A Continuous Max-Flow Approach to Minimal Partitions with Label Cost Prior

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Abstract. This paper investigates a convex relaxation approach for minimum description length (MDL) based image partitioning or labeling, which proposes an energy functional regularized by the spatial smoothness prior joint with a penalty for the total number of appearences or labels, the so-called *label cost prior*. As common in recent studies of convex relaxation approaches, the total-variation term is applied to encode the spatial regularity of partition boundaries and the auxiliary label cost term is penalized by the sum of convex infinity norms of the labeling functions. We study the proposed such convex MDL based image partition model under a novel continuous flow maximization perspective, where we show that the label cost prior amounts to a relaxation of the flow conservation condition which is crucial to study the classical duality of max-flow and min-cut! To the best of our knowledge, it is new to demonstrate such connections between the relaxation of flow conservation and the penalty of the total number of active appearences. In addition, we show that the proposed continuous max-flow formulation also leads to a fast and reliable max-flow based algorithm to address the challenging convex optimization problem, which significantly outperforms the previous approach by direct convex programming, in terms of speed, computation load and handling large-scale images. Its numerical scheme can by easily implemented and accelerated by the advanced computation framework, e.g. GPU.

1 Introduction

In this work, we study image labeling with the minimum description length principle (MDL) which naturally leads to regularities on both the spatial features, e.g. the minimum perimeter, and the total number of 'appearence' models. The MDL principle provides both an important concept of information theory and powerful tool to compress data, which states that 'the best hypothesis for a given set of data is the one that leads to the best compression of the data' (we refer to

[22] for a detailed review). It naturally leads to use fewer symbols or models to describe the given data [14]. In fact, such requirement of model reduction have been considered in model fitting problems of computer vision for a long history, e.g. image segmentation [18, 31, 25], motion segmentation [20, 24] etc.

In image segmentation or partitioning, it boils down to the penalization of the total number of appearence models or segments in addition to fitting data and regularities of segmentation boundaries. For the given n models/labels l_i, $i = 1 \ldots n$, Zhu and Yuille [31] proposed to partition images based on the minimization of the following energy function:

$$\min_{\Omega_i} \sum_{i=1}^{n} \Big\{ \int_{\Omega_i} \rho(l_i, x)\, dx \,+\, \lambda \int_{\partial \Omega_i} ds \Big\} \,+\, \gamma M \,, \tag{1}$$

where Ω_i, $i = 1, \ldots, n$, are homogeneous partitions corresponding to l_i, $M = \#\{\, 1 \le i \le n \,|\, \Omega_i \neq \emptyset \}$ gives the number of nonempty partitions, i.e. the so-called label cost prior. The data fidelity function $\rho(l_i, x) = -\log P(I_x | l_i)$ is a negative log-likelihood for model l_i at pixel x. The second term in (1) describes the total perimeter of the partitions and favors spatially regular boundaries with minimum length. Zhu and Yuille applied a local searching method, namely region competition, to approximate the highly nonconvex optimization problem (1). Their method slowly converges to a local minimum. Such MDL principle was further developed in the evolution of level sets to assist merging, e.g. [17, 6, 1]. On the other hand, the label cost prior was also considered in the recent developments of graph cuts: Hoeim et al [16] introduced a technique of $\alpha-$expansion combined with MDL to the application of object recognition; Delong et al [9] independently developed another $\alpha-$expansion method which can efficiently optimize more general energy functions with incorporated label cost prior.

