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Computational Science in the eighteenth Century Test Cases for the methods of Newton, Raphson and Halley: 1685 to 1745

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Abstract This is an overview of examples and problems posed in the late 1600s up to the mid 1700s for the purpose of testing or explaining the two different implementations of the Newton-Raphson method, Newton's method as described by Wallis in 1685, Raphson's method from 1690 and Halley's method from 1694 for solving nonlinear equations. It is demonstrated that already in 1745, it was shown that the methods of Newton and Raphson were the same but implemented in different ways.

Keywords Newton-Raphson iteration · Nonlinear equations.

1 Introduction

We consider finding a root of the nonlinear function $f : \mathbb{R} \mapsto \mathbb{R}$. The methods we consider are iterative and generate a sequence of iterates that terminates after a finite number of steps to reach a certain accuracy. The Newton-Raphson method for finding a root as we know it today generates a sequence of iterates from an initial x_0 as follows

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1 \ldots . \tag{1}$$

Newton's method is first published in print in 1685 [27, Chapter 94]. The method appears in the manuscript *De analysi per æquationes numero terminorum infinitas* (in the following *De analysi* for brevity) by Isaac Newton around 1669 but for the historical background, it is only *circumstantial evidence* [33, pg. 165]. However, it was printed in 1711 in an edition edited by William Jones [15, pp.8–10]. It is in the commentary section of the translation

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to English of this work that Stewart [25] in 1745 observes that the methods of Newton and Raphson are the same. A minor corrected version of the method appears in the manuscript *De methodis fluxionum et serierum infinitarum* (in the following *De methodis* for brevity) written in 1671 by Isaac Newton and translated and commented by Colson in 1736 [16]. Based on an error in the last iteration presented in *De analysi* but corrected in *De methodis* and the material *De methodis* and in [27, Chapter 94] it is reasonable to assume that Wallis based his printed presentation in 1685 on a transcript of the manuscript from 1671.

Today we may regard the difference between Newton's method as explained in Wallis Algebra from 1685 [27, Chapter 94] and Raphson's method [18] from 1690 as two different implementations of the same method. However, it took over 100 years before it was generally accepted that these two methods are the same. Cajori [2, pg.32] writes

> Nearly all eighteenth century writers and most of the early writers of the nineteenth century carefully discriminated between the method of Newton and that of Raphson.

In the following, m is the number of iterations and x_0 is the starting point for the iterations.

Newton

Let $g_0(s) = f(x_0 + s)$
For $k = 0, 1, 2, \ldots, m-1$
  Compute $\tilde{g}_k(s) \approx g_k(s)$
  Solve for s_k in $\tilde{g}_k(s) = 0$
  Let $g_{k+1}(s) = g_k(s_k + s)$

$x \approx x_0 + \sum_{k=0}^{m-1} s_k$

Raphson and Halley

For $k = 0, 1, 2, \ldots, m-1$
  Compute $\tilde{g}_k(s) \approx f(x_k + s)$
  Solve for s_k in $\tilde{g}_k(s) = 0$
  Let $x_{k+1} = x_k + s_k$

$x \approx x_m$

Newton annd Raphson consider linear approximations $\tilde{g}_k(s) = g(x_k) + g'(x_k)s$, and $\tilde{g}_k(s) = f(x_k) + f'(x_k)s$ while Halley considers a quadratic approximation

$$\tilde{g}_k(s) = f(x_k) + f'(x_k)s + \frac{1}{2}f''(x_k)s^2.$$

Solving the quadratic equation $\tilde{g}_k(s) = 0$ will give what is called Halley's irrational method or using the approximation

$$\sqrt{a^2 - b} \approx a - \frac{ab}{2a^2 - \frac{1}{2}b}$$

where a and b are real numbers giving Halley's rational method.

Method	Update s_k
Raphson	$s_k = -\frac{f(x_k)}{f'(x_k)}$
Halley Rational	$s_k = -\frac{f'(x_k)f(x_k)}{f'(x_k)^2 - \frac{1}{2}f''(x_k)f(x_k)}$
Halley Irrational	$s_k = -\frac{f'(x_k) - \sqrt{f'(x_k)^2 - 2f(x_k)f''(x_k)}}{f''(x_k)}$

Consider the function $f(x) = x^3 - 2x - 5$ used by Newton in the manuscripts *De analysi*, *De methodis* and, by Wallis [27, Ch.94, pg. 338] to illustrate Newton's method. Figure 1 shows the equation in Newton's handwriting in the manuscript *De methodis*[1]. The starting point is $x_0 = 2$ and the number of

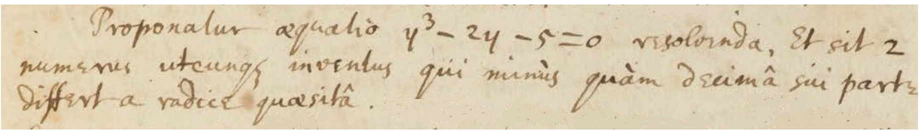

Fig. 1 *Proponatur æquatio* $y^3 - 2y - 5 = 0$ *resolvenda, Et sit 2 numerus utcunque inventus qui minùs quàm decimâ sui parte differt a radice quæsitâ.*
Let the equation $y^3 - 2y - 5 = 0$ be proposed for solution and let the number 2 be found, one way or other, which differs from the required root by less than its tenth part (translation in [33, pg.43]).

