Paper B

A local streamline Eulerian-Lagrangian method for two-phase flow *

A Local Streamline Eulerian-Lagrangian Method for Two-Phase Flow

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ABSTRACT: A method for solving the saturation equation for two-phase flow is presented. The method may be viewed as an operator splitting method or as an Eulerian-Lagrangian backtracking procedure or as a modified method of characteristics. For each time step, the method consists of an advection step and a diffusion step. The advection step requires the tracking of streamlines locally around certain integration points. On each streamline we have to solve a nonlinear hyperbolic equation. This is done using a front tracking method. The solution from the tracking step is projected back onto a fixed grid, and we finally have to solve a parabolic heat-type equation.

1 INTRODUCTION

The numerical solution of advective-diffusive transport problems arise in many important applications in science and engineering, e.g. oil reservoir flow, transport of solutes in ground water and surface water, the movement of aerosols and trace gases in the atmosphere, to mention a few examples. The difficulty of solving such problems, especially if advection dominates, have long been recognized as one of the more challenging tasks in scientific computing, see (Morton 1996) for an overview.

An important class of numerical schemes for solving such problems are the Eulerian-Lagrangian localized adjoint methods (ELLAM) (Celia et al. 1990; Herrera et al. 1993; Wang et al. 1999). These schemes have been successfully applied to linear transport problems of various types, and give a framework for devising Eulerian-Lagrangian type methods that are both mass conservative and able to handle boundary conditions in a fairly systematic way.

A more difficult problem is the solution of multiphase transport processes. This leads to nonlinear advection which greatly complicates the tracking part of the algorithm. Usually this difficulty has been overcome by some kind of linearization, e.g., (Dahle et al. 1995; Douglas et al. 1997). However, such linearizations may put artificial restrictions on the time steps that are not dictated by the physical processes investigated.

The tracking algorithm is a major part of any Eulerian-Lagrangian type method. Essentially, this part consists of tracing flow lines from certain integration points through a fixed spatial grid. Usually the velocity field is calculated as a part of the model, and the calculation of streamlines may then be done efficiently by analytical methods (Pollock 1988; Goode 1990; Datta-Gupta & King 1995; Lu 1994; Russell & Healy 2000), or less efficiently by ode-solvers like the Runge-Kutta methods. On the other hand, standard ode-solvers do not depend on any particular representation of the velocity field, and are a lot easier to implement.

A number of Eulerian-Lagrangian type methods have been suggested to solve the saturation equation for two-phase immiscible flow, e.g., (Dahle et al. 1990; Dahle et al. 1992; Dawson 1991; Douglas et al. 1997; Espedal & Karlsen 1999). In petroleum research, streamline/streamtube-methods have become very popular for solving this type of problems (Datta-Gupta & King 1995; Bratvedt et al. 1996; Hewett & Yamada 1997; King & Datta-Gupta 1998). The main difficulties encountered are the nonlinearity of the fractional flow (or flux) function f (typically s-shaped), which leads to self-sharpening fronts, and the possible degenerate nature of the small capillary-diffusion term.

Here we focus on the first difficulty and will present a scheme based on the following steps: *Lagrangian-step*: Calculate the streamlines locally around certain integration points on a fixed grid. Since the transport equation is nonlinear, the calculation of the foot of the streamline, i.e. the point where the streamline cross the previous time level, becomes a nonlinear hyper-

bolic equation. Thus, to advect the solution along the local streamlines we use front-tracking (Holden et al. 1988) as a fast hyperbolic solver. This step involves a projection of the solution from the previous time step, onto the streamline. *Eulerian-step*: The advected solution is projected back onto a fixed grid, and then used as initial condition for a heat-type equation. This parabolic equation is solved on a fixed grid, using a finite difference scheme.

In the following, we first introduce a simple model for two-phase flow. Then, a fairly detailed description of the numerical method is given, followed by some preliminary numerical results and conclusions.