Recently, Yuan & Boykov [29] studied the MDL based image partitioning problem (1) in the spatially continuous setting such that

$$\min_{u_i(x) \in \{0,1\}} \sum_{i=1}^{n} \Big\{ \int_{\Omega} u_i(x) \rho(l_i, x)\, dx \,+\, \lambda \int_{\Omega} |\nabla u_i| \; dx \Big\} \,+\, \gamma M \,, \tag{2}$$

subject to $\sum_{i=1}^{n} u_i(x) = 1$, where $u_i(x) \in \{0, 1\}$, $i = 1 \ldots n$, is the indicator function of $\Omega_i \subset \Omega$. Here M is the total number of 'active' partitions and the total-variation terms encode the total perimeter of partitions. The authors [29] proposed a convex relaxation formulation of (2) as

$$\min_{u_i(x)} \sum_{i=1}^{n} \Big\{ \int_{\Omega} u_i(x) \rho(l_i, x)\, dx \,+\, \lambda \int_{\Omega} |\nabla u_i(x)| \; dx \Big\} \,+\, \gamma \sum_{i=1}^{n} \max_{x \in \Omega} u_i(x) \tag{3}$$

$$\text{s.t.} \sum_{i=1}^{n} u_i(x) \,=\, 1 \,,\; u_i(x) \,\geq\, 0 \,;\; \forall x \in \Omega \tag{4}$$

where the labeling functions $u_i(x)$, $i = 1 \ldots n$, are relaxed by the pixelwise simplex constraint (4) and the label cost term in (2) is encoded by the sum of convex infinity norms of $u_i(x)$ instead. (3) proposes the minimization of a convex energy function over a convex constraint. It was optimized globally by a direct convex programming based solver in [29], which is not feasible to handle large-scale image data and highly time-consuming.

Actually, the first two terms of (3) together with the pixelwise convex constraint (4) correspond to the convex relaxation formulation of the minimal partition model, i.e. Potts model,

$$\min_{u_i(x)} \sum_{i=1}^{n} \int_{\Omega} \left\{ u_i(x)\rho(l_i, x) \, + \, \lambda \left| \nabla u_i \right| \right\} dx \, , \quad \text{subject to (4)} \, . \tag{5}$$

(5) was actively studied during the last years, e.g. [7, 19, 2, 3, 27], and fast algorithms were developed at the same time, upon standard theories of convex optimization. It is well-known that the regularities of the partition boundaries, i.e. the second term of (3) helps to smooth out small-scale partitions, hence reduce the total number of 'appearences' implicitly. However, only considering such smoothness prior often fails to recover correct labeling results and often leads to either over-partition or over-smoothness (see Fig. 1). This is in contrast to the model (3) which explicitly couples the label cost prior. Its result possesses optimalities of both geometry and model simplicity. We show this by Fig. 1.

Fig. 1. (a) shows the given image. (b) shows the image partition result of (3) computed by the proposed method in this paper. It gives only two segments left along with properly smoothed boundaries! (c)-(d) show the partition results computed by the Potts model (5) without the label cost prior, which give the results either oversegmented (more labels) or oversmoothed. In this example we have used 11 evenly spaced labels.

Contributions: we focus on the convex relaxation model (3) of the MDL based image partition and propose a novel flow maximization perspective, i.e. the continuous max-flow formulation which is dual to (3). We show that the label cost prior in (3) just corresponds to a new flexible flow conservation constraint on the proposed continuous max-flow formulation, i.e. relaxation of flow conservation amounts to minimizing the number of 'active' labels! This is new to the best knowledge of the authors. It is in contrast to the crucial flow conservation condition of the classical max-flow models, where the flow excess given at each image node or pixel strictly vanishes, e.g. [27, 26]. Moreover, we derive an efficient and reliable max-flow based algorithm which significantly outperforms the direct convex programming based method proposed by [29] in terms of speed, memory load and handling large-scale data. Compared to graph-cut based approaches [16, 9], our continuous max-flow approach comes with an elegant mathematical theory and is computed in the spatially continuous setting, which properly avoids metrication errors and can be easily implemented and accelerated on the advanced computation environment, e.g. GPU.

2 Previous Works

2.1 Convex Relaxation Approaches

Image labeling subject to the minimum perimeter, i.e. the Potts model, was intensively studied in both graph configuration [5] and spatially continuous settings [7, 27] etc. Current studies [19, 7, 3, 27, 30] focus on computing the associated convex relaxation formulation (5) in the spatially continuous context, which avoid directly tackling the non-convex energies, as level-sets or active-contour method, and can be solved efficiently.

