T_{(w,h)}(W \times H) &\longmapsto \mathfrak{g} \\ (v,y) &\longmapsto Ad(h^{-1})\theta(v) + \omega_H(y). \end{aligned}$$

We need to see that these isomorphisms fit together smoothly as we vary the gauge, such that we get a $\mathfrak{g}$-valued 1-form on P. We need to look at the transition functions of the form

$$\begin{aligned} id \times f : W \times H &\longrightarrow W \times H \\ (w,h) &\longmapsto (w, k(w)^{-1}h). \end{aligned}$$

The differential of this function gives a map

$$\begin{aligned} (id \times f)_* : T_{(w,h)}(W \times H) &\longrightarrow T_{(w,k(w)^{-1}h)}(W \times H) \\ (v,y_1) &\longmapsto (v,y_2). \end{aligned}$$

Now we have $f(w,h) = k(w)^{-1}h$ and $f_*(v,y_1) = y_2$.

Claim 3: If $\imath : G \to G$ by $g \mapsto g^{-1}$, then

$$\imath^*\omega_G(v) = -Ad(g)\omega_G(v)$$

for $v \in T_gG$, see Proposition A.0.2 (ii).

Claim 4: If $f_1, f_2 : M \to G$, and $h(x) = f_1(x)f_2(x)$, then

$$h^*\omega_G = Ad(f_2(x)^{-1})f_1^*\omega_G + f_2^*\omega_G.$$

See Corollary A.0.3. Using the claims above, we get

$$\begin{aligned}
\omega_H(y_2) &= \omega_H(f_*(v, y_1)) \\
&= f^*\omega_H(v, y_1) \\
&= Ad(h^{-1})(\imath \circ k)^*\omega_H(v) + h^*\omega_H(y_1) && \text{By Claim 4} \\
&= Ad(h^{-1})k^*\imath^*\omega_H(v) + h^*\omega_H(y_1) \\
&= Ad(h^{-1})\imath^*\omega_H(k_*v) + h^*\omega_H(y_1) \\
&= Ad(h^{-1})(-Ad(k)k^*\omega_H(v) + h^*\omega_H(y_1)) && \text{By Claim 3} \\
&= -Ad(h^{-1}k)k^*\omega_H(v) + h^*\omega_H(y_1) \\
&= -Ad(h^{-1}k)k^*\omega_H(v) + \omega_H(L_{h*}y_1) \\
&= -Ad(h^{-1}k)k^*\omega_H(v) + \omega_H(y_1).
\end{aligned}$$

Here we also used the fact that ω_H is left invariant to get the last equality. We need the diagrams bellow to commute:

$$\begin{tikzcd}
T_{(w,h)}(W \times H) \arrow[r, "\omega"] \arrow[d, "\sim"] & \mathfrak{g} \arrow[d, "id"] \\
T_{(w,k(w)^{-1}h)}(W \times H) \arrow[r, "\omega"] & \mathfrak{g}
\end{tikzcd}$$

$$\begin{tikzcd}
(v, y_1) \arrow[r, mapsto, "\omega"] \arrow[d, mapsto, "\sim"] & Ad(h^{-1})\theta_1(v) + \omega_H(y_1) \arrow[d, mapsto, "id"] \\
(v, y_2) \arrow[r, mapsto, "\omega"] & Ad(h^{-1}k)\theta_2(v) + \omega_H(y_2)
\end{tikzcd}$$

Now, using the relation $\theta_2 = Ad(k^{-1})\theta_1 + k^*\omega_H$ from Definition 2.4.2 (ii), we try to show the relation on the right column above:

$$\begin{aligned}
Ad(h^{-1}k)\theta_2(v) + \omega_H(y_2) &= Ad(h^{-1}k)(Ad(k^{-1})\theta_1(v) + k^*\omega_H(v)) + \omega_H(y_2) \\
&= Ad(h^{-1})\theta_1(v) + Ad(h^{-1}k)k^*\omega_H(v) - Ad(h^{-1}k)k^*\omega_H(v) + \omega_H(y_1) \\
&= Ad(h^{-1})\theta_1(v) + \omega_H(y_1),
\end{aligned}$$

which means that the Cartan connection is well-defined independent of the choice of gauge.

If we look at an element $X \in \mathfrak{h}$, recall that by the Maurer-Cartan form in Definition 2.1.3 we can get a left invariant vector field on H by $\omega_H^{-1}(X) \in \mathcal{T}(H)$. We can use this to create a vector field on P in the following way: Let $X^\sharp_{(w,h)} = (0, \omega_H^{-1}(X)_h) \in T_{(w,h)}(W \times H)$. This gives a vector field on $W \times H$, but we need these to fit together on the intersections, i.e., since $(w,h) \sim (w, k(w)^{-1}h)$ we need $X^\sharp_{(w,k(w)^{-1}h)} = L(k(w)^{-1})_* X^\sharp_{(w,h)}$. This works since $\omega_H^{-1}(X)$ is left invariant:

$$\begin{aligned}
L(k(w)^{-1})_* X^\sharp_{(w,h)} &= L(k(w)^{-1})_*(0, \omega_H^{-1}(X))_{(w,h)} \\
&= (0, L(k(w)^{-1})_*\omega_H^{-1}(X))_{(w,k(w)^{-1}h)} \\
&= (0, \omega_H^{-1}(X))_{(w,k(w)^{-1}h)} \\
&= X^\sharp_{(w,k(w)^{-1}h)}
\end{aligned}$$

Proposition 2.4.5. *[Sha00, Prop 5.2.4, p.182] The Cartan connection ω on P with values in $\mathfrak{g}$ has the following properties*

(i) for each point $p \in P$, the linear map $\omega_p : T_pP \to \mathfrak{g}$ is an isomorphism.

(ii) $(R_h)^*\omega = Ad(h^{-1})\omega$,

(iii) $\omega(X^\sharp) = X$ for all $X \in \mathfrak{h}$.

Proof. (i) We know that $\dim P = \dim M + \dim H = \dim \mathfrak{g}/\mathfrak{h} + \dim \mathfrak{h} = \dim \mathfrak{g}$, so it is sufficient to show that $\omega_p : T_pP \to \mathfrak{g}$ is linear and injective for each $p \in P$. Recall that the Cartan connection is given by

$$\omega(v, y) = Ad(h^{-1})\theta(v) + \omega_H(y)$$

with $(v, y) \in T_{(w,h)}(W \times H) \cong T_pP$. Notice that the gauges, the adjoint map and the Maurer-Cartan form are all linear, so the Cartan connetion is agin linear. We need to prove injectivity, thus is it sufficient to show that $\omega(v, y) = 0$ implies $(v, y) = 0$. If $\omega(v, y) = 0$ we have $Ad(h^{-1})\theta(v) = -\omega_H(y)$, but $\omega_H(y)$ lies in $\mathfrak{h}$. Since $Ad(h)g$ lies in $\mathfrak{h}$ if and only if g lies in $\mathfrak{h}$ we know that $\theta(v)$ must lie in $\mathfrak{h}$. Since θ is a Cartan gauge, we know that $\bar{\theta} : T_uU \to \mathfrak{g}/\mathfrak{h}$ is an isomorphism, hence if $\bar{\theta}(v) = 0$, then $v = 0$. But $\bar{\theta}(v) = 0$ if and only if $\theta(v) \in \mathfrak{h}$, hence $v = 0$. Then we must have $\theta(v) = 0$, which means that $\omega_H(y) = 0$, hence $y = 0$ and we have proven that ω_p is an isomorphism.

(ii) We need the following diagrams to commute.

$$\begin{tikzcd}
T_{(w,h)}(W \times H) \arrow[r, "\omega"] \arrow[d, "R_{r*}"] & \mathfrak{g} \arrow[d, "Ad(r^{-1})"] \\
T_{(w,hr)}(W \times H) \arrow[r, "\omega"] & \mathfrak{g}
\end{tikzcd}$$

$$\begin{tikzcd}
(v, y) \arrow[r, mapsto, "\omega"] \arrow[d, "R_{r*}"] & Ad(h^{-1})\theta(v) + \omega_H(y) \arrow[d, dashed, "Ad(r^{-1})"] \\
(v, R_{r*}y) \arrow[r, mapsto, "\omega"] & Ad((hr)^{-1})\theta(v) + \omega_H(R_{r*}y)
\end{tikzcd}$$

We have $Ad((hr)^{-1})\theta(v) + \omega_H(R_{r*}y) = Ad(r^{-1})(Ad(h^{-1})\theta(v) + \omega_H(y))$, hence the diagrams commute.

(iii) Recall that $X^\sharp = (0, \omega_H^{-1}(X))$ on any $W \times H$, hence

$$\omega(X^\sharp) = \omega(0, \omega_H^{-1}(X)) = Ad(h^{-1})\theta(0) + \omega_H(\omega_H^{-1}(X)) = X$$

□

Notice that the Cartan connection is not a principle connection as in Definition 2.3.6. Even though the two definitions are similar, there are some important differences. The principal connections takes values in the Lie algebra of the Lie group that is acting on the principal bundle, which can be identified with the Lie algebra of the fibers. Also, this 1-form is not in general an isomorphism when restricted to a point of the principal bundle. This is clear since the dimension of the tangent space at a point of the principal bundle is equal to the dimension of the Lie algebra of the fibers plus the dimension of the manifold, so this can be an isomorphism only if the manifold have dimension zero.

There are an interesting point to be made when comparing principal connections with Cartan connections. If the Lie algebra $\mathfrak{g}$ of the model geometry Klein pair $(\mathfrak{g}, \mathfrak{h})$ used in the model geometry is reductive, we can write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$. In this case we can project the Cartan connection to each of the components $\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{g}/\mathfrak{h}}$, and then $\omega_{\mathfrak{h}}$ is a principal connection. See Appendix A in [Sha00] for the proof and for more details on the topic.
Now we give another definition of the Cartan geometry by relating it to a principal bundle over M.

Definition 2.4.6. A Cartan geometry (P, ω) on M modeled on the Klein pair $(\mathfrak{g}, \mathfrak{h})$ with H as the Lie group with Lie algebra $\mathfrak{h}$ consists of:

(a) a smooth manifold M,

(b) a principle bundle $P \to M$ with H acting on P from the right,

(c) a $\mathfrak{g}$-valued 1-form ω on P satisfying

   (i) $\omega_p : T_pP \to \mathfrak{g}$ is a linear isomorphism for each point $p \in P$,

   (ii) $R_h^*\omega = Ad_{h^{-1}}\omega$ for all $h \in H$,

   (iii) $\omega(X^\sharp) = X$ for all $X \in \mathfrak{h}$.

♠

Now, let us check that any Klein geometry can be associated uniquely with a Cartan geometry.

Example 2.4.7. Let (G, H) be a Klein geometry. Then we have

(a) a smooth manifold $M = G/H$,

(b) a principal bundle $G \to M$ with H acting on G from the right,

(c) the Maurer-Cartan form ω_G satisfying

   (i) $(\omega_G)_g : T_gG \to \mathfrak{g}$ is a linear isomorphism,

   (ii) $R_h^*\omega = Ad_{h^{-1}}\omega$ for all $h \in H$,

   (iii) $\omega(X^\sharp) = X$ for all $X \in \mathfrak{h}$, the Lie algebra of H.

We have already checked in Example 2.3.3 that a Klein geometry does indeed give a principle bundle. That the Maurer-Cartan form is a linear isomorphism on each fiber comes from the fact that any tangent vector can be extended to a left-invariant vector field, and the last two properties we checked in Example 2.3.7. This shows that Cartan geometry is a generalization of Klein geometry. ♣

2.5 Cartan Curvature

Notice that in many ways the Cartan connection in Cartan geometry fulfills an analogous role as the Levi-Civita connection does in Riemannian geometry, and just like the Levi-Civita connection gives a canonical definition of curvature in a Riemannian geometry, we can analogously use the Cartan connection to give a canonical definition of curvature in a Cartan geometry.

Definition 2.5.1. Let (P, ω) be a Cartan geometry on the manifold M. The $\mathfrak{g}$-valued 2-form defined by

$$K = d\omega + \frac{1}{2}[\omega, \omega]$$

is called the curvature form of the Cartan geometry (P, ω), or just the Cartan curvature. Let $\rho : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ be the canonical projection, then $\rho(K)$ is called the torsion of the curvature. ♠

This is the definition of curvature we will use in the later chapters. Notice that it resembles the structural equation of a Lie group. From Example 2.4.7 we know that a Klein geometry (G, H) induces a Cartan geometry with the Maurer-Cartan form ω_G as the Cartan connection. In this case the curvature becomes

$$K = d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0$$

since this is exactly the structural equation of a Lie group G. We may conclude that any Cartan geometry induced by a Klein geometry has zero curvature. We will include one important result about the Cartan curvature form.

Lemma 2.5.2. *[Sha00, Cor. 5.3.10, p.187] The Cartan curvature form $K(u, v)$ of a Cartan geometry (P, ω) on a manifold M vanishes whenever u or v is tangent to the fiber.*

Recall that the tangent space to a point in a principal bundle P can be decomposed into a horizontal space and a vertical space

$$T_pP \cong \mathcal{H}_p \oplus \mathcal{V}_p.$$

Here the vertical space $\mathcal{V}_p$ is exactly the space of vectors tangent to the fibers. For the horizontal space we have

$$\mathcal{H}_p = T_pP/\ker(d\pi_p) \cong \pi^*(T_{\pi(p)}M),$$

where π is the principle bundle projection $\pi : P \to M$. This means that we can write any tangent vector $v \in T_pP$ as $v_\mathcal{H} + v_\mathcal{V}$ where $v_\mathcal{H} \in \mathcal{H}_p$ and $v_\mathcal{V} \in \mathcal{V}_p$, and by the lemma above we get

$$\begin{aligned} K(u, v) &= K(u_\mathcal{H} + u_\mathcal{V}, v_\mathcal{H} + v_\mathcal{V}) \\ &= K(u_\mathcal{H}, v_\mathcal{H}) + K(u_\mathcal{H}, v_\mathcal{V}) + K(u_\mathcal{V}, v_\mathcal{H}) + K(u_\mathcal{V}, v_\mathcal{V}) \\ &= K(u_\mathcal{H}, v_\mathcal{H}) \end{aligned}$$

which means that the Cartan curvature can be regarded as a 2-form on the pullback of TM to TP.

There is also a function called the curvature function that is associated with the Cartan curvature, and it will be usefull when doing computations later.

Definition 2.5.3. Let (P, ω) be a Cartan geometry on the manifold M. Define the a map $\kappa : P \to \hom(\wedge^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$ by the formula

$$\kappa_p(X_1, X_2) = K_p(\omega_p^{-1}(X_1), \omega_p^{-1}(X_2)).$$

This function is called the curvature function of the Cartan geometry (P, ω), or the Cartan curvature function. ♠

Notice that the curvature function is well-defined; if $Z_i = X_i + Y_i$ with $X_i \in \mathfrak{g}/\mathfrak{h}$ and $Y_i \in \mathfrak{h}$ for $i = 1, 2$, we get

$$\kappa_p(Z_1, Z_2) = K_p(\omega_p^{-1}(Z_1), \omega_p^{-1}(Z_2)) = K_p(\omega_p^{-1}(X_1), \omega_p^{-1}(X_2)) = \kappa_p(X_1, X_2)$$

since $\omega_p^{-1}(V_i)$ is tangent to the fiber. If $K_p = 0$ we see that the curvature function will be the zero homomorphism $\kappa_p = 0_{hom}$, or equivalently $\kappa_p = 0 \in \wedge^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$.

Example 2.5.4. Let M be a Riemannain manifold of dimension n. Let $G = E(n)$ and $H = O(n)$ be the Lie groups from the Klein geometry in Example 2.2.2 (i) with the same representations. Then we get the Klein pair $(\mathfrak{g}, \mathfrak{h})$ with

$$\mathfrak{g} = \mathfrak{euc}(n) = \{ \begin{pmatrix} 0 & 0 \\ t & A \end{pmatrix} \mid A + A^T = 0, \, t \in \mathbb{R}^n \}$$
$$\mathfrak{h} = \mathfrak{o}(n) = \{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \mid A + A^T = 0 \}.$$

Notice that we can write

$$\mathfrak{p} = \mathfrak{g}/\mathfrak{h} = \{ \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \mid t \in \mathbb{R}^n \},$$

and that $Ad(H)\mathfrak{p} \subset \mathfrak{p}$, $Ad(H)\mathfrak{h} \subset \mathfrak{h}$. This means that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ with both components invariant under the adjoint action of H. Moreover, for $u, v \in \mathfrak{p}$ and $A, B \in \mathfrak{h}$ we have the following brackets:

$$[A, B] = AB - BA \in \mathfrak{h}$$
$$[A, u] = Au \in \mathfrak{p}$$
$$[u, v] = 0.$$

We might use this Klein pair and the adjoint representation of H as a model geometry to construct a Cartan geometry on M. To do this we use orthonormal frame bundle $P \to M$ (see Section 3.4). For now, just notice that it makes sense to talk about the orthogonal group acting on the tangent spaces of M since we have a metric that would be preserved under orthogonal transformations. This gives a Cartan connection

$$\omega : TP \longrightarrow \mathfrak{h} \oplus \mathfrak{p}$$

with $\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{p}}$. We can give the curvature $K = d\omega + \frac{1}{2}[\omega, \omega]$, and the torsion is given by

$$\rho(K) = \rho(d\omega + \frac{1}{2}[\omega, \omega]) = d\omega_{\mathfrak{p}} + [\omega_{\mathfrak{h}}, \omega_{\mathfrak{p}}]$$

since by the bracket relations, the rest of the Cartan curvature lies in $\mathfrak{h}$ and will hence be in the kernel of the projection. A Cartan geometry modeled on this Klein pair is called a

Euclidean geometry and we will define a Riemannian geometry as a torsion free Euclidean geometry. In this case the 1-form $\omega_{\mathfrak{h}}$ is called the Levi-Civita connection. Notice that the Levi-Civita is a principal connection. This example can be found in [Mor08], or with more details in [Sha00, Chapter 6]. ♣

We could have been more precise in the example above and said that this is the riemannian geometry modeled on $\mathbb{R}^n$, since the model geometry we chose was exactly the Klein geometry of $\mathbb{R}^n$. This is just one of the three possible model spaces for Riemannian geometry, the other being the sphere S^n and the hyperbolic space $\mathbb{H}^n$. Recall from Example 2.2.2 (ii) that the subgroup of the Klein geometry $H = O(n)$ was the same group as in (i), which is the Klein geometry we used when constructing the Euclidean geometry. This means that the same principal bundle $O(n) \to P \to M$ for some Riemannian manifold M could be given different Cartan geometries modeled on either $(E(n), O(n))$ or $(O(n+1), O(n))$ which would give very different outcomes for the Cartan curvature [Ča17, Example 2.8]. In the Euclidean case, the curvature would measure how different our manifold is from $\mathbb{R}^n$, in the sense that $\mathbb{R}^n$ would have zero curvature. In the spherical case we would measure how different our manifold is from S^n, in the sense that S^n would have zero curvature. A similar statement could be made for hyperbolic space.

Chapter 3

Curvature of Sub-Riemannian Manifolds with Constant Sub-Riemannian Symbol

Recall from Example 2.5.4 that we used the Klein pair $(\mathfrak{euc}(n), \mathfrak{o}(n))$ coming from the Euclidean group and the orthogonal group respectively to construct a Cartan geometry we called the Euclidean geometry. In the special case where we choose a torsion free Cartan connection we called it a Riemannian geometry. It needs to be emphasised that this works because we started out with a Riemannian manifold, that is, because we had a Riemannian metric to be preserved under the group action of $O(n)$. Exactly how will be made clear in Section 3.4. In this chapter we will consider sub-Riemannian manifolds, which means we will only have a metric on some subbundle of the tangent bundle. Our goal in this chapter will be to define a canonical way of constructing a Cartan geometry on sub-Riemannian manifolds with constant sub-Riemannian symbol as defined in 3.3.2, which would let us give a canonical expression for the Cartan curvature. We will see that the canonical construction will finally rely upon Theorem 3.5.5 by Morimoto [Mor08]. The method we will use to explain how to actually construct this canonical Cartan geometry will follow the lines of [AMS19], while much of the theory is explained in [Ča17].