2 MODEL

The problem to be considered here is the saturation equation for two-phase flow (Chavent & Jaffre 1986):

$$u_t + \nabla \cdot (f(u)\mathbf{v}) = \epsilon \nabla \cdot (D(u)\nabla u), \tag{1}$$

where $u \in [0,1]$ is the saturation of the wetting phase. For ease of presentation, we restrict the problem to a two-dimensional domain. Furthermore, we have neglected gravity in this model, since gravity gives rise to an additional advection term (Karlsen et al. 1998). The velocity field $\mathbf{v}$ is normally calculated as part of the model, involving Darcy's law and a compressibility condition. For the purpose of this work, we shall simply assume that $\mathbf{v}$ is a given vector-valued function which satisfy:

$$\nabla \cdot \mathbf{v} = 0. \tag{2}$$

The fractional flow (or flux) function f is given as the relative permeability of the wetting phase divided by the sum of the relative permeabilities. A simple analytic expression for this function is given by

$$f(u) = \frac{u^2}{u^2 + (1-u)^2}. \tag{3}$$

Note that $f' \geq 0$ for $u \in [0,1]$. The capillary diffusion coefficient $D(u)$ is generally a nonlinear (bell-shaped) function of u which becomes zero at the end points $u = 0, 1$. Thus, (1) is an example of a parabolic degenerate equation. This degeneracy add extra difficulties that will be avoided here by setting $D = 1$. Finally, the parameter ϵ determines the relative importance of advective and diffusive forces, and is small for advection dominated problems.

To close this model we need to specify an initial state given by

$$u(x, y, 0) = u_0(x, y), \tag{4}$$

and boundary conditions. Generally, boundary conditions add greatly to the complexity of any numerical scheme, and is one reason why the ELLAM-methodology was originally devised. However, since the main purpose of this work is to demonstrate a

tracking concept for nonlinear transport problems, we shall assume that the saturation u is defined on an infinite domain. The numerical solver will then be supplied with zero Dirichlet- or Neumann-conditions, whenever required.

3 NUMERICAL FORMULATION

Let the computational domain be discretized by a rectangular Cartesian grid, and assume that an approximate solution, U_{ij}^{n-1}, of equation (1), is given at each node $\mathbf{x}_{ij}$ on the grid and time t^{n-1}. The problem is to find a new approximation at time-level t^n, where $t^n - t^{n-1} = \Delta t$. To do so, observe that a directional derivative

$$\frac{d}{d\xi} = \mathbf{v} \cdot \nabla, \tag{5}$$

may be defined along streamlines

$$\frac{d\mathbf{r}}{d\xi} = \mathbf{v}. \tag{6}$$

It follows that the hyperbolic part of the problem will greatly simplify if we can construct a new orthogonal coordinate system based on streamlines and velocity equipotentials, see e.g. (King & Datta-Gupta 1998).

The approach taken here is to split equation (1) into a *hyperbolic part*:

$$u_t + \mathbf{v} \cdot \nabla f(u) \equiv u_t + f_\xi(u) = 0, \tag{7}$$

using (2) and (5), and a *parabolic part*:

$$u_t = \epsilon \nabla \cdot (D \nabla u). \tag{8}$$

The procedure is then first to advect the solution from one time level to the next by solving (7) with the solution at the previous time level as initial condition. Secondly, this solution is diffused by solving (8) to obtain a final solution at the new time level. This operator-splitting algorithm is analyzed by (Karlsen & Risebro 1997).

3.1 *Calculating Streamlines*

Let $\mathbf{x}_q$ denote fixed integration points. The placement and number of integration points are somewhat arbitrary, but is chosen to be the cell-centers in this work. We have to approximate streamlines backwards from $\mathbf{x}_q$ since $f' \geq 0$. An ode-solver denoted $RK(\cdot)$ is used to solve equation (6). Let $\bar{\mathbf{x}}_q = \bar{\mathbf{x}}_q(\xi)$ be the (approximate) streamline such that $\bar{\mathbf{x}}_q(0) = \mathbf{x}_q$. Furthermore, let $\lambda = \max_u |f'|$, and $\Delta \xi_q$ be the (largest) Runge-Kutta step. The streamline has to be traced for $\xi \in [-\lambda \Delta t, 0]$. The following quasi-algorithm describes the tracking:

Algorithm 1:

$\bar{\mathbf{x}}_q = \mathbf{x}_q$;
$\xi = 0$;
while $\xi > -\lambda\Delta t$;
    $\Delta\xi \leftarrow \Delta\xi_q$;
    $\mathbf{x} \leftarrow RK(\Delta\xi, \bar{\mathbf{x}}_q)$;
    while $\mathbf{x} \notin neigh(\bar{\mathbf{x}}_q)$;
        $\Delta\xi \leftarrow \Delta\xi/2$;
        $\mathbf{x} \leftarrow RK(\Delta\xi, \bar{\mathbf{x}})$;
    end while;
    $\xi \leftarrow \xi - \Delta\xi$;
    $\bar{\mathbf{x}}_q \leftarrow \mathbf{x}$;
end while;

The second while-loop is introduced to avoid that a streamline is tracked more than one grid-cell in each Runge-Kutta step. Here, $neigh(\mathbf{x})$ denote the grid-cell to which $\mathbf{x}$ belongs, and the eight neighbour cells to this grid-cell.

Algorithm 1 produces points $\bar{\mathbf{x}}_q^k = \bar{\mathbf{x}}_q(\xi_k) = (x_q^k, y_q^k)$, $k = 1, 2, \ldots, K$, on the streamline. The entire streamline is constructed by introducing straight lines between such points. Thus,

$$\bar{\mathbf{x}}_q(\xi) = \frac{1}{\Delta\xi_k}\left(\bar{\mathbf{x}}_q^{k-1}(\xi_k - \xi) - \bar{\mathbf{x}}_q^k(\xi - \xi_{k-1})\right),$$

for $\xi_k \leq \xi \leq \xi_{k-1}$. It is now easy to determine points ξ_c^l, $l = 1, 2, \ldots, L$, where the streamline crosses grid-lines. Assume that $x_q^{k-1} \leq x_i \leq x_q^k$, where x_i is the i-th grid-line orthogonal to the x-axis, then

$$\xi^c = \frac{\Delta\xi_k}{\Delta x_k}x_i + x_q^k\xi_{k-1} - x_q^{k-1}\xi_k.$$

The points where a streamline crosses grid-lines orthogonal to the y-axis, are found by a similar calculation. The ordered set of all such points on a given streamline, $\{\xi_c^l\}$, is of course generated as part of Algorithm 1.

3.2 *Advection*
We wish to determine the solution of (7) at the integration points $\mathbf{x}_q$ and time-level t^n, given the approximate solution U^{n-1}.

Equation (7) is a one-dimensional hyperbolic conservation law. The basic waves of the scalar equation are rarefaction waves and shocks. There exist numerous efficient solver for such problems, see for example (Toro 1999). Here, we will use a front tracking approach. The front tracking method was first presented in (Dafermos 1972), and later developed into a numerical method by (Holden et al. 1988).

The front tracking method requires that the data is replaced by piecewise constants along the streamline. Here, we choose to place the discontinuities at the crossing points $\{\xi_c^l\}$. Then the initial data v_0 can be computed by

$$v_0(\xi) = \frac{1}{\Delta\xi^l}\int_{\xi_c^l}^{\xi_c^{l-1}} U^{n-1}(\xi)d\xi, \tag{9}$$

for $\xi_c^l < \xi < \xi_c^{l-1}$. To simplify the last calculation, U^{n-1} is replaced by some cell average $\tilde{U}^{n-1}$ on each grid-cell, so that $v_0(\xi) = \tilde{U}^n$ on the segment of the streamline contained in the respective grid-cell. Note that the way the streamlines are defined, states to the left of ξ_q^K and right of $\xi_q^0 = 0$ do not interact with the solution at $\mathbf{x}_q$ for $t \in [t^{n-1}, t^n]$. These states are therefore replaced by the values at the endpoints.

Front tracking proceeds by calculating the position of the discontinuities (fronts) at time t^n. This also imply the solution values at integration points $\mathbf{x}_q$ to be used in the diffusion step.