Let the convex set S denote the pixelwise simplex constraint (4) of $u(x) = (u_1(x), \dots, u_n(x))^{\mathrm{T}}$. [30, 19] proposed an optimization method which involves two substeps within each iteration: one explores the pointwise simplex constraint $u(x) \in S$ and the other tackles the total-variation term. In [7, 23], Pock et al introduced a variant implementation of the constraint $u(x) \in S$, i.e. a tighter relaxation based on a multi-layered configuration, and gives a more complex constraint on the concerning dual variable p to avoid multiple counting. In contrast to the works of [30, 19, 7, 23] which tried to compute the labeling functions $u(x)$ of (5) directly, Bae et al [3] proposed to solve (5) based on its equivalent dual formulation. The nonsmooth dual formulation can then be efficiently approximated by a smooth convex energy function.

Max-Flow and Flow Conservation: In the very recent studies of [26, 28, 27], Yuan et al proposed the new continuous max-flow model which regards (5) as its dual formulation in the spatially continuous setting. As the hard constraint of the proposed max-flow model, the flow conservation condition should be strictly satisfied.

For the Potts model (5), i.e. $n \geq 3$, the spatially continuous flow configurations are given as [27]: Let Ω_i, $i = 1 \dots n$, be the n copies of the image domain Ω. For each $x \in \Omega$, the source flow $p_s(x)$ streams from the source s to the same position x of each Ω_i, $i = 1 \dots n$, simultaneously. For each $x \in \Omega$, the sink flow $p_i(x)$ is directed from x of each Ω_i, $i = 1 \dots n$, to the sink t. The spatial flow fields $q_i(x)$, $i = 1 \dots n$, are defined within each Ω_i, $i = 1 \dots n$.

The sink and spatial flow fields $p_i(x)$ and $q_i(x)$, $i = 1 \dots n$, are constrained by the capacities such that

$$|q_i(x)| \leq C_i(x)\,,\; p_i(x) \leq \rho(l_i, x)\,;\; i = 1 \dots n\,. \tag{6}$$

Especially, at each $x \in \Omega$, the source flow $p_s(x)$, the sink and spatial flows $p_i(x)$ and $q_i(x)$, $i = 1 \dots n$, satisfy the exact flow conservation conditions:

$$\operatorname{div} q_i(x) - p_s(x) + p_i(x) = 0\,, \quad i = 1 \dots n\,. \tag{7}$$

Likewise, the continuous max-flow problem is formulated as [27]:

$$\max_{p_s, p, q} \int_{\Omega} p_s(x)\, dx \tag{8}$$

subject to (6) and (7). [27] proved that (8) is dual to (5). Clearly, the three types of flow fields $p_s(x)$, $p_i(x)$ and $q_i(x)$, $i = 1 \dots n$, are connected by the flow conservation constraints (7). The labeling functions $u_i(x)$, $i = 1 \dots n$, just amount to the Lagrangian multipliers to the crucial flow-conservations [27].

Clearly, the flow conservation condition (7) plays the central role in the studies of the continuous max-flow model (8), and so for the theories of continuous max-flow and min-cut [26, 28].

2.2 Convex Relaxed MDL Approach

Now we review the convex relaxation approach [29] to the challenging nonconvex problem (2): Given n labels $\{l_1, \dots, l_n\}$, if the maximum of the labeling function $u_k(x) \in \{0, 1\}$, $1 \le k \le n$, over the whole image domain Ω is 1, there must be some pixel $x \in \Omega$ which is labeled by l_k, i.e. the label l_k must present in the final image labeling result. Hence we can apply the sum of labeling functions' infinity norms $\sum_{i=1}^n \max_{x \in \Omega} u_i(x)$ to denote the total number M of 'active' models. Then, (2) can be equivalently reformulated by

$$\min_{u_i(x) \in \{0,1\}} \sum_{i=1}^n \Big\{ \int_{\Omega} u_i(x) \rho(l_i, x)\, dx \, + \, \lambda \int_{\Omega} |\nabla u_i| \, dx \Big\} \, + \, \gamma \sum_{i=1}^n \max_{x \in \Omega} u_i(x) \tag{9}$$

$$\text{s.t.} \quad \sum_{i=1}^n u_i(x) \, = \, 1 \, , \quad \forall x \in \Omega \, .$$