3.1 Smooth Sub-Riemannian Geometry

Definition 3.1.1. Let M be a smooth manifold. A map

$$D : x \longmapsto D_x \subset T_xM$$

is called a distribution of M. The distribution is smooth if for every $q \in M$ there is a neighbourhood $q \in U$ and smooth linearly independent vector fields $\{X_1, ..., X_k\}$ such that $D_x = \text{span}\{X_1(x), ..., X_k(x)\}$ for all $x \in U$. ♠

Any vector $V_x \in T_xM$ is called horizontal if $V_x \in D_x$, and the vector field V is called horizontal if $V_x \in D_x$ for all $x \in M$. A smooth curve $c : I \to M$ is called horizontal if $\dot{c}(t) \in D_{c(t)}$ for all $t \in I \subset \mathbb{R}$. Let $\mathcal{D}^{-1}$ denote the set of smooth vector fields on M such that if $V \in \mathcal{D}^{-1}$, then $V_x \in D_x$ for all $x \in M$, and define

$$\mathcal{D}^{-j} = \mathcal{D}^{-j+1} \cup \{[X,Y] \,|\, X \in \mathcal{D}^{-j+1}, Y \in \mathcal{D}^{-1}\}, \quad j = 1, 2, 3, \ldots.$$

This induces the subsets $D^{-j} \subset TM$ by

$$D_x^{-j} = \mathcal{D}_x^{-j} = \{V_x \in T_xM \mid V \in \mathcal{D}^{-j}\}.$$

Definition 3.1.2. Let M be a smooth manifold with a smooth distribution D. We say that the distribution D is bracket generating if for every $x \in M$ there is an integer $k > 0$ such that $\mathcal{D}_x^{-k} = T_xM$. If there is a k such that this is true for all $x \in M$ and this is the lowest integer with this property, we say that the distribution D is k-step bracket generating. ♠

If we have a smooth manifold M with a smooth k-step bracket generating distibution D we get a filtration on the tangen bundle

$$D = D^{-1} \subset D^{-2} \subset \ldots \subset D^{-k} = TM.$$

Even if we have a k-step bracket generating distribution, the number of steps needed to generate T_xM might depend on the point x, as we shall see in Example 3.1.4 (ii).

Definition 3.1.3. A distribution D on a sub-Riemannian manifold M is equiregular if it is k-step generating for some $k \geq 1$ and the dimension of D_x^{-i} is independent of $x \in M$ for all $i = 1, \ldots, k$. ♠

Example 3.1.4. Lets consider two different distributions on $\mathbb{R}^3$:

(i) Let $D = \operatorname{span}\{\partial_x, \partial_y + x\partial_z\}$. Then we have $D^{-2} = \operatorname{span}\{\partial_x, \partial_y + x\partial_z, \partial_z\} = T\mathbb{R}^3$ since $[\partial_x, \partial_y + x\partial_z] = \partial_z$. This means that D is equiregular 2-step bracket generating.

(ii) Let $D = \operatorname{span}\{\partial_x, \partial_y + x^2\partial_z\}$. Then we have $D^{-2} = \operatorname{span}\{\partial_x, \partial_y + x^2\partial_z, x\partial_z\}$ since $[\partial_x, \partial_y + x^2\partial_z] = x\partial_z$. This gives

$$\begin{aligned} x \neq 0 \quad &\Rightarrow \quad D^{-2}_{(x,y,z)} = T_{(x,y,z)}\mathbb{R}^3, \\ x = 0 \quad &\Rightarrow \quad D^{-2}_{(0,y,z)} = \operatorname{span}\{\partial_x, \partial_y\} \neq T_{(0,y,z)}\mathbb{R}^3. \end{aligned}$$

However, since $[\partial_x, x\partial_z] = \partial_z$, we have $D^{-3} = T\mathbb{R}^3$, so D is 3-step bracket generating, but not equiregular.

♣

Recall that a Riemannian metric g on a manifold M is induced by a map

$$g : \mathcal{T}(M) \times \mathcal{T}(M) \longrightarrow \mathcal{C}^\infty(M)$$

where $\mathcal{T}(M)$ is the space of smooth sections on M, such that $g|_x$ is a symmetric bilinear positive definite map from $T_xM \times T_xM$ to $\mathbb{R}$ for any $x \in M$. Such a metric gives a way to measure distance on the manifold.

Definition 3.1.5. Let M be a Riemannian manifold with a metric g, and let $q_0, q_1 \in M$ be two points on the manifold. We define the distance between q_0 and q_1 to be

$$d(q_0, q_1) = \inf_c \{\int_0^1 (g(\dot{c}(t), \dot{c}(t)))^{1/2} dt\}$$

where the infimum is taken over all piecewise smooth curves $c : [0,1] \to M$ with $c(0) = q_0$, $c(1) = q_1$. ♠

We want to define a metric on the distribution of a smooth manifold in a similar manner.

Definition 3.1.6. Let $\mathcal{D}^{-1}$ be the space of smooth sections on a distribution D on a smooth manifold M. A map

$$S : \mathcal{D}^{-1} \times \mathcal{D}^{-1} \longrightarrow \mathcal{C}^{\infty}(M)$$

such that $S|_x$ is a symmetric bilinear positive definite map from $D_x \times D_x$ to $\mathbb{R}$ for all $x \in M$ is a metric on the distribution D.

♠

Definition 3.1.7. Let M be a smooth manifold. A smooth distribution D of M together with a metric S on D is a sub-Riemannian structure on M, and the triplet (M, D, S) is called a sub-Riemannian manifold. ♠

An important question is when will this metric provide a way of measuring the distance between any two points on the manifold? We want to define the distance in a similar manner to how it is defined for Riemannian manifolds in Defintion 3.1.5, but this formula will not be defined for general smooth curves $c : [0, 1] \to M$, since the metric on a sub-Riemannian manifold is only defined on the distribution.

Theorem 3.1.8. *Let (M, D, S) be a connected sub-Riemannian manifold and let $p, q \in M$ be any two points. If the distribution D is bracket generating, then we can find a horizontal path $c : [0, 1] \to M$ such that $c(0) = p$ and $c(1) = q$.*

This theorem was proved by P.K. Rashevskiï [Ras38] and W.L. Chow [Cho39] independently. The proof can be found in [Mon06, Chapter 2] or [AS10, Chapter 5].

Definition 3.1.9. Let (M, D, S) be a bracket generating connected sub-Riemannian manifold. Define the Carnot-Carathéodory distance

$$d_{c-c}(q_0, q_1) = \inf_{c} \{ \int_0^1 (S(\dot{c}(t), \dot{c}(t)))^{1/2} dt \}$$

where $q_0, q_1 \in M$ and $c : [0, 1] \to M$ is a horizontal curve with $c(0) = q_0$ and $c(1) = q_1$. ♠

3.2 Carnot Groups

In this section we will study Carnot groups which are highly symmetrical Lie groups. Recall that $\mathbb{R}^n$, S^n and the hyperbolic spaces $\mathbb{H}^n$ are highly symmetrical Riemannian manifolds that are the established model spaces for Riemannian geometry, see for example [Lee97, Theorem 11.12]. We intend to use the Carnot groups as model spaces when determining the curvature of sub-Riemannian manifolds with constant sub-Riemannian symbol, see Definition 3.3.2. More details about the topics in this section can be found in [Don16].

Definition 3.2.1. Let $\mathfrak{g}$ be a Lie algebra. A negative grading on $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_{-i}$$

such that $[\mathfrak{g}_{-i}, \mathfrak{g}_{-j}] \subset \mathfrak{g}_{-(i+j)}$ for all $i, j = 0, 1, 2, \ldots$, and such that $\mathfrak{g}_{-i} = 0$ for all but finitely many values of i. A Lie algebra equipped with a negative grading is called a negatively graded Lie algebra. ♠

Definition 3.2.2. Let $\mathfrak{g}$ be a Lie algebra. A stratification on $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}$$

such that $[\mathfrak{g}_{-1}, \mathfrak{g}_{-j}] = \mathfrak{g}_{-(1+j)}$ for all $j = 0, 1, 2, \ldots$ such that $1 + j \leq k$ and $[\mathfrak{g}_{-1}, \mathfrak{g}_{-j}] = 0$ for $1 + j > k$. A Lie algebra equipped with a stratification is called stratified. ♠

We could equivalently define a stratified Lie algebra as a negatively graded Lie algebra $\mathfrak{g}$ where $\mathfrak{g}_{-1}$ generates $\mathfrak{g}$, i.e. $\mathfrak{g} = \oplus_{i=1}^{k} \mathfrak{g}_{-1}^{i}$ where $\mathfrak{g}_{-1}^{i}$ is defined inductively by $\mathfrak{g}_{-1}^{i} = [\mathfrak{g}_{-1}^{(i-1)}, \mathfrak{g}_{-1}]$. In particular, any stratified Lie algebra is also negatively graded.

Definition 3.2.3. A Lie group G is called negatively graded if G is connected, simply connected and the Lie algebra of G is negatively graded. A Lie group G is called stratified if it is connected, simply connected and the Lie algebra of G is stratified. ♠

Definition 3.2.4. Let λ be a positive real number. A dilation δ_λ on a negatively graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}$ is a linear map $\delta_\lambda : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that for any $X \in \mathfrak{g}_{-i}$ we have $\delta_\lambda(X) = \lambda^i X$. ♠

Any dilation of a Lie algebra is a Lie algebra automorphism, i.e. a linear isomorphism that preserves the bracket. To see this, notice that δ_λ is by definition a linear map and has the dilation $\delta_{1/\lambda}$ as inverse. Also, for $X \in \mathfrak{g}_{-i}$ and $Y \in \mathfrak{g}_{-j}$, we have

$$\delta_\lambda([X, Y]) = \lambda^{i+j}[X, Y] = [\lambda^i X, \lambda^j Y] = [\delta_\lambda X, \delta_\lambda Y]$$

hence the bracket is preserved. It is also worth noting that for any two dilations δ_λ and δ_η we have

$$\delta_\lambda \circ \delta_\eta = \delta_{\lambda\eta}.$$

Definition 3.2.5. Let G be a negatively graded Lie group and let $\lambda > 0$ be a real number. A dilation Δ_λ on G is a Lie group automorphism $\Delta_\lambda : G \to G$ such that $d\Delta_\lambda = \delta_\lambda$ is a dilation on the negatively graded Lie algebra $\mathfrak{g}$ of G. ♠

Definition 3.2.6. A Lie group G is called a Carnot group if G is stratified and the stratified Lie algebra $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}$ of G has a metric $\langle ., . \rangle_{\mathfrak{g}_{-1}}$ on $\mathfrak{g}_{-1}$. ♠

Now, let G be a Carnot group. Then we can use left-translation to generate a distribution on G by

$$D_g = L_{g*}(\mathfrak{g}_{-1}) = \operatorname{span}\{L_{g*}(X) \mid X \in \mathfrak{g}_{-1}\}.$$

Since any two vectors $u, v \in D_g$ can be written as $u = L_{g*}(X)$ and $v = L_{g*}(Y)$ for some elements $X, Y \in \mathfrak{g}_{-1}$, we can give a metric S on the distribution by

$$S(u, v) = \langle X, Y \rangle_{\mathfrak{g}_{-1}}.$$

Writing $D : g \to D_g \subset T_g G$, we see that this is the structure of a sub-Riemannian manifold (G, D, S). Since the Lie algebra is generated by $\mathfrak{g}_{-1}$ we get a bracket generating distribution D, which means we have a distance between any two points on our manifold, namely the Carnot-Carathéodory distance from Definition 3.1.9. This means that Carnot groups are metric spaces. Let us consider the distance $d_{c-c}(\Delta_\lambda p, \Delta_\lambda q)$ for two points p, q on G. Let $c : [0,1] \to G$ be a horizontal path with $c(0) = p$ and $c(1) = q$, and let $X(t) = \omega_G(c(t))$.

$$\begin{aligned}
d_{c-c}(\Delta_\lambda p, \Delta_\lambda q) &= \inf_c \{ \int_0^1 (S(d(\Delta_\lambda c)(t), d(\Delta_\lambda c)(t)))^{1/2} dt \} \\
&= \inf_c \{ \int_0^1 (S(d\Delta_\lambda \circ dc(t), d\Delta_\lambda \circ dc(t)))^{1/2} dt \} \\
&= \inf_c \{ \int_0^1 (S(\delta_\lambda \circ dc(t), \delta_\lambda \circ dc(t)))^{1/2} dt \} \\
&= \inf_c \{ \int_0^1 (\langle \delta_\lambda X(t), \delta_\lambda X(t) \rangle_{\mathfrak{g}_{-1}})^{1/2} dt \} \\
&= \inf_c \{ \int_0^1 (\lambda^2 \langle X(t), X(t) \rangle_{\mathfrak{g}_{-1}})^{1/2} dt \} \\
&= \inf_c \{ \lambda \int_0^1 (\langle X(t), X(t) \rangle_{\mathfrak{g}_{-1}})^{1/2} dt \} \\
&= \inf_c \{ \lambda \int_0^1 (S(dc(t), dc(t)))^{1/2} dt \} \\
&= \lambda d_{c-c}(p, q)
\end{aligned}$$

which gives us the result

$$d(\Delta_\lambda p, \Delta_\lambda q) = \lambda d(p, q).$$

There is a slightly more general type of Lie groups called sub-Finsler-Carnot groups which consist of a stratified Lie group with a norm $\| \ . \ \|_{\mathfrak{g}_{-1}}$ on $\mathfrak{g}_{-1}$. Since any metric provides a norm, it is clear that a Carnot group is also a sub-Finsler-Carnot group. It is also intuitively clear that all we need is a norm, since we only use the norm when defining the Carnot-Carathéodory distance. The reason we mention this is to state the theorem bellow, but we need to define some terms first.

Definition 3.2.7. A metric space X is:

(i) Geodesic if for any two points $p, q \in X$ we have

$$d(p, q) = \inf_c \{ \sup_{\mathfrak{P}} \{ \sum d(c(t_i), c(t_{i+1})) \} \}$$

where $c : [0,1] \to X$ runs over all curves from p to q, and $\mathfrak{P}$ runs over all partitions $0 = t_0 < \ldots < t_k = 1$ of $[0,1]$.

(ii) Isometrically homogeneous if for any two points $p, q \in X$ there is an isometry $f : X \to X$ such that $f(p) = q$.

(iii) Self-similar if there exist a real number $\lambda > 1$ and a homeomorphism $f : X \to X$ such that

$$d(f(p), f(q)) = \lambda d(p, q)$$

for all $p, q \in X$.

♠

Recall that any metric space is also a topological space, since the metric induces a topology, hence the definition above makes sense. We have seen that a Carnot group will be self-similar by the function Δ_λ. It will be isometrically homogeneous since it is a Lie group, so for any two elements $g, h \in G$ we can look at $R_h \circ R_{g^{-1}} : g \mapsto gg^{-1}h = h$. Any space that has distance defined by the infimum of the length of curves is geodesic, since the sum over any partition will be the same.

Theorem 3.2.8. *[LD15] Sub-Finlser-Carnot groups are the only metric spaces that are*

(i) geodesic,

(ii) isometrically homogeneous,

(iii) self-similar, and

(iv) locally compact.

For a proof, follow the reference. The purpose we introduce Carnot groups is because we want to use them as model spaces for Cartan geometries. This theorem shows that Cartnot groups are highly symmetric which is what we want for model space.

Example 3.2.9. It is worth noting that $\mathbb{R}^n$ is a Carnot group with vector addition as group operator and the commutative Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1}$. The sphere S^n is however does not satisfy the properties of the theorem above; it lacks self-similarity. ♣

3.3 Constant Sub-Riemannian Symbol

Let (M, D, S) be a sub-Riemannian manifold as in Definition 3.1.7. Let $\mathcal{D}^{-1}$ denote the space of all vector fields on D and recall

$$\mathcal{D}^j = \mathcal{D}^{j+1} \cup \{[X,Y] \,|\, X \in \mathcal{D}^{j+1}, Y \in \mathcal{D}^{-1}\}, \quad j = -2, -3, \ldots.$$

If the distribution D is bracket generating, then at each point $x \in M$ we get a filtration of subspaces $T_xM = D_x^{-k} \supset \ldots \supset D_x^{-1}$. Using this, we can introduce a grading on T_xM:

$$gr(T_xM) = D_x^{-k}/D_x^{-k+1} \oplus D_x^{-k+1}/D_x^{-k+2} \oplus \ldots \oplus D_x^{-2}/D_x^{-1} \oplus D_x^{-1}.$$

Let $(T_xM)^{-i} = D_x^{-i}/D_x^{-i+1}$ so that

$$gr(T_xM) = \bigoplus_{i=1}^{k} (T_xM)^{-i}.$$

A priori this is just a graded vector space, that is a decomposition of the vector space into a direct sum of vector subspaces. We want to inroduce a bracket on the the graded tangent space, following the lines of [Ča17, Sec.2]. Let $\xi \in \mathcal{D}^{-i}$ and $\eta \in \mathcal{D}^{-j}$, then by construction we have $[\xi, \eta] \in \mathcal{D}^{-(i+j)}$. Let $q_i(x) : D_x^{-i} \to gr(T_xM)^{-i}$ be the quotient map at $x \in M$. This map induces a map $q_i : D^{-i} \longrightarrow gr(TM)^{-i}$. Consider the map

$$\begin{aligned} \vartheta_{i,j} : \mathcal{D}^{-i} \times \mathcal{D}^{-j} &\longrightarrow (T_xM)^{-(i+j)}, \\ (\xi, \eta) &\longmapsto q_{i+j}([\xi, \eta]_x). \end{aligned}$$

This map is only dependent on the value of ξ and η at x. To see this, let $f \in \mathcal{C}^\infty(M)$ be a real valued smooth function on M and consider

$$\begin{aligned} q_{i+j}([\xi, f\eta]_x) &= q_{i+j}(\xi(f)\eta_x + (f[\xi,\eta])_x) \\ &= q_{i+j}(f(x)[\xi,\eta]_x) \\ &= f(x)q_{i+j}([\xi,\eta]_x). \end{aligned}$$

Here we used the fact that $\eta_x \in D^{-j}$ which gets killed by the quotient map $q_{i+j}(D^{-j}) = 0$ since $j \le i+j-1$. Moreover, ϑ only depends on the values of $q_i(x)(\xi_x)$ and $q_j(x)(\eta_x)$. To see this, let $\xi = \xi^{-i} + \xi'$ with $\xi' \in \mathcal{D}^{-i+1}$ and $\xi^{-i} \in \mathcal{D}^{-i}$. Then $[\xi,\eta] = [\xi^{-i},\eta] + [\xi',\eta]$ where $[\xi',\eta] \in \mathcal{D}^{-i-j+1}$ so that $q_{i+j}([\xi',\eta]) = 0$. This lets us define a Lie bracket on the graded tangent space in the following way: For any $v \in gr(T_xM)^{-i}$ there is a representative $\xi \in \mathcal{D}^{-i}$ with $q_i(x)(\xi) = v$. Define

$$\begin{aligned} \mathcal{L}_x^{i,j} : gr(T_xM)^{-i} \times gr(T_xM)^{-j} &\longrightarrow gr(T_xM)^{-(i+j)} \\ (v,w) &\longmapsto \vartheta(\xi,\eta), \end{aligned}$$

where ξ and η are representatives of v and w. We know by the examination above that this map is well-defined since $\vartheta_{i,j}$ only depends on the value the representatives chosen in the relevant quotient space. Adding these together for different values of i and j gives a Lie bracket.

Definition 3.3.1. Let $v = (v_{-k}, \dots, v_{-1}) \in gr(T_xM)$ and $w = (w_{-k}, \dots, w_{-1}) \in gr(T_xM)$. Define the Levi bracket by

$$\begin{aligned} \mathcal{L}_x : gr(T_xM) \times gr(T_xM) &\longrightarrow gr(T_xM) \\ (v,w) &\longmapsto \sum_{i,j} \mathcal{L}_x^{i,j}(v_{-i}, w_{-j}). \end{aligned}$$

♠

The Levi bracket makes the graded tangent space at a point $x \in M$ into a Lie algebra. This algebra is called the symbol algebra of M at x.

Definition 3.3.2. Let (M, D, S) be a sub-Riemannian manifold with a metric on the distribution. We say that (M, D, S) has constant sub-Riemannian symbol if D is bracket generating and there is a startified Lie algebra

$$\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1}$$

with a fixed metric $\langle .,. \rangle_{\mathfrak{g}_{-1}}$ on $\mathfrak{g}_{-1}$ such that for any $x \in M$ there is a linear isomorphism

$$f : \mathfrak{g} \longrightarrow gr(T_xM)$$

with the properties that

(i) for any $X, Y \in \mathfrak{g}$, f preserves the bracket in the sense $f([X,Y]) = [f(X), f(Y)]$ where the later one is the Levi bracket on the graded tangent space, and

(ii) for any $X, Y \in \mathfrak{g}_{-1}$, f preserves the metric of $\mathfrak{g}_{-1}$ onto the distribution in the sense that $\langle X, Y \rangle_{\mathfrak{g}_{-1}} = S(f(X), f(Y))$.

