

Sub-semi-Riemannian geometry on Heisenberg-type groups

ANNA KOROLKO

Dissertation for the degree of Philosophiae Doctor (Ph.D.)



Department of Mathematics
University of Bergen

October 2010

Preface

The structure of the thesis is as follows:

Chapter 1. We give an abbreviated review covering the history and applications of sub-Riemannian geometry and semi-Riemannian geometry, since those are the areas we make a start from. We introduce also H (eisenberg)-type groups with a natural left-invariant Riemannian metric. We provide basic definitions and facts from designated research areas and give a short overview of our contribution to the subject.

Chapter 2. With the above tools in hands, we are prepared to turn to the main results of the thesis. We introduce principal notions in sub-semi-Riemannian geometry based on ideas of sub-Riemannian and semi-Riemannian geometries, emphasizing the difference between them. We construct various examples of H -type groups with left-invariant semi-Riemannian metrics and describe their geodesics, our main object of interest. We start from the Heisenberg group with sub-Lorentzian metric and then, in order to exhibit more features of sub-semi-Riemannian geometry, we pass to higher dimensional examples. Further, we present general H -type groups with nondegenerate metric of an arbitrary index and the detailed description of geodesics. We finish the survey with a summary and a list of open questions related to sub-semi-Riemannian geometry.

Chapter 3. We include five papers, two of which are published, one accepted, one submitted and one is in preparation. The last article is somewhat independent of the rest of the thesis. It concerns numerical integration on the discrete nonholonomic systems on sub-Riemannian Heisenberg-type groups.

Acknowledgements

First of all, I wish to thank my principal supervisor Professor Irina Markina for her invaluable scientific and non-scientific guidance, enormous patience and enthusiasm, and for always sparing time and energy for discussions with me. At the same time I am very grateful to my co-supervisor Professor Alexander Vasiliev for all the care and warm support I received from him.

I want to thank my colleagues: Mauricio Godoy, Georgy Ivanov and Pavel Gumenyuk for their responsive attention and for the lavishly friendly atmosphere in the office. In particular, I am indebted to Mauricio Godoy who provided me valuable assistance during this project. I thank also Erlend Grong for his occasional illuminations. The help and friendship of Martin Stolz is also very much appreciated. Many thanks go to all members of the analysis group in University of Bergen for interesting collaboration and nice environment.

I would also like to thank Professor Robert McLachlan for the warm hospitality during my 2-months visit to his research group at Massey University in New Zealand.

I am grateful to the University of Bergen for the financial support. Special thanks to my former supervisor Professor Sergey Vodopyanov for his positive encouragement to apply for this Ph.D. position. Thanks to helpful administrative staff of the mathematical institute and to everyone else who in some way participated in my work under the thesis.

Finally, I express my deep gratitude to my family for their kind support and constant interest in the process even from the long distance, to my dear Andreas Sandvin for his love and making my life very happy, and to his warm-hearted family who have all been very supportive during these years.

Anna Korolko
Bergen, October 2010

Contents

1	Introduction	7
1.1	History and motivation	7
1.1.1	Sub-Riemannian geometry and its applications	7
1.1.2	Semi-Riemannian Geometry	8
1.1.3	Heisenberg-type groups	8
1.1.4	Motivation	8
1.2	Overview of the thesis	9
1.3	Necessary background	10
2	Presentation of main results	13
2.1	Semi-Riemannian geometry with constraint	13
2.2	Sub-Lorentzian Heisenberg group	15
2.3	Sub-Lorentzian Quaternion group and its physical interpretation	18
2.4	Sub-semi-Riemannian Quaternion group with the metric of index 2	20
2.5	General sub-semi-Riemannian Heisenberg-type group	21
2.6	Summary and open problems	24
3	Papers A-E	29
3.1	Paper A	31
3.2	Paper B	59
3.3	Paper C	89
3.1	Paper D	121
3.1	Paper E	147

Chapter 1

Introduction

The introductory chapter concerns preliminary knowledge about sub-Riemannian and semi-Riemannian geometries and Heisenberg-type groups. In the first section we provide a glimpse of the history of the question and motivation. Section 1.2. is devoted to a brief overview of the thesis and our contribution to the development of the theory of sub-semi-Riemannian geometry. The necessary notations and background are contained in Section 1.3.

1.1 History and motivation

1.1.1 Sub-Riemannian geometry and its applications

Interest to sub-Riemannian geometry was awakened in the past by many authors from different research domains, such as control theory, classical mechanics, Riemannian geometry, electromagnetism, robotics, quantum physics, neurobiology, finances and others. But only in the last decade of the past century sub-Riemannian geometry has been recognized as a possible common abstract geometrical framework for all these topics. The subject has received popularity and has been studied by a number of different investigators, more or less independently, from a number of different viewpoints and under a number of different names (*Carnot-Carathéodory geometry, geometric control theory, nonholonomic geometry*). Now it is a fully-fledged research domain and is still very much the subject of active investigation [BR96, CC09, Mon02, Str86, CDPT07, LD09, BR08].

Sub-Riemannian geometry is a certain type of generalization of the Riemannian geometry. Roughly speaking, a sub-Riemannian manifold is a Riemannian manifold together with constraints on the admissible directions of motion.

A classical example of sub-Riemannian geometry is the well-known 3-dimensional Heisenberg group \mathbb{H}^1 . Topologically it is the space \mathbb{R}^3 . The constraint on curves is given by the so-called *horizontal* distribution which assigns smoothly to each point of the underlying space a 2-dimensional plane in the tangent space $T\mathbb{R}^3$. The admissible, or *horizontal*, curves would be those that are tangent to such distribution of planes.

By the Chow-Rashevskii theorem [Cho39, Ras38] each pair of points on a sub-Riemannian manifold with a completely nonholonomic distribution can be connected by at least one piecewise smooth horizontal curve. Then the distance between two points p and q can be defined as an infimum of the lengths, defined with respect to a metric given on the distribution, of all those horizontal curves joining p to q . Moreover, for any two points on a sub-Riemannian

manifold locally there is a minimizing curve joining these points, i. e. the distance between these points equals to the length of the curve.

Sub-Riemannian manifolds often occur in the study of constrained systems in classical mechanics, such as the motion of vehicles on a surface, the motion of robot arms, and the orbital dynamics of satellites. The functional mechanisms of the human visual cortex may be clarified with the help of the sub-Riemannian geometry. In another words, it has applications in those subjects where the movements are constrained but the manifold of reachable positions should not be decreased.

1.1.2 Semi-Riemannian Geometry

Semi-Riemannian geometry studies smooth manifolds furnished with a nondegenerate metric of an arbitrary index [O'N83]. Two principal cases of Semi-Riemannian Geometry are Riemannian and Lorentzian Geometries, where the index of the metric is 0 and 1 respectively [dC92, PR84, BEE96]. For many years these two geometries have developed almost independently. More recently, in the 1970's, progress on causality theory, singularity theory and black holes in General Relativity, described in the influential work of S. W. Hawking and G. F. R. Ellis [HE73], resulted in a resurgence of interest in global Lorentzian Geometry. A formulation of Einstein's General Theory of Relativity was given in terms of mathematical model of space-time using prerequisites of general topology and differential geometry. This model is a manifold with a metric of Lorentzian signature, which was later generalized to a semi-Riemannian one. Thus, the mathematical model and physical phenomenon met and gave seeds to the mathematically rigorous theory of semi-Riemannian geometry. Most of the notions in semi-Riemannian geometry originate in Riemannian and Lorentzian geometries.

1.1.3 Heisenberg-type groups

Heisenberg-type groups, also known as generalized Heisenberg groups, or H -type groups, form a subclass of the simply connected two-step nilpotent Lie groups having additional symmetries. Generalized Heisenberg groups endowed with a left-invariant Riemannian metric provide a framework in which interesting examples in geometry and analysis were constructed [BTV95, CDKR91, Kap81, Kap83]. For instance, they appear in the Iwasawa decomposition of semisimple Lie groups [CDKR91]. In fact, H -type groups can be regarded as special model spaces for sub-Riemannian geometry in a similar way as \mathbb{R}^n among all Riemannian manifolds.