3.3 *Diffusion*
The solution of Equation (8), given the solution of (7) at the integration points, are open for many choices of discretization. Based on the ELLAM-methodology finite-element or finite-volume type techniques are natural choices. Here, a simple second order accurate explicit central-difference scheme have been used, since the main purpose of this work is to test the tracking concepts. The initial data $\bar{U}_{ij}^{n-1}$ of (8) at nodes $\mathbf{x}_{ij}$ is taken to be

$$\bar{U}_{ij}^{n-1} = \frac{1}{4}\sum_{k=1}^{4}\bar{U}^{n-1}(\mathbf{x}_{q_k}),$$

where $\mathbf{x}_{q_k}$, $k = 1, \ldots, 4$, are the integration points of the four cells that have $\mathbf{x}_{ij}$ as a common vertex.

Note that when solving (8) using an explicit difference scheme, there is a stability constraint on the time step. Consequently, we may have to take many substeps to propagate the solution of (8) from time t^{n-1} to t^n.
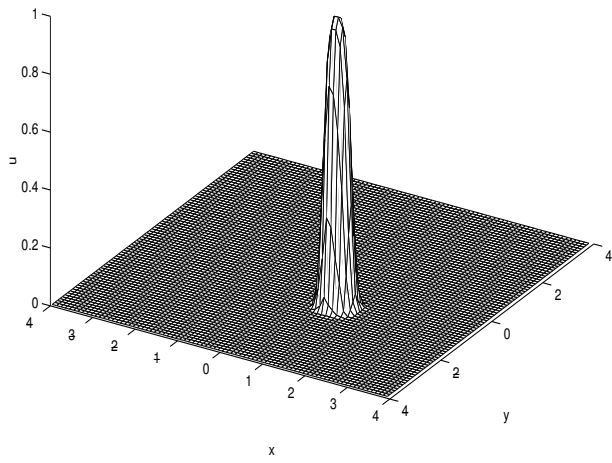
4 NUMERICAL EXAMPLES
Some preliminary experiments are performed with a rotating velocity field
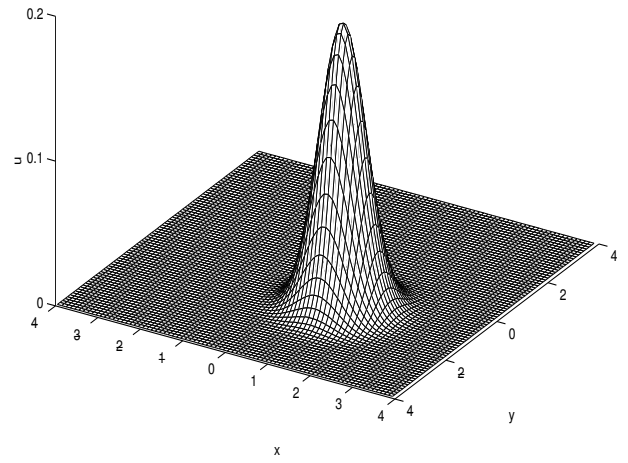
$$\mathbf{v}(\mathbf{x}) = 2\pi[y, -x].$$

The initial condition (4) is chosen to be the cylindrical profile:

$$u_0(x) = \begin{cases} 0, & (x-1)^2 + (y+1)^2 > 0.3^2, \\ 1, & (x-1)^2 + (y+1)^2 \leq 0.3^2. \end{cases}$$
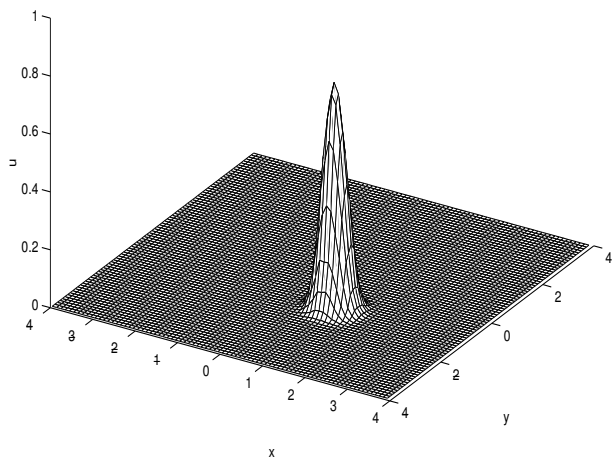
The computational domain is set to be the square $[-4, 4] \times [-4, 4]$. This domain is discretized using a uniform grid of 160 grid-cells in each direction ($\Delta x = \Delta y = 0.05$).
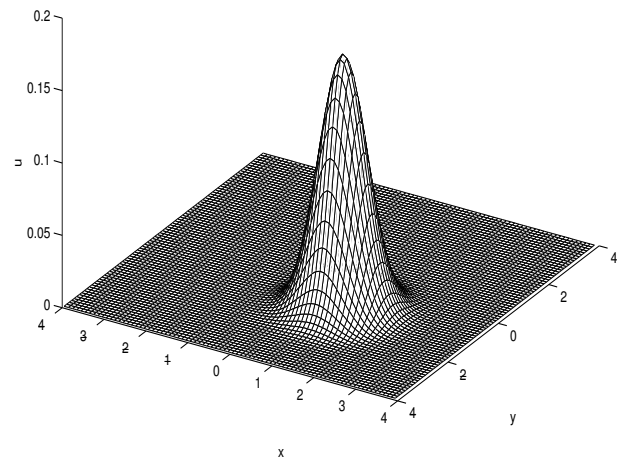
3

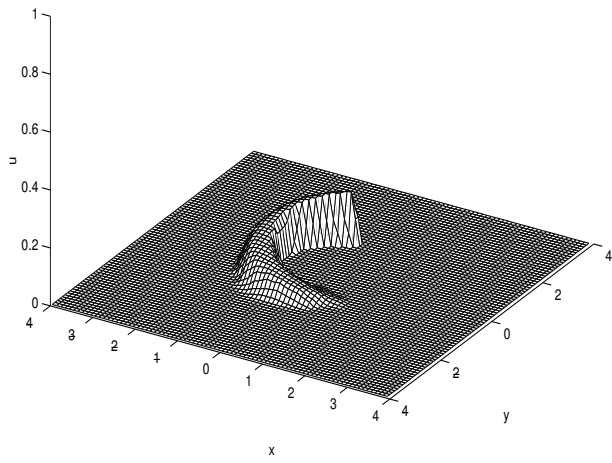(a) $\Delta t = 0.2$
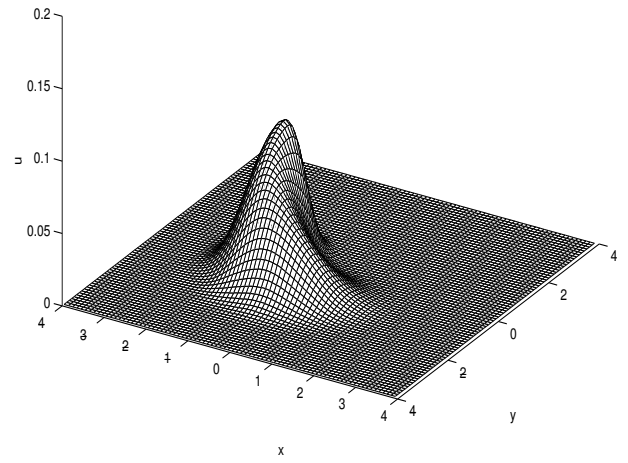
(a) $\Delta t = 0.2$

(b) $\Delta t = 0.05$

(b) $\Delta t = 0.05$

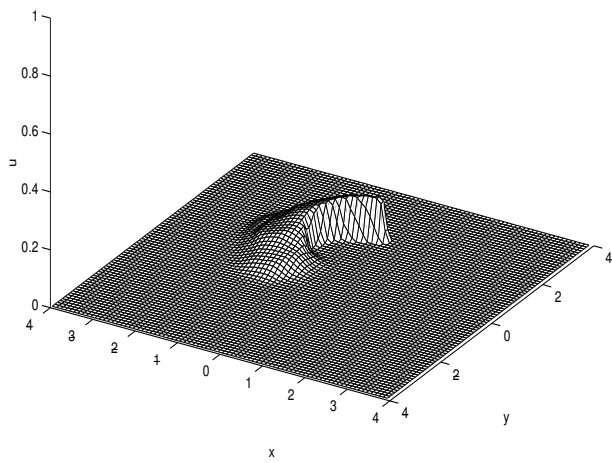Figure 1: Solutions with linear flux, $\epsilon = 0$.

