Classical and Stochastic Slit Löwner Evolution

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Abstract

The 2-dimensional discrete random walk is one of the most simple and most important lattice processes in the plane. It is the well-known that its scaling limit, as the mesh size approaches zero, is given by the 2-dimensional Brownian motion. An important property of the 2-dimensional Brownian motion is its conformal invariance. In statistical physics, however, there is a number of other important lattice models in two dimensions expected to have conformally invariant scaling limits. Until recently, describing those limits was considered an extremely challenging task.

In 2000, Oded Schramm combined the Löwner equation, a well-known tool in the geometric function theory, with stochastic calculus in order to tackle this problem. This approach was highly successful, and the resulting 2-dimensional process, known today as the Schramm-Löwner evolution, has since become one of the most researched topics in probability.

The underlying deterministic Löwner equation has also attracted renewed interest in the last decade. In particular, the so-called general Löwner theory was created, allowing to treat three versions of the Löwner equation (radial, chordal and dipolar) in a uniform way.

The general theory describes a vast class of evolutions of complex domains. However, several crucial features common to radial, chordal and dipolar equation were lost in generalization. In this work, we study a subclass of general Löwner equations preserving those features. In particular, the evolutions have slit geometry and in the stochastic case lead to holomorphic stochastic flows.

The thesis is organized as follows. In Section 1.1, we cover most of the preliminaries in complex analysis and probability beyond the graduate course level, which are needed for understanding the main results of the thesis. In Section 1.2 we give a brief introduction to the Löwner theory, with an emphasis on the general Löwner theory. Section 1.3 describes the Gaussian free field, a random object through which Schramm-Löwner evolution interacts with the conformal field theory (CFT).

We include four research papers, preceded by their brief descriptions. Paper A deals with geometrical aspects of the deterministic radial and chordal Löwner equations. Paper B studies a subclass of general Löwner equations with a “boundary attracting point”, which leads to holomorphic stochastic flows in the stochastic case. In Paper C we define general slit Löwner chains and general slit holomorphic stochastic flows. In Paper D we look at how the Gaussian free field is related to general slit holomorphic stochastic flows.

Contents

Part I

Introduction and Main Results

Chapter 1

Introduction

1.1 Preliminaries

1.1.1 Topics in complex analysis

Riemann mapping theorem and theory of prime ends

Let D be a simply connected open subset of the compactified complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, such that the complement $\hat{\mathbb{C}} \setminus D$ contains infinitely many points. A classical result in complex analysis, the Riemann mapping theorem, states that D is conformally isomorphic to the unit disk $\mathbb{D} = \{z : |z|< 1\}$, i.e., there exists a holomorphic bijection $f : \mathbb{D} \to D$. It is usual to say in this case that D is a simply connected *hyperbolic domain*.

Let $\bar{\mathbb{D}} = \mathbb{D} \cup \partial\mathbb{D}$, and $\bar{D} = D \cup \partial D$. According to the Carathéodory extension theorem (see [Pom75]), the mapping f can be extended to a homeomorphism $f : \bar{\mathbb{D}} \to \bar{D}$ if and only if the boundary ∂D is a Jordan curve (i.e., a non-self-intersecting continuous loop).

It is possible, however, to generalize the notion of boundary points in a way that such a homeomorphic extension of f can be constructed for any simply connected hyperbolic domain, no matter how complicated its boundary is. These generalized boundary points are called *prime ends*, and they are defined as equivalence classes of so-called fundamental chains of crosscut neighborhoods. This construction is well-known and can be found in many sources, for instance, in [Mil06, §17].

We use $P(D)$ to denote the set of prime ends of D, and write $\hat{D} = D \cup P(D)$. Then $\hat{D}$ can be given a topology such that $f : \mathbb{D} \to D$ extends to a homeomorphism $\hat{f} : \bar{\mathbb{D}} \to \hat{D}$. The set $\hat{D}$ endowed with this topology is called the *Carathéodory compactification* of D, and $\bar{\mathbb{D}}$ is canonically homeomorphic to $\hat{\mathbb{D}}$.

With this terminology in hand we can formulate the following version of the Riemann mapping theorem.

Theorem (the Riemann mapping theorem)**.** *Let $D \subset \hat{\mathbb{C}}$ be a simply connected hyperbolic domain, let $a \in D$, and let the prime ends b_1, b_2, $b_3 \in P(D)$ be distinct and ordered anticlockwise. Then there exists a* *conformal isomorphism*

$$f : \mathbb{D} \to D,$$

which can be uniquely extended to a homeomorphism of Carathéodory compactifications

$$\hat{f} : \hat{\mathbb{D}} \to \hat{D}.$$

The map f can be specified uniquely by imposing any of the following three normalization conditions

1. $\hat{f}(0) = a$ *and* $\hat{f}'(0) > 0$,

2. $\hat{f}(0) = a$ *and* $\hat{f}(1) = b_1$,

3. $\hat{f}(-i) = b_1$, $\hat{f}(1) = b_2$, *and* $\hat{f}(i) = b_3$.

Holomorphic semiflows in the unit disk

This section contains a brief overview of the most basic results in the theory of holomorphic semiflows. Detailed explanations and proofs can be found, e.g., in [Sho01].

Let $\mathrm{Hol}(\mathbb{D}, \mathbb{D})$ denote the set of holomorphic maps of $\mathbb{D}$ into itself, and let $\mathrm{Aut}(\mathbb{D}) \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ be the set of holomorphic automorphisms of $\mathbb{D}$.

Definition 1. A family $\{g_t\}_{t\geq 0} \subset \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ is called a *holomorphic semiflow* in $\mathbb{D}$ (or, alternatively, a one-parameter continuous semigroup of holomorphic self-mappings of $\mathbb{D}$) if

1. $g_0 = \mathrm{id}_{\mathbb{D}}$,

2. $g_{t+s} = g_t \circ g_s$, for $s, t \geq 0$,

3. $\lim_{t\to 0^+} g_t(z) = z$ for each $z \in \mathbb{D}$.

These conditions imply that the family g_t is continuous in the local uniform topology (the topology of uniform convergence on compact subsets of $\mathbb{D}$). Moreover, for every fixed $z \in \mathbb{D}$, the map $t \mapsto g_t(z)$ is differentiable for all $t \in [0, +\infty)$.

In the case when $g_t \in \mathrm{Aut}(\mathbb{D})$ for all $t \geq 0$, the semiflow $\{g_t\}_{t\geq 0}$ can be extended to a *flow* $\{g_t\}_{t\in\mathbb{R}}$ by setting $g_{-t} := g_t^{-1}$.

For every holomorphic semiflow $\{g_t\}_{t\geq 0}$ there exists a unique holomorphic function $V : \mathbb{D} \to \mathbb{C}$, such that

$$\begin{cases} \frac{\partial}{\partial t} g_t(z) = V(g_t(z)), \\ g_0(z) = z, \end{cases} \qquad t \geq 0,\ z \in \mathbb{D}. \tag{1.1}$$

The function V is called the *infinitesimal generator* of the semiflow $\{g_t\}_{t\geq 0}$. Infinitesimal generators of flows and semiflows are often called *complete* and *semicomplete holomorphic vector fields*, respectively.

There is a simple representation formula for semicomplete vector fields. A holomorphic function $V(z)$ is a semicomplete holomorphic vector field if and only if it can be written in the form

$$V(z) = V(0) - z\,q(z) - \overline{V(0)}\,z^2, \tag{1.2}$$

where $q(z)$ is a holomorphic function with $\operatorname{Re} q(z) \geq 0$. Moreover, $V(z)$ is complete if and only if $q(z) = ib$, $b \in \mathbb{R}$.

By the Schwarz lemma, a map $\phi \in \operatorname{Hol}(\mathbb{D}, \mathbb{D})$, $\phi \neq \operatorname{id}_{\mathbb{D}}$, may have at most one fixed point τ in $\mathbb{D}$. In this case, ϕ is said to be a mapping of *elliptic* (or *dilation*) type. Otherwise, by the Denjoy-Wolff theorem, there exists a unique point $\tau \in \partial\mathbb{D}$, such that $\angle \lim_{z\to\tau} \phi(z) = \tau$, and the angular derivative $\alpha = \angle \lim_{z\to\tau} \frac{\phi(z)-\tau}{z-\tau}$ exists with $0 < \alpha \leq 1$. If $0 < \alpha < 1$, then ϕ is of *hyperbolic* type, and if $\alpha = 1$, then ϕ is of *parabolic type*.

In all the three cases, the point τ is called the *Denjoy-Wolff point* of ϕ.