H -type groups with a left-invariant Riemannian metric were introduced by A. Kaplan [Kap80, Kap81, Kap83] about thirty years ago. They were defined as Riemannian nilmanifolds attached to Clifford modules. In many ways H -type groups constitute a natural generalization of the Heisenberg group and represent a first large class of objects with similar features which can be described using the language of sub-Riemannian geometry. In [Kap81] some of their geometry were studied and exact formulae for geodesics on H -type groups with Riemannian metric were obtained. The work in this direction was continued further in, for example, [CC09, CM06, CCM09].

1.1.4 Motivation

It was first mentioned in [Str86] that sub-Lorentzian geometry would be an interesting line of research, but there are only few works devoted to this subject [BG01, Gro02, Gro04, Gro06,

Gro09, CMV08].

In this project we attempt to give a coherent introduction to the general theory of sub-semi-Riemannian geometry, taking a point of view that the subject is a generalization of sub-Riemannian and semi-Riemannian geometries. Besides the relationship between these two geometries and sub-semi-Riemannian geometry, the latter one may have some interest on its own.

The goal behind this thesis is to understand how the properties of nonholonomic manifolds change when we substitute a Riemannian metric on a smooth completely nonholonomic distribution with a semi-Riemannian one. More precisely, we are interested in giving an accurate definition of sub-semi-Riemannian manifolds. The idea is to begin to study the model examples of this kind of geometry, that is why we restrict our consideration to 2-step nilpotent Lie groups equipped with a nondegenerate left-invariant metric. These model examples represent analogues of Heisenberg-type groups introduced in [Kap80, Kap81, Kap83]. In contrast to Kaplan we equip horizontal distributions of our H -type groups with a nondegenerate indefinite metric instead of a positively definite one.

1.2 Overview of the thesis

This thesis is about sub-semi-Riemannian geometry on H -type groups equipped with nondegenerate left-invariant metric. To begin the study we construct three examples of such groups: 3-dimensional Heisenberg group with Lorentzian metric, 7-dimensional Quaternion H -type group with Lorentzian metric and 7-dimensional Quaternion H -type group with nondegenerate metric of index 2.

The first interest concerns the existence of geodesics and the setting of explicit formulae for them on H -type groups with semi-Riemannian metric. We define geodesics as projections of the solution of the associated Hamiltonian system onto the underlying manifold. We consider different positions of points and study the cardinality of the set of geodesics connecting these points. Since the nondegeneracy of the metric brings in addition the causality property to the configuration space, the resulting geodesics appear to be causal, i. e. timelike, spacelike or lightlike. In sub-Lorentzian manifolds we define also a time orientation and focus on reachable sets by geodesics of different causal types and their physical interpretation. Furthermore, we frequently compare and contrast the results and techniques of sub-semi-Riemannian geometry to those of sub-Riemannian geometry to alert the reader to the basic differences between these two geometries.

To develop the theory of sub-semi-Riemannian geometry we introduce and study such basic notions as the exponential map, the Christoffel symbols and other differential operators. We show the differential properties of the exponential map, namely, that it possesses an analogue of “local diffeomorphism” property, although it is not a diffeomorphism at some points.

After the consideration of the aforementioned examples and getting a first feeling of the subject we proceed to the general case of H -type groups furnished with a nondegenerate metric of an arbitrary index. In particular, we give a description of geodesics on sub-semi-Riemannian H -type groups depending on the index of the given nondegenerate metric. We find geodesics as a solution to a geodesic equation derived with the help of the analogue of Levi-Civita connection for the sub-semi-Riemannian case. An essential role in this description plays the skew-symmetry of arising objects with respect to the semi-Riemannian product on distribution. The evenness or oddness of the index of the metric influences significantly the

geometric properties of the resulting causal geodesics.

1.3 Necessary background

We start from the basic definitions of sub-Riemannian geometry.

Let M be a connected k -dimensional, $k \geq 3$, C^∞ -manifold. Let $T_x M$ denote the tangent space at a point $x \in M$. The tangent bundle is denoted by $TM = \cup_{x \in M} T_x M$. Fix an integer n , such that $1 < n < k$. Let \mathcal{D} be a fixed subbundle of the tangent bundle TM , $\mathcal{D} = \cup_{x \in M} \mathcal{D}_x$, \mathcal{D}_x be the fibre over x , of dimension n . The subbundle \mathcal{D} will be called *bracket generating*, or *completely nonholonomic*, if the vector fields which are sections of \mathcal{D} , together with all brackets span $T_x M$ at each $x \in M$. The bracket generating subbundle \mathcal{D} is called the *horizontal subbundle*, or *horizontal distribution*, and a curve $\gamma(t)$ satisfying $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ a. e. is called the *horizontal curve*. A result of Chow and Rashevskii [Cho39, Ras38] says that any two points on M can be connected by a piecewise smooth horizontal curve because of the bracket generating property of \mathcal{D} .

A sub-Riemannian metric Q on \mathcal{D} is a smoothly varying in x positively definite quadratic form Q_x on \mathcal{D}_x . The triple (M, \mathcal{D}, Q) is called a *sub-Riemannian manifold*.

Below we present an example of sub-Riemannian manifold – an H -type group with a Riemannian left-invariant metric on a distribution.

Definition 1 *The H -type homogeneous groups are simply connected 2-step nilpotent Lie groups G whose Lie algebras \mathcal{G} are graded and carry an inner product $Q_{\mathcal{G}}$ such that*

- \mathcal{G} is the orthogonal direct sum of the generating space V and the center U : $\mathcal{G} = V \oplus U$, $[V, V] = U$, $[V, U] = 0$,
- the endomorphisms $j(u): V \rightarrow V$, $u \in U$, defined by

$$Q_{\mathcal{G}}(j(u)v, w) = Q_{\mathcal{G}}(u, [v, w]), \quad v, w \in V,$$

satisfy the equation

$$j^2(u) = -|u|^2 I_V, \quad u \in U. \quad (1.3.1)$$

Here $Q_{\mathcal{G}}(\cdot, \cdot)$ is an inner product on \mathcal{G} defined as a sum of two positively definite non-degenerating quadratic forms on V and U respectively: $Q_{\mathcal{G}}(\cdot, \cdot) = Q_V(\cdot, \cdot) + Q_U(\cdot, \cdot)$, and $|u|^2 = Q_{\mathcal{G}}(u, u)$, I_V denotes the identity mapping on V . Recall that the Clifford algebra $\text{Cl}(U, m)$ over the m -dimensional space U with its quadratic form $Q_U(\cdot, \cdot)$ is defined as the free associative unitary algebra modulo the relations $u^2 = -|u|^2 I_V$. Then the linear mapping $j: U \rightarrow \text{End}(V)$ extends to a representation of $\text{Cl}(U, m)$ on V , i. e. V becomes a Clifford module over $\text{Cl}(U, m)$. If there exists such an endomorphism $j(u): V \rightarrow V$, the algebra $\mathcal{G} = V \oplus U$ is called an *H -type algebra*. It is known that unless m equals 3 or 7 modulo 8, $\text{Cl}(U, m)$ is a real, Heisenberg or quaternionic matrix algebra. When m is 3 or 7 modulo 8, the algebra $\text{Cl}(U, m)$ is a direct sum of two real or quaternionic matrix algebras (see, for example, [Lou01]).

The Lie algebra \mathcal{G} is identified with the tangent space to G at the identity $T_e G$. Push-forward allows to define the inner product on the entire tangent bundle $TG = \cup_{\sigma} T_{\sigma} G$ as:

$$Q_{\mathcal{G}}(\cdot, \cdot)(\sigma) = Q_{\mathcal{G}}(dL_{\sigma^{-1}} \cdot, dL_{\sigma^{-1}} \cdot),$$

where L_σ denotes the left translation on G by the element $\sigma \in G$. Since $V \subset T_e G$, the left translation of V by element σ is $dL_\sigma(V) \subset T_\sigma G$. The mapping $\mathcal{D}: \sigma \mapsto dL_\sigma(V)$ represents a horizontal distribution and any vector field v such that $v(\sigma) \in dL_\sigma(V)$ is a horizontal vector at the point σ .

The Lie algebra \mathcal{G} is isomorphic to the set of left-invariant vector fields, i. e. vector fields $v \in \mathcal{G}$ such that $v(\sigma) = dL_\sigma v(e)$ for any $\sigma \in G$.