Relax the binary constraint of the labeling functions $u_i(x) \in \{0, 1\}$ together with $\sum_{i=1}^n u_i(x) = 1$ to the pointwise simplex constraint (4), i.e. $u(x) := (u_1(x), \dots, u_n(x))^{\mathrm{T}} \in S$. The nonconvex optimization problem (9) can then be written as the continuous convex optimization problem (3), i.e.

$$\min_{u(x) \in S} \sum_{i=1}^n \{ \int_{\Omega} u_i(x) \rho(l_i, x)\, dx \, + \, \lambda \int_{\Omega} |\nabla u_i(x)| \, dx \} \, + \, \gamma \sum_{i=1}^n \max_{x \in \Omega} u_i(x) \tag{10}$$

where S denotes the pointwise simplex constraint (4). Obviously, the convex constrained convex optimization problem (10) can be solved globally. Its third term penalizes the infinity norm of each labeling function $u_i(x)$, $i = 1 \dots n$, which amounts to convex relaxation of the label cost prior.

3 Continuous Max-Flow Approach

In this section, we adopt the flow setting proposed in [27] and introduce a novel continuous max-flow formulation which is dual to the convex relaxed MDL-based labeling model (3) or (10). We show the label cost term is reduced to new flexible flow conservation constraints.

3.1 Continuous Max-Flow Formulation

In this section, we adopt the flow-maximization configurations and notations proposed in [27] and follow discussions in the above section.

By virtue of such continuous flow settings, the flow capacity constraints of flows $p_i(x)$ and $q_i(x)$, at $x \in \Omega$, are given in the same way as (6).

The flow conservation condition is formulated in a new flexible way:

$$\big(\operatorname{div} q_i - p_s + p_i\big)(x) \, \in \, R_i^\gamma \, , \; R_i^\gamma \, := \, \{ r_i(x) \, | \, \int_\Omega |r_i(x)| \, dx \, \leq \, \gamma \, \} \, ; \; i = 1 \ldots n \, . \quad (11)$$

Note: The new flow conservation condition (11) proposes that at each $x \in \Omega$, the total in-coming flow is not balanced by the total out-going flow. However, the total absolute flow excesses associated with each label l_i, $i = 1 \ldots n$, is controlled below γ as (11). This is in contrast to the exact flow conservation condition (7) in the classical max-flow theory, where the total in-coming flow should be strictly balanced by the total out-going flow.

We propose our new continuous max-flow model such that

$$\max_{p_s, p, q, r} \; \Big\{ P(p_s, p, q) \, := \, \int_\Omega p_s(x) \, dx \Big\} \quad (12)$$

subject to (6) and (11). In the following section, we study the equivalence between the proposed continuous max-flow formulation (12) and the convex relaxed MDL-based labeling model (3) or (10), especially for the case where $C(x) = \lambda$.

3.2 Equivalent Primal-Dual Model

We introduce the multiplier functions $u_i(x)$, $i = 1, \ldots, n$, to the new flow conservation condition (11). Therefore, we have its equivalent primal-dual model:

$$\max_{p_s, p, q, r} \min_u \Big\{ E(p_s, p, q, r; u) \, := \, \int_\Omega p_s \, dx \, + \, \sum_{i=1}^n \int_\Omega u_i (\operatorname{div} q_i - p_s + p_i - r_i) \, dx \Big\} \quad (13)$$

$$\text{s.t.} \;\; p_i(x) \, \leq \, \rho(l_i, x) \, , \; |q_i(x)| \, \leq \, C_i(x) \, ; \quad \int_\Omega |r_i(x)| \, dx \, \leq \, \gamma \, ; \; i = 1 \ldots n \, .$$