In a semiflow $\{g_t\}_{t\geq 0}$, all functions except for $g_0 = \operatorname{id}_{\mathbb{D}}$ are mappings of the same type, and they all have the same Denjoy-Wolff point τ. We can thus call τ the Denjoy-Wolff point of the semiflow $\{g_t\}_{t\geq 0}$.

The notion of the Denjoy-Wolff point leads to another important representation formula for semicomplete vector fields, which was found by Berkson and Porta [BP78]. A holomorphic function $V(z)$ is a semicomplete holomorphic vector field if and only if there exists a point $\tau \in \overline{\mathbb{D}}$ and a holomorphic function $p : \mathbb{D} \to \mathbb{C}$, with $\operatorname{Re} p \geq 0$, such that

$$V(z) = (z - \tau)(\bar{\tau}\,z - 1)\,p(z), \quad z \in \mathbb{D}. \tag{1.3}$$

This representation is unique if $V(z) \not\equiv 0$. The point τ in this representation is precisely the Denjoy-Wolff point of the generated semiflow $\{g_t\}_{t\geq 0}$. We call the function $p(z)$ the *Herglotz function*, and the point τ the *attracting point* of the field $V(z)$. The term "attracting point" is motivated by the fact that $g_t(z) \to \tau$ locally uniformly in $\mathbb{D}$ as $t \to +\infty$, except for the case when $\{g_t\}_{t\geq 0}$ is a semiflow consisting of elliptic automorphisms.

Pushforwards of holomorphic vector fields

It is easy to extend the definition of holomorphic semiflows to an arbitrary simply-connected hyperbolic domain D, which amounts to replacing $\mathbb{D}$ by D in Definition 1.

Let $f : D_1 \to D_2$ be a conformal isomorphism of two simply connected hyperbolic domains. If $\{g_t^1\}_{t\geq 0}$ is a semiflow in D_1, then $\{g_t^2\}_{t\geq 0} = \{\phi \circ g_t^1 \circ \phi^{-1}\}_{t\geq 0}$ is a semiflow in D_2. Let V_1 and V_2 be the infinitesimal generators of $\{g_t^1\}_{t\geq 0}$ and $\{g_t^2\}_{t\geq 0}$, respectively. Then we say that the vector fields V_1 and V_2 are *ϕ-related*, and we call V_2 the *pushforward* of V_1 with respect to ϕ. We write $V_2 = \phi_* V_1$. Explicitly,

$$\phi_* V_1(z) = \frac{1}{\phi^{-1\prime}(z)}\,V_1(\phi^{-1}(z)).$$

Lie derivatives of complex functions

Let f be a C^1-smooth complex-valued function in a domain D, and let $z = x + iy$. Then it is common to write

$$\partial f = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right) f,$$

$$\bar{\partial} f = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right) f.$$

Let V be a holomorphic vector field in a domain D. Then V acts on a complex function $f \in C^1(D, \mathbb{C})$ according to the rule

$$f \mapsto \left(V\partial + \bar{V}\bar{\partial}\right) f.$$

We denote this operator by $\mathcal{L}_V$, so that

$$\mathcal{L}_V = V\,\partial + \bar{V}\,\bar{\partial}.$$

It is a special case of the *Lie derivative*. Geometrically, $\mathcal{L}_V$ represents the derivative of f in the direction of the semiflow generated by V.

Some of its basic properties are listed below.

1. $\mathcal{L}_V f$ is linear in f and V;
2. $\mathcal{L}_V \bar{f} = \overline{\mathcal{L}_V f}$, and, consequently, $\mathcal{L}_V \operatorname{Re} f = \operatorname{Re} \mathcal{L}_V f$, $\mathcal{L}_V \operatorname{Im} f = \operatorname{Im} \mathcal{L}_V f$;
3. $\mathcal{L}_V(\partial f) = \partial \mathcal{L}_V f$, $\mathcal{L}_V(\bar{\partial} f) = \bar{\partial} \mathcal{L}_V f$.

This definition of the Lie derivative can be extended to the case of several complex variables, that is, when V is a vector field in a domain $D \subset \mathbb{C}^n$, and $f : \mathbb{C}^n \to \mathbb{C}$. We write $z = (z_1, \ldots, z_n)$, $\partial = \left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\right)$, $\bar{\partial} = \left(\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}\right)$, $V(z) = (V_1(z), \ldots, V_n(z))$, and then the derivative of f along the flow generated by V is given by the sum of dot products

$$\mathcal{L}_V f(z) = \left(V(z) \cdot \partial + \overline{V(z)} \cdot \bar{\partial}\right) f(z). \tag{1.4}$$

1.1.2 Topics in stochastic analysis

Unless specified otherwise, the detailed formulations and proofs of the results mentioned in this section can be found in [Øks03] or [RY99].

Stochastic integrals of Itô and Stratonovich

The two most widely used types of stochastic integrals are those of Itô and Stratonovich. The corresponding constructions are fairly complicated; it is possible however to give intuitively clear definitions if the integrands are adapted continuous processes.

Let X_t be a continuous process adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of the standard Brownian motion B_t, and let $T > 0$. Then the *Itô integral* of X_t

with respect to B_t on $[0,T]$ can be defined as the limit in probability of the Riemann-Stieltjes sums

$$\int_0^T X_t\,dB_t := \lim_{\Delta t\to 0}\sum_{j=1}^n X_{t_{j-1}}\left(B_{t_j}-B_{t_{j-1}}\right), \tag{1.5}$$

where $0=t_0<t_1<\ldots<t_n=T$, and $\Delta t := \max_{j=1,\ldots,n}(t_j-t_{j-1})$.

Similarly, the *Stratonovich integral* can be defined as the limit in probability

$$\int_0^T X_t\circ dB_t := \lim_{\Delta t\to 0}\sum_{j=1}^n X_{\frac{t_{j-1}+t_j}{2}}\left(B_{t_j}-B_{t_{j-1}}\right). \tag{1.6}$$

The definitions of the Itô and Stratonovich stochastic integrals can be extended to more complicated integrands and integrators, but the intuitive formulas (1.5) and (1.6) will not hold in general.

For a fixed $T>0$, the Itô integral $\int_0^T X_s\,dB_s$ is a random variable, and it is defined up to a set of measure zero. If we allow the upper integral limit vary, then $\int_0^t X_s\,dB_s$ may be regarded as a stochastic process. More precisely, it can be proved that there exists a continuous process Y_t adapted to $\{\mathcal{F}_t\}_{t\geq 0}$such that

$$\mathbb{P}\left(Y_t=\int_0^t X_s\,dB_s\right)=1 \quad \text{for all } t\geq 0.$$

Similarly, the Stratonovich integral $\int_0^t X_s\circ dB_s$ with a variable upper limit also defines a continuous stochastic process.

If $\mathsf{E}\left(\int_0^T X_s^2\,ds\right)<\infty$ for all $T>0$, then $\int_0^t X_s\,dB_s$ is a martingale. If we relax this condition to $\mathbb{P}\left(\int_0^T X_s^2\,ds<\infty\right)=1$, then we can only guarantee that Y_t is a local martingale. Moreover, by the martingale representation theorem, every continuous $\{\mathcal{F}\}_{t\geq 0}$-adapted local martingale M_t can be written as $M_t=M_0+\int_0^t X_s\,dB_s$ for some $\mathcal{F}_t$-adapted process X_t.

Quadratic covariation

Recall that the *quadratic covariation* of two processes X_t and Y_t is defined as the limit in probability

$$\langle X_T, Y_T\rangle = \langle X, Y\rangle_T := \lim_{\Delta t\to 0}\sum_{j=1}^n (X_{t_j}-X_{t_{j-1}})(Y_{t_j}-Y_{t_{j-1}})$$

(if it exists). As before, $0=t_0<t_1<\ldots<t_n=T$, and $\Delta t := \max_{j=1,\ldots,n}(t_j-t_{j-1})$.