The H -type group G is introduced in such a way, that the horizontal distribution \mathcal{D} possesses a 2-step bracket generating property. In this case by Chow-Rashevskii theorem any two points in G can be connected by horizontal curves tangent to the distribution \mathcal{D} . The triple (G, \mathcal{D}, Q_V) is a sub-Riemannian manifold. Due to left translation we can work further with the distribution at the point e , i. e. with $dL_e(V) = V$, instead of \mathcal{D} .

We define the exponential map as a function $\exp: \mathcal{G} \rightarrow G$ in the usual way in the Lie groups theory.

The most studied examples of H -type groups are the Euclidean k -dimensional space \mathbb{R}^k , the Heisenberg group and the quaternion H -type group. These groups satisfy the j^2 condition: for any $v \in V$ and any $u_1, u_2 \in U$ with $Q_U(u_1, u_2) = 0$ there exists $u_3 \in U$ such that

$$j(u_1)j(u_2)v = j(u_3)v.$$

The Euclidean space. The space \mathbb{R}^k is a trivial example of an H -type group since all commutative relations vanish. The horizontal space V is identified with \mathbb{R}^k via the exponential map which is identity in this case. The center U is the empty set.

The Heisenberg group. We restrict our consideration to the 3-dimensional Heisenberg group \mathbb{H}^1 which is a graded 2-step nilpotent Lie group whose underlying manifold is \mathbb{R}^3 with the noncommutative group law

$$L_{(x,y,z)}(x', y', z') = (x, y, z) \circ (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - yx')).$$

The Heisenberg algebra \mathcal{G} is identified with the set of left-invariant vector fields and has the following properties:

- $\mathcal{G} = V \oplus U$, where V has dimension 2 and U has dimension 1, and
- $[V, V] = U$, $[V, U] = 0$ and $[U, U] = 0$.

A left-invariant basis written in normal coordinates is

$$X = \partial_x + \frac{1}{2}y\partial_z, \quad Y = \partial_y - \frac{1}{2}x\partial_z, \quad \text{and } Z = \partial_z = [X, Y],$$

and the horizontal distribution is $V = \text{span}\{X, Y\}$.

The Heisenberg group with a Riemannian metric on the distribution is a sub-Riemannian manifold and will be denoted by \mathbb{H}_R^1 to accentuate the metric considered. The description of higher-dimensional analogues of the Heisenberg group can be found, for example, in [CDPT07].

Quaternion H -type group. As in the previous case, we limit ourselves to the 1-dimensional case. The multidimensional analogue can be seen, for instance, in [CCM09]. We will consider the quaternion H -type group \mathbb{Q} defined by (1.3.1) with V associated with the space of quaternions and U as a three-dimensional center. A quaternion can be represented in a matrix way

by $x = aI + bI_1 + cI_2 + dI_3$, where

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad I_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$I_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

in the basis of quaternion numbers given by real (4×4) -matrices and $a, b, c, d \in \mathbb{R}$. Let us take the background manifold as \mathbb{R}^7 and define the noncommutative law with the help of quaternion multiplication rule:

$$L_{(x,z)}(x', z') = (x, z) \circ (x', z') = (x + x', z + z' + \frac{1}{2}\text{Im}(\bar{x} * x')),$$

for (x, z) and (x', z') belong to $\mathbb{R}^4 \times \mathbb{R}^3$. Here $\text{Im}(\bar{x} * x')$ is the imaginary part of the product $\bar{x} * x'$ of the conjugate quaternion \bar{x} to x by another quaternion x' . Let $\partial_{x_0}, \dots, \partial_{x_3}, \partial_{z_1}, \dots, \partial_{z_3}$ be a standard basis of the tangent space $T_e\mathbb{Q}$ to \mathbb{Q} at $e \in \mathbb{Q}$. The basic left-invariant vector fields are obtained by the action of the tangent map $dL_{(x,z)}$ of $L_{(x,z)}$ to the standard basis as $dL_{(x,z)}\left(\frac{\partial}{\partial x_i}\right) = X_i(x, z)$, $dL_{(x,z)}\left(\frac{\partial}{\partial z_k}\right) = Z_k(x, z)$. Then the vector fields

$$\begin{aligned} X_0 &= \partial_{x_0} + \frac{1}{2}(+x_1\partial_{z_1} - x_3\partial_{z_2} - x_2\partial_{z_3}), \\ X_1 &= \partial_{x_1} + \frac{1}{2}(-x_0\partial_{z_1} - x_2\partial_{z_2} + x_3\partial_{z_3}), \\ X_2 &= \partial_{x_2} + \frac{1}{2}(+x_3\partial_{z_1} + x_1\partial_{z_2} + x_0\partial_{z_3}), \\ X_3 &= \partial_{x_3} + \frac{1}{2}(-x_2\partial_{z_1} + x_0\partial_{z_2} - x_1\partial_{z_3}), \end{aligned} \tag{1.3.2}$$

span a 4-dimensional horizontal distribution V . The left-invariant vector fields $Z_\beta = \partial_{z_\beta}$, $\beta = 1, 2, 3$ form a basis of the complement U to V in $T_e\mathbb{Q}$. The commutation relations are as follows

$$\begin{aligned} [X_0, X_1] &= -Z_1, & [X_0, X_2] &= Z_3, & [X_0, X_3] &= Z_2, \\ [X_1, X_2] &= Z_2, & [X_1, X_3] &= -Z_3, & [X_2, X_3] &= -Z_1. \end{aligned}$$

Therefore, $\{X_0, \dots, X_3\}$ and their commutators span the entire tangent space $T_{(x,z)}\mathbb{Q}$ at each point $(x, z) \in \mathbb{R}^4 \times \mathbb{R}^3$. This property makes the distribution V to be bracket-generating of step 2. The Lie algebra with basis $\{X_0, \dots, X_3, Z_1, Z_2, Z_3\}$ is nilpotent of step 2.

Let us define the Riemannian metric Q_V on the distribution V in such a way that $Q_V(X_i, X_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. With this, we get the sub-Riemannian manifold $\mathbb{Q}_R = (\mathbb{R}^7, V, Q_V)$.

Chapter 2

Presentation of main results

In this chapter we introduce the notion of sub-semi-Riemannian geometry. We give examples, showing the complexity of this geometry, and announce new results. More detailed presentation of main results can be found in the papers A-D from Chapter 3.

2.1 Semi-Riemannian geometry with constraint

In contrast to sub-Riemannian geometry we furnish the horizontal distribution with a semi-Riemannian, i. e. nondegenerate indefinite metric.

Let M be a smooth k -dimensional manifold, let T_xM and T_x^*M denote the tangent and cotangent spaces at a point $x \in M$, and let $\langle W, \xi \rangle$ be a pairing between them, $W \in T_xM$, $\xi \in T_x^*M$. As before, let D be a smooth n -dimensional, $n < k$, bracket generating subbundle of $TM = \cup_{x \in M} T_xM$, $D = \cup_{x \in M} D_x$. A *sub-semi-Riemannian metric* Q on D is a smoothly varying in x nondegenerate form Q_x on D_x .

We remind that the index p of a metric is the maximal dimension of the space $S_x \subset D_x$, where the form Q_x is negative. If $p = 1$, then we call such a metric the *sub-Lorentzian* metric following the tradition in semi-Riemannian geometry. The sub-semi-Riemannian metric with index $p = 0$ is just a sub-Riemannian metric.

Given Q_x , we may define a linear mapping $g_x: T_x^*M \rightarrow T_xM$ by

(i) image of T_x^*M under g_x is D_x ,

(ii) g_x and Q_x are related by the identity

$$Q_x(W, g_x\xi) = \langle W, \xi \rangle \text{ for all } W \in D_x. \quad (2.1.1)$$

The map g_x is called a cometric. We understand the action of the cometric g on $T^*M \times T^*M \rightarrow \mathbb{R}$ (omitting x) as follows: $g(\xi, \psi) = \langle g\xi, \psi \rangle = Q(g\xi, g\psi)$ for any two covectors ξ and ψ from T^*M . Let D_x^\perp denote the kernel of g_x , and $D^\perp \subseteq T^*M$ be the subbundle with fibers D_x^\perp . Then $g_x: T_x^*M/D_x^\perp \rightarrow D_x$ is a bijection. The elements from D^\perp will be called annihilators.

Remark. If Q_x is symmetric, nondegenerate and has index p on D_x , then g_x is symmetric, nondegenerate on and has index p on T_x^*M/D_x^\perp .