iterations is $m = 3$. Using the notation above we have:

- $g_0(s) = f(2+s) = s^3 + 6s^2 + 10s - 1$ and $s_0 = \frac{1}{10}$.
- $g_1(s) = g_0(\frac{1}{10} + s) = s^3 + \frac{63}{10}s^2 + \frac{1123}{100}s + \frac{61}{1000}$ and $s_1 = -\frac{61}{11230}$.
- $g_2(s) = g_1(-\frac{61}{11230} + s) = s^3 + \frac{35\ 283}{5\ 615}s^2 + \frac{351\ 906\ 913}{31\ 528\ 225}s + \frac{32\ 878\ 756}{177\ 030\ 983\ 375}$,

  and $s_2 = -\frac{32\ 878\ 756}{1\ 975\ 957\ 316\ 495}$.
- $x \approx 2 + \frac{1}{10} - \frac{61}{11230} - \frac{32\ 878\ 756}{1\ 975\ 957\ 316\ 495} = \frac{4\ 138\ 744\ 325\ 037}{1\ 975\ 957\ 316\ 495}$.

Raphson's method ($x_0 = 2$):

- $s_0 = -\frac{f(2)}{f'(2)} = \frac{-1}{10}$ and $x_1 = \frac{21}{10}$.
- $s_1(s) = -\frac{f(\frac{21}{10})}{f'(\frac{21}{10})} = -\frac{\frac{61}{1000}}{\frac{1123}{100}}$ and $x_2 = \frac{21}{10} - \frac{61}{11230} = \frac{11761}{5615}$.
- $s_2 = -\frac{f(\frac{11761}{5615})}{f'(\frac{11761}{5615})} = -\frac{\frac{32\ 878\ 756}{177\ 030\ 983\ 375}}{\frac{351\ 906\ 913}{31\ 528\ 225}} = -\frac{32\ 878\ 756}{1\ 975\ 957\ 316\ 495}$.
- $x \approx x_3 = x_2 + s_2 = \frac{4\ 138\ 744\ 325\ 037}{1\ 975\ 957\ 316\ 495}$.

Thus giving the sequence $2, 2.1, 2.094568121104185, 2.094551481698199$ and the error in the last iterate is $1.6\ 10^{-10}$. To simplify the computation Wallis is truncating 61/11230 to 0.0054. This makes computing s_2 by hand easier. The sequence of corrections or updates is then 0.1, -0.0054 and -0.00004852 with final iterate $x = 2.09455148$ where all digits are correct. Raphson is generating the sequence of iterates 2.1, 2.0946, 2.094551483, and 2.0945514815427104141

[1] MS Add. 3960.14, Cambridge University Library, Cambridge, UK.

with an error of 3.8 10^{-13} in the final iterate. The only real root of the equation is 2.094551481542327 (correctly rounded to 15 decimal digits).

Halley's rational formula and two iterations generate the sequence 2, 111/53, and 1 090 082 546 191/52 079 190 773 or 2, 2.094339622641509, and 2.094551481540164 with an error 2.2 10^{-12} in the final iterate. To see this note that

- $s_0 = -\frac{f'(2)f(2)}{f'(2)^2 - \frac{1}{2}f''(2)f(2)} = \frac{5}{53}$ and $x_1 = \frac{111}{53}$,
- and using that $f(\frac{111}{53}) = \frac{2^5 \cdot 11}{53^3}$, $f'(\frac{111}{53}) = \frac{5 \cdot 6269}{53^2}$ and $f''(\frac{111}{53}) = \frac{2 \cdot 3^2 \cdot 37}{53}$, then $s_1 = \frac{11033440}{52079190773}$ and x_2 is given as above.

2 Test examples

2.1 Test examples of Wallis 1685

Wallis [27, Chapter 94] demonstrates Newton's method using the function $x^3 - 2x - 5$ with starting point $x_0 = 2$ and using 3 iterations.

Name	$f(x)$
Ch.94, pg. 338	$x^3 - 2x - 5$
Ch.62, pg. 231	$-x^4 + 80x^3 - 1998x^2 + 14937x - 5000$

The two problems are also used by Raphson in 1690 [18] and the second problem is used by Halley [6] and presented in [32, Appendix II,pg.214] and in *Livre VI. De l'approximation des équations numeriques* by Reyneau in 1708 [20, pg.335].

2.2 The test examples of Raphson 1690

Raphson in 1690 [18] and in the second edition from 1697 [19] illustrates his method on more than 30 examples. The problems are all polynomials and most have integer coefficients. For the first eight problems, the starting point x_0 of the iteration and the number of iterations are included in the table. Most of the examples used by Raphson in 1690 and 1697 had not appeared in the literature, but what is evident is that many authors have used Raphson's examples with or without references.