We may also say that such a manifold has a constant sub-Riemannian symbol $(\mathfrak{g}_-, \langle .,. \rangle_{\mathfrak{g}_{-1}})$, clarifying the fixed Lie algebra and the metric. ♠

A sub-Riemannian manifold with constant sub-Riemannian symbol must also have an equiregular distribution, see Definition 3.1.3. This is clear since if the distribution is not equiregular, then there exist two points $x_0, x_1 \in M$ such that the dimension of $D^{-i}_{x_0}$ is different from the dimension of $D^{-i}_{x_1}$ for some i. By letting i be the lowest integer that gives different dimensions, we see that the dimensions of $(T_{x_0}M)^{-i}$ and $(T_{x_1}M)^{-i}$ must be different. This means that $gr(T_{x_0}M)$ and $gr(T_{x_1}M)$ can not be isomorphic to the same stratified Lie algebra, hence the manifold can not have constant sub-Riemannian symbol.

Notice that a Lie group G realizing the stratified Lie algebra $\mathfrak{g}$ in Definition 3.3.2 will be a Carnot group as in Definition 3.2.6. Moreover, this Carnot group G will have the same dimension as the sub-Riemannian manifold M and the induced sub-Riemannian structure on G will mirror the one in M in the sense that the induced distribution on G will have the same dimension as D and generate G in the same amount of steps as D generate M. This suggest that the Carnot group G is a naturally fit space to use as a comparison for M.

Definition 3.3.3. Let $\mathfrak{g}_-$ be such a stratified Lie algebra with a metric $\langle .,. \rangle_{\mathfrak{g}_{-1}}$ on $\mathfrak{g}_{-1}$. Define G_0 to be the Lie group of all automorphisms on $\mathfrak{g}_-$ preserving the grading and preserving the metric on $\mathfrak{g}_{-1}$, that is, G_0 is the Lie group of elements g such that

(i) $g : \mathfrak{g}_- \to \mathfrak{g}_-$ is a Lie algebra automorphism, which means that it is a linear automorphism of a vector space that preserves the bracket: $g([X,Y]) = [g(X), g(Y)]$ for all $X, Y \in \mathfrak{g}_-$,

(ii) if $X \in \mathfrak{g}_{-i}$ then $g(X) \in \mathfrak{g}_{-i}$ for $i = 1, \dots, k$,

(iii) if $X, Y \in \mathfrak{g}_{-1}$, then $\langle g(X), g(Y) \rangle_{\mathfrak{g}_{-1}} = \langle X, Y \rangle_{\mathfrak{g}_{-1}}$.

We define $\mathfrak{g}_0$ to be the Lie algebra of the Lie group G_0, i.e $\mathfrak{g}_0 = T_e G_0$ where $e \in G_0$ is the identity element. ♠

It is worth noting that these properties means that G_0 is determined by how it acts on $\mathfrak{g}_{-1}$. This follows from the fact that $\mathfrak{g}_-$ is stratified, so any element $Y \in \mathfrak{g}_-$ can be written as $[X_1, [\dots, [X_{s-1}, X_s]\dots]$, hence $g(Y)$ is determined by $g(X_i)$ for all $g \in G_0$ by property (i). Since G_0 preserves the metric on $\mathfrak{g}_{-1}$, we have that $G_0 \subset O(\mathfrak{g}_{-1}$. By the properties of the exponential map we know that any element $\phi \in \mathfrak{g}_0$ has $\phi = \frac{d}{dt} e^{t\phi}|_{t=0}$ with $e^{t\phi} \in G_0$. If we let X and Y be elements of $\mathfrak{g}_-$, we get

$$\begin{aligned}
\phi([X,Y]) &= \frac{d}{dt} e^{t\phi}([X,Y])|_{t=0} \\
&= \frac{d}{dt} [e^{t\phi}(X), e^{t\phi}(Y)]|_{t=0} \\
&= \frac{d}{dt} e^{t\phi}(X) e^{t\phi}(Y) - e^{t\phi}(Y) e^{t\phi}(X)|_{t=0} \\
&= \phi(X) e^{t\phi}(Y)|_{t=0} + e^{t\phi}(X)|_{t=0} \phi(Y) - (\phi(Y) e^{t\phi}(X)|_{t=0} + e^{t\phi}(Y)|_{t=0} \phi(X)) \\
&= (\phi(X) Y - Y \phi(X)) + (X \phi(Y) - \phi(Y) X) \\
&= [\phi(X), Y] + [X, \phi(Y)].
\end{aligned}$$

Hence the elements of $\mathfrak{g}_0$ works as a derivation on $\mathfrak{g}_-$. This lets us define a new Lie algebra

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$$

where the bracket is defined as $[\phi, X] = \phi(X)$ where $\phi \in \mathfrak{g}_0$ and $X \in \mathfrak{g}_-$. Since the elements of $\mathfrak{g}_0$ are derivations on $\mathfrak{g}_-$, the Jacobi identity is satisfied naturally

$$\begin{aligned}[][\phi, [X, Y]] + [X, [Y, \phi]] + [Y, [\phi, X]] &= \phi([X, Y]) + [X, -\phi(Y)] + [Y, \phi(X)] \\ &= [\phi(X), Y] + [X, \phi(Y)] - [X, \phi(Y)] - [\phi(X), Y] \\ &= 0.\end{aligned}$$

If $\phi, \theta \in \mathfrak{g}_0$, let $[\phi, \theta] \in \mathfrak{g}_0$ be such that the Jacobi identity is satisfied, i.e. for $X \in \mathfrak{g}_-$, we have

$$[\phi, [\theta, X]] + [\theta, [X, \phi]] + [X, [\phi, \theta]] = \phi(\theta(X)) - \theta(\phi(X)) - [\phi, \theta](X) = 0$$

such that

$$[\phi, \theta](X) = \phi(\theta(X)) - \theta(\phi(X))$$

or in other notation

$$[\phi, \theta] = \phi \circ \theta - \theta \circ \phi$$

just like we would expect from the bracket.

Since the action of G_0 on $\mathfrak{g}_-$ preserves the grading, the action of $\mathfrak{g}_0$ on $\mathfrak{g}_-$ also preserves grading, so that $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ is a graded Lie algebra. Now we have a Klein pair $(\mathfrak{g}, \mathfrak{g}_0)$, and clearly this Klein pair is effective since if $\phi \in \mathfrak{g}_0$ and $X \in \mathfrak{g}_{-j}$, then $[\phi, X] \in \mathfrak{g}_{-j}$ as well, so the kernel of the Klein pair is the only ideal contained in $\mathfrak{g}_0$, i.e. the trivial ideal $\mathfrak{k} = \{0\}$. Since we also have the Lie group G_0 such that $\mathfrak{g}_0$ is the Lie algebra of G_0 and we have the adjoint representation of G_0 acting on the whole Lie algebra $\mathfrak{g}$. This means we have a model geometry for a Cartan geometry.

3.4 Frame Bundles

Let $E \to M$ be a smooth vector bundle of rank k. A frame at a point $x \in M$ is an ordered basis for the vector space E_x, and can be evaluated as a linear isomorphism

$$f : \mathbb{R}^k \longrightarrow E_x.$$

Let F_x denote the space of all frames at x. The general linear group $GL(k, \mathbb{R})$ acts on F_x from the right by composition to give a new frame

$$f \circ g : \mathbb{R}^k \longrightarrow E_x, \quad f \in F_x,\, g \in GL(k, \mathbb{R}).$$

This action is simple transitive on F_x. Now, let F^E denote the disjoint union of all the F_x:

$$F^E := \coprod_{x \in M} F_x.$$

The set F^E clearly has a projection

$$\begin{aligned}\pi_F : F^E &\longrightarrow M \\ (f, x) &\longmapsto x.\end{aligned}$$

If (U_i, ϕ_i) is a local trivialization of E,

$$\begin{tikzcd} \pi_E^{-1}(U_i) \arrow[r, "\phi_i"] \arrow[dr, "\pi_E"'] & U_i \times \mathbb{R}^k \arrow[d, "pr"] \\ & U_i, \end{tikzcd}$$

then we have linear isomorphisms

$$\phi_i|_x : E_x \longrightarrow \mathbb{R}^k$$

for all $x \in U_i$. We will use this to construct a local trivialization of F^E:

$$\begin{aligned} \psi_i : \pi_F^{-1}(U_i) &\longrightarrow U_i \times GL(k, \mathbb{R}) \\ (x, f) &\longmapsto (x, \phi_i|_x \circ f), \end{aligned}$$

which makes $\xi = (F^E, M, \pi_F, GL(k, \mathbb{R}))$ into a smooth fiber bundle. $GL(k, \mathbb{R})$ acts simply transitively on F^E on the right by $(x, f)g = (x, f \circ g)$, giving $(\xi, GL(k, \mathbb{R}))$ the structure of a principal $GL(k, \mathbb{R})$-bundle. Any such principal bundle arising from the set of frames on a vector bundle is called a frame bundle.

Definition 3.4.1. A frame bundle of a vector bundle E is the fiber bundle $\xi = (F^E, M, \pi, GL(k, \mathbb{R}))$ with fibers F_x.

♠

In the case where M is a k-dimensional Riemannian manifold, we can look at the frame bundle of the tangent bundle consisting of only orthogonal frames. In this case, the elements of F_x will be isometries

$$f : \mathbb{R}^k \longrightarrow T_xM,$$

where $\mathbb{R}^k$ is equipped with the standard Euclidean metric. The map

$$\begin{aligned} \phi : T_xM &\longrightarrow \mathbb{R}^k \\ u_i &\longmapsto e_i \end{aligned}$$

where $\{u_1, \ldots, u_k\}$ is an orthonormal basis of T_xM and $\{e_1, \ldots, e_k\}$ is the standard orthonormal basis of $\mathbb{R}^k$ is an isomorphism. Since the tangent space T_xM is isomorphic to $\mathbb{R}^k$, we know that the group of isometries $g : \mathbb{R}^k \to \mathbb{R}^k$ preserving the origin is the orthogonal group

$$O(k) = \{X \in GL(k, \mathbb{R}) \,|\, X^T X = XX^T = I\}$$

where X^T is the transpose matrix of X. This will be a principal $O(k)$-bundle by a construction similar to the one above, except the fibers will now be the Lie group $O(k)$. The fiber bundle arising this way is called an orthogonal frame bundle or a Riemannian frame bundle. It is worth noting that the orthogonal frame bundle is a subbundle of the frame bundle since $O(k) \subset GL(k, \mathbb{R})$; the orthogonal frame bundle is the subbundle of the frame bundle consisting only of frames preserving the metric.

If we have an orientation on M we might also want to consider the frame bundle of positively oriented orthonormal frames. In this case the fibers will consist of the isometries that preserve orientation and will be isomorphic to the special orthogonal group

$$SO(k) = \{X \in GL(k, \mathbb{R}) \,|\, X^T X = XX^T = I, \, \det(X) = 1\}.$$

This will be a principal $SO(k)$-bundle by the same construction. This fiber bundle is called the oriented orthogonal frame bundle or the oriented Riemannian frame bundle. The oriented frame bundle is a subset of the orthogonal frame bundle since $SO(n) \subset O(n)$, and it is the fiber bundle consisting of frames preserving the metric and the orientation.

Example 3.4.2. Let S^n be the sphere as a Riemannian manifold. Then we have $T_xS^n \cong \mathbb{R}^n$ for $x \in S^n$, and we can let $f : \mathbb{R}^n \to T_xS^n$ be a frame sending the standard orthonormal basis of $\mathbb{R}^n$ to an orthonormal basis of T_xS^n. As we have seen above, this means that any isometric frame on T_xS^n can be written as $f \circ g$ for some $g \in O(n)$. Since this is valid for all $x \in S^n$, we get a frame bundle over S^n with structure group $O(n)$ acting from the right. From example 2.3.3 we have see that the orthogonal frame bundle of S^n is in fact the principal bundle $O(n) \to O(n+1) \to S^n$.

From the same example we see immediately that the orthogonal frame bundle of $\mathbb{R}^n$ is the principal bundle $O(n) \to E(n) \to \mathbb{R}^n$. ♣

When we have a (smooth) manifold we will always have a frame bundle built on the tangent bundle, and we have seen above that if we have additional structure on the manifolds that could be preserved by the frames, we can restrict the frame bundle to the subbundle consisting only of frames preserving the structure. Let (M, D, S) be a bracket generating sub-Riemannian manifold with constant sub-Riemannian symbol as in Definition 3.3.2. In section 3.3 we have seen how the bracket generating distribution D induces a grading on the tangent space. The graded tangent space is a vector bundle over the manifold $gr(TM) \to M$, and we might evaluate the frames from the stratified Lie algebra $\mathfrak{g}_-$ associated with the manifold to the graded tangent space $gr(T_xM)$. Moreover we might consider only frames that preserve the grading, that would be all frames

$$f : \mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1} \longrightarrow gr(T_xM)$$

such that $f(\mathfrak{g}_{-i}) = (T_xM)^{-i}$ and $f([X,Y]) = [f(X), f(Y)]$ for all $i = 1, \ldots, k$ and $X, Y \in \mathfrak{g}_-$. We call these frames graded frames. In this case the group $\mathrm{Aut}_{gr}(\mathfrak{g}_-)$ of graded automorphisms on $\mathfrak{g}_-$ will act from the right on the set of graded frames to give a principal bundle. Recall that the stratified Lie algebra $\mathfrak{g}_-$ associated with a sub-Riemannian manifold with constant sub-Riemannian symbol is equipped with a metric $\langle .,. \rangle_{\mathfrak{g}_{-1}}$ on $\mathfrak{g}_{-1}$. This means that we might consider only frames that preserves the metric in the sense that $\langle X, Y \rangle_{\mathfrak{g}_{-1}} = S(f(X), f(Y))$ for all $X, Y \in \mathfrak{g}_{-1}$. The group acting on the Lie algebra satisfying all these properties is exactly the group $G_0 \subset \mathrm{Aut}_{gr}(\mathfrak{g}_-)$ from Definition 3.3.3.

Definition 3.4.3. Let (M, D, S) be a sub-Riemannian manifold with constant sub-Riemannian symbol. Define the principal G_0-bundle of M to be the principal bundle $(\mathcal{G}, M, \pi, G_0)$ where the elements $f \in \mathcal{G}_x$ are the graded frames $f : \mathfrak{g}_- \to gr(T_xM)$ with the following properties:

(i) f is a Lie algebra isomorphism; that is it preserves the bracket: $f([X,Y]) = [f(X), f(Y)]$ for all $X, Y \in \mathfrak{g}_-$,

(ii) if $X \in \mathfrak{g}_{-i}$ then $f(X) \in (T_xM)^{-i}$ for $i = 1, \ldots, k$,

(iii) if $X, Y \in \mathfrak{g}_{-1}$, then $\langle X, Y \rangle_{\mathfrak{g}_{-1}} = S(f(X), f(Y))$.

♠

Notice how the principal G_0-bundle of a sub-Riemannian manifold is canonically constructed, and it comes with a naturally associated Klein pair $(\mathfrak{g}, \mathfrak{g}_0)$, where $\mathfrak{g}_0$ is the Lie algebra of G_0 and $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0$. This means that if we can find a Cartan connection $\omega : T\mathcal{G} \to \mathfrak{g}$ we would have a Cartan geometry as in Definition 2.4.6.

Definition 3.4.4. Let $G \to \mathcal{F} \xrightarrow{\pi} M$ be a frame bundle of the tangent bundle TM, let $x \in M$ and let $f \in \mathcal{F}_x$ such that $f : \mathbb{R}^n \longrightarrow T_xM$ is a linear isomorphism. The soldering 1-form θ is a $\mathbb{R}^n$ valued 1-form on $T\mathcal{F}$ defined by

$$\theta_f(v) = f^{-1}d\pi(v)$$

where $v \in T_f\mathcal{F}$ and $f^{-1} : T_xM \to \mathbb{R}^n$ is the inverse of f. The map $d\pi : T\mathcal{F} \to TM$ is the differential of the projection π. ♠

The soldering 1-form will be our tool to construct a Cartan connection on the principal G_0-bundle of a sub-Riemannian manifold with constant sub-Riemannian symbol.

Lemma 3.4.5. *The soldering 1-form θ of a frame bundle is zero on the vertical space $\mathcal{V} = ker(d\pi)$ and it satisfies*

$$R_g^*\theta(v) = Ad_{g^{-1}}\theta(v).$$

where $g \in G$ is an element in the group acting on the fibers of the principal bundle induced by the frame bundle.

Proof. Let $v \in T_f\mathcal{F}$. By definition we have $\theta(v) = f^{-1}(d\pi(v))$, so if $v \in \mathcal{V}_f$ we get

$$\theta(v) = f^{-1}(d\pi(v)) = f^{-1}(0) = 0$$

since $\mathcal{V}$ is exactly the kernel of $d\pi$. We get

$$\begin{aligned} R_g^*\theta(v) &= \theta(R_{g*}v) \\ &= (fg)^{-1}(d\pi(v)) \\ &= g^{-1} \circ f^{-1}(d\pi(v)) \\ &= g^{-1}(\theta(v)) \\ &= Ad_{g^{-1}}\theta(v). \end{aligned}$$

□

Corollary 3.4.6. *Let θ be the soldering 1-form of a frame bundle $H \to \mathcal{F} \to M$ and let ω be a principal connection of the same frame bundle. Then $\varpi = \theta + \omega$ satisfy all the properties of a Cartan connection as in Proposition 2.4.5.*

Proof. (i) We need to show that $\varpi : T_f\mathcal{F} \to \mathfrak{g}$ is a linear isomorphism where $\mathfrak{g} = \mathbb{R}^n \oplus \mathfrak{h}$. But we know that by defining $\mathcal{V} = ker(d\pi)$ and $\mathcal{H} = ker(\omega)$ we get $T_f\mathcal{F} = \mathcal{H}_f \oplus \mathcal{V}_f$ and compute

$$\varpi(v) = \varpi(v_{\mathcal{H}} + v_{\mathcal{V}}) = \theta(v_{\mathcal{H}}) + \omega(v_{\mathcal{V}}).$$

Since $\theta|_{\mathcal{H}} : \mathcal{H} \to \mathbb{R}^n$ and $\omega|_{\mathcal{V}} : \mathcal{V} \to \mathfrak{h}$ are isomorphisms, we conclude that ϖ is a linear isomorphism.

(ii) Need to show that $R_g^*\varpi = Ad_{g^{-1}}\varpi$, but since both θ and ω have this property, it follows directly from linearity.

(iii) We need to show that $\varpi(X^\sharp) = X$ when $X \in \mathfrak{h}$, but since $X^\sharp \in \mathcal{V}$ we get

$$\varpi(X^\sharp) = \theta(X^\sharp) + \omega(X^\sharp) = 0 + X.$$

□

An analogue of the soldering 1-form is the graded soldering 1-form associated with the graded frame bundle of a sub-Riemannian manifold with constant sub-Riemannian symbol. Let $\mathcal{G} \to M$ be the graded frame bundle of such a manifold. We have $f : \mathfrak{g}_- \to gr(T_xM)$, and can define

$$\theta(v) = f^{-1}gr(d\pi(v))$$

for $v \in T_f\mathcal{G}$, where $gr : T_xM \to gr(T_xM)$ maps the tangent space of the manifold into the graded tangent space of the manifold. The problem is that this is not canonical, or more precicely; the map gr is only canonical when restricted to the distribution $gr|_D : D \to (T_xM)^{-1}$. For the next step, we need to have a projection $pr_{-2} : D^{-2} \to D$ so that we can write $D^{-2} = ker(pr_{-2}) \oplus D$, but this projection needs to be chosen. Choosing projections for each D^{-i} in the sense $pr_{-i} : D^{-i} \to D^{-i+1}$ gives $T_xM = ker(pr_{-k}) \oplus \ldots \oplus ker(pr_{-2}) \oplus D$, and then we can define the grading function gr by letting $gr(ker(pr_{-i})) = (T_xM)^{-i}$. With this grading function defined, all the results above for the soldering 1-form will also apply to the graded soldering 1-form as well. This is easily seen, since the grading function is an isomorphism, and the graded frames are just the subset of frames that preserve the additional structure induced by this isomorphim, so all the proofs will be exactly the same.