Rearranging the energy function $E(p_s, p, q, r; u)$ of (13), we have

$$E \, = \, \int_\Omega (1 - \sum_{i=1}^n u_i) \, p_s \, dx + \sum_{i=1}^n \Big\{ \int_\Omega u_i \, p_i \, dx - \int_\Omega u_i \, r_i \, dx + \int_\Omega u_i \operatorname{div} q_i \, dx \Big\} \, . \quad (14)$$

For the primal-dual model (13), the conditions of the minimax theorem [11] are all satisfied. That is, the constraints of flows are convex, and the energy function is linear to both the multiplier u and the flow functions p_s, p and q, hence convex l.s.c. for fixed u and concave u.s.c. for fixed p_s, p and q. This confirms the strong dualities of (13) and the existence of at least one saddle point [11, 12]. It follows that the min and max operators of (13) can be interchanged:

$$\max_{p_s, p, q, r} \big\{ \min_u \, E(p_s, p, q, r; u) \big\} \, = \, \min_u \big\{ \max_{p_s, p, q, r} \, E(p_s, p, q, r; u) \big\} \, . \quad (15)$$

3.3 Equivalent Dual Model

Now we consider the optimization of (13) by switching the max-min order of the left hand side of (15), i.e. first maximizing $E(p_s, p, q, r; u)$ over the functions $p_s(x)$, $p(x)$, $q(x)$ and $r(x)$. Then we have

Proposition 1 *The maximization of* (13) *over the flow functions* $p_s(x)$, $p(x)$, $q(x)$ *and* $r(x)$ *amounts to the following dual model:*

$$\min_{u(x) \in S} D(u) := \sum_{i=1}^{n} \Big\{ \int_\Omega u_i(x)\, \rho(l_i, x)\, dx + \int_\Omega C_i(x)\, |\nabla u_i| \, dx \Big\} + \gamma \sum_{i=1}^{n} \max_{x \in \Omega} u_i(x) \quad (16)$$

which is equivalent to (3) *and* (10) *for the special case when* $C(x) = \lambda$.

To see Prop. 1, we follow the same analyzes as [26, 28] , which gives

$$\max_{p_i(x) \leq \rho(l_i, x)} \int_\Omega u_i p_i \, dx = \int_\Omega u_i(x) \rho(l_i, x)\, dx \quad (17)$$

together with $u_i(x) \geq 0$, $i = 1 \dots n$.

For the maximization of (14) over q_i and r_i, $i = 1 \dots n$, it is well-known [13, 15] that

$$\max_{|q_i(x)| \leq C_i(x)} \int_\Omega u_i \operatorname{div} q_i \, dx = \int_\Omega C_i(x)\, |\nabla u_i| \, dx \,, \quad (18)$$

and by the symmetry of the L_1-ball R_i^γ, we have

$$\max_{r_i(x) \in R_i^\gamma} - \int_\Omega u_i r_i \, dx = \gamma \max_{x \in \Omega} u_i(x) \,. \quad (19)$$

Moreover, observe the source flow $p_s(x)$ is unconstrained, then the maximization of (14) over p_s gives $1 - \sum_{i=1}^{n} u_i(x) = 0$, $\forall x \in \Omega$. Therefore, we have

Proposition 2 *The continuous max-flow model* (12)*, the primal-dual model* (13) *and the dual model* (16) *are equivalent to each other.*

4 Fast Continuous Max-Flow Algorithm

Observe that the energy function of the primal-dual model (13) is nothing but the Lagrangian function of the proposed max-flow formulation (12) and the labeling functions $u_i(x)$, $i = 1 \dots n$, give the corresponding multipliers to the introduced new flow conservation constraints (11). Observe this, we derive the new algorithm for (3) based on its equivalent continuous max-flow model (12).