Name	$f(x)$	x_0	Iterations
Problem I	$x^2 - 2$	1	4
Problem II	$x^3 - 37945$	33	3
Problem III	$x^4 - 2741583974$	229	2
Problem IV	$x^5 - 2327834559873$	298	1
Problem V	$x^2 + 587x - 987459$	746	2
Problem VI	$x^2 - 5x - 31$	8	3
Problem VII	$-x^2 + 8x - 14$	2	4
Problem VIII	$x^3 + 24x - 587914$	83	2

The Newton-Raphson method (1) is invariant under scaling. However, with the rounding or truncation in the computation the method depends on the scaling. After performing the scaling, Raphson is truncating the coefficients to integers. When doing hand calculation this saves arithmetic operations.

Specific comments on the problems:

- Problem I: Ward [29, Ch.VIII,pg.42] uses three iterations with the same starting point but writes

  > ...if more accuracy be required, it may be called a new g (i.e., new iterate), for a fourth operation; and by repeating the operations, you may have as many places in the root as you please.

- Problem II: Observe $37945 \approx 37 \cdot 10^3$. Rescale and do one iteration using (1) with the function $x^3 - 37$ and starting point 3 which gives $3 - (27 - 37)/27 \approx 3 + 0.3$. The new starting point for the function in Problem II is then $(3 + 0.3) \cdot 10 = 33$. Ward [29] uses the same problem and the same starting point but terminates the iteration (1) after 2 iterations.
- Problem III: As for Problem II observe $2741583974 \approx 274158 \cdot 10^4 \approx (27 \cdot 10^4) \cdot 10^4$. Rescale and do one iteration using (1) using the function $x^4 - 27$ and starting point 2 which gives $2 - (16 - 27)/32 \approx 2.3$. Now consider the function $x^4 - 274158$ with the starting point $2.3 \cdot 10^1$ which gives

  $$23 - (23^4 - 274158)/4 \cdot 23^3 = 23 - 5683/48668 \approx 23 - 0.1.$$

  The starting point for the function in Problem III is then $22.9 \cdot 10^1$.
- Problem IV: As for Problems II and III, observe $2327834559873 \approx 232 \cdot 10^{10}$. Rescale and do one iteration with (1) using the function $x^5 - 232$ and starting point 3 which gives $3 - (243 - 232)/405 \approx 2.98$. Starting point for the function in Problem IV is then $2.98 \cdot 10^2$.
- Problem V: To find a starting point Raphson replaces the function in Problem V first with $x^2 + 5x - 98$ and then with $x^2 + 58x - 9874$. For the first function the variable $x \leftarrow x/100$ and the coefficients are truncated to integers. For the second function $x \leftarrow x/10$. Do one iteration with (1) using the function $x^2 + 5x - 98$ and 8 as initial value. This gives $8 - 6/21 \approx 7.8$. Now use 78 as initial value using the second function and (1). This gives

$78-(734/214) \approx 74.6$. The starting point for the function in problem V is then 746. We find this problem in Parsons and Wastell [17, Book 2, Ch.21] and Sault [22, Example I,pg.49].
- Problem VI: The first step $s_0 = \frac{7}{11}$ is truncated to 0.6 which simplifies the hand calculation. This problem is used with (1) in Wolff 1713 [34, Problema 163 pg. 359] and Chambers 1728 [3, Approximation pg. 123] with the same starting point and number of iterations. It is interesting to note that Hutton in his dictionary from 1795 [9, Approximation pg.132] demonstrates both Newton's way to compute the corrections and Raphson's way to compute the iterates on this problem.
- Problem VII: The second step $s_1 = \frac{1}{12}$ is truncated to 0.083.
- Problem VIII: To find a starting point, Raphson replaces the variable $x \leftarrow x/10$ and scales the function so that the leading term is 1. The constants are rounded down to nearest integer which gives the function $x^3 - 587$. Do one iteration with initial value 8 which gives $8+75/192 \approx 8.3$. The starting point is then taken to be 83. We find this problem in Wells [32, Appendix II, pg.213], and [30, pg.238]

The next tables contains the remaining test examples and we see that the only test example where not all coefficients are integer is problem 22.