At this point we are able to clearly formulate a strategy to reach our goal. For any given sub-Riemannian manifold with constant sub-Riemannian symbol our goal is to find a canonical Cartan geometry that will provide an expression for the Cartan curvature on the given manifold. We have seen how we can associate a principal G_0-bundle with our manifold, and this construction is canonical. In addition we have an associated graded soldering 1-form taking values in the stratified Lie algebra $\mathfrak{g}_-$. By Corollary 3.4.6 we know that the soldering 1-form can create a connection satisfying the properties of a Cartan connection when added with a principal connection. When we have such a connection we have the full structure of a Cartan geometry. The part that is non-cannonical at this point is the choice of principal connection $\omega : T\mathcal{G} \to \mathfrak{h}$ and the choice of a decomposition of the tangent space to determine the graded soldering 1-form $\theta(v) = f^{-1}gr(d\pi(v))$, $v \in T_f\mathcal{G}$. this method for determining the Cartan connection is the same as the one described in [AMS19].

3.5 The Normalizing Condition; Extension of Metric and the Exterior Differential

As we saw in the last section, we have ways to construct Cartan geometries on a fixed sub-Riemannian manifold with constant sub-Riemannian symbol, but the construction is not canonical. The plan going forward is to state a normalizing condition on the curvature that will restrict our freedom when constructing a Cartan geometry. If we have a normalizing condition that yields a unique Cartan geometry by the construction above, the construction will be canonical. Let M be a sub-Riemannian manifold with constant sub-Riemannian symbol, and consider all the possible Cartan geometries that could be constructed on M by using the principal G_0-bundle and choosing a Cartan connection based on the graded soldering 1-form. The goal of this section is to state a normalizing condition on the Cartan curvature that will be satisfied by exactly one of the Cartan geometries considered. We will let the curvature arising from this unique Cartan geometry be the canonically defined Cartan curvature of a sub-Riemannian manifold with constant sub-Riemannian symbol.

Recall from Section 2.5 that a Cartan connection ω gives an expression for the Cartan curvature $K = d\omega + \frac{1}{2}[\omega, \omega]$. We also get a curvature function

$$\kappa : \mathcal{G} \longrightarrow \bigwedge^2 \mathfrak{g}_-^* \oplus \mathfrak{g}$$

as in Definition 2.5.3, where $\mathcal{G} \to M$ is the principal bundle of the Cartan geometry. The normalizing condition will be imposed on the Cartan curvature function κ and it will use the metric and exterior derivative of the tensor space $\wedge^2\mathfrak{g}_-^* \oplus \mathfrak{g}$. Since we a priori have a metric only on $\mathfrak{g}_{-1}$, we need to extend this metric to $\mathfrak{g}$. We will do this similar to [Ča17, Section 3.4, p. 28].

Proposition 3.5.1. *Let $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}$ be a stratified Lie algebra with a metric $\langle ., . \rangle_{\mathfrak{g}_{-1}}$ on $\mathfrak{g}_{-1}$. There is a canonical way to extend this metric to a metric $\langle ., . \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$.*

Proof. Since $\mathfrak{g}_-$ is stratified we know that $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-2}$. Let $x, y \in \mathfrak{g}_{-1}$. Then $[x, y] \in \mathfrak{g}_{-2}$. Define a linear map

$$\begin{aligned} \phi : \bigwedge^2 \mathfrak{g}_{-1} &\longrightarrow \mathfrak{g}_{-2} \\ x \wedge y &\longmapsto [x, y]. \end{aligned}$$

On the space $\bigwedge^2 \mathfrak{g}_{-1}$ we have a metric by extending to the tensor product:

$$\begin{aligned} \langle x_1 \wedge y_1, x_2 \wedge y_2 \rangle_{\bigwedge^2 \mathfrak{g}_{-1}} &= \det \begin{pmatrix} \langle x_1, x_2 \rangle_{\mathfrak{g}_{-1}} & \langle x_1, y_2 \rangle_{\mathfrak{g}_{-1}} \\ \langle y_1, x_2 \rangle_{\mathfrak{g}_{-1}} & \langle y_1, y_2 \rangle_{\mathfrak{g}_{-1}} \end{pmatrix} \\ &= (\langle x_1, x_2 \rangle_{\mathfrak{g}_{-1}} \langle y_1, y_2 \rangle_{\mathfrak{g}_{-1}} - \langle x_1, y_2 \rangle_{\mathfrak{g}_{-1}} \langle y_1, x_2 \rangle_{\mathfrak{g}_{-1}}) \end{aligned}$$

Since ϕ is a surjective linear map, we can rewrite

$$\bigwedge^2 \mathfrak{g}_{-1} = \ker(\phi) \oplus \mathrm{im}(\phi) = \ker(\phi) \oplus \mathfrak{g}_{-2}.$$

Since we have a metric $\langle ., . \rangle_{\bigwedge^2 \mathfrak{g}_{-1}}$ on $\bigwedge^2 \mathfrak{g}_{-1}$, using the decomposition above we get a new metric $\langle ., . \rangle_{\mathfrak{g}_{-2}} = \langle ., . \rangle_{\bigwedge^2 \mathfrak{g}_{-1}}|_{\mathfrak{g}_{-2}}$. this gives us a metric on $\mathfrak{g}_{-2}$. Again, since $\mathfrak{g}_-$ is stratified we know that $[\mathfrak{g}_{-1}, \mathfrak{g}_{-l}] = \mathfrak{g}_{-(l+1)}$, and if we have a metric $\langle ., . \rangle_{\mathfrak{g}_{-l}}$, we can use a construction similar to the one above, and we get a metric $\langle ., . \rangle_{\mathfrak{g}_{-(l+1)}}$ on $\mathfrak{g}_{-(l+1)}$. By induction we have a metric $\langle ., . \rangle_{\mathfrak{g}_{-i}}$ on $\mathfrak{g}_{-i}$ for all $i = 1, 2, ..., k$. This gives us a metric $\langle ., . \rangle_{\mathfrak{g}_-}$ on $\mathfrak{g}_-$ since we have a metric on each of the components:

$$\langle u, v \rangle_{\mathfrak{g}_-} = \langle u_{-1}, v_{-1} \rangle_{\mathfrak{g}_{-1}} + \ldots + \langle u_{-k}, v_{-k} \rangle_{\mathfrak{g}_{-k}}$$

where $u = u_{-1} + \ldots + u_{-k} \in \mathfrak{g}_-$ with $u_{-i} \in \mathfrak{g}_{-i}$, and $v = v_{-1} + \ldots + v_{-k} \in \mathfrak{g}_{-k}$ with $v_{-i} \in \mathfrak{g}_{-i}$. $\square$

Since $G_0 \subset O(\mathfrak{g}_{-1}$ we know that $\mathfrak{g}_0 \subset \mathfrak{so}(\mathfrak{g}_-)$. This allows us to define a positive definite inner product on the Lie algebra $\mathfrak{g}_0$ by using the negative of the Killing form

$$\langle A, B\rangle_{\mathfrak{g}_0} = -\frac{1}{2}\mathrm{Tr}(AB)$$

for any $A, B \in \mathfrak{g}_0$.

We have already seen how we can extend the metric to a wedge product space. In general we can extend to any tensor product space by

$$\langle u_1 \otimes v_1, u_2 \otimes v_2\rangle_{\mathfrak{g}\otimes\mathfrak{h}} = \langle u_1, u_2\rangle_{\mathfrak{g}}\langle v_1, v_2\rangle_{\mathfrak{h}}.$$

If $\{X_i\}_{i=1}^n$ is an orthonorlmal basis for $\mathfrak{g}$, then $\{X_i \otimes X_j\}_{i,j=1}^n$ is an orthonormal basis of $\mathfrak{g}\otimes\mathfrak{g}$

$$\begin{aligned}\langle X_i \otimes X_j, X_k \otimes X_l\rangle_{\mathfrak{g}\otimes\mathfrak{g}} &= \langle X_i, X_k\rangle_{\mathfrak{g}}\langle X_j, X_l\rangle_{\mathfrak{g}} \\ &= \delta_{ik}\delta_{jl}\end{aligned}$$

where $\delta_{ik} = 1$ if $i = k$ and zero otherwise. We also get an orthonormal basis if we extend to the wedge product as in the proof of Proposition 3.5.1. The basis of $\bigwedge^2 \mathfrak{g}$ is given by $\{X_i \wedge X_j\}_{0\le i<j\le n}$, and we have

$$\begin{aligned}\langle X_i \wedge X_j, X_k \wedge X_l\rangle_{\bigwedge^2 \mathfrak{g}} &= \langle X_i, X_k\rangle_{\mathfrak{g}}\langle X_j, X_l\rangle_{\mathfrak{g}} - \langle X_i, X_l\rangle_{\mathfrak{g}}\langle X_j, X_k\rangle_{\mathfrak{g}} \\ &= \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \\ &= \delta_{ik}\delta_{jl}\end{aligned}$$

which means that the basis is orthonormal. The last step follows since $i < j$ and $k < l$, hence either δ_{il} or δ_{jk} must be zero.

We also need to address how to get a metric on the dual space. we can construct a basis of the dual space by using the metric such that if X_i^* is a basis element of $\mathfrak{g}^*$ dual to $X_i \in \mathfrak{g}$, we have

$$X_i^*(u) = \langle X_i, u\rangle_{\mathfrak{g}}.$$

Then we extend the metric to $\mathfrak{g}^*$ by putting

$$\langle X_i^*, X_j^*\rangle_{\mathfrak{g}^*} = \langle X_i, X_j\rangle_{\mathfrak{g}}$$

where $\{X_i\}_{i=1}^n$ is a basis of $\mathfrak{g}$.

In the case of an orthonormal basis $\{X_i\}_{i=1}^n$ of $\mathfrak{g}$, we see that this particular choice of dual basis correspond to the classical way of choosing dual basis by putting

$$X_i^*(X_j) = \delta_{ij},$$

and the resulting dual basis $\{X_i^*\}_{i=1}^n$ is then also orthonormal.

Note that for a Lie group $\mathfrak{g}$ with a metric and an orthonormal basis $\{X_i\}_{i=1}^n$, the spaces $\bigwedge^2 \mathfrak{g}_-^*$ and $(\bigwedge^2 \mathfrak{g}_-)^*$ are formally different, with bases $X_i^* \wedge X_j^*$ and $(X_i \wedge X_j)^*$ respectively. Even if formally different, they are isometric, so later we might consider $X_i^* \wedge X_j^*$ as the basis element dual to $X_i \wedge X_j$.

Definition 3.5.2. Let δ be the Lie algebra cohomology differential of a graded Lie algebra $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0$ be defined by

$$
\begin{aligned}
\delta_k : \bigwedge^k \mathfrak{g}_-^* \otimes \mathfrak{g} &\longrightarrow \bigwedge^{k+1} \mathfrak{g}_-^* \otimes \mathfrak{g} \\
\delta_k(\alpha \otimes Z)(X_0, \dots, X_k) &= \sum_{i=0}^{k} (-1)^i \alpha(X_0, \dots, \hat{X}_i, \dots, X_k)[X_i, Z] \\
&\quad + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k) Z
\end{aligned}
$$

where $X_i \in \mathfrak{g}_-$ for all $i = 0, 1, \dots, k$, $Z \in \mathfrak{g}$ and $\alpha \in \wedge^k \mathfrak{g}_-^*$. Here $\hat{X}_i$ means that X_i is removed from the equation.

♠

We will often omitt the k in the notation and just write $\delta = \delta_k$ if it is clear from context what value the k must have. Recall that $\bigwedge^k \mathfrak{g}_-^* \otimes \mathfrak{g} = \hom(\bigwedge^k \mathfrak{g}_-, \mathfrak{g})$, so we can also think of the operator $\delta = \delta_1$ as

$$
\begin{aligned}
\delta : \hom(\mathfrak{g}_-, \mathfrak{g}) &\longrightarrow \hom(\bigwedge^2 \mathfrak{g}_-, \mathfrak{g}) \\
\delta(\alpha, Z)(X_0, X_1) &= [X_0, \alpha(X_1)Z] - [X_1, \alpha(X_0)Z] - \alpha([X_0, X_1])Z
\end{aligned}
$$

Notice that δ is a linear operator, hence we might describe it as a matrix.

Let (M, D, S) be a sub-Riemannian manifold with constant sub-Riemannian symbol and let let $T_xM \cong \mathfrak{g}_-$ for all $x \in M$ with a metric $\langle ., . \rangle_{\mathfrak{g}_{-1}}$ on $\mathfrak{g}_{-1}$. Let $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0$ where $\mathfrak{g}_0 = T_eG_0$ as described in definition 3.3.3. We have showed that the metric $\langle ., . \rangle_{\mathfrak{g}_{-1}}$ can be extended to a metric $\langle ., . \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$.

If we have a metric on $\mathfrak{g}$, we also have a metric on any tensor space built by $\mathfrak{g}$ and $\mathfrak{g}_-$, hence we have a metric on $\mathfrak{g}_-^* \otimes \mathfrak{g}$ and one on $\wedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$, lets say η and μ respectively. Using these metrices we can find the codifferential of δ.

Definition 3.5.3. Let δ_k be as in Definition 3.5.2, and let $\langle ., . \rangle_k$ and $\langle ., . \rangle_{k+1}$ be metrics on $\wedge^k \mathfrak{g}_-^* \oplus \mathfrak{g}$ and $\wedge^{k+1} \mathfrak{g}_-^* \oplus \mathfrak{g}$ respectively. We define the Lie algebra cohomology codifferential

$$
\delta_{k+1}^* : \bigwedge^{k+1} \mathfrak{g}_-^* \oplus \mathfrak{g} \longrightarrow \bigwedge^k \mathfrak{g}_-^* \oplus \mathfrak{g}
$$

by the relation

$$
\langle u, \delta_k(v) \rangle_{k+1} = \langle \delta_{k+1}^*(u), v \rangle_k
$$

where $u \in \wedge^{k+1} \mathfrak{g}_-^* \oplus \mathfrak{g}$ and $v \in \wedge^k \mathfrak{g}_-^* \oplus \mathfrak{g}$ ♠

If we have an orthonormal basis of $\wedge^k \mathfrak{g}_-^* \oplus \mathfrak{g}$ and an orthonormal basis of $\wedge^{k+1} \mathfrak{g}_-^* \oplus \mathfrak{g}$, then we can represent δ_k as a matrix relative to these bases. In this case we can find a matrix representation of δ_{k+1}^* immediately by the relation $\delta_{k+1}^* = (\delta_k)^T$.

Let $G_0 \to \mathcal{G} \to M$ be a principal G_0-bundle on a sub-Riemannian manifold with constant sub-Riemannian symbol as in Definition 3.4.3, and let $\varpi = \theta + \omega$ be a Cartan connection on this principal bundle were θ is the graded soldering 1-form and ω is a principal connection. Then we can induce a filtration on the tangent space $T_f\mathcal{G}$ by

$$T_f^{-i}\mathcal{G} = \{\xi \in T_f\mathcal{G} \mid \varpi(\xi) \in \sum_{j=0}^{i} \mathfrak{g}_{-j}\}.$$

The smoothness of ϖ lets us combine these subspaces into smooth subbundles

$$T\mathcal{G} = T^{-k}\mathcal{G} \supset T^{-k+1}\mathcal{G} \supset \ldots \supset T^0\mathcal{G}.$$

We are now ready to give a definition of homogeneous k-forms on a principal bundle.

Definition 3.5.4. Let ϕ be a $\mathfrak{g}$-valued n-form on $\mathcal{G}$ where $\mathfrak{g} = \mathfrak{g}_{-k} \ldots \oplus \mathfrak{g}_0$. We say that ϕ is homogeneous of degree $l \geq 0$ if for any tangent fields $\xi_j \in T^{-i_j}\mathcal{G}$ we always have $\phi(\xi_1, \ldots, \xi_n) \in \bigoplus_{r=0}^{i_1+\cdots+i_n+l} \mathfrak{g}_{-r}$. ♠

The curvature K is a $\mathfrak{g}$-valued 2-form on $\mathcal{G}$, hence we can write it as

$$K = \sum_{p\geq 0} K^p$$

where K^p is homogeneous of degree p. We can use this to get the homogeneous variants of κ_f

$$(\kappa^p(f))(u,v) := \kappa_f^p(u,v) = K^p(d\pi_{H_f}^{-1} \circ f(u), d\pi_{H_f}^{-1} \circ f(v)).$$

It is clear that $\kappa_f = \sum_{p\geq 0} \kappa_f^p$, and $\kappa_f^p \in \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$ for all $p \geq 0$. We can now state an important theorem from which we will get a normalizing condition.

Theorem 3.5.5. *[Mor08, Theorem 1] To each sub-Riemannian manifold (M, D, S) having a constant sub-Riemannian symbol $(\mathfrak{g}_-, \langle .,. \rangle_{\mathfrak{g}_{-1}})$, there is canonically associated Cartan connection ϖ on the principle bundle $G_0 \to \mathcal{G} \to M$ such that the associated curvature function κ_f satisfy $\delta^*\kappa_f^p = 0$.*

By this theorem it is evident that using $\delta^*\kappa_f^p = 0$ as a normalizing condition gives us a canonical expression for the curvature of any sub-Riemannian manifold with constant sub-Riemannian symbol.

At this point we must make it clear that we concern ourselves with a special case of the problem of how to associate a canonical Cartan connection. The more general approach is to use the Tanaka prolongation. This method put forth by Noboru Tanaka [Tan70] is an effective way of asociating canonical frames to a broadet category of manifolds than just the ones with constant sub-Riemannian symbol. However, when we have constant sub-Riemannian symbol we can avoid the issue due to this next theorem by Morimoto.

Lemma 3.5.6. *[Mor08, Prop 1][AMS19] If $\mathfrak{g} = \bigoplus_{i=0}^{k} \mathfrak{g}_{-i}$ is a graded Lie algebra and $\mathfrak{g}_0 \subset \mathfrak{so}(\mathfrak{g}_{-1})$ then the Tanaka prolongation is trivial.*

The proof can be found in either of the two cited articles. For more information about Tanaka prolongation, see for example [Zel09].

Chapter 4

Curvature with the Heisenberg Group as Model Space

We have seen in Theorem 3.5.5 that there is a canonical way of finding the Cartan curvature of a sub-Riemannian manifold with constant sub-Riemannian symbol, using Carnot groups as model spaces. In this chapter we will examine sub-Riemannian manifolds that have the Heisenberg Lie algebra as its symbol. The Heisenberg group is a Lie group that can easily be described as a bracket generating sub-Riemannian manifold. In particular, we will see that the Heisenberg group is a Carnot group (Example 4.1.2). This means that the Heisenberg group is fit to be used as a model space when measuring curvature. In Section 4.1 we will introduce the Heisenberg group and examine sub-Riemannian manifolds that have the Heisenberg Lie algebra as its symbol. In Section 4.2 we will give an expression for the canonically associated Cartan connection of a general sub-Riemannian manifold with the Heisenberg Lie algebra as its sub-Riemannian symbol. The connection is presented in Theorem 4.2.2 and is a new result as far as he author knows. In Section 4.3 we will apply some theory to the Heisenberg Lie algebra before we finally prove Theorem 4.2.2 in Section 4.4.

4.1 The Heisenberg Group

Let $\mathfrak{h}$ be a three dimensional Lie algebra with basis elements $\{X, Y, Z\}$ and with the only non-trivial bracket being $[X, Y] = Z$. This Lie algebra is called the Heisenberg Lie algebra. We can also find the corresponding Lie group by using the exponential map. Let H be the space generated by $\{e^X, e^Y, e^Z\}$. By the Baker-Campbell-Hausdorff formula [Hal15, Theorem 5.3] we can determine a group operation on H. Let $V_1 = a_1X + b_1Y + c_1Z$ and $V_2 = a_2X + b_2Y + c_2Z$, and then let $g_1 = e^{V_1} \in H$ and $g_2 = e^{V_2} \in H$. We get

$$\begin{aligned}
g_1g_2 &= e^{V_1}e^{V_2} \\
&= \exp\left(V_1 + V_2 + \frac{1}{2}[V_1, V_2] + \frac{1}{12}[V_1, [V_1, V_2]] - \frac{1}{12}[V_2, [V_1, V_2]] + \cdots\right) \\
&= \exp\left(V_1 + V_2 + \frac{1}{2}[V_1, V_2]\right) \\
&= \exp\left((a_1 + a_2)X + (b_1 + b_2)Y + (c_1 + c_2\frac{1}{2}(a_1b_2 - a_2b_1))Z\right).
\end{aligned}$$

This gives us a group structure on H. We might also write the coordinates of the elements of H as $g_1 = (a_1, b_1, c_1)$ and $g_2 = (a_2, b_2, c_2)$. With these coordinates the group operation would simply look like

$$g_1g_2 = (a_1+a_2, b_1+b_2, c_1+c_2+\frac{1}{2}(a_1b_2-a_2b_1)).$$

These will also be function as a chart giving H the structure of a smooth manifold, hence H is a Lie group called the Heisenberg group.

