Existence of a classical solution of a parabolic PIDE associated with ruin probability

June 18, 2012

Abstract

In this article we will prove existence of a classical solution of the integro-differential equation for ruin probability in finite time stated in Paulsen (2008).

Contents

1 Risk process model **1**

2 PIDE for the ruin probability **2**

3 Existence and uniqueness on a truncated domain **4**

4 Global estimates **24**
- 4.1 Constant coefficients 25
  - 4.1.1 Global estimates for a subproblem with constant coefficients 25
  - 4.1.2 Global estimates for a subproblem with an integral term and constant coefficients 46
- 4.2 Unbounded coefficients 58
  - 4.2.1 Global estimates for a subproblem with unbounded coefficients 58
  - 4.2.2 Regularity estimates for for a subproblem with an integral term and unbounded coefficients 73

5 Existence on an unbounded domain **82**
- 5.1 Constant coefficients 84
- 5.2 Unbounded coefficients 101

1 Risk process model

In Paulsen (2008) the risk model consists of a basic risk process P_t with $P_0 = 0$, and a return on investment generating process R, with $R_0 = 0$. The risk process is defined as

$$Y_t := y + P_t + \int_0^t Y_{s-} dR_s, \tag{1.0.1}$$

with initial value $Y_0 = y$. In the above the stochastic process R_t is assumed to be a diffusion process of the form

$$R_t = rt + \sigma_R W_{R,t}, \tag{1.0.2}$$

where r and σ_R are nonnegative constants and W_R is a Brownian motion. P_t is assumed to be a jump-diffusion process of the form

$$P_t = pt + \sigma_P W_{P,t} - \sum_{i=1}^{N_t} S_i, \tag{1.0.3}$$

where p and σ_P are nonnegative constants and $W_{P,t}$ is a Brownian motion, N_t is a Poisson process with rate λ, and the $\{S_i\}$ are positive, independent and identically distributed random variables with distribution function F. $W_{P,t}, W_{R,t}, N_t$ and the $\{S_i\}$ are assumed to be mutually independent. The time of ruin is defined as the stopping time

$$\tau = \inf\{t : Y_t < 0\}, \tag{1.0.4}$$

with $\tau = \infty$ if Y stays nonnegative. In the case that $\sigma_P > 0$ the infinite variation of the Brownian process $W_{P,t}$ ensures that

$$\inf\{t : Y_t < 0\} = \inf\{t : Y_t \leq 0\}.$$

With τ defined as above the probability of ruin in a given finite time t is defined as

$$\psi(y,t) = P(\tau \leq t | Y_0 = y).$$

2 PIDE for the ruin probability

Let F be the distribution function of a probability measure that assigns no mass to $(-\infty, 0]$. For every $(y,t) \in (0,\infty) \times (0,T]$, let L be the parabolic differential operator

$$Lh(y,t) = \frac{1}{2}(\sigma_P^2 + \sigma_R^2 y^2)\frac{\partial^2 h(y,t)}{\partial y^2} + (p + ry)\frac{\partial h(y,t)}{\partial y},$$

and let A be the integro-differential operator

$$Ah(y,t) = Lh(y,t) + \lambda \int_0^y h(y-z,t)dF(z) - \lambda h(y,t).$$

In Paulsen (2008) it is stated that the ruin probability should be the solution of the following partial integro-differential equation (PIDE):

$$\begin{cases} \psi(y,0) = 0, \quad y > 0 \\ \psi(0,t) = 1, \quad t \in [0,T] \\ \frac{\partial \psi(y,t)}{\partial t} - A\psi(y,t) = \lambda \bar{F}(y), \quad (y,t) \in (0,\infty) \times (0,T]. \end{cases} \tag{2.0.5}$$

In the above $\bar{F}(y) = 1 - F(y)$ is the tail distribution function. Asymptotically a solution of equation (2.0.5) should satisfy

$$\lim_{y \to \infty} \psi(y,t) = 0, \quad t \in [0,T]. \tag{2.0.6}$$

We observe that the operator A is linear and uniformly elliptic, while the initial condition, the boundary condition, and all the coefficients are all analytic for $y > 0$. This suggests that equation (2.0.5) "should" have a smooth solution, at least if the distribution function $F(z)$ is smooth. A closer look, however, reveals a number of properties that violate the standard assumptions in the literature on PDE and PIDE problems.

- The domain is unbounded.
  Some literature, in particular on PDE's, discusses problems with unbounded domains. In general, however, these treatises require that at least the coefficients of the second space derivative be bounded. In our case the coefficient of the second space derivative is

$$\frac{1}{2}(\sigma_P^2 + \sigma_R^2 y^2),$$

  which is obviously not bounded for $y \in (0, \infty)$, when $\sigma_R > 0$.

- Violation of compatibility condition.
  The initial condition dictates that $\lim_{y\downarrow 0} \psi(y, 0) = 0$, whereas the boundary condition dictates that $\lim_{t\downarrow 0} \psi(0, t) = 1 \neq 0$. The initial condition and the boundary condition are thus incompatible. Any solution of (2.0.5) must hence be discontinuous at the origin, which violates the requirement that a classical solution must be continuous at at the boundary.

- Asymptotic boundary condition
  In addition to the difficulties mentioned above we need to verify that, for any $t \in (0, T]$, $\lim_{y\uparrow\infty} \psi(y, t) = 0$.

The upshot of this is that standard theory does not immediately ensure existence and uniqueness of a solution of equation (2.0.5). Instead we have to rely on more indirect methods, and work mostly with an emulation of (2.0.5) on a truncated domain $(0, \kappa) \times (0, 1]$, with the more standard boundary equation $\psi(\kappa, t) = 0$ for $t \in [0, 1]$. Since there can be no *classical* solution we will in this article instead look for a solution that satisfies the requirements of a classical solution, including continuity to the boundary, except at the origin. We will call such a solution a classical solution, except at the origin. The last result in Section 3, Theorem 3.0.4 establishes the existence of such a classical solution, except at the origin, on any truncated domain.

Our objective is to establish existence on an unbounded domain, with the asymptotic boundary condition. For this we will need some estimates which we will obtain in Section 4. To derive these estimates we assume that the coefficients satisfy $\sigma_P > 0$ and either $\sigma_R = r = 0$ or $\sigma_R > 0$ and that the tail distribution $\bar{F}$ satisfies

$$\bar{F}(\zeta) \leq C\,(1 + \zeta)^{-\beta}\,, \quad \zeta \geq 0,$$

for some positive constants C and β.

In the last part of the article, Section 5, we will establish in Theorem 5.1.2 and Theorem 5.2.2 the existence of a classical solution on the original unbounded domain which even satisfies the asymptotic boundary condition.

3 Existence and uniqueness on a truncated domain

In this paper we will be working with the Green spaces defined in chapter VII in Garroni and Menaldi (1992). To be compatible with the definition of these spaces we will henceforth assume that $T = 1$.

In order to standardize equation (2.0.5) with $T \neq 1$ we can just substitute the parameters p, σ_P, σ_R and λ with $pT, \sigma_P\sqrt{T}, \sigma_R\sqrt{T}$ and λT. We can therefore without loss of generality assume that $T = 1$, which we will do in the rest of the paper. In order to have all the coefficients of A bounded we introduce a truncated domain $(0, \kappa)$ for y. The upper boundary condition is now in a standard form.

$$\begin{cases} \psi_\kappa(y,0) = 0, \quad y \in (0,\kappa)\,, \\ \psi_\kappa(0,t) = 1, \quad t \in [0,1]\,, \\ \psi_\kappa\,(\kappa,t) = 0, \quad t \in [0,1]\,, \\ \frac{\partial \psi_\kappa(y,t)}{\partial t} - A\psi_\kappa\,(y,t) = \lambda \bar{F}(y), \quad (y,t) \in (0,\kappa)\times(0,1]. \end{cases} \tag{3.0.7}$$

Taking a cue from Garroni and Menaldi (2002) we will look for a solution $\psi_\kappa(y,t)$ of (3.0.7) by considering the three equations

$$\begin{cases} \psi_{1,\kappa}(y,0) &= 0, \quad y \in (0,\kappa)\,, \\ \psi_{1,\kappa}(0,t) &= 1, \quad t \in [0,1]\,, \\ \psi_{1,\kappa}\,(\kappa,t) &= 0, \quad t \in [0,1]\,, \\ \frac{\partial \psi_{1,\kappa}(y,t)}{\partial t} &= \frac{1}{2}\sigma_P^2 \frac{\partial^2 \psi_{1,\kappa}(y,t)}{\partial y^2} + p\frac{\partial \psi_{1,\kappa}(y,t)}{\partial y}, \quad (y,t) \in (0,\kappa)\times(0,1], \end{cases} \tag{3.0.8}$$

$$\begin{cases} \psi_{2,\kappa}(y,0) &= 0, \quad y \in (0,\kappa)\,, \\ \psi_{2,\kappa}(0,t) &= 0, \quad t \in [0,1]\,, \\ \psi_{2,\kappa}\,(\kappa,t) &= 0, \quad t \in [0,1]\,, \\ \frac{\partial \psi_{2,\kappa}(y,t)}{\partial t} - L\psi_{2,\kappa} &= H_{1,\kappa}\,(y,t)\,, \quad (y,t) \in (0,\kappa)\times(0,1], \end{cases} \tag{3.0.9}$$

where

$$\begin{aligned} H_{1,\kappa}\,(y,t) = \frac{1}{2}\sigma_R^2 y^2 \frac{\partial^2 \psi_{1,\kappa}\,(y,t)}{\partial^2 y^2} + ry\frac{\partial \psi_{1,\kappa}\,(y,t)}{\partial y} - \lambda\psi_{1,\kappa}\,(y,t) \\ + \lambda \int_0^y \psi_{1,\kappa}\,(y-z,t)\,dF(z) + \lambda\bar{F}(y), \end{aligned}$$

and

$$\begin{cases} \psi_{3,\kappa}(y,0) &= 0, \quad y \in (0,\kappa)\,, \\ \psi_{3,\kappa}(0,t) &= 0, \quad t \in [0,1]\,, \\ \psi_{3,\kappa}\,(\kappa,t) &= 0, \quad t \in [0,1]\,, \\ \frac{\partial \psi_{3,\kappa} y,t)}{\partial t} - A\psi_{3,\kappa}\,(y,t) &= H_{2,\kappa}\,(y,t)\,, \quad (y,t) \in (0,\kappa)\times(0,1]. \end{cases} \tag{3.0.10}$$

Here

$$H_{2,\kappa}\,(y,t) = -\lambda\psi_{2,\kappa}(y,t) + \lambda \int_0^y \psi_{2,\kappa}\,(y-z,t)\,dF(z).$$

Now we focus our attention on the first of the above three equations (3.0.8). Existence and regularity of a solution to that equation can be determined from the close relation between this equation and a certain *passage time* of the Brownian motion $W_{p,t}$. Consider the following three equations.

$$\begin{cases} \psi_1^*(y,0) & = 0, \quad y > 0, \\ \psi_1^*(0,t) & = 1, \quad t \in [0,1], \\ \lim_{y\to\infty} \psi_1^*(y,t) & = 0, \quad t \in [0,1], \\ \frac{\partial \psi_1^*(y,t)}{\partial t} & = \frac{1}{2}\sigma_P^2 \frac{\partial^2 \psi_1^*(y,t)}{\partial y^2}, \quad (y,t) \in (0,\infty)\times(0,1]. \end{cases} \tag{3.0.11}$$

$$\begin{cases} \psi_{1,\kappa}^*(y,0) & = 0, \quad y \in (0,\kappa), \\ \psi_{1,\kappa}^*(0,t) & = 1, \quad t \in [0,1], \\ \psi_{1,\kappa}^*(\kappa,t) & = 0, \quad t \in [0,1], \\ \frac{\partial \psi_{1,\kappa}^*(y,t)}{\partial t} & = \frac{1}{2}\sigma_P^2 \frac{\partial^2 \psi_{1,\kappa}^*(y,t)}{\partial y^2}, \quad (y,t) \in (0,\kappa)\times(0,1]. \end{cases} \tag{3.0.12}$$

and

$$\begin{cases} \psi_1(y,0) & = 0, \quad y > 0, \\ \psi_1(0,t) & = 1, \quad t \in [0,1], \\ \lim_{y\to\infty} \psi_1(y,t) & = 0, \quad t \in [0,1], \\ \frac{\partial \psi_1(y,t)}{\partial t} & = \frac{1}{2}\sigma_P^2 \frac{\partial^2 \psi_1(y,t)}{\partial y^2} + p\frac{\partial \psi_1(y,t)}{\partial y}, \quad (y,t) \in (0,\infty)\times(0,1]. \end{cases} \tag{3.0.13}$$

Let

$$\tau_0 = \inf\{t \geq 0 : y + \sigma_P W_{P,t} < 0\},$$
$$\tilde{\tau}_0 = \inf\{t \geq 0 : y + pt + \sigma_P W_{P,t} < 0\},$$
$$\tau_\kappa = \inf\{t \geq 0 : y + \sigma_P W_{P,t} > \kappa\},$$

and let

$$\tilde{\tau}_\kappa(y) = \inf\{t \geq 0 : y + pt + \sigma_P W_{P,t} > \kappa\}.$$

Since $\psi_1^*(y,t)$ is just the probability $P(\tau_0 \leq t)$ it is well known that

$$\psi_1^*(y,t) = \sqrt{\frac{2}{\pi}} \int_{\frac{y}{\sigma_P\sqrt{t}}}^{\infty} e^{-\frac{s^2}{2}}\,ds = \frac{y}{\sigma_P\sqrt{2\pi}} \int_0^t s^{-\frac{3}{2}} e^{-\frac{y^2}{2\sigma_P^2 s}}\,ds$$

is a unique solution of equation (3.0.11). Equation (3.0.12) corresponds to the probability $P(\tau_0 \leq \min(\tau_\kappa, t))$. It is known (see exercise 2.8.11 in Karatzas and Shreve (1991)) that equation (3.0.12) has the unique solution

$$\psi_{1,\kappa}^*(y,t) = \frac{1}{\sigma_P\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} (2n\kappa + y) \int_0^t s^{-\frac{3}{2}} e^{-\frac{(2n\kappa+y)^2}{2\sigma_P^2 s}}\,ds.$$

Similarly, equation (3.0.13) corresponds to the probability $P(\tilde{\tau}_0 \leq t)$ and (3.0.8) corresponds to the probability $P(\tilde{\tau}_0 \leq \min(\tilde{\tau}_\kappa, t))$. Similar applications of Girsanov's theorem, as in section 3.5.C in Karatzas and Shreve (1991)), yield that

$$\psi_1(y,t) = \frac{y}{\sigma_P\sqrt{2\pi}} \int_0^t s^{-\frac{3}{2}} e^{-\frac{(y+ps)^2}{2\sigma_P^2 s}}\,ds \tag{3.0.14}$$

and

$$\psi_{1,\kappa}(y,t) = \frac{1}{\sigma_P\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} (2n\kappa + y) \int_0^t s^{-\frac{3}{2}} e^{-\left[\frac{(2n\kappa+y)^2}{2\sigma_P^2 s} + \hat{p}y + \frac{1}{2}\sigma_P^2\hat{p}^2 s^2\right]} ds, \tag{3.0.15}$$

where

$$\hat{p} = \frac{p}{\sigma_P^2}.$$

We will return to equation (3.0.13) and the solution (3.0.14) later in the article. Unfortunately it will turn out to be much more difficult to establish the existence of a solution of equation (3.0.9). Uniqueness, however, is relatively straightforward to establish, as outlined below.

Theorem 3.0.1. *If*

$$g_1(y,t) \in C^{2,1}((0,\kappa) \times (0,1])$$

and

$$g_2(y,t) \in C^{2,1}((0,\kappa) \times (0,1])$$

are two classical solutions of equation (3.0.9)*, then*

$$g_1(y,t) = g_2(y,t),$$

for every $(y,t) \in [0,\kappa] \times [0,1]$.

Proof. Since $g_1(y,t)$ and $g_1(y,t)$ are assumed to be solutions of equation (3.0.9) this follows from Theorem I.3.1 in Garroni and Menaldi (1992) by considering the differences

$$g_1(y,t) - g_2(y,t).$$

□

Before proceeding to establish existence of a solution of (3.0.9) we will first need to establish some auxiliary results and then introduce the concept of a Green function.

Proposition 3.0.1.

For every $x \in \mathbb{R}$*,* $t > 0$ *and for any* $\alpha, c > 0$ *and* $0 < \theta < c$

$$\sup_{t\in(0,1]} |x|^\alpha \exp\left(-c\frac{x^2}{t}\right) \le Ct^{\frac{\alpha}{2}} \exp\left(-(c-\theta)\frac{x^2}{t}\right),$$

where

$$C = \left(\frac{\frac{\alpha}{2}}{\theta}\right)^{\frac{\alpha}{2}} \exp\left(-\frac{\alpha}{2}\right).$$

Proof. Let $(t,\theta) \in (0,1] \times (0,c)$. We observe that since

$$\left(\frac{\frac{\alpha}{2}}{\theta}\right)^{\frac{\alpha}{2}} \exp\left(-\frac{\alpha}{2}\right) t^{\frac{\alpha}{2}} \exp\left(-(c-\theta)\frac{x^2}{t}\right) > 0,$$

there must exist some $\epsilon \in \left(0, \frac{\frac{\alpha}{2}}{\theta}\right)$ such that

$$|x|^\alpha \exp\left(-c\frac{x^2}{t}\right) < Ct^{\frac{\alpha}{2}} \exp\left(-(c-\theta)\frac{x^2}{t}\right),$$

for every $x \in [0,\epsilon]$. Moreover, for every $x \geq \epsilon$

$$|x|^{\alpha} \exp\left(-\theta \frac{|x|^2}{t}\right) \leq \left(t^{\frac{\alpha}{2}}\right) \left(\sup_{z \in [\frac{\epsilon^2}{t},\infty)} z^{\frac{\alpha}{2}} \exp\left(-\theta z\right)\right).$$

Let $h(z) = z^{\frac{\alpha}{2}} \exp\left(-\theta z\right)$. Differentiating h we get that

$$h'(z) = z^{\frac{\alpha}{2}} \left(-\theta + \frac{\alpha}{2} z^{-1}\right) \exp\left(-\theta z\right),$$

which is positive for $z \in (0, \frac{\frac{\alpha}{2}}{\theta})$, 0 for $z = \frac{\frac{\alpha}{2}}{\theta}$ and negative for $z > \frac{\frac{\alpha}{2}}{\theta}$. Thus

$$\sup_{z \in [\frac{\epsilon^2}{t},\infty)} z^{\frac{\alpha}{2}} \exp\left(-\theta z\right) = \left(\frac{\frac{\alpha}{2}}{\theta}\right)^{\frac{\alpha}{2}} \exp\left(-\frac{\alpha}{2}\right).$$

Since t was arbitrarily chosen the result follows. $\square$

Proposition 3.0.2. *For every* $(x,t,\xi,\vartheta) \in \mathbb{R} \times (0,1] \times \mathbb{R} \times [0,t)$ *and* $p,q,c > 0$,

$$\int_{\vartheta}^{t} (t-s)^{p-1} (s-\vartheta)^{q-1}\, ds = (t-\vartheta)^{p+q-1} \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)},$$

and

$$\int_{\mathbb{R}} \exp\left(-c\left[\frac{|x-z|^2}{t-s} + \frac{|z-\xi|^2}{s-\vartheta}\right]\right) dz$$
$$= \left(\frac{\pi}{c}\right)^{\frac{1}{2}} \left[\frac{(t-s)(s-\vartheta)}{t-\vartheta}\right]^{\frac{1}{2}} \exp\left(-c\frac{(x-\xi)^2}{(t-\vartheta)}\right),$$

where $\Gamma(x)$ *is the Gamma function*

$$\Gamma(x) := \int_0^{\infty} z^{x-1} \exp\left(-z\right) dz, \quad x > 0.$$

Proof. These identities are proven in section 1.1 in Garroni and Menaldi (2002). $\square$

Proposition 3.0.3. *Let* $c > 0$, $d \in \mathbb{R}$, *let* $-\infty < a_1 < a_2 < \infty$, $-\infty < b_1 < b_2 < \infty$ *and let*

$$\mathcal{D}_{ab} := (a_1,a_2) \times (0,1] \times (b_1,b_2) \times [0,t).$$

Let $h(y,t,\xi,\vartheta)$ *be a continuous function on* $\mathcal{D}_{ab}$ *such that* $h(y,t,\xi,\vartheta)$ *is differentiable with respect to* t *on* $\mathcal{D}_{ab}$, *and for some constant* C

$$|h(y,t,\xi,\vartheta)| \leq C(t-\vartheta)^{-d} \exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right) \tag{3.0.16}$$

and

$$\left|\frac{\partial h(y,t,\xi,\vartheta)}{\partial t}\right| \leq C(t-\vartheta)^{-(d+1)} \exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right)$$

for every $(y,t,\xi,\vartheta)\in\mathcal{D}_{ab}$*. Then, for some constant* C

$$|h(y,t,\xi,\vartheta)-h(y,t',\xi,\vartheta)|\leq C|t-t'|^{\alpha}\left[(t-\vartheta)^{-(d+\alpha)}+(t'-\vartheta)^{-(d+\alpha)}\right]$$
$$\times\left(\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right)+\exp\left(-c\frac{(y-\xi)^2}{t'-\vartheta}\right)\right) \tag{3.0.17}$$

and

$$|h(y,t,\xi,\vartheta)-h(y,t',\xi,\vartheta)|\leq C|t-t'|^{\alpha}\left[(t-\vartheta)^{-(d+\alpha)}\exp\left(-\frac{1}{2}c\frac{(y-\xi)^2}{t-\vartheta}\right)\right.$$
$$\left.+(t'-\vartheta)^{-(d+\alpha)}\exp\left(-\frac{1}{2}c\frac{(y-\xi)^2}{t'-\vartheta}\right)\right] \tag{3.0.18}$$

for every $(y,t,\xi,\vartheta)\in\mathcal{D}_{ab}$*, every* $t'\in(\vartheta,1]$*, and every* $\alpha\in[0,1]$*.*

Proof. Let $t_2=\max(t,t')$ and $t_1=\min(t,t')$. Assume first that

$$t_2-t_1\geq t_1-\vartheta.$$

We note that in this case

$$t_2-\vartheta\leq 2(t_2-t_1).$$

Hence, for every $\alpha\in[0,1]$

$$|h(y,t,\xi,\vartheta)-h(y,t',\xi,\vartheta)|\leq|h(y,t,\xi,\vartheta)|+|h(y,t',\xi,\vartheta)|$$

$$\leq 2C|t-t'|^{\alpha}\left[(t-\vartheta)^{-(d+\alpha)}\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right)\right.$$
$$\left.+(t'-\vartheta)^{-(d+\alpha)}\exp\left(-c\frac{(y-\xi)^2}{t'-\vartheta}\right)\right].$$

From the above it is obvious that for this case the inequality (3.0.17) also holds. Now, assume instead that

$$t_2-t_1<t_1-\vartheta.$$

We first observe that under this condition

$$t_2-\vartheta<2(t_1-\vartheta)$$

and hence we only need to prove that the inequality (3.0.17) holds. Moreover, it follows from the mean value theorem that

$$|h(y,t,\xi,\vartheta)-h(y,t',\xi,\vartheta)|\leq C|t-t'|\left[(t-\vartheta)^{-(c+1)}+(t'-\vartheta)^{-(d+1)}\right]$$
$$\times\left(\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right)+\exp\left(-d\frac{(y-\xi)^2}{t'-\vartheta}\right)\right).$$

Thus the required bound (3.0.17), and hence (3.0.18), can be obtained, since

$$|t-t'| \le \min\left(t-\vartheta, t'-\tau\right).$$

□

Corollary 1. *Assume that* $h(y,t,\xi,\vartheta)$ *is differentiable with respect to* ϑ *on* $\mathcal{D}_{ab}$*, that* (3.0.16) *holds and that*

$$\left|\frac{\partial h(y,t,\xi,\vartheta)}{\partial \vartheta}\right| \le C(t-\vartheta)^{-(d+1)}\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right)$$

for every $(y,t,\xi,\vartheta)\in\mathcal{D}_{ab}$*. Then, for some constant* C

$$\begin{aligned}|h(y,t,\xi,\vartheta)-h(y,t,\xi,\vartheta')| \le C|\vartheta-\vartheta'|^{\alpha}\left[(t-\vartheta)^{-(d+\alpha)}+(t-\vartheta')^{-(d+\alpha)}\right]\\ \times\left(\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right)+\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta'}\right)\right).\end{aligned}$$

Hence

$$\begin{aligned}|h(y,t,\xi,\vartheta)-h(y,t,\xi,\vartheta')| \le C|t-t'|^{\alpha}\Bigg[(t-\vartheta)^{-(d+\alpha)}\exp\left(-\frac{1}{2}c\frac{(y-\xi)^2}{t-\vartheta}\right)\\ +(t-\vartheta')^{-(d+\alpha)}\exp\left(-\frac{1}{2}c\frac{(y-\xi)^2}{t-\vartheta'}\right)\Bigg].\end{aligned}$$

Proposition 3.0.4. *Let* $c>0$*,* $d\in\mathbb{R}$*, let* $-\infty<a_1<a_2<\infty$*,* $-\infty<b_1<b_2<\infty$ *and let*

$$\mathcal{D}_{ab} := (a_1,a_2)\times(0,1]\times(b_1,b_2)\times[0,t).$$

Let

$$\mathcal{D}_{\bar{a}b} := [a_1,a_2]\times(0,1]\times(b_1,b_2)\times[0,t)$$

and let $h(y,t,\xi,\vartheta)$ *be a continuous function on* $\mathcal{D}_{\bar{a}b}$ *such that* $h(y,t,\xi,\vartheta)$ *is differentiable with respect to* y *on* $\mathcal{D}_{ab}$*. Assume that, for some constant* C*,*

$$|h(y,t,\xi,\vartheta)| \le C(t-\vartheta)^{-d}\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right) \tag{3.0.19}$$

on $\mathcal{D}_{\bar{a}b}$ *and*

$$\left|\frac{\partial h(y,t,\xi,\vartheta)}{\partial y}\right| \le C(t-\vartheta)^{-\left(d+\frac{1}{2}\right)}\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right) \tag{3.0.20}$$

on $\mathcal{D}_{ab}$*. Then, for some constant* C*,*

$$\begin{aligned}|h(y,t,\xi,\vartheta)-h(y',t,\xi,\vartheta)| \le& C\exp(c)|y-y'|^{\alpha}(t-\vartheta)^{-\left(d+\frac{\alpha}{2}\right)}\\ &\times\left(\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right)+\exp\left(-c\frac{(y'-\xi)^2}{t-\vartheta}\right)\right)\end{aligned} \tag{3.0.21}$$

for every $(y,t,\xi,\vartheta)\in\mathcal{D}_{\bar{a}b}$*, every* $y'\in[a_1,a_2]$*, and for every* $\alpha\in[0,1]$*.*

Proof. Let $y_2 = \max(y, y')$ and $y_1 = \min(y, y')$. Assume first that

$$t - \vartheta \le |y - y'|^2 .$$

We note that in this case

$$(t-\vartheta)^{-d} \le (t-\vartheta)^{-\left(d+\frac{\alpha}{2}\right)} |y - y'|^{\alpha} ,$$

and hence in this case it follows from the bound (3.0.19) that the bound (3.0.21) holds. In the rest of the proof we will assume that

$$t - \vartheta > |y - y'|^2 .$$

Because of the continuity on $\mathcal{D}_{\bar{a}b}$ we can also assume that

$$a_1 < y_1 < y_2 < a_2.$$

We note that in this case

$$|y - y'| (t-\vartheta)^{-\left(d+\frac{1}{2}\right)} \le |y - y'|^{\alpha} (t-\vartheta)^{-\left(d+\frac{\alpha}{2}\right)} . \tag{3.0.22}$$

Assume in addition that $\xi \notin (y_1, y_2)$. For this case it follows from the Middle Value Theorem and the bounds (3.0.20) and (3.0.22) that,

$$\begin{aligned} |h(y,t,\xi,\vartheta) - h(y',t,\xi,\vartheta)| \le & C |y - y'|^{\alpha} (t-\vartheta)^{-\left(d+\frac{\alpha}{2}\right)} \\ & \times \left(\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right) + \exp\left(-c\frac{(y'-\xi)^2}{t-\vartheta}\right)\right). \end{aligned}$$

The last possible case is that (3.0.22) holds and that $\xi \in (y_1, y_2)$. In this case we note that

$$\min\left(\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right), \exp\left(-c\frac{(y'-\xi)^2}{t-\vartheta}\right)\right) \ge \exp(-c) ,$$

and hence it follows from the Middle Value Theorem and the bounds (3.0.20) and (3.0.22) that

$$\begin{aligned} |h(y,t,\xi,\vartheta) - h(y',t,\xi,\vartheta)| \le & C \exp(c) |y - y'|^{\alpha} (t-\vartheta)^{-\left(d+\frac{\alpha}{2}\right)} \\ & \times \left(\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right) + \exp\left(-c\frac{(y'-\xi)^2}{t-\vartheta}\right)\right). \end{aligned}$$

□

Corollary 2. *Assume that $h(y,t,\xi,\vartheta)$ is differentiable with respect to ξ on $\mathcal{D}_{ab}$, that* (3.0.19) *holds and that, for some constant C,*

$$\left|\frac{\partial h(y,t,\xi,\vartheta)}{\partial \xi}\right| \le C (t-\vartheta)^{-\left(d+\frac{1}{2}\right)} \exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right),$$

for every $(y,t,\xi,\vartheta) \in \mathcal{D}$. Then, for some constant C

$$\begin{aligned} |h(y,t,\xi,\vartheta) - h(y,t,\xi',\vartheta)| \le & C \exp(c) |\xi - \xi'|^{\alpha} (t-\vartheta)^{-\left(d+\frac{\alpha}{2}\right)} \\ & \times \left(\exp\left(-c\frac{(y-\xi)^2}{t-\vartheta}\right) + \exp\left(-c\frac{(y-\xi')^2}{t-\vartheta}\right)\right). \end{aligned}$$

Proposition 3.0.5. (i) *If $a, b > 0$, then*

$$\int_0^b a s^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds = \int_{\frac{a^2}{b}}^{\infty} z^{-\frac{1}{2}} \exp(-z)\, dz,$$

$$\frac{\partial}{\partial a} \int_0^b a s^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds = -2b^{-\frac{1}{2}} \exp\left(-\frac{a^2}{b}\right).$$

and

$$\frac{\partial}{\partial b} \int_0^b a s^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds = -ab^{-\frac{3}{2}} \exp\left(-\frac{a^2}{b}\right).$$

(ii) *If $a \in \mathbb{R}$ and $b > 0$, then for some constant C*

$$\int_0^b \left| a s^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) \right| ds \leq C \exp\left(-\frac{a^2}{b}\right),$$

(iii) *If $a \neq 0$ and $b > 0$, then for some constant C*

$$\left| \frac{\partial}{\partial a} \int_0^b a s^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds \right| \leq C b^{-\frac{1}{2}} \exp\left(-\frac{a^2}{b}\right),$$

$$\left| \frac{\partial^2}{\partial a^2} \int_0^b a s^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds \right| \leq C a b^{-\frac{3}{2}} \exp\left(-\frac{a^2}{b}\right),$$

$$\left| \frac{\partial}{\partial b} \int_0^b a s^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds \right| \leq C a b^{-\frac{3}{2}} \exp\left(-\frac{a^2}{b}\right),$$

$$\left| \frac{\partial^3}{\partial a^3} \int_0^b a s^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds \right| \leq C b^{-\frac{3}{2}} \exp\left(-\frac{1}{2}\frac{a^2}{b}\right).$$

Proof. For part (i): This can be calculated using the substitution $z = \frac{a^2}{s}$.

For part (ii): This is obvious if $a = 0$. Assume that $a \neq 0$ and that

$$|a| \geq \sqrt{b}.$$

For this case a simple calculation using the identity given in part (i) yields that the stated claim holds. In the following assume that $a \neq 0$. and that

$$|a| \leq \sqrt{b}. \tag{3.0.23}$$

Consider

$$I_1 := \int_0^{\frac{b}{2}} |a|\, s^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds,$$

and

$$I_2 := \int_{\frac{b}{2}}^{b} |a|\, s^{-\frac{3}{2}} \exp\left(-\frac{a^2}{s}\right) ds.$$

The identity given in part (ii) yields that

$$
\begin{aligned}
I_1 &= \int_{2\frac{a^2}{b}}^{\infty} z^{-\frac{1}{2}} \exp(-z)\, dz \\
&\le \exp\left(-\frac{a^2}{b}\right) \int_{2\frac{a^2}{b}}^{\infty} z^{-\frac{1}{2}} \exp\left(-\frac{1}{2}z\right) dz \\
&\le C \exp\left(-\frac{a^2}{b}\right)
\end{aligned}
$$

for some constant C. Under the assumption (3.0.23) a simple calculation yields that

$$I_2 \le \exp\left(-\frac{a^2}{b}\right).$$

The other bounds follow from Proposition 3.0.1. □

The most important concept in this article is that of a *Green function*, which we will now define, adapted to equation (3.0.9). '

Definition 3.0.1. *A function $G_{L^*,\kappa}(y,t,\xi,\vartheta)$ defined in the domain $\bar{\mathcal{D}}_\kappa$, where*

$$
\begin{cases}
\mathcal{D}_\kappa = \{y,t,\xi,\vartheta : y \in (0,\kappa), \xi \in (0,\kappa), 0 \le \vartheta < t \le 1\}, \\
\partial\mathcal{D}_\kappa = \{y,t,\xi,\vartheta : y \in \{0,\kappa\}, \xi \in (0,\kappa), 0 \le \vartheta < t \le 1\}, \\
\bar{\mathcal{D}}_\kappa = \mathcal{D}_\kappa \cup \partial\mathcal{D}_\kappa
\end{cases}
$$

is called a Green function on $\bar{\mathcal{D}}_\kappa$ for the differential operator L^* with Dirichlet boundary condition if it satisfies:

(i) *$G_{L^*,\kappa}(y,t,\xi,\vartheta)$ is continuous in (y,t), and locally integrable in (ξ,ϑ),*

(ii)

$$
\begin{aligned}
\frac{\partial G_{L^*,\kappa}(y,t,\xi,\vartheta)}{\partial t} &- L^* G_{L^*,\kappa}(y,t,\xi,\vartheta) \\
&= \delta(y-\xi)\,\delta(t-\vartheta), \quad \textit{in } \mathcal{D}_\kappa,
\end{aligned}
$$

(iii)

$$\lim_{t-\vartheta\downarrow 0} G_{L^*,\kappa}(y,t,\xi,\vartheta) = \delta(y-\xi), \quad \textit{in } \mathcal{D}_\kappa, \tag{3.0.24}$$

(iv)

$$G_{L^*,\kappa}(y,t,\xi,\vartheta) = 0, \quad \textit{in } \partial\mathcal{D}_\kappa,$$

In the above $\delta(y,t)$ is the Dirac measure at 0.

In order to derive existence and some regularity of a solution of equation (3.0.9) we want to use use Theorem VI.2.2 in Garroni and Menaldi (1992). This theorem, however, requires the right hand side of the equation (in our case the function $H_{1,\kappa}(y,t)$) to belong to the function space $C^{\alpha,\frac{\alpha}{2}}([0,\kappa]\times[0,1],\mathbb{R})$ defined below.

Definition 3.0.2. *Let $C^0([0,\kappa]\times[0,1],\mathbb{R})$ be the Banach space of bounded, real valued, continuous functions on $[0,\kappa]\times[0,1]$, with the supremum norm.*

Let $g(y,t) \in C^0([0,\kappa]\times[0,1],\mathbb{R})$. We will say that

$$g \in C^{\alpha,\frac{\alpha}{2}}([0,\kappa]\times[0,1],\mathbb{R})$$

or that g is Hölder continuous on $[0,\kappa]\times[0,1]$ with index α
if g has a finite value for the semi norm

$$\inf\left\{C \geq 0 : |g(y,t) - g(y',t)| \leq C|y-y'|^{\alpha}, \forall y,y' \in [0,\kappa] \text{ and } \forall t \in [0,1]\right\}$$
$$+\inf\left\{C \geq 0 : |g(y,t) - g(y,t')| \leq C|t-t'|^{\frac{\alpha}{2}}, \forall y \in [0,\kappa] \text{ and } \forall t,t' \in [0,1]\right\}.$$

Alas, because of the singularity at the origin it is clear that $H_{1,\kappa}(y,t) \notin C^{\alpha,\frac{\alpha}{2}}([0,\kappa]\times[0,1],\mathbb{R})$ and we will have to rely on a more indirect approach. But first we need to explore a bit more the local regularity of $H_{1,\kappa}(y,t)$ on the inner domain, as we do in the next two results.

Definition 3.0.3. *Let*

$$c_0 = \frac{1}{2\sigma_P^2}.$$

Lemma 3.0.1. *There exists a constant C such that for every $(y,t) \in (0,\kappa)\times(0,1]$*

$$0 \leq \psi_{1,\kappa}(y,t) \leq C\exp\left(-c_0\frac{y^2}{t}\right),$$
$$\left|\frac{\partial\psi_{1,\kappa}(y,t)}{\partial y}\right| \leq Ct^{-\frac{1}{2}}\exp\left(-c_0\frac{y^2}{t}\right),$$
$$\left|\frac{\partial\psi_{1,\kappa}(y,t)}{\partial y}\right| \leq Ct^{-\frac{1}{2}}\exp\left(-\frac{1}{2}c_0\frac{y^2}{t}\right),$$
$$\left|\frac{\partial^2\psi_{1,\kappa}(y,t)}{\partial y^2}\right| \leq Ct^{-1}\exp\left(-\frac{1}{2}c_0\frac{y^2}{t}\right),$$
$$\left|\frac{\partial\psi_{1,\kappa}(y,t)}{\partial t}\right| \leq Ct^{-1}\exp\left(-\frac{1}{2}c_0\frac{y^2}{t}\right),$$

and

$$\left|\frac{\partial^3\psi_{1,\kappa}(y,t)}{\partial y^3}\right| \leq Ct^{-\frac{3}{2}}\exp\left(-\frac{1}{2}c_0\frac{y^2}{t}\right).$$

Proof. We first observe that in the formula (3.0.15) the singularity at the origin of $\psi_{1,\kappa}$ is taken care of by the term $n=0$, i.e. the term

$$\psi_1(y,t) = \frac{1}{\sigma_P\sqrt{2\pi}}\int_0^t ys^{-\frac{3}{2}}e^{-c_0\frac{(y+ps)^2}{s}}ds.$$

From Leibniz' rule it follows that

$$\frac{\partial\psi_1(y,t)}{\partial t} = \frac{y}{\sigma_P\sqrt{2\pi}}t^{-\frac{3}{2}}\exp\left(-c_0\frac{(y+pt)^2}{t}\right).$$

Because of Proposition 3.0.1 we conclude that for some constant C

$$\left|\frac{\partial\psi_1(y,t)}{\partial t}\right| \leq Ct^{-1}\exp\left(-\frac{1}{2}c_0\frac{y^2}{t}\right).$$

Moreover, similar calculations as in the proof of Proposition 3.0.5 yield that for some constant C

$$\left|\frac{\partial^l \psi_1(y,t)}{\partial y^l}\right| \leq C t^{-\frac{l}{2}} \exp\left(-\frac{1}{2} c_0 \frac{y^2}{t}\right)$$

for $l \in \{1,2,3\}$. Similar calculations as in the proof of Proposition 3.0.5 yield that the stated bounds hold for this term. In this calculation it is helpful to use the fact that the second derivative with respect to y can be expressed in terms of the derivative with respect to t and the first derivative with respect to y (a consequence of $\psi_1(y,t)$ being a solution of equation (3.0.13). The ratio test shows that the full series expression for $\psi_{1,\kappa}(y,t)$ given in (3.0.15) converges uniformly and thus $\psi_{1,\kappa}(y,t)$ can be differentiated term by term. For $|n| \geq 1$ we note that $(2n-\kappa y)^2 \geq \kappa^2$, so an application of Proposition 3.0.1 yields that all the other terms are smooth and sufficiently bounded for the whole series to obey the stated bounds. □

Lemma 3.0.2. *There exists a constant C such that for every $(y,t) \in (0,\kappa) \times (0,1]$, every $y_1, y_2 \in (0,\kappa)$, every $t_1, t_2 \in (0,1]$ and every $\alpha \in (0,1]$ the following bounds hold:*

$$\int_0^y \psi_{1,\kappa}(y-z,t)dF(z) + \bar{F}(y) \leq C \exp\left(-\frac{1}{4} c_0 \frac{y^2}{t}\right) + \bar{F}\left(\frac{y}{2}\right), \tag{3.0.25}$$

$$\begin{aligned} &\left|\int_0^{y_2} \psi_{1,\kappa}(y_2-z,t)dF(z) + \bar{F}(y_2)\right. \\ &\qquad \left. - \left(\int_0^{y_1} \psi_{1,\kappa}(y_1-z,t)dF(z) + \bar{F}(y_1)\right)\right| \\ &\leq C\left|y_2-y_1\right|^{\alpha} t^{-\frac{\alpha}{2}} \\ &\quad \times \left(\exp\left(-\frac{1}{8} c_0 \frac{y_1^2}{t}\right) + \exp\left(-\frac{1}{8} c_0 \frac{y_2^2}{t}\right)\right. \\ &\qquad \left. + \bar{F}\left(\frac{y_1}{2}\right) + \bar{F}\left(\frac{y_2}{2}\right)\right), \end{aligned} \tag{3.0.26}$$

and

$$\begin{aligned} &\left|\int_0^y \psi_{1,\kappa}(y-z,t_2)dF(z) + \bar{F}(y) - \left(\int_0^y \psi_{1,\kappa}(y-z,t_1)dF(z) + \bar{F}(y)\right)\right| \\ &\qquad \leq C\left|t_2-t_1\right|^{\alpha}\left(t_1^{-\alpha} + t_2^{-\alpha}\right) \\ &\qquad\quad \times \left(\exp\left(-\frac{1}{8} c_0 \frac{y^2}{t_1}\right) + \exp\left(-\frac{1}{8} c_0 \frac{y^2}{t_2}\right) + \bar{F}\left(\frac{y}{2}\right)\right). \end{aligned} \tag{3.0.27}$$

Proof. Let

$$\tilde{\psi}_{1,\kappa}(y,t) := \begin{cases} \psi_{1,\kappa}(y,t), & (y,t) \in [0,\kappa] \times (0,1], \\ 1, \quad , & (y,t) \in (-\infty,0) \times (0,1] \end{cases}$$

We note that, for every $t \in (0,1]$, $\psi_{1,\kappa}(0,t) = 1$, and thus $\tilde{\psi}_{1,\kappa}(y,t)$ is continuous on $(-\infty,\kappa) \times (0,1]$. Moreover, since $F(y)$ is a probability distribution it follows that, for every $(y,t) \in [0,\kappa] \times (0,1]$

$$\int_0^y \psi_{1,\kappa}(y-z,t)dF(z) + \bar{F}(y) = \int_0^\infty \tilde{\psi}_{1,\kappa}(y-z,t)dF(z).$$

Let $\tilde{y} = \min(y_2, y_1)$. From the identity above it follows that

$$\left| \int_0^{y_2} \psi_{1,\kappa}(y_2-z,t)dF(z) - \int_0^{y_1} \psi_{1,\kappa}(y_1-z,t)dF(z) \right| \\ \leq |I_1| + |I_2|$$

where

$$I_1 = \int_0^{\frac{\tilde{y}}{2}} \left(\psi_{1,\kappa}(y_2-z,t) - \psi_{1,\kappa}(y_1-z,t)\right) dF(z),$$

and

$$I_2 = \int_{\left\{z: z > \frac{\tilde{y}}{2}\right\}} \left(\psi_{1,\kappa}(y_2-z,t) - \psi_{1,\kappa}(y_1-z,t)\right) dF(z),$$

The stated bounds (3.0.26) and (3.0.27) can be obtained from considering I_1 and I_2, applying Proposition 3.0.4 and Proposition 3.0.3 and using the bounds given in Lemma 3.0.1. □

Proposition 3.0.6. *There exists a constant C such that the bounds stated below hold for every $y, y_1, y_2 \in (0,\kappa)$ and every $t, t_1, t_2 \in (0,1]$ and every $\alpha \in [0,1]$.*

$$|H_{1,\kappa}(y,t)| \leq C\left(\exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right) + \bar{F}\left(\frac{y}{2}\right)\right),$$

$$\begin{aligned} |H_{1,\kappa}(y_2,t) - H_{1,\kappa}(y_1,t)| \leq{}& C\,|y_2-y_1|^\alpha\, t^{-\frac{\alpha}{2}} \\ &\times \left(\exp\left(-\frac{1}{8}c_0\frac{y_1^2}{t}\right) + \exp\left(-\frac{1}{8}c_0\frac{y_2^2}{t}\right)\right. \\ &\left. + \bar{F}\left(\frac{y_1}{2}\right) + \bar{F}\left(\frac{y_2}{2}\right)\right), \end{aligned} \tag{3.0.28}$$

and

$$\begin{aligned} |H_{1,\kappa}(y,t_2) - H_{1,\kappa}(y,t_1)| \leq{}& C\,|t_2-t_1|^\alpha \left(t_1^{-\alpha} + t_2^{-\alpha}\right) \\ &\times \left(\exp\left(-\frac{1}{8}c_0\frac{y^2}{t_1}\right) + \exp\left(-\frac{1}{8}c_0\frac{y^2}{t_2}\right) + \bar{F}\left(\frac{y}{2}\right)\right). \end{aligned} \tag{3.0.29}$$

Proof. The bounds stated above can be obtained from the bounds given in Lemma 3.0.1 and Lemma 3.0.2 and applying Proposition 3.0.4 and Proposition 3.0.3. □

Since $H_{1,\kappa}(y,t)$ is not Hölder continuous we will instead work with a sequence of Hölder continuous functions that converge to $H_{1,\kappa}(y,t)$.

Definition 3.0.4. *For every $n \in 2, 3, \ldots,$ let*

$$\eta_n(t) := \begin{cases} 0, & t \in \left[0, \frac{1}{2n}\right], \\ \exp\left(\frac{1}{\frac{1}{2n}-t} + \frac{1}{\frac{1}{2n}}\right)\left(1 - \exp\left(\frac{1}{t-\frac{1}{n}}\right)\right), & t \in \left(\frac{1}{2n}, \frac{1}{n}\right), \\ 1, & t \in \left[\frac{1}{n}, 1\right], \end{cases}$$

and let

$$H_{1,\kappa,n}(y,t) := \eta_n(t) H_{1,\kappa}(y,t), \quad (0,t) \in [0,\kappa] \times [0,1] \, .$$

The lemma below states that, for any fixed n, the $H_{1,\kappa,n}(y,t)$ is indeed a Hölder continuous function. Because of this property we can invoke Theorem VI.2.2 in Garroni and Menaldi (1992) to establish existence of a solution of the following equation:

$$\begin{cases} \psi_{2,\kappa,n}(y,0) & = 0, \quad y \in (0,\kappa) \, , \\ \psi_{2,\kappa,n}(0,t) & = 0, \quad t \in [0,1] \, , \\ \psi_{2,\kappa,n}\,(\kappa,t) & = 0, \quad t \in [0,1] \, , \\ \frac{\partial \psi_{2,\kappa,n}(y,t)}{\partial t} & -\frac{1}{2}\left(\sigma_P^2 + \sigma_R^2 y^2\right) \frac{\partial^2 \psi_{2,\kappa,n}(y,t)}{\partial y^2} - (p+ry)\,\frac{\partial \psi_{2,\kappa,n}(y,t)}{\partial y} \\ & = H_{1,\kappa,n}(y,t), \quad (y,t) \in (0,\kappa) \times (0,1]. \end{cases} \tag{3.0.30}$$

Moreover, Theorem VI.2.2 also gives us a representation formula for $\psi_{2,\kappa,n}(y,0)$, which we will later use to show that

$$\lim_{n\to\infty} \psi_{2,\kappa,n}(y,t)$$

is a classical solution of equation (3.0.9).