Suppose that the processes X_t and Y_t can be represented by

$$X_t(\omega)=\int_0^t b_1(\omega,s)\,ds+\int_0^t \sigma_1(\omega,s)\,dB_s(\omega), \tag{1.7}$$

and

$$Y_t(\omega)=\int_0^t b_2(\omega,s)\,ds+\int_0^t \sigma_2(\omega,s)\,dB_s(\omega),$$

where $b_j(\omega,t)$, $\sigma_j(\omega,t)$, $j=1,2$ are $\mathcal{F}_t$-adapted processes satisfying

$$\mathbb{P}\left(\int_0^T |b_j(\omega,s)|\,ds<\infty\right)=1,\quad \mathbb{P}\left(\int_0^T |\sigma_j(\omega,s)|^2\,ds\right)=1,$$

for all $T>0$ and $j=1,2$. Then the quadratic variation can be conveniently written as the following integral

$$\langle X,Y\rangle_t(\omega)=\int_0^t \sigma_1(\omega,s)\,\sigma_2(\omega,s)\,ds.$$

The notion of quadratic covariation provides a simple formula relating the two types of the stochastic integral mentioned above:

$$\int_0^T X_t\circ dB_t=\int_0^T X_t\,dB_t+\frac{1}{2}\langle X,B\rangle_T. \tag{1.8}$$

Complex Ito formula

Let $w_t = X_t^1 + i\,X_t^2$ be a complex-valued process such that the real-valued processes X_t^1, X_t^2 are of the form (1.7). Let $f:\mathbb{C}\to\mathbb{C}$ be a twice differentiable complex function. The complex Itô formula [Ubø87] describes how w_t transforms under such functions:

$$\begin{aligned} f(w_t) &= f(w_0)+\int_0^t \partial f(w_s)\,dw_s+\int_0^t \bar\partial f(w_s)\,d\bar w_s\\ &\quad+\frac{1}{2}\int_0^t \partial^2 f(w_s)\,d\langle w\rangle_s+\frac{1}{2}\int_0^t \bar\partial^2 f(w_s)\,d\langle \bar w\rangle_s\\ &\quad+\int_0^t \partial\bar\partial f(w_s)\,d\langle w,\bar w\rangle_s, \end{aligned} \tag{1.9}$$

which is usually written in the shorter differential form

$$\begin{aligned} df(w_t) &= \partial f(w_t)\,dw_t+\bar\partial f(w_t)\,d\bar w_t\\ &\quad+\frac{1}{2}\,\partial^2 f(w_t)\,d\langle w\rangle_t+\frac{1}{2}\,\bar\partial^2 f(w_t)\,d\langle \bar w\rangle_t+\partial\bar\partial f(w_t)\,d\langle w,\bar w\rangle_t. \end{aligned} \tag{1.10}$$

Suppose w_t satisfies the following Stratonovich integral equation

$$w_t=w_0+\int_0^t b(w_s)\,ds+\sum_{k=1}^n\int_0^t \sigma_k(w_s)\circ dB_s^k, \tag{1.11}$$

with holomorphic coefficients b and σ, and n independent Brownian motions $B_t^1,\ldots,B_t^n$. A more usual, but formal, way to write (1.11) is the Stratonovich stochastic differential equation (SDE)

$$dw_t=b(w_t)\,dt+\sum_{k=1}^n \sigma_k(w_t)\circ dB_t^k. \tag{1.12}$$

It can be deduced from (1.8) and (1.10) that (1.12) is equivalent to the following Itô SDE[1]

$$dw_t = \left(b(w_t) + \frac{1}{2} \sum_{k=1}^{n} \sigma_k(w_t)\, \sigma_k'(w_t) \right) dt + \sum_{k=1}^{n} \sigma_k(w_t)\, dB_t^k,$$

which is shorthand for the Itô integral equation

$$w_t = w_0 + \int_0^t \left(b(w_s) + \frac{1}{2} \sum_{k=1}^{n} \sigma_k(w_s)\, \sigma_k'(w_s) \right) ds + \sum_{k=1}^{n} \int_0^t \sigma_k(w_s)\, dB_s^k.$$

Processes satisfying (1.12) are a special case of diffusion processes. For diffusion processes with holomorphic coefficients, as in (1.12), the complex Itô formula can be elegantly formulated using the Lie derivatives notation. Given a twice differentiable function $f : \mathbb{C} \to \mathbb{C}$, the process $f(w_t)$ satisfies the following SDE

$$\begin{aligned} df(w_t) &= \mathcal{L}_b f(w_t)\, dt + \sum_{k=1}^{n} \mathcal{L}_{\sigma_k} f(w_t) \circ dB_t^k \\ &= \left(\mathcal{L}_b + \frac{1}{2} \sum_{k=1}^{n} \mathcal{L}_{\sigma_k}^2 \right) f(w_t)\, dt + \sum_{k=1}^{n} \mathcal{L}_{\sigma_k} f(w_t)\, dB_t^k. \end{aligned} \tag{1.13}$$

The formula remains valid if the process w_t takes values in $\mathbb{C}^n$, f is a C^2 complex function of n complex variables, and the coefficients $b : \mathbb{C}^n \to \mathbb{C}$, $\sigma_k : \mathbb{C}^n \to \mathbb{C}$ are holomorphic in each variable.

Stochastic flows in the complex plane

In the deterministic case, the flow (semiflow) $\{g_t\}_{t\geq 0}$ of an ordinary autonomous differential equation

$$\frac{\partial}{\partial t} g_t(z) = V(g_t(z)), \quad g_0(z) = z,$$

describes how the ODE's solution depends on the initial condition, so that for a fixed value of t, the map $g_t(\cdot)$ is a certain transformation of the space on which the ODE is considered. For instance, if the equation is considered in the unit disk $\mathbb{D}$ and V is a semicomplete holomorphic vector field in $\mathbb{D}$, then $g_t \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ for each $t \geq 0$, as we have seen above.

In a similar vein, one can study flows of stochastic differential equations. Let $z \in D$, and let $w_t(z)$ denote the solution of (1.12) with the initial condition $w_0 = z$. The family $\{w_t\}_{t\geq 0}$ solving the problem

$$\begin{cases} dw_t(z) = b(w_t(z), t)\, dt + \sum_{k=1}^{n} \sigma_k(w_t(z), t) \circ dB_t^k, \\ w_0(z) = z, \quad z \in D. \end{cases} \tag{1.14}$$

[1] In this thesis, the prime mark always denotes the derivative of a holomorphic function with respect to the complex variable, thus, for example, $\sigma_k'(w)$ denotes $\frac{\partial}{\partial w}\sigma_k(w)$, and $f_t'(z)$ has the same meaning as $\frac{\partial}{\partial z} f_t(z)$.

is called the stochastic flow of (1.14). Comprehensive overviews of the theory of stochastic flows can be found in [Kun84], [Kun97] and [IW89].

For the existence of a unique maximal solution its is sufficient that the vector fields $b(z,t)$, $\sigma_1(z,t), \dots \sigma_n(z,t)$ are C^2 in z, and C^1 in t [Kun84, Theorem II.8.3].

Analytic properties of stochastic flows

Many analytic properties of a stochastic flow can be deduced from the analytic properties of the vector fields determining the flow.

Let $T(z)$ denote the random time when $w_t(z)$, the solution to (1.14) with the initial condition $w_t(z) = z$, ceases to exist. $T(z)$ is usually called the *explosion time* or the *escape time*.

Let D_t denote the random set of initial conditions for which the solution exists at least up to time $t > 0$, i.e., $D_t = \{z \in D : T(z) > t\}$. It is easy to see that $D_s \supseteq D_t$ for $s \le t$.

Let R_t denote the image of D_t with respect to w_t, i.e., $R_t := w_t(D_t)$. Then $w_t : D_t \to R_t$ is a *homeomorphism* for all $t \ge 0$ with probability 1 [Kun84, Theorem II.9.1].

Suppose that the vector field $b(z,t)$ is C^1 in $t > 0$ and C^d in $z \in D$, and the vector fields $\sigma_1(z,t), \dots, \sigma_n(z,t)$ are C^1 in t and C^{d+1} in z. Then $w_t : D_t \to R_t$ is a C^d*-diffeomorphism* for all $t \ge 0$ with probability 1 [Kun84, Theorem II.9.2].

In particular, if the fields $b(z,t)$, $\sigma_1(z,t), \dots \sigma_n(z,t)$ are C^1 in $t > 0$ and C^∞ in $z \in D$, then the $w_t(z) : D_t \to R_t$ is a C^∞*-diffeomorphism* for all $t \ge 0$ with probability 1.

If the vector fields $b(z,t)$, $\sigma_1(z,t), \dots \sigma_n(z,t)$ are C^1 in $t > 0$ and holomorphic in $z \in D$, then the $w_t(z) : D_t \to R_t$ is a *conformal isomorphism* for all $t \ge 0$ with probability 1 [Kun84, Theorem III.5.7].

Consider the flow $w_t(z)$ of the autonomous SDE

$$\begin{cases} dw_t(z) = b(w_t(z))\,dt + \sum_{k=1}^n \sigma_k(w_t(z)) \circ dB_t^k, \\ w_0(z) = z, \quad z \in D, \end{cases} \tag{1.15}$$

where $b(z)$, $\sigma_1(z), \dots, \sigma_n(z)$ are C^∞, complete, time-independent vector fields. Then $w_t(z)$ is a diffeomorphism of D for each $t \ge 0$ with probability 1. If the Lie algebra $\mathfrak{g}$ generated by these vector fields is finite-dimensional, then $w_t(z)$ takes values in the associated Lie group of diffeomorphisms [Kun84, Theorem III.5.1].