Name	$f(x)$
Problem IX	$x^3 - 2x - 5$
Problem X	$x^3 + 6272x - 288512$
Problem XI	$x^3 - 16x - 444$
Problem XII	$x^3 - 50x - 120$
Problem XIII	$-x^3 + 77284x - 8083128$
Problem XIV	$-x^3 + 300x - 1000$
Problem XV	$-x^3 + 9x^2 - 100$
Problem XVI	$-x^3 + 9x^2 - 100$

Specific comments on the problems:

- Problem IX: This is the classical problem where Newton's method is applied. It is used by Wallis [27] using the same starting point but with three iterations while Raphson uses four. The problem is deleted in the second edition from 1697 [19]. We find this problem also in Sault 1694, Wells 1698, Colson 1736, and Stewart 1745 [22,32,16,25].
- Problem X: Raphson is scaling and truncating the problem $x^3+62x-288 = 0$. This gives the starting point for the unscaled equation to be 38 and using three iterations. We find this problem also in Ward 1695 and 1707 [29, pg.66], [30, Problem 15, pg.325], Holliday 1745 [8, Example 10, pg.103].
- Problem XI: Starting point is 8 and three iterations.
- Problem XII: Starting point is 8 and three iterations. This problem is also Problem V in Ward 1695 [29, pg.68] and Problem 16 in [30, pg. 325].

The problem is to determine the diameter of a circle that circumscribes a trapezium with the three given sides a, b, and c and the diameter x is the fourth. Raphson states the equation $x^3 - (a^2 + b^2 + c^2)x - 2abc = 0$. This equation is derived in [29,30] and by Newton [14, pg.108] but without any numerical example. A discussion of Newton's [14] derivation is found in [13].

- Problem XIII: Starting point is 200 and four iterations. The problem is also found in [32, Appendix II, pg.213]. Bailey in 1989 [1] used this test example to illustrate (1) using the same starting point and iterations.
- Problem XIV: Starting point is 3 and four iterations. We find this problem also in Simpson 1740 [23, Example 1, pg.83] and in 1745 [24, pp.149-150].
- Problem XV: Starting point is 3 and five iterations.

Problem XVII	$x^3 + 74x^2 + 8729x - 560783$
Problem XVIII	$x^3 - 65x^2 + 914x - 98746$
Problem XIX	$x^4 + 6808x^2 + 672792x - 43507216$
Problem XX	$-8072x^4 + 501636x^3 - 9856921x^2 + 141873228x - 1096735689$
Problem XXI	$-x^4 + 80x^3 - 1998x^2 + 14937x - 5000$
Problem XXII	$-x^4 + 323609.663689x^2 + 4228931.085087852x + 22540483202.613561987516$
Problem XXIII	$x^4 + 40x^3 + 751x^2 - 9000x - 90000$
Problem XXIV	$-x^5 + 7x^4 - 20x^3 + 155x^2 - 10000$
Problem XXV	$x^5 - 5x^3 + 5x - 1$
Problem XXVI	$x^6 - 5x^3 + 5x - 1.5$
Problem XXVII	$-x^7 + 7x^5 - 14x^3 + 7x - 1.5$
Problem XXVIII	$x^2 + 5x - 646$
Problem XXIX	$-x^3 + 1000x - 174$
Problem XXX	$x^2 + 5x - 646$
Problem XXXI	$x^3 - 430x - 231$
Problem XXXII	$x^4 - 5x^2 + 7x - 291$

- Problem XVII is used in [22, Example III] and in [30, pg.242].
- Problem XX is used in [29, pg.101]
- Problem XXI is from Wallis Algebra [27, Chap.62,pp.231] and used in by Halley [6, Example III pg. 146]. The problem is also presented in [32, Appendix II, pg.213].
- Problem XXIII is also used by Holliday [8, Example 18, pg. 110].
- In Problem XXVIII Raphson [18] computes the correction and truncates and adds the correction to the iterate for each iteration. This leads to the sequence of iterates $1, 92, 49, 30, 24, 23.1, 23.04$.

- Problem XXIX: Two iterations starting from 1.
- Problem XXX is equal to Problem XXVIII. For this problem Raphson computes the iterate directly and then truncate. This leads to the sequence of iterates $1, 92, 48, 29, 24, 23, 23.04$.

2.3 The test example of Wallis 1693

In 1693 came a Latin translation [28] of Wallis [27]. It contains some additional material. In the section *Methodus D. Josephi Raphson* [28, Ch.95,pp.396–397] is Raphson's method demonstrated on the example $-x^4 + 56x^3 - 1680x^2 + 20160x - 40320 = 0$ with the starting point 2 and using three iterations.

2.4 The test examples of Halley 1694

Halley [6] derives the two methods and illustrates the computation on three examples. These three examples are used by Harris in 1710 in the entry *Roots* in *Lexicon Technicum* [7] with the same starting point and number of iterations.

Name	$f(x)$
Example I pg.144	$x^4 - 3x + 75x - 10000$
Example II pg.145	$x^3 - 17x^2 + 54x - 350$
Example III pg.146	$-x^4 + 80x^3 - 1998x^2 + 14937x - 5000$

Specific comments on the problems:

- Example I is solved using the irrational method [6] with the starting point 10 and 2 iterations. The example is used in Harris [7, Roots] with Halley's irrational method. The example is later used by Holliday in 1745 [8, Example 20 pg.112].
- Example II is solved using starting point 10 and 2 iterations with the irrational formula.
- Example III is Problem XXI in Raphson [18] and in Wallis [27, Ch.62]. The equation is scaled $x \leftarrow \frac{1}{10}x$ and the truncated equation is $x^4 - 8x^3 + 20x^2 - 15x + \frac{1}{2} = 0$. One iteration using the irrational formula with starting point 1. New starting point for the original function is then $1.27 \cdot 10$. For this starting point both the rational and irrational formula are used. This problem is also used by Ronayne [21, Book 1, Part XV] in 1717. Harris [7, Roots] shows that *the rule of pointing* (the point above numbers) is just a scaling of x, $x \leftarrow \frac{1}{10}x$.