Lemma 3.0.3. *For every* $n \in 2,3,\ldots,$

(i) $\eta_n(t)$ *is differentiable on* $\left(0,\frac{1}{n}\right)$*, and for every* $t \in [0,1]$

$$0 \le \eta_n(t) \le 1.$$

(ii) *There exists a constant* C_n*, depending on* n*, such that, for every* $\alpha \in (0,1]$*, every* $(y,t) \in [0,\kappa] \times [0,1]$*, every* $y_1, y_2 \in [0,\kappa]$ *and every* $t_1, t_2 \in [0,1]$

$$|H_{1,\kappa,n}(y_1,t) - H_{1,\kappa,n}(y_2,t)| \le C_n \, |y_2 - y_1|^{\alpha}$$

and

$$|H_{1,\kappa,n}(y,t_2) - H_{1,\kappa,n}(y,t_1)| \le C_n \, |t_2 - t_1|^{\alpha} \, .$$

Proof. Without loss of generality we can assume that $t_2 \ge t_1$. It follows from the bounds given in Proposition 3.0.6 that there exists a constant C such that, for every $(y,t) \in [0,\kappa] \times \left[\frac{1}{2n},1\right]$, every $y_1, y_2 \in [0,\kappa]$, and every $t_1, t_2 \in \left[\frac{1}{2n},1\right]$,

$$|H_{1,\kappa}(y_2,t) - H_{1,\kappa}(y_1,t)| \le C n^{-\frac{1}{2}} \, |y_2 - y_1| \, , \tag{3.0.31}$$

and

$$|H_{1,\kappa}(y,t_2) - H_{1,\kappa}(y,t_1)| \le C n^{-1} \, |t_2 - t_1| \, . \tag{3.0.32}$$

Now, for fixed $n \in 2,3,\ldots,$ consider the function $\eta_n(t)$. An inspection yields that

$$0 < \eta_n(t) < 1$$

for every $t \in \left(\frac{1}{2n},\frac{1}{n}\right)$. Since $0 \le \eta_n(t) \le 1$ and since $\eta_n(t)$ vanishes for $t < \frac{1}{2n}$ it follows from the bound (3.0.31) that, for every $y_1, y_2 \in [0,\kappa]$,

$$|H_{1,\kappa,n}(y_2,t) - H_{1,\kappa}(y_1,t)| \le C n^{-\frac{1}{2}} \, |y_2 - y_1| \, .$$

Moreover, $H_{1,\kappa,n}(y,t)$ is a bounded function, thus, for some (other) constant C,

$$|H_{1,\kappa,n}(y_2,t) - H_{1,\kappa}(y_1,t)| \leq Cn^{-\frac{1}{2}} |y_2 - y_1|^{\alpha},$$

for any $\alpha \in (0,1]$. Now, consider $\eta_n(t)$. Taking the limit we observe that

$$\lim_{t \downarrow \frac{1}{2n}} \eta_n(t) = 0,$$

while

$$\lim_{t \uparrow \frac{1}{n}} \eta_n(t) = 1,$$

thus $\eta_n(t)$ is continuous. Moreover, it can be calculated that the limit

$$\lim_{t \uparrow \frac{1}{n}} \eta_n'(t)$$

exists. Hence $\eta_n'(t)$ is bounded by some constant $\hat{C}$ on $\left(0, \frac{1}{n}\right)$. From the bound and the identities above, it follows that, for some other constant K

$$|\eta_n(t_2) - \eta_n(t_1)| \leq K |t_2 - t_1|^{\alpha},$$

for any $\alpha \in (0,1]$, and thus, for some constant C_n

$$\begin{aligned} |H_{1,\kappa,n}(y,t_2) - H_{1,\kappa,n}(y,t_1)| &\leq |H_{1,\kappa}(y,t_2)\left(\eta_n(t_2) - \eta_n(t_1)\right)| \\ &\quad + |(H_{1,\kappa}(y,t_2) - H_{1,\kappa,n}(y,t_1))|\, \eta_n(t_1) \\ &\leq C_n |t_2 - t_1|^{\alpha}. \end{aligned}$$

□

Since $H_{1,\kappa,n}(y,t)$ is Hölder continuous, we get an existence and representation result for equation (3.0.30), as stated in the theorem below.

Theorem 3.0.2. (i) *There exists a unique Green function $G_{L,\kappa}(y,t,\xi,\vartheta)$ associated with the differential operator L and Dirichlet boundary conditions on the domain $\mathcal{D}_\kappa$, i.e. satisfying the conditions in Definition 3.0.1. Furthermore, there exist positive constants C_κ and c_κ, depending on κ, such that, for $l \in \{0,1,2\}$,*

$$\left| \frac{\partial^l G_{L,\kappa}(y,t,\xi,\vartheta)}{\partial y^l} \right| \leq C_\kappa (t-\vartheta)^{-\frac{1+l}{2}} \exp\left(-c_\kappa \frac{(y-\xi)^2}{(t-\vartheta)} \right),$$

and such that

$$\left| \frac{\partial G_{L,\kappa}(y,t,\xi,\vartheta)}{\partial t} \right| \leq C_\kappa (t-\vartheta)^{-\frac{3}{2}} \exp\left(-c_\kappa \frac{(y-\xi)^2}{(t-\vartheta)} \right).$$

(ii) *For any fixed $n \in 2,3,\ldots,$,*

$$\psi_{2,\kappa,n}(y,t) = \int_0^t \int_0^\kappa G_{L,\kappa}(y,t,\xi,\vartheta)\, H_{1,\kappa,n}(\xi,\vartheta) d\xi d\vartheta.$$

is a unique, bounded classical solution of equation (3.0.30).

Proof. This can be shown to follow from Theorem VI.2.1 and Theorem VI.2.2 in Garroni and Menaldi (1992). □

The next result is the first step to prove that $\psi_{2,\kappa,n}(y,t)$ converges to a solution of (3.0.9) of the form given below.

Definition 3.0.5. *Let*

$$\tilde{\psi}_{2,\kappa}(y,t) = \begin{cases} 0, & (y,t) \in (0,\kappa) \times \{0\}, \\ 0, & (y,t) \in \{0,\kappa\} \times [0,1], \\ \int_0^t \int_0^\kappa G_{L,\kappa}(y,t,\xi,\vartheta)\, H_{1,\kappa}(\xi,\vartheta) d\xi d\vartheta, & (y,t) \in (0,\kappa) \times (0,1]. \end{cases}$$

Lemma 3.0.4. *There exists a constant C_κ, depending on κ, such that, for any $(y_0,t_0) \in (0,\kappa) \times (0,1]$,*

(i)

$$\left|\tilde{\psi}_{2,\kappa}(y_0,t_0)\right| \le C_\kappa t_0. \tag{3.0.33}$$

Moreover, for every $(y_1,t_1) \in \{0,\kappa\} \times [0,1]$

$$\lim_{(y,t)\to(y_1,t_1)} \tilde{\psi}_{2,\kappa}(y,t) = 0.$$

(ii)

$$\tilde{\psi}_{2,\kappa}(y,t) \in C^{2,1}\left((0,\kappa) \times (0,1), \mathbb{R}\right).$$

Moreover, for $l \in \{0,1,2\}$ and $n > \frac{2}{t_0}$,

$$\left| \left.\frac{\partial^l \tilde{\psi}_{2,\kappa}(y,t_0)}{\partial y^l}\right|_{y=y_0} - \left.\frac{\partial^l \psi_{2,\kappa,n}(y,t_0)}{\partial y^l}\right|_{y=y_0} \right| \le C_\kappa \frac{t_0^{-\frac{l}{2}}}{n},$$

and

$$\left| \left.\frac{\partial \tilde{\psi}_{2,\kappa}(y_0,t)}{\partial t}\right|_{t=t_0} - \left.\frac{\partial \psi_{2,\kappa,n}(y_0,t)}{\partial t}\right|_{t=t_0} \right| \le C_\kappa \frac{t_0^{-1}}{n}.$$

Proof. For part (i): It follows from the bounds given in Theorem 3.0.2 and the boundedness of $(H_{1,\kappa}(\xi,\vartheta)$ that there exists a constant K_κ, depending on κ, such that

$$|G_{L,\kappa}(y,t,\xi,\vartheta)|\, |H_{1,\kappa}(\xi,\vartheta)| \le K_\kappa (t-\vartheta)^{-\frac{1}{2}}, \tag{3.0.34}$$

for every $(y,t,\xi,\vartheta)\,(0,\kappa) \times (0,1] \times (0,\kappa) \times [0,t)$. A calculation using the bound above yields the bound (3.0.33). Moreover, because of the bound (3.0.34), the Dominated Convergence Theorem can be invoked to yield that

$$\lim_{(y,t)\to(y_1,t_1)} \tilde{\psi}_{2,\kappa}(y,t) = 0,$$

for every $(y_1,t_1) \in \{0,\kappa\} \times [0,1]$.

For part (ii): Let $(y_0, t_0) \in (0, \kappa) \times (0, 1]$, and let

$$n \in \left\lceil \frac{2}{t_0} \right\rceil, \left\lceil \frac{2}{t_0} \right\rceil + 1, \left\lceil \frac{2}{t_0} \right\rceil + 2 \ldots, .$$

We observe that, for every $(y, t) \in (0, \kappa) \times (\frac{t_0}{2}, 1]$,

$$\tilde{\psi}_{2,\kappa}(y, t) = \psi_{2,\kappa,n}(y, t) + I_n(y, t),$$

where

$$I_n(y, t) = \int_0^{\frac{1}{n}} \int_0^{\kappa} G_{L,\kappa}(y, t, \xi, \vartheta) \left(H_{1,\kappa}(\xi, \vartheta) - H_{1,\kappa,n}(\xi, \vartheta)\right) d\xi d\vartheta.$$

It follows from Theorem 3.0.2 that $\psi_{2,\kappa,n}(y, t) \in C^{2,1}((0, \kappa) \times (0, 1), \mathbb{R})$. Furthermore, a similar calculation as in part (i) yields that

$$|I_n(y_0, t_0)| \leq C_\kappa \frac{1}{n},$$

for some constant C_κ, depending on κ.

Moreover, we note that

$$\frac{1}{n} < \frac{t_0}{2},$$

and it can be shown that the function $G_{L,\kappa}(y, t, \xi, \vartheta)$ is sufficiently regular that the partial differential operators $\frac{\partial}{\partial y}, \frac{\partial^2}{\partial y^2}$ and $\frac{\partial}{\partial t}$ can be taken inside the integral. Thus similar calculations as in part (i) yield that, for $l \in \{1, 2\}$,

$$\left| \frac{\partial^l I_n(y, t_0)}{\partial y^l} \right|_{y=y_0} \leq C_\kappa t_0^{-\frac{l}{2}} \frac{1}{n},$$

and

$$\left| \frac{\partial I_n(y_0, t_0)}{\partial t} \right|_{t=t_0} \leq C_\kappa t_0^{-1} \frac{1}{n},$$

for some constant C_κ, depending on κ. □

Theorem 3.0.3. *$\tilde{\psi}_{2,\kappa}(y, t)$ is a unique classical solution of equation 3.0.9. Moreover, $\tilde{\psi}_{2,\kappa}(y, t) \in C([0, \kappa] \times [0, 1], \mathbb{R})$.*

Proof. Let $(y_0, t_0) \in (0, \kappa) \times (0, 1]$, and let

$$E := \left[\frac{y_0}{2}, \frac{y_0 + \kappa}{2}\right] \times \left[\frac{3}{4} t_0, 1\right].$$

We know from Theorem 3.0.2 that, for every $n \in 2, 3, \ldots$, $\psi_{2,\kappa,n}(y, t)$ is a unique, bounded classical solution of equation (3.0.30), and, from Lemma 3.0.4, that $\psi_{2,\kappa,}(y, t) \in C^{2,1}((0, \kappa) \times (0, 1], \mathbb{R})$.

Moreover, similar bounds as those stated in Lemma 3.0.4 yield that the sequences

$$\left\{ \frac{\partial^l \psi_{2,\kappa,n}(y, t)}{\partial y^l} \right\}_{n=0}^{\infty}, \quad l \in \{0, 1, 2\}$$

converge uniformly on E to

$$\frac{\partial^l \tilde{\psi}_{2,\kappa}(y,t)}{\partial y^l}, \quad l \in \{0,1,2\},$$

and that $\frac{\partial \psi_{2,\kappa,n}(y,t)}{\partial t}$ converges uniformly on E to $\frac{\partial \tilde{\psi}_{2,\kappa}(y,t)}{\partial t}$. It follows from the above that, for $(y,t) \in E$

$$\begin{aligned}\frac{\partial \tilde{\psi}_{2,\kappa}(y,t)}{\partial t} &- \left\{ \frac{1}{2}\left(\sigma_P^2 + \sigma_R^2 y^2\right) \frac{\partial^2 \tilde{\psi}_{2,\kappa}(y,t)}{\partial y^2} - (p + r y_1) \frac{\partial \tilde{\psi}_{2,\kappa}(y,t)}{\partial y} \right\} \\ &= \lim_{n\to\infty} H_{1,\kappa,n}(y,t) \\ &= H_{1,\kappa}(y,t).\end{aligned}$$

Since (y_0, t_0) (the point used to define E) was an arbitrarily chosen point in $(0,\kappa) \times (0,1]$ it follows that

$$\begin{aligned}\frac{\partial \tilde{\psi}_{2,\kappa}(y_1,t)}{\partial t} &- \left\{ \frac{1}{2}\left(\sigma_P^2 + \sigma_R^2 y^2\right) \frac{\partial^2 \psi_{2,\kappa,n}(y,t_1)}{\partial y^2} - (p + r y) \frac{\partial \psi_{2,\kappa,n}(y,t)}{\partial y} \right\} \\ &= H_{1,\kappa}(y,t),\end{aligned}$$

on $(y,t) \in (0,\kappa) \times (0,1]$. Lastly, we observe that by definition $\tilde{\psi}_{2,\kappa}(y,t)$ satisfies the initial condition and the boundary condition, and it follows from Lemma 3.0.4 that $\tilde{\psi}_{2,\kappa}(y,t)$ is continuous on $[0,\kappa] \times [0,1]$. □

In the following we will refer to $\tilde{\psi}_{2,\kappa}$ as $\psi_{2,\kappa}$. To obtain existence also of a solution to the last equation (3.0.10) we need $\psi_{2,\kappa}(y,t)$ to be Hölder continuous on $[0,\kappa] \times [0,1]$ with respect to both y and t, not just continuous. To obtain the Hölder continuity in t we first need the result below.

Lemma 3.0.5. *There exists a constant C_κ, depending on κ, such that, for every $t \in [0,1]$, every $y, y' \in [0,\kappa]$, $t, t' \in [0,1]$, and every $\alpha \in [0,1]$*

$$|\psi_{2,\kappa}(y,t) - \psi_{2,\kappa}(y',t)| \leq C_\kappa \sqrt{t}\, |y - y'|^\alpha . \tag{3.0.35}$$

Proof. It is trivial that the bound (3.0.35) holds if $t = 0$. If $t > 0$ the bound follows from the bounds given in Theorem 3.0.2, the boundedness of $H_{1,\kappa}(y,t)$ and Proposition 3.0.4. □

Lemma 3.0.6. *There exists a constant C_κ, depending on κ, such that, for every $t_2, t_1 \in [0,1]$, every $\alpha \in [0,1]$ and every $y \in [0,\kappa]$*

$$|\psi_{2,\kappa}(y,t_2) - \psi_{2,\kappa}(y,t_1)| \leq C_\kappa\, |t_2 - t_1|^{\frac{\alpha}{2}} .$$

Proof. Let $\alpha \in [0,1]$. Without loss of generality we can assume that $t_2 > t_1$.

Assume first that

$$t_1 \leq \frac{1}{2} t_2.$$

For this case it follows from Lemma 3.0.4 and Proposition 3.0.3, that, for some constant C_κ, depending on κ,

$$|\psi_{2,\kappa}(y,t_2) - \psi_{2,\kappa}(y,t_1)| \leq C_\kappa\, (t_2 - t_1)^{\frac{\alpha}{2}} .$$

Assume instead that $t_1 > \frac{1}{2}t_2$. We then have the bound

$$|\psi_{2,\kappa}(y,t_2) - \psi_{2,\kappa}(y,t_1)| \leq |I_1| + |I_2|,$$

where

$$I_1 = \int_{t_1}^{t_2} \int_0^{\kappa} G_{L,\kappa}(y,t_2,\xi,\vartheta)\, H_{1,\kappa}(\xi,\vartheta) d\xi d\vartheta,$$

and

$$I_2 = \int_0^{t_1} \int_0^{\kappa} \left(G_{L,\kappa}(y,t_2,\xi,\vartheta) - G_{L,\kappa}(y,t_1,\xi,\vartheta)\right) H_{1,\kappa}(\xi,\vartheta) d\xi d\vartheta.$$

A similar calculation as in the proof of Lemma 3.0.4 yields that, for some constant C_κ depending on κ,

$$\begin{aligned} |I_1| &\leq C_\kappa (t_2 - t_1) \\ &\leq C_\kappa (t_2 - t_1)^{\frac{\alpha}{2}}. \end{aligned}$$

Lastly, a calculation, using the bound given in Proposition 3.0.3, yields that, for some constants $\hat{C}_\kappa$, C_κ, and c_κ depending on κ,

$$|I_2| \leq C_\kappa (t_2 - t_1)^{\frac{\alpha}{2}}.$$

□

Before proceeding with equation (3.0.10) we will need a regularity result concerning the function $H_{2,\kappa}(y,t)$, which is the right hand side of equation (3.0.10).

Lemma 3.0.7. *There exists a constant C_κ, depending on κ, such that, for every $(y,t) \in (0,\kappa) \times (0,1]$, every $y_1, y_2 \in (0,\kappa)$ every $t_1, t_2 \in (0,1]$, and every $\alpha \in [0,]$, the following bounds hold:*

$$|H_{2,\kappa}(y_2,t) - H_{2,\kappa}(y_1,t)| \leq C_\kappa |y_2 - y_1|^{\alpha},$$

and

$$|H_{2,\kappa}(y,t_2) - H_{2,\kappa}(y,t_1)| \leq C_\kappa |t_2 - t_1|^{\frac{\alpha}{2}}.$$

Proof. Let

$$\tilde{\psi}_{2,\kappa}(y,t) := \begin{cases} \psi_{2,\kappa}(y,t), & y \in [0,\kappa], \\ 0, & y < 0. \end{cases}$$

We observe that, for every $t \in (0,1]$, $\psi_{2,\kappa}(0,t) = 0$, and that, for every $(y,t) \in (0,1]$

$$\lambda \int_0^y \psi_{2,\kappa}(y-z,t) dF(z) = \lambda \int_0^\infty \tilde{\psi}_{2,\kappa}(y-z,t) dF(z).$$

The stated bounds can be calculated using the identity above and the Hölder bounds in y and t for $\tilde{\psi}_{2,\kappa}(y-z,t)$, given in Lemma 3.0.5 and Lemma 3.0.5, respectively. □

In Garroni and Menaldi (1992) they also define Green functions for parabolic integro-differential equations. Below we have adapted definition IV.2.1 from Garroni and Menaldi (1992) to the PIDE (3.0.10). In this section we will not examine this Green function, but later, in Section (4.1.2) we will study this Green function more closely in the special case that $\sigma_R = r = 0$.

Definition 3.0.6. *A function $G_{A,\kappa}(y,t,\xi,\vartheta)$ defined in the domain $\bar{\mathcal{D}}_\kappa$, where*

$$\begin{cases} \mathcal{D}_\kappa = \{y,t,\xi,\vartheta : y \in (0,\kappa), \xi \in (0,\kappa), 0 \le \vartheta < t \le 1\}, \\ \partial\mathcal{D}_\kappa = \{y,t,\xi,\vartheta : y \in \{0,\kappa\}, \xi \in (0,\kappa), 0 \le \vartheta < t \le 1\}, \\ \bar{\mathcal{D}}_\kappa = \mathcal{D}_\kappa \cup \partial\mathcal{D}_\kappa \end{cases}$$

is called a Green function on $\bar{\mathcal{D}}_\kappa$ for the differential operator

$$\frac{\partial}{\partial t} - A,$$

with Dirichlet boundary conditions if it satisfies:

(i) $G_{A,\kappa}(y,t,\xi,\vartheta)$ *is continuous in* (y,t) *and locally integrable in* (ξ,ϑ),

(ii)

$$\frac{\partial G_{A,\kappa}(y,t,\xi,\vartheta)}{\partial t} - AG_{A,\kappa}(y,t,\xi,\vartheta) = \delta(y-\xi)\,\delta(t-\vartheta), \quad \text{in } \mathcal{D}_\kappa,$$

(iii)

$$\lim_{t-\vartheta\downarrow 0} G_{A,\kappa}(y,t,\xi,\vartheta) = \delta(y-\xi), \quad \text{in } \mathcal{D}_\kappa,$$

(iv)

$$G_{A,\kappa}(y,t,\xi,\vartheta) = 0, \quad \text{in } \partial\mathcal{D}_\kappa.$$

Theorem 3.0.4. *There exists a unique Green function $G_{A,\kappa}(y,t,\xi,\vartheta)$ associated with the integro-differential operator $\frac{\partial}{\partial t} - A$ with Dirichlet boundary conditions (i.e., satisfying the requirements of Definition 3.0.6). Let*

$$\psi_{3,\kappa}(y,t) = \begin{cases} 0 & (y,t) \in (0,\kappa) \times \{0\}, \\ 0 & (y,t) \in \{0,\kappa\} \times [0,1], \\ \int_0^t \int_0^\kappa G_{A,\kappa}(y,t,\xi,\vartheta)\, H_{2,\kappa}(\xi,\vartheta)\, d\xi d\vartheta \\ (y,t) \in (0,\kappa) \times (0,1]. \end{cases} \tag{3.0.36}$$

and let

$$\psi_\kappa(y,t) = \sum_{j=1}^{3} \psi_{j,\kappa}(y,t) \quad (y,t) \in [0,\kappa] \times [0,1].$$

With the definition above, for any given $\kappa > 0$ the following holds: $\psi_\kappa(y,t) \in C^{2,1}((0,\kappa) \times (0,1])$ and $\psi_\kappa(y,t)$ is a classical solution except at the origin of the integro-differential equation (3.0.7)*, i.e.,*

$$\begin{cases} \psi_\kappa(y,0) = 0, & y \in (0,\kappa), \\ \psi_\kappa(0,t) = 1, & t \in [0,1], \\ \psi_\kappa(\kappa,t) = 0, & t \in [0,1], \\ \frac{\partial\psi_\kappa y,t)}{\partial t} - A\psi_\kappa y,t) = \lambda\bar{F}(y), & (y,t) \in (0,\kappa) \times [0,1]. \end{cases}$$

Proof. Since we have already established existence and uniqueness of equation (3.0.8) and equation (3.0.9), we only need to consider equation (3.0.10), i.e. the PIDE

$$\begin{cases} \psi_{3,\kappa}(y,0) & = 0, \quad y \in (0,\kappa)\,, \\ \psi_{3,\kappa}(0,t) & = 0, \quad t \in [0,1]\,, \\ \psi_{3,\kappa}(\kappa,t) & = 0, \quad t \in [0,1]\,, \\ \frac{\partial \psi_{3,\kappa}(y,t)}{\partial t} & - A\psi_{3,\kappa}(y,t) = -\lambda\psi_{2,\kappa}(y,t) + \lambda \int_0^y \psi_{2,\kappa}(y-z,t)\, dF(z), \\ & \quad (y,t) \in (0,\kappa) \times [0,1]\,. \end{cases} \tag{3.0.37}$$

It follows from Lemma 3.0.7) that $H_{2,\kappa}(y,t) \in C^{\frac{2}{3},\frac{1}{3}}([0,\kappa]\times[0,1])$. Thus, existence and uniqueness will follow from Theorem VIII.2.1 in Garroni and Menaldi (1992), once we have verified that the conditions (VIII.1.2), (VIII.1.3), (VIII.1.11),(VIII.1.12, (VIII.1.14)) and (VIII.1.15) in Garroni and Menaldi (1992) all hold.

The conditions (VIII.1.2) and (VIII.1.3) concern the coefficients of differential terms of the operator A, while the conditions (VIII.1.11), (VIII.1.12, (VIII.1.14)) and (VIII.1.15)) concern the terms

$$\lambda \int_0^y \psi_{3,\kappa}(y-z,t)dF(z) - \lambda\psi_{3,\kappa}(y,t).$$

We note that none of these coefficients depend on t, and that they are all bounded and Lipschitz continuous in y on the truncated domain $[0,\kappa]\times[0,1]$. It follows that the coefficients of A are in $C^{\alpha,\frac{\alpha}{2}}([0,\kappa]\times[0,1])$ for any $\alpha \in (0,1)$. Since we are assuming that $\sigma_P > 0$ it is obvious that the second order coefficient $\frac{1}{2}\left(\sigma_P^2 + \sigma_R^2 y^2\right)$ is bounded away from 0. From these observations it follows that the conditions (VIII.1.2) and (VIII.1.3) are satisfied.

Let $g(y,t)$ be a Borel-measurable function defined on $[0,\kappa]\times[0,1]$, and let

$$\tilde{g}(y,t) = \begin{cases} g(y,t), \quad y \in [0,\kappa]\,, \\ 0, \quad y < 0, \end{cases}$$

and let π be the finite Borel measure on $[0,\infty)$ defined by

$$\pi\left((a,b]\right) = \lambda\left(F(b) - F(a)\right), \quad b \geq 0, -\infty < a \leq b.$$

Let

$$j\,(y,t,z) = -z, \quad (y,t,z) \in [0,\kappa]\times[0,1]\times(-\infty,\infty)\,,$$

let

$$j\,(y,t,z,\theta) = \theta j\,(y,t,z)\,, \quad (y,t,z,\theta) \in [0,\kappa]\times[0,1]\times(-\infty,\infty)\times[0,1]\,,$$

and let

$$m\,(y,t,z) = 1, \quad (y,t,z) \in [0,\kappa]\times[0,1]\times[0,\infty).$$

Since F is a probability measure that assigns all its mass to $[0, \infty)$ it follows that

$$\lambda \int_0^y g(y-z,t)dF(z) - \lambda g(y,t)$$
$$= \int_0^\infty \left(g(y + j\left(y,t,z\right),t) - g(y,t)\right) m\left(y,t,z\right) d\pi\left(z\right).$$

Since both $j\left(y,t,z,\theta\right)$, and $m\left(y,t,z\right)$ are invariant of y and t it follows that conditions VIII.1.12, VIII.1.14 and condition VIII.1.15 are all satisfied. Since

$$0 \leq m\left(y,t,z\right) \leq 1,$$

and

$$\pi\left([0,\infty)\right) = \lambda,$$

it follows that the last condition, VIII.1.11, is also satisfied. Hence, it follows from Theorem VIII.2.1 in Garroni and Menaldi (1992) that $\psi_{3,\kappa}(y,t)$ as defined in (3.0.36) is a unique solution of the PIDE (3.0.10). □

4 Global estimates

So far we have shown existence and uniqueness of a classical solution except at the origin of equation (3.0.7). However, what we really want is to prove existence and uniqueness of a solution of equation (3.0.7) on the full unbounded domain, subject to an asymptotic upper boundary condition rather than a conventional Dirichlet boundary condition. Unfortunately, since so much of the conventional theory for PDE's and PIDE-s breaks down when the domain is unbounded we will not in this article be able to prove uniqueness of a solution of (3.0.7) on the full unbounded domain. The breakdown of conventional PDE-theory is also the reason we in this section will need to do extensive work with Green functions and representation formulas like the one in Definition 3.0.5. In this article we take the approach of first working with Green functions to obtain regularity bounds on the solutions of equations (3.0.8) and (3.0.9) that are independent of the upper domain boundary constant κ.

In the general case the main problem is that when the domain is not bounded, then both the first and second order coefficients go to infinity as $y \to \infty$. When $\sigma_R^2 > 0$ we deal with this problem by making the change of variable $x = \ln\left(1+y\right)$ and consider the functions

$$\hat{\psi}_{2,\kappa}(x,t) := \psi_{2,\kappa}(e^x - 1, t), \quad x \in [0, \ln\left(1+\kappa\right)] \times [0,1]$$

and

$$\hat{\psi}_{3,\kappa}(x,t) := \psi_{3,\kappa}(e^x - 1, t), \quad x \in [0, \ln\left(1+\kappa\right)] \times [0,1].$$

For now though, we will assume that $\sigma_R = r = 0$ (constant coefficients). Under this assumption regularity bounds not depending on κ can be obtained by working directly with the Green functions $G_{L^*,\kappa}\left(y,t,\xi,\vartheta\right)$ and $G_{A^*,\kappa}\left(y,t,\xi,\vartheta\right)$ and the formulas (3.0.5) and (3.0.36). The case with constant coefficients is much simpler than the other two cases, and some central ideas are considerably easier to understand in this setting. We will later see that several of these results can be recycled for the case when $\sigma_R > 0$.

To make things work on unbounded domain we will for the rest of this article make the assumption that for some $\beta > 0$ and some constant C, the tail distribution $\bar{F}$ satisfies the inequality

$$\bar{F}(\zeta) \leq C\,(1+\zeta)^{-\beta}\,. \tag{4.0.38}$$

The bounds we will obtain at the end will depend on this β. These bounds will not be sharp, but still sufficient to show that the derivatives evaluated at points bounded away from the origin are bounded, that the solution vanishes as the space variable y goes to infinity, and that the asymptotic boundary condition is thus satisfied.

4.1 Constant coefficients

4.1.1 Global estimates for a subproblem with constant coefficients

In this section we will obtain regularity estimates of the PDE (3.0.9) that are independent of the constant γ, for the special case that $\sigma_R = r = 0$. In the next section we will do the same for the PIDE (3.0.10), still assuming that $\sigma_R = r = 0$. In both cases the main tools that we want to use are representations of the solutions of the PDE (3.0.9) and the PIDE (3.0.10) in terms of Green functions. For the PDE the representation formula is given in Theorem VI.2 in Garroni and Menaldi (1992), while for the PIDE the representation formula is given in Theorem VIII.2.1. Unfortunately constructing these Green functions is quite a lot of work. In addition, since the end goal is to prove existence on an unbounded domain, we will need suitable estimates that we can later use to show that the solutions of the PDE (3.0.9) and the PIDE (3.0.10) converge in an appropriate manner, as we let the upper boundary constant γ tend to infinity.

In Garroni and Menaldi (1992) it is suggested to use *fundamental solutions*, a notion defined below, to construct Green functions for PDE problems with Dirichlet boundary conditions. We will follow this approach except that we will first focus on the construction of a Green function associated with the operator $\frac{\partial}{\partial t} - \frac{1}{2}\sigma_P^2 \frac{\partial^2}{\partial y^2}$. After having constructed a Green function associated with this simpler operator we will use Proposition VIII.1.2 to construct a Green function associated with the larger operator $\frac{\partial}{\partial t} - \frac{1}{2}\sigma_P^2 \frac{\partial^2}{\partial y^2} - p\frac{\partial}{\partial y}$. Finally, in section 4.1.2 we will use Proposition VIII.1.2 again to construct the full Green function $G_{A^*,\gamma}\,(y,t,\xi,\vartheta)$, still assuming that $\sigma_R = r = 0$. In section 4.2 we will show that an analogous approach, with a different variable, yields similar results when $\sigma_R > 0$ as in the case when $\sigma_R = r = 0$. The last case, when $\sigma_P = 0$ but $r > 0$, will not be treated in this article. We will use the following definition of a fundamental solution, taken from the definition in chapter IV in Garroni and Menaldi (1992).

Definition 4.1.1. *A function $\Gamma_L\,(y,t,\xi,\vartheta)$ defined in the domain*

$$\mathcal{D} = \{y,t,\xi,\vartheta : y,\xi \in \mathbb{R}, 0 \leq \vartheta < t \leq 1\}$$

is called a fundamental solution for the differential operator

$$\frac{\partial}{\partial t} - L$$

if it satisfies the following:

(i) $\Gamma_L(y,t,\xi,\vartheta)$ *is continuous in* (y,t) *and locally integrable in* (ξ,ϑ),

(ii)

$$\frac{\partial\Gamma_L(y,t,\xi,\vartheta)}{\partial t} - L\Gamma_L(y,t,\xi,\vartheta) = \delta(y-\xi)\,\delta(t-\vartheta), \quad \text{in } \mathcal{D},$$

(iii)

$$\lim_{t-\vartheta\downarrow 0}\Gamma_L(y,t,\xi,\vartheta) = \delta(y-\xi), \quad \text{in } \mathcal{D}.$$

In the above $\delta(y,t)$ *is the Dirac measure at* 0. *As discussed in section IV.1 in Garroni and Menaldi (1992) we need a further* boundedness condition, *like the one given below, to ensure uniqueness of the fundamental solution. In Garroni and Menaldi (1992) a function satisfying this condition in addition to the condition below is referred to as a* principal fundamental solution. *In this article we will, for simplicity, use this condition as part of our definition of a fundamental solution.*

(iv) *For every* $\delta > 0$, *there exists a finite positive constant* M_δ *such that*

$$|\Gamma_L(y,t,\xi,\vartheta)| \le M_\delta, \quad \text{for } |t-\vartheta| + |y-\xi|^2 \ge \delta.$$

Condition (ii) *means that the* volume potential,

$$u(y,t) = \int_0^t \int_{-\infty}^{\infty} \Gamma_L(y,t,\xi,\vartheta)\, f(\xi,\vartheta)\, d\xi d\vartheta$$

is a classical (i.e. $C^{2,1}((-\infty,\infty)\times(0,1],\mathbb{R})$*) solution of the equation*

$$\frac{\partial u(y,t)}{\partial t} - Lu(y,t) = f(y,t), \quad ,\forall y,t \in (0,1],$$

for any smooth function $f(y,t)$ *with compact support in* $\mathbb{R}\times(0,1]$. (iii) *means that for every smooth function* $\phi(y)$ *with compact support in* $\mathbb{R}$ *the* potential

$$w_\vartheta(y,t) = \int_{-\infty}^{\infty} \Gamma_L(y,t,\xi,\vartheta)\,\phi(\xi)\, d\xi$$

is a continuous and bounded function, i.e. in $C^0(\mathbb{R}\times[\vartheta,1),\mathbb{R})$, *and satisfies the limit condition*

$$\lim_{(t-\vartheta)\to 0} w_\vartheta(y,t) = \phi(y), \quad \forall y \in \mathbb{R}.$$

Now, consider the function

$$\Gamma_{\sigma_P}(y,t,\xi,\vartheta) := \frac{1}{\sqrt{2\pi(t-\vartheta)\sigma_P^2}} \exp\left(-\frac{(y-\xi)^2}{2\sigma_P^2(t-\vartheta)}\right), \quad (y,t,\xi,\vartheta) \in \mathcal{D}. \tag{4.1.1}$$

It is easy to verify that this function satisfies the identities and bounds stated in the results below.

Proposition 4.1.1. *For every* $(t, \vartheta) \in (0, 1] \times [0, t))$ *and* $y, \xi \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \Gamma_{\sigma_P}(y, t, \xi, \vartheta)\, d\xi = 1.$$

Proposition 4.1.2. *For every* $(y, t, \xi, \vartheta) \in \mathcal{D}$

$$\begin{aligned}
\Gamma_{\sigma_P}(y, t, \xi, \vartheta) &= \Gamma_{\sigma_P}(y - \xi, t, 0, \vartheta), \\
\Gamma_{\sigma_P}(y, t, \xi, \vartheta) &= \Gamma_{\sigma_P}(y, t - \vartheta, \xi, 0), \\
\frac{\partial \Gamma_{\sigma_P}(y, t, \xi, \vartheta)}{\partial \xi} &= \frac{y - \xi}{(t - \vartheta)\,\sigma_P^2} \Gamma_{\sigma_P}(y, t, \xi, \vartheta), \\
\frac{\partial \Gamma_{\sigma_P}(y, t, \xi, \vartheta)}{\partial y} &= -\frac{\partial \Gamma_{\sigma_P}(y, t, \xi, \vartheta)}{\partial \xi}, \\
\frac{\partial^2 \Gamma_{\sigma_P}(y, t, \xi, \vartheta)}{\partial y^2} &= \frac{\Gamma_{\sigma_P}(y, t, \xi, \vartheta)}{(t - \vartheta)\,\sigma_P^2} \left[-1 + \frac{(y - \xi)^2}{(t - \vartheta)\,\sigma_P^2} \right]. \\
\frac{\partial \Gamma_{\sigma_P}(y, t, \xi, \vartheta)}{\partial t} &= \frac{1}{2} \sigma_P^2 \frac{\partial^2 \Gamma_{\sigma_P}(y, t, \xi, \vartheta)}{\partial y^2} \quad \textit{and} \\
\frac{\partial \Gamma_{\sigma_P}(y, t, \xi, \vartheta)}{\partial \vartheta} &= -\frac{\partial \Gamma_{\sigma_P}(y, t, \xi, \vartheta)}{\partial t}.
\end{aligned}$$

Because of Proposition 3.0.1 we also have the following bounds:

Proposition 4.1.3. *There exists a positive constant* C *such that for every*

$(y,t,\xi,\vartheta) \in \mathcal{D}$ *the following inequalities hold:*

$$\begin{aligned}
\left|\Gamma_{\sigma_P}(y,t,\xi,\vartheta)\right| &\le C(t-\vartheta)^{-\frac{1}{2}} \exp\left(-c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right), \\
\left|\frac{\partial \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y}\right| &\le C|y-\xi|(t-\vartheta)^{-\frac{3}{2}} \exp\left(-c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right), \\
\left|\frac{\partial \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial \xi}\right| &\le C|y-\xi|(t-\vartheta)^{-\frac{3}{2}} \exp\left(-c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right), \\
\left|\frac{\partial^2 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y^2}\right| &\le C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right), \\
\left|\frac{\partial \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial t}\right| &\le C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right), \\
\left|\frac{\partial^2 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y \partial \xi}\right| &\le C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right), \\
\left|\frac{\partial^3 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y^2 \partial \xi}\right| &\le C|y-\xi|(t-\vartheta)^{-\frac{5}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right), \\
\left|\frac{\partial^2 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial t \partial \xi}\right| &\le C|y-\xi|(t-\vartheta)^{-\frac{5}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right). \\
\left|\frac{\partial^3 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y^3}\right| &\le C|y-\xi|(t-\vartheta)^{-\frac{5}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right), \\
\left|\frac{\partial^4 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y^3 \partial \xi}\right| &\le C(t-\vartheta)^{-\frac{5}{2}} \exp\left(-\frac{1}{2}c_0 \frac{(y-\xi)^2}{(t-\vartheta)}\right).
\end{aligned}$$

The most important consequence of the results above is that $\Gamma_{\sigma_P}(y,t,\xi,\vartheta)$ is a fundamental solution in the special case when $\sigma_R = p = r = 0$. Moreover, it follows from Theorem V.3.5 in Garroni and Menaldi (1992) that this fundamental solution is unique. Following the discussion in section VI.1.5 it is clear that the problem of constructing a Green function associated with the operator $\frac{\partial}{\partial t} - \frac{1}{2}\sigma_P^2$ can be reformulated as finding a solution of a PDE, as indicated in the next result.

Lemma 4.1.1. *Let* $g^*_{L_0,\gamma}(y,t,\xi)$ *be the unique classical solution of the equation*

$$\begin{cases}
g^*_{L_0,\gamma}(y,0,\xi) &= 0, \quad y \in [0,\gamma], \\
g^*_{L_0,\gamma}(0,t,\xi) &= \Gamma_{\sigma_P}(0,t,\xi,0), \quad t \in (0,1], \\
g^*_{L_0,\gamma}(\gamma,t,\xi) &= \Gamma_{\sigma_P}(\gamma,t,\xi,0), \quad t \in (0,1], \\
\frac{\partial g^*_{L_0,\gamma}(y,t,\xi)}{\partial t} &= \frac{1}{2}\sigma_P^2 \frac{\partial^2 g^*_{L_0,\gamma}(y,t,\xi)}{\partial y^2} \\
& \quad (y,t) \in (0,\gamma) \times (0,1],
\end{cases} \tag{4.1.2}$$

and let

$$g_{L_0,\gamma}(y,t,\xi,\vartheta) := g^*_{L_0,\gamma}(y,t-\vartheta,\xi), \quad (y,t,\vartheta,\xi) \in \bar{\mathcal{D}}_\gamma.$$

Assume that for any smooth function $f(\xi,\vartheta)$ *with compact support and any* $(y,t)\in(0,\gamma)\times(0,1]$*, and* $l\in\{1,2\}$

$$\frac{\partial^l}{\partial y^l}\int_0^t\int_0^\gamma g_{L_0,\gamma}(y,t,\xi,\vartheta)\,f(\xi,\vartheta)\,d\xi=\int_0^t\int_0^\gamma\frac{\partial^l g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial y^l}f(\xi,\vartheta)\,d\xi,$$
$$\frac{\partial}{\partial t}\int_0^t\int_0^\gamma g_{L_0,\gamma}(y,t,\xi,\vartheta)\,f(\xi,\vartheta)\,d\xi=\int_0^t\int_0^\gamma\frac{\partial g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial t}f(\xi,\vartheta)\,d\xi \tag{4.1.3}$$

and that for any smooth function $\phi(y)$ *with compact support*

$$\lim_{t-\vartheta\to0}\int_0^\gamma g_{L_0,\gamma}(y,t,\xi,\vartheta)\,\phi(\xi)d\xi=0. \tag{4.1.4}$$

Then

$$G_{L_0,\gamma}(y,t,\xi,\vartheta)=\Gamma_{\sigma_P}(y,t,\xi,\vartheta)-g_{L_0,\gamma}(y,t,\xi,\vartheta)$$

is the Green function associated with the differential operator

$$\frac{\partial}{\partial t}-\frac{1}{2}\sigma_P^2\frac{\partial^2}{\partial y^2}$$

with Dirichlet boundary conditions on $(y,t)\in(0,\gamma)\times(0,T]$*.*

Proof. We first observe that, because of Theorem 3.0.2, existence and uniqueness of the Green function is already established.

It follows from the proof of Theorem VI.2.1 in Garroni and Menaldi (1992) that $g_{L_0,\gamma}(y,t,\xi,\vartheta)$ must satisfy the equation below, which is the same as equation VI.2.8 in Garroni and Menaldi (1992) adapted to equation (3.0.9):

$$\begin{cases}\lim_{t\downarrow\vartheta}g_{L_0,\gamma}(y,t,\xi,\vartheta) & =0,\quad y\in(0,\gamma),\\ g_{L_0,\gamma}(0,t,\xi,\vartheta) & =\Gamma_{\sigma_P}(0,t,\xi,\vartheta),\quad t\in(\vartheta,1],\\ g_{L_0,\gamma}(\gamma,t,\xi,\vartheta) & =\Gamma_{\sigma_P}(\gamma,t,\xi,\vartheta),\quad t\in(\vartheta,1],\\ \frac{\partial g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial t} & =\frac{1}{2}\sigma_P^2\frac{\partial^2 g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial y^2},\ \text{for}\ (y,t)\in(0,\gamma)\times(\vartheta,1].\end{cases} \tag{4.1.5}$$

Moreover, it follows from Proposition 3.0.1, Proposition 4.1.2 and Proposition 4.1.3 that for some constant C the following equality and bounds hold, for every $(y,t,\xi,\vartheta)\in\partial\mathcal{D}_\gamma$ and every $t_2,t_1\in(\vartheta,1]$:

$$\Gamma_{\sigma_P}(y,t,\xi,\vartheta)=\Gamma_{\sigma_P}(y,t-\vartheta,\xi,0),$$
$$|\Gamma_{\sigma_P}(y,t,\xi,\vartheta)|\le C\,|y-\xi|^{-3}(t-\vartheta),$$

and

$$|\Gamma_{\sigma_P}(y,t_2,\xi,\vartheta)-\Gamma_{\sigma_P}(y,t_1,\xi,\vartheta)|\le C\,|y-\xi|^{-3}|t_2-t_1|\,.$$

Because of the bounds above and the smoothness of the coefficients (trivial since they are constants) of $\Gamma_{\sigma_P}(\gamma,t-\vartheta,\xi,0)$ it follows from Theorem I.2.1 in Garroni and Menaldi (1992) that, for every fixed $\xi\in(0,\xi)$, there exists a unique classical solution $g^*_{L_0,\gamma}(y,t,\xi)$ of the PDE (4.1.2). Also, this solution satisfies the boundedness condition given in part (iv)) of Definition 4.1.1 (the definition of the corresponding fundamental solution). Because of how $g_{L_0,\gamma}(y,t,\xi,\vartheta)$ was defined it is obvious that $g_{L_0,\gamma}(y,t,\xi,\vartheta)$ also satisfies that boundedness condition.

Lastly, it follows from the symmetry property (in t and ϑ) of $\Gamma_{\sigma_P}(y,t,\xi,\vartheta)$ and the chain rule that $g_{L_0,\gamma}(y,t,\xi,\vartheta)$ is a solution of equation (4.1.5) and satisfies the other requirements in Definition 3.0.1, when $\sigma_R = r = 0$. □

To solve the PDE (4.1.5) we will rely on Theorem V.5.5 in Garroni and Menaldi (1992), which in the theorem below is adapted to our situation.

Definition 4.1.2. *For*

$$g \in C([0,1],\mathbb{R})$$

let

$$P^{(1)}_{g,\gamma}(y,t) := \int_0^t \frac{1}{2}\sigma_P^2 \left.\frac{\partial \Gamma_{\sigma_P}(y,t,\eta,\vartheta)}{\partial \eta}\right|_{\eta=\gamma} g(\vartheta)\,d\vartheta, \quad y \geq 0, t \in [0,1],$$

and

$$P^{(2)}_{g}(y,t) := \int_0^t \frac{1}{2}\sigma_P^2 \left.\frac{\partial \Gamma_{\sigma_P}(y,t,\eta,\vartheta)}{\partial \eta}\right|_{\eta=0} g(\vartheta)\,d\vartheta, \quad y \geq 0, t \in [0,1].$$

For

$$\mathbf{g} = \left(g^{(1)}(t), g^{(2)}(t)\right) \in C\left([0,1],\mathbb{R}^2\right)$$

let

$$P_{\mathbf{g},\gamma}(y,t) := P^{(1)}_{g^{(1)},\gamma}(y,t) - P^{(2)}_{g^{(2)}}(y,t), \quad t \in [0,1].$$

Theorem 4.1.1. *Assume that $\sigma_R = p = r = 0$. Also assume that $\mu(t) = \left(\mu^{(1)}(t), \mu^{(2)}(t)\right) \in C\left([0,1],\mathbb{R}^2\right)$ is a solution of the integral equation*

$$\begin{cases} -\frac{1}{2}\mu^{(1)}(t) + P_{\mu,\gamma}(\gamma,t) = \Gamma_{\sigma_P}(\gamma,t,\xi,0), & t \in (0,1] \\ -\frac{1}{2}\mu^{(2)}(t) + P_{\mu,\gamma}(0,t) = \Gamma_{\sigma_P}(0,t,\xi,0), & t \in (0,1] \end{cases} \tag{4.1.6}$$

such that

$$\lim_{t\downarrow 0} \mu^{(1)}(t) = 0.$$

Then

$$P_{\mu,\gamma}(y,t)$$

is a classical solution of the PDE (4.1.5).

Proof. This follows from Theorem V.5.5 and more generally the discussion in section V.5.2 in Garroni and Menaldi (1992). □

We will proceed to construct a solution of the integral equation above using the *method of successive approximations.* This entails constructing a recursively defined sequence, the sum of which is the solution of the integral equation. It will be clear from the next result that the limit of this sequence exists. It will turn out that in order to obtain regularity estimates of the entire Green function we will need regularity estimates for each "building block". We therefore include bounds for the first two derivatives with respect to t, as well as bounds for $g_{L_0,\gamma}(y,t,\xi,\vartheta)$.