Composition and inversion of stochastic flows

Stochastic differential equations written in the Stratonovich form obey the transformation rules of classical calculus. The chain rule and related formulas can be written in an especially neat form if we use the pushforward notation from Section 1.1.1.

Consider the flows

$$\begin{cases} dw_t(z) = b(w_t(z),t)\,dt + \sum_{k=1}^n \sigma_k(w_t(z),t) \circ dB_t^k, \\ w_0(z) = z, \quad z \in D, \end{cases} \tag{1.16}$$

and

$$\begin{cases} d\tilde{w}_t(z) = \tilde{b}(\tilde{w}_t(z), t)\, dt + \sum_{k=1}^{n} \tilde{\sigma}_k(\tilde{w}_t(z), t) \circ dB_t^k, \\ \tilde{w}_0(z) = z, \quad z \in D, \end{cases} \tag{1.17}$$

and let $T(z)$, $\tilde{T}(z)$ denote the corresponding explosion times.

Consider the composite flow $\{\zeta_t\}_{t\geq 0}$, $\zeta_t := \tilde{w}_t \circ w_t$. The explosion time for the composite flow is then given by $U(z) := \min[\inf\{t > 0 : w_t(z) \notin \tilde{D}_t\}, T(z)]$, where $\tilde{D}_t = \{z \in D : \tilde{T}(z) > t\}$.

The composite flow satisfies

$$\begin{aligned} d\zeta_t(z) = &\left[\tilde{b}(\zeta_t(z), t) + \tilde{w}_{t\,*}\, b(\zeta_t(z), t)\right] dt \\ &+ \sum_{k=1}^{n} \left[\tilde{\sigma}_k(\zeta_t(z), t) + \tilde{w}_{t\,*}\, \sigma_k(\zeta_t(z), t)\right] \circ dB_t^k, \end{aligned} \tag{1.18}$$

see [Kun84, Theorem III.3.1].

The inverse flow $\{\eta_t\}_{t\geq 0}$, $\eta_t = w_t^{-1}$, satisfies

$$\begin{aligned} d\eta_t(z) &= -\left(\eta_{t\,*}\, b\right)(\eta_t(z), t)\, dt - \sum_{k=1}^{n} \left(\eta_{t\,*}\, \sigma_k\right)(\eta_t(z), t) \circ dB_t^k \\ &= -\eta_t{}'(z)\, b(z, t)\, dt - \sum_{k=0}^{n} \eta_t{}'(z)\, \sigma_k(z, t) \circ dB_t^k, \end{aligned} \tag{1.19}$$

see [Kun84, Corollary III.3.4].

1.2 Löwner theory

Ninety years have passed since Karl Löwner (Charles Loewner) introduced his original equation in [Löw23], and today there is at least a dozen of objects referred to as the “Löwner equation”. There are chordal, radial, dipolar Löwner equations, Löwner’s ordinary and partial differential equations, forward and backward (reverse) equations, the Löwner-Kufarev equations, the general Löwner equations etc.

Moreover, there is a rich family of stochastic Löewner equations: chordal, radial and dipolar SLE_κ, as well as $SLE(\kappa, \rho)$. Stochastic Löwner equations are, in fact, the main reason why Löwner theory attracts so much interest nowadays.

To describe this plethora of equations in a concise way, we take the following approach. First, we introduce the main notions of the so-called general Löwner theory, a rather recent framework developed in [BCDM09], [CDMG10], [BCDM12] and [CDMG14]. Then we define all the other equations mentioned above as special cases of the general formulation.

In the simply connected case, the general Löwner theory has been developed for one (the unit disk) and several complex dimensions (complete hyperbolic manifolds). In [CDMG13] and [CDMG11] the general Löwner theory is constructed for the doubly connected one-dimensional case (the annulus).

Only the one-dimensional simply connected theory is relevant to the problems considered in this thesis.

The general Löwner theory has been formulated in two versions, which reflects the fact that the the numerous Löwner equations fall into two classes, so-called "forward" and "backward" equations. The reason for this terminology is that the equations of one class can be obtained as time-reversed versions of the equations from the other class, i.e., by replacing the time variable t by $-t$.

There seems to be a certain degree of disagreement in the terminology. For instance, the words "backward" and "time-reversed" have opposite meanings if we compare, for example, [CDMG14] with [Law09], [DS11], [RZ13]. In this thesis we try to follow the latter convention.

1.2.1 Backward (increasing) general Löwner theory

The equation considered by Löwner in his seminal paper [Löw23] belongs to the realm of the *backward* Löwner theory (or the *increasing* Löwner theory, if we use the terminology of [CDMG14]). Backward theory is closely related to the theory of semigroups of analytic functions, and to a certain extent can be regarded as its non-autonomous generalization. In many cases, backward equations are easier to work with, compared to the forward equations that we consider in the next section.

The three central objects of the general Löwner theory are evolution families, Herglotz vector fields and general Löwner chains.

Throughout this section we assume that the evolution families, Herglotz vector fields and Löwner chains are defined in the unit disk. This is merely a convention, and all the results of the general Löwner theory (except for some representation formulas) translate to the case of an arbitrary simply-connected hyperbolic domain, for instance, the upper half-plane $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$ or the infinite strip $\mathbb{S} = \{z : 0 < \operatorname{Im} z < 1\}$. We call the domain in which the evolution families are defined the *canonical domain*.

Evolution families and Herglotz vector fields

Definition 2. An *evolution family* of order $d \in [1, +\infty]$ is a two-parameter family $\{\phi_{s,t}\}_{0 \le s \le t < +\infty}$ of holomorphic self-maps of the unit disk, such that the following three conditions are satisfied.

- $\phi_{s,s} = id_{\mathbb{D}}$;
- $\phi_{s,t} = \phi_{u,t} \circ \phi_{s,u}$ for all $0 \le s \le u \le t < +\infty$;
- for any $z \in \mathbb{D}$ and $T > 0$ there is a non-negative function $k_{z,T} \in L^d([0, T], \mathbb{R})$, such that

$$|\phi_{s,u}(z) - \phi_{s,t}(z)| \le \int_u^t k_{z,T}(\xi) d\xi \tag{1.20}$$

for all $0 \le s \le u \le t \le T$.

The condition (1.20) implies that for every fixed $z \in \mathbb{D}$, the function $\phi_{s,t}(z)$ is differentiable with respect to t almost everywhere on $[s, +\infty)$ (by absolute continuity). An infinitesimal description of an evolution family is given in terms of a *Herglotz vector field*.

Definition 3. A (generalized) *Herglotz vector field* of order $d \in [1, +\infty]$ is a function $V : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ satisfying the following conditions:

- the function $[0, +\infty) \ni t \mapsto V(z, t)$ is measurable for every $z \in \mathbb{D}$;
- the function $z \mapsto V(z, t)$ is holomorphic in the unit disk for $t \in [0, +\infty)$;
- for any compact set $K \subset \mathbb{D}$ and for every $T > 0$ there exists a non-negative function $k_{K,T} \in L^d([0, T], \mathbb{R})$, such that

$$|V(z, t)| \leq k_{K,T}(z)$$

  for all $z \in K$ and almost every $t \in [0, T]$;
- for almost every $t \in [0, +\infty)$ the vector field $V(\cdot, t)$ is semicomplete.

Theorem 1 ([BCDM12, Theorem 1.1])**.** *For any evolution family $\{\phi_{s,t}\}$ of order $d \geq 1$ in the unit disk there exists an essentially unique Herglotz vector field $V(z, t)$ of order d, such that for all $z \in \mathbb{D}$ and for almost all $t \in [0, +\infty)$*

$$\frac{\partial}{\partial t}\phi_{s,t}(z) = V(\phi_{s,t}(z), t). \tag{1.21}$$

Conversely, for any Herglotz vector field $V(z, t)$ of order $d \geq 1$ in the unit disk there exists a unique evolution family $\{\phi_{s,t}\}_{0 \leq s \leq t < +\infty}$ of order d, such that the equation above is satisfied.

Essential uniqueness in the theorem above means that for any other Herglotz vector field $H(z, t)$ satisfying (1.21), the equality $H(z, t) = V(z, t)$ holds for all $z \in \mathbb{D}$ and almost all $t \in [0, +\infty)$.

Löwner chains

Recall that the term *"univalent"* is shorthand for "injective and holomorphic".