2.5 The test examples of Raphson 1697

Second edition of Raphson *Analysis Equationum Universalis* is from 1697 [19] and contains four new problems and two problems were deleted. The edition from 1702 is merely a reprint of the edition from 1697. The first eight problems are unchanged. Problems XXII to XXVII are the same but in a different order.

Name	$f(x)$
Problem XX(1697)	$-x^4 + 5.36165x^3 - 202.186724x^2 + 107.233013x - 8.98350657$
Problem XXI(1697)	$x^4 + 28x^3 + 200.163879x^2 - 12.5083613x - 184.381270981$
Problem XXVIII(1697)	$x^9 - 9x^7 + 27x^5 - 30x^3 + 9x - c$
Problem XXIX(1697)	$x^{365} - 1.06$

Raphson writes that the new Problem XX and XXI are from Ward. The constant term in Problem XXVIII(1697) [19] is $c = 1.568525312$. Except for a different constant term, this problem is the same as used by Ward in 1695 [29, pg.81)]. Problem XXIX(1697) is used by Ward in 1707 [30, pg.260].

2.6 The test examples from Ward 1695

Ward [29] considers two examples in Chapter VIII *An abstract of Mr. Raphson's method of converging series* from 1695. These examples are two first problems considered by Raphson in 1690.

Name	$f(x)$
Example 1 (pg.42)	$x^2 - 2$
Example 2 (pg.43)	$x^3 - 37945$
Problem I (pg.60)	$x^2 + 34x - 2304$
Problem II (pg.62)	$x^2 - 40x - 196$
Problem III (pg.64)	$-x^2 + 42.93x - 418.8444$
Problem IV (pg.66)	$x^3 + 6272x - 288512$
Problem V (pg.68)	$x^3 - 50x - 120$
Problem VI (pg.70)	$-x^3 + 3x - 1$
Problem VII (pg.74)	$-x^4 + 4x^2 - 1$
Problem VIII (pg.76)	$x^5 - 5x^3 + 5x - 1$
Problem IX (pg.79)	$-x^7 + 7x^5 - 14x^3 + 7x - 1$
Problem X (pg.81)	$x^9 - 9x^7 + 27x^5 - 30x^3 + 9x - c$
Example (pg.96)	$x^{8388608} - 2$
Example (pg.99)	$x^3 + 438x^2 - 7825x - 98508430$
Problem XII (pg.101)	$-2018x^4 + 125409x^3 - 2464230, 25x^2 + 35468307x - 274183922, 25$

Specific comments on the problems:

- For Example 1 Ward [29] uses the method described by Raphson [18] for computing the square root of c and derives the iteration formula $x_{k+1} = \frac{c-x_k^2}{2x_k}$. Starting point is 1 and 3 iterations.
- Similarly, for Example 2, the iteration formula for the cube root of c is found $x_{k+1} = \frac{c-x_k^3}{3x_k^2}$. Ward [29, Ch.VIII,pg.43] formalizes the method of finding the starting point of the iteration by a punctation notation, points above the digits to indicate the scale. To find the first approximate digit of the cube root of 37945 the number will be written as $3\dot{7}94\dot{5}$ and now do one iteration using $x^3 - 37$ with starting point 3. Ward [29] terminates the iteration (1) after 2 iterations.
- Problem IV (pg.66) is Problem X in [18].
- Problem V (pg.68) is also used by Holliday [8, Example 4 pg.98].
- Problem VIII is Problem XXV in Raphson [18].
- Problem IX: In Problem XXVII in Raphson [18] the constant is 1.5, i.e. the function is $-x^7 + 7x^5 - 14x^3 + 7x - 1.5$.
- Problem X: Ward considers $c = 1$ and $c = 0.116289652$. In Problem XXVIII(1697) in Raphson [19], the constant term is $c = 1.568525312$.
- Example pg. 96. $8388608 = 2^{23}$ the 23rd term in the geometric progression.
- Example pg.99. This example is used by Wolf in 1713 [34, §328 pg.361] using Halley rational method.
- Problem XII: The coefficient for x^3 should be 125409 and not 125609 as stated in [29] and this problem is the same as Problem XX [18]. Ward solves $-x^4 + 62.1x^3 - 1221.12x^2 + 17575.969x - 135869.1388 = 0$.

It is only Example 1 and 2 where Ward uses Raphson's method. The other examples and problems are illustrated using a hybrid approach combining the technique "digit-by-digit" and a Raphson/Halley like variant. The test examples are frequently used later.