We define the augmented Lagrangian function

$$L_c(p_s, p, q, r, u) := \int_\Omega p_s \, dx + \sum_{i=1}^{n} \langle u_i, \operatorname{div} q_i - p_s + p_i - r_i \rangle - \frac{c}{2} \sum_{i=1}^{n} \| \operatorname{div} q_i - p_s + p_i - r_i \|^2$$

where $c > 0$ and the auxiliary L_2 penalty term facilitates the vanishing of $\operatorname{div} q_i(x) - p_s(x) + p_i(x) - r_i(x)$ at each $x \in \Omega$.

Now we construct our multiplier-based max-flow algorithm based on the augmented Lagrangian scheme [4]. Each k-th iteration includes the following steps:

– Maximize the energy $L_c(p_s, p, q, r, u)$ over the spatial flows $q_i(x)$, $i = 1 \ldots n$, by fixing other variables, which amounts to:

$$q_i^{k+1} := \arg \max_{\|q_i\|_\infty \leq \lambda} -\frac{c}{2} \left\| \operatorname{div} q_i - C^k(x) \right\|^2 , \tag{20}$$

where

$$C^k(x) \;=\; -p_i^k(x) + p_s^k(x) + r_i^k(x) + u_i^k(x)/c \, .$$

(20) can be approximated by a Chambolle-like projection-descent step [8].

– Maximize the energy $L_c(p_s, p, q, r, u)$ over the sink flows $p_i(x)$, $i = 1 \ldots n$, by fixing other variables, which corresponds to

$$p_i^{k+1} := \arg \max_{p_i(x) \leq \rho(\ell_i, x)} -\frac{c}{2} \left\| p_i - D^k \right\|^2 , \tag{21}$$

where

$$D^k(x) \;=\; -\operatorname{div} q_i^{k+1}(x) + p_s^k(x) + r_i^k(x) + u_i^k(x)/c \, .$$

(21) can be directly computed pointwise at each $x \in \Omega$.

– Maximize the energy $L_c(p_s, p, q, r, u)$ over $r_i(x)$, $i = 1 \ldots n$, by fixing other variables, which amounts to

$$r_i^{k+1} := \arg \max_{r_i(x) \in R_i^\gamma} -\frac{c}{2} \left\| r_i - F^k \right\|^2 \tag{22}$$

where

$$F^k(x) \;=\; \operatorname{div} q_i^{k+1}(x) - p_s^k(x) + p_i^k(x) - u_i^k(x)/c \, .$$

(22) can be addressed by the projection of $F^k(x)$ to the L_1-ball R_i^γ with the fast projection algorithm of linear $O(N)$ complexity [21, 10].

– Optimize the energy $L_c(p_s, p, q, r, u)$ over the unconstrained source flow p_s and

$$p_s^{k+1} := \arg\max_{p_s} \int_\Omega p_s \, dx - \frac{c}{2} \sum_{i=1}^n \left\| p_s - G^k \right\|^2 \tag{23}$$

where

$$G^k(x) \;=\; p_i^{k+1}(x) + \operatorname{div} q_i^{k+1}(x) - r_i^{k+1}(x) - u_i^k(x)/c \, .$$

Finally, update the multiplier functions $u_i(x)$, $i = 1 \ldots n$, as follows

$$u_i^{k+1} = u_i^k - c \left(\operatorname{div} q_i^{k+1} - p_s^{k+1} + p_i^{k+1} \right) . \tag{24}$$

Both (23) and (24) can be computed in a closed form.