Conversely, given a symmetric linear operator $g_x: T_x^*M \rightarrow T_xM$ with image D_x , there is a unique nondegenerate quadratic form Q_x satisfying (2.1.1). The matrix defining the cometric g_x is never invertible. See the paper B for more details on this.

A smooth manifold M with a chosen subbundle D of the tangent bundle and with a given nondegenerate sub-semi-Riemannian metric Q on D will be called the *sub-semi-Riemannian*

manifold. If the index p of Q is 1, then we call the triple (M, D, Q) sub-Lorentzian manifold and in the case of $p = 0$ we get sub-Riemannian manifolds, which have been widely studied in [Gro99, LS95, Mon02, Str86], see also the references cited therein.

We define the causal character of the underlying manifold M as follows. Fix a point $x \in M$. A horizontal vector $v \in D_x$ is called

- *timelike* if $Q(v, v) < 0$,
- *spacelike* if $Q(v, v) > 0$ or $v = 0$,
- *lightlike* if $Q(v, v) = 0$ and $v \neq 0$,
- *nonspacelike* if $Q(v, v) \leq 0$.

A horizontal curve is called timelike if its tangent vector is timelike at each point, similarly for spacelike, lightlike and nonspacelike curves.

If the index of the metric Q is 1, then we can define a time orientation of the sub-Lorentzian manifold. By a *time orientation* of (M, D, Q) we mean a horizontal timelike vector field \mathcal{T} on M . Then \mathcal{T} divides all horizontal vectors into two disjoint classes, called future directed and past directed. Namely, a nonspacelike $v \in D_x$ is said to be *future* (respectively *past*) directed if $Q_x(\mathcal{T}(x), v) < 0$ (respectively $Q_x(\mathcal{T}(x), v) > 0$). Throughout this paper f.d. stands for “future directed”, t. for “timelike”, and nspc. for “nonspacelike”.

For an open set N and fixed $x \in N$, we define two reachable sets: $I^+(x, N)$ (respectively, $J^+(x, N)$) as the set of all points $y \in N$ that can be reached from x along a t.f.d. (respectively, nspc.f.d.) curve contained in N . In Lorentzian geometry $I^+(x, N)$ is called the chronological future of x (with respect to N); similarly, $J^+(x, N)$ is called the causal future of x (with respect to N). The terminology is adapted from the relativity theory. For a nice and complete presentation of the semi-Riemannian geometry see [O’N83].

In this thesis we use two approaches to find geodesics: as a projection of a solution of a Hamiltonian system and as a solution to the geodesic equation. In Sections 2.2–2.4 we use Hamiltonian formalism which is widely applied in nonholonomic geometry. Given the cometric $g_x: T_x^*M \rightarrow D_x$ we form the Hamiltonian function

$$H(x, \xi) = \frac{1}{2} \langle g_x(\xi), \xi \rangle$$

on T^*M . If we have the orthonormal basis $X_1, \dots, X_p, \dots, X_n$ on \mathcal{D} such that X_1, \dots, X_p are timelike and X_{p+1}, \dots, X_n are spacelike, we can write the Hamiltonian function in the form

$$H(x, \xi) = -\frac{1}{2} \sum_{j=1}^p \langle X_j(x), \xi \rangle^2 + \frac{1}{2} \sum_{j=p+1}^n \langle X_j(x), \xi \rangle^2,$$

where p is the index of g_x . Consider the Hamiltonian equations

$$\dot{x}(t) = \frac{\partial H(x, \xi)}{\partial \xi}, \quad \dot{\xi}(t) = -\frac{\partial H(x, \xi)}{\partial x}. \quad (2.1.2)$$

An absolutely continuous curve $\Gamma(t) = (x(t), \xi(t))$ satisfying (2.1.2) is called a *bicharacteristic* of H . Its projection $x(t)$ on M is called *normal geodesic*. Since we work only with normal geodesics we will drop the word “normal” for shortness.

Remark. Bicharacteristics of a Hamiltonian $H \in C^k(T^*M)$ are curves of class C^k along which H is constant. It means that it is not possible that the geodesic $x(t): [a, b] \rightarrow M$ can change its causal character, i. e. $Q_{x(t)}(\dot{x}(t), \dot{x}(t))$ keeps its sign for all $t \in [a, b]$.

Notice also that if we differentiate the first equation of (2.1.2) and substitute the second we will obtain the analogue of the geodesic equation in sub-Riemannian geometry. For details see the paper B.

In Section 2.5 in order to find geodesics we use the notion of Levi-Civita connection on a manifold M which we define as an affine connection compatible with nondegenerate metric $Q(\cdot, \cdot) + \bar{Q}(\cdot, \cdot)$ on TM , where $\bar{Q}(\cdot, \cdot)$ is a Riemannian metric on the complement to D in TM .

Let $\gamma(t)$ be a C^1 -piecewise curve in M for $t \in (a, b)$, where (a, b) is an interval in \mathbb{R} . We remind that a curve $\gamma(t)$ is called horizontal if $\dot{\gamma}(t) \in D_{\gamma(t)}$ for any t where $\dot{\gamma}(t)$ is defined. A section $\xi(t)$ is called a cotangent lift of $\gamma(t)$ if $\xi(t) \in T_{\gamma(t)}^*M$ and $g_{\gamma(t)}\xi(t) = \dot{\gamma}(t)$ for every t where it is defined.

If $x \in M$, let Ω_x be the set of covectors v in T_x^*M such that the geodesic $\gamma_v(t)$ is defined at least on $[0, 1]$ and $\gamma_v(0) = x$, $\xi(0) = v$. The *exponential map* of M at x is the function

$$\exp_x: \Omega_x \rightarrow M, \text{ such that } \exp_x(v) = \gamma_v(1).$$

The set Ω_x is the largest subset of T_x^*M on which \exp_x can be defined.

As in sub-Riemannian geometry the exponential map is always differentiable, but it is not a diffeomorphism at the origin. One of the main results of the Paper B says that the exponential map $\exp_x(v)$ is a diffeomorphism if v is neither a lightlike vector nor an annihilator. The sub-semi-Riemannian analogue of Gauss lemma is also stated and proved in Paper B.

In the next sections we present several examples of sub-semi-Riemannian manifolds, that are Heisenberg-type groups with sub-Lorentzian metric and sub-semi-Riemannian metric of index 2. Then we focus on the generalization of all H -type groups with nondegenerate metric of arbitrary index p .

2.2 Sub-Lorentzian Heisenberg group

Let us consider the following example of sub-Lorentzian manifold that we call the Heisenberg group with sub-Lorentzian metric and provide a description of geodesics on it. We remind that the Heisenberg group \mathbb{H}^1 is the space \mathbb{R}^3 furnished with the noncommutative law of multiplication

$$(x, y, z) \circ (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(yx' - xy')).$$

The two-dimensional horizontal subbundle V is given as a span of the left-invariant vector fields

$$X = \partial_x + \frac{1}{2}y\partial_z, \quad Y = \partial_y - \frac{1}{2}x\partial_z.$$

Moreover, $[X, Y] = Z = \partial_z$. Let us consider the Lorentzian metric Q on the horizontal distribution V . Suppose that it is defined by

$$Q(X, X) = -1, \quad Q(Y, Y) = 1, \quad Q(X, Y) = 0.$$

Ipsa facto, the time orientation is given by the horizontal vector field X . Thus, we call the triple (\mathbb{R}^3, V, Q) Heisenberg group with sub-Lorentzian metric, and to differ it from the classical case we use the notation \mathbb{H}_L^1 .