Definition 4. A family $\{f_t\}_{0 \leq t < \infty}$ of maps $f_t : \mathbb{D} \to \mathbb{C}$ is called a (backward) Löwner chain of order $d \in [1, +\infty]$ if

1. each function f_t is univalent,
2. $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ for $0 \leq s < t < +\infty$,
3. for any compact set $K \subset \mathbb{D}$ and all $T > 0$ there exists a non-negative function $k_{K,T} \in L^d([0, T], \mathbb{R})$ such that

$$|f_s(z) - f_t(z)| \leq \int_s^t k_{K,T}(\xi) d\xi$$

   for all $z \in K$ and all $0 \leq s \leq t \leq T$.

The domain $f_t(\mathbb{D})$ is called the *evolution domain* at the time t.

Every Löwner chain $\{f_t\}_{t\geq 0}$ of order d generates an evolution family $\{\phi_{s,t}\}$ of the same order d defined by

$$\phi_{s,t} = f_t^{-1} \circ f_s. \tag{1.22}$$

This correspondence is, however, not one-to one – there may be many different Löwner chains associated to the given evolution family. Fortunately, they are unique up to normalization and composition with a univalent function, as the following theorem states.

Theorem 2 ([CDMG10, Theorems 1.6-1.7])**.** *For any evolution family $\{\phi_{s,t}\}_{0\leq s\leq t<+\infty}$ of order d, there exists a unique Löwner chain $\{f_t\}_{t\geq 0}$ of the same order d, such that*

1. $\phi_{s,t} = f_t^{-1} \circ f_s$ *for any* $0 \leq s \leq t$;
2. $f(0) = 0$ *and* $f'(0) = 1$;
3. $\Omega := \cup_{t\geq 0} f_t(\mathbb{D})$ *is a disk* $\{z : |z| < R\}$ *of possibly infinite radius* $R \in (0, +\infty]$.

Any other Löwner chain satisfying Condition 1 is of the form $\{g_t\}_{t\geq 0} = \{F \circ f_t\}_{t\geq 0}$, *where* $F : \Omega \to \mathbb{C}$ *is univalent.*

The radius R can be calculated explicitly as $1/\beta_0$, where

$$\beta_0 = \lim_{t\to+\infty} \frac{|\phi'_{0,t}(0)|}{1 - |\phi_{0,t}(z)|^2}.$$

It was also shown [CDMG10, Theorem 4.1] that every Löwner chain $\{f_t\}_{t\geq 0}$ of order d satisfies the generalized Löwner PDE

$$\frac{\partial}{\partial s} f_s(z) = -V(z,s) f'_s(z) \quad \text{(for almost all } s \geq 0), \tag{1.23}$$

where $V(z,s)$ is the Herglotz vector field generating the associated evolution family $\{\phi_{s,t}\}_{0\leq s\leq t<+\infty}$.

Berkson-Porta representation

Due to last condition in Definition 3, one should be able to represent $V(\cdot, t)$ in the form (1.3) for almost every fixed $t \geq 0$. In order to describe how the function p in the representation depends on t, the notion of *generalized Herglotz functions* was introduced.

Definition 5. A Herglotz function of order $d \in [1, +\infty)$ is a function $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$, such that

- the function $t \mapsto p(z,t)$ belongs to $L^d_{\text{loc}}([0, +\infty), \mathbb{C})$ for all $z \in \mathbb{D}$;
- the function $z \mapsto p(z,t)$ is holomorphic in $\mathbb{D}$ for every fixed $t \in [0, +\infty)$;

- $\operatorname{Re} p(z,t) \geq 0$ for all $z \in \mathbb{D}$ and for all $t \in [0,+\infty)$.

This definition lets us extend the Berkson-Porta representation formula from the case of autonomous semicomplete fields to general Herglotz vector fields.

Theorem 3 ([BCDM12, Theorem 1.2])**.** *Given a Herglotz vector field $V(z,t)$ of order $d \geq 1$ in the unit disk, there exists an essentially unique (i. e., defined uniquely for almost all t for which $V(\cdot,t) \neq 0$) measurable function $\tau : [0,+\infty) \to \overline{\mathbb{D}}$ and a Herglotz function $p(z,t)$ of order d, such that for all $z \in \mathbb{D}$ and almost all $t \in [0,+\infty)$*

$$V(z,t) = (z-\tau(t))\,(\overline{\tau(t)}\,z-1)\,p(z,t). \tag{1.24}$$

Conversely, given a measurable function $\tau : [0,+\infty) \to \overline{\mathbb{D}}$ and a Herglotz function $p(z,t)$ of order $d \geq 1$, the vector field defined by the formula above is a Herglotz vector field of order d.

According to Theorem 1, to every evolution family $\{\phi_{s,t}\}$ one can associate an essentially unique Herglotz vector field $V(z,t)$. The pair of functions (p,τ) representing the vector field $V(z,t)$ is called the *Berkson-Porta data* of the evolution family $\{\phi_{s,t}\}_{0\leq s\leq t<+\infty}$.

One of the advantages of representation (1.24) is that it provides geometric intuition about the behavior of the corresponding evolution family. By analogy with the time-independent case, we call the function $\tau(t)$ the *attracting point* of the Herglotz vector field $V(z,t)$ and of the evolution family $\{\phi_{s,t}\}_{0\leq s\leq t<+\infty}$.

1.2.2 Forward (decreasing) Löwner theory

The forward Löwner theory is the preferred tool in the study of Schramm-Löwner evolution (see Section 1.2.7).

Definition 6 ([CDMG14, Definition 1.9])**.** Let $d \in [1,+\infty]$. A family $\{\phi_{s,t}\}_{0\leq s\leq t<+\infty} \subset \operatorname{Hol}(\mathbb{D},\mathbb{D})$ is called a *reverse (or forward) evolution family of order d* if it satisfies the following conditions

1. $\phi_{s,s} = \mathrm{id}_{\mathbb{D}}$,
2. $\phi_{s,t} = \phi_{s,u} \circ \phi_{u,t}$ for all $0 \leq s \leq u \leq t < \infty$,
3. for any $z \in \mathbb{D}$ and any $T > 0$ there exists a non-negative function $k_{z,T} \in L^2([0,T],\mathbb{R})$ such that

$$|\phi_{s,u}(z) - \phi_{s,t}(z)| \leq \int_u^t k_{z,T}(\xi)\,d\xi,$$

for all $s,t,u \in [0,T]$ satisfying inequality $s \leq u \leq t$.

Definition 7 ([CDMG14, Definition 1.6])**.** Let $d \in [1,+\infty]$. A family $\{f_t\}_{t\geq 0} \subset \operatorname{Hol}(\mathbb{D},\mathbb{D})$ is called a *forward (or decreasing) Löwner chain of order d* if it satisfies the following conditions:

1. each function $f_t : \mathbb{D} \to \mathbb{D}$ is univalent,
2. $f_0 = \mathrm{id}_{\mathbb{D}}$, and $f_s(\mathbb{D}) \supset f_t(\mathbb{D})$ for $0 \le s < t < +\infty$,
3. for any compact set $K \subset \mathbb{D}$ and for all $T > 0$ there exists a non-negative function $k_{K,T} \in L^d([0,T], \mathbb{R})$ such that

$$|f_s(z) - f_t(z)| \le \int_s^t k_{K,T}(\xi) d\xi$$

for all $z \in K$ and for all $0 \le s \le t \le T$.

Similarly to the backward case, there is a one-to-one correspondence between forward evolution families and forward Löwner chain. Given a Löwner chain $\{f_t\}_{t \ge 0}$, we can obtain the corresponding evolution family using the following analog of (1.22)

$$\phi_{s,t}(z) = f_s^{-1} \circ f_t, \quad 0 \le s \le t < \infty. \tag{1.25}$$

And conversely, given an evolution family $\{\phi_{s,t}\}_{0 \le s \le t < +\infty}$, we can recover the Löwner chain using the simple formula $f_t(z) := \phi_{0,t}$.

Löwner chains are related to Herglotz vector fields by means of the forward Löwner ODE. The precise statement is given in the following theorem.

Theorem 4 ([CDMG14, Theorem 1.11])**.** *Let V be a Herglotz vector field of order $d \in [1, +\infty]$. Then,*

1. *For every $z \in \mathbb{D}$, there exists a unique maximal solution $g_t(z) \in \mathbb{D}$ to the following initial value problem*

$$\begin{cases} \frac{\partial}{\partial t} g_t(z) = -V(g_t(z), t), \\ g_0(z) = z. \end{cases} \tag{1.26}$$

2. *For every $t \ge 0$, the set D_t of all $z \in \mathbb{D}$, for which $g_t(z)$ is defined at the point t, is a simply connected domain, and the function $g_t(z)$ defined for all $z \in D_t$ maps D_t conformally onto $\mathbb{D}$.*
3. *The functions $f_t := g_t^{-1}$ form a decreasing Löwner chain of order d, which is the unique solution to the following initial value problem for PDE*

$$\begin{cases} \frac{\partial}{\partial t} f_t(z) = f_t'(z)\, V(z,t), \\ f_0 = \mathrm{id}_{\mathbb{D}}. \end{cases}$$

We call the set $D_t = g_t^{-1}(\mathbb{D}) = f_t(\mathbb{D})$ the *evolution domain* at the time t, and the complement $K_t = \mathbb{D} \setminus D_t$ the *Löwner hull* at the time t. The family $\{D_t\}_{t \ge 0}$ is decreasing, and the family $\{K_t\}_{t \ge 0}$ is increasing, i.e., for $0 \le s \le t < \infty$, $D_s \supseteq D_t$ and $K_s \subseteq K_t$.