Example 4.1.1. The Heisenberg group and the Heisenberg Lie algebra could be represented in several different ways. The standard representation is the group of upper triangular 3×3 matrices, i.e.:

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a,b,c \in \mathbb{R} \right\},$$

with Lie algebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mid a,b,c \in \mathbb{R} \right\}.$$

In this case the elements

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

form a basis of $\mathfrak{h}$. It is easily checked that

$$e^X = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e^Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e^Z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

♣

It is worth noticing that in this representation the group operation is going to look different:

$$\begin{pmatrix} 1 & a_1 & c_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 & c_2 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1+a_2 & c_1+c_2+a_1b_2 \\ 0 & 1 & b_1+b_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Even if they look different, the groups defined by these operations are isomorphic, or stated differently, they are just the same group given different coordinates. To see this, let H_1 be the group defined with operation

$$(a_1,b_1,c_1) \cdot_{H_1} (a_2,b_2,c_2) = (a_1+a_2, b_1+b_2, c_1+c_2+\frac{1}{2}(a_1b_2-a_2b_1))$$

and H_2 defined by

$$(\alpha_1,\beta_1,\gamma_1) \cdot_{H_2} (\alpha_2,\beta_2,\gamma_2) = (\alpha_1+\alpha_2, \beta_1+\beta_2, \gamma_1+\gamma_2+\alpha_1\beta_2).$$

Now look at the function

$$\begin{aligned} t: H_2 &\longrightarrow H_1 \\ (\alpha,\beta,\gamma) &\longmapsto (\alpha,\beta,\gamma-\frac{1}{2}\alpha\beta). \end{aligned}$$

Then

$$\begin{aligned}
t((\alpha_1,\beta_1,\gamma_1)\cdot_{H_2}(\alpha_2,\beta_2,\gamma_2)) &= t((\alpha_1+\alpha_2,\beta_1+\beta_2,\gamma_1+\gamma_2+\alpha_1\beta_2))\\
&= (\alpha_1+\alpha_2,\beta_1+\beta_2,\gamma_1+\gamma_2+\alpha_1\beta_2-\frac{1}{2}(\alpha_1+\alpha_2)(\beta_1+\beta_2))\\
&= (\alpha_1+\alpha_2,\beta_1+\beta_2,\gamma_1+\gamma_2+\frac{1}{2}(\alpha_1\beta_2-\alpha_2\beta_1)-\frac{1}{2}(\alpha_1\beta_1+\alpha_2\beta_2))
\end{aligned}$$

while

$$\begin{aligned}
t(\alpha_1,\beta_1,\gamma_1)\cdot_{H_1} t(\alpha_2,\beta_2,\gamma_2) &= (\alpha_1,\beta_1,\gamma_1-\frac{1}{2}\alpha_1\beta_1)\cdot_{H_1}(\alpha_2,\beta_2,\gamma_2-\frac{1}{2}\alpha_2\beta_2)\\
&= (\alpha_1+\alpha_2,\beta_1+\beta_2,\gamma_1+\gamma_2+\frac{1}{2}(\alpha_1\beta_2-\alpha_2\beta_1)-\frac{1}{2}(\alpha_1\beta_1+\alpha_2\beta_2)).
\end{aligned}$$

This shows that it is an isomorphism since it is clearly a bijective group homomorphism. You might also notice that this is also a smooth function from $\mathbb{R}^3$ to $\mathbb{R}^3$, which means we can think of the seemingly different Lie groups H_1 and H_2 as just different coordinates of the same Lie group H, where t is the transition function between the coordinates.

Example 4.1.2. Looking at the Lie algebra $\mathfrak{h}$ of the Heisenberg group we can provide the following grading: Let $\mathfrak{h} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ where

$$\begin{aligned}
\mathfrak{g}_{-2} &= \{tZ \mid t \in \mathbb{R}\}\\
\mathfrak{g}_{-1} &= \{rX + sY \mid r, s \in \mathbb{R}\}.
\end{aligned}$$

Since the only non-trivial bracket relation is $[X,Y] = Z$, this satisfy the definition of a graded Lie algebra, and we can even see that $\mathfrak{g}_{-1}$ generates $\mathfrak{h}$. This means that the Heisenberg Lie algebra is stratifiable as in Definition 3.2.2 and $\mathfrak{h} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is a stratification of $\mathfrak{h}$. If we in addition has a metric $\langle .,. \rangle_{\mathfrak{g}_{-1}}$ on $\mathfrak{g}_{-1}$, we see that the Heisenberg group with this metric is a Carnot group as in Definition 3.2.6. ♣

Lemma 4.1.3. *Let $\mathfrak{g}_-$ be the Heisenberg Lie algebra $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. The structure group G_0 of $\mathfrak{g}_-$ as defined in 3.3.3 is given by*

$$G_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & det(A) \end{pmatrix} \mid A \in O(2) \right\}$$

Proof. We need to show that an element $g \in Aut(\mathfrak{g}_-)$ satisfy the three conditions of Definition 3.3.3 if and only if it can be represented by a matrix of the given form. Let $g \in G_0$ and let $(u,v) \in \mathfrak{g}_-$ such that $(u,v,w) = uX + vY + wZ$. Since $G_0 \subset Aut(\mathfrak{g}_-)$, we know that the element $g : \mathfrak{g}_- \to \mathfrak{g}_-$ must be a linear isomorphism, hence it can be represented as a 3×3 matrix $g = \{m_{i,j}\}$ for $i,j = 1,2,3$. This gives

$$g(u,v,w) = (m_{1,1}u + m_{1,2}v + m_{1,3}w)X + (m_{2,1}u + m_{2,2}v + m_{2,3}w)Y + (m_{3,1}u + m_{3,2}v + m_{3,3}w)Z$$

Recall that g has to preserve the grading. If we let $u, v = 0$, then we see that $m_{1,3}, m_{2,3} = 0$, and similarily, if we let $w = 0$, we see that $m_{3,1}, m_{3,2} = 0$. This means we can write

$$g = \begin{pmatrix} A & 0 \\ 0 & m_{3,3} \end{pmatrix}$$

where A is the 2×2 matrix given by $\{m_{i,j}\}$ for $i, j = 1, 2$. Clearly an automorphism of $\mathfrak{g}_-$ satisfy property (ii) if and only if it can be represented by a matrix on the form above. By property (iii) we need the metric on $\mathfrak{g}_{-1}$ to be preserved. This is true if and only if $A \in O(2)$. This means that $\det(A) = \pm 1$, and in the case $\det(A) = 1$ we get

$$A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

for some real number t. By property (i) we need the bracket to be preserved, which gives

$$\begin{aligned} g(Z) &= g([X, Y]) \\ &= [AX, AY] \\ &= [\cos tX - \sin tY, \sin tX + \cos tY] \\ &= (\cos^2 t + \sin^2 t)Z \\ &= Z. \end{aligned}$$

A similar calculation gives $g(Z) = -Z$ if $\det(A) = -1$. Notice that $g(Z) = m_{3,3}$, so property (i) is satisfied if and only if $m_{3,3} = \det(A)$. $\square$

Now that we have a representation of G_0, we would like to describe its Lie algebra $\mathfrak{g}_0 = T_e G_0$. Notice that $e \in G_0$ is the identity matrix which is part of the connected component where $\det(A) = 1$. This is exactly the same as saying $A \in SO(2) \cong U(1)$ which is a one-dimensional space, and any such matrix is uniquely given by

$$A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

for $t \in \mathbb{R}$. By the definition of $\mathfrak{g}_0$ we know that $\phi \in \mathfrak{g}_0$ has $\phi = \frac{d}{dt} e^{t\phi}|_{t=0}$ with $e^{t\phi} \in G_0$. Look at the matrix

$$\phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Evaluating the exponential of this matrix we get

$$\begin{aligned} e^{t\phi} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{1!}\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} -t^2 & 0 \\ 0 & -t^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 & t^3 \\ -t^3 & 0 \end{pmatrix} + \dots \\ &= \begin{pmatrix} (1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots) & -(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots) \\ (t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots) & (1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots) \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \end{aligned}$$

which is the general form of a matrix in $SO(2)$. We let

$$Q := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then $e^{tQ} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in G_0$, hence $Q \in \mathfrak{g}_0$. Since G_0 is one-dimensional, $\mathfrak{g}_0$ must be one-dimensional as well, hence $\mathfrak{g}_0 = \text{span}\{Q\}$. We want to see how Q work on the other elements of $\mathfrak{g}_-$ by the bracket, and a quick computation gives:

$$\begin{aligned} [Q, X] &= Y \\ [Q, Y] &= -X \\ [Q, Z] &= 0 \end{aligned}$$

4.2 The Canonical Cartan Connection of the Sub-Riemannian Manifolds with the Heisenberg Lie Algebra as Symbol

In this section we will examin sub-Riemannian manifolds that have the Heisenberg Lie algebra as constant symbol. Our goal will be to describe the Cartan curvature of these manifolds depending only on the local structure of the manifold. By Theorem 3.5.5 there is a canonical Cartan connection associated to these sub-Riemannian manifolds.

Example 4.2.1. Let (M, D, S) be a sub-Riemannian manifold with a bracket generating distribution D. Let $D_x = \text{span}\{A_x, B_x\}$ where A, B are two smooth vector fields over $U \subset M$ that would form an orthonormal basis at each point, i.e. $S(A_x, B_x) = 0$ for all $x \in U$. Let $D^{-2} = D \cup [D, D] = TU$, and let $[A, B] = C$. Then we have that $T_xU = \text{span}\{A_x, B_x, C_x\}$ for all $x \in U$. Let the other brackets be given by

$$\begin{aligned} [C, A] &= a_1A + a_2B + a_3C \\ [C, B] &= b_1A + b_2B + b_3C \end{aligned}$$

where $a_i = a_i(x)$ and $b_i = b_i(x)$ for $i = 1, 2, 3$. We want to have a constant sub-Riemannian symbol on this manifold, and we want the graded tangent space to be isomorphic to the Heisenberg Lie algebra. We need an isomorphism

$$\imath : gr(T_xM) \longrightarrow \mathfrak{h} = \mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

such that $\imath$ preserves the grading and for $W_x, V_x \in D_x$ we have $S(W_x, V_x) = \langle \imath(W_x), \imath(W_x) \rangle_{\mathfrak{g}_{-1}}$. This can be done by letting $\imath(A_x) = X$ and $\imath(B_x) = Y$, since A_x and B_x are orthogonal with respect to S, while X, Y are orthogonal with respect to $\langle ., . \rangle_{\mathfrak{g}_{-1}}$. Recall that the graded tangent space is given by

$$gr(T_xM) = D \oplus D^{-2}/D$$

so that the elements of D^{-2}/D are really equivalence classes of elements from $D^{-2} = T_xM$. For $W, V \in D^{-2}$, the equivalence relation is given by

$$W_x \sim_{-2} V_x \quad \Leftrightarrow \quad W_x - V_x = a(x)A + b(x)B$$

for any $a, b \in \mathcal{C}^\infty(U)$, that is; for any $a(x), b(x) \in \mathbb{R}$. We should keep in mind that for $A_x, B_x \in D \subset gr(T_xM)$, we have $[A_x, B_x] = \tilde{C}_x$ where $\tilde{C}_x$ is the equivalence class of $C_x \in D^{-2}$. Here $[., .] = \mathcal{L}_x$ is the Levi bracket defined on the graded tangent space. Since our map is supposed to preserve the grading we must have

$$\imath([A_x, B_x]) = [\imath(A_x), \imath(B_x)] \quad \Rightarrow \quad \imath(\tilde{C}_x) = Z.$$

This also means that $\imath([A_x, \tilde{C_x}]) = [X, Z] = 0$, which gives

$$\imath([A_x, \tilde{C_x}]) = \imath((a_1 A_x + a_2 B_x + a_3 C_x)/\sim_{-3}) = 0.$$

This is an observation coming from how the Levi bracket work. In our example $D_x^{-2} = T_x M$, hence $D_x^{-3} = D_x^{-2} = T_x M$, so that $(T_x M)^{-3} = D_x^{-3}/D_x^{-2} = \{0\}$. We have that $A_x \in D = (T_x M)^{-1}$ and $\tilde{C} \in (T_x M)^{-2}$ we get $[A, \tilde{C}] \in (T_x M)^{-3} = \{0\}$.

♣

For the rest of the section we let (M, D, S) be a sub-Riemannian manifold as in the example above, with constant sub-Riemannian symbol such that the graded tangent space is isomorphic to the Heisenberg Lie algebra. The distribution at each point is spanned by the ortonormal vector fields A and B, and TM is spanned by $\{A, B, [A, B] = C\}$. Let the other brackets be

$$\begin{aligned}
[A, B] &= C \\
[A, C] &= a_1 A + a_2 B + a_3 C \\
[B, C] &= b_1 A + b_2 B + b_3 C.
\end{aligned}$$

This is a general form of a sub-Riemannian manifold with constant sub-Riemannian symbol $(\mathfrak{g}_-, \langle ., . \rangle_{\mathfrak{g}_{-1}})$ where $\mathfrak{g}_-$ is the Heisenberg Lie algebra and $\langle ., . \rangle_{\mathfrak{g}_{-1}}$ is the standard metric on $\mathfrak{g}_{-1}$ for which the basis $\{X, Y\}$ is orthonormal. It is worth noting that the functions a_i and b_i are not independent, that is they are restricted by the Jacobi identity.

Let $G_0 \to \mathcal{G} \xrightarrow{\pi} M$ be the principle G_0 bundle of M and let $T_f\mathcal{G} = \mathrm{span}\{\hat{A}_f, \hat{B}_f, \hat{C}_f, Q^\sharp_f\}$ with $d\pi(\hat{A}_f) = A_{\pi(f)}$, $d\pi(\hat{B}_f) = B_{\pi(f)}$, $d\pi(\hat{C}_f) = C_{\pi(f)}$ and $\ker(d\pi) = \mathrm{span}\{Q^\sharp\}$.

Let $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ be the $\mathfrak{g}_0$-prolongation of the Heisenberg Lie algebra where Q generate $\mathfrak{g}_0$, $\{X, Y\}$ generate $\mathfrak{g}_{-1}$ and $Z = [X, Y]$ generate $\mathfrak{g}_{-2}$.

Theorem 4.2.2. *The canonical Cartan connection $\varpi : T\mathcal{G} \to \mathfrak{g}$ is given by*

$$\begin{aligned}
\varpi_f : T_f\mathcal{G} &\longrightarrow \mathfrak{g} \\
\hat{A} &\longmapsto \cos tX - \sin tY + a_3 Q \\
\hat{B} &\longmapsto \sin tX + \cos tY + b_3 Q \\
\hat{C} &\longmapsto (a_3 \sin t - b_3 \cos t)X + (a_3 \cos t + b_3 \sin t)Y + Z + h_C Q \\
Q^\sharp &\longmapsto Q.
\end{aligned}$$

where the function $h_C \in \mathcal{C}^\infty(M)$ is given by

$$h_C = \frac{1}{3}(a_2 - b_1 + 2(a_3^2 + b_3^2)).$$

and the frame f is given by

$$f = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The principal G_0-bundle over M together with the Cartan connection in the theorem above is a Cartan geomtetry. Recall from Definition 2.5.3 that we can use the Cartan curvature to determine the Cartan curvature function.

Corollary 4.2.3. *Let $\{\beta_i\}$ be the basis of $\wedge^2\mathfrak{g}_-^*\otimes\mathfrak{g}$ given later in Definition 4.3.2. The canonical Cartan curvature function of a sub-Riemannian manifold (M,D,S) with the Heisenberg Lie algebra as constant sub-Riemannian symbol*

$$\begin{aligned} \kappa : \mathcal{G} &\longrightarrow \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g} \\ f &\longmapsto k^i\beta_i \end{aligned}$$

is given by

$$\begin{aligned}
k^1 &= 0\\
k^2 &= 0\\
k^3 &= 0\\
k^4 &= (\hat{A}a_3 - \hat{B}b_3 + h_C)\\
k^5 &= \cos t\hat{A}(\sin ta_3 - \cos tb_3) + \sin t\hat{B}(\sin ta_3 - \cos tb_3)\\
&\quad - \cos t(\cos ta_1 + \sin ta_2) - \sin t(\cos tb_1 - \sin tb_2)\\
k^6 &= h_c - (a_3^2 + b_3^2) + \cos t\hat{A}(\cos ta_3 + \sin tb_3) + \sin t\hat{B}(\cos ta_3 + \sin tb_3)\\
&\quad + \cos t(\sin ta_1 - \cos ta_2) + \sin t(\sin tb_1 - \cos tb_2)\\
k^7 &= 0\\
k^8 &= -\cos t(\hat{A}h_C + \hat{C}b_3 + a_1b_3 + a_2a_3) - \sin t(\hat{B}h_C + \hat{C}a_3 + b_1b_3 + b_2a_3) - (\cos ta_3 + \sin tb_3)(\hat{A}a_3 - \hat{B}b_3)\\
k^9 &= -h_c + (a_3^2 + b_3^2) - \sin t\hat{A}(\sin ta_3 - \cos tb_3) + \cos t\hat{B}(\sin ta_3 - \cos tb_3)\\
&\quad + \sin t(\cos ta_1 + \sin ta_2) - \cos t(\cos tb_1 + \sin tb_2)\\
k^{10} &= -\sin t\hat{A}(\cos ta_3 + \sin tb_3) + \cos t\hat{B}(\cos ta_3 + \sin tb_3)\\
&\quad - \sin t(\sin ta_1 - \cos ta_2) + \cos t(\sin tb_1 - \cos tb_2)\\
k^{11} &= 0\\
k^{12} &= \sin t(\hat{A}h_C + \hat{C}b_3 + a_1b_3 + a_2a_3) - \cos t(\hat{B}h_C + \hat{C}a_3 + b_1b_3 + b_2a_3) + (\sin ta_3 - \cos tb_3)(\hat{A}a_3 - \hat{B}b_3).
\end{aligned}$$

The proof of Theorem 4.2.2 is given in Section 4.4. The corollary above will follow.

Example 4.2.4. One simple case is if the manifold we are examining is the Heisenberg group with a metric on the distribution. This means that $a_i = b_i = 0$ for all i in the bracket equations. We know from Example 4.1.2 that the Heisenberg group with such a metric is a Carnot group, hence it is a model space for our measure of curvature. This means that the curvature should be zero in this case, and indeed, if we let all the a_i's and b_i's be zero we see immediately by Corollary 4.2.3 that the Cartan curvature function will be zero. ♣

From differential geometry we know that the Lie bracket has to satisfy the Jacobi identity [O'N83, Lemma 1.18]. Considering the brackets on M, we get

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$
$$\Rightarrow \begin{cases} Ab_1 - Ba_2 + a_1b_3 - a_3b_1 &= 0\\ Ab_2 - Ba_2 + a_2b_3 - a_3b_2 &= 0\\ Ab_3 - Ba_3 + b_2 + a_1 &= 0 \end{cases}$$

Since these equations must be satisfied, we see that the functions a_i and b_i are dependant on each other and on the derivative of each other with respect to A and B. In particular, if the functions are all constant we get

$$\begin{aligned} a_1b_3 - a_3b_1 &= 0 \\ a_2b_3 - a_3b_2 &= 0 \\ b_2 + a_1 &= 0 \end{aligned}$$

In this case it is clear that $b_2 = -a_1$. If $a_3 \neq 0$ and $b_3 \neq 0$ we can describe the full structure with only three constants. By the first equation we get $b_1 = \frac{a_1b_3}{a_3}$ and by the second we get $a_2 = \frac{a_3b_2}{b_3} = -\frac{a_1a_3}{b_3}$.