We apply the method of Hamiltonian mechanics in order to calculate geodesics $\gamma(t) = (x, y, z)(t)$ of different causal character which pass through the origin with a given initial velocity $(\dot{x}_0, \dot{y}_0, \dot{z}_0)$. The Hamiltonian system in this case can be reduced to the following system (see paper A)

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\theta \\ -\theta & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix},$$

where θ is a real parameter. The solution is of the following form

$$\begin{aligned} x(t) &= \frac{\dot{x}_0}{|\theta|} \sinh(|\theta|t) - \frac{\dot{y}_0}{|\theta|} (\cosh(|\theta|t) - 1), \\ y(t) &= -\frac{\dot{x}_0}{|\theta|} (\cosh(|\theta|t) - 1) + \frac{\dot{y}_0}{|\theta|} \sinh(|\theta|t). \end{aligned} \quad (2.2.1)$$

Using the horizontality condition $\dot{z} = \frac{1}{2}(y\dot{x} - x\dot{y})$, one can get (see also [Gro06])

$$z(t) = \frac{\|v_0\|^2}{2|\theta|^2} (|\theta|t - \sinh(|\theta|t)), \quad \|v_0\|^2 = -\dot{x}_0^2 + \dot{y}_0^2.$$

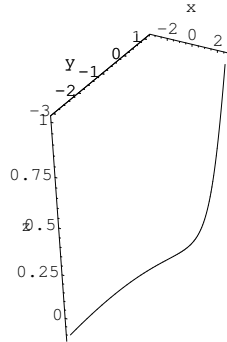


Figure 2.1: The graph of timelike geodesic in \mathbb{R}^3 passing through the origin

The projection of the timelike geodesic on (x, y) -plane is a branch of hyperbola passing through the origin, see Figure (2.2).

We investigate the question of quantity of geodesics connecting the origin and an arbitrary point. For that we calculate $-x^2 + y^2 = \frac{4\|v_0\|^2}{|\theta|^2} \sinh^2\left(\frac{|\theta|t}{2}\right)$ and notice that $\frac{4z}{-x^2 + y^2} = \frac{t}{\sinh^2(|\theta|t)} - \coth(|\theta|t)$.

Remark. The function $\mu(\tau) = \frac{\tau}{\sinh^2(\tau)} - \coth(\tau)$ is strictly decreasing in the interval $(-\infty, \infty)$ from 1 to -1 , see Figure 2.3. Therefore, the equation

$$\frac{4z}{-x^2 + y^2} = \frac{\tau}{\sinh^2(\tau)} - \coth(\tau) \quad (2.2.2)$$

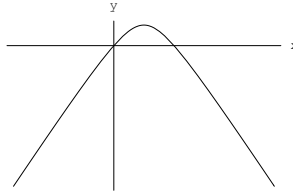


Figure 2.2: The graph of projection of timelike geodesic on (x, y) -plane

has a unique solution τ for given coordinates (x, y, z) of the finite point, if $-1 < \frac{4z}{-x^2+y^2} < 1$, and has no solution otherwise.

We would like to draw the reader’s attention to the function μ . It is an intriguing analogue of the function used by Gaveau [Gav77] to study geodesics and geometry in general on the classical Heisenberg group \mathbb{H}_R^1 . The classical counterpart $\tilde{\mu}$ of the function μ has the form $\tilde{\mu}(\tau) = \frac{\tau}{\sin^2(\tau)} - \cot(\tau)$. The classical Heisenberg analogue of the equation (2.2.2) $\frac{4z}{-x^2+y^2} = \frac{\tau}{\sin^2(\tau)} - \cot(\tau)$ has always more, or equal to, one solution.

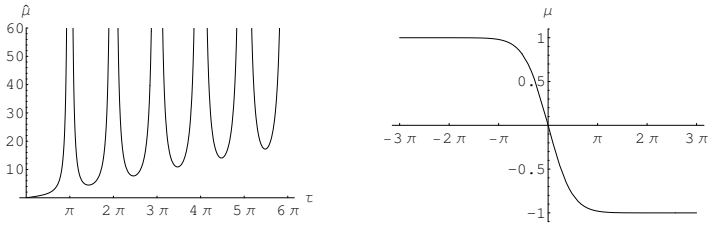


Figure 2.3: The graphs of functions $\tilde{\mu}(\tau)$ and $\mu(\tau)$

We state the following theorem describing the reachable set by geodesics starting from the origin.

Theorem 1 *Let us define the following sets*

$$R_t = \left\{ -x^2 + y^2 < 0, \frac{4|z|}{x^2 - y^2} < 1 \right\},$$

$$R_{sp} = \left\{ -x^2 + y^2 > 0, \frac{4|z|}{-x^2 + y^2} < 1 \right\},$$

$$R_l = \left\{ -x^2 + y^2 = 0, z = 0 \right\}.$$

Then there exists a unique geodesic connecting the point $O = (0, 0, 0)$ with a point $A = (x, y, z)$ that belongs to one of the sets R_t , R_{sp} or R_l . Particularly, if $A \in R_t$, then the geodesic is timelike, if $A \in R_{sp}$, then the geodesic is spacelike, and if $A \in R_l$, then the geodesic is lightlike. If point A does not belong to any of the sets R_t , R_{sp} or R_l , then there are no geodesics of any causal type joining O with A . See Figure 2.4.

In other words, $I^+(0, \mathbb{R}^3) = R_t$, $J^+(0, \mathbb{R}^3) = R_t \cup R_l$. The proof of the theorem and the parametric equations for geodesics can be found in Paper A in Chapter 3.

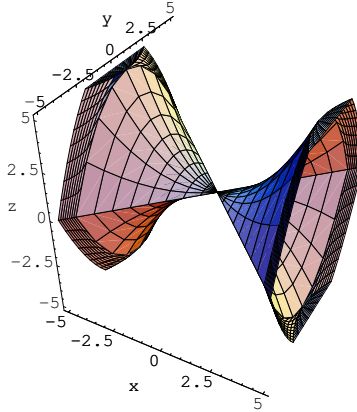


Figure 2.4: The domain reachable from the origin by timelike geodesics R_t

Remark. We would like to stress the difference between sub-Riemannian and sub-Lorentzian cases. Unlike the sub-Riemannian Heisenberg Group \mathbb{H}_R^1 we get the uniqueness of geodesics between two points (if exist at all). Also, in the Heisenberg group with a positively definite metric \mathbb{H}_R^1 the point $(0, 0, z)$, $z > 0$ is connected with the origin $O = (0, 0, 0)$ by uncountably many geodesics [CCG07]. Theorem 1 shows that the same point in the Heisenberg group with Lorentzian metric \mathbb{H}_L^1 can not be connected with origin by any geodesic independently of the causal character.

2.3 Sub-Lorentzian Quaternion group and its physical interpretation

The sub-Lorentzian Quaternion H -type group \mathbb{Q}_L is defined in the same way as the sub-Riemannian Quaternion group \mathbb{Q}_R (see Subsection 1.3), but instead of Riemannian metric on the distribution V we consider a Lorentzian metric Q_V . Denote by η the matrix of its metric tensor in the basis $\{X_0, X_1, X_2, X_3\}$ which is defined by (1.3.2):

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The causal character on \mathbb{Q}_L is defined in a usual way.

We find geodesics on \mathbb{Q}_L as the projections of solutions of the associated Hamiltonian system on \mathbb{Q}_L . The Hamiltonian function with respect to the sub-Lorentzian metric is defined in the following way:

$$H_L(x, \xi) = -\frac{1}{2}\langle X_0, \xi \rangle^2 + \frac{1}{2} \sum_{i=1}^3 \langle X_i, \xi \rangle^2.$$

We roll up the corresponding Hamiltonian system to the following linear system of ordinary

differential equations

$$\ddot{x} = A\dot{x}, \quad (2.3.1)$$

$$\dot{z}_k = \dot{x}^t I_k x, \quad k = 1, 2, 3, \quad (2.3.2)$$

that gives the equations for geodesics on \mathbb{Q}_L . Here the matrices I_k , $k = 1, 2, 3$, are the basis of quaternions in the representation given by the real (4×4) -matrices and A is a skew-symmetric matrix with respect to the Lorentzian metric, i. e. $Q_V(Av, w) = -Q_V(v, Aw)$, where $v, w \in V$:

$$A = \begin{pmatrix} 0 & -\theta_1 & \theta_3 & \theta_2 \\ -\theta_1 & 0 & -\theta_2 & \theta_3 \\ \theta_3 & \theta_2 & 0 & \theta_1 \\ \theta_2 & -\theta_3 & -\theta_1 & 0 \end{pmatrix}, \quad (2.3.3)$$

where $\theta_1, \theta_2, \theta_3$ are real parameters.