Definition 4.1.3. *Let*

$$V^{(1)}_{\xi,0,\gamma}(t) := -2\Gamma_{\sigma_P}(\gamma, t, \xi, 0), \quad (t,\xi) \in [0,1] \times (0,\gamma),$$

$$V^{(2)}_{\xi,0,\gamma}(t) := -2\Gamma_{\sigma_P}(0, t, \xi, 0), \quad (t,\xi) \in [0,1] \times (0,\gamma) \text{ and let}$$

$$\mathbf{V}_{\xi,0,\gamma}(t,\xi) := \left(V^{(1)}_{\xi,0,\gamma}(t), V^{(2)}_{\xi,0,\gamma}(t)\right).$$

For $n \in 0, 1, 2, \dots,$ *and* $(t,\xi) \in [0,1] \times (0,\gamma)$ *define*

$$\mathbf{V}_{\xi,n,\gamma}(t,\xi) = \left(V^{(1)}_{\xi,n,\gamma}(t,\xi), V^{(2)}_{\xi,n,\gamma}(t,\xi)\right)$$

recursively by

$$\begin{aligned} V^{(1)}_{\xi,n+1,\gamma}(t) &:= 2P_{\mathbf{V}_{\xi,n,\gamma},\gamma}(\gamma, t), \\ V^{(2)}_{\xi,n+1,\gamma}(t) &:= 2P_{\mathbf{V}_{\xi,n,\gamma},\gamma}(0, t) \\ \mathbf{V}_{\xi,n+1,\gamma}(t) &:= \left(V^{(1)}_{\xi,n+1,\gamma}(t), V^{(2)}_{\xi,n+1,\gamma}(t)\right). \end{aligned}$$

Let

$$U^{(1)}_{\xi,n,\gamma}(t) := \sum_{k=0}^{n} V^{(1)}_{\xi,k}(t), \quad t \in [0,1], n \in 0, 1, \dots,$$

$$U^{(2)}_{\xi,n,\gamma}(t) := \sum_{k=0}^{n} V^{(2)}_{\xi,k}, \quad n \in 0, 1, \dots,$$

let

$$\mathbf{U}_{\xi,n,\gamma}(t) := \left(U^{(1)}_{\xi,n,\gamma}(t), U^{(2)}_{\xi,n,\gamma}(t)\right), \quad n \in 0, 1, \dots,$$

let

$$U^{(1)}_{\xi,\gamma}(t) := \lim_{n\to\infty} U^{(1)}_{\xi,n,\gamma}(t), \quad t \in [0,1],$$

$$U^{(2)}_{\xi,\gamma}(t) := \lim_{n\to\infty} U^{(2)}_{\xi,n,\gamma}(t), \quad t \in [0,1],$$

and let

$$\mathbf{U}_{\xi,\gamma}(t) := \left(U^{(1)}_{\xi,\gamma}(t), U^{(2)}_{\xi,\gamma}(t)\right).$$

Lemma 4.1.2. (i) *For every* $\mathbf{g} = \left(g^{(1)}, g^{(2)}\right) \in C\left([0,1], \mathbb{R}^2\right)$

$$P_{\mathbf{g},\gamma}(\gamma, t) = -\frac{1}{2}\sigma_P^2 \int_0^t \left.\frac{\partial \Gamma_{\sigma_P}(\gamma, t, \eta, \vartheta)}{\partial \eta}\right|_{\eta=0} g^{(2)}(\vartheta)\, d\vartheta,$$

and

$$P_{\mathbf{g},\gamma}(0, t) = -\frac{1}{2}\sigma_P^2 \int_0^t \left.\frac{\partial \Gamma_{\sigma_P}(0, t, \eta, \vartheta)}{\partial \eta}\right|_{\eta=\gamma} g^{(1)}(\vartheta)\, d\vartheta.$$

(ii) $\left(P_{\mathbf{g},\gamma}(\gamma, t), P_{\mathbf{g},\gamma}(0, t)\right)$ *maps* $C\left([0,1], \mathbb{R}^2\right)$ *to* $C^2\left([0,1], \mathbb{R}^2\right)$*. Moreover, there exists a constant* C *such that for every* $t \in (0,1]$ *and* $l \in \{0,1,2\}$ *the following inequalities are all valid:*

$$\left|\frac{\partial^l P_{\mathbf{g},\gamma}(\gamma, t)}{\partial t^l}\right| \le C \exp\left(-\frac{1}{2} c_0 \frac{\gamma^2}{t}\right) \int_0^t \left|g^{(2)}(\vartheta)\right| d\vartheta$$

and

$$\left|\frac{\partial^l P_{\mathbf{g},\gamma}(0, t)}{\partial t^l}\right| \le C \exp\left(-\frac{1}{2} c_0 \frac{\gamma^2}{t}\right) \int_0^t \left|g^{(1)}(\vartheta)\right| d\vartheta.$$

(iii) *For every $n \in 0, \ldots,$ and every $t \in [0,1]$*

$$-\frac{1}{2}U^{(1)}_{\xi,n,\gamma}(t) + P_{\mathbf{U}_{\xi,n,\gamma}(t)}(\gamma,t) = \Gamma_{\sigma_P}(\gamma,t,\xi,\vartheta) + P_{\mathbf{V}_{\xi,n,\gamma}}(\gamma,t),$$

and

$$-\frac{1}{2}U^{(2)}_{\xi,n,\gamma}(t) + P_{\mathbf{U}_{\xi,n,\gamma},\gamma}(0,t) = \Gamma_{\sigma_P}(0,t,\xi,\vartheta) + P_{\mathbf{V}_{\xi,n,\gamma}}(0,t).$$

(iv) *There exists a sequence $\{k_n\}_{n=0}^{\infty}$ of positive constants, such that*

$$\lim_{n\to\infty}\frac{k_{n+1}}{k_n} = 0,$$

and such that the inequalities

$$\left|V^{(1)}_{\xi,n,\gamma}(t)\right| \le k_n t^{n-\frac{1}{2}} \exp\left(-\frac{1}{2}c_0\frac{\gamma^2}{t}\right), \tag{4.1.7}$$

$$\left|V^{(1)}_{\xi,n,\gamma}{}'(t)\right| \le k_n t^{n-\frac{1}{2}} \exp\left(-\frac{1}{2}c_0\frac{\gamma^2}{t}\right),$$

$$\left|V^{(1)}_{\xi,n,\gamma}{}''(t)\right| \le k_n t^{n-\frac{1}{2}} \exp\left(-\frac{1}{2}c_0\frac{\gamma^2}{t}\right),$$

$$\left|V^{(2)}_{\xi,n,\gamma}(t)\right| \le k_n t^{n-\frac{1}{2}} c_0 \exp\left(-\frac{1}{2}\frac{\gamma^2}{t}\right),$$

$$\left|V^{(2)}_{\xi,n,\gamma}{}'(t)\right| \le k_n t^{n-\frac{1}{2}} c_0 \exp\left(-\frac{1}{2}\frac{\gamma^2}{t}\right),$$

and

$$\left|V^{(2)}_{\xi,n,\gamma}{}''(t)\right| \le k_n t^{n-\frac{1}{2}} c_0 \exp\left(-\frac{1}{2}\frac{\gamma^2}{t}\right),$$

all hold for every $t \in [0,t]$ and every $n \in 1, 2 \ldots,$.

Proof. For (i): This is obvious because of Proposition 4.1.2.

For (ii): It follows from Proposition 3.0.1 and Leibniz' rule that

$$\frac{\partial P_{\mathbf{g},\gamma}(\gamma,t)}{\partial t} = \int_0^t \left.\frac{\partial^2\Gamma_{\sigma_P}(\gamma,t,\eta,\vartheta)}{\partial t\partial\eta}\right|_{\eta=0} g^{(2)}(\vartheta)\,d\vartheta$$

and

$$\frac{\partial P_{\mathbf{g},\gamma}(0,t)}{\partial t} = \int_0^t \left.\frac{\partial^2\Gamma_{\sigma_P}(0,t,\eta,\vartheta)}{\partial t\partial\eta}\right|_{\eta=\gamma} g^{(1)}(\vartheta)\,d\vartheta.$$

The stated bounds in part (ii) can be calculated from the identities above and the identities and bounds given in Proposition 3.0.1, Proposition 4.1.2 and Proposition 4.1.3.

For (iii): The equalities given in part (iii) obviously hold for $n = 0$. Assume that for every $k \in 0, 1, \ldots, n$

$$-\frac{1}{2} U^{(1)}_{\xi,n,\gamma}(t) + P_{\mathbf{U}_{\xi,n,\gamma},\gamma}(\gamma, t) = \Gamma_{\sigma_P}(\gamma, t, \xi, \vartheta) + P_{\mathbf{V}_{n,\gamma}}(\gamma, t) .$$

Since by definition

$$V^{(1)}_{n+1,\gamma}(t) := 2 P_{\mathbf{V}_{n,\gamma}}(\gamma, t) ,$$

it follows that

$$\begin{aligned} & -\frac{1}{2} U^{(1)}_{\xi,n+1,\gamma}(t) + P_{\mathbf{U}_{\xi,n+1,\gamma},\gamma}(\gamma, t) \\ & = -\frac{1}{2} U^{(1)}_{\xi,n,\gamma} + P_{\mathbf{U}_{\xi,n,\gamma},\gamma}(\gamma, t) - \frac{1}{2} V^{(1)}_{\xi,n+1,\gamma} + P_{\mathbf{V}_{\xi,n+1,\gamma}}(\gamma, t) \\ & = \Gamma_{\sigma_P}(\gamma, t, \xi, \vartheta) + P_{\mathbf{V}_{\xi,n+1,\gamma},\gamma}(\gamma, t) . \end{aligned}$$

A similar argument yields that for every $n \in 0, 1, \ldots,$

$$-\frac{1}{2} U^{(2)}_{\xi,n,\gamma}(t) + P_{\mathbf{U}_{\xi,n,\gamma},\gamma}(0, t) = 1 + P_{\mathbf{V}_{\xi,n,\gamma}}(0, t) .$$

For (iv): Let $m_n := \frac{1}{\Gamma\left(n+\frac{1}{2}\right)}$. We first observe that for some constant C

$$\left| V^{(1)}_{\xi,0,\gamma}(t) \right| \leq C t^{-\frac{1}{2}} \exp\left(-c_0 \frac{(\gamma - \xi)^2}{t} \right) .$$

Because of the bounds given in part (ii) and the identity given in Proposition 3.0.2 it can be calculated by induction that, for some (different from above) constant C and $n \in 1, 2, \ldots,$

$$\left| V^{(2)}_{\xi,n,\gamma}(t) \right| \leq C^n \frac{m_{n-1}}{m_n} .$$

Because of Proposition 3.0.2 a simple calculation yields that

$$\lim_{n \to \infty} \frac{C^{n+1} m_n}{C^n m_{n-1}} = c \lim_{n \to \infty} \frac{1}{\frac{1}{2} + n} = 0,$$

yielding the bound (4.1.7). Similar calculations also yield the other bounds given in part (iv). □

In Definition VII.1.1 in Garroni and Menaldi (1992) they define certain function spaces, denoted by $\mathcal{G}^{\alpha,\frac{\alpha}{2}}_k$, that we we will work with in the rest of the article. Specifically we want the function $\frac{\partial P_{\mathbf{U}_{\xi,\gamma},\gamma}(y,t)}{\partial y}$ to be in the function space $\mathcal{G}^{\alpha,\frac{\alpha}{2}}_1$ for every $\alpha \in (0, 1)$. For that we need a few more regularity results given below.

Lemma 4.1.3. (i) *There exists a constant C such that the following inequalities are all valid for every $(y, t, \xi, \vartheta) \in \mathcal{D}_\gamma$ and every $l \in \{0, 1, 2\}$.*

$$\begin{aligned} \frac{\partial^l g_{L_0,\gamma}(y, t, \xi, \vartheta)}{\partial y^l} \leq C (t - \vartheta)^{-\frac{1+l}{2}} \Bigg\{ & \exp\left(-\frac{1}{2} c_0 \left[\frac{(y-\gamma)^2 + (\xi - \gamma)^2}{t - \vartheta} \right] \right) \\ & + \exp\left(-\frac{1}{2} c_0 \left[\frac{y^2 + \xi^2}{t} \right] \right) \Bigg\}, \end{aligned}$$

$$\frac{\partial g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial t} \leq C\left(t-\vartheta\right)^{-\frac{3}{2}}\left\{\exp\left(-\frac{1}{2}c_0\left[\frac{\left(y-\gamma\right)^2+\left(\xi-\gamma\right)^2}{t-\vartheta}\right]\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}c_0\left[\frac{y^2+\xi^2}{t}\right]\right)\right\},$$
$$\frac{\partial^2 g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial y\partial t} \leq C\left(t-\vartheta\right)^{-2}\left\{\exp\left(-\frac{1}{2}c_0\left[\frac{\left(y-\gamma\right)^2+\left(\xi-\gamma\right)^2}{t-\vartheta}\right]\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}c_0\left[\frac{y^2+\xi^2}{t}\right]\right)\right\},$$
$$\frac{\partial g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial \xi} \leq C\left(t-\vartheta\right)^{-1}\left\{\exp\left(-\frac{1}{2}c_0\left[\frac{\left(y-\gamma\right)^2+\left(\xi-\gamma\right)^2}{t-\vartheta}\right]\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}c_0\left[\frac{y^2+\xi^2}{t}\right]\right)\right\},$$

and

$$\frac{\partial^2 g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial y\partial \xi} \leq C\left(t-\vartheta\right)^{-\frac{3}{2}}\left\{\exp\left(-\frac{1}{2}c_0\left[\frac{\left(y-\gamma\right)^2+\left(\xi-\gamma\right)^2}{t-\vartheta}\right]\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}c_0\left[\frac{y^2+\xi^2}{t}\right]\right)\right\},$$

(ii) *Let*

$$G_{L_0,\gamma}\left(y,t,\xi,\vartheta\right)=\Gamma_{\sigma_P}\left(y,t,\xi,\vartheta\right)-g_{L_0,\gamma}\left(y,t,\xi,\vartheta\right).$$

There exists a constant C such that for every $(y,t,\xi\vartheta)\in\mathcal{D}_\gamma$ the following inequalities are all valid:

$$\left|\frac{\partial^l G_{L_0,\gamma}\left(y,t,\xi,\vartheta\right)}{\partial y^l}\right| \leq C\left(t-\vartheta\right)^{-\frac{1+l}{2}}\exp\left(-\frac{1}{4}c_0\frac{\left(y-\xi\right)^2}{t-\vartheta}\right),$$
$$\left|\frac{\partial G_{L_0,\gamma}\left(y,t,\xi,\vartheta\right)}{\partial \xi}\right| \leq C\left(t-\vartheta\right)^{-1}\exp\left(-\frac{1}{4}c_0\frac{\left(y-\xi\right)^2}{t-\vartheta}\right),$$
$$\left|\frac{\partial G_{L_0,\gamma}\left(y,t,\xi,\vartheta\right)}{\partial t}\right| \leq C\left(t-\vartheta\right)^{-\frac{3}{2}}\exp\left(-\frac{1}{4}c_0\frac{\left(y-\xi\right)^2}{t-\vartheta}\right),$$
$$\left|\frac{\partial^2 G_{L_0,\gamma}\left(y,t,\xi,\vartheta\right)}{\partial x\partial \xi}\right| \leq C\left(t-\vartheta\right)^{-\frac{3}{2}}\exp\left(-\frac{1}{4}c_0\frac{\left(y-\xi\right)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial^2 G_{L_0,\gamma}\left(y,t,\xi,\vartheta\right)}{\partial x\partial t}\right| \leq C\left(t-\vartheta\right)^{-\frac{3}{2}}\exp\left(-\frac{1}{4}c_0\frac{\left(y-\xi\right)^2}{t-\vartheta}\right).$$

(iii) *Assume that $\sigma_R=p=r=0$. Then for every $(y,t,\xi,\vartheta)\in\mathcal{D}_\gamma$*

$$G_{L,\gamma}\left(y,t,\xi,\vartheta\right)=G_{L_0,\gamma}\left(y,t,\xi,\vartheta\right).$$

Proof. For (i): It follows from Lemma 4.1.1, Theorem 4.1.1 and Lemma 4.1.2 that for $(y,t,\xi,\vartheta)\in\mathcal{D}_\gamma$

$$g_{L_0,\gamma}(y,t,\xi,\vartheta)=g^*_{L_0,\gamma}(y,t-\vartheta,\xi)=P_{\mathbf{U}_{\xi,\gamma}}(y,t-\vartheta)\,.$$

In the above $g^*_{L_0,\gamma}$ is defined in Lemma 4.1.1. We note that the biggest singularities of $\mathbf{U}_{\xi,\gamma}$ stem from the first term $\mathbf{V}_{\xi,0,\gamma}$. Furthermore the partial derivatives of the integral kernel $\Gamma_{\sigma_P}(y,t,\eta,\vartheta)$ are all interconnected, as indicated in Proposition 4.1.2. The stated bounds can be calculated by means of partial integration. In doing this it is helpful to consider separately the two halves of the domain of integration, corresponding to $0<\vartheta<\frac{t}{2}$ and $\frac{t}{2}<\vartheta<t$ respectively.

For (ii): Since, for any $y,\xi\in[0,\gamma]$,

$$(y-\xi)^2\le\min\left(y^2+\xi^2,(y-\gamma)^2+(\xi-\gamma)^2\right), \tag{4.1.8}$$

this follows from the bounds given in part (i) and the regularity bounds of the function $\Gamma_{\sigma_P}(y,t,\xi,\vartheta)$.

For (iii): What remains for $G_{L_0,\gamma}(y,t,\xi,\vartheta)$ to be a Green function for the special case $\sigma_R=p=r$ is to show that $g_{L_0,\gamma}(y,t,\xi,\vartheta)$ satisfies the requirements (4.1.3) and (4.1.4). Because of the bounds given in part (ii) it follows that for any such smooth $f(\xi,\vartheta)$ there exists a constant C such that

$$\int_0^\gamma\left|g_{L_0,\gamma}(y,t,\xi,\vartheta)f(\xi,\vartheta)\right|d\xi\le C\Bigg(\exp\left(-\frac{1}{2}c_0\frac{(y-\gamma)^2}{t-\vartheta}\right)+\exp\left(-\frac{1}{2}c_0\frac{y^2}{t-\vartheta}\right)\Bigg),$$

and such that

$$\int_0^t\int_0^\gamma\left|\frac{\partial g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial t}f(\xi,\vartheta)\right|d\xi d\vartheta\le C\sqrt{t}\left((\gamma-y)^{-1}+y^{-1}\right)\times\Bigg(\exp\left(-\frac{1}{2}c_0\frac{(y-\gamma)^2}{t-\vartheta}\right)+\exp\left(-\frac{1}{2}c_0\frac{y^2}{t-\vartheta}\right)\Bigg).$$

From these two inequalities it follows that the requirement (4.1.4) is satisfied and that

$$\frac{\partial}{\partial t}\int_0^t\int_0^\gamma g_{L_0,\gamma}(y,t,\xi,\vartheta)f(\xi,\vartheta)\,d\xi d\vartheta=\int_0^t\int_0^\gamma\frac{\partial g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial t}f(\xi,\vartheta)\,d\xi d\vartheta.$$

Similar calculations also yield that

$$\frac{\partial^l}{\partial y^l}\int_0^t\int_0^\gamma g_{L_0,\gamma}(y,t,\xi,\vartheta)f(\xi,\vartheta)\,d\xi d\vartheta=\int_0^t\int_0^\gamma\frac{\partial^l g_{L_0,\gamma}(y,t,\xi,\vartheta)}{\partial y^l}f(\xi,\vartheta)\,d\xi d\vartheta,$$

for $l\in\{1,2\}$.

□

The next step is to solve another integral equation in order to construct a slightly more general Green function corresponding to $p \geq 0$. To this end we will first need to do some preparatory work that is a bit similar to what we did to solve the integral equation (4.1.6).

Definition 4.1.4. *Let*

$$Q_{\kappa,0}(y,t,\xi,\vartheta) := p\frac{\partial G_{L_0,\kappa}(y,t,\xi,\vartheta)(y,t,\xi,\vartheta)}{\partial y}, \quad (y,t,\xi,\vartheta) \in \mathcal{D}_\kappa.$$

Define the sequence of functions $\{Q_{\kappa,n}\}_{n=0}^{\infty}$ *recursively for* $n \in 1,2,\ldots,$ *and* $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ *by*

$$Q_{\kappa,n+1}(y,t,\xi,\vartheta) = \int_\vartheta^t \int_0^\kappa Q_{\kappa,0}(y,t,z,s)\, Q_{\kappa,n}(z,s,\xi,\vartheta)\, dzds,$$

and let

$$Q_\kappa(y,t,\xi,\vartheta) = \sum_{n=0}^{\infty} Q_{\kappa,n}(y,t,\xi,\vartheta).$$

The result below shows that the sequence defined above solves the integral equation (4.1.9). This in turn will turn out to make it possible to conclude that

$$G_{L,\kappa}(y,t,\xi,\vartheta) = G_{L_0,\kappa}(y,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\kappa Q_\kappa(z,s,\xi,\vartheta)\, dzds,$$

in the case that $\sigma_R = r = 0$. In addition to solving the integral equation (4.1.9). we will need some regularity results, also given below, for the limit $Q_\kappa(y,t,z,s)$. These regularity results are a part of the effort in showing that the solution $\psi_{2,\kappa}(y,t)$ has bounded first two derivatives with respect to y, and bounded derivative with respect to t.

Lemma 4.1.4. *Let* $\alpha \in (0,1)$ *and let* $\mathcal{G}_k^{\alpha,\frac{\alpha}{2}}$ *be the Green function spaces defined in Definition VII.1.1 in Garroni and Menaldi (1992).*

(i) $Q_\kappa \in \mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$. *Moreover,* Q_κ *is the unique solution in* $\mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$ *of the integral equation*

$$Q_\kappa(y,t,\xi,\vartheta) = Q_{\kappa,0}(y,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\kappa Q_{\kappa,0}(y,t,z,s)\, Q_\kappa(z,s,\xi,\vartheta)\, dzds. \tag{4.1.9}$$

(ii) *There exists a sequence* $\{k_n\}$ *of positive constants and a constant* C *such that*

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0$$

and such that for every $(y,t,\xi,\vartheta) \in \mathcal{D}$

$$|Q_{\kappa,n}(y,t,\xi,\vartheta)| \leq k_n (t-\vartheta)^{\frac{n-2}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right)$$

and such that

$$|Q_\kappa(y,t,\xi,\vartheta)| \leq C(t-\vartheta)^{-1} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right).$$

(iii) *For every* $(y,t,\xi,\vartheta)\in\bar{\mathcal{D}}_\kappa$

$$Q_\kappa\left(y,t,\xi,\vartheta\right)=Q_\kappa\left(y,t-\vartheta,\xi,0\right). \tag{4.1.10}$$

(iv) *There exists a constant* C *such that, for every* $(y,t,\xi,\vartheta)\in\bar{\mathcal{D}}_\kappa$*, every* $y',\xi'\in(0,\kappa)$*,and every* $t'\in(0,t,)$ *the following inequalities are both valid:*

$$\begin{aligned}\left|Q_\kappa\left(y,t,\xi,\vartheta\right)-Q_\kappa\left(y',t,\xi,\vartheta\right)\right|\le C\left|y-y'\right|^{\frac{1}{2}}\left(t-\vartheta\right)^{-\frac{5}{4}}&\\ \times\exp\left(-\frac{1}{4}c_0\frac{\left(y-\xi\right)^2}{t-\vartheta}\right),&\end{aligned} \tag{4.1.11}$$

and

$$\begin{aligned}\left|Q_\kappa\left(y,t,\xi,\vartheta\right)-Q_\kappa\left(y,t',\xi,\vartheta\right)\right|\le C\left|t-t'\right|^{\frac{1}{4}}\left(t'-\vartheta\right)^{-\frac{5}{4}}&\\ \times\exp\left(-\frac{1}{4}c_0\frac{\left(y-\xi\right)^2}{t-\vartheta}\right),&\end{aligned} \tag{4.1.12}$$

(v)

$$\left|\frac{\partial Q_\kappa\left(y,t,\xi,\vartheta\right)}{\partial\xi}\right|\le C\left(t-\vartheta\right)^{-\frac{3}{2}}\exp\left(-\frac{1}{8}c_0\frac{\left(y-\xi\right)^2}{t-\vartheta}\right).$$

Proof. For part (i): It follows from Lemma VII.1.3 in Garroni and Menaldi (1992), and the bounds given in Lemma 4.1.3, that $\frac{\partial G_{L,\kappa}(y,t,\xi,\vartheta)}{\partial x}\in\mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$, and hence $Q_{\kappa,0}\in\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$. Since $Q_{\kappa,0}\in\mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$ it follows from Proposition VIII.1.2 in Garroni and Menaldi (1992) that Q_κ is the unique solution, in the function space $\mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$, of the integral equation (4.1.9).

For (ii): It can be shown by induction, following the technique outlined in the proof of Lemma V.3.3 in Garroni and Menaldi (1992), that for some constants C and c

$$\left|Q_{\kappa,n}\left(y,t,\xi,\vartheta\right)\right|\le cC^n\frac{1}{\Gamma\left(\frac{1}{2}\left(n+1\right)\right)}\left(t-\vartheta\right)^{\frac{1}{2}n-1}\exp\left(-\frac{1}{4}\frac{\left(y-\xi\right)^2}{t-\vartheta}\right).$$

This yields the stated bounds, since

$$\lim_{n\to\infty}\frac{\Gamma\left(\frac{1}{2}\left(n+1\right)\right)}{\Gamma\left(\frac{1}{2}\left(n+2\right)\right)}=0.$$

,

For part (iii): We first note that it is obvious that

$$Q_{\kappa,0}\left(y,t,\xi,\vartheta\right)=Q_{\kappa,0}\left(y,t-\vartheta,\xi,0\right).$$

Assume that

$$Q_{\kappa,k}\left(y,t,\xi,\vartheta\right)=Q_{\kappa,k}\left(y,t-\vartheta,\xi,0\right),$$

for $k \in 0, 1, \ldots, n$. It then follows, using the substitution $\varrho = s - \vartheta$, that

$$\begin{aligned} Q_{p,n+1}(y,t,\xi,\vartheta) &= \int_{\vartheta}^{t}\int_{0}^{\kappa} Q_{\kappa,0}(y,t,z,s)\, Q_{p,n}(z,s,\xi,\vartheta)\, dzds \\ &= \int_{0}^{t-\vartheta}\int_{0}^{\kappa} Q_{\kappa,0}(y,t-\vartheta,z,\varrho)\, Q_{p,n}(z,\varrho,\xi,0)\, dzd\varrho, \end{aligned}$$

and hence

$$Q_{p,n}(y,t,\xi,\vartheta) = Q_{p,n}(y,t-\vartheta,\xi,0)$$

for any $n \in 0, 1, 2, \ldots,$. Since for any $\epsilon > 0$ we can pick an N such that for every $(y,t,\xi,\vartheta) \in \mathcal{D}$

$$\sum_{k=N}^{\infty} |Q_{p,n}(y,t,\xi,\vartheta)| < \epsilon$$

we conclude that the identity (4.1.10) holds for every $(y,t,\xi,\vartheta) \in \mathcal{D}$.

For part (iv): We observe that, if

$$t - t' \geq t' - \vartheta,$$

then the inequality (4.1.12) follows from the bounds given in part (ii). Assume instead that

$$t - t' < t' - \vartheta. \tag{4.1.13}$$

We conclude from the regularity of $G_{L_0,\kappa}(y,t,\xi,\vartheta)$ given in Lemma 4.1.3 and the auxiliary result Proposition 4.1.3 that the inequality (4.1.12) holds for $n = 0$. Let $n \in 1, 2, \ldots,$. It is obvious that

$$Q_{\kappa,n}(y,t,\xi,\vartheta) - Q_{\kappa,n}(y,t',\xi,\vartheta) = I_{1,n} + I_{2,n},$$

where

$$I_{1,n} = \int_{\vartheta}^{t'}\int_{0}^{\kappa} \left(Q_{\kappa,0}(y,t,z,s) - Q_{\kappa,0}(y,t',z,s)\right) Q_{\kappa,n-1}(z,s,\xi,\vartheta)\, dzds$$

and

$$I_{2,n} = \int_{t'}^{t}\int_{0}^{\kappa} Q_{\kappa,0}(y,t,z,s)\, Q_{\kappa,n-1}(z,s,\xi,\vartheta)\, dzds.$$

Let $\{k_n\}$ be the sequence from the bound given in part (ii). It follows from the regularity of $G_{L_0,\kappa}(y,t,\xi,\vartheta)$ and Proposition 3.0.2 that, for some constants K and C, not depending on n

$$\begin{aligned} |I_{1,n}| &\leq K k_n (t-t')^{\frac{1}{4}} \int_{\vartheta}^{t'}\int_{0}^{\kappa} (t'-s)^{-\frac{5}{4}} (s-\vartheta)^{\frac{1}{2}(n-1)n-1} \\ &\quad \times \exp\left(-\frac{1}{4}c_0\left[\frac{(y-z)^2}{t-s} + \frac{(z-\xi)^2}{s-\vartheta}\right]\right) dzds \\ &\leq C k_n (t-t')^{\frac{1}{4}} (t-\vartheta)^{\frac{1}{2}n-\frac{1}{4}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right). \end{aligned}$$

Similar calculations yield that $|I_{2,n}|$ satisfies an inequality of the form given in equation (4.1.12), and that

$$Q_{\kappa}(y,t,\xi,\vartheta) - Q_{\kappa}(y',t,\xi,\vartheta)$$

satisfies an inequality of the form given in equation (4.1.11).

For (v): The real problem here is to obtain an appropriate bound for the second function in the sequence, i.e. $\frac{\partial Q_{\kappa,1}(y,t,\xi,\vartheta)}{\partial \xi}$, which we do below. For $n > 1$ we can obtain appropriate estimates using induction and similar calculations as in part (ii) and in the proof of Lemma V.3.1 in Garroni and Menaldi (1992). To accomplish the needed bound for $\frac{\partial Q_{\kappa,1}(y,t,\xi,\vartheta)}{\partial \xi}$ the most important idea is to split the domain of integration into appropriate parts. This technique is used throughout the book Garroni and Menaldi (1992) and we will tacitly (and sometimes explicitly) make use of it to obtain other bounds later on. We note that

$$\int_{\vartheta}^{t} \left| \int_{0}^{\kappa} Q_{\kappa,0}(y,t,z,s) \frac{\partial Q_{\kappa,0}(z,s,\xi,\vartheta)}{\partial \xi} dz \right| ds \leq \sum_{j=1}^{4} I_j,$$

where

$$I_1 = \int_{\vartheta}^{\frac{t}{2}} \left| \int_{0}^{\kappa} \left(Q_{\kappa,0}(y,t,z,s) - Q_{\kappa,0}(y,t,\xi,s) \right) \frac{\partial Q_{\kappa,0}(z,s,\xi,\vartheta)}{\partial \xi} dz \right| ds,$$

$$I_2 = \int_{\vartheta}^{\frac{t}{2}} \left| Q_{\kappa,0}(y,t,\xi,s) - Q_{\kappa,0}\left(y,t,\xi,\frac{t}{2}\right) \right| \left| \int_{0}^{\kappa} \frac{\partial Q_{\kappa,0}(z,s,\xi,\vartheta)}{\partial \xi} dz \right| ds,$$

$$I_3 = \left| Q_{\kappa,0}\left(y,t,\xi,\frac{t}{2}\right) \right| \int_{\vartheta}^{\frac{t}{2}} \left| \int_{0}^{\kappa} \frac{\partial Q_{\kappa,0}(z,s,\xi,\vartheta)}{\partial \xi} dz \right| ds$$

and

$$I_4 = \int_{\frac{t}{2}}^{t} \left| \int_{0}^{\kappa} Q_{\kappa,0}(y,t,\xi,s) \frac{\partial Q_{\kappa,0}(z,s,\xi,\vartheta)}{\partial \xi} dz \right| ds.$$

Because of the local Hölder-continuity of $Q_{\kappa,0}(z,s,\xi,\vartheta)$, an application of Proposition 3.0.1 and Proposition 3.0.2 yields that, for some constants C and K,

$$\begin{aligned} I_1 &\leq \int_{\vartheta}^{t} \int_{0}^{\kappa} K (t-s)^{-\frac{5}{4}} (s-\vartheta)^{-\frac{5}{4}} \exp\left(-\frac{1}{8} c_0 \left[\frac{(y-z)^2}{s-\vartheta} \right] \right) dz ds \\ &\leq C (t-\vartheta)^{-1} \exp\left(-\frac{1}{8} c_0 \frac{(y-\xi)^2}{t-\vartheta} \right). \end{aligned}$$

For I_2 and I_3 we recall that $\frac{\partial Q_{\kappa,0}(z,s,\xi,\vartheta)}{\partial \xi} = p \frac{\partial^2 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial \xi \partial \xi}$ and apply Proposition 3.0.5 to obtain that these terms also obey a bound of the form stated in part (v). For I_4 there are no strong singularities and the stated bound can be obtained from a straightforward calculation. We conclude from the above that the differential operator can be taken inside the integral (the order of differentiation and integration can be interchanged) and that for some constant C

$$\frac{\partial Q_{\kappa,1}(y,t,\xi,\vartheta)}{\partial \xi} \leq C (t-\vartheta)^{-1} \exp\left(-\frac{1}{8} c_0 \frac{(y-\xi)^2}{t-\vartheta} \right).$$

For $n > 1$ a similar induction as in the proof of Lemma V.3.1 in Garroni and Menaldi (1992) yields that there exists a sequence of constants $\{k_n\}$ such that $\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0$ and such that

$$\left| \frac{\partial Q_{\kappa,n}(y,t,\xi,\vartheta)}{\partial \xi} \right| \leq k_n (t-\vartheta)^{\frac{n-3}{2}}.$$

Because of this property we then conclude that the sequence

$$S_n(y,t,\xi,\vartheta) = \sum_{j=0}^{n} \frac{\partial Q_{\kappa,0}(y,t,\xi,\vartheta)}{\partial \xi}$$

converges uniformly on $\mathcal{D}$, which justifies differentiating the sequence term by term. □

Definition 4.1.5. *Let*

$$G_{L_1,\kappa}(y,t,\xi,\vartheta) = \int_{\vartheta}^{t}\int_{0}^{\kappa} G_{L_0,\kappa}(y,t,z,s)\, Q_\kappa(z,s,\xi,\vartheta)\, dzds.$$

Lemma 4.1.5. *There exists a constant C such that following identities and bounds are valid for every $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ and every $l \in \{0,1,2\}$:*

(i)

$$G_{L_1,\kappa}(y,t,\xi,\vartheta) = G_{L_1,\kappa}(y,t-\vartheta,\xi,0).$$

(ii)

$$\frac{\partial^l G_{L_1,\kappa}(y,t,\xi,\vartheta)}{\partial y^l} = \int_{\vartheta}^{t}\int_{0}^{\kappa} \frac{\partial^l G_{L_0,\kappa}(y,t,z,s)}{\partial x^l} Q_\kappa(z,s,\xi,\vartheta)\, dzds, \tag{4.1.14}$$

$$\frac{\partial G_{L_1,\kappa}(y,t,\xi,\vartheta)}{\partial \xi} = \int_{\vartheta}^{t}\int_{0}^{\kappa} G_{L_0,\kappa}(y,t,z,s)\, Q_\kappa(z,s,\xi,\vartheta)\, dzds,$$

$$\begin{aligned}\frac{\partial G_{L_1,\kappa}(y,t,\xi,\vartheta)}{\partial t} &= Q_\kappa(y,t,\xi,\vartheta) \\ &\quad + \int_{\vartheta}^{t}\int_{0}^{\kappa} \frac{\partial G_{L_0,\kappa}(y,t,z,s)}{\partial t} Q_\kappa(z,s,\xi,\vartheta)\, dzds,\end{aligned} \tag{4.1.15}$$

$$\begin{aligned}\left|\frac{\partial^l G_{L_1,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}\right| &\le C(t-\vartheta)^{-\frac{l}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right), \\ \left|G_{L_1,\kappa}(y,t,\xi,\vartheta)\frac{\partial}{\partial t}\right| &\le C(t-\vartheta)^{-1} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right), \\ \left|\frac{\partial G_{L_1,\kappa}(y,t,\xi,\vartheta)}{\partial \xi}\right| &\le C(t-\vartheta)^{-\frac{1}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),\end{aligned} \tag{4.1.16}$$

and

$$\left|\int_{-\infty}^{\infty} G_{L_1,\kappa}(y,t,\xi,\vartheta)\, d(y-\xi)\right| \le C\sqrt{t}.$$

(iii)

$$G_{L,\kappa}(y,t,\xi,\vartheta) = G_{L_0,\kappa}(y,t,\xi,\vartheta) + G_{L_1,\kappa}(y,t,\xi,\vartheta).$$

Proof. For (i): This follows from making the substitution $\varrho = t - s$.

For (ii): As in the proof of Lemma 4.1.4 these results can be proven by splitting the domains of integration into appropriate parts. To obtain the identity (4.1.15) we will consider the functions

$$I_1(y,t,\xi,s,\vartheta) = \int_0^\kappa G_{L_0,\kappa}(y,t,z,s)\left(Q_\kappa(z,s,\xi,\vartheta) - Q_\kappa(y,s,\xi,\vartheta)\right)dz,$$

$$I_2(y,t,\xi,s,\vartheta) = \left(Q_\kappa(y,s,\xi,\vartheta) - Q_\kappa(y,t,\xi,\vartheta)\right)\int_0^\kappa G_{L_0,\kappa}(y,t,z,s)\,dz,$$

$$I_3(y,t,\xi,s,\vartheta) = Q_\kappa(y,t,\xi,\vartheta)\int_0^\kappa \Gamma_{\sigma_P}(y,t,z,s)\,dz,$$

and

$$I_4(y,t,\xi,s,\vartheta) = -Q_\kappa(y,t,\xi,\vartheta)\int_0^\kappa g_{L_0,\kappa}(y,t,\xi,s)\,dz.$$

Because of the way $G_{L_0,\kappa}(y,t,z,s)$ was constructed it is obvious that

$$\int_0^\kappa G_{L_0,\kappa}(y,t,z,s)\,Q_\kappa(z,s,\xi,\vartheta)\,dz = \sum_{j=1}^{4} I_j(y,t,\xi,s,\vartheta).$$

Because of the local Hölder continuity of $Q_\kappa(y,t,\xi,\vartheta)$, the bounds obeyed by $g_{L_0,\kappa}(y,t,\xi,\vartheta)$ and application of Proposition 3.0.2, we see that, for some constant C

$$I_1(y,t,\xi,\vartheta) \le C(t-\vartheta)^{-\frac{1}{2}}(t-s)^{\frac{1}{4}}(s-\vartheta)^{-\frac{3}{4}}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

$$I_2(y,t,\xi,\vartheta) \le C(t-\vartheta)^{-\frac{1}{2}}(t-s)^{\frac{1}{4}}(s-\vartheta)^{-\frac{5}{4}}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

and

$$\begin{aligned} I_4(y,t,\xi,\vartheta) \le C(t-\vartheta)^{-1}(t-s)^{\frac{1}{4}}&\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right)\\ &\times\Bigg(\exp\left(-\frac{1}{4}c_0\frac{(y-\kappa)^2}{t-\vartheta}\right)\\ &\quad+\exp\left(-\frac{1}{4}c_0\frac{y^2}{t-\vartheta}\right)\Bigg). \end{aligned}$$

From these identities it is clear that, for any fixed $y,\xi\in(0,\xi)$

$$\lim_{s\to t}\left[I_1(y,t,\xi,s,\vartheta) + I_2(y,t,\xi,s,\vartheta) + I_4(y,t,\xi,s,\vartheta)\right] = 0.$$

For the last term $I_3(y,t,\xi,\vartheta)$ we get from the substitution $w = \sqrt{2c_0}\frac{y-z}{\sqrt{t-s}}$ that

$$\begin{aligned}\lim_{s\to t} I_3(y,t,\xi,s,\vartheta) &= Q_\kappa(y,t,\xi,\vartheta)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}w^2\right)dw\\ &= Q_\kappa(y,t,\xi,\vartheta).\end{aligned}$$

Similar calculations as above and as in the proof of Lemma 4.1.4 yield that, for some constant C, the following inequalities are all valid for $(y,t,\xi,\vartheta)\in\mathcal{D}_\kappa$ and $l\in\{0,1,2\}$:

$$\begin{aligned}\int_\vartheta^t\left|\int_0^\kappa \frac{\partial^l G_{L_0,\kappa}(y,t,z,s)}{\partial y^l} Q_\kappa(z,s,\xi,\vartheta)\,dz\right|ds&\\ \leq C(t-\vartheta)^{-\frac{l}{2}}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right)&,\\ \int_\vartheta^t\left|\int_0^\kappa \frac{\partial G_{L_0,\kappa}(y,t,z,s)}{\partial \xi} Q_\kappa(z,s,\xi,\vartheta)\,dz\right|ds&\\ \leq C(t-\vartheta)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right)&,\end{aligned}\tag{4.1.17}$$

and

$$\begin{aligned}\int_\vartheta^t\left|\int_0^\kappa \frac{\partial G_{L_0,\kappa}(y,t,z,s)}{\partial t} Q_\kappa(z,s,\xi,\vartheta)\,dz\right|ds&\\ \leq C(t-\vartheta)^{-1}\exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right)&.\end{aligned}\tag{4.1.18}$$

The stated identity (4.1.15) follows from the discussion above and the bound (4.1.18). The other stated bounds follow from the bound (4.1.17).

For part (iii): Since $G_{L_0,\kappa}(y,t,\xi,\vartheta)$ is the Green function associated with the differential operator $\frac{\partial}{\partial t}-\frac{1}{2}\sigma_P^2\frac{\partial^2}{\partial y^2}$ and Dirichlet boundary conditions, and Q_κ is a solution of the integral equation (4.1.9), this follows from the bounds given in part (ii). □

We are now in position to get some regularity results for the solution $\psi_{2,\kappa}(y,t)$ of the PDE (3.0.9). The representation formula given in (3.0.5) depends on the jump measure F as well as the Green function. In this article we will assume that the measure F satisfies the bound (4.0.38), and that the regularity results we get for $\psi_{2,\kappa}(y,t)$ and $\psi_{3,\kappa}(y,t)$ will depend on the values of β for which this inequality is satisfied. In this article we will not discuss what happens if we let $\beta\to\infty$.

Lemma 4.1.6. (i) *Assume that $\sigma_R=r=0$. There exists a constant C_β, depending on the β from* (4.0.38)*, such that for $l\in\{0,1\}$*

$$\left|\frac{\partial\psi_{2,\kappa}(y,t)}{\partial y^l}\right|\leq C_\beta t^{\frac{2-l}{2}}(1+y)^{-\beta}.$$

(ii) *Let $H_{1,\kappa}(y,t)$ be as in section 3.*

$$\frac{\partial^2\psi_{2,\kappa}(y,t)}{\partial y^2}=\int_0^t\int_0^\kappa\frac{\partial^2 G_{L,\kappa}(y,t,\xi,\vartheta)}{\partial y^2}H_{1,\kappa}(\xi,\vartheta)\,d\xi d\vartheta$$

and

$$\frac{\partial\psi_{2,\kappa}(y,t)}{\partial t}=H_{1,\kappa}(y,t)+\int_0^t\int_0^\kappa\frac{\partial G_{L,\kappa}(y,t,\xi,\vartheta)}{\partial t}H_{1,\kappa}(\xi,\vartheta)\,d\xi d\vartheta.$$

(iii) *There exists a constant C_β, depending on the β such that the following inequalities are all valid:*

$$\left|\int_0^t \int_0^\kappa \frac{\partial^2 \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial y^2} H_{1,\kappa}(\xi,\vartheta)\, d\xi d\vartheta\right| \le C_\beta (1+y)^{-\beta},$$
$$|H_{1,\kappa}(y,t)| + \left|\int_0^t \int_0^\kappa \frac{\partial \Gamma_{\sigma_P}(y,t,\xi,\vartheta)}{\partial t} H_{1,\kappa}(\xi,\vartheta)\, d\xi d\vartheta\right| \le C_\beta (1+y)^{-\beta},$$
$$\left|\int_0^t \int_0^\kappa \frac{\partial^2 G_{1,\kappa}(y,t,\xi,\vartheta)}{\partial y^2} H_{1,\kappa}(\xi,\vartheta)\, d\xi d\vartheta\right| \le C_\beta \sqrt{t}\,(1+y)^{-\beta}, \tag{4.1.19}$$

and

$$\left|\int_0^t \int_0^\kappa \frac{\partial G_{1,\kappa}(y,t,\xi,\vartheta)}{\partial t} H_{1,\kappa}(\xi,\vartheta)\, d\xi d\vartheta\right| \le C_\beta \sqrt{t}\,(1+y)^{-\beta}. \tag{4.1.20}$$

Proof. Because of the representation formula given in Definition 3.0.5 and the local Hölder continuity of the function $H_{1,\kappa}(\xi,\vartheta)$, these identities and bounds follow from similar calculations as in the proof of Lemma 4.1.5. □

It is a bit more technical to obtain appropriate estimates of $\frac{\partial^2 \psi^*_{2,\kappa}(y,t)}{\partial y^2}$ and $\frac{\partial \psi^*_{2,\kappa}(y,t)}{\partial t}$. In particular the proof of the next result involves a change in the order of integration.

Lemma 4.1.7. *Assume that $\sigma_R = r = 0$ and that the tail distribution of the jumps satisfies the bound (4.0.38) for some $\beta > 0$. Then there exists a constant C_β, depending on β, such that, for every $(y,t) \in (0,\kappa) \times (0,1]$, every $y' \in (y,\kappa)$, every $t' \in (0,t)$ and every $\alpha \in (0,1]$ the following inequalities are all valid:*

$$\left|\frac{\partial \psi_{2,\kappa}(y,t)}{\partial t}\right| \le C_\beta (1+y)^{-\beta}, \tag{4.1.21}$$

$$|\psi_{2,\kappa}(y,t) - \psi_{2,\kappa}(y',t)| \le C_\beta (y'-y)^\alpha\, t^{\frac{2-\alpha}{2}} (1+y)^{-\beta},$$

and

$$|\psi_{2,\kappa}(y,t) - \psi_{2,\kappa}(y,t')| \le C_\beta (t-t')^\alpha\, t^{1-\alpha} (1+y)^{-\beta}.$$

Proof. Thanks to Lemma 4.1.6 we only need to show that the integrals

$$\int_0^t \int_0^\kappa \frac{\partial g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial t} d\xi d\vartheta \tag{4.1.22}$$

and

$$\int_0^t \int_0^\kappa \frac{\partial^2 g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial y^2} H_{1,\kappa}(\xi,\vartheta)\, d\xi d\vartheta \tag{4.1.23}$$

satisfy the stated bounds. We will do that by first showing that the order of integration can be interchanged as explained below.