5 Numerical experiments

Experiments demonstrate the advantages of the label cost model over Potts model and the superior efficiency of the new max-flow algorithm over the previous SOCP method [29]. Gray scale image segmentation can be modeled as (1) with the data term

$$\rho(l_i, x) \, = \, |f(x) - l_i|^p \ , \quad i = 1 \ldots n \, ; \quad p = 1 \text{ or } 2$$

where $l_1, ..., l_n$ are predefined gray values, for instance the gaussian distribution model of images. For colour image segmentation, the labels are instead colour vectors $(l_1^j ... l_n^j)$, where $j \in \{r, g, b\}$. The data term is modeled as

$$\rho(\ell_i, x) \, = \sum_{j \in \{r,g,b\}} \left| f(x) - l_i^j \right|^p \ , \quad i = 1, \ldots, n \, ; \quad p = 1 \text{ or } 2 \, .$$

In the experiments of Fig. 2 - 3, $\ell_1, ...\ell_n$ are chosen as evenly spaced gray values in the interval between the smallest and largest gray value. For the color image, $\ell_1, ..., \ell_n$ are evenly spaced color vectors. The results of Potts model are shown in the 2nd coloumns. It may produce more labels than desired. On the other hand, the label cost prior, 3rd and 4th coloumn, greatly greatly helps to generate less labels along with properly smoothed edges, such that the objects are more clearly distinguished. The label cost model allows to reduce the number of labels without simultaneously oversmoothing the partition boundaries, as Potts model does (see also Fig. 1).

The efficiency of the proposed max-flow algorithm is significantly superior to the SOCP implementation in [29]. Whereas [29] requires several hours to converge for even one small input image (150×150), the proposed max-flow algorithm converges around 2 minutes for a large image (about 500×500) (serial matlab implementation). The convergence is just a little slower than the max-flow algorithm [27] without label cost prior, due to the projections onto the L^1-ball (22). The algorithm [29] even fails to converge when the problem size is too big, due to the intense memory requirement. For instance, the problem in Fig. 3 bottom (21 labels) could not be handled by [29] for the Ubuntu desktop we used (Intel Xeon 3.06G, 16G Memory).

6 Conclusions

We studied a convex relaxed MDL based labeling model (3) in this work, and showed its effectiveness for image partitioning in minimizing both the total number of 'active' labels and the perimeter of partitions [29]. More specially, we proposed and investigated a novel continuous max-flow model which is dual to (3). We showed that the label cost prior introduced in (3) just corresponds to the new flexible constraint of flow conservation under the flow-maximization perspective. This is in contrast to the strict flow balance for the classical max-flow/min-cut theories. In numerics, the proposed continuous max-flow model naturally leads