In paper C we show that the equation (2.3.1) coincides with the Lorentz force law $\frac{dU}{dt} = \eta F U$, where $U = \dot{x}$ is the world velocity of a charged particle, $\frac{dU}{dt}$ is the world momentum (assuming that the charge and the mass of the particle equals 1) and F is an electromagnetic tensor field which corresponds to a linear skew-symmetric transformation A : $A = \eta F$. Thus, we conclude that the equation (2.3.1) describes the motion of a particle of unit charge in the constant electromagnetic field with electric field $\vec{E} = (\theta_2, \theta_3, -\theta_1)$ and magnetic field $\vec{B} = -\vec{E}$.

The matrix A in (2.3.3) can be transformed to a canonical form

$$\tilde{A} = \begin{pmatrix} 0 & |\theta| & 0 & 0 \\ |\theta| & 0 & 0 & 0 \\ 0 & 0 & 0 & -|\theta| \\ 0 & 0 & |\theta| & 0 \end{pmatrix},$$

where $|\theta| = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}$. If M is a (4×4) -matrix, such that $\tilde{A} = M^{-1}AM$, then $\tilde{x} = Mx$ and $\tilde{z}_k = \dot{x}^t M I_k M^{-1} \tilde{x}$, $k = 1, 2, 3$. The solution $\tilde{x}(t) = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3)(t)$ to the system $\ddot{\tilde{x}} = \tilde{A}\dot{\tilde{x}}$ can be written in a matrix form $\tilde{x}(t) = W(t)\dot{\tilde{x}}(0)$, where

$$W(t) = \frac{1}{|\theta|} \begin{pmatrix} \sinh |\theta|t & \cosh |\theta|t - 1 & 0 & 0 \\ \cosh |\theta|t - 1 & \sinh |\theta|t & 0 & 0 \\ 0 & 0 & \sin |\theta|t & \cos |\theta|t - 1 \\ 0 & 0 & 1 - \cos |\theta|t & \sin |\theta|t \end{pmatrix},$$

and $\dot{\tilde{x}}(0)$ is an initial velocity.

Remark 1. The projection of the geodesic onto the $(\tilde{x}_0, \tilde{x}_1)$ -plane is a branch of hyperbola with canonical equation

$$\left(\tilde{x}_0 + \frac{\dot{\tilde{x}}_1(0)}{|\theta|} \right)^2 - \left(\tilde{x}_1 + \frac{\dot{\tilde{x}}_0(0)}{|\theta|} \right)^2 = \frac{-\dot{\tilde{x}}_0^2(0) + \dot{\tilde{x}}_1^2(0)}{|\theta|^2}.$$

Remark 2. The projection of the geodesic onto the $(\tilde{x}_2, \tilde{x}_3)$ -plane is a circle with center at $\left(-\frac{\dot{\tilde{x}}_3(0)}{|\theta|}, \frac{\dot{\tilde{x}}_2(0)}{|\theta|} \right)$ and of radius $\frac{\sqrt{\dot{\tilde{x}}_2^2(0) + \dot{\tilde{x}}_3^2(0)}}{|\theta|}$.

Explicit formulae for the vertical part $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)$ can be found in Paper C.

We have the following results regarding the number and causal character of geodesics connecting the origin of \mathbb{Q}_L and the point $P = (\tilde{x}, \tilde{z}) = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3) \in \mathbb{Q}_L$, where $|\tilde{x}|_L^2 = Q_V(\tilde{x}, \tilde{x}) = 0$.

- Given a point $P = (\tilde{x}, 0)$, $|\tilde{x}|_L = 0$, $\tilde{x}_0^2 + \tilde{x}_1^2 \neq 0$, there is a unique lightlike geodesic joining the origin and P which is a straight line.
- Given a point $P = (0, \tilde{z})$ there are uncountably many spacelike geodesics connecting the origin with P .
- Given a point $P = (\tilde{x}, \tilde{z})$, $|\tilde{x}|_L = 0$, $\tilde{x}_0 = \pm\tilde{x}_1 \neq 0$, $|\tilde{z}| \neq 0$, there are uncountably many spacelike geodesics connecting the origin with P .
- If $\theta_1 = \theta_2 = \theta_3 = 0$, then the system (2.3.1)-(2.3.2) has a straight line as a solution which is timelike/lightlike/spacelike if so is the initial velocity.
- Let $P = (\tilde{x}, \tilde{z})$, $|\tilde{x}|_L = 0$, $\tilde{x}_1^2 < \tilde{x}_0^2$ and m_0 is a global maximum of the function $\bar{\mu}(\tau) = \frac{\cosh(\tau) - \cos(\tau) - \sin(\tau) \sinh(\tau)}{\sin^2 \frac{\tau}{2} (\sinh(\tau) - \tau) - \sinh^2 \frac{\tau}{2} (\sin(\tau) - \tau)}$ (see Figure 2.5). If $\left| \frac{\tilde{z}_2^2 + \tilde{z}_3^2}{2\tilde{z}_1 \tilde{x}_0^2} \right| < m_0$, then there exist more than one spacelike geodesics joining the origin with P . If $\left| \frac{\tilde{z}_2^2 + \tilde{z}_3^2}{2\tilde{z}_1 \tilde{x}_0^2} \right| > m_0$, then there are no geodesics of any causal type joining 0 and P .
- Given a point $P = (\tilde{x}, \tilde{z})$ on the surface $|\tilde{x}|_L = 0$, $\tilde{z} \neq 0$, there are no timelike geodesics joining 0 to P .

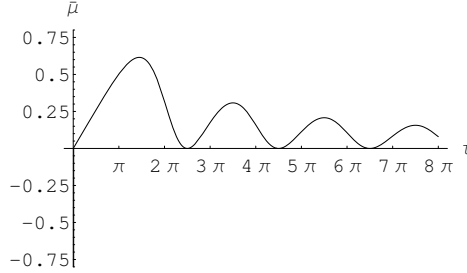


Figure 2.5: The graph of function $\bar{\mu}(\tau)$

Making a parallel between the sub-Lorentzian Quaternion group \mathbb{Q}_L and the sub-Lorentzian Heisenberg group \mathbb{H}_L^1 we notice that in the latter one the uniqueness of geodesics joining the origin and $P \in R_l$ happens due to lower dimension of the “spacelike” part of \mathbb{H}_L^1 .

Another interesting observation is that the horizontal part of the geodesic on sub-Lorentzian Quaternion H -type group \mathbb{Q}_L contains horizontal parts of both geodesics on sub-Riemannian and sub-Lorentzian Heisenberg groups \mathbb{H}_R^1 and \mathbb{H}_L^1 .

2.4 Sub-semi-Riemannian Quaternion group with the metric of index 2

The sub-semi-Riemannian Quaternion H -type group \mathbb{Q}_{sR} with metric of index 2 is defined similarly to the previous case of \mathbb{Q}_L , but with the metric Q_V now being an index 2 semi-Riemannian metric on the distribution V .

The Hamiltonian function with respect to the semi-Riemannian metric is defined in the following way:

$$H_{sR}(x, \xi) = -\frac{1}{2}\langle X_0, \xi \rangle^2 - \frac{1}{2}\langle X_1, \xi \rangle^2 + \frac{1}{2}\langle X_2, \xi \rangle^2 + \frac{1}{2}\langle X_3, \xi \rangle^2,$$

where $X_k, k = 0, \dots, 3$, span V and $(x, \xi) \in T^*\mathbb{Q}_{sR}$.

We reduce the corresponding Hamiltonian system to the following linear system of ordinary differential equations

$$\ddot{x} = A\dot{x}, \tag{2.4.1}$$

$$\dot{z}_k = \dot{x}^t I_k x, \quad k = 1, 2, 3, \tag{2.4.2}$$

that gives the equations for geodesics on \mathbb{Q}_{sR} . Here the matrix A is skew-symmetric with respect to semi-Riemannian product Q_V . Notice that the equations (2.4.2) are exactly the same as in case of \mathbb{Q}_L and, in fact, the same as in case of \mathbb{Q}_R , since the horizontality condition does not depend on the metric.

In paper B one can find the explicit formulae for geodesics on \mathbb{Q}_{sR} and their homogeneous norms.

2.5 General sub-semi-Riemannian Heisenberg-type group

It is known that the sub-Riemannian task of finding length-minimizing curves can be formulated in terms of optimal control theory. In the case of sub-Lorentzian geometry it is also possible reformulate the problem of finding length-maximizing curves as a solution to an affine control system [Gro09]. But in the general case of sub-semi-Riemannian geometry the task of finding geometrically optimal curves can not be reformulated in terms of control theory. That is why the tools from differential geometry, not available to control theory provide an important novelty and make it possible to work with the subject.