Let $U^{(1)}_{\xi,\kappa}(t)$ and $U^{(2)}_{\xi,\kappa}(t)$ be the limits defined in definition 4.1.3, let $U^{(1)}_{\kappa}(\xi,t) = U^{(1)}_{\xi,\kappa}(t)$ (i.e. $U^{(1)}_{\xi,\kappa}(t)$ considered as a function of ξ as well as t) and likewise let $U^{(2)}_{\kappa}(\xi,t) = U^{(2)}_{\xi,\kappa}(t)$. Let

$$B^{(1)}_{\kappa}(s) = \int_0^{\kappa} U^{(1)}_{\kappa}(\xi,s)d\xi, \quad s \in (0,1],$$

and let

$$B^{(2)}_{\kappa}(s) = \int_0^{\kappa} U^{(2)}_{\kappa}(\xi,s)d\xi, \quad s \in (0,1].$$

We note that

$$\int_0^t \int_0^{\kappa} \frac{\partial g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial t} d\xi d\vartheta = I_1 - I_2,$$

where

$$I_1 = \frac{1}{2}\sigma_P^2 \int_0^t \int_0^{\kappa} \int_0^{t-\vartheta} \left.\frac{\partial^2 \Gamma_{\sigma_P}(y,t-\vartheta,z,s)}{\partial t \partial z}\right|_{z=\kappa} U^{(1)}_{\kappa}(\xi,s) ds d\xi d\vartheta,$$

and

$$I_2 = \frac{1}{2}\sigma_P^2 \int_0^t \int_0^{\kappa} \int_0^{t-\vartheta} \left.\frac{\partial^2 \Gamma_{\sigma_P}(y,t-\vartheta,z,s)}{\partial t \partial z}\right|_{z=0} U^{(2)}_{\kappa}(\xi,s) ds d\xi d\vartheta.$$

We observe that the function $\Gamma_{\sigma_P}(y,t,z,s)$ is independent of the variable ξ and that $B^{(1)}_{\kappa}(s)$ and $B^{(2)}_{\kappa}(s)$ do not depend on ϑ. Moreover, because of the bounds given in Proposition 4.1.3 and Lemma 4.1.2 we are, for fixed $(y,t) \in (0,\kappa)\times(0,1]$, free to interchange the order of integration, as in the calculation below.

$$\begin{aligned}
I_1 &= \frac{1}{2}\sigma_P^2 \int_0^t \int_0^{t-\vartheta} B^{(1)}_{\kappa}(s) \left.\frac{\partial^2 \Gamma_{\sigma_P}(y,t-\vartheta,z,s)}{\partial t \partial z}\right|_{z=\kappa} ds d\vartheta \\
&= -\frac{1}{2}\sigma_P^2 \int_0^t B^{(1)}_{\kappa}(s) \int_0^{t-s} \left.\frac{\partial^2 \Gamma_{\sigma_P}(y,t-s,z,\vartheta)}{\partial z \partial \vartheta}\right|_{z=\kappa} d\vartheta ds \\
&= \frac{1}{2}\sigma_P^2 \int_0^t B^{(1)}_{\kappa}(s) \left.\frac{\partial \Gamma_{\sigma_P}(y,t-s,z,0)}{\partial z}\right|_{z=\kappa} ds.
\end{aligned}$$

In the last step above we have also used the symmetry property between the second and fourth variables of the fundamental solution. A similar calculation yields that

$$I_2 = \frac{1}{2}\sigma_P^2 \int_0^t B^{(2)}_{\kappa}(s) \left.\frac{\partial \Gamma_{\sigma_P}(y,t-\vartheta,z,s)}{\partial z}\right|_{z=0} ds.$$

Since we also have that

$$\frac{1}{2}\sigma_P^2 \frac{\partial^2 \Gamma(y,t,\xi,\vartheta)}{\partial y^2}) = \frac{\partial \Gamma(y,t,\xi,\vartheta)}{\partial t},$$

we get the following identities:

$$\begin{aligned}
&\int_0^t \int_0^{\kappa} \frac{\partial g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial t} d\xi d\vartheta \\
&= \int_0^t B^{(1)}_{\kappa}(s) \left.\frac{\partial \Gamma_{\sigma_P}(y,t-s,z,0)}{\partial z}\right|_{z=\kappa} ds \\
&\quad - \int_0^t B^{(2)}_{\kappa}(s) \left.\frac{\partial \Gamma_{\sigma_P}(y,t-\vartheta,z,s)}{\partial z}\right|_{z=0} ds,
\end{aligned}$$

and

$$\int_0^t \int_0^\kappa \frac{\partial g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial t} d\xi d\vartheta$$
$$= \frac{1}{2}\sigma_P^2 \int_0^t B_\kappa^{(1)}(s) \frac{\partial \Gamma_{\sigma_P}(y,t-s,z,0)}{\partial z}\bigg|_{z=\kappa} ds$$
$$- \frac{1}{2}\sigma_P^2 \int_0^t B_\kappa^{(2)}(s) \frac{\partial \Gamma_{\sigma_P}(y,t-\vartheta,z,s)}{\partial z}\bigg|_{z=0} ds.$$

Because of the bounds given in Proposition 4.1.3 and Lemma 4.1.2 and the inequality (4.1.8) it is straightforward to calculate that $B_\kappa^{(1)}(t)$ and $B_\kappa^{(2)}(t)$ are both bounded functions and, hence, for some constant C

$$\left| \int_0^t \int_0^\kappa \frac{\partial g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial t} d\xi d\vartheta \right| \le C,$$

and

$$\left| \int_0^t \int_0^\kappa \frac{\partial^2 g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial y^2} d\xi d\vartheta \right| \le C.$$

Because of this boundedness and the bounds and Hölder continuity of $H_{1,\kappa}(\xi,\vartheta)$, similar calculations as in the proof of Lemma 4.1.5 yield that the stated bounds are valid for the integrals (4.1.22) and (4.1.23). □

Lemma 4.1.8. *Assume that $\sigma_R = r = 0$ and that the tail distribution satisfies the bound* (4.0.38)*. Then, for some constant C_β, depending on β, the bounds stated below all hold for every $0 < y < y' < \kappa$, every $0 \le t' < t \le 1$ and every $\alpha \in (0,1]$:*

$$|H_{2,\kappa}(y,t)| \le C_\beta t (1+y)^{-\beta},$$
$$|H_{2,\kappa}(y,t) - H_{2,\kappa}(y',t)|^\alpha \le C_\beta |y-y'|^\alpha t^{\frac{2-\alpha}{2}} (1+y)^{-\beta} \tag{4.1.24}$$

and

$$|H_{2,\kappa}(y,t) - H_{2,\kappa}(y,t')|^\alpha \le C_\beta (t-t')^\alpha t^{1-\alpha} (1+y)^{-\beta}. \tag{4.1.25}$$

Proof. Let

$$\tilde{\psi}_{2,\kappa}(y,t) := \begin{cases} \psi_{2,\kappa}(y,t), & (y,t) \in [0,\kappa] \times (0,1], \\ 0, & (y,t) \in (-\infty,0) \times (0,1]. \end{cases}$$

We note that for every $t \in (0,1]$

$$\lim_{y \downarrow 0} \psi_{2,\kappa}(0,t) = 0,$$

that $\tilde{\psi}_{2,\kappa}(y,t)$ is continuous on $(-\infty,\kappa) \times (0,1]$, and that

$$\int_0^y \psi_{2,\kappa}(y-z,t) dF(z) = \int_0^\infty \tilde{\psi}_{2,\kappa}(y-z,t) dF(z).$$

A similar calculation as in the proof of Lemma 3.0.2, using the identity above, the bounds given in Lemma 4.1.6 and Lemma 4.1.7, as well as the auxiliary results Proposition 3.0.4 and Proposition 3.0.3, yields that all the stated bounds hold. □

4.1.2 Global estimates for a subproblem with an integral term and constant coefficients

In the remaining part of this section we will obtain regularity estimates of the PIDE (3.0.10) that are independent of the constant κ, still assuming that $\sigma_R = r = 0$. Analogous to the previous section we will do that by working with the Green function

$$G_{A,\kappa}\left(y,t,\xi,\vartheta\right)$$

defined in Definition 3.0.6. The main idea is to construct this Green function from the Green function $G_{L,\kappa}\left(y,t,\xi,\vartheta\right)$, using the parametrix method. The first step is to construct the Green function defined below. It is known to exist and be unique because of Theorem VI.1.10 in Garroni and Menaldi (1992).

Definition 4.1.6. *Let L_λ be the differential operator*

$$L_\lambda = L - \lambda,$$

and let $G_{L_\lambda,\kappa}\left(y,t,\xi,\vartheta\right)$ be the Green function associated with L_λ.

We will do this by first looking for a function $Q_{\lambda,\kappa}$ that solves the integral equation given in the next lemma. Also, because of the next lemma, the sequence of functions defined below is well defined.

Definition 4.1.7. *Let*

$$Q_{\lambda,\kappa,0}\left(y,t,\xi,\vartheta\right) = -\lambda G_{L,\kappa}\left(y,t,\xi,\vartheta\right),$$

and let the sequence of functions $\{Q_{\lambda,\kappa,n}\}_{n=0}^{\infty}$ be defined recursively for $n \in 1,2,\ldots,$ and $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$, by

$$Q_{\lambda,\kappa,n+1}\left(y,t,\xi,\vartheta\right) = \int_\vartheta^t \int_0^\kappa Q_{\lambda,\kappa,0}\left(y,t,z,s\right) Q_{\lambda,\kappa,n}\left(z,s,\xi,\vartheta\right) dzds,$$

and let

$$Q_{\lambda,\kappa}\left(y,t,\xi,\vartheta\right) = \sum_{n=0}^{\infty} Q_{\lambda,\kappa,n}\left(y,t,\xi,\vartheta\right).$$

Lemma 4.1.9. *Assume that $\sigma_R = r = 0$ and let $\alpha \in (0,1)$ and $\mathcal{G}_k^{\alpha,\frac{\alpha}{2}}$ be the Green function spaces defined in Definition VII.1.1 in Garroni and Menaldi (1992).*

(i) $Q_{\lambda,\kappa,0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ *and* $Q_{\lambda,\kappa} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$. *Moreover $Q_{\lambda,\kappa}$ is the unique solution in $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ of the integral equation*

$$\begin{aligned} Q_{\lambda,\kappa}\left(y,t,z,\vartheta\right) = &-\lambda G_{L,\kappa}\left(y,t,\xi,\vartheta\right) \\ &-\lambda \int_\vartheta^t \int_0^\kappa G_{L,\kappa}\left(y,t,z,s\right) Q_{\lambda,\kappa}\left(z,s,\xi,\vartheta\right) dzds. \end{aligned} \tag{4.1.26}$$

(ii) $Q_{\lambda,\kappa}\left(y,t,\xi,\vartheta\right)$ *is differentiable with respect to all four variables on $\mathcal{D}_\kappa$. Furthermore, there exists a constant C such that for every $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ the following identity and inequalities are all valid for $l \in \{0,1\}$:*

$$Q_{\lambda,\kappa}\left(y,t,\xi,\vartheta\right) = Q_{\lambda,\kappa}\left(y,t-\vartheta,\xi,0\right),$$

$$\left|\frac{\partial^l Q_{\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}\right| \leq C(t-\vartheta)^{-\frac{1+l}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial Q_{\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial \xi}\right| \leq C(t-\vartheta)^{-1} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial Q_{\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial t}\right| \leq C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right).$$

Proof. For part (i): It follows from Lemma VII.1.3 in Garroni and Menaldi (1992) and the bounds given in Proposition 4.1.3, Lemma 4.1.3 and Lemma 4.1.5, that $G_{L,\kappa}(y,t,\xi,\vartheta) \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ and hence $Q_{\lambda,\kappa,0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$. Since $Q_{\lambda,\kappa,0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ it follows from Proposition VIII.1.2 in Garroni and Menaldi (1992) that $Q_{\lambda,\kappa}$ is the unique solution in the function space $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$, of the integral equation (4.1.26).

For part (ii): This can be shown using the same calculations and reasoning, based on induction and uniform convergence, as in the proof of Lemma 4.1.4 and the proof of Lemma V.3.1 in Garroni and Menaldi (1992). □

Lemma 4.1.10. *Assume that*

$$\sigma_R = r = 0.$$

(i) *For every* $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ *and* $l \in \{0,1,2\}$

$$\int_\vartheta^t \int_0^\kappa G_{L,\kappa}(y,t,z,s)\, Q_{\lambda,\kappa}(z,s,\xi,\vartheta)\, dzds$$
$$= \int_0^{t-\vartheta} \int_0^\kappa G_{L,\kappa}(y,t-\vartheta,z,s)\, Q_{\lambda,\kappa}(z,s,\xi,0)\, dzds,$$

$$\frac{\partial^l}{\partial y^l}\int_\vartheta^t \int_0^\kappa G_{L,\kappa}(y,t,z,s)\, Q_{\lambda,\kappa}(z,s,\xi,\vartheta)\, dzds$$
$$= \int_\vartheta^t \int_0^\kappa \frac{\partial^l G_{L,\kappa}(y,t,z,s)}{\partial y^l} Q_{\lambda,\kappa}(z,s,\xi,\vartheta)\, dzds,$$

$$\frac{\partial}{\partial t}\int_\vartheta^t \int_0^\kappa G_{L,\kappa}(y,t,z,s)\, Q_{\lambda,\kappa}(z,s,\xi,\vartheta)\, dzds$$
$$= Q_{\lambda,\kappa}(y,t,\xi,\vartheta)$$
$$+ \int_\vartheta^t \int_0^\kappa \frac{\partial G_{L,\kappa}(y,t,z,s)}{\partial t} Q_{\lambda,\kappa}(z,s,\xi,\vartheta)\, dzds.$$

Furthermore, for some constant C

$$\left|\frac{\partial^l}{\partial y^l}\int_\vartheta^t \int_0^\kappa G_{L,\kappa}(y,t,z,s)\, Q_{\lambda,\kappa}(z,s,\xi,\vartheta)\, dzds\right|$$
$$\leq C(t-\vartheta)^{\frac{1-l}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial}{\partial t}\int_{\vartheta}^{t}\int_{0}^{\kappa}G_{L,\kappa}\left(y,t,z,s\right)Q_{\lambda,\kappa}\left(z,s,\xi,\vartheta\right)dzds\right|$$
$$\leq C\left(t-\vartheta\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}c_0\frac{\left(y-\xi\right)^2}{t-\vartheta}\right).$$

(ii) *For every* $(y,t,\xi,\vartheta)\in\mathcal{D}_{\kappa}$

$$G_{L_{\lambda},\kappa}\left(y,t,\xi,\vartheta\right)=G_{L,\kappa}\left(y,t,\xi,\vartheta\right)$$
$$+\int_{\vartheta}^{t}\int_{0}^{\kappa}G_{L,\kappa}\left(y,t,z,s\right)Q_{\lambda,\kappa}\left(z,s,\xi,\vartheta\right)dzds.$$

Proof. For (i): These identities and bounds can be derived from similar calculations as in the proof of Lemma 4.1.5.

For (ii): Since $G_{L,\kappa}\left(y,t,\xi,\vartheta\right)$ is the Green function associated with the differential operator L and Dirichlet boundary conditions, and $Q_{\lambda,\kappa}\left(y,t,\xi,\vartheta\right)$ satisfies the integral equation (4.1.26), this can be derived from the identities and bounds given in part (i). It follows from the way the Green function was constructed that it satisfies the boundary conditions. □

After the next result we will begin the process of constructing the Green function associated with the entire operator A.

Definition 4.1.8. *Let*

$$\psi_{3,a,\kappa}(y,t):=\int_{0}^{t}\int_{0}^{\kappa}G_{L_{\lambda},\kappa}\left(y,t,\xi,\vartheta\right)H_{2,\kappa}\left(\xi,\vartheta\right)d\xi d\vartheta.$$

Lemma 4.1.11. *Assume that* $\sigma_R=r=0$. *There exists a constant* C_β, *depending on* β, *such that, for every* $(y,t)\in(0,\kappa)\times(0,1]$ *and every* $l\in\{0,1,2\}$ *the following inequalities are all valid:*

$$\left|\frac{\partial^l\psi_{3,a,\kappa}(y,t)}{\partial y^l}\right|\leq C_\beta t^{\frac{4-l}{2}}\left(1+y\right)^{-\beta}$$

and

$$\left|\frac{\partial\psi_{3,a,\kappa}(y,t)}{\partial t}\right|\leq C_\beta t\left(1+y\right)^{-\beta}.$$

Proof. This follows from the inequalities given in Lemma 4.1.9, Lemma 4.1.10 and Lemma 4.1.8 by making similar calculations as in the proofs of Lemma 4.1.4 and Lemma 4.1.5.

□

Definition 4.1.9. *Let*

$$Q_{I,\kappa,0}\left(y,t,\xi,\vartheta\right):=\lambda\int_{0}^{y}G_{L_{\lambda},\kappa}\left(y-\zeta,t,\xi,\vartheta\right)dF\left(\zeta\right).$$

Let the sequence of functions

$$\left\{Q_{I,\kappa,n}\right\}_{n=0}^{\infty}$$

be defined inductively by

$$Q_{I,\kappa,n}\left(y,t,\xi,\vartheta\right) = \int_{\vartheta}^{t}\int_{0}^{\kappa} Q_{I,\kappa,0}\left(y,t,z,s\right) Q_{I,\kappa,n-1}\left(z,s,\xi,\vartheta\right) dzds,$$
$$n \in 1, 2, \ldots,$$

let

$$Q_{I,\kappa}\left(y,t,\xi,\vartheta\right) = \sum_{n=0}^{\infty} Q_{I,\kappa,n}\left(y,t,\xi,\vartheta\right),$$

let

$$G_{I_\lambda,\kappa,n}\left(y,t,\xi,\vartheta\right) := \int_{\vartheta}^{t}\int_{0}^{\kappa} G_{L_\lambda,\kappa}\left(y,t,z,s\right) Q_{I,\kappa,n}\left(z,s,\xi,\vartheta\right) dzds,$$

and let

$$G_{I_\lambda,\kappa}\left(y,t,\xi,\vartheta\right) := \int_{\vartheta}^{t}\int_{0}^{\kappa} G_{L_\lambda,\kappa}\left(y,t,z,s\right) Q_{I,\kappa}\left(z,s,\xi,\vartheta\right) dzds.$$

Lemma 4.1.12. *Assume that the tail distribution of the claims satisfies the inequality* (4.0.38) *and let* $\mathcal{G}_k^{\alpha,\frac{\alpha}{2}}$ *be the Green function space defined in Definition VII.1.1 in Garroni and Menaldi (1992).*

(i) *For every* $\alpha \in (0,1)$
$$Q_{I,\kappa,0}\left(y,t,\xi,\vartheta\right) \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}.$$

(ii) *For every* $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$
$$\begin{aligned} Q_{I,\kappa}\left(y,t,\xi,\vartheta\right) = \lambda \int_{[0,y]} G_{L_\lambda,\kappa}\left(y-\zeta,t,\xi,\vartheta\right) dF\left(\zeta\right) \\ + \lambda \int_{[0,y]} G_{I_\lambda,\kappa}\left(y-\zeta,t,\xi,\vartheta\right) dF\left(\zeta\right). \end{aligned} \tag{4.1.27}$$

(iii) *There exists a sequence* $\{k_n\}_{n=0}^{\infty}$ *such that*
$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0,$$
and such that, for every finite $n \in 0, 1, \ldots,$, *and every* $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ *the following inequalities are valid:*
$$\begin{aligned} \left|Q_{I,\kappa,n}\left(y,t,\xi,\vartheta\right)\right| \le k_n \left(t-\vartheta\right)^{n-\frac{1}{2}} \\ \times \int_0^\infty \cdots \int_0^\infty \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^{n}\zeta_j\right)^2}{t-\vartheta}\right) \\ \times dF(\zeta_0)dF(\zeta_1),\ldots,dF(\zeta_n), \end{aligned} \tag{4.1.28}$$

$$
\begin{aligned}
&\left|Q_{I,\kappa,n}\left(y,t,\xi,\vartheta\right)-Q_{I,\kappa,n}\left(y',t,\xi,\vartheta\right)\right| \\
&\qquad \le Ck_n\left|y-y'\right|\left(t-\vartheta\right)^{n-1} \\
&\qquad\quad \times\left(\exp\left(-\frac{1}{4}c_0\frac{\left(y-\xi-\sum_{j=0}^{n}\zeta_j\right)^2}{t-\vartheta}\right)\right. \\
&\qquad\quad \left.+\exp\left(-\frac{1}{4}c_0\frac{\left(y'-\xi-\sum_{j=0}^{n}\zeta_j\right)^2}{t-\vartheta}\right)\right) \\
&\qquad\quad \times dF(\zeta_1),\ldots,dF(\zeta_n),
\end{aligned}
\tag{4.1.29}
$$

and

$$
\begin{aligned}
&\left|Q_{I,\kappa,n}\left(y,t,\xi,\vartheta\right)-Q_{I,\kappa,n}\left(y,t',\xi,\vartheta\right)\right| \\
&\le Ck_n\left|t-t'\right|^{\frac{1}{4}}\left(\tilde{t}-\vartheta\right)^{n-\frac{3}{4}} \\
&\quad \times\exp\left(-\frac{1}{4}c_0\frac{\left(y-\xi-\sum_{j=0}^{n}\zeta_j\right)^2}{t-\vartheta}\right) \\
&\quad \times dF(\zeta_1),\ldots,dF(\zeta_n).
\end{aligned}
\tag{4.1.30}
$$

(iv)

$$
\begin{aligned}
\left|Q_{I,\kappa}\left(y,t,\xi,\vartheta\right)\right| &\le C\left(t-\vartheta\right)^{-\frac{1}{2}}, \\
\left|Q_{I,\kappa}\left(y,t,\xi,\vartheta\right)-Q_{I,\kappa}\left(y',t,\xi,\vartheta\right)\right| &\le C\left|y-y'\right|\left(t-\vartheta\right)^{-1},
\end{aligned}
$$

and

$$
\left|Q_{I,\kappa}\left(y,t,\xi,\vartheta\right)-Q_{I,\kappa}\left(y,t',\xi,\vartheta\right)\right|\le C\left|t-t'\right|^{\frac{1}{4}}\left(t'-\vartheta\right)^{-\frac{3}{4}}.
$$

Proof. For part (i): Let $\alpha\in(0,1)$. We first observe that it follows from Lemma VII.1.3 in Garroni and Menaldi (1992) and the bounds given in Lemma 4.1.10 that $G_{L_\lambda,\kappa}\in\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$. Moreover, using similar arguments as in the proof of Theorem 3.0.4, it can be shown that all the requirements of Lemma VII.3.2 hold and hence

$$
-\lambda G_{L_\lambda,\kappa}\left(y,t,\xi,\vartheta\right)+Q_{I,\kappa,0}\left(y,t,\xi,\vartheta\right)\in\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}.
$$

Since $Q_{I,\kappa,0}\left(y,t,\xi,\vartheta\right)$ is the difference between two functions that are both in the space $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ it is trivial to show that $Q_{I,\kappa,0}\left(y,t,\xi,\vartheta\right)$ is also in $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$.

For (ii): It follows from Proposition VIII.1.2 in Garroni and Menaldi (1992) and part (i) that, for any $\alpha\in(0,1)$, $Q_{I,\kappa}\left(y,t,\xi,\vartheta\right)$ is a solution in the function space $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ of the integral equation

$$
\begin{aligned}
Q_{I,\kappa}\left(y,t,\xi,\vartheta\right) &= Q_{I,\kappa,0}\left(y,t,\xi,\vartheta\right) \\
&\quad +\int_\vartheta^t\int_0^\kappa Q_{I,\kappa,0}\left(y,t,z,s\right)Q_{I,\kappa}\left(z,s,\xi,\vartheta\right)dsdz.
\end{aligned}
$$

Since $Q_{I,\kappa}\left(y,t,\xi,\vartheta\right)\in\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ it follows from the Fubini Theorem that we are allowed to change the order of integration, yielding the identity (4.1.27).

For (iii): We first observe that since, for every $t > \vartheta$,

$$\lim_{y \downarrow 0} G_{L_\lambda,\kappa}(y,t,\xi,\vartheta) = 0,$$

the stated bound (4.1.28) holds for $n = 0$. A similar induction as in the proof of Lemma 4.1.4 and Lemma V.3.1 in Garroni and Menaldi (1992) yields that (4.1.28) holds for any finite n. The most important difference is that this time we need to also invoke Fubini's theorem in order to change the order of integration.

Next, we observe that, if

$$t - t' \geq t' - \vartheta,$$

then the inequality (4.1.30) follows from the bounds given in part (iii). Assume instead that

$$t - t' > t' - \vartheta.$$

Because of the regularity of $G_{L_{\kappa,\kappa}}$ and Proposition 3.0.3, it is trivial that under this assumption the inequality (4.1.30) holds for $n = 0$. Let $n \in 1, 2, \ldots,$. It is obvious that

$$Q_{I,\kappa,n}(y,t,\xi,\vartheta) - Q_{I,\kappa,n}(y,t',\xi,\vartheta) = I_{1,n} + I_{2,n},$$

where

$$I_{1,n} = \int_\vartheta^{t'} \int_0^\kappa \left(Q_{I,\kappa,0}(y,t,z,s) - Q_{I,\kappa,0}(y,t',z,s)\right) Q_{I,\kappa,n-1}(z,s,\xi,\vartheta)\, dzds$$

and

$$I_{2,n} = \int_{t'}^{t} \int_0^\kappa Q_{I,\kappa,0}(y,t,z,s)\, Q_{I,\kappa,n-1}(z,s,\xi,\vartheta)\, dzds.$$

Let $\{k_n\}$ be the sequence from the bound (4.1.28). It follows from the regularity of $G_{L_{\kappa,\kappa}}$, Proposition 3.0.3, the bound (4.1.28), Fubini's theorem (to allow the changing of the order of integration) and Proposition 3.0.2, that, for some constants K and C, not depending on n,

$$\begin{aligned}
|I_{1,n}| \leq{}& K k_n (t-t')^{\frac{1}{4}} \int_0^\infty \ldots, \int_0^\infty \int_\vartheta^{t'} (t'-s)^{-\frac{3}{4}} (s-\vartheta)^{n-\frac{1}{2}} \\
&\times \int_0^\kappa \int_0^\infty \exp\left(-\frac{1}{4}c_0 \frac{(y-z-\zeta)^2}{t-s}\right) \\
&\times \exp\left(-\frac{1}{4}c_0 \frac{\left(z-\xi-\sum_{j=0}^{n-1}\zeta_j\right)^2}{s-\vartheta}\right) dzds \\
&\times dF(\zeta)\, dF(\zeta_0) \ldots, dF(\zeta_{n-1}) \\
\leq{}& C 2^n k_n |t-t'|^{\frac{1}{4}} (t'-\vartheta)^{n-\frac{3}{4}} \\
&\times \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^{n}\zeta_j\right)^2}{t-\vartheta}\right) \\
&\times dF(\zeta_0), \ldots, dF(\zeta_n).
\end{aligned}$$

Similar calculations yield that

$$|I_{2,n}| \le C 2^n k_n |t-t'|^{\frac{1}{4}} (t'-\vartheta)^{n-\frac{3}{4}}$$
$$\times \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^{n}\zeta_j\right)^2}{t-\vartheta}\right)$$
$$\times dF(\zeta_0),\ldots,dF(\zeta_n)$$

and that

$$Q_{I,\kappa,n}(y,t,\xi,\vartheta) - Q_{I,\kappa,n}(y',t,\xi,\vartheta)$$

satisfies an inequality of the form given in part (4.1.29).

For part (iv): Since F is a probability distribution this follows from the bounds given in part (iii). □

Lemma 4.1.13. *There exists a sequence $\{k_n\}_{n=0}^{\infty}$ such that*

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0,$$

and such that, for every finite $n \in 0,1,\ldots,$ every $l \in \{0,1,2\}$ and every $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ the following identities and inequalities are valid:

(i)

$$\frac{\partial^l G_{I_\lambda,\kappa,n}(y,t,\xi,\vartheta)}{\partial y^l} = \int_\vartheta^t \int_0^\kappa \frac{\partial^l G_{L_\lambda,\kappa}(y,t,z,s)}{\partial y^l} Q_{I,\kappa,n}(z,s,\xi,\vartheta)\,dzds,$$

and

$$\frac{\partial G_{I_\lambda,\kappa,n}(y,t,\xi,\vartheta)}{\partial t} = Q_{I,\kappa,n}(y,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\kappa \frac{\partial G_{L_\lambda,\kappa}(y,t,z,s)}{\partial t} Q_{I,\kappa,n}(z,s,\xi,\vartheta)\,dzds.$$

(ii)

$$\left|\frac{\partial^l G_{I_\lambda,\kappa,n}(y,t,\xi,\vartheta)}{\partial y^l}\right| \le k_n (t-\vartheta)^{n+\frac{1-l}{2}}$$
$$\times \int_0^\infty \cdots \int_0^\infty \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^{n}\zeta_j\right)^2}{t-\vartheta}\right)$$
$$\times dF(\zeta_1),\ldots,dF(\zeta_n),$$

$$\left|\frac{\partial G_{I_\lambda,\kappa,n}(y,t,\xi,\vartheta)}{\partial t}\right| \le k_n (t-\vartheta)^{n-\frac{1}{2}}$$
$$\times \int_0^\infty \cdots \int_0^\infty \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^{n}\zeta_j\right)^2}{t-\vartheta}\right)$$
$$\times dF(\zeta_1),\ldots,dF(\zeta_n),$$

and

$$
\begin{aligned}
\left|G_{I_\lambda,\kappa,n}(y,t,\xi,\vartheta)\right| &\le k_n \left(\min\left(y,\kappa-y\right)\right)^{\frac{1}{2}}(t-\vartheta)^{n+\frac{1}{4}} \\
&\times \int_0^\infty \cdots \int_0^\infty \left(\exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta} \right) \right. \\
&+ \exp\left(-\frac{1}{4}c_0 \frac{\left(\kappa-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta} \right) \\
&+ \left. \exp\left(-\frac{1}{4}c_0 \frac{\left(\xi+\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta} \right) \right) \\
&\times dF(\zeta_1),\ldots,dF(\zeta_n).
\end{aligned}
$$

(iii)

$$
\frac{\partial^l G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial y^l} = \int_\vartheta^t \int_0^\kappa \frac{\partial^l G_{L_\lambda,\kappa}(y,t,z,s)}{\partial y^l} Q_{I,\kappa}(z,s,\xi,\vartheta)\, dzds,
$$

and

$$
\begin{aligned}
\frac{\partial G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial t} &= Q_{I,\kappa}(y,t,\xi,\vartheta) \\
&+ \int_\vartheta^t \int_0^\kappa \frac{\partial G_{L_\lambda,\kappa}(y,t,z,s)}{\partial t} Q_{I,\kappa}(z,s,\xi,\vartheta)\, dzds,
\end{aligned}
$$

Furthermore, there exists a constant C such that for every (y,t,ξ,ϑ), and every $l \in \{0,1,2\}$

$$
\left|\frac{\partial^l G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}\right| \le C\,(t-\vartheta)^{\frac{1-l}{2}},
$$

and

$$
\left|\frac{\partial G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial t}\right| \le C\,(t-\vartheta)^{-\frac{1}{2}}.
$$

(iv) *For every $\xi \in (0,\kappa)$ and $0 \le \vartheta < t \le 1$*

$$
G_{A,\kappa}(y,t,\xi,\vartheta) = G_{L_\lambda,\kappa}(y,t,\xi,\vartheta) + G_{I_\lambda,\kappa}(y,t,\xi,\vartheta).
$$

Proof. For part (i): These identities follow from similar calculations as in the proof of Lemma 4.1.5, using the bounds given in Lemma 4.1.12.

For part (ii): It follows from the identities in part (i) that, for every finite n

$$
\frac{\partial^2 G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial y^2} = \sum_{j=1}^3 I_{j,n},
$$

where

$$I_{1,n} = \int_{\vartheta}^{t} \int_{0}^{\kappa} \frac{\partial^2 G_{L_\lambda,\kappa}(y,t,z,s)}{\partial y^2} \left(Q_{I,\kappa,n}(z,s,\xi,\vartheta) - Q_{I,\kappa,n}(y,s,\xi,\vartheta)\right) dzds,$$

$$I_{2,n} = \int_{\vartheta}^{t} \left(Q_{I,\kappa,n}(y,s,\xi,\vartheta) - Q_{I,\kappa,n}(y,t,\xi,\vartheta)\right) \int_{0}^{\kappa} \frac{\partial^l G_{L_\lambda,\kappa}(y,t,z,s)}{\partial y^l} dzds,$$

and

$$I_{3,n} = Q_{I,\kappa,n}(y,t,\xi,\vartheta) \int_{\vartheta}^{t} \int_{0}^{\kappa} \frac{\partial^l G_{L_\lambda,\kappa}(y,t,z,s)}{\partial y^l} dzds.$$

A calculation using the bounds given in Lemma 4.1.10, Lemma 4.1.12 and Proposition 3.0.2, and invoking the Fubini's theorem to change the order of integration, yields that, for some constant C, not depending on n

$$\begin{aligned}
|I_{1,n}| &\leq Ck_n (t-\vartheta)^{-\frac{1}{2}} \int_0^\infty \cdots \int_0^\infty \exp\left(-\frac{1}{4}c_0 \frac{\left(y - \sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right) \\
&\quad \times \int_{\vartheta}^{t} (t-s)^{-\frac{1}{2}} (s-\vartheta)^{n-\frac{1}{2}} dsdF(\zeta_1)\ldots dF(\zeta_n), \\
&\leq C\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} k_n (t-\vartheta)^{n-\frac{1}{2}} \\
&\quad \times \int_{\vartheta}^{t} (t-s)^{-\frac{1}{2}} \exp\left(-\frac{1}{4}c_0 \frac{\left(y - \sum_{j=0}^n \zeta_j\right)}{t-\vartheta}\right) \\
&\quad \times dF(\zeta_1)\ldots dF(\zeta_n),
\end{aligned}$$

where $\{k_n\}_{n=0}^{\infty}$ is a sequence of positive constants, such that

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0.$$

It follows from the above that the stated bound for

$$\left|\frac{\partial^2 G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial y^2}\right|$$

is valid for $|I_{1,n}|$. Similar calculations yield that bounds of the form given in the claim are also valid for $|I_{2,n}|$ and $|I_{3,n}|$, and thus the stated bound for

$$\left|\frac{\partial^2 G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial y^2}\right|$$

is valid. Other calculations along these lines also yield that the stated bounds for

$$\left|\frac{\partial^l G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}\right|, \quad l \in \{0,1\},$$

and

$$\left|\frac{\partial G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial t}\right|,$$

are also valid.

For part (iii): This follows from uniform convergence and similar considerations as in the proof of Lemma 4.1.5.

For part iv: Since $G_{L_\lambda,\kappa}(y,t,\xi,\vartheta)$ is the Green function associated with the differential operator $G_{I_\lambda,\kappa}(y,t,\xi,\vartheta)$, with Dirichlet boundary conditions, and because of the properties given in part (iii), the only property that remains to be shown is that, for every $\xi \in (0,\kappa)$ and $0 \leq \vartheta < t \leq 1$,

$$\lim_{y\to 0} G_{I_\lambda,\kappa}(y,t,\xi,\vartheta) = \lim_{y\to\kappa} G_{I_\lambda,\kappa}(y,t,\xi,\vartheta) = 0.$$

Since $G_{L_\lambda,\kappa}(y,t,\xi,\vartheta)$ is continuous and vanishes at $y=0$ and $y=\kappa$, a similar calculation as in the proof Proposition 3.0.4 yields that, for some constant C, the following bound is valid for every $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$:

$$|G_{L_\lambda,\kappa}(y,t,\xi,\vartheta)| \leq C \min\left(\sqrt{y}, \sqrt{\kappa - y}\right)(t-\vartheta)^{-\frac{3}{4}}.$$

Because of this inequality and the bound on $Q_{I,\kappa}(y,t,\xi,\vartheta)$, the Dominated Convergence Theorem can be applied to yield the inequality stated in part (iv). □

Theorem 4.1.2. *Assume that $\sigma_R = r = 0$ and that the bound (4.0.38) on the tail distribution function $\bar{F}$ holds. Then there exist constants C and C_β, where C_β depends on β, such that for every $n \in 0,1,\ldots,$ and every $l \in \{0,1,2\}$ the following inequalities are all valid:*

$$\begin{aligned} \left|\frac{\partial^l \psi_{3,\gamma}(y,t)}{\partial y^l}\right| &\leq C_\beta t^{\frac{4-l}{2}}(1+y)^{-\beta}, \\ \left|\frac{\partial \psi_{3,\gamma}(y,t)}{\partial t}\right| &\leq C_\beta t(1+y)^{-\beta}, \end{aligned} \tag{4.1.31}$$

$$|\psi_{3,\gamma}(y,t)| \leq C t^{\frac{7}{4}} \min(y, \gamma - y)^{\frac{1}{2}}, \tag{4.1.32}$$

$$\begin{aligned} \left|\frac{\partial^l \psi_{3,\gamma}(y,t)}{\partial y^l}\right| &\leq C_\beta t^{\frac{4-l}{2}}(1+y)^{-\beta}, \\ \left|\frac{\partial \psi_{3,\gamma}(y,t)}{\partial t}\right| &\leq C_\beta t(1+y)^{-\beta}, \end{aligned} \tag{4.1.33}$$

$$\begin{aligned} \left|\frac{\partial^l \psi_{\gamma}(y,t)}{\partial y^l}\right| &\leq C\left(t^{-\frac{l}{2}}\exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right) + t^{\frac{2-l}{2}} C_\beta (1+y)^{-\beta}\right) \quad \textit{and} \\ \left|\frac{\partial \psi_{\gamma}(y,t)}{\partial t}\right| &\leq C\left(t^{-1}\exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right) + C_\beta (1+y)^{-\beta}\right). \end{aligned} \tag{4.1.34}$$

Proof. For every $n \in 0,1,\ldots,$ and $(y,t) \in [0,\gamma]\times[0,1]$ let

$$\psi_{3,b,\gamma,n}(y,t) = \int_0^t \int_0^\gamma G_{I_\lambda,\gamma,n}(y,t,\xi,\vartheta)\, H_{2,\gamma}(\xi,\vartheta)\, d\xi d\vartheta$$

and let

$$\psi_{3,b,\gamma}(y,t) = \int_0^t \int_0^\gamma G_{I_\lambda,\gamma}(y,t,\xi,\vartheta)\, H_{2,\gamma}(\xi,\vartheta)\, d\xi d\vartheta.$$

It follows from Lemma 4.1.11 that the inequalities given in equation (4.1.31) are valid for $\psi_{3,a,\gamma}(y,t)$ (defined in Definition 4.1.8). Thus, to establish these bounds, what remains is to show that they also hold for $\psi_{3,b,\gamma}(y,t)$. Once this is done, it will follow from the already established regularity properties of $\psi_{1,\gamma}(y,t)$ and $\psi_{2,\gamma}(y,t)$ that the inequalities given in equation (4.1.34) are all valid. This can be done in 3 steps.

The first step is to use the bounds given in Lemma 4.1.13 to show that, for every $(y,t) \in (0,\gamma) \times (0,1]$, every finite $n \in 0,1,\ldots$, and every $l \in \{0,1,2\}$

$$\frac{\partial^l \psi_{3,b,\gamma,n}(y,t)}{\partial y^l} = \int_0^t \int_0^\gamma \frac{\partial^l G_{I_\lambda,\gamma,n}}{\partial y^l} H_{2,\gamma}(\xi,\vartheta)\, d\xi d\vartheta \tag{4.1.35}$$

and

$$\frac{\partial \psi_{3,b,\gamma,n}(y,t)}{\partial t} \int_0^t \int_0^\gamma \frac{\partial G_{I_\lambda,\gamma,n}}{\partial t} H_{2,\gamma}(\xi,\vartheta)\, d\xi d\vartheta. \tag{4.1.36}$$

The second step is to establish that there exists a sequence $\{k_n\}$ of positive numbers and a constant C_β, depending on β, such that

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0,$$

and such that, for every $(y,t) \in (0,\gamma) \times (0,1]$, every $n \in 0,1,\ldots$, and every $l \in \{0,1,2\}$

$$\left| \frac{\partial^l \psi_{3,b,\gamma,n}(y,t)}{\partial y^l} \right| \leq C_\beta t^{n+\frac{6-l}{2}} (1+y)^{-\beta}, \tag{4.1.37}$$

and

$$\left| \frac{\partial \psi_{3,b,\gamma,n}(y,t)}{\partial t} \right| \leq C_\beta t^{n+2} (1+y)^{-\beta}. \tag{4.1.38}$$

The last step is to establish that

$$\frac{\partial^l \psi_{3,b,\gamma}(y,t)}{\partial y^l} = \sum_{n=0}^{\infty} \frac{\partial^l \psi_{3,b,\gamma,n}(y,t)}{\partial y^l} \tag{4.1.39}$$

and that

$$\frac{\partial \psi_{3,b,\gamma}(y,t)}{\partial t} = \sum_{n=0}^{\infty} \frac{\partial \psi_{3,b,\gamma,n}(y,t)}{\partial t}. \tag{4.1.40}$$

It follows from the identity (4.1.36), the regularity bounds obeyed by $G_{I_\lambda,\gamma,n}$ stated in Lemma 4.1.13, and Fubini's theorem, that there exists a sequence $\{k_n\}_{n=0}^{\infty}$ and a constant C_β, depending on β, such that

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0.$$

Also

$$\left|\frac{\partial\psi_{3,\gamma,b,n}(y,t)}{\partial t}\right| \le k_n \int_0^\infty \cdots \int_0^\infty \int_0^t (t-\vartheta)^{n-\frac{1}{2}}$$
$$\times \int_0^\gamma \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right) H_{2,\gamma}(\xi,\vartheta)$$
$$\times\, d\xi d\vartheta dF(\zeta_0)\dots dF(\zeta_n)$$
$$\le I_{1,n} + I_{2,n} + I_{3,n},$$

where

$$I_{1,n} = C_\beta k_n (1+y)^{-\beta} \int_0^\infty \cdots \int_0^\infty \int_0^t (t-\vartheta)^{n-\frac{1}{2}} \vartheta$$
$$\times \int_{\frac{y}{2}}^\gamma \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right)$$
$$\times\, d\xi d\vartheta dF(\zeta_0)\dots dF(\zeta_n),$$

$$I_{2,n} = k_n \exp\left(-\frac{1}{128}c_0\frac{y^2}{t}\right)\int_{\left\{\zeta_0,\dots,\zeta_n\ge 0:\sum_{j=0}^n \zeta_j \le \frac{y}{4}\right\}}$$
$$\times \int_0^t (t-\vartheta)^{n-\frac{1}{2}}\vartheta$$
$$\times \int_0^{\frac{y}{2}} \exp\left(-\frac{1}{8}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right)$$
$$\times\, d\xi d\vartheta dF(\zeta_0)\dots dF(\zeta_n),$$

and

$$I_{3,n} = k_n t^{n+2}\int_{\left\{\zeta_0,\dots,\zeta_n\ge 0:\sum_{j=0}^n \zeta_j > \frac{y}{4}\right\}} dF(\zeta_0)\dots dF(\zeta_n).$$

From the above it is clear that the stated bounds holds for $I_{1,n}$ and $I_{2,n}$. Moreover, we observe that it follows from the assumption (4.0.38) on the tail distribution function $\bar{F}$ that, for every $\zeta_0,\zeta_1,\dots,\zeta_n \ge 0$,

$$\Pi_{j=0}^\infty \bar{F}(\zeta_0)\bar{F}(\zeta_1)\dots,\bar{F}(\zeta_n) \le C^n\left[(1+\zeta_0)(1+\zeta_1)\cdots\times(1+\zeta_n)\right]^{-\beta}$$
$$\le C^n\left(1+\sum_{j=0}^n \zeta_j\right)^{-\beta}.$$

Since $\lim_{n\to\infty}\frac{k_{n+1}}{k_n}=0$, it follows from this inequality that the stated bound (4.1.38) also holds for $I_{3,n}$. Similar calculations also yield that the bounds given in equation (4.1.37) and (4.1.32) also hold. Similar reasoning, also based on uniform convergence, as in the proof of Lemma 4.1.5, yields that the differentiation can be done term by term, as indicated in the identities (4.1.39) and (4.1.40). □

4.2 Unbounded coefficients

4.2.1 Global estimates for a subproblem with unbounded coefficients

In this section we will study the equation (3.0.9)) when $\sigma_R > 0$. We will assume that σ_R is positive, and not look into the case $\sigma_R = 0, r > 0$. In much the same way as we did in Section 4.1.1 and Section 4.1.2, we will do this by obtaining bounds for some Green functions, denoted $\hat{G}_{\hat{L},\kappa}$ and $\hat{G}_{\hat{A},\kappa}$, with the above assumption. Because the coefficients of L are not bounded on $(0,\infty)$ it is very hard to prove directly the existence of the fundamental solution associated with L. This is only one of the number of problems that arise when σ_R is positive. Instead of working with the original Green function we will work with something we call an *auxiliary* Green function.

The basic idea is to consider the function

$$\hat{\psi}_{2,\hat{\kappa}}(x,t) := \psi_{2,\kappa}(e^x - 1, t), \quad (x,t) \in [0, \ln(\kappa+1)].$$

From the definition above it is obvious that

$$\psi_{2,\kappa}(y,t) = \hat{\psi}_{2,\hat{\kappa}}(x,t) \quad (y,t) \in [0,\kappa].$$

and the chain rule yields the result below.

Definition 4.2.1. *Let*

$$\hat{a}_{1,1}(x) := \frac{1}{2}\left(\sigma_p^2 e^{-2x} + \sigma_R^2\left(1-e^{-x}\right)^2\right), \quad x \geq 0,$$

let

$$\hat{a}_1(x) := \left(pe^{-x} + r\left(1-e^{-x}\right)\right) - \hat{a}_{1,1}(x), \quad x \geq 0,$$

and let

$$\hat{L} := \left(\hat{a}_{1,1}(x)\frac{\partial^2}{\partial x^2} + \hat{a}_1(x)\frac{\partial}{\partial x}\right), \quad x \geq 0.$$

Lemma 4.2.1. *Let the function $H_{1,\kappa}$ be as in Section 3. $\hat{\psi}_{2,\hat{\kappa}}(x,t)$ is the unique solution of the PDE*

$$\begin{cases} \hat{\psi}_{2,\hat{\kappa}}(x,0) & = 0, \quad x \in (0, \ln(1+\kappa)), \\ \hat{\psi}_{2,\hat{\kappa}}(0,t) & = 0, \quad t \in [0,1], \\ \hat{\psi}_{2,\hat{\kappa}}(ln(1+\kappa), t) & = 0, \quad t \in [0,1], \\ \frac{\partial \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial t} - \hat{L}\hat{\psi}_{2,\hat{\kappa}}(x,t) & = H_{1,\kappa}(e^x - 1, t), \quad (x,t) \in (0,\ln(1+\kappa)) \times (0,1]. \end{cases} \tag{4.2.1}$$

Proof. Let

$$x = \ln(1+y).$$

From the definition and the chain rule it follows that

$$\frac{\partial \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial x} = e^x \frac{\partial \psi_{2,\kappa}(y,t)}{\partial y}\bigg|_{y=e^x-1},$$

and that

$$\frac{\partial^2 \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial x^2} = e^{2x}\left(\left.\frac{\partial \psi_{2,\kappa}(y,t)}{\partial y}\right|_{y=e^x-1} + \left.\frac{\partial^2 \psi_{2,\kappa}(y,t)}{\partial y^2}\right|_{y=e^x-1}\right).$$

The claim follows from the identities above, the maximum theorem (similar to the uniqueness of the PDE (3.0.9)) and $\psi_{2,\kappa}(y,t)$ being a solution of the PDE (3.0.9). □

Crucially the coefficients of the differential operator $\hat{L}$ are bounded on $(0,\infty)$. As we shall see this property enables us to obtain regularity estimates for $\hat{\psi}_{2,\hat{\kappa}}(x,t)$ similar to those we obtained for the PDE (3.0.9), where we assumed constant coefficients.

Starting with the representation formula below, much of what will follow will resemble the discussion in sections 4.1.1.

Definition 4.2.2. *Let*

$$\hat{\kappa} := \ln(1+\kappa),$$

and let

$$\begin{cases} \mathcal{D}_{\hat{\kappa}} = \{x,t,\xi,\vartheta : x,\xi \in (0,\hat{\kappa}), 0 \le \vartheta < t \le \}, \\ \partial\mathcal{D}_{\hat{\kappa}} = \{x,t,\xi,\vartheta : x,\xi \in \{0,\hat{\kappa}\}, 0 \le \vartheta < t \le 1\}, \\ \bar{\mathcal{D}}_{t,\hat{\kappa}} = \mathcal{D}_{\hat{\kappa}} \cup \partial\mathcal{D}_{\hat{\kappa}}. \end{cases}$$

Theorem 4.2.1. *There exists a unique Green function $\hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta)$ associated with the differential operator $\hat{L}$ and Dirichlet boundary conditions on the domain $\mathcal{D}_{\hat{\kappa}}$, i.e. satisfying the conditions in Definition 3.0.1 with L replaced by $\hat{L}$ and κ replaced by $\hat{\kappa}$. Furthermore, for every $(y,t) \in (0,\hat{\kappa}) \times (0,1]$*

$$\hat{\psi}_{2,\hat{\kappa}}(x,t) = \int_0^t \int_0^{\hat{\kappa}} \hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta)\, H_{1,\kappa}(e^{\xi}-1,\vartheta) d\xi d\vartheta.$$

Proof. This can be shown using arguments similar to those that lead to the result in Theorem 3.0.3. □

In the next section we will discuss the regularity of a function we will refer to as the *auxiliary* fundamental solution. Similar to what we did in Section 4.1.1 and Section 4.1.2, we will use this function to construct the Green function $\hat{G}_{\hat{L},\hat{\kappa}}$. After that a similar calculation as in Section 4.1.2 will yield estimates of the derivatives of $\hat{\psi}_{2,\hat{\kappa}}(x,t)$, which in turn can be used to obtain estimates of the derivatives of $\psi_{2,\kappa}(x,t)$. We will construct the Green function $\hat{G}_{\hat{L},\hat{\kappa}}$ by first constructing a fundamental solution and a Green function associated with, not the differential operator $\hat{L}$, but a "smaller" equation, that only includes the second order term. We then use the Green function associated with the second order term as building material for $\hat{G}_{\hat{L},\hat{\kappa}}$, similar to our construction of the Green function for the whole operator A from a simpler equation (assuming constant coefficients) in section 4.1.2. For technical reasons (we want to invoke Theorem V.1.3.5 and Theorem V.5.5) we will define a fundamental solution associated with an extension of the second order term $\hat{a}_{1,1}(x)\frac{\partial^2}{\partial x^2}$ to the whole line, that preserves the differentiability, uniform ellipticity and boundedness of the coefficients.

Definition 4.2.3. *Let*

$$\hat{a}_{1,1}^{*}(x) = \begin{cases} \hat{a}_{1,1}(x) & x \geq 0, \\ \frac{1}{2}\sigma_p^2 + \left[(1-e^x)\sigma_P^2 + \frac{1}{2}\sigma_P^2 x^2 e^x\right] + \frac{1}{2}\left(\sigma_R^2 + 2\sigma_P^2\right) x^2 e^x, & x < 0, \end{cases},$$

let

$$\hat{a}_{1}^{*}(x) = \begin{cases} \hat{a}_{1}(x) & x \geq 0, \\ \hat{a}_{1}(0) & x < 0, \end{cases},$$

let

$$\hat{L}_0 := \hat{a}_{1,1}^{*}(x)\frac{\partial^2}{\partial x}$$

and let

$$\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)$$

be the fundamental solution associated with the differential operator $\hat{L}_0$.

It can be calculated that the extended second order coefficient $\hat{a}_{1,1}^{*}$ and the first order coefficient $\hat{a}_1$ (restricted to $x > 0$) are smooth, uniformly elliptic and bounded. We state these properties in the next result without giving a proof.