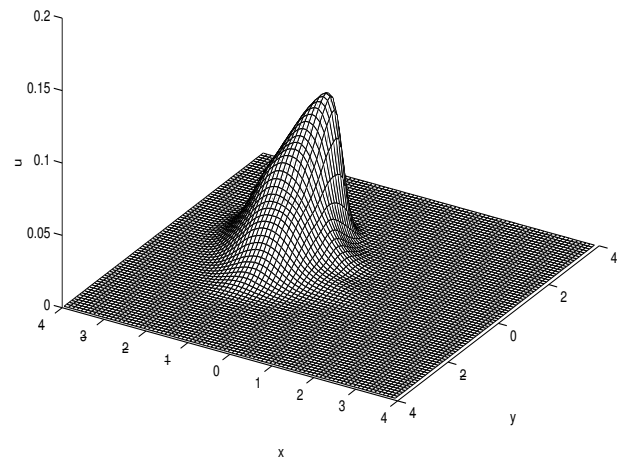Figure 2: Solutions with linear flux, $\epsilon = 0.1$.

(a) $\Delta t = 0.2$, $\epsilon = 0$

(a) $\Delta t = 0.2$, $\epsilon = 0.1$

(b) $\Delta t = 0.05$, $\epsilon = 0$

(b) $\Delta t = 0.05$, $\epsilon = 0.1$

Figure 3: Solutions with nonlinear flux, $\epsilon = 0$.

Figure 4: Solutions with nonlinear flux, $\epsilon = 0.1$.

5

We have run two sets of experiments. The first with linear advection, $f(u) \equiv 1$, and the second with a nonlinear flux given by Equation (3). For each experiment the final time is $T = 1$ (one rotation in the linear case), and we have varied the time step as $\Delta t = 0.05, 0.2$. The diffusion parameter is set to be $\epsilon = 0$ (no diffusion) and $\epsilon = 0.1$.

The experiments show that some numerical diffusion is introduced by the projection step, see Figure 1 and Figure 3. In particular, we see that the nonlinear hyperbolic waves are slightly distorted by the number of projection steps, see Figure 3. However, the front-tracking method is itself diffusion free, and reproduces the waves correctly.

The introduction of a diffusion term, seems to alleviate the effect of the projection steps, see Figure 2 and Figure 4 and note that the u-axis is scaled differently form Figures 1 and 3. In fact, the combination of front tracking and local streamlines seems to advect the solution accurately and without grid orientation effects.

5 CONCLUSIONS

The experiments reported here are in agreement with what should be expected: The calculation of local streamlines leads to negligible grid orientation effects and allow us to use fast hyperbolic solvers. Some numerical diffusion is introduced by the projection steps, whereas the front tracker introduce no numerical diffusion.

The method seems to be fairly flexible. However, before any definite conclusions can be drawn, more extensive experiments have to be performed. In particular, semianalytical tracking methods should be implemented, and different projection strategies should be investigated. Comparisons with other methods and/or analytical results must also be done.

Work is in progress on the following modifications/extensions of the method: Most important, a finite-element or finite-volume type method will be implemented for the parabolic step. This should allow us to use concepts from the ELLAM-methodology and enable the method to be tested on more realistic problems. Furthermore, the calculation of streamlines using ode-solvers seems to be computationally very time consuming and will be replaced by semianalytical methods. Finally, splitting errors caused by the nonlinearity of the flux will appear when too large time-steps are taken, see (Karlsen & Risebro ; Karlsen et al. 1998; Brusdal et al. 1998). The use of a front tracker allow us to construct correction terms that may compensate for splitting errors.

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