Let U and V be real vector spaces of dimensions m and n respectively with positively definite product $Q_U(\cdot, \cdot)$ on U and nondegenerate product $Q_V(\cdot, \cdot)$ of signature (p, q) , $p+q = n$, on V . Denote by η the metric tensor of the product on V , which is a diagonal matrix with first p negative and then q positive unities, i. e.

$$\eta = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}, \tag{2.5.1}$$

where I_p and I_q are usual identity matrices of dimensions $(p \times p)$ and $(q \times q)$ respectively. The corresponding metric is a nondegenerate metric of index p and $Q_V(v, w) = w^t \eta v$, where $v, w \in V$ and w^t denotes the vector transposed to w .

We give a new definition of H -type group that carries a sub-semi-Riemannian metric as follows.

Definition 2 Consider a mapping $A: V \rightarrow V$, which is assigned to each $u \in U$ and satisfies the following properties:

- i) $AA^t = A^t A = |u|^2 I_V$,
- ii) $Q_V(Av, w) = -Q_V(v, Aw)$,

where $v, w \in V$, $u \in U$ and I_V is the identity mapping in V . Then we can define the bilinear skew-symmetric map $[\cdot, \cdot]: V \times V \rightarrow U$ by

$$Q_U(u, [v, w]) = Q_V(Av, w).$$

We can now define a Lie algebra $\mathcal{G} = U \oplus V$ with a Lie structure on it:

$$[u + v, r + w] = [v, w] \quad (u, r \in U, \quad v, w \in V).$$

Constructed in this way Lie algebra is graded and 2-step nilpotent with center U . We shall call such algebras *H-type algebras with nondegenerate metric* $Q_V(\cdot, \cdot)$. We shall refer to associated Lie group G as *H-type group with nondegenerate metric* $Q_V(\cdot, \cdot)$.

This definition is equivalent to Definition 1 of Kaplan, but written in terms of the matrix A which is skew-symmetric with respect to the semi-Riemannian product $Q_V(\cdot, \cdot)$. As before, the Lie algebra \mathcal{G} is identified with the tangent space to G at the identity $T_e G$. Let L_σ denote the left translation on G by an element $\sigma \in G$. Then the mapping $D: \sigma \mapsto dL_\sigma(V)$ is a horizontal distribution. Push-forward allows to define the scalar product on the entire D :

$$Q_D(\cdot, \cdot)(\sigma) = Q_V(dL_{\sigma^{-1}\cdot}, dL_{\sigma^{-1}\cdot})(e).$$

The triple (G, D, Q_D) is a sub-semi-Riemannian manifold and will be referred to as a sub-semi-Riemannian Heisenberg-type group.

We investigate the problem concerning finding causal geodesics passing through the identity with a given initial vector on such groups. Let $\{V_i, U_\alpha\}$, $i = 1, \dots, n$, $\alpha = 1, \dots, m$ be an orthonormal left-invariant basis on V and U respectively. By geodesic we mean a curve $\gamma: [0, 1] \rightarrow G$ such that $\dot{\gamma} \in \mathcal{D}$ and $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$, where ∇ is a Levi-Civita connection that is compatible with the semi-Riemannian metric obtained by left translations of $Q_V(\cdot, \cdot) + Q_U(\cdot, \cdot)$. A curve $t \mapsto \gamma(t) \in G$ will be described by means of the vector-valued functions $t \mapsto v(t) \in \mathcal{D}$ and $t \mapsto u(t) \in dL_{\gamma(t)}U$ by $\gamma(t) = \exp(v(t), u(t))$. Then in terms of global coordinates on G one gets $\dot{\gamma} = \sum_{i=1}^n \dot{v}_i V_i + \sum_{\alpha=1}^m \dot{u}_\alpha U_\alpha \in T_\gamma G$. The equation $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ for the geodesic through the identity with the initial vector $\dot{\gamma}(0) = (\dot{v}^0, \dot{u}^0)$ is equivalent to the following system with initial conditions $(v(0), u(0)) = (0, 0)$, $(\dot{v}(0), \dot{u}(0)) = (\dot{v}^0, \dot{u}^0)$

$$\begin{cases} \ddot{v} - A\dot{v} = 0, \\ \dot{u} + \frac{1}{2}[\dot{v}, v] = \dot{u}^0, \\ v(0) = 0, \quad u(0) = 0 \\ \dot{v}(0) = \dot{v}^0, \quad \dot{u}(0) = \dot{u}^0. \end{cases} \quad (2.5.2)$$

The causal character of the geodesic is identical to the causal character of the initial vector. Observe that the first equation of the system (2.5.2) coincides with the one for sub-Riemannian geodesics when A is skew-symmetric with respect to Riemannian product on V (see [Kap81]) and the second equation is the condition for the curve γ to be horizontal.

We notice several important properties of the matrix A which will allow us to describe qualitatively solutions of the system (2.5.2):

- The matrix A has n mutually orthogonal unit eigenvectors and is a diagonalizable matrix (over \mathbb{C} in general).

- All eigenvalues of A have the same absolute value $|\dot{u}^0|$.
- If λ is an eigenvalue of A , then $-\lambda$ is also an eigenvalue of A .
- If $\lambda = \alpha + i\beta$ is an eigenvalue of A , then $\bar{\lambda} = \alpha - i\beta$ is also an eigenvalue of A .
- The following decomposition of A is possible:

$$A = PDP^{-1}, \tag{2.5.3}$$

where P is an orthogonal matrix and D is a block-diagonal matrix with (2×2) and (4×4) blocks of the following possible forms:

$$\begin{pmatrix} 0 & |\dot{u}^0| \\ |\dot{u}^0| & 0 \end{pmatrix}, \tag{2.5.4}$$

$$\begin{pmatrix} 0 & |\dot{u}^0| \\ -|\dot{u}^0| & 0 \end{pmatrix}, \tag{2.5.5}$$

and

$$\begin{pmatrix} \alpha_k & \beta_k & & 0 \\ -\beta_k & \alpha_k & & \\ & 0 & -\alpha_k & -\beta_k \\ & & \beta_k & -\alpha_k \end{pmatrix}, \quad k = 1, \dots, \frac{n}{4} - \frac{r}{2},$$

where $\alpha_k^2 + \beta_k^2 = |\dot{u}^0|^2$, $k = 1, \dots, \frac{n}{4} - \frac{r}{2}$, and r is a number of cells of types (2.5.4) and (2.5.5) together.

- We may represent the matrix A as a linear combination of matrices ηj_α , $\alpha = 1, \dots, m$:

$$A = \dot{u}_1^0 \eta j_1 + \dots + \dot{u}_m^0 \eta j_m,$$

where the matrix η is defined in (2.5.1) and $j_\alpha: V \rightarrow V$, $\alpha = 1, \dots, m$, are standard generators for the representation j of the Clifford algebra $\text{Cl}(U, m)$ on V from Definition 1, such that $A = \eta j(\dot{u}^0)$. Matrices corresponding to endomorphisms j_α are skew-symmetric, anti-commuting and can be viewed as square roots of $-I_V$.

- If the index p of Q_V is even, then A has no real eigenvalues and there are no cells of type (2.5.4) in the decomposition (2.5.3). If the index p is odd, then A has two real eigenvalues $\pm|\dot{u}^0|$ and there is only one cell of type (2.5.4) in the decomposition (2.5.3).

Recall that if λ is an eigenvalue of A then there exists a subspace V_λ of V such that $A(V_\lambda) \subset V_\lambda$. In paper *D* we show that if λ is real, then the projection of the curve $v(t)$ onto the 2-dimensional space $V_\lambda \cup V_{-\lambda}$ is a branch of hyperbola. The amazing part is that this projection is identical to the horizontal part (2.2.1) of the geodesic in sub-Lorentzian Heisenberg group \mathbb{H}_L^1 .

If λ is purely imaginary, we show that the projection of $v(t)$ onto the corresponding 2-dimensional space $V_\lambda \cup V_{-\lambda}$ is a circle as in case of geodesics of the sub-Riemannian Heisenberg

group \mathbb{H}_R^1 . This is a new, though not surprising result since in the sub-Riemannian case the matrix A is skew-symmetric with respect to the positively definite inner product and has only purely imaginary eigenvalues, and as it is known, the geodesic on \mathbb{H}_R^1 between the origin in \mathbb{R}^3 and the point (x, y, z) is the lift of a circular arc joining the origin with (x, y) , whose convex hull has area z .