Example 4.2.5. Let us consider the sub-Riemannian manifold M with the Heisenberg group as constant sub-Riemannian symbol, and where the structural functions are given by $a_2 = -1$, $b_1 = 1$ and the rest are zero. This gives us the bracket equations

$$[A, B] = C, \qquad [B, C] = A, \qquad [C, A] = B$$

which is exactly the brackets of the Lie algebra $\mathfrak{su}(2)$ of the special unitary group $SU(2)$, see [Hal15, Example 3.27]. $SU(2)$ is diffeomorphic to the sphere S^3 as a manifold [Hal15, Exercise 1.5]. We may let $M = S^3$ with a sub-Riemannian structure, that is the distribution $D_x = \operatorname{span}\{A_x, B_x\}$ and a metric S on D. We can use Corollary 4.2.3 to determine the Cartan curvature function of S^3, which gives $k^4 = -\frac{2}{3}$, $k^6 = \frac{1}{3}$ and $k^9 = -\frac{1}{3}$. The rest of the k^i's are zero. this means that

$$\kappa_f = -\frac{2}{3}X^* \wedge Y^* \otimes Q + \frac{1}{3}X^* \wedge Z^* \otimes Y - \frac{1}{3}Y^* \wedge Z^* \otimes X \in \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g},$$

see Definition 4.3.2. It is clear that the Cartan curvature function is constant, that is independent on $f \in \mathcal{G}$ where $G_0 \to \mathcal{G} \xrightarrow{\pi} S^3$ is the principal G_0-bundle. This is expected as S^n is a very symmetric space. Notice that in Riemannian geometry we can use S^3 as a model space, in which case S^3 would have zero curvature as a Riemannian manifold. Recall from Example 3.2.9 that the spheres S^n are not self-similar. Even though S^3 is a special case of a sphere on which we can get a group structure inherited from $SU(2)$, under our restrictions we could not use S^3 as a model space since it is still not a Carnot group. ♣

4.3 Metric and Exterior Differential of the Heisenberg Lie Algebra

In this Section we will apply the theory developed in Section 3.5 to the $\mathfrak{g}_0$-prolongation of the Heisenberg Lie algebra. Let $\mathfrak{g}_-$ be the Heisenberg Lie algebra with a metric on $\mathfrak{g}_{-1}$. By Proposition 3.5.1 we can construct a canonical metric on $\mathfrak{g}_{-2}$.

Proposition 4.3.1. *Let $\mathfrak{g}$ be the $\mathfrak{g}_0$-prolongation of the Heisenberg Lie algebra with a metric $\langle.,.\rangle_{\mathfrak{g}_{-1}}$ on $\mathfrak{g}_{-1}$ under which the basis $\{X, Y\}$ of $\mathfrak{g}_{-1}$ is orthonormal. Then there is a canonical extended metric $\langle.,.\rangle_{\mathfrak{g}}$ for which the basis $\{X, Y, Z, Q\}$ is orthonormal.*

Proof. Recall that $\mathfrak{g}_{-2}$ is spanned by the single element $[X,Y]=Z$. For $a,b\in\mathbb{R}$ we get

$$\begin{aligned}\langle aZ,bZ\rangle_{\mathfrak{g}_{-2}} &= \langle a(X\wedge Y), b(X\wedge Y)\rangle_{\bigwedge^2\mathfrak{g}_{-1}}\\ &= ab(\langle X,X\rangle_{\mathfrak{g}_{-1}}\langle Y,Y\rangle_{\mathfrak{g}_{-1}} - \langle X,Y\rangle_{\mathfrak{g}_{-1}}\langle Y,X\rangle_{\mathfrak{g}_{-1}})\\ &= ab(1-0)\\ &= ab.\end{aligned}$$

In Section 4.1, we calculated $\mathfrak{g}_0$ for the Heisenberg Lie algebra. It is spanned by a single element Q. Recall how we could construct a metric on $\mathfrak{g}_0$ by using the trace

$$\langle aQ,bQ\rangle_{\mathfrak{g}_0} = -\frac{1}{2}\mathrm{Tr}(aQ\cdot bQ) = -\frac{1}{2}ab(-2) = ab$$

where $a,b\in\mathbb{R}$. Now we have a metric on each of the components of the $\mathfrak{g}_0$-prolongation of the Heisenberg Lie algebra $\mathfrak{g}=\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0$, which gives us a metric on $\mathfrak{g}$

$$\begin{aligned}&\langle(x_1X+y_1Y++z_1Z+q_1Q),(x_2X+y_2Y+z_2Z+q_2Q)\rangle_{\mathfrak{g}}\\ &=\langle(x_1X+y_1Y),(x_2X+y_2Y)\rangle_{\mathfrak{g}_{-1}}+\langle z_1Z,z_2Z\rangle_{\mathfrak{g}_{-2}}+\langle q_1Q,q_2Q\rangle_{\mathfrak{g}_0}\\ &=x_1x_2+y_1y_2+z_1z_2+q_1q_2.\end{aligned}$$

□

Let $\mathfrak{g}=\mathfrak{g}_-\oplus\mathfrak{g}_0$ be the $\mathfrak{g}_0$-prolongation of the Heisenberg Lie algebra, and let $\mathfrak{g}^*$ be the dual space to $\mathfrak{g}$. We have the basis $\{X,Y,Z,Q\}$ of $\mathfrak{g}$, and let $\{X^*,Y^*,Z^*,Q^*\}$ be the dual basis. We note that this correspond to the duals with respect to the metric $\langle.,.\rangle_{\mathfrak{g}}$, so that

$$\begin{aligned}X^*(u)&=\langle X,u\rangle_{\mathfrak{g}}\\ Y^*(u)&=\langle Y,u\rangle_{\mathfrak{g}}\\ Z^*(u)&=\langle Z,u\rangle_{\mathfrak{g}}\\ Q^*(u)&=\langle Q,u\rangle_{\mathfrak{g}}.\end{aligned}$$

This gives us a metric $\langle.,.\rangle_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$, and since it is dual to an orthonormal basis on $\mathfrak{g}$, this basis as well must be orthonormal. This also ensures us that we have orthonormal bases for $\mathfrak{g}_-^*\otimes\mathfrak{g}$ and $\bigwedge^2\mathfrak{g}_-\otimes\mathfrak{g}$ as well, see Section 3.5.

Let $\mathfrak{g}_-$ be the Heisenberg Lie algebra and $\mathfrak{g}=\mathfrak{g}_-\oplus\mathfrak{g}_0$ where $\mathfrak{g}_0$ is the Lie algebra of the structure group G_0. We want to find the linear operator $\delta:\mathfrak{g}_-^*\otimes\mathfrak{g}\to\bigwedge^2\mathfrak{g}_-^*\otimes\mathfrak{g}$ and describe it as a matrix. Since we know all the bracket-relations on the Heisenberg Lie algebra, we can find this matrix by evaluation of $\delta(e_i\otimes f_j)$, $i=1,2,3$, $j=1,2,3,4$, on two general elements of $\mathfrak{g}_-$, where $e_i\in\mathfrak{g}_-^*$ and $f_j\in\mathfrak{g}$ are the basis elements of $\mathfrak{g}_-^*$ and $\mathfrak{g}$ respectively. If we order the basis of $\mathfrak{g}_-^*\otimes\mathfrak{g}$ by choosing

$$\begin{aligned}\alpha_1&=X^*\otimes X, &\alpha_5&=Y^*\otimes X, &\alpha_9&=Z^*\otimes X,\\ \alpha_2&=X^*\otimes Y, &\alpha_6&=Y^*\otimes Y, &\alpha_{10}&=Z^*\otimes Y,\\ \alpha_3&=X^*\otimes Z, &\alpha_7&=Y^*\otimes Z, &\alpha_{11}&=Z^*\otimes Z,\\ \alpha_4&=X^*\otimes Q, &\alpha_8&=Y^*\otimes Q, &\alpha_{12}&=Z^*\otimes Q,\end{aligned}$$

where $\{X^*,Y^*,Z^*\}$ is the dual elements of $\{X,Y,Z\}$ respectively. Also order the basis of $\bigwedge^2\mathfrak{g}_-^*\otimes\mathfrak{g}$ in a similar way:

$$\begin{aligned}\beta_1&=X^*\wedge Y^*\otimes X, &\beta_5&=X^*\wedge Z^*\otimes X, &\beta_9&=Y^*\wedge Z^*\otimes X,\\ \beta_2&=X^*\wedge Y^*\otimes Y, &\beta_6&=X^*\wedge Z^*\otimes Y, &\beta_{10}&=Y^*\wedge Z^*\otimes Y,\\ \beta_3&=X^*\wedge Y^*\otimes Z, &\beta_7&=X^*\wedge Z^*\otimes Z, &\beta_{11}&=Y^*\wedge Z^*\otimes Z,\\ \beta_4&=X^*\wedge Y^*\otimes Q, &\beta_8&=X^*\wedge Z^*\otimes Q, &\beta_{12}&=Y^*\wedge Z^*\otimes Q.\end{aligned}$$

Then δ will be a 12×12 matrix sending vectors of the form $a_1\alpha_1 + \ldots + a_{12}\alpha_{12} \in \mathfrak{g}_-^* \otimes \mathfrak{g}$ to vectors $b_1\beta_1 + \ldots + b_{12}\beta_{12} \in \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$.

Definition 4.3.2. We will call $\{\alpha_i\}$ the standard orthonormal basis of $\mathfrak{g}_-^* \otimes \mathfrak{g}$ and $\{\beta_i\}$ the standard orthonormal basis of $\wedge^2\mathfrak{g}_-^* \otimes \mathfrak{g}$. ♠

Proposition 4.3.3. *Let $\mathfrak{g}$ be the $\mathfrak{g}_0$-prolongation of the Heisenberg Lie algebra. The Lie algebra differential*

$$\delta : \mathfrak{g}_-^* \oplus \mathfrak{g} \longrightarrow \bigwedge^2 \mathfrak{g}_-^* \oplus \mathfrak{g}$$

can be represented as the matrix

$$\delta = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

relative to the bases $\{\alpha_i\}$ of $\mathfrak{g}_-^* \oplus \mathfrak{g}$ and $\{\beta_i\}$ of $\wedge^2\mathfrak{g}_-^* \oplus \mathfrak{g}$.

Proof. Let $u = u_1X + u_2Y + u_3Z = (u_1, u_2, u_3) \in \mathfrak{g}_-$ and $v = v_1X + v_2Y + v_3Z = (v_1, v_2, v_3) \in \mathfrak{g}_-$. Then we have

$$\begin{aligned} \delta(X^* \otimes X)(u, v) &= [u, X^*(v)X] - [v, X^*(u)X] - X^*([u, v])X \\ &= [u, v_1X] - [v, u_1X] - X^*((u_1v_2 - u_2v_1)Z)X \\ &= (-u_2v_1 + u_1v_2)Z - 0 \\ &= (u_1v_2 - u_2v_1)Z. \end{aligned}$$

Recall that $u \wedge v \in \bigwedge^2 \mathfrak{g}_-$ and if we write out the coordinates we get

$$u \wedge v = (u_1v_2 - u_2v_1)X \wedge Y + (u_1v_3 - u_3v_1)X \wedge Z + (u_2v_3 - u_3v_2)Y \wedge Z.$$

Since $\delta(X^* \otimes X) \in \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$ is a $\mathfrak{g}$-linear function on $\bigwedge^2 \mathfrak{g}_-$ sending $u \wedge v$ to $(u_1v_2 - u_2v_1)Z$, it is clear that $\delta(X^*, X) = X^* \wedge Y^* \otimes Z$. By doing this computation for all the basis elements

α_i we determine the matrix.

$$\begin{aligned}
\delta(X^* \otimes X)(u,v) &= (u_1v_2 - u_2v_1)Z \\
\delta(X^* \otimes Y)(u,v) &= 0 \\
\delta(X^* \otimes Z)(u,v) &= 0 \\
\delta(X^* \otimes Q)(u,v) &= -(u_1v_2 - u_2v_1)X \\
\delta(Y^* \otimes X)(u,v) &= 0 \\
\delta(Y^* \otimes Y)(u,v) &= (u_1v_2 - u_2v_1)Z \\
\delta(Y^* \otimes Z)(u,v) &= 0 \\
\delta(Y^* \otimes Q)(u,v) &= -(u_1v_2 - u_2v_1)Y \\
\delta(Z^* \otimes X)(u,v) &= -(u_2v_3 - u_3v_2)Z - (u_1v_2 - u_2v_1)X \\
\delta(Z^* \otimes Y)(u,v) &= (u_1v_3 - u_3v_1)Z - (u_1v_2 - u_2v_1)Y \\
\delta(Z^* \otimes Z)(u,v) &= -(u_1v_2 - u_2v_1)Z \\
\delta(Z^* \otimes Q)(u,v) &= (u_2v_3 - u_3v_2)X - (u_1v_3 - u_3v_1)Y - (u_1v_2 - u_2v_1)Q.
\end{aligned}$$

Putting all this information together we get exactly the proposed matrix. □

4.4 Proof of Theorem 4.2.2

Let (M, D, S) be a sub-Riemannian manifold with the Heisenberg group as constant sub-Riemannian symbol. Let $\pi : \mathcal{G} \to M$ be graded frame bundle of M. The frames $f \in \mathcal{G}_x$ works by

$$f : \mathfrak{g}_- \longrightarrow gr(T_xM).$$

From Lemma 4.1.3 we know that the group G_0 acting on the fibers is isomorphic to $O(2)$, and this means that the Lie algebra of G_0 is $\mathfrak{so}(2)$, which gives

$$T\mathcal{G} \cong TM \times \mathfrak{so}(2).$$

Here $\mathfrak{so}(2) = \text{span}\{Q^\sharp\}$ is one dimensional with $Q^\sharp = \frac{\partial}{\partial t}$, coming from the fact that G_0 is one dimensional and parametrized by t. Recall that $T\mathcal{G}$ is spanned by $\{\hat{A}, \hat{B}, \hat{C}, Q^\sharp\}$ with the projection

$$\begin{aligned}
d\pi(\hat{A}) &= A, \\
d\pi(\hat{B}) &= B, \\
d\pi(\hat{C}) &= C,
\end{aligned}$$

and we can evaluate how the Cartan connection works on $T_f\mathcal{G}$. The Cartan connection is the sum of the graded soldering 1-form $\theta : T\mathcal{G} \to \mathfrak{g}_-$ and a principal connection $\omega : T\mathcal{G} \to \mathfrak{g}_0$. From Section 3.4 we have the graded soldering 1-form $\theta_f = f^{-1} \circ gr \circ d\pi$, where the grading function gr is yet to be determined by the normalizing condition. Let $f = f(t)$ be a graded frame with positive determinant. Then

$$f^{-1} = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

such that if $A^{gr}, B^{gr} \in (T_xM)^{-1}$ and $C^{gr} \in (T_xM)^{-2}$ denote an orthonormal basis of the graded tangent space at $x \in M$, then f sends a vector in this basis to a vector of $\mathfrak{g}_-$ in the basis $\{X, Y, Z\}$. To evaluate the grading map, we give it a general form by writing the projection

$$pr_{-2}(x) : D_x^{-2} = T_xM \longrightarrow D, \qquad \ker(pr_{-2}(x)) = \operatorname{span}\{C + \alpha A + \beta B\}$$

where $\alpha, \beta \in \mathcal{C}^\infty(M)$ are real valued functions yet to be determined. This gives us the grading function by mapping D_x to $(T_xM)^{-1}$ by the identity and $\ker(pr_{-2}(x))$ to $(T_xM)^{-2}$ by letting $r(C+\alpha A+\beta B) \mapsto rC^{gr}$. We can now evaluate how the grading function works on an element

$$\begin{aligned} gr : T_xM &\longrightarrow gr(T_xM) \\ v_1A + v_2B + v_3C &\longmapsto (v_1 - \alpha v_3)A^{gr} + (v_2 - \beta v_3)B^{gr} + v_3C^{gr}. \end{aligned}$$

Adding the information together we can see how the graded soldering 1-form acts on the basis elements of $T_f\mathcal{G}$:

$$\begin{aligned} \theta_f(\hat{A}) &= \cos tX - \sin tY \\ \theta_f(\hat{B}) &= \sin tX + \cos tY \\ \theta_f(\hat{C}) &= -\alpha(\cos tX - \sin tY) - \beta(\sin tX + \cos tY) + Z \\ \theta_f(Q^\sharp) &= 0. \end{aligned}$$

Alternatively we can represent θ_f as a matrix

$$\theta_f|_{\mathcal{H}} = \begin{pmatrix} \cos t & \sin t & -\alpha\cos t - \beta\sin t \\ -\sin t & \cos t & \alpha\sin t - \beta\cos t \\ 0 & 0 & 1 \end{pmatrix}$$

Here we have written the matrix that represent the restriction $\theta_f|_{\mathcal{H}}$ of the graded soldering 1-form to the horizontal space $\mathcal{H}_f$, but it is clear that this determine the whole θ_f, since it is zero on the vertical space $\mathcal{V}$. Now we need to find a general form of the Cartan connection $\varpi = \theta + \omega$. We can write

$$\varpi = \theta_X \otimes X + \theta_Y \otimes Y + \theta_Z \otimes Z + \omega_Q \otimes Q$$

where $\theta_X, \theta_Y, \theta_Z$ and ω_Q are real valued 1-forms. We can evaluate these on the basis elements of $T_f\mathcal{G}$

$$\begin{array}{cccc} \theta_X(\hat{A}) = \cos t, & \theta_X(\hat{B}) = \sin t, & \theta_X(\hat{C}) = -(\alpha\cos t + \beta\sin t)X, & \theta_X(Q^\sharp) = 0 \\ \theta_Y(\hat{A}) = -\sin t, & \theta_Y(\hat{B}) = \cos t, & \theta_Y(\hat{C}) = (\alpha\sin t - \beta\cos t)Y, & \theta_Y(Q^\sharp) = 0, \\ \omega_Q(\hat{A}) = -h_A, & \omega_Q(\hat{B}) = -h_B, & \omega_Q(\hat{C}) = -h_C, & \omega_Q(Q^\sharp) = 1. \end{array}$$

Here we already know what the functions θ_X, θ_Y and θ_Z are, since they all come from the soldering 1-form. We keep ω_Q general, where $h_A, h_B, h_C \in \mathcal{C}^\infty(M)$. Our goal is to use Theorem 3.5.5 to determine the Cartan connection ϖ canonically. This is the same as determining functions α, β, h_A, h_B and h_C. According to the theorem the Cartan connection ϖ will be canonical in the sense that it will be the unique Cartan connection satisfying $\delta^*\kappa_f^p = 0$ for all $p \geq 0$.