Proposition 4.2.1. *The extended second order coefficient* $\hat{a}_{1,1}^{*}$ *and the restricted* $\hat{a}_1(x)$ *are bounded and two times continuously differentiable, on the real line and for positive* x*, respectively. Furthermore, for every* $x \geq 0$

$$\frac{1}{2}\left[\frac{\sigma_p^2\sigma_R^2}{\sigma_p^2+\sigma_R^2}\right] \leq \hat{a}_{1,1}(x) \leq \frac{1}{2}\max\left(\sigma_P^2, \sigma_R^2\right)$$

and, for some constant C*, the following inequalities are valid for* $x > 0$*:*

$$|\hat{a}_1(x)| \leq C$$
$$|\hat{a}_{1,1}{}'(x)| \leq C$$
$$|\hat{a}_1{}'(x)| \leq C$$
$$|\hat{a}_{1,1}{}''(x)| \leq C$$
$$|\hat{a}_1{}''(x)| \leq C.$$

Definition 4.2.4. *For* $(x,t,\xi,\vartheta) \in \mathcal{D}$ *let*

$$\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta) := \frac{1}{\sqrt{2\pi(t-\vartheta)\hat{a}_{1,1}^{*}(\xi)}}\exp\left(-\frac{(x-\xi)^2}{4\hat{a}_{1,1}^{*}(\xi)(t-\vartheta)}\right), \qquad (4.2.2)$$

and let

$$\hat{c}_0 := \frac{1}{2}\max\left(\sigma_P^2, \sigma_R^2\right).$$

Basic calculations yield that the function defined above has certain properties that we state in the next two results without giving a proof. Because of these basic properties it follows that $\hat{\Gamma}_{\hat{L}_0}$ is a fundamental solution associated with the extended second order term, as stated in Lemma 4.2.2 below. The main idea that be inferred from these results is that the principal term can be split into two terms, where the first term behaves very much like the principal term in the case of constant coefficients, while the second term has a weaker singularity than the first. We will use these properties primarily when, as part of the effort to construct the auxiliary Green function, we want do integration by parts similar to what we relied on in the proof of Lemma 4.1.3.

Proposition 4.2.2. *For every $(x,t,\xi,\vartheta)\in\mathcal{D}$*

$$
\begin{aligned}
\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta) &= \hat{\Gamma}_{\hat{L}_0}(x,t-\vartheta,\xi,0),\\
\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x} &= -\frac{x-\xi}{2\hat{a}^*_{1,1}(\xi)(t-\vartheta)}\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta),\\
\frac{\partial^2\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^2} &= \frac{\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{2\hat{a}^*_{1,1}(\xi)(t-\vartheta)}\left[-1+\frac{(x-\xi)^2}{2\hat{a}^*_{1,1}(\xi)(t-\vartheta)}\right],\\
\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial t} &= \hat{a}^*_{1,1}(\xi)\frac{\partial^2\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^2},\\
\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial \xi} &= -\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x}\\
&\quad+\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)\frac{\hat{a}^{*}_{1,1}{}'(\xi)}{\hat{a}^*_{1,1}(\xi)}\left[-\frac{1}{2}+\frac{1}{4\hat{a}^*_{1,1}(\xi)}\frac{(x-\xi)^2}{t-\vartheta}\right],\\
\frac{\partial^3\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^2\partial\xi} &= \frac{1}{\hat{a}^*_{1,1}(\xi)}\left[\frac{\hat{a}^{*}_{1,1}{}'(\xi)}{\hat{a}^*_{1,1}(\xi)}\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial\vartheta}-\frac{\partial^2\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial\xi\partial\vartheta}\right].
\end{aligned}
$$

Proposition 4.2.3. *There exists a positive constant C such that, for every $(x,t,\xi,\vartheta)\in\mathcal{D}\times\mathbb{R}$, the following inequalities hold:*

$$
\begin{aligned}
\left|\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)\right| &\le C(t-\vartheta)^{-\frac{1}{2}}\exp\left(-\hat{c}_0\frac{(x-\xi)^2}{(t-\vartheta)}\right),\\
\left|\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x}\right| &\le C|x-\xi|(t-\vartheta)^{-\frac{3}{2}}\exp\left(-\hat{c}_0\frac{(x-\xi)^2}{(t-\vartheta)}\right),\\
\left|\frac{\partial^2\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^2}\right| &\le C(t-\vartheta)^{-\frac{3}{2}}\exp\left(-\hat{c}_0\frac{(x-\xi)^2}{(t-\vartheta)}\right),\\
\left|\frac{\partial^2\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial t^2}\right| &\le C(t-\vartheta)^{-\frac{5}{2}}\exp\left(-\hat{c}_0\frac{(x-\xi)^2}{(t-\vartheta)}\right),
\end{aligned}
$$

$$
\left|\int_{-\infty}^{\infty}\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)\,d(x-\xi)-1\right|\le C\sqrt{t-\vartheta},
$$

$$
\begin{aligned}
\left|\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial\xi}+\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x}\right| &\le C(t-\vartheta)^{-\frac{1}{2}}\\
&\quad\times\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\xi)^2}{(t-\vartheta)}\right),\\
\left|\frac{\partial^2\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x\partial\xi}+\frac{\partial^2\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^2}\right| &\le C|x-\xi|(t-\vartheta)^{-\frac{3}{2}}\\
&\quad\times\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\xi)^2}{(t-\vartheta)}\right),
\end{aligned}
$$

and

$$\left|\frac{\partial^3\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial\xi\partial x^2} - \frac{1}{\hat{a}^*_{1,1}(x)}\left\{\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial\xi\partial\vartheta} + \frac{\hat{a}^*_1(x)}{\hat{a}^*_{1,1}(x)}\frac{\partial\hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial\vartheta}\right\}\right|$$
$$\leq C\,(t-\vartheta)^{-\frac{1}{2}}\left(1+\frac{|x-\xi|}{t-\vartheta}\right)\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\xi)^2}{(t-\vartheta)}\right).$$

Lemma 4.2.2. $\hat{\Gamma}_{\hat{L}_0}$ *is the unique (principal) Fundamental solution associated with the equation* $\hat{L}_0$*.*

Proof. This can be calculated using the identities and inequalities given in Proposition 4.2.2 and Proposition 4.2.3. ☐

Analogous to the construction of the Green function $G_{L,\hat{\kappa}}$ in Section 4.1.1 (where we assumed constant coefficients), we can construct the Green function associated with just the second order term by solving the PDE given in the next result below.

Lemma 4.2.3. *Let* $\hat{g}^*_{\hat{L}_0,\hat{\kappa}}(x,t,\xi)$ *be the unique classical solution of the equation*

$$\begin{cases} \hat{g}^*_{\hat{L}_0,\hat{\kappa}}(0,0,\xi) & = 0, \quad x\in[0,\hat{\kappa}]\,, \\ \hat{g}^*_{\hat{L}_0,\hat{\kappa}}(0,t,\xi) & = \hat{\Gamma}_{\hat{L}_0}(0,t,\xi,0)\,, \quad t\in(0,1], \\ \hat{g}^*_{\hat{L}_0,\hat{\kappa}}(\kappa,t,\xi,0) & = \hat{\Gamma}_{\hat{L}_0}(\kappa,t,\xi,0)\,, \quad t\in(0,1], \\ \frac{\partial\hat{g}^*_{\hat{L}_0,\hat{\kappa}}(x,t,\xi)}{\partial t} & = \hat{a}_{1,1}(x)\frac{\partial^2\hat{g}^*_{\hat{L}_0,\hat{\kappa}}}{\partial x^2}(x,t,\xi)\,, \quad (x,t)\in(0,\hat{\kappa})\times(0,1]. \end{cases} \tag{4.2.3}$$

Let

$$\hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta) := \hat{g}^*_{\hat{L}_0,\hat{\kappa}}(x,t-\vartheta,\xi), \quad (x,t,\vartheta,\xi)\in\bar{\mathcal{D}}_{\hat{L}_0,\hat{\kappa}}.$$

Assume in addition that for any smooth function $f(\xi,\vartheta)$ *with compact support, any* $(x,t)\in(0,\kappa)\times(0,1]$*, and* $l\in\{1,2\}$

$$\frac{\partial^l}{\partial x^l}\int_0^t\int_0^{\hat{\kappa}}\hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)\,f(\xi,\vartheta)\,d\xi = \int_0^t\int_0^{\hat{\kappa}}\frac{\partial^l\hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial x^l}f(\xi,\vartheta)\,d\xi,$$
$$\frac{\partial}{\partial t}\int_0^t\int_0^{\hat{\kappa}}\hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)\,f(\xi,\vartheta)\,d\xi = \int_0^t\int_0^{\hat{\kappa}}\frac{\partial\hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial t}f(\xi,\vartheta)\,d\xi, \tag{4.2.4}$$

and that for any smooth function $\phi(y)$ *with compact support*

$$\lim_{t-\vartheta\to 0}\int_0^{\hat{\kappa}}\hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)\,\phi(\xi)d\xi = 0. \tag{4.2.5}$$

Then

$$\hat{G}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta) = \hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta) - \hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)$$

is the unique Green function associated with the differential operator $\hat{L}_0$ *and Dirichlet boundary conditions.*

Proof. Because of the symmetry property between the variables t and ϑ this follows from reasoning similar to the proof of Lemma 4.1.1. ☐

It follows from the lemma following after the definitions below that the sequences and series in the definitions below are actually well defined.

Definition 4.2.5. *For*

$$g \in C\left([0,1],\mathbb{R}\right)$$

let

$$\hat{P}^{(1)}_{g,\hat{\gamma}}\left(x,t\right) := \int_0^t \hat{a}_{1,1}\left(\hat{\gamma}\right) \frac{\partial \hat{\Gamma}_{\hat{L}_0}\left(x,t,\eta,\vartheta\right)}{\partial \eta}\Bigg|_{\eta=\hat{\gamma}} g\left(\vartheta\right) d\vartheta, \quad y \geq 0, t \in [0,1]$$

and

$$\hat{P}^{(2)}_{g}\left(x,t\right) := \int_0^t \hat{a}_{1,1}\left(0\right) \frac{\partial \hat{\Gamma}_{\hat{L}_0}\left(x,t,\eta,\vartheta\right)}{\partial \eta}\Bigg|_{\eta=0} g\left(\vartheta\right) d\vartheta, \quad y \geq 0, t \in [0,1],$$

and for

$$\mathbf{g} = \left(g^{(1)}\left(t\right), g^{(2)}\left(t\right)\right) \in C\left([0,1],\mathbb{R}^2\right)$$

let

$$\hat{P}_{\mathbf{g},\hat{\gamma}}\left(x,t\right) := \hat{P}^{(1)}_{g^{(1)},\gamma}\left(x,t\right) - \hat{P}^{(2)}_{g^{(2)}}\left(x,t\right), \quad t \in [0,1].$$

Definition 4.2.6. *Let*

$$\hat{V}^{(1)}_{\xi,0,\gamma}\left(t\right) := -2\hat{\Gamma}_{\hat{L}_0}\left(\hat{\gamma},t,\xi,0\right), \quad (t,\xi) \in [0,1] \times (0,\hat{\gamma}),$$

$$\hat{V}^{(2)}_{\xi,0,\hat{\gamma}}\left(t\right) := -2\hat{\Gamma}_{\hat{L}_0}\left(0,t,\xi,0\right), \quad (t,\xi) \in [0,1] \times (0,\hat{\gamma}), \text{ and}$$

$$\hat{\mathbf{V}}_{\xi,0,\hat{\gamma}}\left(t,\xi\right) := \left(\hat{V}^{(1)}_{\xi,0,\gamma}\left(t\right), \hat{V}^{(2)}_{\xi,0,\hat{\gamma}}\left(t\right)\right).$$

For $n \in 0,1,2,\ldots,$ *define*

$$\hat{\mathbf{V}}_{\xi,n,\hat{\gamma}} = \left(\hat{V}^{(1)}_{\xi,n,\hat{\gamma}}\left(t,\xi\right), \hat{V}^{(2)}_{\xi,n,\hat{\gamma}}\left(t,\xi\right)\right)$$

recursively by

$$\begin{aligned}
&\hat{V}^{(1)}_{\xi,n+1,\hat{\gamma}}\left(t\right) := 2\hat{P}_{\hat{\mathbf{V}}_{\xi,n,\hat{\gamma}},\hat{\gamma}}\left(\hat{\gamma},t\right), \quad t \in [0,1], \\
&\hat{V}^{(2)}_{\xi,n+1,\hat{\gamma}}\left(t\right) := 2\hat{P}_{\hat{\mathbf{V}}_{\xi,n,\hat{\gamma}},\hat{\gamma}}\left(0,t\right). \quad t \in [0,1], \quad n \in 0,1,\ldots, \quad t \in [0,1], \\
&\hat{\mathbf{V}}_{\xi,n+1,\hat{\gamma}}\left(t\right) := \left(\hat{V}^{(1)}_{\xi,n+1,\hat{\gamma}}\left(t\right), \hat{V}^{(2)}_{\xi,n+1,\hat{\gamma}}\left(t\right)\right), \quad n \in 0,1,\ldots, \quad t \in [0,1].
\end{aligned}$$

Let

$$\hat{U}^{(1)}_{\xi,n,\hat{\gamma}}\left(t\right) := \sum_{k=0}^{n} \hat{V}^{(1)}_{\xi,k}\left(t\right), \quad t \in [0,T], n \in 0,1,\ldots,$$

$$\hat{U}^{(2)}_{\xi,n,\hat{\gamma}}\left(t\right) := \sum_{k=0}^{n} \hat{V}^{(2)}_{\xi,k}, \quad n \in 0,1,\ldots,$$

let

$$\hat{\mathbf{U}}_{\xi,n,\hat{\gamma}}\left(t\right) := \left(\hat{U}^{(1)}_{\xi,n,\hat{\gamma}}\left(t\right), \hat{U}^{(2)}_{\xi,n,\hat{\gamma}}\left(t\right)\right), \quad n \in 0,1,\ldots,$$

let

$$\hat{U}^{(1)}_{\xi,\hat{\gamma}}\left(t\right) := \lim_{n\to\infty} \hat{U}^{(1)}_{\xi,n,\hat{\gamma}}(t), \quad t \in [0,1],$$

$$\hat{U}^{(2)}_{\xi,\hat{\gamma}}(t) := \lim_{n\to\infty} \hat{U}^{(2)}_{\xi,n,\hat{\gamma}}(t), \quad t \in [0,1],$$

and let

$$\hat{\mathbf{U}}_{\xi,\hat{\gamma}}(t) := \left(\hat{U}^{(1)}_{\xi,\hat{\gamma}}(t), \hat{U}^{(2)}_{\xi,\hat{\gamma}}(t)\right).$$

Lemma 4.2.4. (i) *For every $n \in 0, 1, \ldots$, $\hat{V}^{(1)}_{\xi,n,\hat{\gamma}}(t)$ and $\hat{V}^{(2)}_{\xi,n,\hat{\gamma}}(t)$ are continuous on $[0,1]$ and differentiable on $(0,1]$, and the same holds for $\hat{U}^{(1)}_{\xi,\hat{\gamma}}$ and $\hat{U}^{(2)}_{\xi,\hat{\gamma}}$. Furthermore, there exists a sequence of positive constants $\{k_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} k_n < \infty$, and a constant C, such that for every $t \in (0,1]$ and $l \in \{0,1,2\}$ the identities and inequalities stated below are all valid:*

$$-\frac{1}{2}\hat{U}^{(1)}_{\xi,n,\hat{\gamma}}(t) + \hat{P}_{\hat{\mathbf{U}}_{\xi,n,\hat{\gamma}}}(\hat{\gamma}, t) = \hat{\Gamma}_{\hat{L}_0}(\hat{\gamma}, t, \xi, 0) + \hat{P}_{\hat{\mathbf{V}}_{\xi,n,\hat{\gamma}}}(\hat{\gamma}, t),$$

$$-\frac{1}{2}\hat{U}^{(2)}_{\xi,n,\hat{\gamma}}(t) + \hat{P}_{\hat{\mathbf{U}}_{\xi,n,\hat{\gamma}}}(0, t) = \hat{\Gamma}_{\hat{L}_0}(0, t, \xi, 0) + \hat{P}_{\hat{\mathbf{V}}_{\xi,n,\hat{\gamma}}}(0, t),$$

$$\left|\frac{\partial^l \hat{V}^{(1)}_{\xi,n,\hat{\gamma}}(t)}{\partial t^l}\right| \le k_n t^{\frac{n-1}{2}-l} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(\hat{\gamma}-\xi)^2}{t}\right),$$

$$\left|\frac{\partial^l \hat{V}^{(2)}_{\xi,n,\hat{\gamma}}(t)}{\partial t^l}\right| \le k_n t^{\frac{n-1}{2}-l} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{\xi^2}{t}\right),$$

$$\left|\frac{\partial^l \hat{U}^{(1)}_{\xi,\hat{\gamma}}(t)}{\partial t^l}\right| \le C t^{-\frac{1}{2}-l} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(\hat{\gamma}-\xi)^2}{t}\right),$$

$$\left|\frac{\partial^l \hat{U}^{(1)}_{\xi,\hat{\gamma}}(t)}{\partial t^l}\right| \le C t^{-\frac{1}{2}-l} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(\hat{\gamma}-\xi)^2}{t}\right),$$

$$\left|\frac{\partial^l \hat{U}^{(2)}_{\xi,\hat{\gamma}}(t)}{\partial t^l}\right| \le k_n t^{-\frac{1}{2}-l} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{\xi^2}{t}\right).$$

(ii) *For every fixed $\xi \in (0,\hat{\gamma})$, $\hat{\mathbf{U}}_{\xi,\hat{\gamma}}(t)$ is a solution of the integral equation*

$$\begin{cases} -\frac{1}{2}\hat{\mathbf{U}}^{(1)}_{\xi,\hat{\gamma}}(t) + \hat{P}_{\hat{\mathbf{U}}_{\xi,\hat{\gamma}},\hat{\gamma}}(\hat{\gamma}, t) = \hat{\Gamma}_{\hat{L}_*}(\hat{\gamma}, t, \xi, 0), & t \in (0,1], \\ -\frac{1}{2}\hat{\mathbf{U}}^{(2)}_{\xi,\hat{\gamma}}(t) + \hat{P}_{\hat{\mathbf{U}}_{\xi,\hat{\gamma}},\hat{\gamma}}(0, t) = \hat{\Gamma}_{\hat{L}_*}(0, t, \xi, 0), & t \in (0,1]. \end{cases} \tag{4.2.6}$$

(iii) *There exists a constant C such that for every $(x,t,\xi,\vartheta) \in \bar{\mathcal{D}}_{\hat{\gamma}}$*

$$\left|\hat{P}_{\hat{\mathbf{U}}_{\xi,\hat{\gamma}},\hat{\gamma}}(x,t)\right| \le C(t-\vartheta)^{-\frac{1}{2}}\left(\exp\left(-\frac{1}{2}\hat{c}_0 \frac{(x-\hat{\gamma})^2 + \frac{1}{2}(\xi-\hat{\gamma})^2}{t-\vartheta}\right)\right.$$
$$\left. + \exp\left(-\frac{1}{2}\hat{c}_0 \frac{x^2 + \frac{1}{2}\xi^2}{t-\vartheta}\right)\right).$$

(iv) *$P_{\hat{\mathbf{U}}_{\xi,\hat{\gamma}}}(x,t)$ is the classical solution of the PDE (4.2.3).*

Proof. Because of the symmetry property between the variables t and ϑ and the bounds given in Proposition 4.2.2 and Proposition 4.2.3, the lemma follows from Theorem V.5.5 in Garroni and Menaldi (1992) and similar calculations as in the proof of Lemma 4.1.2. □

Definition 4.2.7. *For every $n \in 0, 1, \ldots,$ and $(\xi, t) \in (0, \gamma) \times (0, 1]$ let*

$$\hat{V}^{(1)}_{\hat{\gamma}}(\xi, t) := \hat{V}^{(1)}_{\xi,n,\hat{\gamma}}(t),$$

let

$$\hat{V}^{(2)}_{\hat{\gamma}}(\xi, t) := \hat{V}^{(2)}_{\xi,n,\hat{\gamma}}(t),$$

let

$$\hat{U}^{(1)}_{\hat{\gamma}}(\xi, t) := \hat{U}^{(1)}_{\xi,\hat{\gamma}}(t),$$

and let

$$\hat{U}^{(2)}_{\hat{\gamma}}(\xi, t) := \hat{U}^{(2)}_{\xi,\hat{\gamma}}(t).$$

Proposition 4.2.4. (i) *For every $n \in 0, 1, 2, \ldots,$ $\hat{V}^{(1)}_{\xi,n,\hat{\gamma}}(\xi, t)$ and $\hat{V}^{(1)}_{\xi,n,\hat{\gamma}}(\xi, t)$ are differentiable with respect to ξ on $(\xi, t) \in (0, \hat{\gamma}) \times (0, 1]$. Furthermore, there exists a constant C, and a sequence of positive constants $\{k_n\}_{n=0}^{\infty}$ such that,*

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0,$$

and such that, for every $(\xi, t) \in (0, \gamma) \times (0, 1]$, and $l \in \{0, 1\}$ the following inequalities are all valid:

$$\left| \frac{\partial \hat{V}^{(1)}_{\xi,0,\gamma}(\xi, t)}{\partial \xi} \right| \le C t^{-\frac{1}{2}} \left(1 + \frac{|\hat{\gamma} - \xi|}{t}\right) \exp\left(-\hat{c}_0 \frac{(\hat{\gamma} - \xi)^2}{t}\right),$$

$$\left| \frac{\partial \hat{V}^{(2)}_{\xi,0,\hat{\gamma}}(\xi, t)}{\partial \xi} \right| \le C t^{-\frac{1}{2}} \left(1 + \frac{\xi}{t}\right) \exp\left(-\hat{c}_0 \frac{\xi^2}{t}\right),$$

$$\left| \frac{\partial^{1+l} \hat{V}^{(1)}_{\xi,n,\hat{\gamma}}(\xi, t)}{\partial \xi \partial t^l} \right| \le k_n t^{\frac{n}{2}-(1+l)} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(\hat{\gamma} - \xi)^2}{t}\right)$$

and

$$\left| \frac{\partial^{1+l} \hat{V}^{(2)}_{\xi,n,\hat{\gamma}}(\xi, t)}{\partial \xi \partial t^l} \right| \le k_n t^{\frac{n}{2}-(1+l)} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{\xi^2}{t}\right).$$

(ii) *$\hat{U}^{(1)}_{\hat{\gamma}}(\xi, t)$ and $\hat{U}^{(2)}_{\hat{\gamma}}(\xi, t)$ are differentiable with respect to ξ on $(\xi, t) \in (0, \gamma) \times (0, T)$. Moreover, there exists a constant C such that, for every $(\xi, t) \in (0, \gamma) \times (0, 1]$,*

$$\left| \frac{\partial \hat{U}^{(1)}_{\hat{\gamma}}(\xi, t)}{\partial \xi} \right| \le C t^{-\frac{1}{2}} \left(1 + \frac{|\hat{\gamma} - \xi|}{t}\right) \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(\hat{\gamma} - \xi)^2}{t}\right),$$

$$\left| \frac{\partial \hat{U}^{(2)}_{\hat{\gamma}}(\xi, t)}{\partial \xi} \right| \le C t^{-\frac{1}{2}} \left(1 + \frac{\xi}{t}\right) \exp\left(-\frac{1}{4}\hat{c}_0 \frac{\xi^2}{t}\right),$$

$$\left| \frac{\partial^2 \hat{U}^{(1)}_{\hat{\gamma}}(\xi, t)}{\partial \xi \partial t} \right| \le C t^{-2} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(\hat{\gamma} - \xi)^2}{t}\right)$$

and

$$\left| \frac{\partial^2 \hat{U}^{(2)}_{\hat{\gamma}}(\xi, t)}{\partial \xi \partial t} \right| \le C t^{-2} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{\xi^2}{t}\right).$$

Proof. For part (i): It follows from Proposition 4.2.2 that a bound of this form holds for $n = 0$. The claim can be established by exploiting the symmetry property between t and ϑ, and doing a similar induction as in Lemma 4.1.2. The main problem is the singularity at $t = \vartheta$, but this is only a problem for the first few terms in the sequence.

For part (ii): This can be established from part (i) and the uniform convergence of the derivatives of $\hat{U}_{\hat{\gamma}}^{(n)}(\xi, t)$ as $n \to \infty$. $\square$

Lemma 4.2.5. *There exists a constant C such that, for every $(x, t, \vartheta, \xi) \in \bar{\mathcal{D}}_{\hat{\gamma}}$, and every $l \in \{0, 1, 2\}$ the following inequalities are all valid:*

(i)

$$\left|\frac{\partial^l \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial x^l}\right| \le C(t-\vartheta)^{-\frac{1+l}{2}}\left(\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\hat{\gamma})^2+\frac{1}{2}(\xi-\hat{\gamma})^2}{t-\vartheta}\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2+\frac{1}{2}\xi^2}{t-\vartheta}\right)\right).$$

$$\left|\frac{\partial \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial \xi}\right| \le C(t-\vartheta)^{-1}\left(\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\hat{\gamma})^2+\frac{1}{2}(\xi-\hat{\gamma})^2}{t-\vartheta}\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2+\frac{1}{2}\xi^2}{t-\vartheta}\right)\right).$$

$$\left|\frac{\partial \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial t}\right| \le C(t-\vartheta)^{-\frac{3}{2}}\left(\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\hat{\gamma})^2+\frac{1}{2}(\xi-\hat{\gamma})^2}{t-\vartheta}\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2+\frac{1}{2}\xi^2}{t-\vartheta}\right)\right).$$

$$\left|\frac{\partial^2 \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial x \partial t}\right| \le C(t-\vartheta)^{-2}\left(\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\hat{\gamma})^2+\frac{1}{2}(\xi-\hat{\gamma})^2}{t-\vartheta}\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2+\frac{1}{2}\xi^2}{t-\vartheta}\right)\right).$$

$$\left|\frac{\partial^2 \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial \xi \partial x}\right| \le C(t-\vartheta)^{-\frac{3}{2}}\left(\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\hat{\gamma})^2+\frac{1}{2}(\xi-\hat{\gamma})^2}{t-\vartheta}\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2+\frac{1}{2}\xi^2}{t-\vartheta}\right)\right).$$

(ii) *For every $(x, t, \xi, \vartheta) \in \bar{\mathcal{D}}_{\hat{\gamma}}$*

$$\hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta) = \hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta) - \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)$$

is the Green function associated with the differential operator $\hat{L}_0$ and Dirichlet boundary conditions.

(iii)

$$\left|\frac{\partial^l \hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial y^l}\right| \le C(t-\vartheta)^{-\frac{1+l}{2}} \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial \hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial \xi}\right| \le C(t-\vartheta)^{-1} \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial \hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial t}\right| \le C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial^2 \hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial x\partial \xi}\right| \le C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial^2 \hat{G}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial x\partial t}\right| \le C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right).$$

Proof. For (i): It follows from the inequalities in Proposition 4.2.3 that the derivatives of $\hat{\Gamma}_{\hat{L}_{0,*}}(x,t,\xi,\vartheta)$ can be written as the sum of terms which behave like the fundamental solution with constant coefficients discussed in Section 4.1.1. Thus we can calculate bounds for the derivatives using integration by parts as in the proof of Lemma 4.1.3. Some extra terms have a weaker singularity. A calculation along these lines yields the stated inequalities.

For part (ii): Because of Lemma 4.2.2 this follows from similar calculations as in the proof of part (iii) of Lemma 4.1.3.

For part (iii): Since, for any $x,\xi \in [0,\hat{\gamma}]$,

$$(x-\xi)^2 \le \min\left(x^2+\xi^2, (x-\hat{\gamma})^2 + (\xi-\hat{\gamma})^2\right),$$

this follows from the bounds given in part (i) and the regularity bounds of the function $\hat{\Gamma}_{\hat{L}_0}$. □

Proposition 4.2.5. *There exists a constant C such that, for every $(x,t) \in (0,\hat{\gamma})\times(0,1]$:*

$$\left|\int_0^t\int_0^{\hat{\gamma}} \frac{\partial^2 \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial x^2} d\xi d\vartheta\right| \le C\Bigg(\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\hat{\gamma})^2}{t}\right) + \exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2}{t}\right)\Bigg)$$

and

$$\left|\int_0^t\int_0^{\hat{\gamma}} \frac{\partial \hat{g}_{\hat{L}_0,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial t} d\xi d\vartheta\right| \le C\Bigg(\exp\left(-\frac{1}{2}\hat{c}_0\frac{(x-\hat{\gamma})^2}{t}\right) + \exp\left(-\frac{1}{2}\hat{c}_0\frac{x^2}{t}\right)\Bigg).$$

Proof. Let

$$\hat{B}^{(1)}_{\hat{\gamma}}(s) = \int_0^{\hat{\gamma}} U^{(1)}_{\hat{\gamma}}(\xi, s) d\xi, \quad s \in (0, 1],$$

and let

$$\hat{B}^{(2)}_{\hat{\gamma}}(s) = \int_0^{\hat{\gamma}} U^{(2)}_{\hat{\gamma}}(\xi, s) d\xi, \quad s \in (0, 1].$$

A similar calculation as in the proof of Lemma 4.1.7 yields that

$$\int_0^t \int_0^{\hat{\gamma}} \frac{\partial \hat{g}_{\hat{L}_0, \hat{\gamma}}(x, t, \xi, \vartheta)}{\partial t} d\xi d\vartheta = I_1 - I_2,$$

where

$$I_1 = \int_0^t B^{(1)}(s) \int_0^{t-\vartheta} \hat{a}_{1,1}(\hat{\gamma}) \left. \frac{\partial^2 \hat{\Gamma}_{\hat{L}_0}(x, t-\vartheta, \eta, s)}{\partial t \partial \eta} \right|_{\eta = \hat{\gamma}} d\vartheta ds,$$

and

$$I_2 = \int_0^t B^{(2)}(s) \int_0^{t-\vartheta} \hat{a}_{1,1}(0) \left. \frac{\partial^2 \hat{\Gamma}_{\hat{L}_0}(x, t-\vartheta, \eta, s)}{\partial t \partial \eta} \right|_{\eta = 0} d\vartheta ds.$$

Calculating the integrals above, using the bounds given in Proposition 4.2.3, yields that the stated inequalities are valid. □

Analogous to what we did in Section 4.1.1, we will construct the Green function $\hat{G}_{\hat{L}, \hat{\gamma}}$, associated with the entire differential operator $\hat{L}$ and Dirichlet boundary condition, by solving an integral equation.

Definition 4.2.8. *Let*

$$\hat{Q}_{\hat{\gamma},0}(x, t, \xi, \vartheta) := \hat{a}_1^*(x) \frac{\partial \hat{G}_{\hat{L}_0, \hat{\gamma}}(x, t, \xi, \vartheta)}{\partial x}, \quad (x, t, \xi, \vartheta) \in \bar{\mathcal{D}}_{\hat{\gamma}}.$$

Let the sequence of functions $\left\{ \hat{Q}_{\hat{\gamma},n} \right\}_{n=0}^{\infty}$ *be defined recursively for* $n \in 1, 2, \ldots,$ *and* $(x, t, \xi, \vartheta) \in \mathcal{D}_\gamma$ *by*

$$\hat{Q}_{\hat{\gamma},n+1}(x, t, \xi, \vartheta) = \int_\vartheta^t \int_0^{\hat{\gamma}} \hat{Q}_{\hat{\gamma},0}(x, t, z, s) \hat{Q}_{\hat{\gamma},n}(z, s, \xi, \vartheta) dz ds,$$

and let

$$\hat{Q}_{\hat{\gamma}}(x, t, \xi, \vartheta) = \sum_{n=0}^{\infty} \hat{Q}_{\hat{\gamma},n}(x, t, \xi, \vartheta).$$

Lemma 4.2.6. *Assume that* $\sigma_R > 0$. *Let* $\alpha \in (0, 1)$ *and let* $\mathcal{G}_k^{\alpha, \frac{\alpha}{2}}$ *be the Green function spaces defined in Definition VII.1.1 in Garroni and Menaldi (1992).*

(i) $\hat{Q}_{\hat{\gamma}} \in \mathcal{G}_1^{\alpha, \frac{\alpha}{2}}$. *Moreover,* $\hat{Q}_{\hat{\gamma}}$ *is the unique solution in* $\mathcal{G}_1^{\alpha, \frac{\alpha}{2}}$ *of the integral equation*

$$\hat{Q}_{\hat{\gamma}}(x, t, \xi, \vartheta) = \hat{Q}_{\hat{\gamma},0}(x, t, \xi, \vartheta) + \int_\vartheta^t \int_0^{\hat{\gamma}} \hat{Q}_{\hat{\gamma},0}(x, t, z, s) \hat{Q}_{\hat{\gamma}}(z, s, \xi, \vartheta) dz ds. \tag{4.2.7}$$

(ii) *There exists a constant C such that, for every $(x,t,\xi,\vartheta) \in \bar{\mathcal{D}}_{\hat{\gamma}}$, every $x', \in (0,\hat{\gamma})$, and every $t' \in (0,t,)$ the following identities and inequalities are all valid:*

$$\hat{Q}_{\hat{\gamma}}(x,t,\xi,\vartheta) = \hat{Q}_{\hat{\gamma}}(x,t-\vartheta,\xi,0),$$

$$\left|\hat{Q}_{\hat{\gamma}}(x,t,\xi,\vartheta)\right| \le C(t-\vartheta)^{-1} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(x-\xi)^2}{t-\vartheta}\right),$$

$$\left|\hat{Q}_{\hat{\gamma}}(x,t,\xi,\vartheta) - \hat{Q}_{\hat{\gamma}}(x',t,\xi,\vartheta)\right| \le C|x-x'|^{\frac{1}{2}}(t-\vartheta)^{-\frac{5}{4}} \times \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(x-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\hat{Q}_{\hat{\gamma}}(x,t,\xi,\vartheta) - \hat{Q}_{\hat{\gamma}}(x,t',\xi,\vartheta)\right| \le C|t-t'|^{\frac{1}{4}}(t'-\vartheta)^{-\frac{5}{4}} \times \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(x-\xi)^2}{t-\vartheta}\right).$$

(iii)

$$\left|\frac{\partial \hat{Q}_{\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial \xi}\right| \le C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(x-\xi)^2}{t-\vartheta}\right).$$

Proof. For part (i): It follows from Lemma VII.1.3 in Garroni and Menaldi (1992), and the bounds given in Lemma 4.2.5, that $\frac{\partial \hat{G}_{L,\hat{\gamma}}(x,t,\xi,\vartheta)}{\partial x} \in \mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$, and hence $\hat{Q}_{\hat{\gamma},0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$. Since $\hat{Q}_{\hat{\gamma},0} \in \mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$ it follows from Proposition VIII.1.2 in Garroni and Menaldi (1992) that $\hat{Q}_{\hat{\gamma}}$ is the unique solution in the function space $\mathcal{G}_1^{\alpha,\frac{\alpha}{2}}$ of the integral equation (4.2.7).

For part (ii): It follows from similar calculations as in the proofs of Lemma 4.1.12, that these regularity bounds hold for the function $\sum_{j=0}^{n} \hat{Q}_{\hat{\gamma},j}(x,t,\xi,\vartheta)$, for any n. Furthermore, it can be shown that these sums converge uniformly.

For part (iii): We first observe that this bound holds for $\frac{\partial \hat{Q}_{\hat{\gamma},0}(x,t,\xi,\vartheta)}{\partial \xi}$. A similar calculation as in the proof of Lemma 4.1.4 part (v) yields that, for some constant C

$$\left|\frac{\partial \hat{Q}_{\hat{\gamma},1}(x,t,\xi,\vartheta)}{\partial \xi}\right| \le C(t-\vartheta)^{-1} \exp\left(-\frac{1}{4}\hat{c}_0 \frac{(x-\xi)^2}{t-\vartheta}\right).$$

For $n \in 2,3,\ldots$ it can be shown by induction that the functions $\frac{\partial \hat{Q}_{\hat{\gamma},n}(x,t,\xi,\vartheta)}{\partial \xi}$ are less singular, and that the sum $\sum_{j=0}^{n} \frac{\partial \hat{Q}_{\hat{\gamma},n}(x,t,\xi,\vartheta)}{\partial \xi}$ converges uniformly on $\mathcal{D}_{\hat{\gamma}}$, thus allowing the sum to be differentiated term by term. □

Lemma 4.2.7. *There exists a constant C such that following identities and bounds are valid for every $(x,t,\xi,\vartheta) \in \mathcal{D}_{\hat{\gamma}}$ and every $l \in \{0,1,2\}$:*

(i)

$$\begin{aligned}
&\int_{\vartheta}^{t}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}\left(x,t,z,s\right)\hat{Q}_{\hat{\gamma}}\left(z,s,\xi,\vartheta\right)dzds\\
&\qquad=\int_{0}^{t-\vartheta}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}\left(x,t-\vartheta,z,s\right)\hat{Q}_{\hat{\gamma}}\left(z,s,\xi,0\right)dzds,\\
&\frac{\partial^l}{\partial x^l}\int_{\vartheta}^{t}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}\left(x,t,z,s\right)\hat{Q}_{\hat{\gamma}}\left(z,s,\xi,\vartheta\right)dzds\\
&\qquad=\int_{\vartheta}^{t}\int_{0}^{\gamma}\frac{\partial^l\hat{G}_{\hat{L}_0,\hat{\gamma}}\left(x,t,z,s\right)}{\partial x^l}\hat{Q}_{\hat{\gamma}}\left(z,s,\xi,\vartheta\right)dzds,\\
&\frac{\partial}{\partial t}\int_{\vartheta}^{t}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}\left(x,t,z,s\right)\hat{Q}_{\hat{\gamma}}\left(z,s,\xi,\vartheta\right)dzds\\
&\qquad=\hat{Q}_{\hat{\gamma}}\left(x,t,\xi,\vartheta\right)\\
&\qquad\quad+\int_{\vartheta}^{t}\int_{0}^{\gamma}\frac{\partial\hat{G}_{\hat{L}_0,\hat{\gamma}}\left(x,t,z,s\right)}{\partial t}\hat{Q}_{\hat{\gamma}}\left(z,s,\xi,\vartheta\right)dzds,\\
&\left|\frac{\partial^l}{\partial x^l}\int_{\vartheta}^{t}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}\left(x,t,z,s\right)\hat{Q}_{\hat{\gamma}}\left(z,s,\xi,\vartheta\right)dzds\right|\\
&\qquad\le C\left(t-\vartheta\right)^{-\frac{l}{2}}\exp\left(-\frac{1}{4}\hat{c}_0\frac{\left(x-\xi\right)^2}{t-\vartheta}\right),\\
&\left|\frac{\partial}{\partial t}\int_{\vartheta}^{t}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}\left(x,t,z,s\right)\hat{Q}_{\hat{\gamma}}\left(z,s,\xi,\vartheta\right)dzds\right|\\
&\qquad\le C\left(t-\vartheta\right)^{-1}\exp\left(-\frac{1}{4}\hat{c}_0\frac{\left(x-\xi\right)^2}{t-\vartheta}\right),\\
&\left|\frac{\partial}{\partial \xi}\int_{\vartheta}^{t}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}\left(x,t,z,s\right)\hat{Q}_{\hat{\gamma}}\left(z,s,\xi,\vartheta\right)dzds\right|\\
&\qquad\le C\left(t-\vartheta\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}\hat{c}_0\frac{\left(x-\xi\right)^2}{t-\vartheta}\right),\\
&\left|\int_{-\infty}^{\infty}\int_{\vartheta}^{t}\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\hat{\gamma}}\left(x,t,z,s\right)\hat{Q}_{\hat{\gamma}}\left(z,s,\xi,\vartheta\right)dzdsd\left(x-\xi\right)\right|\\
&\qquad\le C\sqrt{t}.
\end{aligned}$$

(ii)

$$\begin{aligned}
\hat{G}_{\hat{L},\hat{\gamma}}\left(x,t,\xi,\vartheta\right)&=\hat{G}_{\hat{L}_0,\hat{\gamma}}\left(x,t,\xi,\vartheta\right)\\
&\quad+\int_{\vartheta}^{t}\int_{0}^{\hat{\gamma}}\hat{G}_{\hat{L}_0,\hat{\gamma}}\left(x,t,z,s\right)\hat{Q}_{\hat{\gamma}}\left(z,s,\xi,\vartheta\right)dzds.
\end{aligned}$$

Proof. For (i): These identities and bounds can be derived from similar calculations as in Lemma 4.1.5.

For part (ii): Since $\hat{G}_{\hat{L}_0,\hat{\gamma}}$ is the Green function associated with the differential operator $\hat{L}_0$ and Dirichlet boundary conditions, and $\hat{Q}_{\hat{\gamma}}$ is a solution of the integral equation (4.2.7), this follows from the bounds given in part (i). □

After the next result we will finally be ready to obtain regularity bounds on $\psi_2(y,t)$ (using the original variable y).

Proposition 4.2.6. *Assume that $\sigma_R > 0$, and that the tail distribution $\bar{F}$ satisfies the bound* (4.0.38). *Let the function $H_{1,\kappa}$ be as in section 3. Then there exists a constant C_β, depending on β, such that, for every $x' > x > 0$ and every $1 \geq t > t' > 0$, the following inequalities are valid:*

(i)
$$\left|H_{1,\kappa}\left(e^x - 1, t\right)\right| \leq C_\beta e^{-\beta x},$$

and for every $\alpha \in (0,1]$

$$\left|H_{1,\kappa}\left(e^x - 1, t\right) - H_{1,\kappa}\left(e^{x'} - 1, t'\right)\right| \leq C_\beta \left(t - t'\right)^\alpha \left(t'\right)^{-\alpha} e^{-\beta x}.$$

(ii) *For every $\alpha \in (0, \min(1,\beta))$*

$$\left|H_{1,\kappa}\left(e^x - 1, t\right) - H_{1,\kappa}\left(e^x - 1, t\right)\right| \leq C_\beta \left|x - x'\right|^\alpha t^{-\frac{\alpha}{2}} \exp\left(-\left(\beta - \alpha\right)\right).$$

Proof. For part (i): These inequalities follow trivially from the bounds given in Proposition 3.0.6.

For part (ii): Assume first that

$$x' - x \geq \frac{1}{2}.$$

For this case the stated inequality is trivially true because of the bound on the function $H_{1,\kappa}$ itself given in Proposition 3.0.6. Assume instead that

$$x' - x < \frac{1}{2}.$$

We observe that in this case

$$e^{x'} - e^x \leq \frac{4}{3} e^x \left(x' - x\right).$$

Because of this bound and the bounds in Proposition 3.0.6 it can be calculated that the stated bound holds even for this case. □

Lemma 4.2.8. *There exists a constant C_β, depending on β, such that, for every $(x,t) \in (0,\hat{\kappa}) \times (0,1]$ and every $y \in (0,\kappa)$ the following inequalities are valid:*

(i)

$$\int_0^t \left| \int_0^{\hat{\kappa}} \frac{\partial^2 \hat{\Gamma}_{\hat{L}_{*,0}}\left(x,t,\xi,\vartheta\right)}{\partial x^2} H_{1,\kappa}\left(e^\xi - 1, \vartheta\right) d\xi \right| d\vartheta \leq C_\beta \exp\left(-\frac{1}{2}\beta x\right),$$

$$\int_0^t \left| \int_0^{\hat{\kappa}} \frac{\partial \hat{\Gamma}_{\hat{L}_{*,0}}\left(x,t,\xi,\vartheta\right)}{\partial t} H_{1,\kappa}\left(e^\xi - 1, \vartheta\right) d\xi \right| d\vartheta \leq C_\beta \exp\left(-\frac{1}{2}\beta x\right),$$

$$\left| \int_0^t \int_0^{\kappa} \frac{\partial^2 \hat{g}_{\hat{L}_{*,0},\hat{\kappa}}\left(x,t,\xi,\vartheta\right)}{\partial x^2} H_{1,\kappa}\left(e^\xi - 1, \vartheta\right) d\xi d\vartheta \right| \leq C_\beta \exp\left(-\frac{1}{2}\beta x\right),$$

and

$$\left|\int_0^t\int_0^\kappa \frac{\partial \hat{g}_{\hat{L}_{*,0},\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial t} H_{1,\kappa}\left(e^\xi-1,\vartheta\right)d\xi d\vartheta\right| \le C_\beta \exp\left(-\frac{1}{2}\beta x\right).$$

(ii) *For every $l\in\{1,2\}$, the following identities are all valid:*

$$\frac{\partial^l \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial x^l} = \int_0^t\int_0^{\hat{\kappa}} \frac{\partial^l \hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial x^l} H_{1,\kappa}\left(e^\xi-1,\vartheta\right)d\xi d\vartheta,$$

$$\frac{\partial \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial t} = H_{1,\kappa}\left(e^x-1,t\right) + \int_0^t\int_0^{\hat{\kappa}} \frac{\partial \hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial t} H_{1,\kappa}\left(e^\xi-1,\vartheta\right)d\xi d\vartheta.$$

(iii) *For every $l\in\{0,1\}$, every $(x',t')\in(x,\kappa)\times(0,t)$ and every $\alpha\in(0,\frac{1}{2}]$ the following bounds are all valid:*

$$\left|\frac{\partial^l \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial x^l}\right| \le C_\beta t^{\frac{2-l}{2}}\exp\left(-\beta x\right),$$

$$\left|\frac{\partial^2 \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial x^2}\right| \le C_\beta t^{\frac{2-l}{2}}\exp\left(-\frac{1}{2}\beta x\right),$$

$$\left|\frac{\partial \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial t}\right| \le C_\beta \exp\left(-\frac{1}{2}\beta x\right),$$

$$\left|\hat{\psi}_{2,\hat{\kappa}}(x,t)-\hat{\psi}_{2,\hat{\kappa}}(x',t)\right| \le C_\beta\left(x'-x\right)^\alpha t^{\frac{2-\alpha}{2}}\exp\left(-\beta x\right),$$

$$\left|\hat{\psi}_{2,\hat{\kappa}}(x,t)-\hat{\psi}_{2,\hat{\kappa}}(x,t')\right| \le C_\beta\left(t-t'\right)^\alpha t^{1-\alpha}\exp\left(-\beta x\right),$$

and

$$\left|\hat{\psi}_{2,\hat{\kappa}}(x,t)\right| \le C_\beta\left(\min\left(x,\hat{\kappa}-x\right)\right)^{\frac{1}{2}} t^{\frac{3}{4}}.$$

(iv) *There exists a constant C_β such that for every $(y,t)\in(0,\kappa)\times(0,T]$ and every $l\in\{0,1\}$*

$$\left|\frac{\partial^l \psi_{2,\kappa}(y,t)}{\partial y^l}\right| \le C_\beta t^{\frac{2-l}{2}}\left(1+y\right)^{-(l+\beta)},$$

$$\left|\frac{\partial^2 \psi_{2,\kappa}(y,t)}{\partial y^2}\right| \le C_\beta\left(1+y\right)^{-\frac{1}{2}\beta},$$

$$\left|\frac{\partial \psi_{2,\kappa}(y,t)}{\partial t}\right| \le C_\beta\left(1+y\right)^{-\frac{1}{2}\beta}.$$

Proof. For parts (i)-(ii): We first observe that for any $a,b>0$ there exists a constant C depending on a and b such that for any $x,\xi>0$ and $0\le\vartheta<t\le 1$

$$\exp\left(-a\frac{(x-\xi)^2}{t-\vartheta}\right)\exp\left(-b\xi\right) \le C\exp\left(-\frac{1}{2}a\frac{(x-\xi)^2}{t-\vartheta}\right)\exp\left(-bx\right). \quad (4.2.8)$$

The identities and bounds given in part (i) and part (ii) follow from the bound above and similar calculations as in the proof of Lemma 4.1.6 and Lemma 4.1.5.

For part (iii): This follows from the bound (4.2.8), the bounds given in part (i), and the identities and bounds given in Proposition 4.2.2, Proposition 4.2.3, Lemma 4.2.5, Lemma 4.2.7 and Proposition 4.2.5.

For part (iv): Since

$$\psi_{2,\kappa}(y,t) = \hat{\psi}_{2,\hat{\kappa}}(\ln(1+y), t),$$

this follows from the bounds given in part (ii) and the chain rule. □

Lemma 4.2.9. *Assume that $\sigma_R > 0$ and that the tail distribution satisfies the bound* (4.0.38)*. Let the function $H_{2,\kappa}$ be as in section 3. Then, for some constant C_β, depending on β, the bounds stated below all hold for every $0 < x < x' < \hat{\kappa}$, every $0 \le t' < t \le 1$ and every $\alpha \in (0, \frac{1}{2}\min(\beta, 1)]$:*

$$|H_{2,\kappa}(e^x - 1, t)| \le C_\beta t \exp(-\beta x),$$

$$|H_{2,\kappa}(e^x - 1, t) - H_{2,\kappa}(e^x - 1, t')| \le C_\beta (t - t')^\alpha t^{1-\alpha} \exp(-\beta x),$$

and

$$\left|H_{2,\kappa}(e^x - 1, t) - H_{2,\kappa}\left(e^{x'} - 1, t\right)\right| \le C_\beta (x' - x)^\alpha t^{\frac{2-\alpha}{2}} \exp(-(\beta - \alpha) x).$$

Proof. It follows from similar calculations as in Proposition 4.2.6 that the stated inequalities are valid for $\psi_{2,\kappa}(y,t)$. Similar calculations as in Lemma 4.1.8 yield that the bound inequalities are valid for $H_{2,\kappa}(e^x - 1, t)$. □

4.2.2 Regularity estimates for for a subproblem with an integral term and unbounded coefficients

In this section we will consider the function

$$\hat{\psi}_{3,\hat{\kappa}}(x,t) := \psi_{3,\kappa}(e^x - 1, t), \quad (x,t) \in [0, \ln(\kappa+1)], \quad (x,t) \in [0, \hat{\kappa}] \times [0,1],$$

where as before $\hat{\kappa} = \ln(1+\kappa)$. Since $\psi_{3,\kappa}(e^x - 1, t)$ is a classical solution of the PIDE 3.0.10 it follows from the chain rule that the function $\hat{\psi}_{3,\hat{\kappa}}(x,t)$ is a solution of a different PIDE defined in the result below.