2.7 The test examples of Sault 1694

Richard Sault in *Of Infinite Approximations, or a Numeral Converging Series for all Adfected Equations whatever* [22] uses six of Raphson's [18] problems.

Name	$f(x)$	x_0	Iterations
Example I pg.49	$x^2 + 587x - 987459$	746	2
Example II pg.49	$x^3 - 2x - 5$	2	3
Example III pg.50	$x^3 + 74x^2 + 8729x - 560783$	41	2
Example IV pg.50	$-x^5 + 7x^4 - 20x^3 + 155x^2 - 10000$	-5	2
Example V pg.51	$-x^7 + 7x^5 - 14x^3 + 7x - 1.5$	0.2	3
Example pg.51	$x^2 + 5x - 646$	1	5

Specific comments on the problems:

- Example I is Problem V in Raphson [18]. Sault is using the punctation or point notation. $x^2 + \dot{5}\dot{8}\dot{7}x - 9\dot{8}7\dot{4}5\dot{7}9$ and apply one iteration of (1) in sequence for the two problems $x^2 + 5x - 98 = 0$ and $x^2 + 58x - 9874 = 0$ with the starting point 8, 78, respectively, and two iterations for $f(x) = x^2 + 587x - 987459$ with the starting point 746. This is the same sequence of operations as for Problem V in Raphson [18].
- Example II is Problem IX in [18] and also used in [27]. Sault is using three iterations of (1).
- Example III is Problem XVII in [18]. Sault is using the punctuation notation on the digits $x^3 + \dot{7}\dot{4}x^2 + 8\dot{7}2\dot{9}x - 56\dot{0}78\dot{3}$ and use one iteration of (1) on $x^3 + 7x^2 + 87x - 560$ using the starting point 4 and two iterations using $x^3 + 74x^2 + 8729x - 560783$ with starting point 41.
- Example IV is Problem XXIV in [18]. Starting point is -5 and two iterations performed.
- Example V is Problem XXVII in [18]. Starting point is .2 and three iterations.
- The additional example on page 51 is problem XXVIII/XXX used to show that punctuation is not necessary and the starting point can be far away from the solution. Sault illustrates this with the sequence 1, 92, 48, 29, 23, and 23.03.

2.8 The test examples of Parsons and Wastell 1704

In the book from 1704 Parsons and Wastell [17, Book 2, Ch.21] show that there is a general "formula" for all polynomials when using the formulation in [18] and applies the method to four examples.

Name	$f(x)$	x_0	Iterations
Book 2 pg. 140	$x^2 - 2$	1	4
Book 2 pg. 141	$x^4 - 2839.8241$	10	3
		5	4
Book 2 pg. 141-142	$x^2 + 587x - 987459$	746	2
Book 2 pg. 142	$x^2 - 20x - 53482$	250	4

Specific comments on the problems:

- Example Book 2 pg. 140 is Problem I in Raphson [18].
- Example Book 2 pg. 141 is shown with two different starting points 10 and 5 and writes

  > That it matters not, whether N (*starting point*) be taken above or below the Root, nor how far from it.

  We find the same example and wording in the dictionary [7, Infinite].
- Example Book 2 pg. 141–142 is Problem V in Raphson [18] using the same scaling or punctation; first solve $x^2 + 5x - 98$ and then $x^2 + 58x - 9874$.

– For last the example in Book 2 pg. 142, Parsons and Wastell [17, Book 2, Ch.21] and writes

  > From these two last it is plain; First, That there is no absolute necessity for Punctation. Secondly, That Punctation does nevertheless shorten the Work, where it can be done.

  We find the same example and wording in the dictionary [7, Infinite].

2.9 Example of Jones 1706

While Raphson makes a iteration formula for polynomials of a specific degree, Jones [10, pp.189–197] gives a general formula for all polynomials of different degrees and applies the method to Problem XXIX(1697) in [19] $x^{365} - 1.06$. Jones [10] use Halley's rational and irrational formula for a problem from Ward [29, Example pg.99] using the starting point $x_0 = 300$ and 2 iterations.

2.10 Examples from Ward 1707

The Young Mathematician's Guide. Being a Plain and Easie Introduction to the Mathematicks by John Ward from 1707 was one of the most popular mathematical textbooks in Georgian Britain [31]. Ward writes [30, pg.350]

> But I shall here shew how to find the Natural Sine (and consequently the Natural Tangent) of any proposed Arch or Angle, by Two æquations, without the help of any preceedent Sine as usual; which I did some Years ago communicate to the Ingenious Mr. Joseph Raphson, and he so well approved of them, as to make then the 20 and 21 Problems in the Second Edition of his *Analysis Æquationum Universalis.*

The equation used in [19,30] is

$$-x^4 + 12px^3 - (36p^2 + 195)x^2 + 24px - 45p^2 = 0,$$

where p is a parameter and $p = 0.4468042$ in [19, Problem XX(1697)] and $p = 0.3353940946$ in [30, pg. 350]. The second equation is

$$x^4 + 28x^3 + (36p^2 + 195)x^2 + (108p - 28)x - 196 + 81p = 0,$$

where the parameter $p = 0.143441099$ in [19, Problem XXI(1697)] and $p = 0.06375172518$ in [30, pg. 352].