Determining the Lie algebra codifferential δ^*: Consider the $\mathfrak{g}_0$-prolongation of the Heisenberg Lie algebra $\mathfrak{g}$. By Proposition 4.3.1 we have a canonically defined, orthonormal metric on $\mathfrak{g}$. This extends canonically to an orthonormal metric on both $\mathfrak{g}_-^* \otimes \mathfrak{g}$ and $\wedge^2\mathfrak{g}_-^* \otimes \mathfrak{g}$ with basis $\{\beta_i\}$ and $\{\alpha_i\}$ respectively as in Section 4.3. By Proposition 4.3.3 we have a matrix representation of the Lie algebra differential $\delta : \mathfrak{g}_-^* \otimes \mathfrak{g} \to \wedge^2\mathfrak{g}_-^* \otimes \mathfrak{g}$ relative to the bases $\{\beta_i\}$ and $\{\alpha_i\}$. Recall from Definition 3.5.3 that the codifferential δ^* is determined by

$$\langle \delta(\xi), \eta \rangle_{\wedge^2\mathfrak{g}_-^* \oplus \mathfrak{g}} = \langle \xi, \delta^*(\eta) \rangle_{\mathfrak{g}_-^* \oplus \mathfrak{g}}$$

where $\xi \in \mathfrak{g}_-^* \oplus \mathfrak{g}$ and $\eta \in \wedge^2\mathfrak{g}_-^* \oplus \mathfrak{g}$. Since this matrix is written in relation to orthonormal bases, we can find δ^* by transposing δ

$$\delta^* = \delta^T.$$

Determining the Cartan curvature function κ: At this point we have a principal bundle over a smooth manifold $G_0 \to \mathcal{G} \to M$ with a $\mathfrak{g}$-valued Cartan connection ϖ, hence by Defintion 2.4.6 we have a Cartan geometry. This means that we can compute the Cartan curvature K from Definition 2.5.1. Recall that the differential of a $\mathfrak{g}$-valued 1-form γ is given by

$$d\gamma(U, V) = U\gamma(V) - V\gamma(U) - \gamma([U, V]).$$

By Lemma 2.5.2 K will vanish on the vertical vectors, hence it is sufficient to consider only horizontal vectors.

$$\begin{aligned}
K(\hat{A}, \hat{B}) &= d\varpi(\hat{A}, \hat{B}) + [\varpi(\hat{A}), \varpi(\hat{B})] \\
&= \hat{A}(\sin tX + \cos tY - h_B Q) - \hat{B}(\cos tX - \sin tY - h_A Q) \\
&\quad + (\alpha \cos t + \beta \sin t)X - (\alpha \sin t - \beta \cos t)Y \\
&\quad - Z + h_C Q + [\cos tX - \sin tY - h_A Q, \sin tX + \cos tY - h_B Q] \\
&= (\cos t(\alpha + h_A) + \sin t(\beta + h_B))X \\
&\quad + ((-\sin t(\alpha + h_A) + \cos t(\beta + h_B))Y \\
&\quad + (-\hat{A}h_B + \hat{B}h_A + h_C)Q
\end{aligned}$$

$$\begin{aligned}
K(\hat{A}, \hat{C}) = ((&-\cos ta_1 - \sin ta_2) + (\sin t\alpha - \cos t\beta)h_A + \sin th_C \\
&+ \hat{A}(-\cos t\alpha - \sin t\beta) + a_3(\cos t\alpha + \sin t\beta))X \\
&+ ((\sin ta_1 - \cos ta_2) + (\cos t\alpha + \sin t\beta)h_A + \cos th_C \\
&+ \hat{A}(\sin t\alpha - \cos t\beta) - a_3(\sin t\alpha - \cos t\beta))Y \\
&+ (-\beta - a_3)Z \\
&+ (-\hat{A}h_C + \hat{C}h_A + a_1 h_A + a_2 h_B + a_3 h_C)Q
\end{aligned}$$

$$\begin{aligned}
K(\hat{B}, \hat{C}) = ((&-\cos tb_1 - \sin tb_2) + (\sin t\alpha - \cos t\beta)h_B - \cos th_C \\
&+ \hat{B}(-\cos t\alpha - \sin t\beta) + b_3(\cos t\alpha + \sin t\beta))X \\
&+ ((\sin tb_1 - \cos tb_2) + (\cos t\alpha + \sin t\beta)h_B + \sin th_C \\
&+ \hat{B}(\sin t\alpha - \cos t\beta) - b_3(\sin t\alpha - \cos t\beta))Y \\
&+ (\alpha - b_3)Z \\
&+ (-\hat{B}h_C + \hat{C}h_B + b_1 h_A + b_2 h_B + b_3 h_C)Q
\end{aligned}$$

The Cartan curvature function κ is defined by evaluating the Cartan curvature on the horizontal lift of $\mathfrak{g}_-$, see Definition 2.5.3. To determine κ_f we need to evaluate how the basis elements of $\mathfrak{g}_-$ are lifted to $\mathcal{H}_f$:

$$
\begin{aligned}
\pi_{\mathcal{H}_f}^{-1} \circ gr^{-1} \circ f(X) &= \pi_{\mathcal{H}_f}^{-1}(\cos tA + \sin tB) = \cos t\hat{A} + \sin t\hat{B} + (\cos th_A + \sin th_B)Q^\sharp \\
\pi_{\mathcal{H}_f}^{-1} \circ gr^{-1} \circ f(Y) &= \pi_{\mathcal{H}_f}^{-1}(-\sin tA + \cos tB) = -\sin t\hat{A} + \cos t\hat{B} + (-\sin th_A + \cos th_B)Q^\sharp \\
\pi_{\mathcal{H}_f}^{-1} \circ gr^{-1} \circ f(Z) &= \pi_{\mathcal{H}_f}^{-1}(C + \alpha A + \beta B) = \hat{C} + \alpha\hat{A} + \beta\hat{B} + (h_C + \alpha h_A + \beta h_B)Q^\sharp.
\end{aligned}
$$

Using the definition of the Cartan curvature function we can compute how it works on the basis elements of $\wedge^2\mathfrak{g}_-$

$$
\begin{aligned}
\kappa_f(X \wedge Y) &= K((\cos t\hat{A} + \sin t\hat{B} + (\cos th_A + \sin th_B)Q^\sharp), (-\sin t\hat{A} + \cos t\hat{B} + (-\sin th_A + \cos th_B)Q^\sharp)) \\
&= \cos^2 tK(\hat{A}, \hat{B}) + \sin^2 tK(\hat{A}, \hat{B}) \\
&= K(\hat{A}, \hat{B}) \\
\kappa_f(X \wedge Z) &= K((\cos t\hat{A} + \sin t\hat{B} + (\cos th_A + \sin th_B)Q^\sharp), \hat{C} + \alpha\hat{A} + \beta\hat{B} + (h_C + \alpha h_A + \beta h_B)Q^\sharp) \\
&= \cos tK(\hat{A}, \hat{C}) + \sin tK(\hat{B}, \hat{C}) - (\sin t\alpha - \cos t\beta)K(\hat{A}, \hat{B}) \\
\kappa_f(Y \wedge Z) &= K((-\sin t\hat{A} + \cos t\hat{B} + (-\sin th_A + \cos th_B)Q^\sharp), \hat{C} + \alpha\hat{A} + \beta\hat{B} + (h_C + \alpha h_A + \beta h_B)Q^\sharp) \\
&= -\sin tK(\hat{A}, \hat{C}) + \cos tK(\hat{B}, \hat{C}) - (\cos t\alpha + \sin t\beta)K(\hat{A}, \hat{B}).
\end{aligned}
$$

Since κ_f is an element of the 12-dimensional space $\bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$, we can write $\kappa_f = k^i\beta_i$, where $\{\beta_i\}_{i=1}^{12}$ is the basis given in Section 4.3. Let us consider $\delta^*\kappa_f = \delta^*(k^i\beta_i)$:

$$
\delta^*\kappa_f = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\begin{pmatrix} k^1 \\ k^2 \\ k^3 \\ k^4 \\ k^5 \\ k^6 \\ k^7 \\ k^8 \\ k^9 \\ k^{10} \\ k^{11} \\ k^{12} \end{pmatrix}
=
\begin{pmatrix} k^3 \\ 0 \\ 0 \\ -k^1 \\ 0 \\ k^3 \\ 0 \\ -k^2 \\ -k^1 - k^{11} \\ -k^2 + k^7 \\ -k^3 \\ -k^4 - k^6 + k^9 \end{pmatrix}
$$

The information we get from the normalizing condition only depend on the k^i's that are present in $\delta^*\kappa_f$. We will compute these, explaining in detail for k^1. We know that k^1 is the coefficient of $\beta_1 = X^* \wedge Y^* \otimes X$. This means that k^1 is the coefficient we get in from of X when considering $\kappa_f(X \wedge Y)$. We get

$$\kappa_f(X \wedge Y) = k^1X + k^2Y + k^3Z + k^4Q.$$

By the calculations above we know that $\kappa_f(X \wedge Y) = K(\hat{A}, \hat{B})$ which we have calculated, and the coefficient in front of X for $K(\hat{A}, \hat{B})$ is given by $(\cos t(\alpha + h_A) + \sin t(\beta + h_B))$. Solving for the other k^i's we get

$$
\begin{aligned}
k^1 &= (\cos t(\alpha + h_A) + \sin t(\beta + h_B)) \\
k^2 &= ((-\sin t(\alpha + h_A) + \cos t(\beta + h_B)) \\
k^3 &= 0 \\
k^4 &= (-\hat{A} h_B + \hat{B} h_A + h_C) \\
k^6 &= \cos t((\sin t a_1 - \cos t a_2) + (\cos t\alpha + \sin t\beta) h_A + \cos t h_C \\
&\quad + \hat{A}(\sin t\alpha - \cos t\beta) - a_3(\sin t\alpha - \cos t\beta)) \\
&\quad + \sin t((\sin t b_1 - \cos t b_2) + (\cos t\alpha + \sin t\beta) h_B + \sin t h_C \\
&\quad + \hat{B}(\sin t\alpha - \cos t\beta) - b_3(\sin t\alpha - \cos t\beta)) \\
&\quad - (\sin t h_A - \cos t h_B)(-\sin t(\alpha + h_A) + \cos t(\beta + h_B) \\
k^7 &= (\sin t(\alpha - b_3) - \cos t(-\beta - a_3)) \\
k^9 &= -\sin t((-\cos t a_1 - \sin t a_2) + (\sin t\alpha - \cos t\beta) h_A + \sin t h_C \\
&\quad + \hat{A}(-\cos t\alpha - \sin t\beta) + a_3(\cos t\alpha + \sin t\beta)) \\
&\quad + \cos t((-\cos t b_1 - \sin t b_2) + (\sin t\alpha - \cos t\beta) h_B - \cos t h_C \\
&\quad + \hat{B}(-\cos t\alpha - \sin t\beta) + b_3(\cos t\alpha + \sin t\beta)) \\
&\quad - (\cos t\alpha + \sin t\beta)(\cos t(\alpha + h_A) + \sin t(\beta + h_B)) \\
k^{11} &= (\cos t(\alpha - b_3) + \sin t(-\beta - a_3))
\end{aligned}
$$

By Theorem 3.5.5 we know that there is a unique Cartan connection ϖ satisfying $\delta^*\kappa_f = 0$. We want to determine the functions $\alpha, \beta, h_A, h_B, h_C$ such that this normalizing condition is satisfied. Notice that these functions determine the Cartan connection. We expect the result to depend on the structure of the manifold, so the functions we determine should depend on the structural functions a_i, b_i for $i = 1, 2, 3$. Recall $\delta^*\kappa_f \in \mathfrak{g}_-^* \oplus \mathfrak{g}$ is a twelve dimensional vector, and we can write it as $\kappa_f = r^i \alpha_i$ where α_i is the basis of $\mathfrak{g}_-^* \oplus \mathfrak{g}$ introduced earlier. The equation $\delta^*\kappa_f = 0$ gives the twelve equations $r^i = 0$ which give us the following information:

(i) Consider r^4 and r^8. This gives $k^1 = k^2 = 0$, and we get

$$
\begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \begin{pmatrix} \alpha + h_A \\ \beta + h_B \end{pmatrix} = 0 \quad \Rightarrow \quad h_A = -\alpha, \quad h_B = -\beta,
$$

(ii) Consider r^9 and r^{10}. Since we saw that k^1 and k^2 must be zero, we get $k^7 = k^{11} = 0$, which gives

$$
\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} \alpha - b_3 \\ -\beta - a_3 \end{pmatrix} = 0 \quad \Rightarrow \quad b_3 = \alpha, \quad a_3 = -\beta
$$

(iii) Consider r^{12}. Here we get the equation

$$
k^9 - k^6 - k^4 = 0.
$$

We fill in the values for k^4, k^6 and k^9, simplified by the relations above. This gives

$$
\begin{aligned}
&-\sin t\Big((-\cos ta_1 - \sin ta_2) + (\sin tb_3 + \cos ta_3)(-b_3) + \sin th_C +\\
&\hat{A}(-\cos tb_3 + \sin ta_3) + a_3(\cos tb_3 - \sin ta_3)\Big)\\
&+\cos t\Big((-\cos tb_1 - \sin tb_2) + (\sin tb_3 + \cos ta_3)a_3 - \cos th_C\\
&+\hat{B}(-\cos tb_3 + \sin ta_3) + b_3(\cos tb_3 - \sin ta_3)\Big)\\
&-\cos t\Big((\sin ta_1 - \cos ta_2) + (\cos tb_3 - \sin ta_3)(-b_3) + \cos th_C\\
&+\hat{A}(\sin tb_3 + \cos ta_3) - a_3(\sin tb_3 + \cos ta_3)\Big)\\
&-\sin t\Big((\sin tb_1 - \cos tb_2) + (\cos tb_3 - \sin ta_3)a_3 + \sin th_C\\
&+\hat{B}(\sin tb_3 + \cos ta_3) - b_3(\sin tb_3 + \cos ta_3)\Big)\\
&-(-\hat{A}a_3 - \hat{B}b_3 + h_C)\\
&= 0.
\end{aligned}
$$

Most of the terms will cancel, if we collect all the terms involving $\hat{A}$, we get

$$
\begin{aligned}
& -\sin t\hat{A}(-\cos tb_3 + \sin ta_3) - \cos t\hat{A}(\sin tb_3 + \cos ta_3) + \hat{A}a_3\\
= \;& \sin t\cos t\hat{A}b_3 - \sin^2 t\hat{A}a_3 - \cos t\sin t\hat{A}b_3 - \cos^2 t\hat{A}a_3 + \hat{A}a_3\\
= \;& 0.
\end{aligned}
$$

A similar calculation show that terms involving $\hat{B}$ also cancel. The terms that involve a_i and b_i for $i = 1, 2$ can be sorted out the same way, and it gives

$$
\begin{aligned}
& -\sin t(-\cos ta_1 - \sin ta_2) + \cos t(-\cos tb_1 - \sin tb_2)\\
& -\cos t(\sin ta_1 - \cos ta_2) - \sin t(\sin tb_1 - \cos tb_2)\\
= \;& \sin t\cos ta_1 + \sin^2 ta_2 - \cos^2 tb_1 - \sin t\cos tb_2\\
& -\sin t\cos ta_1 + \cos^2 ta_2 - \sin^2 tb_1 + \sin t\cos tb_2\\
= \;& a_2 - b_1.
\end{aligned}
$$

The terms involving h_C gives

$$-\sin^2 th_C - \cos^2 th_c - \cos^2 th_c - \sin^2 th_C - h_C = -3h_C,$$

and the rest gives

$$
\begin{aligned}
& -\sin t\Big((\sin tb_3+\cos ta_3)(-b_3)+a_3(\cos tb_3-\sin ta_3)\Big)\\
& +\cos t\Big((\sin tb_3+\cos ta_3)a_3+b_3(\cos tb_3-\sin ta_3)\Big)\\
& -\cos t\Big((\cos tb_3-\sin ta_3)(-b_3)-a_3(\sin tb_3+\cos ta_3)\Big)\\
& -\sin t\Big((\cos tb_3-\sin ta_3)a_3-b_3(\sin tb_3+\cos ta_3)\Big)\\
= \quad & \sin^2 tb_3^2+\sin t\cos ta_3b_3-\sin t\cos ta_3b_3+\sin^2 ta_3^2\\
& \cos t\sin ta_3b_3+\cos^2 ta_3^2+\cos^2 tb_3^2-\sin t\cos ta_3b_3\\
& \cos^2 tb_3^2-\sin t\cos ta_3b_3+\sin t\cos ta_3b_3+\cos^2 ta_3^2\\
& -\sin t\cos ta_3b_3+\sin^2 ta_3^2+\sin^2 tb_3^2+\sin t\cos ta_3b_3\\
= \quad & 2(a_3^2+b_3^2).
\end{aligned}
$$

We have no simplified the equation to

$$k^9-k^6-k^4=-3h_C+a_2-b_1+2(a_3^2+b_3^2)=0.$$

We solve for h_C and get

$$h_C=\frac{1}{3}(a_2-b_1+2(a_3^2+b_3^2)).$$

By inserting these values into the general form of the Cartan connection ϖ we see that the Cartan connection is given uniquely depending only on the structure of the manifold. This proves Theorem 4.2.2. Corollary 4.2.3 follows by computing the rest of the k^i's and simplifying for the now determined functions.

Appendix A

Properties of Cartan Gauges

The goal of this section is to complete the theory that is needed to justify the claims that was used in Section 2.4.

Let G be a Lie group and let $\mu : G \times G \to G$ be the group multiplication. Let $\pi_{1*} \times \pi_{2*} : T(G \times G) \to T(G) \times T(G)$ be the diffeomorphism made from the projections to the first and second coordinate of $G \times G$ respectively.

Proposition A.0.1. *[Sha00, Prop 3.1.2, p.97] Let ν be defined by the commutativity of the diagram bellow.*

$$\begin{tikzcd}
T(G \times G) \arrow[rr, "\pi_{1*}\times\pi_{2*}"] \arrow[dr, "\mu_*"] & & T(G) \times T(G) \arrow[dl, "\nu"] \\
& T(G) &
\end{tikzcd}$$

Then $\nu((g,u) \times (h,v)) = (gh, R_{h*}u + L_{g*}v)$.

Proof. First we check that ν is properly defined, i.e. that we can add together the terms in the second coordinate: We have $u \in T_gG$, hence $R_{h*}u \in T_{gh}G$, and $v \in T_hG$ so that $L_{g*}v \in T_{gh}G$, so the definition makes sense.
The spaces $T(G \times G), T(G) \times T(G)$ and $T(G)$ are all vector bundles and the maps between them are vector bundle homomorphisms.

$$\begin{tikzcd}
T(G \times G) \arrow[rr] \arrow[dr, "\mu_*"] \arrow[dd] & & T(G) \times T(G) \arrow[dl, "\nu"] \arrow[dd] \\
& T(G) \arrow[dd] & \\
G \times G \arrow[rr] \arrow[dr, "\mu"] & & G \times G \arrow[dl, "\mu"] \\
& G &
\end{tikzcd}$$

This means that ν preserves the fibers and is a linear map on each fiber, so it is sufficient to verify the formula for ν restricted to an arbitrary fiber:

$$
\begin{aligned}
\nu|_{(g,h)} : T_g(G) \times T_h(G) &\longrightarrow T_{gh}(G) \\
(u,v) &\longmapsto A(u) + B(v)
\end{aligned}
$$

where A and B are linear. Define two maps

$$
\begin{aligned}
\lambda_h : G &\longrightarrow G \times G \\
g &\longmapsto (g,h)
\end{aligned}
$$

$$
\begin{aligned}
\rho_g : G &\longrightarrow G \times G \\
h &\longmapsto (g,h)
\end{aligned}
$$

Notice that $R_h = \mu \circ \lambda_h$ and $L_g = \mu \circ \rho_g$.

By the chain rule we get $R_{h*} = \mu_* \circ \lambda_{h*}$. By letting $c_h : G \to G$ be the constant map $c_h(g) = h$, such that $\lambda_h = id \times c_h$, the diagram bellow commutes.

$$
\begin{tikzcd}
 & T(G) \arrow[dl, "id_* \times c_{h*}"'] \arrow[d, "\lambda_{h*}"] \arrow[dr, "R_{h*}"] & \\
 & T(G \times G) \arrow[dl, "\cong"] \arrow[dr, "\mu_*"'] & \\
T(G) \times T(G) \arrow[rr, "\nu"'] & & T(G)
\end{tikzcd}
$$

Looking at an element $(g,u) \in T(G)$ we have $id_* \times c_{h*}(g,u) = ((g,u) \times (h,0))$, and by commutation of the diagram, $\nu((g,u) \times (h,0)) = R_{h*}(g,u)$, hence $A(u) + B(0) = A(u) = R_{h*}u$. The proof to show that $B(v) = L_{g*}v$ is similar.

$\square$

Proposition A.0.2. *[Sha00, Prop 3.4.10, p.113] Let $\mu : G \times G \to G$ be multiplication and $\imath : G \to G$ be the inverse map on a Lie group G. Then*

(i) $\mu^*\omega_G(w) = Ad(h^{-1})(\omega_G(\pi_{1*}w)) + \omega_G(\pi_{2*}w)$ *for* $w \in T_{(g,h)}(G \times G)$,

(ii) $\imath^*\omega_G(v) = -Ad(g)\omega_G(v)$ *for* $v \in T_g(G)$.

Proof. Let $w \in T(G \times G)$ be such that $\pi_{1*} \times \pi_{2*}(w) = (g,u) \times (h,v) \in T(G) \times T(G)$, and let ν be as in Proposition A.0.1.

$$
\begin{aligned}
(\mu^*\omega_G)w &= \omega_G(\mu_* w) \\
&= \omega_G \nu((\pi_{1*} \times \pi_{2*})w) \\
&= \omega_G \nu((g,u) \times (h,v)) \\
&= \omega_G(gh, R_{h*}u + L_{g*}v) \\
&= L_{(gh)^{-1}}(R_{h*}u + L_{g*}v) \\
&= L_{h^{-1}*}L_{g^{-1}*}R_{h*}u + L_{h^{-1}*}L_{g^{-1}*}L_{g*}v \\
&= L_{h^{-1}*}R_{h*}L_{g^{-1}*}u + L_{h^{-1}*}v \\
&= L_{h^{-1}*}R_{h*}\omega_G(u) + \omega_G(v) \\
&= Ad(h^{-1})\omega_G(u) + \omega_G(v) \\
&= Ad(h^{-1})(\omega_G(\pi_{1*}w)) + \omega_G(\pi_{2*}w)
\end{aligned}
$$

This proves (i). To prove (ii), look at

$$
\begin{aligned}
\Lambda : G &\xrightarrow{\Delta} G \times G \xrightarrow{id \times \imath} G \times G \xrightarrow{\mu} G \\
g &\longmapsto (g,g) \longmapsto (g, g^{-1}) \longmapsto e.
\end{aligned}
$$

This is the constant map, so $\Lambda^*\omega_G(v_g) = \omega_G(\Lambda_* v_g) = \omega_G(0_e) = 0$. Here $v_g \in T_gG$ and $0_e \in T_eG$ is the origin of the tangent space at the identity element e of G. By (i) we get

$$
\begin{aligned}
0 &= \Lambda^*\omega_G \\
&= ((id \times \imath)\Delta)^*\mu^*\omega_G \\
&= ((id \times \imath)\Delta)^*(\pi_1^* Ad(g)\omega_G + \pi_2^*\omega_G) \\
&= (\pi_1(id \times \imath)\Delta)^*(Ad(g)\omega_G) + (\pi_2(id \times \imath)\Delta)^*\omega_G) \\
&= Ad(g)\omega_G + \imath^*\omega_G
\end{aligned}
$$

Which finishes the proof. Notice how we get $Ad(g)$ in the calculation, since μ^* is in this case working on $T_{(g,g^{-1})}(G \times G)$, so h from (i) is g^{-1} here. Also, we used that $(\pi_1(id \times \imath)\Delta) = id$ and $(\pi_2(id \times \imath)\Delta) = \imath$.