Definition 4.2.9. *Let the operator $\hat{A}$ be defined for any function $g(x,t) \in C^{2,1}((0,\hat{\kappa}) \times (0,1])$ as*

$$\hat{A}g(x,t) = \hat{L}g(x,t) - \lambda g(x,t) + \lambda \int_0^{e^x - 1} g(\ln(e^x - \zeta), t)\, dF(\zeta).$$

Lemma 4.2.10. *Let the function $H_{2,\kappa}$ be as in section 3. $\hat{\psi}_{3,\hat{\kappa}}(x,t)$ is a classical solution of the PIDE*

$$\begin{cases} \hat{\psi}_{3,\hat{\kappa}}(x,0) & = 0, \quad x \in (0,\hat{\kappa}), \\ \hat{\psi}_{3,\hat{\kappa}}(0,t) & = 0, \quad t \in [0,1], \\ \hat{\psi}_{3,\hat{\kappa}}(\hat{\kappa},t) & = 0, \quad t \in [0,1], \\ \frac{\partial \hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial t} & -\hat{A}\hat{\psi}_{3,\hat{\kappa}}(x,t) \\ & = H_{2,\kappa}(e^x - 1, t), \quad (x,t) \in (0, \ln(1+\kappa)) \times (0,1]. \end{cases} \tag{4.2.9}$$

Proof. This is similar to Lemma 4.2.1. □

Theorem 4.2.2. *$\hat{\psi}_{3,\hat{\kappa}}(x,t)$ is a unique classical solution of the PIDE (4.2.9)). Furthermore, there exists a unique Green function $\hat{G}_{\hat{A},\hat{\kappa}}(x,t,\xi,\vartheta)$ associated with the differential operator $\hat{L}$ and Dirichlet boundary conditions on the domain $\mathcal{D}_{\hat{\kappa}}$, i.e. satisfying the conditions in Definition 3.0.1 with L replaced by $\hat{L}$ and κ replaced by $\hat{\kappa}$. Furthermore, for every $(x,t) \in (0,\hat{\kappa}) \times (0,1]$*

$$\hat{\psi}_{3,\hat{\kappa}}(x,t) = \int_0^t \int_0^{\hat{\kappa}} \hat{G}_{\hat{A},\hat{\kappa}}(x,t,\xi,\vartheta)\, H_{2,\kappa}(e^{\xi}-1,\vartheta) d\xi d\vartheta.$$

Proof. This can be shown using similar arguments as those that lead to the result in Theorem 3.0.3. The most important difference is that in this case we define the function $j(x,t,\zeta)$ as

$$j(x,t,\zeta) = \begin{cases} -x + \ln(e^x - \zeta), & (x,t,\zeta) \in [0,\hat{\kappa}] \times [0,1] \times [0, e^x - 1] \\ -x + e^x, & (x,t,\zeta) \in [0,\hat{\kappa}] \times [0,1] \times (e^x - 1, \infty). \end{cases}$$

It can be shown that $j(x,t,\zeta)$ is continuously differentiable with respect to x on $[0,\hat{\kappa}]$, and that, for $x \in [0,\hat{\kappa}]$,

$$0 \le \frac{\partial j(x,t,\zeta)}{\partial x} < \infty,$$

thus satisfying the requirement (VIII.1.23) in Garroni and Menaldi (1992). □

Analogous to what we did in Section 4.1.2 we will construct the Green function $\hat{G}_{\hat{A},\hat{\kappa}}(x,t,\xi,\vartheta)$ in two steps. The first step is to use the Green function and Proposition VIII.1.2 to construct a Green function $\hat{G}_{\hat{L}_\lambda,\hat{\kappa}}$ associated with the differential operator

$$\hat{L} - \lambda,$$

and the second step is to do the same once again to construct the full Green function from $\hat{G}_{\hat{L}_\lambda,\hat{\kappa}}$, as was the case in Section 4.1.2.

Definition 4.2.10. *Let*

$$\hat{Q}_{\lambda,\hat{\kappa},0} = -\lambda \hat{G}_{\hat{L},\hat{\kappa}},$$

and let the sequence of function $\left\{\hat{Q}_{\lambda,\hat{\kappa},n}\right\}_{n=0}^{\infty}$ be defined inductively for $n \in 1,2,\ldots,$ and $(x,t,\xi,\vartheta) \in \mathcal{D}_\kappa$, by

$$\hat{Q}_{\lambda,\hat{\kappa},n}(x,t,\xi,\vartheta) = \int_\vartheta^t \int_0^{\hat{\kappa}} \hat{Q}_{\lambda,\hat{\kappa},0}(x,t,z,s)\, \hat{Q}_{\lambda,\hat{\kappa},n}(z,s,\xi,\vartheta)\, dzds,$$

and let

$$\hat{Q}_{\lambda,\hat{\kappa}}(x,t,\xi,\vartheta) = \sum_{n=0}^{\infty} \hat{Q}_{\lambda,\hat{\kappa},n}(x,t,\xi,\vartheta).$$

Lemma 4.2.11. *Assume that $\sigma_R > 0$. Let $\alpha \in (0,1)$ and let $\mathcal{G}_k^{\alpha,\frac{\alpha}{2}}$ be the Green function spaces defined in Definition VII.1.1 in Garroni and Menaldi (1992).*

(i) $\hat{Q}_{\lambda,\hat{\kappa},0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ *and* $\hat{Q}_{\lambda,\hat{\kappa}} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$. *Moreover* $\hat{Q}_{\lambda,\hat{\kappa}}$ *is the unique solution in* $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ *of the integral equation*

$$\begin{aligned}\hat{Q}_{\lambda,\hat{\kappa}}(x,t,z,\vartheta) = &-\lambda \hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta) \\ &-\lambda \int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}} \hat{G}_{\hat{L},\hat{\kappa}}(x,t,z,s)\,\hat{Q}_{\lambda,\hat{\kappa}}(z,s,\xi,\vartheta)\,dzds.\end{aligned} \tag{4.2.10}$$

(ii) $\hat{Q}_{\lambda,\hat{\kappa}}(x,t,\xi,\vartheta)$ *is differentiable with respect to all four variables. Furthermore, there exists a constant* C, *such that, for every* $(x,t,\xi,\vartheta) \in \bar{\mathcal{D}}_{\hat{\kappa}}$, *the following identities and inequalities are all valid:*

$$\hat{Q}_{\lambda,\hat{\kappa}}(x,t,\xi,\vartheta) = \hat{Q}_{\lambda,\hat{\kappa}}(x,t-\vartheta,\xi,0),$$

$$\left|\frac{\partial \hat{Q}_{\lambda,\hat{\kappa}}}{\partial x}(x,t,\xi,\vartheta)\right| \le C(t-\vartheta)^{-1}\exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial \hat{Q}_{\lambda,\hat{\kappa}}}{\partial \xi}(x,t,\xi,\vartheta)\right| \le C(t-\vartheta)^{-1}\exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial \hat{Q}_{\lambda,\hat{\kappa}}}{\partial t}(x,t,\xi,\vartheta)\right| \le C(t-\vartheta)^{-\frac{3}{2}}\exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right).$$

Proof. For part (i): It follows from Lemma VII.1.3 in Garroni and Menaldi (1992), and the bounds given in Lemma 4.2.5 and Lemma 4.2.7, that $\hat{G}_{L,\hat{\kappa}}(x,t,\xi,\vartheta) \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ and hence $\hat{Q}_{\lambda,\hat{\kappa},0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$. Since $\hat{Q}_{\lambda,\hat{\kappa},0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ it follows from Proposition VIII.1.2 in Garroni and Menaldi (1992) that $\hat{Q}_{\lambda,\hat{\kappa}}$ is the unique solution in the function space $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ of the integral equation (4.2.10).

For part (ii): This can be shown using the same calculations and reasoning as in the proof of Lemma 4.1.4, based on induction, the symmetry property between the t and ϑ variable, and uniform convergence. □

Lemma 4.2.12. *Assume that* $\sigma_R > 0$.

(i) *For every* $(x,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ *and* $l \in \{0,1,2\}$

$$\begin{aligned}&\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}} \hat{G}_{\hat{L},\hat{\kappa}}(x,t,z,s)\,\hat{Q}_{\lambda,\hat{\kappa}}(z,s,\xi,\vartheta)\,dzds \\ &\qquad = \int_{0}^{t-\vartheta}\int_{0}^{\hat{\kappa}} \hat{G}_{\hat{L},\hat{\kappa}}(x,t-\vartheta,z,s)\,\hat{Q}_{\lambda,\hat{\kappa}}(z,s,\xi,0)\,dzds,\end{aligned}$$

$$\begin{aligned}&\frac{\partial^l}{\partial x^l}\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}} \hat{G}_{\hat{L},\hat{\kappa}}(x,t,z,s)\,\hat{Q}_{\lambda,\hat{\kappa}}(z,s,\xi,\vartheta)\,dzds \\ &\qquad = \int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}} \frac{\partial^l \hat{G}_{\hat{L},\hat{\kappa}}(x,t,z,s)}{\partial x^l}\hat{Q}_{\lambda,\hat{\kappa}}(z,s,\xi,\vartheta)\,dzds,\end{aligned}$$

$$\frac{\partial}{\partial t}\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,z,s\right)\hat{Q}_{\lambda,\hat{\kappa}}\left(z,s,\xi,\vartheta\right)dzds$$
$$=\hat{Q}_{\lambda,\hat{\kappa}}\left(x,t,\xi,\vartheta\right)$$
$$+\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\frac{\partial\hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,z,s\right)}{\partial t}\hat{Q}_{\lambda,\hat{\kappa}}\left(z,s,\xi,\vartheta\right)dzds.$$

Furthermore, for some constant C

$$\left|\frac{\partial^{l}}{\partial x^{l}}\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,z,s\right)\hat{Q}_{\lambda,\hat{\kappa}}\left(z,s,\xi,\vartheta\right)dzds\right|$$
$$\leq C\left(t-\vartheta\right)^{\frac{1-l}{2}}\exp\left(-\frac{1}{4}\hat{c}_{0}\frac{\left(x-\xi\right)^{2}}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial}{\partial t}\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,z,s\right)\hat{Q}_{\lambda,\hat{\kappa}}\left(z,s,\xi,\vartheta\right)dzds\right|$$
$$\leq C\left(t-\vartheta\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}\hat{c}_{0}\frac{\left(x-\xi\right)^{2}}{t-\vartheta}\right).$$

(ii) *For some constant* C

$$\left|\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,z,s\right)\hat{Q}_{\lambda,\hat{\kappa}}\left(z,s,\xi,\vartheta\right)dzds\right|\leq C\left(\min\left(x,\hat{\kappa}-x\right)\right)^{\frac{1}{2}}$$
$$\times\left(t-\vartheta\right)^{\frac{1}{4}}$$
$$\times\left(\exp\left(-\frac{1}{4}\hat{c}_{0}\frac{\left(x-\xi\right)^{2}}{t-\vartheta}\right)\right.$$
$$+\exp\left(-\frac{1}{4}\hat{c}_{0}\frac{\xi^{2}}{t-\vartheta}\right)$$
$$\left.+\exp\left(-\frac{1}{4}\hat{c}_{0}\frac{\left(\hat{\kappa}-\xi\right)^{2}}{t-\vartheta}\right)\right).$$

(iii)

$$\hat{G}_{\hat{L}_{\lambda},\hat{\kappa}}\left(x,t,\xi,\vartheta\right)=\hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,\xi,\vartheta\right)$$
$$+\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,z,s\right)\hat{Q}_{\lambda,\hat{\kappa}}\left(z,s,\xi,\vartheta\right)dzds.$$

Proof. For (i): These identities and bounds can be derived from similar calculations as in Lemma 4.1.4 and Lemma 4.1.5.

For part (ii): This can be calculated from the bounds given in Lemma 4.2.7.

For part (iii): Since $\hat{G}_{\hat{L},\hat{\kappa}}$ is the Green function associated with the differential operator $\hat{L}$ it can be derived from the bounds given in part (i) and (ii) that

$$\hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta)+\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{G}_{\hat{L},\hat{\kappa}}(x,t,z,s)\,\hat{Q}_{\lambda,\hat{\kappa}}(z,s,\xi,\vartheta)\,dzds$$

is the unique (principal) Green function associated with the differential operator

$$\hat{L}-\lambda$$

and Dirichlet boundary conditions. □

Definition 4.2.11. *For $(x,t,\xi,\vartheta)\in\mathcal{D}_{\hat{\kappa}}$ let*

$$\hat{Q}_{I,\hat{\kappa},0}(x,t,\xi,\vartheta)=\lambda\int_{0}^{e^{x}-1}\hat{G}_{\hat{L}_{\lambda},\hat{\kappa}}(\ln(e^{x}-\zeta),t,\xi,\vartheta)\,dF(\zeta),$$

and let the sequence of functions

$$\left\{\hat{Q}_{I,\hat{\kappa},n}\right\}_{n=0}^{\infty}$$

be defined inductively by

$$\hat{Q}_{I,\hat{\kappa},n}(x,t,\xi,\vartheta)=\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{Q}_{I,\hat{\kappa},0}(x,t,z,s)\,\hat{Q}_{I,\hat{\kappa},n-1}(z,s,\xi,\vartheta)\,dzds,\quad n\in 1,2,\ldots.$$

Let

$$\hat{Q}_{I,\hat{\kappa}}(x,t,\xi,\vartheta)=\sum_{n=0}^{\infty}\hat{Q}_{I,\hat{\kappa},n}(x,t,\xi,\vartheta)$$

and

$$\hat{G}_{I,\hat{\kappa}}(x,t,\xi,\vartheta)=\int_{\vartheta}^{t}\int_{0}^{\hat{\kappa}}\hat{G}_{\hat{L}_{\lambda},\hat{\kappa}}(x,t,z,s)\,\hat{Q}_{I,\hat{\kappa}}(z,s,\xi,\vartheta)\,dzds. \tag{4.2.11}$$

Proposition 4.2.7. *Assume that $\sigma_R>0$ and that the tail distribution $\bar{F}$ satisfies the inequality (4.0.38). Then there exists a constant C_β, depending on β, such that the following inequalities are valid for every $(x,t,\xi,\vartheta)\in\mathcal{D}_{\hat{\kappa}}$, every $(x',t')\in(x,\hat{\kappa})\times[0,t)$ and every $\alpha\in\left(0,\min\left(\frac{1}{2},\frac{\beta}{2}\right)\right)$:*

$$\left|\hat{Q}_{I,\hat{\kappa},0}(x,t,\xi,\vartheta)\right|\le C_\beta(t-\vartheta)^{-\frac{1}{2}}\times\left(\exp\left(-\left(\frac{1}{32}c_0(x-\xi)^2+2\beta|x-\xi|\right)\right)+\exp(-\beta x)\right). \tag{4.2.12}$$

$$\begin{aligned}&\left|\hat{Q}_{I,\hat{\kappa},0}(x,t,\xi,\vartheta)-\hat{Q}_{I,\hat{\kappa},0}(x,t',\xi,\vartheta)\right|\\&\quad\le C_\beta(t-t')^{\frac{1}{4}}(t'-\vartheta)^{-\frac{3}{4}}\\&\qquad\times\left(\exp\left(-\left(\frac{1}{32}c_0(x-\xi)^2+2\beta|x-\xi|\right)\right)\right.\\&\qquad\left.+\exp(-\beta x)\right).\end{aligned} \tag{4.2.13}$$

Also

$$\begin{aligned}&\left|\hat{Q}_{I,\hat{\kappa},0}\left(x,t,\xi,\vartheta\right)-\hat{Q}_{I,\hat{\kappa},0}\left(x',t,\xi,\vartheta\right)\right|\\&\quad\leq C_{\beta}\left|x-x'\right|^{\alpha}\left(t-\vartheta\right)^{-\frac{1+\alpha}{2}}\left(\exp\left(-\left(\frac{1}{32}c_0\left(x-\xi\right)^2+2\beta\left|x-\xi\right|\right)\right)\right.\\&\qquad+\exp\left(-\left(\frac{1}{32}c_0\left(x'-\xi\right)^2+2\beta\left|x'-\xi\right|\right)\right)\\&\qquad\left.+\exp\left(-\left(\beta-\alpha\right)x\right)\right).\end{aligned}\tag{4.2.14}$$

Proof. It is obvious that the stated bounds hold if

$$|x-\xi|\leq 1.$$

Furthermore, it follows from the bounds given in Lemma 4.2.5, Lemma 4.2.7 and Lemma 4.2.12 that there exists a constant C such that, for every $\zeta\in[0,e^x-1]$ and $(y,t,\xi,\vartheta)\in\mathcal{D}_{\hat{\kappa}}$,

$$\left|\hat{G}_{\hat{L}_\lambda,\hat{\kappa}}\left(\ln\left(e^x-\zeta\right),t,\xi,\vartheta\right)\right|\leq C\left(t-\vartheta\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}\hat{c}_0\left(\ln\left(e^x-\zeta\right)-\xi\right)^2\right).\tag{4.2.15}$$

Thus a simple calculation yields that, if

$$x\leq\xi-1\tag{4.2.16}$$

then for some (other) constants K and C and a constant C_β, depending on β,

$$\begin{aligned}\left|\hat{G}_{\hat{L}_\lambda,\hat{\kappa}}\left(\ln\left(e^x-\zeta\right),t,\xi,\vartheta\right)\right|&\leq K\left(t-\vartheta\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}\hat{c}_0\frac{\left(\ln\left(e^x-\zeta\right)-\xi\right)^2}{t-\vartheta}\right)\\&\leq C\left(t-\vartheta\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}\hat{c}_0\left(x-\xi\right)^2\right).\end{aligned}$$

Assume that

$$x>\xi+1.\tag{4.2.17}$$

Another simple calculation yields that, if (4.2.17) holds and

$$\zeta\leq e^x-e^{\frac{1}{2}(x+\xi)},$$

then

$$\left(\ln\left(e^x-\zeta\right)-\xi\right)^2\geq\frac{1}{4}\left(x-\xi\right)^2,$$

while for any ζ such that

$$\zeta>e^x-e^{\frac{1}{2}(x+\xi)},$$

it follows from the assumed inequalities (4.2.17) and (4.0.38) that

$$\bar{F}\left(\zeta\right)\leq C_\beta e^{-\beta x}.$$

From the inequalities above it is clear that the inequality (4.2.12) holds. The inequalities (4.2.13) and (4.2.14) follow from similar calculations as above and as in the proof of Proposition4.2.6. □

Lemma 4.2.13. *Assume that $\sigma_R > 0$ and that the tail distribution $\bar{F}$ obeys the bound* (4.0.38)*.*

(i) *Let $\alpha \in (0,1)$ and let $\mathcal{G}_k^{\alpha,\frac{\alpha}{2}}$ be the Green function spaces defined in Definition VII.1.1 in Garroni and Menaldi (1992). Then $\hat{Q}_{I,\hat{\kappa},0} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ and $\hat{Q}_{I,\hat{\kappa}} \in \mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$. Moreover $\hat{Q}_{I,\hat{\kappa}}$ is the unique solution in $\mathcal{G}_2^{\alpha,\frac{\alpha}{2}}$ of the integral equation*

$$\begin{aligned}\hat{Q}_{I,\hat{\kappa}}\left(x,t,\xi,\vartheta\right) =& \lambda \int_0^{e^x-1} \hat{G}_{\hat{L}_\lambda,\hat{\kappa}}\left(\ln\left(e^x-\zeta\right),t,\xi,\vartheta\right) dF\left(\zeta\right) \\ &+ \lambda \int_0^{e^x-1} \hat{G}_{I,\hat{\kappa}}\left(\ln\left(e^x-\zeta\right),t,\xi,\vartheta\right) dF\left(\zeta\right).\end{aligned} \tag{4.2.18}$$

(ii) *There exists a constant C_β, depending on β such that, for every $(x,t,\xi,\vartheta) \in \mathcal{D}_{\hat{\kappa}}$, and every $(x',t) \in (x,\hat{\kappa}) \times (\vartheta,t)$*

$$\left|\hat{Q}_{I,\hat{\kappa}}\left(x,t,\xi,\vartheta\right)\right| \le C_\beta \left(t-\vartheta\right)^{-\frac{1}{2}} \exp\left(-\beta\left|x-\xi\right|\right), \tag{4.2.19}$$

$$\left|\hat{Q}_{I,\hat{\kappa}}\left(x,t,\xi,\vartheta\right)\right| \le C_\beta \left(t-\vartheta\right)^{-\frac{1}{2}} \times \Bigg(\exp\left(-2\beta\left|x-\xi\right|\right) + \exp\left(-\beta x\right)\Bigg), \tag{4.2.20}$$

$$\begin{aligned}\left|\hat{Q}_{I,\hat{\kappa}}\left(x,t,\xi,\vartheta\right) - \hat{Q}_{I,\hat{\kappa}}\left(x,t',\xi,\vartheta\right)\right| \le& C_\beta \left(t-t'\right)^{\frac{1}{4}} \left(t'-\vartheta\right)^{-\frac{3}{4}} \\ &\times \Bigg(\exp\left(-2\beta\left|x-\xi\right|\right) \\ &+ \exp\left(-\beta x\right)\Bigg),\end{aligned} \tag{4.2.21}$$

and

$$\begin{aligned}\left|\hat{Q}_{I,\hat{\kappa}}\left(x,t,\xi,\vartheta\right) - \hat{Q}_{I,\hat{\kappa}}\left(x',t,\xi,\vartheta\right)\right| \le& C_\beta \left|x-x'\right|^\alpha \left(t-\vartheta\right)^{-\frac{1+\alpha}{2}} \\ &\times \Bigg(\exp\left(-2\beta\left|x-\xi\right|\right) \\ &+ \exp\left(-2\beta\left|x'-\xi\right|\right) \\ &+ \exp\left(-\left(\beta-\alpha\right)x\right)\Bigg).\end{aligned} \tag{4.2.22}$$

(iii) *For every $(x,t,\xi,\vartheta) \in \mathcal{D}$ and every $l \in \{0,1,2\}$ the following identities are all valid:*

$$\frac{\partial^l \hat{G}_{I,\hat{\kappa}}\left(x,t,\xi,\vartheta\right)}{\partial x^l} = \int_\vartheta \int_0^{\hat{\kappa}} \frac{\partial^l \hat{G}_{\hat{L}_\lambda,\hat{\kappa}}\left(x,t,z,\vartheta\right)}{\partial x^l} \hat{Q}_{I,\hat{\kappa}}\left(z,s,\xi,\vartheta\right) dz ds,$$

and

$$\begin{aligned}\frac{\partial \hat{G}_{I,\hat{\kappa}}\left(x,t,\xi,\vartheta\right)}{\partial t} =& \hat{Q}_{I,\hat{\kappa}}\left(x,t,\xi,\vartheta\right) \\ &+ \int_\vartheta \int_0^{\hat{\kappa}} \frac{\partial \hat{G}_{\hat{L}_\lambda,\hat{\kappa}}\left(x,t,z,\vartheta\right)}{\partial t} \hat{Q}_{I,\hat{\kappa}}\left(z,s,\xi,\vartheta\right) dz ds.\end{aligned}$$

(iv) *There exists a constant* C_β *depending on* β *such that for every* $(x,t,\xi,\vartheta)\in\mathcal{D}$ *and every* $l\in\{0,1\}$ *the following inequalities are all valid:*

$$\left|\frac{\partial^l \hat{G}_{I,\hat{\kappa}}\left(x,t,\xi,\vartheta\right)}{\partial x^l}\right| \le C_\beta\left(t-\vartheta\right)^{\frac{1-l}{2}}\left(\exp\left(-2\beta\left|x-\xi\right|\right)+\exp\left(-\beta x\right)\right),$$

$$\left|\frac{\partial^2 \hat{G}_{I,\hat{\kappa}}\left(x,t,\xi,\vartheta\right)}{\partial x^2}\right| \le C_\beta\left(t-\vartheta\right)^{-\frac{1}{2}}\left(\exp\left(-2\beta\left|x-\xi\right|\right)+\exp\left(-\frac{1}{2}\beta x\right)\right),$$

$$\left|\frac{\partial \hat{G}_{I,\hat{\kappa}}\left(x,t,\xi,\vartheta\right)}{\partial t}\right| \le C_\beta\left(t-\vartheta\right)^{-\frac{1}{2}}\left(\exp\left(-2\beta\left|x-\xi\right|\right)+\exp\left(-\frac{1}{2}\beta x\right)\right)$$

and

$$\begin{aligned}\left|\hat{G}_{I,\hat{\kappa}}\left(x,t,\xi,\vartheta\right)\right| \le C_\beta \min\left(x,\hat{\kappa}-x\right)^{\frac{1}{4}}\left(t-\vartheta\right)^{\frac{1}{4}}\Bigg(&\exp\left(-\frac{1}{2}\beta\left|x-\xi\right|\right)\\ &+\exp\left(-\frac{1}{2}\beta\xi\right)\\ &+\exp\left(-\frac{1}{2}\beta\left(\hat{\kappa}-\xi\right)\right)\Bigg).\end{aligned}$$

(v)

$$\hat{G}_{\hat{A},\hat{\kappa}}\left(x,t,\xi,\vartheta\right)=\hat{G}_{\hat{L}_\lambda,\hat{\kappa}}\left(x,t,\xi,\vartheta\right)+\hat{G}_{I,\hat{\kappa}}\left(x,t,\xi,\vartheta\right).$$

Proof. For part (i): Because of the bounds obeyed by $\hat{G}_{\hat{L}_\lambda,\hat{\kappa}}$ this follows from a similar argument as in the proof of Lemma 4.1.12.

For part (ii): Because of the bound (4.2.8) and the bounds given in Proposition 4.2.7, this follows from similar calculations, based on induction and uniform convergence, as in the proof of Lemma 4.1.12. □

For part (iii) *and part* (iv)*: This follows from similar calculations as in the proofs of Lemma 4.1.5 and Lemma 4.2.12, and using the bounds given in part* (ii) *and the bound* (4.2.8).

For part (v)*: Since* $\hat{G}_{\hat{L}_\lambda,\hat{\kappa}}$ *is the Green function associated with the differential operator*

$$\hat{L}-\lambda,$$

this follows from the bounds and identities given in part (i)-(iv).

Theorem 4.2.3. *Assume that* $\sigma_R>0$ *and that the tail distribution* $\bar{F}$ *satisfies the bound* (4.0.38).

(i) *For every* $(x,t)\in(0,\hat{\kappa})\times(0,1]$ *and every* $l\in\{1,2\}$

$$\frac{\partial^l\hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial x^l}=\int_0^t\int_0^\infty\frac{\partial^l\hat{G}_{\hat{A},\hat{\kappa}}\left(x,t,\xi,\vartheta\right)}{\partial x^l}H_{2,\hat{\kappa}}\left(e^\xi-1,\vartheta\right)d\xi d\vartheta$$

and

$$\frac{\partial\hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial t} = H_{2,\hat{\kappa}}\left(e^{x}-1,\vartheta\right) + \int_0^t\int_0^\infty \frac{\partial\hat{G}_{\hat{A},\hat{\kappa}}\left(x,t,\xi,\vartheta\right)}{\partial t} H_{2,\hat{\kappa}}\left(e^{\xi}-1s,\vartheta\right)d\xi d\vartheta.$$

(ii) *There exists a constant C_β, depending on β, such that, for every $(x,t)\in(0,\hat{\kappa})\times(0,1]$ and every $l\in\{0,1\}$ the following inequalities are all valid*

$$\left|\frac{\partial^l\hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial x^l}\right| \le C_\beta t^{\frac{4-l}{2}}\exp\left(-\beta x\right),$$

$$\left|\frac{\partial^2\hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial x^2}\right| \le C_\beta t\exp\left(-\frac{1}{2}\beta x\right),$$

$$\left|\frac{\partial\hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial t}\right| \le C_\beta t\exp\left(-\frac{1}{2}\beta x\right)$$

and

$$\left|\hat{\psi}_{3,\hat{\kappa}}(x,t)\right| \le C_\beta t\min\left(x,\hat{\kappa}-x\right).$$

(iii) *There exists a constant C_β, depending on β, such that, for every $(y,t)\in(0,\kappa)\times(0,1]$ and every $l\in\{0,1\}$*

$$\left|\frac{\partial^l\psi_{3,\kappa}\left(y,t\right)}{\partial y^l}\right| \le C_\beta t^{\frac{4-l}{2}}\left(1+y\right)^{-(\beta+l)}$$

$$\left|\frac{\partial^2\psi_{3,\kappa}\left(y,t\right)}{\partial y^2}\right| \le C_\beta t\left(1+y\right)^{-\left(\frac{1}{2}\beta+2\right)}$$

and

$$\left|\frac{\partial\psi_{3,\kappa}\left(y,t\right)}{\partial t}\right| \le C_\beta t\left(1+y\right)^{-\frac{1}{2}\beta}$$

(iv) *There exists a constant C and a constant C_β, depending on β, such that, for every $(y,t,)\in(0,\kappa)\times(0,1]$ and $l\in\{0,1\}$,*

$$\left|\psi_{3,\kappa}\left(x,t\right)\right| \le C_\beta t\min\left(y,\kappa-y\right),$$

$$\left|\frac{\partial^l\psi_{\kappa}\left(y,t\right)}{\partial y^l}\right| \le Ct^{-\frac{l}{2}}\exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right) + C_\beta t^{\frac{2-l}{2}}\left(1+y\right)^{-(\beta+l)},$$

$$\left|\frac{\partial^2\psi_{\kappa}\left(y,t\right)}{\partial y^2}\right| \le Ct^{-1}\exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right) + C_\beta\left(1+y\right)^{-\left(\frac{1}{2}\beta+2\right)},$$

and

$$\left|\frac{\partial\psi_{\kappa}\left(y,t\right)}{\partial t}\right| \le Ct^{-1}\exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right) + C_\beta\left(1+y\right)^{-\frac{1}{2}\beta}.$$

Proof. For part (i)-(ii): This can be calculated from the representation formula given in Theorem 4.2.2 the bounds on $H_{2,\kappa}$ given in Lemma 4.1.8 and the bounds on the Green functions given in Lemma 4.2.12 and Lemma 4.2.13.

For part-(iii)-(iv): These bounds follows from the bounds given in part (ii), the bounds already obtained for for $\psi_{1,\kappa}$ and $\psi_{2,\kappa}$, the Middle value theorem and the chain rule. $\square$

5 Existence on an unbounded domain

In this section we will finally prove the existence of a classical solution, except at the origin, of the equation

$$\begin{cases} \psi(y,0)=0, \quad y>0, \\ \psi(0,t)=1, \quad t\in[0,1], \\ \lim_{y\to\infty}\psi(y,t)=0, \quad t\in[0,1], \\ \frac{\partial\psi(y,t)}{\partial t}-A\psi(y,t)=\lambda\bar{F}(y), \quad (y,t)\in(0,\infty)\times(0,1]. \end{cases} \tag{5.0.23}$$

Analogous to what we did in Section 3 we will look for a solution $\psi(y,t)$ of (5.0.23) by considering the three equations

$$\begin{cases} \psi_1(y,0) & =0, \quad y>0, \\ \psi_1(0,t) & =1, \quad t\in[0,1], \\ \lim_{y\to\infty}\psi_1(y,t) & =0, \quad t\in[0,1], \\ \frac{\partial\psi_1(y,t)}{\partial t} & =\frac{1}{2}\sigma_P^2\frac{\partial^2\psi_1(y,t)}{\partial y^2}+p\frac{\partial\psi_1(y,t)}{\partial y}, \quad (y,t)\in(0,\infty)\times(0,1], \end{cases} \tag{5.0.24}$$

$$\begin{cases} \psi_2(y,0) & =0, \quad y>0, \\ \psi_2(0,t) & =0, \quad t\in[0,1], \\ \lim_{y\to\infty}\psi_2(y,t) & =0, \quad t\in[0,1], \\ \frac{\partial\psi_2(y,t)}{\partial t}-L\psi_2 & =H_1(y,t), \quad (y,t)\in(0,\infty)\times(0,1], \end{cases} \tag{5.0.25}$$

where

$$H_1(y,t)=\frac{1}{2}\sigma_R^2y^2\frac{\partial^2\psi_1(y,t)}{\partial^2y^2}+ry\frac{\partial\psi_1(y,t)}{\partial y}-\lambda\psi_1(y,t) + \lambda\int_0^y\psi_1(y-z,t)\,dF(z)+\lambda\bar{F}(y),$$

and

$$\begin{cases} \psi_3(y,0) & =0, \quad y>0, \\ \psi_3(0,t) & =0, \quad t\in[0,1], \\ \lim_{y\to\infty}\psi_3(y,t) & =0, \quad t\in[0,1], \\ \frac{\partial\psi_3 y,t)}{\partial t}-A\psi_3(y,t) & =H_2(y,t), \quad (y,t)\in(0,\infty)\times(0,1], \end{cases} \tag{5.0.26}$$

where

$$H_2(y,t)=-\lambda\psi_2(y,t)+\lambda\int_0^y\psi_2(y-z,t)\,dF(z).$$

As discussed in section (3) we already have a solution for the first equation, given as

$$\psi_1(y,t)=\sqrt{\frac{2}{\pi}}\int_{\frac{y}{\sigma_P\sqrt{t}}}^{\infty}e^{-\frac{s^2}{2}}ds=\frac{y}{\sigma_P\sqrt{2\pi}}\int_0^t s^{-\frac{3}{2}}e^{-\frac{(y+\hat{p})^2}{2\sigma_P^2 s}}ds. \tag{5.0.27}$$

Since we also have the representation formula (3.0.15) we immediately get the regularity result given below.

Lemma 5.0.14. (i) *There exists a constant C such that for every $(y,t) \in (0,\infty)\times(0,1]$, every $l \in \{0,1,2,3\}$ and every $m \in \{0,1,2\}$, the following identity and inequalities are all valid:*

$$\frac{\partial \psi_1(y,t)}{\partial t} = \frac{1}{2}\sigma_P^2 \frac{\partial^2 \psi_1(y,t)}{\partial y^2} + p\frac{\partial \psi_1(y,t)}{\partial y},$$

$$0 < \psi_1(y,t) < 1,$$

$$\left|\frac{\partial^l \psi_1(y,t)}{\partial y^l}\right| \le Ct^{-\frac{l}{2}} \exp\left(-\frac{1}{2}c_0\frac{y^2}{t}\right),$$

$$\left|\frac{\partial \psi_1(y,t)}{\partial t}\right| \le Ct^{-1} \exp\left(-\frac{1}{2}c_0\frac{y^2}{t}\right),$$

$$\left|y^m\frac{\partial^l \psi_1(y,t)}{\partial y^l}\right| \le Ct^{-\frac{l-m}{2}} \exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right),$$

$$\left|\frac{\partial^{1+m} y^m \psi_1(y,t)}{\partial t \partial y^m}\right| \le Ct^{-1} \exp\left(-\frac{1}{4}c_0\frac{y^2}{t}\right).$$

(ii) *There exists a constant C such that for every $(y,t) \in (0,\kappa)\times(0,1]$ and every $l \in \{0,1,2,3\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l \psi_1(y,t)}{\partial y^l} - \frac{\partial^l \psi_{1,\kappa}(y,t)}{\partial y^l}\right| \le C\exp\left(-\frac{1}{4}c_0\frac{\kappa^2}{t}\right),$$

$$\left|\frac{\partial \psi_1(y,t)}{\partial t} - \frac{\partial^l \psi_{1,\kappa}(y,t)}{\partial t}\right| \le C\exp\left(-\frac{1}{4}c_0\frac{\kappa^2}{t}\right).$$

Proof. For (i): This follows from similar calculations as described in Lemma 3.0.1.

For (ii): For every $(y,t) \in (0,\kappa)\times(0,1]$ the symmetry properties of the function $\Gamma_{\sigma_P,p}$ yield that

$$\begin{aligned}\psi_{1,\kappa}(y,t) - \psi_1(y,t) = \sigma_P^2 \Bigg\{ &\int_0^t \left.\frac{\partial \Gamma_{\sigma_P,p}(y-\xi,s,0,0)}{\partial \xi}\right|_{\xi=\kappa} U_\kappa^{(1)}(t-s)\, d\vartheta \\ &- \int_0^t \left.\frac{\partial \Gamma_{\sigma_P,p}(y-\xi,s,0,0)}{\partial \xi}\right|_{\xi=0} \\ &\times \Big(U_\kappa^{(2)}(t-s) - U(t-s)\Big)\, ds \Bigg\},\end{aligned} \tag{5.0.28}$$

from which the stated bounds can be calculated using integration by parts. □

In a way that is analogous to the discussion in section 4 we will need regularity results for the functions $H_1(y,t)$ and $H_2(y,t)$. Because of the result above we immediately get the regularity result below, which is very similar to Proposition 3.0.6.

Lemma 5.0.15. *Assume that the tail distribution $\bar{F}$ satisfies the inequality* (4.0.38)*.*

(i) *There exists a constant C_β, depending on β, such that, for every $(y,t) \in (0,\infty)$, every $y' > 0$, every $t' \in (0,t)$ and every $\alpha \in (0,1]$ the following inequalities are all valid:*

$$\begin{aligned} |H_1(y,t)| &\leq C_\beta \left(1+y\right)^{-\beta}, \\ |H_1(y,t) - H_1(y',t)| &\leq C_\beta \left|y-y'\right|^\alpha t^{-\frac{\alpha}{2}} \left(1+y\right)^{-\beta}, \\ |H_1(y,t) - H_1(y,t')| &\leq C_\beta \left(t-t'\right)^\alpha t'^{-\alpha} \left(1+y\right)^{-\beta}, \\ |H_1(y,t) - H_1(y,t')| &\leq C_\beta \left(t-t'\right)^\alpha t'^{-\alpha} \left(1+y\right)^{-\beta}, \end{aligned}$$

and, for every $\alpha \in \left(0, \min\left(1, \frac{\beta}{2}\right)\right)$, and $x, x' > 0$

$$\left|H_1\left(e^x - 1, t\right) - H_1\left(e^{x'} - 1, t\right)\right| \leq C_\beta \left|x - x'\right|^\alpha t^{-\frac{\alpha}{2}} \exp\left(-\frac{\beta x}{2}\right).$$

(ii) *There exists a constant C, such that, for every $(y,t) \in (0,\kappa)$, every $y' \in (0,\kappa)$, every $t' \in (t,1)$ and every $\alpha \in (0,1]$*

$$\begin{aligned} |(H_1(y,t) - H_{1,\kappa}(y,t))| &\leq C \exp\left(-\frac{1}{4} c_0 \frac{\kappa^2}{t}\right), \\ |(H_1(y,t) - H_{1,\kappa}(y,t)) - (H_1(y',t) - H_{1,\kappa}(y',t))| &\leq C\left|y - y'\right|^\alpha \\ &\quad \times \exp\left(-\frac{1}{8} c_0 \frac{\kappa^2}{t}\right), \\ |(H_1(y,t) - H_{1,\kappa}(y,t)) - (H_1(y,t') - H_{1,\kappa}(y,t'))| &\leq C\left(t - t'\right)^\alpha \\ &\quad \times \exp\left(-\frac{1}{8} c_0 \frac{\kappa^2}{t}\right). \end{aligned}$$

and, for every $x, x' \in (0, \ln(1+\kappa))$,

$$\begin{aligned} &\left|\left(H_1\left(e^x - 1, t\right) - H_{1,\kappa}\left(e^x - 1, t\right)\right) - \left(H_1\left(e^{x'} - 1, t\right) - H_{1,\kappa}\left(e^{x'} - 1, t\right)\right)\right| \\ &\qquad \leq C\left|x - x'\right|^\alpha \exp\left(-\frac{1}{16} c_0 \frac{\kappa^2}{t}\right). \end{aligned}$$

Proof. For (i): This follows from the bounds given in 5.0.14 and similar calculations as in Lemma 3.0.2 and Proposition 4.2.6.

For (ii): This follows from the bounds given in Lemma 5.0.14 and similar calculations as in Lemma 4.1.8 and Proposition 4.2.6. □

5.1 Constant coefficients

In this section we will again assume that $\sigma_R = r = 0$. The main idea is to show that, for any sequence $\{\kappa_n\}_{n=0}^\infty$ such that

$$\lim_{n\to\infty} \kappa_n = \infty,$$

the sequences of functions $\{\psi_{2,\kappa_n}\}_{n=0}^{\infty}$ and $\{\psi_{3,\kappa_n}\}_{n=0}^{\infty}$ and their derivatives converge uniformly to solutions ψ_2 and ψ_3 and their derivatives of equations (5.0.25) and (5.0.26), respectively.

Definition 5.1.1. *For $\xi > 0$ and $t \in (0,1]$ let*

$$V_{\xi,0}(t) := -2\Gamma_{\sigma_P}(0,t,\xi,0),$$

and for $n \in \{0,1,2\dots,\}$, let

$$V_{\xi,n+1}(t) := -2P^{(2)}_{V_{\xi,n}}(0,t).$$

Let

$$U_\xi(t) := \sum_{n=0}^{\infty} V_{\xi,n}(t).$$

Lemma 5.1.1. *Assume that $\sigma_R = r = 0$.*

(i) *U_ξ is differentiable. Furthermore, there exists a constant C such that for every $t \in (0,1]$, every $\xi > 0$ the following identity and inequalities are all valid:*

$$-\frac{1}{2}U_\xi(t) - P^{(2)}_{U_{\xi,n}} = \Gamma_{\sigma_P}(0,t,\xi,0),$$

$$|U_\xi(t)| \le Ct^{-\frac{1}{2}} \exp\left(-\frac{1}{2}c_0\frac{\xi^2}{t}\right)$$

and

$$\left|U'_\xi(t)\right| \le Ct^{-\frac{3}{2}} \exp\left(-\frac{1}{2}c_0\frac{\xi^2}{t}\right).$$

(ii) *There exists a constant C such that for every $t \in (0,1]$, every $\xi \in (0,\kappa)$*

$$\left|U_\xi(t) - U^{(2)}_{\xi,\kappa}(t)\right| \le C\exp\left(-\frac{1}{4}c_0\frac{\kappa^2}{t}\right),$$

and

$$\left|U'_\xi(t) - U^{(2)\prime}_{\xi,\kappa}(t)\right| \le C\exp\left(-\frac{1}{4}c_0\frac{\kappa^2}{t}\right).$$

(iii) *For every fixed $\xi > 0$*

$$g^*_{L_0}(y,t) := -P^{(2)}_{U_\xi}(y,t)$$

is a classical solution of the PDE

$$\begin{cases} g^*_{L_0}(y,0,\xi) & = 0, \quad y>0, \\ g^*_{L_0}(0,t,\xi) & = \Gamma_{\sigma_P}(0,t,\xi,0), \quad t\in(0,1], \\ \lim_{y\to\infty} g^*_{L_0}(y,t,\xi) & = 0, \\ \frac{\partial g^*_{L_0}(y,t,\xi)}{\partial t} & = Lg^*_{L_0}(y,t,\xi), \quad (y,t)\in(0,\infty)\times(0,1], \end{cases}$$

Proof. For parts (i) and (ii): We observe that

$$V_{\xi,0}(t) = V^{(2)}_{\xi,0,\kappa}(t).$$

The stated identity and inequalities follow from similar calculations, based on induction and uniform convergence, as in Lemma 4.1.2.