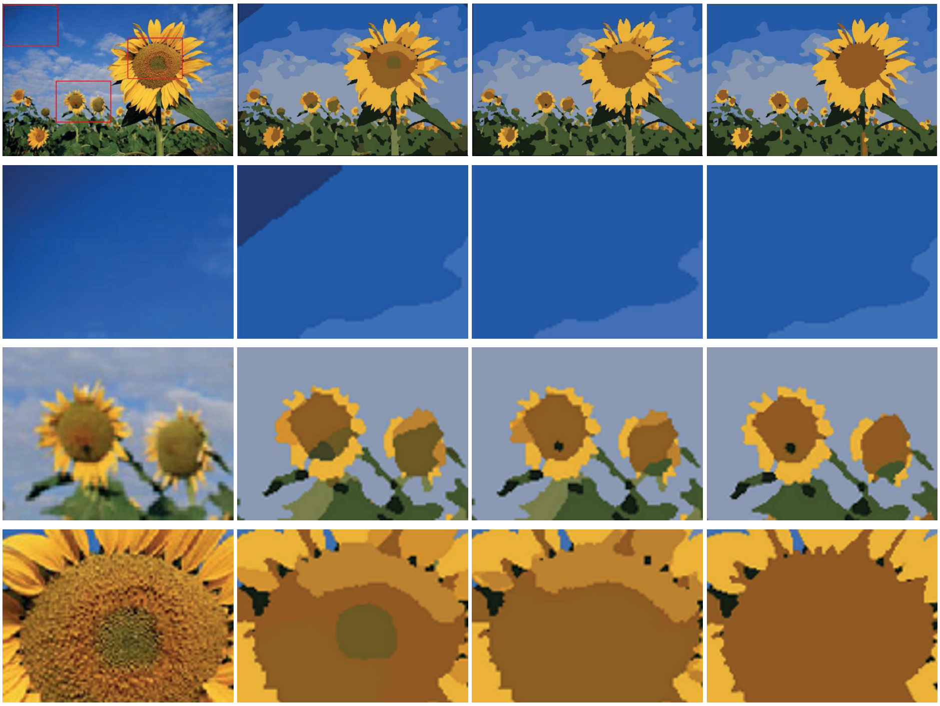

Fig. 2. Image segmentation with 15 labels. **From left to right:** input image (508×336); labeling with Potts model; label cost model with $\gamma = 250$; label cost model with $\gamma = 1000$. *Top:* Full image. *2nd - 4th row:* zoomed parts (red-line croped areas of the input image). Visible differences can be clearly noticed in the zoomed images.

to a new fast max-flow based algorithm, which greatly outperforms the direct convex programming method proposed in [29] in terms of efficiency, computation load, implementation on GPUs, and handling large-scale image data.

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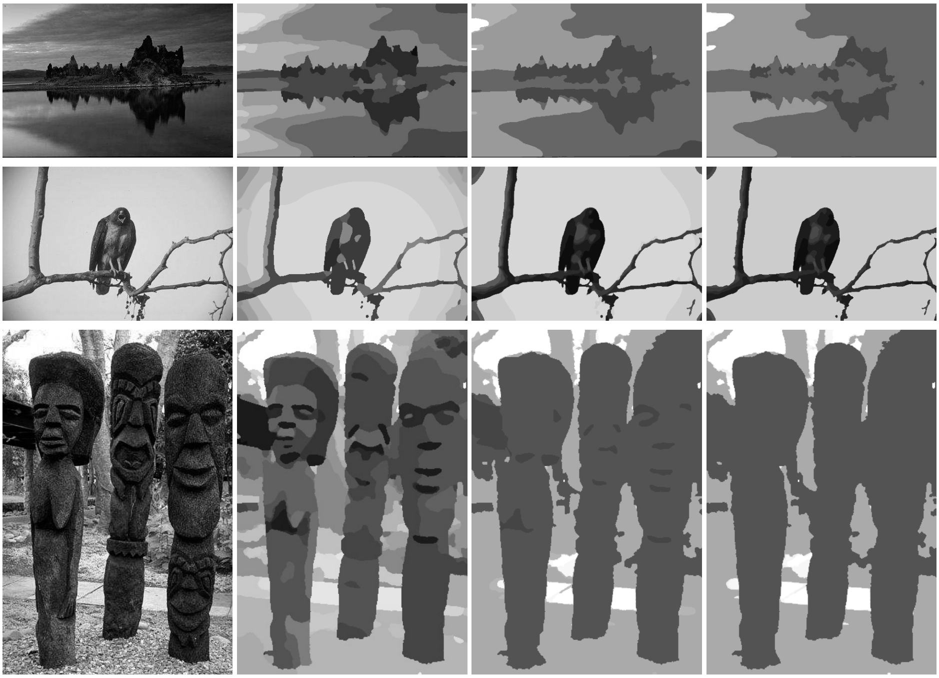

Fig. 3. Top: labeling by 11 labels. From left to right: input (481×321); Potts model $\lambda = 0.1$; MDL model $\lambda = 0.1, \gamma = 2000$; MDL model $\lambda = 0.1, \gamma = 3000$. **Middle:** labeling by 21 labels. From left to right: input (321×481); Potts model $\lambda = 0.05$; MDL model $\lambda = 0.05, \gamma = 3000$; MDL model $\lambda = 0.05, \gamma = 8000$, **Bottom:** labeling by 21 labels. From left to right: input (321×481); Potts model $\lambda = 0.15$; MDL model $\lambda = 0.15, \gamma = 2000$; MDL model $\lambda = 0.15, \gamma = 3000$.

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