1.2.3 Radial Löwner theory

The radial, chordal and dipolar Löwner equations, as well as their probabilistic counterparts, can be obtained from the general Löwner theory by choosing a particular Herglotz vector field. We call these three cases *classical* in contrast to other evolution families and Löwner chains that can be described by the general Löwner theory.

By the *radial Löwner theory* we understand the case when $\tau(t) \equiv \tau_0$ in the representation (1.24) for some $\tau_0 \in \mathbb{D}$, i.e., when the attracting point of the Herglotz vector field is a fixed interior point of the canonical domain. For simplicity one usually takes $\tau_0 = 0$. It is also common to impose the normalization condition $p(0,t) \equiv 1$ on the Herglotz function $p(z,t)$.

Thus, the radial Löwner theory is the study of evolution families and Löwner chains corresponding to Herglotz vector fields of the form

$$V(z,t) = -z\, p(z,t), \quad p(z,t) \text{ is a Herglotz function, } p(0,t) \equiv 1. \tag{1.27}$$

The radial theory was developed in [Löw23], [Gol39], [Kuf43], [Pom65] and [Pom66]. During the 20th century only the backward theory was studied, however, after the introduction of the Schramm-Löwner evolution in [Sch00], the focus shifted towards forward Löwner equations.

The normalization condition $p(0,t) \equiv 1$ ensures that backward radial evolution families have the Taylor expansion

$$\phi_{s,t}(z) = e^{s-t}\, z + \dots, \quad z \in \mathbb{D},$$

and normalized backward radial Löwner chains have the expansion

$$f_t(z) = e^{t}\, z + \dots, \quad z \in \mathbb{D}.$$

In the forward case the first coefficient in the expansion of normalized Löwner chains is different:

$$f_t(z) = e^{-t}\, z + \dots, \quad z \in \mathbb{D}.$$

There is an important relationship between backward evolution families $\{\phi_{s,t}\}_{0\le s\le t<+\infty}$ and backward Löwner chains $\{f_t\}_{t\ge 0}$, namely

$$\lim_{t\to\infty} e^{t}\, \phi_{0,t}(z) = f_0(z) \quad \text{locally uniformly.}$$

Radial Löwner theory and slit geometry

In the original paper [Löw23], Löwner was particularly interested in a special case of (1.27), namely

$$V(z,t) = -z\, \frac{e^{iu(t)} + z}{e^{iu(t)} - z}, \quad u(t) \text{ is continuous and real-valued.} \tag{1.28}$$

To distinguish between (1.27) and (1.28), the more general case (1.27) is sometimes referred to as the *radial Löwner-Kufarev theory*, because it was extensively studied in Kufarev's work [Kuf43].

Recall that a *crosscut* of a domain D is an open Jordan arc C in D such that $\bar{C} \setminus C$ consists of one or two points on ∂D.

Pommerenke [Pom66] gave the following geometric characterization of backward Löwner chains satisfying (1.23) with a Herglotz vector field of the form (1.28).

A normalized backward Löwner chain $\{f_t\}_{t \geq 0}$ satisfies

$$\frac{\partial}{\partial t} f_t(z) = z\, \frac{e^{iu(t)} + z}{e^{iu(t)} - z}\, f_t'(z) \tag{1.29}$$

for some real-valued continuous $u(t)$ if and only if $f_t'(0) = e^t$ and for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $s, t \geq 0$ and $0 \leq t - s \leq \delta$, some crosscut of $f_t(\mathbb{D})$ with spherical diameter less than ϵ separates 0 from $f_t(\mathbb{D}) \setminus f_s(\mathbb{D})$. We can refer to this condition as the *Pommerenke condition.*

A large and important class of backward radial Löwner chains satisfying this condition are the so-called chains *generated by a curve.* A backward radial Löwner chain $\{f_t\}_{t \geq 0}$ with $f_t'(0) = e^t$ is said to be generated by a curve $\Gamma : [0, +\infty) \to \mathbb{C}$ if for every $t \geq 0$ the image $f_t(\mathbb{D})$ is the connected component of $\mathbb{C} \setminus \Gamma[t, +\infty)$ containing 0. We also say that the curve Γ is generated by the chain $\{f_t\}_{t \geq 0}$.

One can check that in this case the Pommerenke condition is fulfilled and, consequently, all normalized radial chains generated by curves satisfy (1.29) for some continuous driving function $u(t)$. There are, however, counterexamples, i.e., normalized radial Löwner chains that are not generated by a curve, but nevertheless satisfy Pommerenke's criterion and (1.29) for some driving function $u(t)$. See, for instance, the chain shown in [Pom66, Figure 3] or the example in [MR05].

Löwner's parametric method

The class of univalent functions in the unit disk normalized by the conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1$$

is denoted by S. The class S is compact with respect to the local uniform topology.

Due to the normalization conditions above, each $f \in S$ has a Taylor expansion

$$f(z) = z + a_2\, z^2 + a_3\, z^3 + \ldots, \quad z \in \mathbb{D}.$$

Bieberbach [Bie16] conjectured that $|a_n| \leq n$ for functions in S. He prove it for $n = 2$. He also showed that the equality $|a_2| = 2$ is attained if and only if $f(z) = e^{-i\theta}\, k(e^{i\theta}\, z)$, where $\theta \in [0, 2\pi)$ and $k(z)$ is the *Koebe function*

$$k(z) = \frac{z}{(1 - z)^2} = z + 2\, z^2 + 3\, z^3 + \ldots, \quad z \in \mathbb{D}.$$

Löwner introduced his namesake equation in an attempt to prove the Bieberbach conjecture, but only succeeded for $n = 3$. After the conjecture had been proved for $n = 4, 5$ and 6 by various mathematicians, de Branges [dB85] used the radial Löwner equation to prove it for all natural n. In general, if the equality $|a_n| = n$ is attained for some $f \in S$ and some $n \in \mathbb{N}$, then, necessarily, $f(z) = e^{-i\theta} k(e^{i\theta} z)$ for some $\theta \in [0, 2\pi)$.

A *slit map* is a univalent function $f \in S$ such that $f(\mathbb{D}) = \mathbb{C} \setminus \Gamma$ for some Jordan arc Γ. The Koebe function is an important example of a slit map with $\Gamma = (-\infty, -1/4]$.

Löwner's parametric method is based on the following three observations proved in [Löw23].

1. Slit maps are dense in S with respect to the local uniform topology;
2. Every slit map f can be embedded in a backward radial Löwner chain $\{f_t\}_{t \geq 0}$ satisfying (1.29), in such a way that $f(z) = f_0(z)$;
3. If $\{f_t\}_{t \geq 0}$ is a backward radial Löwner chain, and $\{\phi_{s,t}\}_{0 \leq s \leq t < +\infty}$ is the corresponding evolution family, then

$$\lim_{t \to \infty} e^t \phi_{0,t}(z) = f_0(z) \quad \text{locally uniformly.} \tag{1.30}$$

Now, suppose $F : S \to \mathbb{C}$ is a functional on S continuous with respect to the local uniform topology. Then finding its extrema over S is equivalent to finding its infimum and supremum over slit maps in S, and for that problem we can use representation (1.30).

One example of such a functional is the absolute value of n-th coefficient in the Taylor expansion of a function in S. Proving that its maximal value equals n is equivalent to proving Bieberbach's conjecture. The method's applications, however, are not limited to proving Bieberbach's conjecture. In fact, Löwner's parametric method is considered to be one of the most powerful techniques in the theory of univalent functions and it leads to relatively easy, direct proofs of numerous inequalities for the class S.

An application of optimal control theory to the Löwner equation turned out to be extremely fruitful in solving extremal problems for univalent functions, also unifying variational methods with Löwner's parametric method. This approach first appeared in the PhD thesis of Goodman [Goo68], and was essentially developed by Friedman, Schiffer, Aleksandrov, Popov, Prokhorov and Roth, see [Rot00, Pro90, Pro02] and references therein. The main idea is to view the Löwner equation as a control system, with control $u(t)$. Since the reachable set is dense in S, the Pontryagin maximum principle can be applied directly to finding the extrema of functionals on S.