For part (iii): Let $\{\kappa_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that

$$\lim_{n\to\infty} \kappa_n = \infty,$$

and consider the sequence of functions

$$\left\{g^*_{L_0,\kappa_n}(y,t,\xi)\right\}_{n=0}^{\infty}.$$

It follows from the bounds given in Lemma 4.1.3 that there exists a constant C such that, for any n such that $\kappa_n > \xi + 1$

$$\left|g^*_{L_0,\kappa_n}(y,t,\xi)\right| \le C\left(1+\xi^{-2}\right)\exp\left(-\frac{1}{8}c_0\frac{\xi^2}{t}\right).$$

Thus, $g^*_{L_0,\kappa_n}$ satisfies the initial condition. Also, for any $t_0 \in [0,1]$

$$\begin{aligned}\lim_{(y,t)\to(0,t_0)} g^*_{L_0}(y,t,\xi) &= \lim_{n\to\infty}\lim_{(y,t)\to(0,t_0)} g^*_{L_0,\kappa_n}(y,t,\xi)\\ &= \begin{cases}\Gamma_{\sigma_P}(0,t_0,\xi,0), & t_0>0\\ 0, & t_0=0.\end{cases}\end{aligned}$$

The uniqueness follows from Theorem I.3.1 in Garroni and Menaldi (1992) and similar arguments as in the proof of Theorem 3.0.1. □

Definition 5.1.2. *For every $y,\xi>0$ and $0\le\vartheta<t\le1$ let*

$$g_{L_0}(y,t,\xi,\vartheta) = -P^{(2)}_{U_\xi}(y,t-\vartheta).$$

Lemma 5.1.2. *Assume that $\sigma_R = r = 0$.*

(i) *There exists a constant C such that for every $y,\xi>0$ and $0\le\vartheta<t\le1$ and $l\in\{0,1,2\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l g_{L_0}(y,t,\xi,\vartheta)}{\partial y^l}\right| \le C(t-\vartheta)^{-\frac{1+l}{2}}\exp\left(-\frac{1}{2}c_0\frac{y^2+\frac{1}{2}\xi^2}{t-\vartheta}\right)$$

and

$$\left|\frac{\partial g_{L_0}(y,t,\xi,\vartheta)}{\partial t}\right| \le C(t-\vartheta)^{-\frac{3}{2}}\exp\left(-\frac{1}{2}c_0\frac{y^2+\frac{1}{2}\xi^2}{t-\vartheta}\right).$$

(ii) *There exists a constant C such that for every $(y,t,\xi,\vartheta)\in\mathcal{D}_\kappa$ and $l\in\{0,1,2\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l g_{L_0}(y,t,\xi,\vartheta)}{\partial y^l} + \frac{\partial^l}{\partial y^l}P^{(2)}_{U^{(2)}_{\xi,\kappa}}(y,t-\vartheta)\right| \le C(t-\vartheta)^{-\frac{l}{2}}\exp\left(-\frac{1}{8}c_0\frac{\kappa^2+\xi^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial g_{L_0}(y,t,\xi,\vartheta)}{\partial t} + \frac{\partial}{\partial t}P^{(2)}_{U^{(2)}_{\xi,\kappa}}(y,t-\vartheta)\right| \le C(t-\vartheta)^{-\frac{l}{2}}\exp\left(-\frac{1}{8}c_0\frac{\kappa^2+\xi^2}{t-\vartheta}\right),$$

$$\begin{aligned}&\left|\frac{\partial^l g_{L_0}(y,t,\xi,\vartheta)}{\partial y^l} - \frac{\partial^l g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}\right|\\ &\le Ct^{-\frac{1+l}{2}}\exp\left(-\frac{1}{2}c_0\frac{(\kappa-y)^2+\frac{1}{2}(\kappa-\xi)^2}{t-\vartheta}\right),\end{aligned}$$

and

$$\left|\frac{\partial g_{L_0}(y,t,\xi,\vartheta)}{\partial t}-\frac{\partial g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial t}\right|$$
$$\leq Ct^{-\frac{3}{2}}\exp\left(-\frac{1}{2}c_0\frac{(\kappa-y)^2+\frac{1}{2}(\kappa-\xi)^2}{t-\vartheta}\right).$$

(iii) *There exists a constant C such that for every $y,\xi>0$ and $0\leq\vartheta<t\leq 1$ the following inequalities are all valid:*

$$\int_0^t\int_\kappa^\infty\left|\frac{\partial^2 g_{L_0}(y,t,\xi,\vartheta)}{\partial y^2}\right|d\xi d\vartheta\leq Ct\exp\left(-\frac{1}{4}c_0\frac{\kappa^2}{t}\right),$$

and

$$\int_0^t\int_\kappa^\infty\left|\frac{\partial g_{L_0}(y,t,\xi,\vartheta)}{\partial t}\right|d\xi d\vartheta\leq Ct\exp\left(-\frac{1}{4}c_0\frac{\kappa^2}{t}\right).$$

(iv) *There exists a constant C such that for every $(y,t,\xi,\vartheta)\in\mathcal{D}_\kappa$ and $l\in\{0,1,2\}$*

$$\left|\int_0^t\int_0^\kappa\left(\frac{\partial^l g_{L_0}(y,t,\xi,\vartheta)}{\partial y^l}-\frac{\partial^l g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}\right)d\xi d\vartheta\right|$$
$$\leq Ct^{\frac{2-l}{2}}\exp\left(-\frac{1}{2}c_0\frac{(\kappa-y)^2}{t}\right),$$

and such that

$$\left|\int_0^t\int_0^\kappa\left(\frac{\partial g_{L_0}(y,t,\xi,\vartheta)}{\partial t}-\frac{\partial g_{L_0,\kappa}(y,t,\xi,\vartheta)}{\partial t}\right)d\xi d\vartheta\right|$$
$$\leq C\exp\left(-\frac{1}{2}c_0\frac{(\kappa-y)^2}{t}\right).$$

Proof. This follows from similar calculations as in the proof of Lemma 4.1.3 and also the bounds given in Lemma 4.1.3. □

Definition 5.1.3. *For every $y,\xi>0$ and $0\leq\vartheta<t\leq 1$ let*

$$G_{L_0}(y,t,\xi,\vartheta)=\Gamma_{\sigma_P}(y,t,\xi,\vartheta)-g_{L_0}(y,t,\xi,\vartheta),$$

let

$$Q_0(y,t,\xi,\vartheta):=p\frac{\partial G_{L_0}(y,t,\xi,\vartheta)}{\partial x},$$

let the sequence of function $\{Q_n\}_{n=0}^\infty$ be defined inductively for $n\in 1,2,\ldots,$ by

$$Q_n(y,t,\xi,\vartheta)=\int_\vartheta^t\int_0^\infty Q_0(y,t,z,s)\,Q_{n-1}(z,s,\xi,\vartheta)\,dzds,$$

and let

$$Q(y,t,\xi,\vartheta)=\sum_{n=0}^\infty Q_n(y,t,\xi,\vartheta).$$

Lemma 5.1.3. *Assume that $\sigma_R > 0$ and let $\alpha \in (0,1)$.*

(i) *Q solves the integral equation*

$$Q(y,t,\xi,\vartheta) = Q_0(y,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\infty Q_0(y,t,z,s)\, Q(z,s,\xi,\vartheta)\, dzds. \tag{5.1.1}$$

(ii) *There exists a constant C, such that, for every $y, y', \xi > 0$ and every $0 \le \vartheta < t' < t \le 1$ the following identities and inequalities are all valid:*

$$Q(y,t,\xi,\vartheta) = Q(y,t-\vartheta,\xi,0),$$

$$|Q(y,t,\xi,\vartheta)| \le C(t-\vartheta)^{-1} \exp\left(-\frac{1}{4}c_0 \frac{(y-\xi)^2}{t-\vartheta}\right),$$

$$|Q(y,t,\xi,\vartheta) - Q(y',t,\xi,\vartheta)| \le C|y-y'|^{\frac{1}{2}}(t-\vartheta)^{-\frac{5}{4}} \times \exp\left(-\frac{1}{4}c_0 \frac{(y-\xi)^2}{t-\vartheta}\right),$$

and

$$|Q(y,t,\xi,\vartheta) - Q(y,t',\xi,\vartheta)| \le C|t-t'|^{\frac{1}{4}}(t'-\vartheta)^{-\frac{5}{4}} \times \exp\left(-\frac{1}{4}c_0 \frac{(y-\xi)^2}{t-\vartheta}\right),$$

(iii) *There exists a constant C, such that, for every $(y,t,\xi,\vartheta) \in \bar{\mathcal{D}}_\kappa$, every $y', \in (0,\kappa)$, and every $t' \in (0,t,)$ the following identities and inequalities are all valid:*

$$|Q(y,t,\xi,\vartheta) - Q_\kappa(y,t,\xi,\vartheta)| \le C(t-\vartheta)^{-1} \times \exp\left(-\frac{1}{4}c_0 \frac{(\kappa-y)^2 + (\kappa-\xi)^2}{t-\vartheta}\right),$$

$$|Q(y,t,\xi,\vartheta) - Q(y',t,\xi,\vartheta) - (Q_\kappa(y,t,\xi,\vartheta) - Q_\kappa(y',t,\xi,\vartheta))| \le C|y-y'|^{\frac{1}{2}}(t-\vartheta)^{-\frac{5}{4}} \exp\left(-\frac{1}{8}c_0 \frac{(\kappa-y)^2 + (\kappa-\xi)^2}{t-\vartheta}\right),$$

and

$$|Q(y,t,\xi,\vartheta) - Q(y,t',\xi,\vartheta) - (Q_\kappa(y,t,\xi,\vartheta) - Q_\kappa(y,t',\xi,\vartheta))| \le C|t-t'|^{\frac{1}{4}}(t-\vartheta)^{-\frac{5}{4}} \exp\left(-\frac{1}{8}c_0 \frac{(\kappa-y)^2 + (\kappa-\xi)^2}{t-\vartheta}\right),$$

Proof. This follows from similar calculations and reasoning as in the proofs of Lemma 4.1.4. □

Proposition 5.1.1. (i) *For every* $y, \xi > 0$ *and* $0 \leq \vartheta < t \leq 1$ *and* $l \in \{0,1,2\}$

$$\begin{aligned}\frac{\partial^l}{\partial y^l} & \int_\vartheta^t \int_0^\infty G_{L_0}(y,t,z,s)\, Q(z,s,\xi,\vartheta)\, dzds \\ &= \int_\vartheta^t \int_0^\infty \frac{\partial^l G_{L_0}(y,t,z,s)}{\partial y^l} Q(z,s,\xi,\vartheta)\, dzds,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t} & \int_\vartheta^t \int_0^\infty G_{L_0}(y,t,z,s)\, Q(z,s,\xi,\vartheta)\, dzds \\ &= Q(y,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\infty \frac{\partial G_{L_0}(y,t,z,s)}{\partial t} Q(z,s,\xi,\vartheta)\, dzds.\end{aligned}$$

(ii) *There exists a constant* C *such that for every* $y, \xi > 0$ *and every* $0 \leq \vartheta < t \leq 1$

$$\begin{aligned}\left| \frac{\partial^l}{\partial y^l} \int_\vartheta^t \int_0^\infty G_{L_0}(y,t,z,s)\, Q(z,s,\xi,\vartheta)\, dzds \right| \\ \leq C(t-\vartheta)^{-\frac{l}{2}} \exp\left(-\frac{1}{4} c_0 \frac{(y-\xi)^2}{t-\vartheta} \right),\end{aligned}$$

and

$$\begin{aligned}\left| \frac{\partial}{\partial t} \int_\vartheta^t \int_0^\infty G_{L_0}(y,t,z,s)\, Q(z,s,\xi,\vartheta)\, dzds \right| \\ \leq C(t-\vartheta)^{-1} \exp\left(-\frac{1}{4} c_0 \frac{(y-\xi)^2}{t-\vartheta} \right),\end{aligned}$$

(iii) *There exists a constant* C *such that for every* $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ *and every* $l \in \{0,1,2\}$

$$\begin{aligned}\Bigg| \frac{\partial^l}{\partial y^l} \int_\vartheta^t \Bigg(& \int_0^\infty G_{L_0}(y,t,z,s)\, Q(z,s,\xi,\vartheta)\, dzds \\ & - \int_0^\kappa G_{L_0,\kappa}(y,t,z,s)\, Q_\kappa(z,s,\xi,\vartheta)\, dz \Bigg) \Bigg| \\ & \leq C(t-\vartheta)^{-\frac{l}{2}} \exp\left(-\frac{1}{8} c_0 \frac{(\kappa-y)^2 + (\kappa-\xi)^2}{t-\vartheta} \right),\end{aligned}$$

and

$$\begin{aligned}\Bigg| \frac{\partial}{\partial t} \int_\vartheta^t \Bigg(& \int_0^\infty G_{L_0}(y,t,z,s)\, Q(z,s,\xi,\vartheta)\, dz \\ & - \int_0^\kappa G_{L_0,\kappa}(y,t,z,s)\, Q_\kappa(z,s,\xi,\vartheta)\, dz \Bigg) ds \Bigg| \\ & \leq C(t-\vartheta)^{-1} \exp\left(-\frac{1}{8} c_0 \frac{(\kappa-y)^2 + (\kappa-\xi)^2}{t-\vartheta} \right),\end{aligned}$$

Proof. Because of the regularity bounds obeyed by Q, given in Lemma 5.1.3, this follows from similar calculations as in the proof of Lemma 4.1.5. □

Definition 5.1.4. *Let*

$$G_L(y,t,\xi,\vartheta) := G_{L_0}(y,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\infty G_{L_0}(y,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dzds,$$

and let

$$\psi_2(y,t) := \int_0^t \int_0^\infty G_L(y,t,\xi,\vartheta)\,H_{1,\kappa}(y,\vartheta)\,d\xi d\vartheta.$$

Theorem 5.1.1. *Assume that $\sigma_R = r = 0$ and that the tail distribution $\bar{F}$ satisfies the inequality* (4.0.38).

(i) *For every $y > 0$, every $t \in (0,1]$ and every $l \in \{1,2\}$*

$$\frac{\partial^l \psi_2(y,t)}{\partial y^l} = \int_0^t \int_0^\infty \frac{\partial^l G_L(y,t,\xi,\vartheta)}{\partial y^l} H_1(\xi)\,d\xi d\vartheta$$

and

$$\frac{\partial \psi_2(y,t)}{\partial t} = H_1(y,t) + \int_0^t \int_0^\infty \frac{\partial G_L(y,t,\xi,\vartheta)}{\partial t} H_1(\xi,\vartheta)\,d\xi d\vartheta.$$

(ii) *There exists a constant C_β, depending on β, such that, for every $y > 0$ and $t \in (0,1]$ and every $l \in \{0,1,2\}$ the following bounds are all valid:*

$$\left|\frac{\partial^l \psi_2(y,t)}{\partial y^l}\right| \le C_\beta t^{\frac{2-l}{2}}(1+y)^{-\beta},$$

$$\left|\frac{\partial \psi_2(y,t)}{\partial t}\right| \le C_\beta (1+y)^{-\beta}$$

and, for every $y' > y$, $t' \in (0,t)$ and $\alpha \in (0,1]$

$$|\psi_2(y,t) - \psi_2(y',t)| \le C_\beta (y'-y)^\alpha t^{\frac{2-\alpha}{2}}(1+y)^{-\beta},$$

and

$$|\psi_2(y,t) - \psi_2(y,t')| \le C_\beta (t-t')^\alpha t^{1-\alpha}(1+y)^{-\beta}.$$

(iii) *There exists a constant C_β, depending on β, such that, for every $(y,t) \in (0,\kappa) \times (0,1]$ and every $l \in \{0,1,2\}$ the following bounds are all valid:*

$$\left|\frac{\partial^l \psi_2(y,t)}{\partial y^l} - \frac{\partial^l \psi_{2,\kappa}(y,t)}{\partial y^l}\right| \le C_\beta t^{\frac{2-l}{2}}(1+\kappa)^{-\beta}\exp\left(-\frac{1}{16}c_0\frac{(\kappa-y)^2}{t}\right),$$

$$\left|\frac{\partial \psi_2(y,t)}{\partial t}\right| \le C_\beta (1+\kappa)^{-\beta}\exp\left(-\frac{1}{16}c_0\frac{(\kappa-y)^2}{t}\right).$$

Also, for every $y' \in (y, \kappa)$, $t' \in (0,t)$ and $\alpha \in (0,1]$

$$\begin{aligned}&\left|\left(\psi_2(y,t) - \psi_{2,\kappa}(y,t)\right) - \left(\psi_2(y',t) - \psi_{2,\kappa}(y',t)\right)\right| \\ &\quad \le C_\beta (y'-y)^\alpha t^{\frac{2-\alpha}{2}} (1+\kappa)^{-\beta} \exp\left(-\frac{1}{16} c_0 \frac{(\kappa - y')^2}{t}\right),\end{aligned}$$

and

$$\begin{aligned}&\left|\left(\psi_2(y,t) - \psi_{2,\kappa}(y,t)\right) - \left(\psi_2(y,t') - \psi_{2,\kappa}(y,t')\right)\right| \\ &\quad \le C_\beta (t-t')^\alpha t^{1-\alpha} (1+\kappa)^{-\beta} \exp\left(-\frac{1}{16} c_0 \frac{(\kappa - y)^2}{t}\right).\end{aligned}$$

(iv) *$\psi_2(y,t)$ is the unique classical solution of the PDE* (5.0.25)*.*

Proof. For (i) and (ii): These follow from the regularity bounds obeyed by $H_1(y,t)$ and $H_1(y,t) - H_{1,\kappa}(y,t)$, given in Lemma 5.0.15 and similar calculations as in the proof of Lemma 4.1.5.

For (iii): This can be calculated from the bounds given in Lemma 5.1.2, that are obeyed by $H_1(y,t) - H_{1,\kappa}(y,t)$, and examining the three cases

$$\xi \le \frac{1}{2}\kappa,$$

$$\frac{1}{2}\kappa < \xi \le \kappa,$$

and

$$\xi > \kappa.$$

For (iv): It follows from Lemma 5.1.1 and part (i) that $\psi_2(y,t)$ satisfies the equation (5.0.25) on the inner domain. It follows from part (ii) that $\psi_2(y,t)$ satisfies the asymptotic boundary condition. Similar reasoning as in the proof of Lemma 5.1.1 yields that, for every $t_0 \in (0,1]$,

$$\lim_{(y,t)\to(0,t_0)} \psi_2(y,t) = 0.$$

□

Definition 5.1.5. *For $y > 0$ and $t \in (0,1]$ let*

$$H_2(y,t) := -\lambda \psi_2(y,t) + \lambda \int_0^y \psi_2(y-z,t)\, dF(z).$$

Proposition 5.1.2. *Assume that $\sigma_R = r = 0$ and that the tail distribution $\bar{F}$ satisfies the inequality* (4.0.38)*.*

(i) *Then there exists a constant C_β, depending on β, such that, for every $y > 0$, $t, \alpha \in (0,1]$ and $y' > 0$*

$$|H_2(y,t)| \le C_\beta (1+y)^{-\beta},$$
$$|H_2(y,t) - H_2(y',t)| \le C_\beta (y'-y)^\alpha t^{\frac{2-\alpha}{2}} (1+y)^{-\beta},$$

and such that

$$|H_2(y,t) - H_2(y,t')| \leq C_\beta \left(t-t'\right)^\alpha {t'}^{1-\alpha} \left(1+y\right)^{-\beta}.$$

(ii) *There exists a constant C_β, depending on β, such that, for every $(y,t) \in (0,\kappa) \times (0,1]$ every $(y',t') \in (y,\kappa) \times (0,t)$ and $\alpha \in (0,1]$*

$$|H_2(y,t) - H_{2,\kappa}(y,t)| \leq C_\beta t \left(1+\kappa\right)^{-\beta} \exp\left(-\frac{1}{16} c_0 \frac{\left(\kappa-y\right)^2}{t}\right),$$

$$\begin{aligned}&|(H_2(y,t) - H_{2,\kappa}(y,t)) - (H_2(y',t) - H_{2,\kappa}(y',t))| \\ &\quad \leq C_\beta \left(y'-y\right)^\alpha t^{\frac{2-\alpha}{2}} \left(1+\kappa\right)^{-\beta} \exp\left(-\frac{1}{16} c_0 \frac{\left(\kappa-y'\right)^2}{t}\right),\end{aligned}$$

and

$$\begin{aligned}&|(H_2(y,t) - H_{2,\kappa}(y,t)) - (H_2(y,t') - H_{2,\kappa}(y,t'))| \\ &\quad \leq C_\beta \left(t-t'\right)^\alpha t^{1-\alpha} \left(1+\kappa\right)^{-\beta} \exp\left(-\frac{1}{16} c_0 \frac{\left(\kappa-y\right)^2}{t}\right).\end{aligned}$$

Proof. This follows from the bounds given in Theorem 5.1.1 and similar calculations as in Lemma 4.1.8. □

We will now proceed to show existence of the equation (5.0.25) analogous to the results in Section 4.1.2.

Definition 5.1.6. *Let*

$$Q_{\lambda,0}\left(y,t,\xi,\vartheta\right) = -\lambda G_L\left(y,t,\xi,\vartheta\right),$$

and let the sequence of functions $\{Q_{\lambda,n}\}_{n=0}^\infty$ be defined inductively for $n \in 1,2,\ldots,$ and $y,\xi > 0$ and $0 \leq \vartheta < t \leq 1$ by

$$Q_{\lambda,n}\left(y,t,\xi,\vartheta\right) = \int_\vartheta^t \int_0^\infty Q_{\lambda,0}\left(y,t,z,s\right) Q_{\lambda,n-1}\left(z,s,\xi,\vartheta\right) dz ds,$$

and let

$$Q_\lambda\left(y,t,\xi,\vartheta\right) = \sum_{n=0}^\infty Q_{\lambda,n}\left(y,t,\xi,\vartheta\right).$$

Lemma 5.1.4. (i) *Q_λ is a solution of the integral equation*

$$\begin{aligned}Q_\lambda\left(y,t,z,\vartheta\right) &= -\lambda G_L\left(y,t,\xi,\vartheta\right) \\ &\quad - \lambda \int_\vartheta^t \int_0^\infty G_L\left(y,t,z,s\right) Q_\lambda\left(z,s,\xi,\vartheta\right) dz ds.\end{aligned} \tag{5.1.2}$$

(ii) *$Q_\lambda\left(y,t,\xi,\vartheta\right)$ is differentiable with respect to y,t and ϑ on $(0,\infty) \times (0,1] \times (0,\infty) \times [0,t)$. Furthermore, there exists a constant C such that for every*

$y, \xi > 0$ and $0 \leq \vartheta < t \leq 1$ the following identity and inequalities are all valid for $l \in \{0,1\}$

$$Q_\lambda(y,t,\xi,\vartheta) = Q_\lambda(y,t-\vartheta,\xi,0),$$

$$\left|\frac{\partial^l Q_\lambda(y,t,\xi,\vartheta)}{\partial y^l}\right| \leq C(t-\vartheta)^{-\frac{1+l}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial Q_\lambda(y,t,\xi,\vartheta)}{\partial t}\right| \leq C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right).$$

(iii) *There exists a constant C such that for every $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ and every $l \in \{0,1\}$ the following inequalities are all valid:*

$$\begin{aligned}\left|\frac{\partial^l Q_\lambda(y,t,\xi,\vartheta)}{\partial y^l} - \frac{\partial Q_{\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}\right| &\leq C(t-\vartheta)^{-\frac{1+l}{2}} \\ &\times \exp\left(-\frac{1}{8}c_0\frac{(\kappa-y)^2}{t-\vartheta}\right) \\ &\times \exp\left(-\frac{1}{8}c_0\frac{(\kappa-\xi)^2}{t-\vartheta}\right),\end{aligned} \tag{5.1.3}$$

and

$$\begin{aligned}\left|\frac{\partial Q_\lambda(y,t,\xi,\vartheta)}{\partial t} - \frac{\partial Q_{\lambda,\kappa}(y,t,\xi,\vartheta)}{\partial t}\right| &\leq C(t-\vartheta)^{-\frac{3}{2}} \\ &\times \exp\left(-\frac{1}{8}c_0\frac{(\kappa-y)^2}{t-\vartheta}\right) \\ &\times \exp\left(-\frac{1}{8}c_0\frac{(\kappa-\xi)^2}{t-\vartheta}\right).\end{aligned} \tag{5.1.4}$$

Proof. For part (i): Similar calculations as in Lemma 4.1.4, based on induction and uniform convergence yield that the inequalities given in part (i) are all valid. In particular, it can be shown that there exists a sequence $\{k_n\}_{n=0}^{\infty}$ of positive constants such that $\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0$ and such that

$$|Q_{\lambda,n}(y,t,\xi,\vartheta)| \leq k_n (t-\vartheta)^{\frac{2n-1}{2}} \exp\left(-\frac{1}{4}c_0\frac{(y-\xi)^2}{t-\vartheta}\right).$$

Also by induction it can be shown that, for every $n \in 1,2,\ldots$,

$$\begin{aligned}\sum_{j=0}^{n} Q_{\lambda,j}(y,t,z,\vartheta) &= -\lambda G_L(y,t,\xi,\vartheta) \\ &\quad - \lambda \int_\vartheta^t \int_0^\infty G_L(y,t,z,s) \sum_{j=0}^{n} Q_{\lambda,j}(z,s,\xi,\vartheta)\, dz ds \\ &\quad + \lambda \int_\vartheta^t \int_0^\infty G_L(y,t,z,s)\, Q_{\lambda,n}(z,s,\xi,\vartheta)\, dz ds.\end{aligned}$$

Because of the bounds obeyed by $Q_{\lambda,n}(z,s,\xi,\vartheta)$ it follows that

$$\sum_{j=0}^{n} Q_{\lambda,j}(y,t,\xi,\vartheta)$$

converges uniformly to a solution of the integral equation (5.1.2).

For part (ii): It follows from the regularity bounds obeyed by the Green function $G_L(y,t,\xi,\vartheta)$ derived in section 4.1.1 that the stated bounds hold for $Q_{\lambda,0}(y,t,\xi,\vartheta)$ and $Q_{\lambda,0}(y,t,\xi,\vartheta)-Q_{\lambda,\kappa,0}(y,t,\xi,\vartheta)$. Similar calculations, based on induction and uniform convergence as in the proof of Lemma 4.1.4 yields that these bounds also hold for the limits $Q_\lambda(y,t,\xi,\vartheta)$ and $Q_\lambda(y,t,\xi,\vartheta)-Q_{\lambda,\kappa}(y,t,\xi,\vartheta)$.

For part (iii): We first note that it follows from the bounds given in that, for some constant C, $(y,t,\xi,\vartheta)\in\mathcal{D}_\kappa$ and $l\in\{0,1\}$

$$\left|\frac{\partial^l Q_{\lambda,0}(y,t,\xi,\vartheta)}{\partial y^l}-\frac{\partial^l Q_{\lambda,0,\kappa}(y,t,\xi,\vartheta)}{\partial y^l}\right|\le C(t-\vartheta)^{-\frac{1+l}{2}} \times\exp\left(-\frac{1}{2}c_0\frac{(\kappa-y)^2+\frac{1}{2}(\kappa-\xi)^2}{t-\vartheta}\right).$$

Furthermore, an application of Proposition 3.0.2 yields that, for some constant C

$$\int_\kappa^\infty \exp\left(-\frac{1}{4}c_0\left(\frac{(y-z)^2}{t-s}+\frac{(z-\xi)^2}{s-\vartheta}\right)\right)dz \le C\exp\left(-\frac{1}{8}c_0\frac{(\kappa-y)^2+(\kappa-\xi)^2}{t-\vartheta}\right)(t-\vartheta)^{-\frac{1}{2}}(t-s)^{\frac{1}{2}}(s-\vartheta)^{\frac{1}{2}}.$$

Similar calculations as in Lemma 4.1.4, based on induction, uniform convergence the symmetry property between there t and ϑ and the bound above, yield that the stated bounds (5.1.3) and (5.1.4) all hold. □

Definition 5.1.7. *For $y,\xi>0$ and $0\le\vartheta<t\le1$ let*

$$G_{L_\lambda}(y,t,\xi,\vartheta):=G_L(y,t,\xi,\vartheta)+\int_\vartheta^t\int_0^\infty G_L(y,t,z,s)\,Q_\lambda(z,s,\xi,\vartheta)\,dzds.$$

Lemma 5.1.5. *Assume that $\sigma_R=r=0$.*

(i) *For every $y,\xi>0$ and $0\le\vartheta<t\le1$ and $l\in\{1,2\}$*

$$\frac{\partial^l}{\partial y^l}\int_\vartheta^t\int_0^\infty G_L(y,t,z,s)\,Q_\lambda(z,s,\xi,\vartheta)\,dzds =\int_\vartheta^t\int_0^\infty\frac{\partial^l G_L(y,t,z,s)}{\partial y^l}Q_\lambda(z,s,\xi,\vartheta)\,dzds,$$

and

$$\frac{\partial}{\partial t}\int_\vartheta^t\int_0^\infty G_L(y,t,z,s)\,Q_\lambda(z,s,\xi,\vartheta)\,dzds =Q_\lambda(y,t,\xi,\vartheta)+\int_\vartheta^t\int_0^\infty\frac{\partial G_L(y,t,z,s)}{\partial t}Q_\lambda(z,s,\xi,\vartheta)\,dzds.$$

(ii) *There exists a constant C such that, for every $y, \xi > 0$ and $0 \le \vartheta < t \le 1$ the following inequalities are all valid:*

$$\left| \frac{\partial^l}{\partial y^l} \int_\vartheta^t \int_0^\infty G_L(y,t,z,s)\, Q_\lambda(z,s,\xi,\vartheta)\, dz ds \right| \le C (t-\vartheta)^{\frac{1-l}{2}} \exp\left(-\frac{1}{4} c_0 \frac{(y-\xi)^2}{t-\vartheta} \right),$$

and

$$\left| \frac{\partial}{\partial t} \int_\vartheta^t \int_0^\infty G_L(y,t,z,s)\, Q_\lambda(z,s,\xi,\vartheta)\, dz ds \right| \le C (t-\vartheta)^{-\frac{1}{2}} \exp\left(-\frac{1}{4} c_0 \frac{(y-\xi)^2}{t-\vartheta} \right).$$

(iii) *There exists a constant C such that, for every $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ and $l \in \{0,1,2\}$ the following inequalities are all valid:*

$$\left| \frac{\partial^l}{\partial y^l} \int_\vartheta^t \left(\int_0^\infty G_L(y,t,z,s)\, Q_\lambda(z,s,\xi,\vartheta)\, dz - \int_0^\infty G_{L,\kappa}(y,t,z,s)\, Q_\lambda(z,s,\xi,\vartheta)\, dz \right) ds \right| \le C (t-\vartheta)^{\frac{1-l}{2}} \exp\left(-\frac{1}{2} c_0 \frac{(\kappa-y)^2 + \frac{1}{2}(\kappa-\xi)^2}{t-\vartheta} \right),$$

and

$$\left| \frac{\partial}{\partial t} \int_\vartheta^t \left(\int_0^\infty G_L(y,t,z,s)\, Q_\lambda(z,s,\xi,\vartheta)\, dz - \int_0^\infty G_{L,\kappa}(y,t,z,s)\, Q_\lambda(z,s,\xi,\vartheta)\, dz \right) ds \right| \le C (t-\vartheta)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} c_0 \frac{(\kappa-y)^2 + \frac{1}{2}(\kappa-\xi)^2}{t-\vartheta} \right).$$

Proof. This follows from the bounds given in Lemma 5.1.4 and Lemma 5.1.2 and similar calculations as in the proof of Lemma 4.1.5. □

Definition 5.1.8. *Let*

$$Q_{I,0}(y,t,\xi,\vartheta) := \lambda \int_0^y G_{L_\lambda}(y-\zeta,t,\xi,\vartheta)\, dF(\zeta).$$

Let the sequence of functions

$$\{Q_{I,n}(y,t,\xi,\vartheta)\}_{n=0}^\infty$$

be defined inductively by

$$Q_{I,n}(y,t,\xi,\vartheta) = \int_\vartheta^t \int_0^\infty Q_{I,0}(y,t,z,s)\, Q_{I,n-1}(z,s,\xi,\vartheta)\, dz ds, \quad n \in 1,2,\ldots.$$

Let

$$Q_I(y,t,\xi,\vartheta) = \sum_{n=0}^{\infty} Q_{I,n}(y,t,\xi,\vartheta).$$

Let

$$G_{I_\lambda,n}(y,t,\xi,\vartheta) := \int_\vartheta^t \int_0^\infty G_{L_\lambda}(y,t,z,s)\, Q_{I,n}(z,s,\xi,\vartheta)\, dzds,$$

and let

$$G_{I_\lambda}(y,t,\xi,\vartheta) := \int_\vartheta^t \int_0^\infty G_{L_\lambda}(y,t,z,s)\, Q_I(z,s,\xi,\vartheta)\, dzds.$$

Lemma 5.1.6. *Assume that $\sigma_R = r = 0$ and that the inequality (4.0.38) holds.*

(i) *There exists a sequence $\{k_n\}_{n=0}^{\infty}$ such that*

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0,$$

and such that, for every finite $n \in 0,1,\ldots,$ every $y,\xi > 0$ and $0 \le \vartheta < t \le 1$,

$$\begin{aligned} |Q_{I,n}(y,t,\xi,\vartheta)| &\le k_n (t-\vartheta)^{n-\frac{1}{2}} \\ &\times \int_0^\infty \ldots \int_0^\infty \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right) \\ &\times dF(\zeta_0)dF(\zeta_1),\ldots,dF(\zeta_n), \end{aligned} \tag{5.1.5}$$

$$\begin{aligned} &\Big|Q_{I,n}(y,t,\xi,\vartheta) - Q_{I,n}(y',t,\xi,\vartheta)\Big| \\ &\le Ck_n |y-y'| (t-\vartheta)^{n-1} \\ &\times \left(\exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right)\right. \\ &\left. + \exp\left(-\frac{1}{4}c_0 \frac{\left(y'-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right)\right) \\ &\times dF(\zeta_1),\ldots,dF(\zeta_n). \end{aligned} \tag{5.1.6}$$

Also

$$\begin{aligned} |Q_{I,n}(y,t,\xi,\vartheta) - Q_{I,n}(y,t',\xi,\vartheta)| &\le Ck_n |t-t'|^{\frac{1}{4}} \left(\tilde{t}-\vartheta\right)^{n-\frac{3}{4}} \\ &\times \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right) \\ &\times dF(\zeta_1),\ldots,dF(\zeta_n), \end{aligned} \tag{5.1.7}$$

(ii)

$$|Q_I(y,t,\xi,\vartheta)| \leq C(t-\vartheta)^{-\frac{1}{2}},$$

$$|Q_I(y,t,\xi,\vartheta) - Q_I(y',t,\xi,\vartheta)| \leq C|y-y'|(t-\vartheta)^{-1},$$

and

$$|Q_I(y,t,\xi,\vartheta) - Q_I(y,t',\xi,\vartheta)| \leq C|t-t'|^{\frac{1}{4}}(t'-\vartheta)^{-\frac{3}{4}}.$$

(iii) *For every* (y,t,ξ,ϑ)

$$\begin{aligned} Q_I(y,t,\xi,\vartheta) = \lambda \int_0^y G_{L_\lambda}(y-\zeta,t,\xi,\vartheta)\,dF(\zeta) \\ + \lambda \int_0^y G_{I_\lambda}(y-\zeta,t,\xi,\vartheta)\,dF(\zeta). \end{aligned} \tag{5.1.8}$$

Proof. For part (i)-(ii): This follows from similar calculations as in Lemma 4.1.12.

For part (iii): This follows from similar calculations as in the proof of Lemma 5.1.4. □

Lemma 5.1.7. *Assume that* $\sigma_R = r = 0$. *There exists a constant* C *such that for every* $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ *and every* $(y',t') \in (y,\kappa)\times(0,t)$ *the following inequalities are all valid:*

$$\begin{aligned} |Q_I(y,t,\xi,\vartheta) - Q_{I,\kappa}(y,t,\xi,\vartheta)| \leq C(t-\vartheta)^{-\frac{1}{2}} \\ \times \exp\left(-\frac{1}{8}c_0\frac{(\kappa-y)^2+(\kappa-\xi)^2}{t-\vartheta}\right), \end{aligned} \tag{5.1.9}$$

$$\begin{aligned} &|(Q_I(y,t,\xi,\vartheta) - Q_{I,\kappa}(y,t,\xi,\vartheta)) - (Q_I(y',t,\xi,\vartheta) - Q_{I,\kappa}(y',t,\xi,\vartheta))| \\ &\quad \leq C(y'-y)^{\frac{1}{2}}(t-\vartheta)^{-\frac{3}{4}}\exp\left(-\frac{1}{8}c_0\frac{(\kappa-y')^2+(\kappa-\xi)^2}{t-\vartheta}\right), \end{aligned} \tag{5.1.10}$$

and

$$\begin{aligned} &|(Q_I(y,t,\xi,\vartheta) - Q_{I,\kappa}(y,t,\xi,\vartheta)) - (Q_I(y,t',\xi,\vartheta) - Q_{I,\kappa}(y,t',\xi,\vartheta))| \\ &\quad \leq C(t-t')^{\frac{1}{4}}t'^{-\frac{3}{4}}\exp\left(-\frac{1}{8}c_0\frac{(\kappa-y)^2+(\kappa-\xi)^2}{t-\vartheta}\right). \end{aligned} \tag{5.1.11}$$

Proof. For $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ and $n \in 0,1,\ldots$, let

$$\Delta Q_{I,n}(y,t,\xi,\vartheta) := Q_{I,n}(y,t,\xi,\vartheta) - Q_{I,\kappa}(y,t,\xi,\vartheta).$$

and let

$$\Delta Q_I(y,t,\xi,\vartheta) := Q_I(y,t,\xi,\vartheta) - Q_{I,\kappa}(y,t,\xi,\vartheta).$$

Because of the bounds given in Lemma 5.1.2 similar calculations as in Proposition 3.0.3 and Proposition 3.0.4 yield that

$$\Delta Q_{I,0}(y,t,\xi,\vartheta),$$

$$\Delta Q_{I,0}\,(y,t,\xi,\vartheta) - \Delta Q_{I,0}\,(y',t,\xi,\vartheta)$$

and

$$\Delta Q_{I,0}\,(y,t,\xi,\vartheta) - \Delta Q_{I,0}\,(y,t',\xi,\vartheta)$$

obey bounds of the stated form (5.1.10). Similar calculations, based on induction and uniform convergence, as in the proofs of Lemma4.1.4 and Lemma (5.1.4) part (iii), yield that the stated regularity bounds for $\Delta Q_I\,(y,t,\xi,\vartheta)$ all hold. □

Lemma 5.1.8. *Assume that $\sigma_R = r = 0$. There exists a sequence $\{k_n\}_{n=0}^{\infty}$ such that*

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 0,$$

and such that, for every finite $n \in 0,1,\ldots,,$ every $l \in \{0,1,2\}$, $y,\xi > 0$ and $0 \le \vartheta < t \le 1$ the following identities and inequalities are valid:

(i)

$$\frac{\partial^l G_{I_\lambda,n}\,(y,t,\xi,\vartheta)}{\partial y^l} = \int_\vartheta^t \int_0^\infty \frac{\partial^l G_{L_\lambda}\,(y,t,z,s)}{\partial y^l} Q_{I,n}\,(z,s,\xi,\vartheta)\,dzds,$$

and

$$\frac{\partial G_{I_\lambda,n}\,(y,t,\xi,\vartheta)}{\partial t} = Q_{I,n}\,(y,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\infty \frac{\partial G_{L_\lambda}\,(y,t,z,s)}{\partial t} Q_{I,n}\,(z,s,\xi,\vartheta)\,dzds.$$

(ii)

$$\left|\frac{\partial^l G_{I_\lambda,n}\,(y,t,\xi,\vartheta)}{\partial y^l}\right| \le k_n\,(t-\vartheta)^{n+\frac{1-l}{2}} \times \int_0^\infty \cdots \int_0^\infty \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right) \times dF(\zeta_1),\ldots,dF(\zeta_n),$$

$$\left|\frac{\partial G_{I_\lambda,n}\,(y,t,\xi,\vartheta)}{\partial t}\right| \le k_n\,(t-\vartheta)^{n-\frac{1}{2}} \times \int_0^\infty \cdots \int_0^\infty \exp\left(-\frac{1}{4}c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta}\right) \times dF(\zeta_1),\ldots,dF(\zeta_n)$$

and

$$
\begin{aligned}
\left|G_{I_\lambda,n}\left(y,t,\xi,\vartheta\right)\right| \le{}& k_n y^{\frac{1}{2}} \left(t-\vartheta\right)^{n+\frac{1}{4}} \\
&\times \int_0^\infty \ldots \int_0^\infty \left(\exp\left(-\frac{1}{4} c_0 \frac{\left(y-\xi-\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta} \right) \right. \\
&\left. + \exp\left(-\frac{1}{4} c_0 \frac{\left(\xi+\sum_{j=0}^n \zeta_j\right)^2}{t-\vartheta} \right) \right) \\
&\times dF(\zeta_1),\ldots,dF(\zeta_n).
\end{aligned}
$$

(iii)

$$
\frac{\partial^l G_{I_\lambda}\left(y,t,\xi,\vartheta\right)}{\partial y^l} = \int_\vartheta^t \int_0^\infty \frac{\partial^l G_{L_\lambda}\left(y,t,z,s\right)}{\partial y^l} Q_{I,\kappa}\left(z,s,\xi,\vartheta\right) dzds
$$

and

$$
\begin{aligned}
\frac{\partial G_{I_\lambda}\left(y,t,\xi,\vartheta\right)}{\partial t} ={}& Q_{I,\kappa}\left(y,t,\xi,\vartheta\right) \\
&+ \int_\vartheta^t \int_0^\infty \frac{\partial G_{L_\lambda}\left(y,t,z,s\right)}{\partial t} Q_{I,\kappa}\left(z,s,\xi,\vartheta\right) dzds.
\end{aligned}
$$

Furthermore, there exists a constant C such that for every (y,t,ξ,ϑ), and every $l \in \{0,1,2\}$

$$
\left| \frac{\partial^l G_{I_\lambda}\left(y,t,\xi,\vartheta\right)}{\partial y^l} \right| \le C\left(t-\vartheta\right)^{\frac{1-l}{2}},
$$

and

$$
\left| \frac{\partial G_{I_\lambda}\left(y,t,\xi,\vartheta\right)}{\partial t} \right| \le C\left(t-\vartheta\right)^{-\frac{1}{2}}.
$$

(iv) *For every $(y,t,\xi,\vartheta) \in \mathcal{D}_\kappa$ and $l \in \{0,1,2\}$*

$$
\begin{aligned}
&\left| \frac{\partial^l G_{I_\lambda}\left(y,t,\xi,\vartheta\right)}{\partial y^l} - \frac{\partial^l G_{I_{\lambda,\kappa}}\left(y,t,\xi,\vartheta\right)}{\partial y^l} \right| \\
&\qquad \le C\left(t-\vartheta\right)^{\frac{1-l}{2}} \exp\left(-\frac{1}{8} c_0 \frac{\left(\kappa-y\right)^2+\left(\kappa-\xi\right)^2}{t-\vartheta} \right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| \frac{\partial G_{I_\lambda}\left(y,t,\xi,\vartheta\right)}{\partial t} - \frac{\partial G_{I_{\lambda,\kappa}}\left(y,t,\xi,\vartheta\right)}{\partial t} \right| \\
&\qquad \le C\left(t-\vartheta\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{8} c_0 \frac{\left(\kappa-y\right)^2+\left(\kappa-\xi\right)^2}{t-\vartheta} \right).
\end{aligned}
$$

Proof. This follows from the bounds given in Lemma 5.1.6 and Lemma 5.1.7 and similar calculations as in the proof of Lemma 4.1.13. □

Definition 5.1.9. *For $y, \xi > 0$ and $0 \le \vartheta < t \le 1$ let*

$$G_A\left(y,t,\xi,\vartheta\right) := G_{L_\lambda}\left(y,t,\xi,\vartheta\right) + G_{I_\lambda}\left(y,t,\xi,\vartheta\right)$$

and let

$$\psi_3(y,t) = \int_0^t \int_0^\infty G_A\left(y,t,\xi,\vartheta\right) H_2\left(\xi,\vartheta\right) d\xi d\vartheta.$$

Theorem 5.1.2. *Assume that $\sigma_R = r = 0$ and that the bound (4.0.38) on the tail distribution function $\bar{F}$ holds.*

(i) *Then $\psi_3(y,t)$ is a classical solution of the PIDE (5.0.25). Furthermore, there exists a constant C_β, depending on β, such that for every $y > 0$ and $t \in (0,1]$ the following inequalities are all valid:*

$$\left|\frac{\partial^l \psi_3(y,t)}{\partial y^l}\right| \le C_\beta t^{\frac{4-l}{2}}\left(1+y\right)^{-\beta},$$

$$\left|\frac{\partial \psi_3(y,t)}{\partial t}\right| \le C_\beta t\left(1+y\right)^{-\beta},$$

$$|\psi_3(y,t)| \le C t^{\frac{7}{4}}\sqrt{y},$$

$$\left|\frac{\partial^l \psi_3(y,t)}{\partial y^l}\right| \le C_\beta t^{\frac{4-l}{2}}\left(1+y\right)^{-\beta},$$

$$\left|\frac{\partial \psi_3(y,t)}{\partial t}\right| \le C_\beta t\left(1+y\right)^{-\beta},$$

$$\left|\frac{\partial^l \psi(y,t)}{\partial y^l}\right| \le C\left(t^{-\frac{l}{2}}\exp\left(-\frac{1}{4}c_0 T\frac{y^2}{t}\right) + t^{\frac{2-l}{2}} C_\beta\left(1+y\right)^{-\beta}\right) \text{ and}$$

$$\left|\frac{\partial \psi(y,t)}{\partial t}\right| \le C\left(t^{-1}\exp\left(-\frac{1}{4}c_0 T\frac{y^2}{t}\right) + C_\beta\left(1+y\right)^{-\beta}\right).$$

(ii) *There exists a constant C_β, depending on β, such that, for every $y \in \left(0, \frac{1}{2}\kappa\right)$, $t \in (0,1]$ and $l \in \{0,1,2\}$ the following bounds are all valid:*

$$\left|\frac{\partial^l \psi_3(y,t)}{\partial y^l} - \frac{\partial^l \psi_{3,\kappa}}{\partial y^l}(y,t)\right| \le C_\beta t^{\frac{4-l}{2}}\left(1+\kappa\right)^{-\beta}\exp\left(-\frac{1}{128}c_0\frac{\left(\kappa-y\right)^2}{t}\right),$$

$$\left|\frac{\partial \psi_3(y,t)}{\partial t} - \frac{\partial \psi_{3,\kappa}}{\partial t}(y,t)\right| \le C_\beta t\left(1+\kappa\right)^{-\beta}\exp\left(-\frac{1}{128}c_0\frac{\left(\kappa-y\right)^2}{t}\right).$$

Proof. This follows from the identities and bounds given in Lemma 5.1.5 and Lemma 4.1.13, the bounds obeyed by $H_1\left(y,t\right)$ given in Proposition 5.1.2 and similar calculations as in the proof of Theorem 4.1.2. For part (ii) it is helpful to consider separately the cases $\xi \le \frac{\kappa+y}{2}$ and $\xi > \frac{\kappa+y}{2}$. □

5.2 Unbounded coefficients

In this section we will prove the existence of a classical solution of the PDE (5.0.25) and the PIDE (5.0.26) under the assumption that $\sigma_R > 0$. Quite similar to what we did in Section 4.2, the main idea is to consider a transformed equation of equation (5.0.25), using the change of variables $x = \ln(1+y)$, and look for a solution $\hat{\psi}_2(x,t)$ of the equations

$$\begin{cases} \hat{\psi}_2(x,0) & = 0, \quad x > 0 \\ \lim_{x\to\infty} \hat{\psi}_2(x,t) & = 0, \\ \hat{\psi}_2(0,t) & = 0, \quad t \in [0,1], \\ \frac{\partial \hat{\psi}_2(x,t)}{\partial t} - \hat{L}\hat{\psi}_2(x,t) & = H_1(e^x - 1, t) \quad x > 0, t \in (0,1], \end{cases} \tag{5.2.1}$$

and

$$\begin{cases} \hat{\psi}_3(x,0) & = 0, \quad x > 0, \\ \hat{\psi}_3(0,t) & = 0, \quad t \in [0,1], \\ \lim_{x\to\infty} \hat{\psi}_3(x,t) & = 0, \quad t \in [0,1], \\ \frac{\partial \hat{\psi}_3(x,t)}{\partial t} - \hat{A}\hat{\psi}_3(x,t) & = H_2(e^x - 1, t), \quad x > 0, t \in (0,1], \end{cases} \tag{5.2.2}$$

where $H_2(y,t)$ is defined in Definition 5.1.5. As we did in Section 5.1 we will also consider the convergence of the solutions as $\gamma \to \infty$. For the PDE (5.2.1) the first step is to establish bounds on a Green function $\hat{G}_{\hat{L}}(x,t,\xi,\vartheta)$ that is very similar to the auxiliary Green function $\hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta)$ except that, instead of satisfying

$$\lim_{x\to\hat{\kappa}} \hat{G}_{\hat{L},\hat{\kappa}}(x,t,\xi,\vartheta) = 0,$$

$\hat{G}_{\hat{L}}(x,t,\xi,\vartheta)$ will satisfy the asymptotic condition

$$\lim_{x\to\infty} \hat{G}_{\hat{L}}(x,t,\xi,\vartheta) = 0.$$

Definition 5.2.1. *For $\xi > 0$ and $t \in (0,1]$ let $\hat{P}^{(2)}_g$ be the operator defined in Definition 4.2.5. Let*

$$\hat{V}_{\xi,0}(t) := -2\hat{\Gamma}_{\hat{L}_0}(0,t,\xi,0).$$

For $n \in 1,2,\ldots,$ define $\hat{V}_{\xi,n}(t)$ recursively as

$$\hat{V}_{\xi,n}(t) = -2\hat{P}^{(2)}_{\hat{V}_{\xi,n-1}}(t),$$

and let

$$\hat{U}_\xi(t) := \sum_{n=0}^{\infty} \hat{V}_{\xi,n}(t).$$

Lemma 5.2.1. *Assume that $\sigma_R > 0$.*

(i) *There exists a constant C such that for every $t \in (0,1]$ and every $\xi \in (0,\hat{\kappa})$*

$$\left|\hat{U}_\xi(t) - \hat{U}^{(2)}_{\xi,\hat{\kappa}}(t)\right| \le C\exp\left(-\frac{1}{4}\hat{c}_0\frac{\hat{\kappa}^2}{t}\right),$$

and

$$\left|\hat{U}'_\xi(t) - \hat{U}^{(2)\prime}_{\xi,\hat{\kappa}}(t)\right| \le C\exp\left(-\frac{1}{4}\hat{c}_0\frac{\hat{\kappa}^2}{t}\right).$$

(ii) *$\hat{U}_\xi(t)$ is differentiable on $(0,1]$. $\hat{U}_\xi(t)$ is a solution of the integral equation*

$$-\frac{1}{2}\hat{U}_\xi(t) - \hat{P}^{(2)}_{U_{\xi,n}} = \hat{\Gamma}_{\hat{L}_0}(0,t,\xi,0).$$

Let

$$\hat{g}^*_{\hat{L}_0}(x,t,\xi) := -P^{(2)}_{\hat{U}_\xi}(x,t).$$

Then $\hat{g}^*_{\hat{L}_0}(x,t,\xi)$ is a classical solution of the PDE

$$\begin{cases} \hat{g}^*_{\hat{L}_0}(x,0,\xi) & = 0, \quad x > 0, \\ \hat{g}^*_{\hat{L}_0}(0,t,\xi) & = \hat{\Gamma}_{\hat{L}_0}(0,t,\xi,0), \quad t \in (0,1], \\ \lim_{x\to\infty} \hat{g}^*_{\hat{L}_0}(x,t,\xi) & = 0, \quad t \in (0,1], \\ \frac{\partial \hat{g}^*_{\hat{L}_0}(x,t,\xi)}{\partial t} & = \hat{a}_{1,1}(x)\frac{\partial^2 \hat{g}^*_{\hat{L}_0}(x,t,\xi)}{\partial x^2}, \quad (x,t) \in (0,\infty)\times(0,1]. \end{cases}$$

Proof. This follows from similar reasoning and calculation also making use of the bounds given in Lemma 5.2.2, as in the proof of Lemma 5.1.1) below, . □

Definition 5.2.2. *Define*

$$\hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta) := \hat{g}^*_{\hat{L}_0}(x,t-\vartheta,\xi).$$

Lemma 5.2.2. *Assume that $\sigma_R > 0$.*

(i) *There exists a constant C such that for every $x,\xi > 0$ and $0 \le \vartheta < t \le 1$ and $l \in \{0,1,2\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^l}\right| \le C(t-\vartheta)^{-\frac{1+l}{2}} \exp\left(-\frac{1}{2}\hat{c}_0 \frac{x^2+\frac{1}{2}\xi^2}{t-\vartheta}\right)$$

and

$$\left|\frac{\partial \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial t}\right| \le C(t-\vartheta)^{-\frac{3}{2}} \exp\left(-\frac{1}{2}\hat{c}_0 \frac{x^2+\frac{1}{2}\xi^2}{t-\vartheta}\right).$$

(ii) *There exists a constant C such that for every $(x,t,\xi,\vartheta) \in \mathcal{D}_{\hat{\kappa}}$ and $l \in \{0,1,2\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^l} + \frac{\partial^l}{\partial x^l}\hat{P}^{(2)}_{\hat{U}^{(2)}_{\xi,\hat{\kappa}}}(x,t-\vartheta)\right| \le C\exp\left(-\frac{1}{8}\hat{c}_0\frac{\hat{\kappa}^2+\xi^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial t} + \frac{\partial}{\partial t}\hat{P}^{(2)}_{\hat{U}^{(2)}_{\xi,\hat{\kappa}}}(x,t-\vartheta)\right| \le C\exp\left(-\frac{1}{8}\hat{c}_0\frac{\hat{\kappa}^2+\xi^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial^l \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^l} - \frac{\partial^l \hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial x^l}\right|$$
$$\le Ct^{-\frac{1+l}{2}} \exp\left(-\frac{1}{2}\hat{c}_0\frac{(\hat{\kappa}-y)^2+\frac{1}{2}(\hat{\kappa}-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial t} - \frac{\partial \hat{g}_{\hat{L}_0,\kappa}(x,t,\xi,\vartheta)}{\partial t}\right|$$
$$\leq Ct^{-\frac{3}{2}} \exp\left(-\frac{1}{2}\hat{c}_0 \frac{(\hat{\kappa}-y)^2 + \frac{1}{2}(\hat{\kappa}-\xi)^2}{t-\vartheta}\right).$$

(iii) *There exists a constant C such that for every $y, \xi > 0$ and $0 \leq \vartheta < t \leq 1$ the following inequalities are all valid:*

$$\int_0^t \int_{\hat{\kappa}}^{\infty} \left|\frac{\partial^2 \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial y^2}\right| d\xi d\vartheta \leq Ct \exp\left(-\frac{1}{4}\hat{c}_0 \frac{\hat{\kappa}^2}{t}\right),$$

and

$$\int_0^t \int_{\hat{\kappa}}^{\infty} \left|\frac{\partial \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial t}\right| d\xi d\vartheta \leq Ct \exp\left(-\frac{1}{4}\hat{c}_0 \frac{\hat{\kappa}^2}{t}\right).$$

(iv) *There exists a constant C such that for every $(x,t,\xi,\vartheta) \in \mathcal{D}_{\hat{\kappa}}$ and $l \in \{0,1,2\}$*

$$\left|\int_0^t \int_0^{\infty} \left(\frac{\partial^l \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x^l} - \frac{\partial^l \hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial x^l}\right) d\xi d\vartheta\right|$$
$$\leq Ct^{\frac{2-l}{2}} \exp\left(-\frac{1}{2}\hat{c}_0 \frac{(\hat{\kappa}-y)^2}{t}\right),$$

and such that

$$\left|\int_0^t \left(\int_0^{\infty} \frac{\partial \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial t} d\xi - \int_0^{\hat{\kappa}} \frac{\partial \hat{g}_{\hat{L}_0,\hat{\kappa}}(x,t,\xi,\vartheta)}{\partial t} d\xi\right) d\vartheta\right|$$
$$\leq C \exp\left(-\frac{1}{2}\hat{c}_0 \frac{(\hat{\kappa}-y)^2}{t}\right).$$

Proof. Because of the bounds given in Lemma 5.2.1 this follows from similar reasoning and calculations as in the proof of Lemma 5.1.2 and the proof of Proposition 4.2.5. □

In the coming results we will establish the existence of a function $\hat{G}_{\hat{L}}(x,t,\xi,\vartheta)$ that has very similar properties on the entire unbounded domain $(y,t) \in (0,\infty) \times (0,1]$ as the Green function $\hat{G}_{\hat{L},\gamma}(x,t,\xi,\vartheta)$, does on the truncated domain. Moreover, we will show that the function $\hat{G}_{\hat{L},\gamma}(x,t,\xi,\vartheta)$ will converge to $\hat{G}_{\hat{L}}(x,t,\xi,\vartheta)$ if we let γ tend towards infinity.