2.11 The dictionary problem 1713

Christian Wolff in his mathematical dictionary from 1713 [34, Vol 1 pp. 359–362] applied Raphson's method to Problem VI [18] with the same starting point and iterations, and Halley's rational method to Example (pg.99) in [29] using the starting point 300 and two iterations. In addition, the function

$$f(x) = x^3 + 2x^2 - 23x - 70$$

with 5 as initial value and two iterations with Raphson's method. In the abridged and translated dictionary to English, only Raphson's method is used on Problem VI [18] and the above function [35, pp.221–222]. The last problem is also found in the dictionary of Chambers [3, Approximation pg.124], in the expanded French translation in 1751 [4, Approximation pg.559]. Diderot uses Raphson's method but adds

> Cette méthode pour approcher les racines des équations numériques, est dûe à M. Newton.

The problem is also used in Vellnagel 1743 [26, pg.572].

2.12 Test examples of Ronayne 1717

Ronayne [21] derives Raphson's method in Book 1 Part XV Chapter II but demonstrates the method in Part IV Chapter I.

Name	$f(x)$
Example 1 pg.63	$x^3 - 231$
Example 2 pg.64	$x^4 - 10x - 1000$
Example I pg. 237	$x^2 - 2$
Example II pg.237	$x^4 - 4x^3 - 13824$
Example I pg. 239	$x^3 - 2$
Example II pg. 241	$x^3 + 438x^2 - 7825x - 98508430$
Example III pg. 242	$x^4 - 80x^3 + 1998x^2 - 14937x + 5000$

Specific comments on the problems:

- Example 1 pg.63: Computation shown in [21, pg.66]. Starting value is 6 and three iterations using (1).
- Example 2 pg.64: Discussion of Raphson's method on this problem [21, pg.66] compared with a "digit-by-digit" computation.
- Example II pg.237 is used by Holliday [8, Example 21, pg.113]. Starting point is 10 and one iteration with (1) and truncation yields the exact solution.
- Example II pg.241 is found in Ward [29, Example pg.99] using (1). Starting value is 300 and two iterations with the two Halley formulas.
- Example III pg. 242. This example is used by [27,18,6,32]. The variable in Example III pg. 242 is scaled $x \leftarrow \frac{x}{10}$ and the coefficients are rounded $x^4 - 8x^3 + 20x^2 - 15x + \frac{1}{2}$. This is the formulation also used in [8, Example 22, pg.113]. The starting point is 1 with two iterations for the two Halley methods.

2.13 Testexample in Colson 1736

Colson's commentary section [16, pg.188] of Newton's Latin version *De methodis fluxionum et serierum infinitarum* writes:

> And thus our Author's Method proceeds, for finding the Roots of affected Equations in Numbers. Long after this was wrote, Mr. Raphson publish'd his Analysis *Æquationum universalis*, containing a Method for the Solution of Numeral Equations, not very much different from this of our Author (i.e., Newton), as may appear by the following Comparison.

The comparison is the 'standard' problem $x^3 - 2x - 5 = 0$ used in [27,18] and others and Colson continue [16, pg.189].

> By this process we may see how nearly these two Methods agree, and wherein they differ. For the difference is only this, that our Author constantly prosecutes the Residual or Supplemental Equations, to find the first, second, third, &c. Supplements to the Root: But Mr. Raphson: continually corrects the Root itself from the same supplemental Equations, which are formed by substituting the corrected Roots in the Original Equation. And the Rate of Convergency will be the same in both.

Colson shows in the commentary section that a general expression for Raphson's formulation can be made for all polynomials and by assuming all coefficients are rational numbers he derives a general formula. He shows that with rational arithmetic the sequence of iterates will be

$$2,\ \frac{21}{10},\ \frac{11761}{5615},\ \frac{4\,138\,744\,325\,037}{1\,975\,957\,316\,495},\ \ldots$$

applying Raphson's formulation and $f(x) = x^3 - 2x - 5$[16, pg.191].

2.14 Test examples of Simpson 1740

In the essay *A new method for the solution of equations in numbers* Simpson [23, pp.81–86] presents five examples. The first example is Problem XIV in Raphson [18] $-x^3 + 300x - 1000 = 0$ using 2 iterations starting from 3.5. The remaining problems are the first examples with algebraic functions and functions with two unknowns and two equations. Further in this essay Simpson is using fluxion-notation (derivative notation).