1.2.4 Chordal Löwner theory

By the *chordal Löwner theory* we understand the case when the attracting point of the Herglotz vector field (1.24) is a fixed boundary point of the canonical domain, i.e., $\tau(t) \equiv \tau_0$ for some boundary point $\tau_0 \in \mathbb{D}$.

It is possible to study the chordal Löwner theory in the unit disk $\mathbb{D}$ and choose, for example, $\tau_0 = -1$, so that the Herglotz field takes form

$$V(z,t) = -(z+1)^2\, p(z,t). \tag{1.31}$$

From a technical point of view, however, it is more practical to use the upper half-plane $\mathbb{H}$ as the canonical domain, and ∞ as the fixed boundary point τ_0.

The vector field (1.31) can be pushed forward from $\mathbb{D}$ to $\mathbb{H}$ by the conformal isomorphism

$$z \mapsto 2\,i\,\frac{1-z}{1+z},$$

so that the Herglotz vector field in the half plane becomes

$$V^{\mathbb{H}}(z,t) = 4\,i\,p\left(\frac{2i-z}{2i+z}, t\right) = i\,\tilde{p}(z,t),$$

where $\tilde{p}(z,t) := 4\,p\left(\frac{2i-z}{2i+z}, t\right)$ is holomorphic in $z \in \mathbb{H}$, measurable in $t \geq 0$, and $\operatorname{Re}\tilde{p}(z,t) \geq 0$ for $z \in \mathbb{H}$.

The simplest Herglotz vector field of order ∞ with a moving singularity on the boundary is

$$V^{\mathbb{H}}(z,t) = \frac{1}{u(t)-z}, \tag{1.32}$$

where $u(t)$ is a real-valued continuous function of t, and in this case,

$$\tilde{p}(z,t) = \frac{i}{z-u(t)},$$

The first works where the corresponding evolution families were studied include [Kuf46] and [KSS68].

Today, it is usual to scale the vector field (1.32) by the factor of 2, so that

$$V^{\mathbb{H}}(z,t) = \frac{2}{u(t)-z}. \tag{1.33}$$

The equation for the forward Löwner chain corresponding to (1.33),

$$\begin{cases} \frac{\partial}{\partial t} f_t(z) = \frac{2}{f_t(z)-u(t)}, \\ f_0(z) = z, \quad z \in \mathbb{H}, \end{cases} \tag{1.34}$$

is the most commonly used form of the Löwner equation nowadays. Lawler's book [Law05] contains a comprehensive analysis of the properties of chordal Löwner chains.

1.2.5 Dipolar Löwner theory

The study of the *dipolar (or strip) Löwner equation* was initiated independently in [Zha04] and [BB04, BBH05], where the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} f_t(z) = \frac{1}{\tanh[(f_t(z)-u(t))/2]}, \\ f_0(z) = z, \end{cases} \quad z \in \mathbb{S}, \tag{1.35}$$

was considered. Here, as usual, $u(t) : [0, +\infty) \to \mathbb{R}$ is a continuous real-valued function (the driving function), and $\mathbb{S}$ is the infinite strip $\{z : 0 < \operatorname{Im} z < \pi\}$.

To put (1.35) into the context of the general Löwner theory, we rewrite it in the unit disk, by considering the the family $\{\tilde{f}_t\}_{t\geq 0}$, with $\tilde{f}_t = \phi \circ f_t \circ \phi^{-1}$, where $\phi : \mathbb{S} \to \mathbb{D}$ is the conformal isomorphism given by

$$\phi(z) := i\,\frac{e^z - i}{e^z + i}.$$

The map ϕ sends 0 to 1, the prime end $-\infty$ to i, and the prime end $+\infty$ to i.

The forward Löwner chain $\{\tilde{f}_t\}_{t\geq 0}$ satisfies the equation

$$\begin{cases} \frac{\partial}{\partial t}\tilde{f}_t(z) = -V(\tilde{f}_t(z), t), \\ \tilde{f}_0(z) = z, \quad z \in \mathbb{D} \end{cases}$$

where

$$V(z,t) = \frac{1}{2}\,(1+z^2)\,\frac{1 - i\,z + e^{u(t)}\,(z-i)}{i + z - e^{u(t)}\,(1 + i\,z)} <$$

is the pushforward of the vector field

$$V(z,t) = -\frac{1}{\tanh[(z - u(t))/2]} \tag{1.36}$$

by ϕ.

Note, that $V(i,t) = V(-i,t) = 0$ for $t \geq 0$, hence the points $\pm i$ are common fixed points of the family $\{\tilde{f}_t\}_{t\geq 0}$. Neither of these points, however, is the attracting point of the Herglotz vector field. The attracting point in this case is time-dependent and is given by

$$\tau(t) = -\operatorname{sech} u(t) + i\,\tanh u(t).$$

1.2.6 Geometric properties of the slits

In [Kuf46], Kufarev formulates the following problem: characterize the geometric properties of evolution families and Löwner chains given the analytic properties of the driving function $u(t)$. One result proved by Kufarev in the same paper is that existence of a uniformly bounded derivative of $u(t)$ ensures that for the corresponding evolution family $\{\phi_{s,t}\}_{0\leq s\leq t<+\infty}$, the domain $\phi_{s,t}(\mathbb{D})$ is a slit disk for all $0 \leq s \leq t < +\infty$. This is equivalent to the fact that the corresponding Löwner chain is generated by a Jordan arc.

A *quasiarc* is the image of $[0, \infty)$ under a quasiconformal homeomorphism of $\mathbb{C}$. Examples of quasiarcs include, e.g., piecewise-smooth curves without zero-angle cusps. Marshall and Rohde [MR05] showed that if a Löwner chain is generated by a quasiarc then necessarily its driving function $u(t)$ is Lip(1/2) (i.e., Hölder continuous with exponent 1/2). Together with Lind [MR05, Lin05] they also proved that the condition $\|u\|_{1/2_{loc}} < 4$ is sufficient for the chain to be generated by a quasiarc. Here,

$$\|u\|_{1/2_{loc}} := \inf_{\epsilon>0} \sup_{|t-s|<\epsilon} \frac{|u(t)-u(s)|}{\sqrt{|t-s|}}.$$

The limit $\lim\limits_{t\to s} \dfrac{|u(t)-u(s)|}{\sqrt{|t-s|}}$ plays an important role for the geometry of the slit generating the chains. In [LMR10] it was shown how the values of this limit are related to the angle at which the slit intersects itself. In [Sch12, WD13] it was shown how in the case of forward Löwner chains this limit describes the angle at which the slit approaches the boundary of the canonical domain at the initial moment.

1.2.7 Schramm-Löwner evolution

The terms radial, chordal and dipolar *Schramm-Löwner evolution* (SLE_κ) refer to the forward random Löwner chains corresponding to the Herglotz fields (1.28), (1.33) and (1.36), respectively, with the driving function given by $u(t) = \sqrt{\kappa}\, B_t$. Here, as usual, B_t is a standard Brownian motion, and $\kappa > 0$.

Thus, the radial SLE_κ is defined by

$$\frac{\partial}{\partial t} f_t(z) = f_t(z) \frac{e^{i\sqrt{\kappa}\, B_t} + f_t(z)}{e^{i\sqrt{\kappa}\, B_t} - f_t(z)}, \quad f_0(z) = z, \quad z \in \mathbb{D}, \tag{1.37}$$

the chordal SLE_κ by

$$\frac{\partial}{\partial t} f_t(z) = \frac{2}{f_t(z) - \sqrt{\kappa}\, B_t}, \quad f_0(z) = z, \quad z \in \mathbb{H},$$

and the dipolar SLE_κ by

$$\frac{\partial}{\partial t} f_t(z) = \frac{2}{\tanh[(f_t(z) - \sqrt{\kappa}\, B_t)/2]}, \quad f_0(z) = z, \quad z \in \mathbb{S}.$$

The hulls of the corresponding chains are almost surely generated by a random curve γ. The curve γ is called SLE_κ *trace*, SLE_κ *curve* or, by abuse of terminology, simply SLE_κ.

The properties of SLE traces have been intensively studied since the introduction of the Schramm-Löwner evolution. It is known that, for a fixed value of $\kappa > 0$, the laws of radial, chordal and dipolar SLE_κ are locally absolutely continuous with respect to each other, see [LSW01, Proposition 4.2] and [Zha04, Theorem 2.3.1]. In [RS05] it is shown that with probability 1 the random curve γ is simple for $\kappa \in [0,4]$, has self-intersections for $\kappa \in (4,8)$ and is space-filling for $\kappa \in [8,\infty)$. The Hausdorff dimension of a trace of SLE_κ is $\min(2, 1+\kappa/8)$ with probability 1, see [Bef08].