$\square$

Corollary A.0.3. *[Sha00, Cor 3.4.11, p.114] Let $f_1, f_2 : M \to G$, and set $h(x) = f_1(x)f_2(x)^{-1}$, then*

$$h^*\omega_G = Ad(f_2(x))[f_1^*\omega_G - f_2^*\omega_G].$$

Proof. We can write $h(x)$ as a composite

$$
\begin{array}{ccccccccc}
M & \xrightarrow{\Delta} & M \times M & \xrightarrow{f_1 \times f_2} & G \times G & \xrightarrow{id \times \imath} & G \times G & \xrightarrow{\mu} & G \\
x & \longmapsto & (x,x) & \longmapsto & (f_1(x), f_2(x)) & \longmapsto & (f_1(x), f_2(x)^{-1}) & \longmapsto & f_1(x)f_2(x)^{-1}.
\end{array}
$$

Then we get

$$\begin{aligned}
h^*\omega_G &= ((id \times \imath)(f_1 \times f_2)\Delta)^*\mu^*\omega_G \\
&= ((id \times \imath)(f_1 \times f_2)\Delta)^*(\pi_1^*(Ad(f_2)\omega_G) + \pi_2^*\omega_G) \\
&= (\pi_1(id \times \imath)(f_1 \times f_2)\Delta)^* Ad(f_2)\omega_G + (\pi_2(id \times \imath)(f_1 \times f_2)\Delta)^*\omega_G \\
&= f_1^* Ad(f_2)\omega_G + f_2^*\imath^*\omega_G \\
&= f_1^* Ad(f_2)\omega_G - f_2^* Ad(f_2)\omega_G \\
&= Ad(f_2)[f_1^*\omega_G - f_2^*\omega_G].
\end{aligned}$$

□

Notice that if we have another function $g(x) = f_1(x)f_2(x)$, then we can simplify the proof above, and we get

$$\begin{aligned}
g^*\omega_G &= ((f_1 \times f_2)\Delta)^*\mu^*\omega_G \\
&= ((f_1 \times f_2)\Delta)^*(\pi_1^*(Ad(f_2^{-1})\omega_G) + \pi_2^*\omega_G) \\
&= (\pi_1(f_1 \times f_2)\Delta)^* Ad(f_2^{-1})\omega_G + (\pi_2(f_1 \times f_2)\Delta)^*\omega_G \\
&= f_1^* Ad(f_2^{-1})\omega_G + f_2^*\omega_G \\
&= f_1^* Ad(f_2^{-1})\omega_G + f_2^*\omega_G \\
&= Ad(f_2^{-1}) f_1^*\omega_G + f_2^*\omega_G.
\end{aligned}$$

This proves **Claim 4** from Section 2.4.

Lemma A.0.4. *[Sha00, Lemma 5.1.4, p.175] Suppose that (U, θ_i) are Cartan gauges for $i = 1, 2, 3$. Then*

(i) $\theta_1 \Rightarrow_{id} \theta_1$,

(ii) $\theta_1 \Rightarrow_k \theta_2$ *implies* $\theta_2 \Rightarrow_{k^{-1}} \theta_1$,

(iii) $\theta_1 \Rightarrow_k \theta_2$ *and* $\theta_2 \Rightarrow_r \theta_3$ *imply* $\theta_1 \Rightarrow_{kr} \theta_3$.

Recall the equation $(\theta_2)_x = Ad(k(x)^{-1})(\theta_1)_x + (k^*)_x\omega_H$ from Definition 2.4.2, and that we write this relation by $\theta_1 \Rightarrow_k \theta_2$. Here θ_i is a $\mathfrak{g}$-valued 1-form on U and $k : U \to H$ is a smooth function where H is a Lie group with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$.

Proof. (i) The function $id(x) = e$ is constantly equal the identity element $e \in H$ for all $x \in U$, $Ad(id^{-1})$ becomes the identity map, and $id^*\omega_G = 0$ since id is constant, which makes the result obvious.

(ii) The equation $\theta_2 = Ad(k^{-1})\theta_1 + k^*\omega_H$ implies

$$\begin{aligned}
\theta_1 &= Ad(k)(\theta_2 - k^*\omega_H) \\
&= Ad(k)\theta_2 - Ad(k)(k^*\omega_H)
\end{aligned}$$

If we look at $k^*\omega_H$ and think of $k(x) = id(x)k(x)$ where $id(x) = e$ is the constant map sending U to the identity $e \in H$. Now we can use Corollary A.0.3 and get

$$\begin{aligned} k^*\omega_H &= Ad(k^{-1})[id^*\omega_H - (k^{-1})^*\omega_H] \\ &= -Ad(k^{-1})(k^{-1})^*\omega_H \end{aligned}$$

which implies $Ad(k)k^*\omega_G = -(k^{-1})^*\omega_H$. Thus we get

$$\begin{aligned} \theta_1 &= Ad(k)\theta_2 - Ad(k)(k^*\omega_H) \\ &= Ad(k)\theta_2 + (k^{-1})^*\omega_H \end{aligned}$$

which by definition means that $\theta_2 \Rightarrow_{k^{-1}} \theta_1$.

(iii) Here we have $\theta_2 = Ad(k^{-1})\theta_1 + k^*\omega_H$ and $\theta_3 = Ad(r^{-1})\theta_2 + r^*\omega_H$, which gives us

$$\begin{aligned} \theta_3 &= Ad(r^{-1})(Ad(k^{-1})\theta_1 + k^*\omega_H) + r^*\omega_H \\ &= Ad((kr)^{-1})\theta_1 + Ad(r^{-1})k^*\omega_H + r^*\omega_H. \end{aligned}$$

Evaluate the composition

$$\begin{aligned} U &\xrightarrow{\Delta} U \times U \xrightarrow{k\times r} H \times H \xrightarrow{\mu} H \\ u &\longmapsto (u,u) \longmapsto (k,r) \longmapsto kr. \end{aligned}$$

By Proposition **??** part (ii) we get

$$\begin{aligned} (kr)^*\omega_H &= ((k\times r)\Delta)^*\mu^*\omega_H \\ &= ((k\times r)\Delta)^*(\pi_1^*(Ad(r^{-1})\omega_G + \pi_2^*\omega_G) \\ &= (\pi_1(k\times r)\Delta)^*Ad(r^{-1})\omega_G + (\pi_2(k\times r)\Delta)^*\omega_G \\ &= Ad(r^{-1})k^*\omega_H + r^*\omega_H \end{aligned}$$

hence we get

$$\begin{aligned} \theta_3 &= Ad((kr)^{-1})\theta_1 + Ad(r^{-1})k^*\omega_H + r^*\omega_H \\ &= Ad((kr)^{-1})\theta_1 + (kr)^*\omega_H \end{aligned}$$

which by definition means that $\theta_1 \Rightarrow_{kr} \theta_3$.

$\square$

Definition A.0.5. Let $N \subset H$ be groups. We say that N is a normal subgroup of H if for any $h \in H$ and $n \in N$ the element hnh^{-1} is in N.

Lemma A.0.6. *[Sha00, Lemma 4.4.3, p.160] Let $\mathfrak{n} \subset \mathfrak{h} \subset \mathfrak{g}$ be Lie algebras and $N \subset H$ Lie groups realizing the inclusion $\mathfrak{n} \subset \mathfrak{h}$. Assume N is normal in H, and let*

$$N' = \{h \in H \mid Ad(h)X - X \in \mathfrak{n} \textit{ for all } X \in \mathfrak{g}\}.$$

Then N' is also a normal subgroup of H.

Proof. First we need to show that N' is a group:

1. **Identity element:** Clearly $e \in N'$ since $Ad(e)X - X = 0 \in \mathfrak{n}$ for any $X \in \mathfrak{g}$.
2. **Inverse element:** Let $\alpha \in N'$, then $Ad(\alpha)X - X \in \mathfrak{n}$ for all $X \in \mathfrak{g}$ which implies $X - Ad(\alpha^{-1})X \in Ad(\alpha^{-1})\mathfrak{n}$. Since $\alpha^{-1} \in N \subset H$ and N is normal in H we know that $Ad(\alpha^{-1})\mathfrak{n} \subset \mathfrak{n}$. This means that $\alpha^{-1} \in N'$.
3. **Closure:** If $\alpha, \beta \in N'$ we get

   $$Ad(\alpha\beta)X - X = Ad(\alpha)(Ad(\beta)X - X) + (Ad(\alpha)X - X) \in Ad(\alpha)\mathfrak{n} + \mathfrak{n} = \mathfrak{n} \text{ for all } X \in \mathfrak{g},$$

   Hence $\alpha\beta \in N'$

Associativity follows automatically from associativity in H, hence N' is a group contained in H. We need to verify that N' is normal in H. Let $\alpha \in N'$ and $h \in H$, then

$$Ad(h^{-1}\alpha h)X - X = Ad(h^{-1})[Ad(\alpha)(Ad(h)X) - (Ad(h)X)] \in Ad(h^{-1})\mathfrak{n} = \mathfrak{n} \text{ for all } X \in \mathfrak{g}.$$

□

Lemma A.0.7. *[Sha00, Lemma 4.4.4, p.161] Let $\mathfrak{h} \subset \mathfrak{g}$ be Lie algebras and H a Lie group corresponding to $\mathfrak{h}$. Define a sequence of subgroups of H inductively by*

$$\begin{aligned} N_0 &= H, \\ N_1 &= \{h \in H \mid Ad(h)X - X \in \mathfrak{n}_0 \textit{ for all } X \in \mathfrak{g}\}, \\ N_2 &= \{h \in H \mid Ad(h)X - X \in \mathfrak{n}_1 \textit{ for all } X \in \mathfrak{g}\}, \\ &\dots \\ N_k &= \{h \in H \mid Ad(h)X - X \in \mathfrak{n}_{k-1} \textit{ for all } X \in \mathfrak{g}\}, \end{aligned}$$

then $N_0 \supset N_1 \supset N_2 \supset \ldots \supset N_k \supset \ldots$ are all Lie groups that are closed and normal in H and, after finitely many steps, the group stabilizes at a group N_∞ whose Lie algebra $\mathfrak{n}_\infty$ is an ideal in $\mathfrak{g}$ and satisfies

$$N_\infty = \{h \in H \mid Ad(h)X - X \in \mathfrak{n}_\infty \textit{ for all } X \in \mathfrak{g}\}.$$

Here $\mathfrak{n}_i$ is the Lie algebra of N_i.

Proof. By Lemma A.0.6 all the groups N_i are normal subgroups of H. Now, clearly $N_0 \supset N_1$. If we assume $N_i \supset N_{i+1}$, then clearly $\mathfrak{n}_i \supset \mathfrak{n}_{i+1}$. Let $n \in N_{i+2}$, then

$$\begin{aligned} & Ad(n)X - X \in \mathfrak{n}_{j+1} \text{ for all } X \in \mathfrak{g}, \\ \Rightarrow\ & Ad(n)X - X \in \mathfrak{n}_j \text{ for all } X \in \mathfrak{g}, \\ \Rightarrow\ & n \in N_{j+1}, \end{aligned}$$

Hence we have $N_{j+1} \supset N_{j+2}$. By induction we have $N_0 \supset N_1 \supset N_2 \supset \ldots \supset N_k \supset \ldots$. Now, set $N_\infty = \cap_i N_i$. Since N_i is normal in H for all i, if $n \in N_\infty$, then $n \in N_i$ for all i, hence for any $h \in H$ we have $hnh^{-1} \in N_i$ for all i which is the same as saying $hnh^{-1} \in N_\infty$, that is; N_∞ is normal in H.

Notice that if $\mathfrak{n}_i = \mathfrak{n}_{i+1}$ then $\mathfrak{n}_k = \mathfrak{n}_i$ for all $k \geq i$. This is clear from the way these Lie algebras are defined. Also notice that $\mathfrak{n}_i$ is a finite dimensional vector space, so the chain of proper containment can't be longer than the dimension of $\mathfrak{n}_0 = \mathfrak{h}$. Since N_i is defined by using $\mathfrak{n}_{i-1}$, the N_i's must stabilize whenever the $\mathfrak{n}_i$'s stabilize, which is after a finite number of steps. Thus N_∞ is a Lie group. Let k be the fist symbol such that $N_k = N_{k+1}$, then clearly $N_\infty = N_k = N_{k+1}$ and $\mathfrak{n}_\infty = \mathfrak{n}_k$, so since

$$N_{k+1} = \{h \in H \mid Ad(h)X - X \in \mathfrak{n}_k \text{ for all } X \in \mathfrak{g}\},$$

we get

$$N_\infty = \{h \in H \mid Ad(h)X - X \in \mathfrak{n}_\infty \text{ for all } X \in \mathfrak{g}\}.$$

Since N_∞ is a normal subgroup of H, its Lie algebra $\mathfrak{n}_\infty$ is an ideal in the Lie algebra $\mathfrak{h}$ of H. This is a known property of Lie groups that we will show in Lemma A.0.8 bellow. to see that $\mathfrak{n}_\infty$ is an ideal in $\mathfrak{g}$, notice that for any $b \in N_\infty$ we have $Ad(b)X - X \in \mathfrak{n}_\infty$ by the definition of N_∞. This gives $ad(B)X \in \mathfrak{n}_\infty$ by the differential map, where $B \in \mathfrak{n}_\infty$. Recall that $[B, X] = ad(B)X$, so that $\mathfrak{n}_\infty$ is an ideal in $\mathfrak{g}$.

□

Lemma A.0.8. *[Sha00, Exercise 3.4.7, p.112]*

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $H \subset G$ be a closed subgroup with Lie algebra $\mathfrak{h}$, then

(i) if H is normal in G, then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$,

(ii) if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and if H and G is connected, then H is normal in G,

(iii) if H is normal in G, then $(Ad(H) - I)\mathfrak{g} \subset \mathfrak{h}$.

Proof. (i) Let $Y \in \mathfrak{h}$ and $X \in \mathfrak{g}$, let $s, t \in \mathbb{R}$, and let $\sigma = \exp tX$. This gives

$$\sigma(\exp sY)\sigma^{-1} = \exp Ad_\sigma(sY) = \exp s[(\exp ad_{tX})(Y)].$$

Since H is normal, $\sigma(\exp sY)\sigma^{-1} \in H$, hence $(\exp ad_{tX})(Y) \in \mathfrak{h}$ for all $t \in \mathbb{R}$. Then we see that

$$\begin{aligned}(\exp ad_{tX})(Y) &= (\exp t(ad_X))(Y)\\ &= Y + t[X, Y] + \frac{t^2}{2!}[X, [X, Y]] + \cdots\end{aligned}$$

is a smooth curve in $\mathfrak{h}$ with tangent vector $[X, Y]$ at $t = 0$. Thus $[X, Y] \in \mathfrak{h}$ which means that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

(ii) Let $Y \in \mathfrak{h}$ and $X \in \mathfrak{g}$, and let $\sigma = \exp X$. We have

$$\begin{aligned}\sigma(\exp Y)\sigma^{-1} &= \exp Ad_\sigma(Y)\\ &= \exp((\exp ad_X)(Y))\\ &= \exp(Y + [X, Y] + \frac{(ad)^2X}{2!}(Y) + \cdots).\end{aligned}$$

Since $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, the series in the last term above converges to an element of $\mathfrak{h}$, hence $\sigma(\exp Y)\sigma^{-1} \in H$. This together with the fact that H is generated by elements of the form e^Y and G is generated by elements of the form e^X proves that H is a normal subgroup of G.

(iii) Let $h \in H$ and $X \in \mathfrak{g}$ such that $e^X = g \in G$. Then we get

$$\begin{aligned}\exp\left((Ad_h - I)X\right) &= \exp\left(hXh^{-1} - X\right)\\ &= \exp\left(hXh^{-1}\right)\exp\left(-X\right)\\ &= he^x h^{-1} e^{-X}\\ &= hgh^{-1}g^{-1}\\ &= hh' \in H\end{aligned}$$

which means that $(Ad(h) - I)X \in \mathfrak{h}$, and since h and X are arbitrary, this proves the statement. Here $h' = gh^{-1}g^{-1} \in H$ since H is normal in G.

□

Proposition A.0.9. *[Sha00, Prop 5.2.1, p.178] Let U support a Cartan geometry modeled on $(\mathfrak{g}, \mathfrak{h})$ with the Lie group H. Let K be the kernel and let θ_j, $j = 1, 2$, be two compatible Cartan gauges on U. Then $\theta_1 \Rightarrow \theta_2$ for a smooth function $k : U \to H$ that is unique up to multiplication with a smooth function $l : U \to K$. In particular, if the Cartan geometry is effective, then k is unique.*

Proof. By Lemma A.0.4 we know that if $\theta_1 \Rightarrow_{k_1} \theta_2$ and $\theta_1 \Rightarrow_{k_2} \theta_2$, then we have

$$\theta_1 \Rightarrow_{k_1 k_2^{-1}} \theta_1$$

so that it is enough to show that any $k : U \to H$ satisfying $\theta \Rightarrow_k \theta$ takes values in $K \subset H$. Recall from Lemma A.0.7 that we have a series of closed and normal subgroups of H

$$H = N_0 \supset N_1 \supset \ldots \supset N_n \supset \ldots$$

and that these groups stabilize after finitely many steps at some group N_∞. The Lie algebra $\mathfrak{n}_\infty$ of N_∞ is then an ideal in $\mathfrak{h}$. By Definition 2.4.1 the Klein pair $(\mathfrak{g}, \mathfrak{h})$ is effective, and by Lemma A.0.7 we know that $\mathfrak{n}_\infty$ is an ideal in $\mathfrak{g}$ that is contained in $\mathfrak{h}$. Thus we have $\mathfrak{n}_\infty = 0$, which means that N_∞ is discrete. We get

$$\begin{aligned}N_\infty &= \{h \in H \,|\, Ad(h)X - X = 0 \text{ for all } X \in \mathfrak{g}\}\\ &= \ker(Ad : H \to Aut(\mathfrak{g}))\\ &= K.\end{aligned}$$

We want to show by induction that k takes values in N_s for all $s \geq 0$. Since $N_0 = H$, it is clear that k takes values in N_0. Assume $k : U \to N_s$. Fix $u \in U$ and write

$$Ad(k^{-1})\theta - \theta = -\omega_H k_*$$

which is the structure equation of gauges form Definition 2.4.2. Let $X \in \theta(T_u U)$, then

$$Ad(k^{-1})X - X \in \text{image}((\omega_H k_*)_u) \subset \mathfrak{n}_s$$

by assumption. Since N_s is normal in H we get $Ad(k^{-1})Y - Y \in \mathfrak{n}_s$ for all $Y \in \mathfrak{h}$ by Lemma A.0.8 (iii). Recall that by teh definition of gauges $\theta(T_u U) \cong \mathfrak{g}/\mathfrak{h}$, so we can write $\mathfrak{g} = \mathfrak{h} \oplus \theta(T_u U)$, so we get

$$Ad(k(u)^{-1})Z - Z \in \mathfrak{n}_\infty$$

for all $Z \in \mathfrak{g}$. this means that $k(u) \in N_{s+1}$ by the definition of N_{s+1}. Since $u \in U$ was arbitrary here, we get $k : U \to N_{s+1}$, and by induction k takes values in $N_\infty = K$. □

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