Name	$f(x)$	x_0	Iterations
Example II	$\sqrt{1-x}+\sqrt{1-2x^2}+\sqrt{1-3x^3}-2$	.5	2
Example III	$\begin{pmatrix} y+\sqrt{y^2-x^2}-10 \\ x+\sqrt{y^2+x}-12 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 6 \end{pmatrix}$	2
Example IV	$\begin{pmatrix} 49\left(x-\frac{x}{(x+y)^2}\right)-25(1-\frac{x^2}{(1+y)^2}) \\ 81(1-\frac{x^2}{(1+y)^2})-49(\frac{x}{y}-\frac{xy}{(1+x)^2}) \end{pmatrix}$	$\begin{pmatrix} .8 \\ .6 \end{pmatrix}$	2
Example V	$\begin{pmatrix} x^x+y^y-1000 \\ x^y+y^x-100 \end{pmatrix}$	$\begin{pmatrix} 4.5 \\ 2.5 \end{pmatrix}$	2

Example II is used by Simpson in 1745 [24, pg.161]. Example II to IV are used by Joseph Fenn [5, Ch LXI, pp.264-265] using the same starting points and number of iterations.

2.15 Kepler's equation

In 1740 Simpson derived Kepler's equation in the essay *From the mean anomaly of a planet given; to find its place in its orbit* [23, pp.41–51]. By using the time after passing of the aphelion (farthest point in the elliptic orbit) Simpson derives the equation and the iterative solution technique

$$E+e\sin(E)=D, \quad E_{k+1}=E_k+\frac{D-E_k-e\sin E_k}{1+e\cos E_k},$$

where E_0 is *estimated pretty near the truth* [23, pg.42]. This is the Newton–Raphson method (1) derived by geometric considerations. In Example I [23, pg.50] Simpson chooses $D=72.21$, and $e=2.86479$. For $E_0=70^\circ$ and two iterations working with degrees in (1) the iterates are given by

$$E_{k+1}=E_k+\frac{D-E_k-e\sin(E_k\pi/180)}{1+e\cos(E_k\pi/180)\pi/180}, \quad k=0,1,\ldots$$

or 70, 69.5261, and 69.5262. By measuring the distance from perihelion (nearest point) we have the classical formulation of the Kepler equation $x-e\sin(x)=M$. In the second and third editions of Newton's *Philosophiae Naturalis Principia Mathematica* published in 1722 and 1726 Kepler's equation is derived [36], but no numerical example is given.

2.16 Selected test examples of Simpson 1745

In Simpson's Algebra [24] from 1745 we find some additional non-polynomial equations. In Simpson [24] he is not using the fluxion notation, but use approximations like

$$\frac{1}{A+B}\approx\frac{1}{A}-\frac{B}{A^2}, \quad \sqrt{A+B}\approx\sqrt{A}+\frac{B}{2\sqrt{A}}.$$

Name	$f(x)$	x_0	Iterations
Algebra pg. 160	$\sqrt{1+x^2}+\sqrt{2+x^2}+\sqrt{3+x^2}-10$	3	2
Algebra pg. 161	$\frac{20x}{\sqrt{16+5x+x^2}}+\frac{x\sqrt{5+x^2}}{25}-34$	20	1
Algebra pg. 161–162	$(1+x)^{\frac{1}{2}}+(1+x^2)^{\frac{1}{3}}+(1+x^3)^{\frac{1}{4}}-6.5$	3	2
Algebra pg. 163	$\begin{pmatrix} x^4+y^4-10000 \\ x^5-y^5-25000 \end{pmatrix}$	$\begin{pmatrix} 9 \\ 8 \end{pmatrix}$	2
Algebra pg. 164	$\begin{pmatrix} (20x+xy^2)^{\frac{1}{3}}+(8x)^{\frac{1}{2}}-12 \\ \sqrt{x^2+y^2}+\frac{xy}{\sqrt{x^2-y^2}}-13 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 4 \end{pmatrix}$	1

2.17 The test examples of Stewart 1745

In 1745 Stewart [25] made a translation and extensive commentaries of Newton's text *De analysi* transcribed and edited by Jones in 1711 [15]. Stewart is the first to point out that the methods of Newton and Raphson are the same [25, pg.395]:

> And as to Mr. Raphson's Method of Approximation in the extracting the Roots of Equations, published in his *Analysis Equationum universalis*, it is, in effect, the very same with our Author's Method here laid down; which proceeds by assuming only the two last Terms of the supplementary Equations, at each new Operation.

He also derives Halley rational and irrational formulas.

3 Final comments

It took an additional 50 years before it was generally accepted that the methods of Raphson and Newton were identical methods, but implemented differently.

Joseph Lagrange in 1798 derives (1) and writes that the Newton's method and Raphson's method are the same but presented differently and Raphson's method is *plus simple que celle de Newton* [12, note V, pg. 138].

Writers like Euler, Laplace, Lacroix, and Legendre all derive the Newton-Raphson method, but use no names or only Newton. The immense popularity of Fourier's writing led to the universal adoption of the name "Newton's method" [2, pg.32]. As is evident from the test examples presented here that Simpson [23, Essay 6] is the first to consider algebraic functions in an iterative method on the form (1). Kollerstrom [11] writes *None the less, one is driven to conclude that neither Raphson, Halley nor anyone else prior to Simpson applied fluxions to an iterative approximation technique.*

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