For complex eigenvalue $\lambda = \alpha + i\beta$ we show that the projection of $v(t)$ onto the corresponding 2-dimensional space $V_\lambda \cup V_{-\lambda}$ is a logarithmic spiral (see Figure 2.6) and coincides with the horizontal part of the geodesic in sub-semi-Riemannian Quaternion group \mathbb{Q}_{sR} with the metric of index 2.

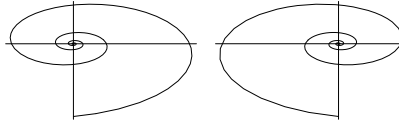


Figure 2.6: Graphics of logarithmic spirals for $\alpha + i\beta$ and $\alpha - i\beta$ respectively.

Thus, the horizontal part of a geodesic on a sub-semi-Riemannian H -type group is built from the horizontal parts of the geodesics on \mathbb{H}_R^1 , \mathbb{H}_L^1 and \mathbb{Q}_{sR} . The reason to this is in the underlying Clifford module structure of H -type groups that keeps popping up in different places throughout mathematics and physics showing its multiple significance.

2.6 Summary and open problems

We summarize that we computed normal geodesics for sub-semi-Riemannian H -type groups. The arised geometry exhibit features of both sub-Riemannian and semi-Riemannian geometries. We hope that with the basic principles and theorems obtained in the thesis the main business of sub-semi-Riemannian theory is ready to begin, namely, to develop a rich theory generalizing and integrating the magnificent results of classical sub-Riemannian and semi-Riemannian theories.

Open problems:

- Generalization to groups with nilpotency of higher steps than 2
- Abnormal extremals in sub-Lorentzian geometry and sub-semi-Riemannian geometry
- Formulation of the problem of finding geodesic in terms of control theory
- Uncovered possible hidden applications to fields such as physics
- Next step could be to consider U and V with both nondegenerate metrics $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_V$.
- Connectivity in case of index 2 of the metric.
- Connectivity on \mathbb{Q}_L (the rest of the cases).

Bibliography

- [BEE96] John K. Beem, Paul E. Ehrlich, and Kevin L. Easley. *Global Lorentzian geometry*, volume 202 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, second edition, 1996.
- [BG01] Valerii Berestovskii and Viktor Gichev. Metrized semigroups. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 279(Geom. i Topol. 6):24–60, 246–247, 2001.
- [BR96] André Bellaïche and Jean-Jacques Risler, editors. *Sub-Riemannian geometry*, volume 144 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1996.
- [BR08] Ugo Boscain and Francesco Rossi. Invariant Carnot-Carathéodory metrics on S^3 , $SO(3)$, $SL(2)$, and lens spaces. *SIAM J. Control Optim.*, 47(4):1851–1878, 2008.
- [BTV95] Jürgen Berndt, Franco Tricerri, and Lieven Vanhecke. *Generalized Heisenberg groups and Damek-Ricci harmonic spaces*, volume 1598 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995.
- [CC09] Ovidiu Calin and Der-Chen Chang. *Sub-Riemannian geometry*, volume 126 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2009. General theory and examples.
- [CCG07] Ovidiu Calin, Der-Chen Chang, and Peter Greiner. *Geometric analysis on the Heisenberg group and its generalizations*, volume 40 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2007.
- [CCM09] Ovidiu Calin, Der-Chen Chang, and Irina Markina. Geometric analysis on H -type groups related to division algebras. *Math. Nachr.*, 282(1):44–68, 2009.
- [CDKR91] Michael Cowling, Anthony H. Dooley, Adam Korányi, and Fulvio Ricci. H -type groups and Iwasawa decompositions. *Adv. Math.*, 87(1):1–41, 1991.
- [CDPT07] Luca Capogna, Donatella Danielli, Scott D. Pauls, and Jeremy T. Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, volume 259 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.
- [Cho39] Wei-Liang Chow. Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung. *Math. Ann.*, 117:98–105, 1939.

- [CM06] Der-Chen Chang and Irina Markina. Geometric analysis on quaternion \mathbb{H} -type groups. *J. Geom. Anal.*, 16(2):265–294, 2006.
- [CMV08] Der-Chen Chang, Irina Markina, and Alexander Vasil’ev. Sub-Lorentzian geometry on anti-de Sitter space. *J. Math. Pures Appl. (9)*, 90(1):82–110, 2008.
- [dC92] Manfredo Perdigão do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.
- [Gav77] Bernard Gaveau. Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents. *Acta Math.*, 139(1-2):95–153, 1977.
- [Gro99] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.
- [Gro02] Marek Grochowski. Geodesics in the sub-Lorentzian geometry. *Bull. Polish Acad. Sci. Math.*, 50(2):161–178, 2002.
- [Gro04] Marek Grochowski. On the Heisenberg sub-Lorentzian metric on \mathbb{R}^3 . In *Geometric singularity theory*, volume 65 of *Banach Center Publ.*, pages 57–65. Polish Acad. Sci., Warsaw, 2004.
- [Gro06] Marek Grochowski. Reachable sets for the Heisenberg sub-Lorentzian structure on \mathbb{R}^3 . An estimate for the distance function. *J. Dyn. Control Syst.*, 12(2):145–160, 2006.
- [Gro09] Marek Grochowski. Properties of reachable sets in the sub-Lorentzian geometry. *J. Geom. Phys.*, 59(7):885–900, 2009.
- [HE73] Stephen W. Hawking and George F. R. Ellis. *The large scale structure of space-time*. Cambridge University Press, London, 1973. Cambridge Monographs on Mathematical Physics, No. 1.
- [Kap80] Aroldo Kaplan. Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. *Trans. Amer. Math. Soc.*, 258(1):147–153, 1980.
- [Kap81] Aroldo Kaplan. Riemannian nilmanifolds attached to Clifford modules. *Geom. Dedicata*, 11(2):127–136, 1981.
- [Kap83] Aroldo Kaplan. On the geometry of groups of Heisenberg type. *Bull. London Math. Soc.*, 15(1):35–42, 1983.
- [LD09] Enrico Le Donne. Lecture notes on sub-riemannian geometry. Available at <http://www.math.ethz.ch/~ledonnee>, 2009.
- [Lou01] Pertti Lounesto. *Clifford algebras and spinors*, volume 286 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, second edition, 2001.

- [LS95] Wensheng Liu and Héctor J. Sussman. Shortest paths for sub-Riemannian metrics on rank-two distributions. *Mem. Amer. Math. Soc.*, 118(564):x+104, 1995.
- [Mon02] Richard Montgomery. *A tour of subriemannian geometries, their geodesics and applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.
- [O’N83] Barrett O’Neill. *Semi-Riemannian geometry*, volume 103 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983. With applications to relativity.
- [PR84] Roger Penrose and Wolfgang Rindler. *Spinors and space-time. Vol. 1*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1984. Two-spinor calculus and relativistic fields.
- [Ras38] P. K. Rashevskii. About connecting two points of complete nonholonomic space by admissible curve. *Uch. Zapiski Ped. Inst. K. Liebnicht*, (2):83–94, 1938.
- [Str86] Robert S. Strichartz. Sub-Riemannian geometry. *J. Differential Geom.*, 24(2):221–263, 1986.

Chapter 3

Papers A-E

- **Paper A:** A. Korolko, I. Markina. *Nonholonomic Lorentzian geometry on some H-type groups*. Journal of Geometric Analysis: Volume 19, Issue 4 (2009), 864–889.
- **Paper B:** A. Korolko, I. Markina. *Semi-Riemannian geometry with nonholonomic constraints*. To appear in December Issue (2011) of Taiwanese Journal of Mathematics.
- **Paper C:** A. Korolko, I. Markina. *Geodesics on H-type quaternion groups with sub-Lorentzian metric and their physical interpretation*. Complex Analysis and Operator Theory: Volume 4, Issue 3 (2010), 589–618.
- **Paper D:** A. Korolko. *Sub-semi-Riemannian geometry on H-type groups*. Submitted to Mathematica Scandinavica.
- **Paper E:** A. Korolko, R. McLachlan. *Symplectic integrators for nonholonomic systems*. In preparation.