Definition 5.2.3. *For $x, \xi > 0$ and $0 \leq \vartheta < t \leq 1$ let*

$$\hat{G}_{\hat{L}_0}(x,t,\xi,\vartheta) := \hat{\Gamma}_{\hat{L}_0}(x,t,\xi,\vartheta) - \hat{g}_{\hat{L}_0}(x,t,\xi,\vartheta),$$

let

$$\hat{Q}_0(x,t,\xi,\vartheta) := \hat{a}_1^*(x) \frac{\partial \hat{G}_{\hat{L}_0}(x,t,\xi,\vartheta)}{\partial x}$$

and let the sequence of functions $\left\{\hat{Q}_n\right\}_{n=0}^{\infty}$ *be defined inductively for* $n \in 1, 2, \ldots,$ *by*

$$\hat{Q}_n(x,t,\xi,\vartheta) = \int_{\vartheta}^{t}\int_{0}^{\infty} \hat{Q}_0(x,t,z,s)\,\hat{Q}_{n-1}(z,s,\xi,\vartheta)\,dzds,$$

and let

$$\hat{Q}(x,t,\xi,\vartheta) = \sum_{n=0}^{\infty} \hat{Q}_n(x,t,\xi,\vartheta).$$

Lemma 5.2.3. *Assume that* $\sigma_R > 0$ *and let* $\alpha \in (0,1)$.

(i) $\hat{Q}$ *solves the integral equation*

$$\hat{Q}(x,t,\xi,\vartheta) = \hat{Q}_0(x,t,\xi,\vartheta) + \int_{\vartheta}^{t}\int_{0}^{\infty} \hat{Q}_0(x,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dzds. \tag{5.2.3}$$

(ii) *There exists a constant* C *such that, for every* $x, x', \xi > 0$ *and every* $0 \leq \vartheta < t' < t \leq 1$ *the following identities and inequalities are all valid:*

$$\hat{Q}(x,t,\xi,\vartheta) = \hat{Q}(x,t-\vartheta,\xi,0),$$

$$\left|\hat{Q}(x,t,\xi,\vartheta)\right| \leq C(t-\vartheta)^{-1}\exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

$$\left|\hat{Q}(x,t,\xi,\vartheta) - \hat{Q}(x',t,\xi,\vartheta)\right| \leq C|x-x'|^{\frac{1}{2}}(t-\vartheta)^{-\frac{5}{4}} \times \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\hat{Q}(x,t,\xi,\vartheta) - \hat{Q}(x,t',\xi,\vartheta)\right| \leq C|t-t'|^{\frac{1}{4}}(t'-\vartheta)^{-\frac{5}{4}} \times \exp\left(-\frac{1}{4}\hat{c}_0\frac{(x-\xi)^2}{t-\vartheta}\right).$$

(iii) *There exists a constant* C *such that, for every* $(x,t,\xi,\vartheta) \in \bar{\mathcal{D}}_{\hat{\kappa}}$, *every* $x', \in (0,\hat{\kappa})$*, and every* $t' \in (0,t,)$ *the following identities and inequalities are all valid:*

$$\left|\hat{Q}(x,t,\xi,\vartheta) - \hat{Q}_{\hat{\kappa}}(x,t,\xi,\vartheta)\right| \leq C(t-\vartheta)^{-1} \times \exp\left(-\frac{1}{4}\hat{c}_0\frac{(\hat{\kappa}-x)^2+(\hat{\kappa}-\xi)^2}{t-\vartheta}\right),$$

$$\left|\hat{Q}(x,t,\xi,\vartheta) - \hat{Q}(x',t,\xi,\vartheta) - \left(\hat{Q}_{\hat{\kappa}}(x,t,\xi,\vartheta) - \hat{Q}_{\hat{\kappa}}(x',t,\xi,\vartheta)\right)\right| \leq C|x-x'|^{\frac{1}{2}}(t-\vartheta)^{-\frac{5}{4}}\exp\left(-\frac{1}{8}\hat{c}_0\frac{(\hat{\kappa}-x)^2+(\hat{\kappa}-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\hat{Q}(x,t,\xi,\vartheta)-\hat{Q}(x,t',\xi,\vartheta)-\left(\hat{Q}_{\hat{\kappa}}(x,t,\xi,\vartheta)-\hat{Q}_{\hat{\kappa}}(x,t',\xi,\vartheta)\right)\right|$$
$$\leq C\left|t-t'\right|^{\frac{1}{4}}(t-\vartheta)^{-\frac{5}{4}}\exp\left(-\frac{1}{8}\hat{c}_0\frac{(\hat{\kappa}-x)^2+(\hat{\kappa}-\xi)^2}{t-\vartheta}\right).$$

Proof. For (i)-(ii): This follows from similar calculations and reasoning as in the proofs of Lemma 4.2.6 and Lemma 5.1.4.

For (iii): This follows from similar calculations as in the proof of Lemma (5.1.4) part (iii), and Lemma 5.1.7. □

Proposition 5.2.1. (i) *For every $x,\xi>0$ and $0\leq\vartheta<t\leq 1$ and $l\in\{0,1,2\}$*

$$\frac{\partial^l}{\partial x^l}\int_\vartheta^t\int_0^\infty \hat{G}_{\hat{L}_0}(x,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dzds$$
$$=\int_\vartheta^t\int_0^\infty \frac{\partial^l\hat{G}_{\hat{L}_0}(x,t,z,s)}{\partial x^l}\hat{Q}(z,s,\xi,\vartheta)\,dzds,$$

and

$$\frac{\partial}{\partial t}\int_\vartheta^t\int_0^\infty \hat{G}_{\hat{L}_0}(x,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dzds=\hat{Q}(x,t,\xi,\vartheta)$$
$$+\int_\vartheta^t\int_0^\infty \frac{\partial\hat{G}_{\hat{L}_0}(x,t,z,s)}{\partial t}\hat{Q}(z,s,\xi,\vartheta)\,dzds.$$

(ii) *There exists a constant C such that for every $x,\xi>0$, every $0\leq\vartheta<t\leq 1$ and every and $l\in\{0,1,2\}$*

$$\left|\frac{\partial^l}{\partial x^l}\int_\vartheta^t\int_0^\infty \hat{G}_{\hat{L}_0}(x,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dzds\right|$$
$$\leq C(t-\vartheta)^{-\frac{l}{2}}\exp\left(-\frac{1}{4}c_0\frac{(x-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial}{\partial t}\int_\vartheta^t\int_0^\infty \hat{G}_{\hat{L}_0}(x,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dzds\right|$$
$$\leq C(t-\vartheta)^{-1}\exp\left(-\frac{1}{4}c_0\frac{(x-\xi)^2}{t-\vartheta}\right).$$

(iii) *There exists a constant C such that for every $(x,t,\xi,\vartheta)\in\mathcal{D}_{\hat{\kappa}}$ and every $l\in\{0,1,2\}$*

$$\left|\frac{\partial^l}{\partial x^l}\int_\vartheta^t\left(\int_0^\infty \hat{G}_{\hat{L}_0}(x,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dzds\right.\right.$$
$$\left.\left.-\int_0^{\hat{\kappa}} \hat{G}_{\hat{L}_0,\hat{\kappa}}(x,t,z,s)\,\hat{Q}_{\hat{\kappa}}(z,s,\xi,\vartheta)\,dz\right)\right|$$
$$\leq C(t-\vartheta)^{-\frac{l}{2}}\exp\left(-\frac{1}{8}\hat{c}_0\frac{(\hat{\kappa}-x)^2+(\hat{\kappa}-\xi)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial}{\partial t}\int_{\vartheta}^{t}\left(\int_{0}^{\infty}\hat{G}_{\hat{L}_0}(x,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dz\right.\right.$$
$$\left.\left.-\int_{0}^{\gamma}\hat{G}_{\hat{L}_0,\gamma}(x,t,z,s)\,\hat{Q}_{\hat{\kappa}}(z,s,\xi,\vartheta)\,dz\right)ds\right|$$
$$\leq C(t-\vartheta)^{-1}\exp\left(-\frac{1}{8}\hat{c}_0\frac{(\hat{\kappa}-x)^2+(\hat{\kappa}-\xi)^2}{t-\vartheta}\right).$$

Proof. For (i)-(ii): Because of the regularity bounds obeyed by $\hat{Q}(x,t,\xi,\vartheta)$, given in Lemma 5.2.3, and the regularity bounds obeyed by $\hat{G}_{\hat{L}_0}$, this follows from similar calculations as in the proof of Lemma 4.1.5.

For part (iii): Because of the bounds given in Lemma 5.2.3 this follows from similar calculations as in the proof of Lemma (5.1.4) part (iii). □

Definition 5.2.4. *Let*

$$\hat{G}_{\hat{L}}(x,t,\xi,\vartheta):=\hat{G}_{\hat{L}_0}(x,t,\xi,\vartheta)+\int_{\vartheta}^{t}\int_{0}^{\infty}\hat{G}_{\hat{L}_0}(x,t,z,s)\,\hat{Q}(z,s,\xi,\vartheta)\,dzds.$$

and let

$$\hat{\psi}_2(x,t):=\int_0^t\int_0^\infty\hat{G}_{\hat{L}}(x,t,\xi,\vartheta)\,H_{1,\gamma}\left(e^{\xi}-1,\vartheta\right)d\xi d\vartheta.$$

Theorem 5.2.1. *Assume that $\sigma_R>0$ and that the tail distribution satisfies the inequality* (4.0.38)*.*

(i) *For every $x>0, t\in(0,1]$ and $l\in\{1,2\}$*

$$\frac{\partial^l\hat{\psi}_2(x,t)}{\partial x^l}=\int_0^t\int_0^\infty\frac{\partial^l\hat{G}_{\hat{L}}(x,t,\xi,\vartheta)}{\partial x^l}H_{1,\gamma}\left(e^{\xi}-1,\vartheta\right)d\xi d\vartheta,$$

and

$$\frac{\partial\hat{\psi}_2(x,t)}{\partial t}=H_{1,\gamma}\left(e^{x}-1,t\right)+\int_0^t\int_0^\infty\frac{\partial\hat{G}_{\hat{L}}(x,t,\xi,\vartheta)}{\partial t}H_{1,\gamma}\left(e^{\xi}-1,\vartheta\right)d\xi d\vartheta.$$

(ii) *There exists a constant C_β, depending on β, such that for every $x>0, t\in(0,1]$ and every $l\in\{0,1\}$ the following bounds are valid:*

$$\left|\frac{\partial^l\hat{\psi}_2(x,t)}{\partial x^l}\right|\leq C_\beta t^{\frac{2-l}{2}}\exp(-\beta x),$$
$$\left|\frac{\partial^2\hat{\psi}_2(x,t)}{\partial x^2}\right|\leq C_\beta\exp\left(-\frac{1}{2}\beta x\right),$$
$$\left|\frac{\partial\hat{\psi}_2(x,t)}{\partial t}\right|\leq C_\beta\exp\left(-\frac{1}{2}\beta x\right),$$

and

$$\left|\hat{\psi}_2(x,t)\right| \le C_\beta t^{\frac{3}{4}} \min\left(x^{\frac{1}{2}}, t^{\frac{1}{4}} \exp\left(-\beta x\right)\right).$$

(iii) $\hat{\psi}_2(x,t)$ *is a classical solution of the PDE* (5.2.1) *and*

$$\psi_2(y,t) := \hat{\psi}_2(\ln\left(1+y\right), t)$$

is a classical solution of the PDE (5.0.25).

(iv) *For every* $x' > x, t' \in (0,t)$ *and* $\alpha \in \left(0, \frac{1}{2}\right)$ *the following bounds hold:*

$$\left|\hat{\psi}_2(x,t) - \hat{\psi}_2(x',t)\right| \le C_\beta \left(x'-x\right)^\alpha t^{\frac{2-\alpha}{2}} \exp\left(-\beta x\right) \text{ and}$$
$$\left|\hat{\psi}_2(x,t) - \hat{\psi}_2(x,t')\right| \le C_\beta \left(t-t'\right)^\alpha t^{1-\alpha} \exp\left(-\beta x\right).$$

(v) *There exists a constant* C_β*, depending on* β*, such that for every* $(x,t) \in (0,\hat{\kappa}) \times (0,1]$ *and every* $l \in \{0,1\}$

$$\left|\frac{\partial^l \hat{\psi}_2(x,t)}{\partial x^l} - \frac{\partial^l \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial x^l}\right| \le C_\beta t^{\frac{2-l}{2}} \exp\left(-\beta\hat{\kappa}\right) \exp\left(-\frac{1}{8}\hat{c_0}\frac{\left(\hat{\kappa}-x\right)^2}{t}\right),$$
$$\left|\frac{\partial^2 \hat{\psi}_2(x,t)}{\partial x^2} - \frac{\partial^2 \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial x^2}\right| \le C_\beta \exp\left(-\frac{1}{2}\beta\hat{\kappa}\right) \exp\left(-\frac{1}{8}\hat{c_0}\frac{\left(\hat{\kappa}-x\right)^2}{t}\right),$$

and

$$\left|\frac{\partial \hat{\psi}_2(x,t)}{\partial t} - \frac{\partial^l \hat{\psi}_{2,\hat{\kappa}}(x,t)}{\partial t}\right| \le C_\beta \exp\left(-\frac{1}{2}\beta\hat{\kappa}\right) \exp\left(-\frac{1}{8}\hat{c_0}\frac{\left(\hat{\kappa}-x\right)^2}{t}\right).$$

(vi) *There exists a constant* C_β*, depending on* β*, such that for every* $(y,t) \in (0,\gamma) \times (0,1]$ *and every* $l \in \{0,1\}$

$$\left|\frac{\partial^l \psi_2(y,t)}{\partial y^l} - \frac{\partial^l \psi_{2,\gamma}(y,t)}{\partial y^l}\right| \le C_\beta t^{\frac{2-l}{2}} \left(1+\gamma\right)^{-\beta}) \times \exp\left(-\frac{1}{8}\hat{c_0}\left[\ln\left(\frac{1+\gamma}{1+y}\right)\right]^2\right),$$
$$\left|\frac{\partial^2 \psi_2(y,t)}{\partial y^2} - \frac{\partial^2 \psi_{2,\gamma}(y,t)}{\partial y^2}\right| \le C_\beta t^{\frac{2-l}{2}} \left(1+\gamma\right)^{-\frac{1}{2}\beta} \left(1+y\right)^{-2} \times \exp\left(-\frac{1}{8}\hat{c_0}\left[\ln\left(\frac{1+\gamma}{1+y}\right)\right]^2\right),$$

and

$$\left|\frac{\partial \psi_2(y,t)}{\partial t} - \frac{\partial \psi_{2,\gamma}(y,t)}{\partial t}\right| \le C_\beta t^{\frac{2-l}{2}} \left(1+\gamma\right)^{-\frac{1}{2}\beta} \times \exp\left(-\frac{1}{8}\hat{c_0}\left[\ln\left(\frac{1+\gamma}{1+y}\right)\right]^2\right).$$

(vii) *For every* $0 < x < x' < \hat{\kappa}$, *every* $t' \in (0,t)$ *and* $\alpha \in (0, \frac{1}{2}]$ *the following bounds hold:*

$$\left|\left(\hat{\psi}_2(x,t) - \hat{\psi}_{2,\hat{\kappa}}(x,t\right) - \left(\hat{\psi}_2(x',t) - \hat{\psi}_{2,\hat{\kappa}}(x',t)\right)\right|$$
$$\leq C_\beta \left(x'-x\right)^\alpha t^{\frac{2-\alpha}{2}} \exp\left(-\beta\hat{\kappa}\right) \exp\left(-\frac{1}{8}\hat{c}_0 \frac{\left(\hat{\kappa}-x'\right)^2}{t}\right),$$
$$\left|\left(\hat{\psi}_2(x,t) - \hat{\psi}_{2,\hat{\kappa}}(x,t\right) - \left(\hat{\psi}_2(x,t') - \hat{\psi}_{2,\hat{\kappa}}(x,t')\right)\right|$$
$$\leq C_\beta \left(t-t'\right)^\alpha t^{1-\alpha} \exp\left(-\frac{1}{2}\beta\hat{\kappa}\right) \exp\left(-\frac{1}{8}\hat{c}_0 \frac{\left(\hat{\kappa}-x'\right)^2}{t}\right).$$

Proof. This follows from similar considerations and calculations as in the proof of Lemma 4.1.5 using the bound given in Lemma 4.2.8, Theorem 5.1.1 and Proposition 5.2.1. We also need to use the chain rule and consider the change of variable

$$y = e^x - 1.$$

The bounds given in part (vii) follow from the bounds given in part (v), Proposition 3.0.3, Proposition 3.0.4 and considering the function

$$\Delta\psi_{2,\gamma}(x,t) = \psi_2(x,t) - \psi_{2,\gamma}(x,t).$$

□

Lemma 5.2.4. *Assume that* $\sigma_R > 0$ *and that the tail distribution satisfies the bound* (4.0.38). *Let the function* H_2 *be as in Definition 5.1.5.*

(i) *There exists a constant* C_β, *depending on* β, *such that the bounds stated below all hold for every* $x' > x > 0$, *and every* $0 \leq t' < t \leq 1$ *and every* $\alpha \in (0, \frac{\min(\beta,1)}{2}]$

$$\left|H_2\left(e^x - 1, t\right)\right| \leq C_\beta t \exp\left(-\beta x\right),$$
$$\left|H_2\left(e^x - 1, t\right) - H_2\left(e^x - 1, t'\right)\right| \leq C_\beta \left(t-t'\right)^\alpha t^{1-\alpha} \exp\left(-\beta x\right),$$

and

$$\left|H_2\left(e^x - 1, t\right) - H_2\left(e^{x'} - 1, t\right)\right| \leq C_\beta \left(x'-x\right)^\alpha t^{\frac{2-\alpha}{2}} \exp\left(-\left(\beta-\alpha\right)x\right).$$

(ii) *Let* $\gamma = e^{\hat{\kappa}} - 1$. *There exists a constant* C_β, *depending on* β, *such that the bounds stated below all hold for every* $0 < x < x' < \hat{\kappa}$, *and every* $0 \leq t' < t \leq 1$ *and every* $\alpha \in (0, \frac{\min(\beta,1)}{2}]$

$$\left|H_2\left(e^x - 1, t\right) - H_{2,\gamma}\left(e^x - 1, t\right)\right| \leq C_\beta t \gamma^{-\beta} \exp\left(-\beta\hat{\kappa}\right)$$
$$\times \exp\left(-\frac{1}{8}\hat{c}_0 \frac{\left(\hat{\kappa}-x\right)^2}{t}\right),$$

$$\left|\left(H_2\left(e^x - 1, t\right) - H_{2,\gamma}\left(e^x - 1, t'\right)\right) - \left(H_2\left(e^x - 1, t\right) - H_{2,\gamma}\left(e^x - 1, t'\right)\right)\right|$$
$$\leq C_\beta \left(x'-x\right)^\alpha t^{\frac{2-\alpha}{2}} \exp\left(-\frac{1}{2}\beta\hat{\kappa}\right) \exp\left(-\frac{1}{8}\hat{c}_0 \frac{\left(\hat{\kappa}-x\right)^2}{t}\right),$$

and

$$\begin{aligned}&\left|\left(H_2\left(e^x-1,t\right)-H_2\left(e^x-1,t'\right)\right)-\left(H_{2,\gamma}\left(e^x-1,t\right)-H_{2,\gamma}\left(e^x-1,t'\right)\right)\right|\\&\quad\le C_\beta\left(t-t'\right)^\alpha t^{1-\alpha}\exp\left(-\frac{1}{2}\beta\hat{\kappa}\right)\exp\left(-\frac{1}{8}\hat{c}_0\frac{\left(\hat{\kappa}-x\right)^2}{t}\right).\end{aligned}$$

Proof. This follows from the bounds given in Theorem 5.2.1 and similar calculations as in Lemma 4.2.9. □

The last part of this article will be a discussion on the PIDE (5.0.26) (transformed to the PIDE (5.2.2)) for the case $\sigma_R > 0$. Most of this discussion will be analogous to the discussions in Section 4.2.2.

Definition 5.2.5. *For $x,\xi > 0$ and $0 \le \vartheta < t \le 1$ let*

$$\hat{Q}_{\lambda,0}\left(x,t,\xi,\vartheta\right)=-\lambda\hat{G}_{\hat{L}}\left(x,t,\xi,\vartheta\right),$$

and let the sequence of functions $\left\{\hat{Q}_{\lambda,n}\right\}_{n=0}^{\infty}$ be defined inductively for $n \in 1,2,\ldots,$ by

$$\hat{Q}_{\lambda,n}\left(x,t,\xi,\vartheta\right)=\int_\vartheta^t\int_0^\infty\hat{Q}_{\lambda,0}\left(x,t,z,s\right)\hat{Q}_{\lambda,n-1}\left(z,s,\xi,\vartheta\right)dzds,$$

and let

$$\hat{Q}_\lambda\left(x,t,\xi,\vartheta\right)=\sum_{n=0}^{\infty}\hat{Q}_{\lambda,n}\left(x,t,\xi,\vartheta\right).$$

Lemma 5.2.5. *Assume that $\sigma_R > 0$. Let $\alpha \in (0,1)$.*

(i) *$\hat{Q}_\lambda$ is a solution of the integral equation*

$$\begin{aligned}\hat{Q}_\lambda\left(x,t,z,\vartheta\right)=&-\lambda\hat{G}_{\hat{L}}\left(x,t,\xi,\vartheta\right)\\&-\lambda\int_\vartheta^t\int_0^\infty\hat{G}_{\hat{L}}\left(x,t,z,s\right)\hat{Q}_\lambda\left(z,s,\xi,\vartheta\right)dzds.\end{aligned}\tag{5.2.4}$$

(ii) *$\hat{Q}_\lambda\left(x,t,\xi,\vartheta\right)$ is differentiable with respect to all four variables. Furthermore, there exists a constant C, such that, for every $x,\xi > 0$ and $0 \le \vartheta < t \le 1$ the following identities and inequalities are all valid:*

$$\hat{Q}_\lambda\left(x,t,\xi,\vartheta\right)=\hat{Q}_\lambda\left(x,t-\vartheta,\xi,0\right),$$

$$\left|\frac{\partial\hat{Q}_\lambda}{\partial x}\left(x,t,\xi,\vartheta\right)\right|\le C\left(t-\vartheta\right)^{-1}\exp\left(-\frac{1}{4}\hat{c}_0\frac{\left(x-\xi\right)^2}{t-\vartheta}\right),$$

$$\left|\frac{\partial\hat{Q}_\lambda}{\partial \xi}\left(x,t,\xi,\vartheta\right)\right|\le C\left(t-\vartheta\right)^{-1}\exp\left(-\frac{1}{4}\hat{c}_0\frac{\left(x-\xi\right)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial\hat{Q}_\lambda}{\partial t}\left(x,t,\xi,\vartheta\right)\right|\le C\left(t-\vartheta\right)^{-\frac{2}{3}}\exp\left(-\frac{1}{4}\hat{c}_0\frac{\left(x-\xi\right)^2}{t-\vartheta}\right).$$

(iii) *There exists a constant* C *such that for every* $(x,t,\xi,\vartheta) \in \mathcal{D}_{\hat{\kappa}}$ *and every* $l \in \{0,1\}$ *the following inequalities are all valid:*

$$\left| \frac{\partial^l \hat{Q}_\lambda (x,t,\xi,\vartheta)}{\partial x^l} - \frac{\partial \hat{Q}_{\lambda,\hat{\kappa}} (x,t,\xi,\vartheta)}{\partial x^l} \right| \leq C (t-\vartheta)^{-\frac{1+l}{3}} \times \exp\left(-\frac{1}{8} \hat{c}_0 \frac{(\hat{\kappa}-x)^2}{t-\vartheta} \right) \times \exp\left(-\frac{1}{8} \hat{c}_0 \frac{(\hat{\kappa}-\xi)^2}{t-\vartheta} \right), \tag{5.2.5}$$

and

$$\left| \frac{\partial \hat{Q}_\lambda (x,t,\xi,\vartheta)}{\partial t} - \frac{\partial \hat{Q}_{\lambda,\hat{\kappa}} (x,t,\xi,\vartheta)}{\partial t} \right| \leq C (t-\vartheta)^{-\frac{2}{3}} \times \exp\left(-\frac{1}{8} \hat{c}_0 \frac{(\hat{\kappa}-x)^2}{t-\vartheta} \right) \times \exp\left(-\frac{1}{8} \hat{c}_0 \frac{(\hat{\kappa}-\xi)^2}{t-\vartheta} \right). \tag{5.2.6}$$

Proof. For part (5.2.4) and part (ii): This follows from similar calculations as in the proof of Lemma 5.2.5.

For part (iii): This follows from similar calculations as in the proof of Lemma 5.1.4. □

Definition 5.2.6. *For* $x, \xi > 0$ *and* $0 \leq \vartheta < t \leq 1$ *let*

$$\hat{G}_{\hat{L}_\lambda} (x,t,\xi,\vartheta) := \hat{G}_{\hat{L}} (x,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\infty \hat{G}_{\hat{L}} (x,t,z,s) \hat{Q}_\lambda (z,s,\xi,\vartheta) \, dzds.$$

Lemma 5.2.6. *Assume that* $\sigma_R > 0$.

(i) *For every* $x, \xi > 0$ *and* $0 \leq \vartheta < t \leq 1$ *and* $l \in \{1,2\}$

$$\frac{\partial^l}{\partial x^l} \int_\vartheta^t \int_0^\infty \hat{G}_{\hat{L}} (x,t,z,s) \hat{Q}_\lambda (z,s,\xi,\vartheta) \, dzds = \int_\vartheta^t \int_0^\infty \frac{\partial^l \hat{G}_{\hat{L}} (x,t,z,s)}{\partial x^l} \hat{Q}_\lambda (z,s,\xi,\vartheta) \, dzds,$$

and

$$\frac{\partial}{\partial t} \int_\vartheta^t \int_0^\infty \hat{G}_{\hat{L}} (x,t,z,s) \hat{Q}_\lambda (z,s,\xi,\vartheta) \, dzds = \hat{Q}_\lambda (x,t,\xi,\vartheta) + \int_\vartheta^t \int_0^\infty \frac{\partial \hat{G}_{\hat{L}} (x,t,z,s)}{\partial t} \hat{Q}_\lambda (z,s,\xi,\vartheta) \, dzds.$$

(ii) *There exists a constant C such that, for every $x, \xi > 0$ and $0 \le \vartheta < t \le 1$ the following inequalities are all valid:*

$$\left|\frac{\partial^l}{\partial x^l}\int_\vartheta^t\int_0^\infty \hat{G}_{\hat{L}}\left(x,t,z,s\right)\hat{Q}_\lambda\left(z,s,\xi,\vartheta\right)dzds\right| \le C\left(t-\vartheta\right)^{\frac{1-l}{3}}\exp\left(-\frac{1}{4}\hat{c}_0\frac{\left(x-\xi\right)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial}{\partial t}\int_\vartheta^t\int_0^\infty \hat{G}_{\hat{L}}\left(x,t,z,s\right)\hat{Q}_\lambda\left(z,s,\xi,\vartheta\right)dzds\right| \le C\left(t-\vartheta\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{4}\hat{c}_0\frac{\left(x-\xi\right)^2}{t-\vartheta}\right).$$

(iii) *There exists a constant C such that, for every $(x,t,\xi,\vartheta) \in \mathcal{D}_{\hat{\kappa}}$ and $l \in \{0,1,2\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l}{\partial x^l}\int_\vartheta^t\left(\int_0^\infty \hat{G}_{\hat{L}}\left(x,t,z,s\right)\hat{Q}_\lambda\left(z,s,\xi,\vartheta\right)dz - \int_0^\infty \hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,z,s\right)\hat{Q}_\lambda\left(z,s,\xi,\vartheta\right)dz\right)ds\right| \le C\left(t-\vartheta\right)^{\frac{1-l}{3}}\exp\left(-\frac{1}{2}\hat{c}_0\frac{\left(\hat{\kappa}-x\right)^2+\frac{1}{2}\left(\hat{\kappa}-\xi\right)^2}{t-\vartheta}\right),$$

and

$$\left|\frac{\partial}{\partial t}\int_\vartheta^t\left(\int_0^\infty \hat{G}_{\hat{L}}\left(x,t,z,s\right)\hat{Q}_\lambda\left(z,s,\xi,\vartheta\right)dz - \int_0^\infty \hat{G}_{\hat{L},\hat{\kappa}}\left(x,t,z,s\right)\hat{Q}_\lambda\left(z,s,\xi,\vartheta\right)dz\right)ds\right| \le C\left(t-\vartheta\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\hat{c}_0\frac{\left(\hat{\kappa}-x\right)^2+\frac{1}{2}\left(\hat{\kappa}-\xi\right)^2}{t-\vartheta}\right).$$

Proof. Because of the bounds on $\hat{G}_{\hat{L}}$ and on $\hat{G}_{\hat{L}} - \hat{G}_{\hat{L},\hat{\kappa}}$ given in Lemma 5.2.5, this follows from similar calculations as in the proof of Lemma 5.1.5. □

Definition 5.2.7. *For $x, \xi > 0$ and $0 \le \vartheta < t \le 1$ let*

$$\hat{Q}_{I,0}\left(x,t,\xi,\vartheta\right) = \lambda\int_0^{e^x-1}\hat{G}_{\hat{L}_\lambda}\left(\ln\left(e^x-\zeta\right),t,\xi,\vartheta\right)dF\left(\zeta\right).$$

Let the sequence of functions

$$\left\{\hat{Q}_{I,n}\right\}_{n=0}^\infty$$

be defined inductively by

$$\hat{Q}_{I,n}\left(x,t,\xi,\vartheta\right) = \int_\vartheta^t\int_0^\infty \hat{Q}_{I,0}\left(x,t,z,s\right)\hat{Q}_{I,n-1}\left(z,s,\xi,\vartheta\right)dzds,$$
$$n \in 1,2,\ldots.$$

Let

$$\hat{Q}_I(x,t,\xi,\vartheta) = \sum_{n=0}^{\infty} \hat{Q}_{I,n}(x,t,\xi,\vartheta).$$

Let

$$\hat{G}_I(x,t,\xi,\vartheta) = \int_{\vartheta}^{t}\int_{0}^{\infty} \hat{G}_{\hat{L}_\lambda}(x,t,z,s)\,\hat{Q}_I(z,s,\xi,\vartheta)\,dzds. \tag{5.2.7}$$

Lemma 5.2.7. *Assume that $\sigma_R > 0$ and that the tail distribution $\bar{F}$ obeys the bound* (4.0.38).

(i) *There exists a constant C_β, depending on β such that, for every $(x,t,\xi,\vartheta) \in \mathcal{D}$ and every $(x',t) \in (x,\infty)\times(\vartheta,t)$*

$$\left|\hat{Q}_I(x,t,\xi,\vartheta)\right| \le C_\beta (t-\vartheta)^{-\frac{1}{2}} \exp(-\beta|x-\xi|), \tag{5.2.8}$$

$$\left|\hat{Q}_I(x,t,\xi,\vartheta)\right| \le C_\beta (t-\vartheta)^{-\frac{1}{2}} \times \Big(\exp(-2\beta|x-\xi|) + \exp(-\beta x)\Big), \tag{5.2.9}$$

$$\left|\hat{Q}_I(x,t,\xi,\vartheta) - \hat{Q}_I(x,t',\xi,\vartheta)\right| \le C_\beta (t-t')^{\frac{1}{4}} (t'-\vartheta)^{-\frac{3}{4}} \times \Big(\exp(-2\beta|x-\xi|) + \exp(-\beta x)\Big), \tag{5.2.10}$$

and

$$\left|\hat{Q}_I(x,t,\xi,\vartheta) - \hat{Q}_I(x',t,\xi,\vartheta)\right| \le C_\beta |x-x'|^{\alpha} (t-\vartheta)^{-\frac{1+\alpha}{3}} \times \Big(\exp(-2\beta|x-\xi|) + \exp(-2\beta|x'-\xi|) + \exp(-(\beta-\alpha)x)\Big). \tag{5.2.11}$$

(ii) *$\hat{Q}_I$ is a solution of the integral equation*

$$\hat{Q}_I(x,t,\xi,\vartheta) = \lambda\int_{0}^{e^x-1} \hat{G}_{\hat{L}_\lambda}(\ln(e^x-\zeta),t,\xi,\vartheta)\,dF(\zeta) + \lambda\int_{0}^{e^x-1} \hat{G}_I(\ln(e^x-\zeta),t,\xi,\vartheta)\,dF(\zeta). \tag{5.2.12}$$

(iii) *For every $(x,t,\xi,\vartheta) \in \mathcal{D}$ and every $l \in \{0,1,2\}$ the following identities are all valid:*

$$\frac{\partial^l \hat{G}_I(x,t,\xi,\vartheta)}{\partial x^l} = \int_{\vartheta}\int_{0}^{\hat{\kappa}} \frac{\partial^l \hat{G}_{\hat{L}_\lambda}(x,t,z,\vartheta)}{\partial x^l}\hat{Q}_I(z,s,\xi,\vartheta)\,dzds,$$

and

$$\frac{\partial \hat{G}_I(x,t,\xi,\vartheta)}{\partial t} = \hat{Q}_I(x,t,\xi,\vartheta) + \int_\vartheta \int_0^{\hat{\kappa}} \frac{\partial \hat{G}_{\hat{L}_\lambda}(x,t,z,\vartheta)}{\partial t} \hat{Q}_I(z,s,\xi,\vartheta)\, dz ds.$$

(iv) *There exists a constant C_β depending on β such that for every $(x,t,\xi,\vartheta) \in \mathcal{D}$ and every $l \in \{0,1\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l \hat{G}_I(x,t,\xi,\vartheta)}{\partial x^l}\right| \le C_\beta (t-\vartheta)^{\frac{1-l}{3}} \Big(\exp(-2\beta|x-\xi|) + \exp(-\beta x)\Big),$$

$$\left|\frac{\partial^2 \hat{G}_I(x,t,\xi,\vartheta)}{\partial x^2}\right| \le C_\beta (t-\vartheta)^{-\frac{1}{2}} \left(\exp(-2\beta|x-\xi|) + \exp\left(-\frac{1}{2}\beta x\right)\right),$$

$$\left|\frac{\partial \hat{G}_I(x,t,\xi,\vartheta)}{\partial t}\right| \le C_\beta (t-\vartheta)^{-\frac{1}{2}} \left(\exp(-2\beta|x-\xi|) + \exp\left(-\frac{1}{2}\beta x\right)\right),$$

and

$$\left|\hat{G}_I(x,t,\xi,\vartheta)\right| \le C_\beta \min(x,1)^{\frac{1}{4}} (t-\vartheta)^{\frac{1}{4}} \times \left(\exp\left(-\frac{1}{2}\beta|x-\xi|\right) + \exp\left(-\frac{1}{2}\beta\xi\right)\right).$$

Proof. This follows from similar calculations as in the proof of Lemma 4.2.13. □

Lemma 5.2.8. *Assume that $\sigma_R > 0$ and that the tail distribution $\bar{F}$ satisfies the bound (4.0.38). There exists a constant C such that for every $(x,t,\xi,\vartheta) \in \mathcal{D}_{\hat{\kappa}}$, and every $(x',t') \in (x,\hat{\kappa}) \times (0,t)$ the following inequalities are all valid:*

$$|Q_I(x,t,\xi,\vartheta) - Q_{I,\kappa}(x,t,\xi,\vartheta)| \le C(t-\vartheta)^{-\frac{1}{2}} \times \exp\left(-\frac{1}{8}\hat{c}_0 \frac{(\hat{\kappa}-y)^2 + (\hat{\kappa}-\xi)^2}{t-\vartheta}\right), \tag{5.2.13}$$

$$|(Q_I(x,t,\xi,\vartheta) - Q_{I,\kappa}(x,t,\xi,\vartheta)) - (Q_I(x',t,\xi,\vartheta) - Q_{I,\kappa}(x',t,\xi,\vartheta))| \le C(y'-y)^{\frac{1}{2}} (t-\vartheta)^{-\frac{3}{4}} \exp\left(-\frac{1}{8}\hat{c}_0 \frac{(\hat{\kappa}-y')^2 + (\hat{\kappa}-\xi)^2}{t-\vartheta}\right), \tag{5.2.14}$$

and

$$|(Q_I(x,t,\xi,\vartheta) - Q_{I,\kappa}(x,t,\xi,\vartheta)) - (Q_I(x,t',\xi,\vartheta) - Q_{I,\kappa}(x,t',\xi,\vartheta))| \le C(t-t')^{\frac{1}{4}} t'^{-\frac{3}{4}} \exp\left(-\frac{1}{8}\hat{c}_0 \frac{(\hat{\kappa}-y)^2 + (\hat{\kappa}-\xi)^2}{t-\vartheta}\right), \tag{5.2.15}$$

Proof. Because of the bounds given in Lemma 5.2.7 this follows from similar calculations as in the proof of Lemma 5.1.7 □

Definition 5.2.8. *For $x, \xi > 0$ and $0 \le \vartheta < t \le 1$ define*

$$\hat{G}_{\hat{A}}(x,t,\xi,\vartheta) := G_{\hat{L}_\lambda}(x,t,\xi,\vartheta) + \hat{G}_I(x,t,\xi,\vartheta),$$

define

$$\hat{\psi}_3(x,t) = \int_0^t \int_0^\infty \hat{G}_{\hat{A}}(x,t,\xi,\vartheta) H_2\left(e^\xi - 1s, \vartheta\right) d\xi d\vartheta,$$

and for $y \ge 0$ define

$$\psi_3(y,t) = \hat{\psi}_3(\ln(1+y), t).$$

Lemma 5.2.9. (i) *For every $x > 0$ and $t \in (0,1]$ and $l \in \{1,2\}$*

$$\frac{\partial^l \hat{\psi}_3(x,t)}{\partial x^l} = \int_0^t \int_0^\infty \frac{\partial^l \hat{G}_{\hat{A}}(x,t,\xi,\vartheta)}{\partial x^l} H_2\left(e^\xi - 1s, \vartheta\right) d\xi d\vartheta$$

and

$$\frac{\partial \hat{\psi}_3(x,t)}{\partial t} = H_2(e^x - 1, \vartheta) + \int_0^t \int_0^\infty \frac{\partial \hat{G}_{\hat{A}}(x,t,\xi,\vartheta)}{\partial t} H_2\left(e^\xi - 1s, \vartheta\right) d\xi d\vartheta.$$

(ii) *There exists a constant C_β, depending on β, such that, for every $x > 0$ and $t \in (0,1]$ and every $l \in \{0,1\}$ the following inequalities are all valid:*

$$\left|\frac{\partial^l \hat{\psi}_3(x,t)}{\partial x^l}\right| \le C_\beta t^{\frac{4-l}{3}} \exp(-\beta x),$$

$$\left|\frac{\partial^2 \hat{\psi}_3(x,t)}{\partial x^2}\right| \le C_\beta t \exp\left(-\frac{1}{2}\beta x\right),$$

$$\left|\frac{\partial \hat{\psi}_3(x,t)}{\partial t}\right| \le C_\beta t \exp\left(-\frac{1}{2}\beta x\right),$$

and

$$\left|\hat{\psi}_3(x,t)\right| \le C_\beta t \min(x,1).$$

(iii) *There exists a constant C_β, depending on β, such that, for every $y > 0$, every $t \in (0,1]$ and every $l \in \{0,1\}$*

$$\left|\frac{\partial^l \psi_3(y,t)}{\partial y^l}\right| \le C_\beta t^{\frac{4-l}{3}} (1+y)^{-(\beta+l)},$$

$$\left|\frac{\partial^2 \psi_3(y,t)}{\partial y^2}\right| \le C_\beta t (1+y)^{-\left(\frac{1}{2}\beta+2\right)},$$

$$\left|\frac{\partial \psi_3(y,t)}{\partial t}\right| \le C_\beta t (1+y)^{-\frac{1}{2}\beta}$$

and

$$|\psi_3(x,t)| \le C_\beta t \min(y, \kappa - y).$$

(iv) *There exists a constant C and a constant C_β, depending on β, such that, for every $(y,t,) \in (0,\kappa) \times (0,1]$*

$$\left|\frac{\partial^l \psi(y,t)}{\partial y^l}\right| \leq Ct^{-\frac{l}{3}} \exp\left(-\frac{1}{4}\hat{c}_0\frac{y^2}{t}\right) + C_\beta t^{\frac{2-l}{3}} (1+y)^{-(\beta+l)},$$

$$\left|\frac{\partial^2 \psi(y,t)}{\partial y^2}\right| \leq Ct^{-1} \exp\left(-\frac{1}{4}\hat{c}_0\frac{y^2}{t}\right) + C_\beta (1+y)^{-\left(\frac{1}{2}\beta+2\right)},$$

and

$$\left|\frac{\partial \psi(y,t)}{\partial t}\right| \leq Ct^{-1} \exp\left(-\frac{1}{4}\hat{c}_0\frac{y^2}{t}\right) + C_\beta (1+y)^{-\frac{1}{2}\beta}.$$

(v) *There exists a constant C_β, depending on β, such that for every $(x,t) \in (0,\hat{\kappa}) \times (0,1]$ and every $l \in \{0,1\}$*

$$\left|\frac{\partial^l \hat{\psi}_3(x,t)}{\partial x^l} - \frac{\partial^l \hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial x^l}\right| \leq C_\beta t^{\frac{2-l}{3}} \exp(-\beta\hat{\kappa}) \exp\left(-\frac{1}{128}\hat{c}_0\frac{(\hat{\kappa}-x)^2}{t}\right),$$

$$\left|\frac{\partial^2 \hat{\psi}_3(x,t)}{\partial x^2} - \frac{\partial^2 \hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial x^2}\right| \leq C_\beta \exp\left(-\frac{1}{2}\beta\hat{\kappa}\right) \exp\left(-\frac{1}{128}\hat{c}_0\frac{(\hat{\kappa}-x)^2}{t}\right)$$

and

$$\left|\frac{\partial \hat{\psi}_3(x,t)}{\partial t} - \frac{\partial^l \hat{\psi}_{3,\hat{\kappa}}(x,t)}{\partial t}\right| \leq C_\beta \exp\left(-\frac{1}{2}\beta\hat{\kappa}\right) \exp\left(-\frac{1}{128}\hat{c}_0\frac{(\hat{\kappa}-x)^2}{t}\right).$$

(vi) *There exists a constant C_β, depending on β, such that for every $(y,t) \in (0,\kappa) \times (0,1]$ and every $l \in \{0,1\}$*

$$\left|\frac{\partial^l \psi_3(y,t)}{\partial y^l} - \frac{\partial^l \psi_{3,\kappa}(y,t)}{\partial y^l}\right| \leq C_\beta t^{\frac{2-l}{3}} (1+\kappa)^{-\beta}) \times \exp\left(-\frac{1}{128}\hat{c}_0\left[\ln\left(\frac{1+\kappa}{1+y}\right)\right]^2\right),$$

$$\left|\frac{\partial^2 \psi_3(y,t)}{\partial y^2} - \frac{\partial^2 \psi_{3,\kappa}(y,t)}{\partial y^2}\right| \leq C_\beta t^{\frac{2-l}{3}} (1+\kappa)^{-\frac{1}{2}\beta} (1+y)^{-2} \times \exp\left(-\frac{1}{128}\hat{c}_0\left[\ln\left(\frac{1+\kappa}{1+y}\right)\right]^2\right)$$

and

$$\left|\frac{\partial \psi_3(y,t)}{\partial t} - \frac{\partial \psi_{3,\kappa}(y,t)}{\partial t}\right| \leq C_\beta t^{\frac{2-l}{3}} (1+\kappa)^{-\frac{1}{2}\beta} \times \exp\left(-\frac{1}{8}\hat{c}_0\left[\ln\left(\frac{1+\kappa}{1+y}\right)\right]^2\right).$$

Proof. This follows from similar calculations as in the proof of Theorem 4.2.3. □

We are now finally in position to establish existence on unbounded domain for the main case $\sigma_R > 0$.

Theorem 5.2.2. *Assume that $\sigma_R > 0$ and that the tail distribution $\bar{F}$ satisfies the bound* (4.0.38). *$\hat{\psi}_3(x,t)$ is a classical solution of the PIDE* (5.2.2) *and $\psi_3(y,t)$ is a classical solution of the PIDE* (5.0.26).

Proof. It follows from the identities given in Lemma 5.2.6 and Lemma 5.2.9 that $\hat{\psi}_3(x,t)$ satisfies the PIDE (5.2.2) on the inner domain, i.e for $y > 0$ and $t \in (0,1]$. Similar arguments as in the proof of Lemma 5.1.1 yield that $\hat{\psi}_3(x,t)$ satisfies the initial condition and the boundary conditions. Since $\hat{\psi}_3(x,t)$ is a classical solution of the PIDE (5.2.2) it follows from the chain rule that $\psi_3(y,t) = \hat{\psi}_3(\ln(1+y), t)$ is a solution of the PIDE (5.0.26). □

References

Garroni, M. and Menaldi, J. (1992). *Green functions for second order parabolic integro-differential problems.* Longman Group UK Limited.

Garroni, M. and Menaldi, J. (2002). *Second Order Elliptic Integro-Differential Problems.* Chapman and Hall.

Karatzas, I. and Shreve, S. (1991). *Brownian Motion and Stochastic Calculus.* Springer.

Paulsen, J. (2008). Ruin Models with investment income. *Probability Surveys*, 5:416–434.