Conformal invariance and domain Markov property

From the point of view of statistical physics, the two most important properties of SLE are *conformal invariance* and the *domain Markov property*. These properties are explained in detail in Paper C, but here we can give an informal explanation for the case of radial SLE.

The random radial Löwner chain defined by (1.37) generates a random non-self-crossing SLE_κ curve that begins at 1 and tends to 0. In fact, this random curve induces a probability measure on the set of non-self-crossing curves connecting 1 and 0. We can denote this measure by $SLE_\kappa(\mathbb{D}, 1, 0)$.

Given any other domain D, a prime end a of D and an interior point $b \in D$, there exists a unique conformal isomorphism $\phi : \mathbb{D} \to D$ such that $\phi(0) = b$ and $\hat{\phi}(1) = a$.

We can define a measure $SLE_\kappa(D, a, b) = SLE_\kappa(\phi(\mathbb{D}), \hat{\phi}(1), \phi(0))$ on the family of non-self-crossing curves lying in D and connecting a and b in a *conformally invariant* way: if the law of γ is given by $SLE_\kappa(\mathbb{D}, 1, 0)$, then the law of $\phi \circ \gamma$ is given by $SLE_\kappa(\phi(\mathbb{D}), \hat{\phi}(1), \phi(0))$.

The domain Markov property means that conditioned on an initial part of the curve $\gamma[0, t]$, the remaining curve has the law $SLE_\kappa(D_t, \gamma(t), b)$. Here, D_t is the unique connected component of $D \setminus \gamma[0, t]$ containing b, and the measure $SLE_\kappa(D_t, \gamma(t), b)$ agrees with the original law of γ in a conformally invariant way.

There is a close connection between the domain Markov property and the fact that equations defining SLE can be reformulated as diffusion equations. For instance, in the case of the radial SLE_κ, the change of variables $w_t(z) = f_t(z)/e^{i\sqrt{\kappa}B_t}$ transforms (1.37) into

$$dw_t(z) = w_t(z)\frac{1 + w_t(z)}{1 - w_t(z)}\,dt - i\sqrt{\kappa}\,w_t \circ dB_t, \quad w_0(z) = z, \quad z \in \mathbb{D}.$$

1.3 Gaussian free field

Standard references on the distribution theory are [AF03] and [Str03]. The Gaussian free field, the most important random distribution in two dimensions, is covered in detail in [She07], [SS13] and [KM13]. A comprehensive treatment of Hilbert spaces, an important tool in the theory of the Gaussian free field, is [Jan97].

1.3.1 Schwartz distributions in the complex plane

Let D be a simply connected domain in the complex plane. Let $C_0^\infty(D)$ denote the space of infinitely differentiable real-valued functions with compact support in D. A sequence of functions $\{\mathfrak{p}_n\}_{n=1}^\infty \subset C_0^\infty(D)$ is said to converge *in the sense of test functions* to the function $\mathfrak{p} \in C_0^\infty(D)$ if

(i) there exists a compact subset K such that $\operatorname{supp}(\mathfrak{p}_n - \mathfrak{p}) \subseteq K$ for all $n = 1, 2, \ldots$,

(ii) $\frac{\partial^{m+p}}{\partial^m x\, \partial^p y}\mathfrak{p}_n \to \frac{\partial^{m+p}}{\partial^m x\, \partial^p y}\mathfrak{p}$, as $n \to \infty$, uniformly on K for all $m, p = 1, 2, \ldots$

There exists a topology on $C_0^\infty(D)$ such that a linear functional T on $C_0^\infty(D)$ is continuous if and only if $T(\mathfrak{p}_n) \to T(\mathfrak{p})$ whenever $\mathfrak{p}_n \to \mathfrak{p}$ in the sense of test functions. The space $C_0^\infty(D)$ endowed with this topology is called the *space of test functions* on D, and is denoted by $\mathscr{D}(D)$.

The dual space endowed with the weak-star topology is denoted by $\mathscr{D}'(D)$, and its elements are called *(Schwartz) distributions*.

Let A denote the area measure on D, and let $(\,\cdot\,,\,\cdot\,)$ denote the $L^2(D, A)$ inner product, so that

$$(\mathfrak{p}, \mathfrak{q}) := \int_D \mathfrak{p}(z)\, \mathfrak{q}(z)\, dA(z),$$

for real-valued functions $\mathfrak{p}$ and $\mathfrak{q}$.

Let $L^1_{loc}(D, A)$ denote the so-called space of *locally integrable* functions in D. By definition, a function h belongs to $L^1_{loc}(D, A)$ if $h \in L^1(U)$ for every open set U with its closure $\overline{U}$ lying in D. The space $L^1_{loc}(D)$ is quite rich, and the inclusion $L^p(D, A) \subset L^1_{loc}(D, A)$ holds for any $p \geq 1$.

Let $h \in L^1_{loc}(D, A)$. Even though it is not true in general that $h \in L^2(D, A)$, the inner product $(h, \mathfrak{p})$ is well-defined for any test function $\mathfrak{p} \in \mathscr{D}(D)$. Moreover, h defines a continuous distribution on D by

$$\mathfrak{p} \mapsto (h, \mathfrak{p}), \quad \mathfrak{p} \in \mathscr{D}(D),$$

and with this identification we can write $L^1_{loc}(D) \subset \mathscr{D}'(D)$.

Not every Schwartz distribution on D can be represented by a locally integrable function as above. Nevertheless, by abuse of notation, we always denote by $(h, \mathfrak{p})$ the result of action of a distribution h on a test function $\mathfrak{p}$.

1.3.2 Green's function

By the Riemann mapping theorem, given a hyperbolic domain D, for each $w \in D$ there exists a unique conformal isomorphism $f_w : D \to \mathbb{D}$, such that $f_w(w) = 0$, and $f_w'(w) > 0$. The *Green's function* of D is defined as

$$G_D(z, w) = -\log|f_w(z)|.$$

The Green's function is symmetric and harmonic in both variables.

It is easy to check that for the unit disk, $G_{\mathbb{D}}(z, w) = \log\left|\frac{1-\bar{w}\,z}{z-w}\right|$, and for the upper half-plane $G_{\mathbb{H}}(z, w) = \log\left|\frac{z-\bar{w}}{z-w}\right|$.

For a test function $\mathfrak{p} \in \mathscr{D}(D)$, Green's third identity implies

$$-\frac{1}{2\pi}\int_D G_D(z, w)\, \Delta\mathfrak{p}(w)\, dA(w) = \mathfrak{p}(z). \tag{1.38}$$

We can rewrite this identity using the formal language of distributional (weak) derivatives (see [AF03]) in the following form

$$\begin{cases} -\Delta_w G_D(z, w) = 2\pi\delta(z - w), \quad z \in D, \\ G_D(z, w) = 0, \quad z \in \partial D. \end{cases} \tag{1.39}$$

One can use (1.39) as an alternative definition of Green's function.

The Laplacian Δ is an injective linear operator in $\mathscr{D}(D)$ (because $\ker\Delta = \{0\}$ in $\mathscr{D}(D)$), and (1.38) yields the following formula for the inverse of Δ

$$\Delta^{-1}\mathfrak{p}(z) = -\frac{1}{2\pi}\int_D G_D(z,w)\,\mathfrak{p}(w)\,dA(w). \tag{1.40}$$

1.3.3 Dirichlet energy and electrostatic potential energy

We use three different inner products on the space $\mathscr{D}(D)$. One of them, the $L^2(D,A)$ inner product, has already been mentioned. We denote it by $(\,\cdot\,,\,\cdot\,)$, or, when the domain is not clear from the context, $(\,\cdot\,,\,\cdot\,)_{L^2(D,A)}$. For two functions $\mathfrak{p},\mathfrak{q}\in\mathscr{D}(D)$,

$$(\mathfrak{p},\mathfrak{q}) := \int_D \mathfrak{p}(z)\,\mathfrak{q}(z)\,dA(z).$$

The *Dirichlet inner product* of two functions $\mathfrak{p},\mathfrak{q}\in\mathscr{D}(D)$ is defined as

$$(\mathfrak{p},\mathfrak{q})_\nabla := \int_D \nabla\mathfrak{p}(z)\cdot\nabla\mathfrak{q}(z)\,dA(z).$$

Yet another inner product on $\mathscr{D}(D)$ that we frequently use is

$$(\mathfrak{p},\mathfrak{q})_{\mathcal{E}(D)} := \int_{D\times D} 2\,G_D(z_1,z_2)\,\mathfrak{p}(z_1)\,\mathfrak{q}(z_2)\,dA(z_1)\,dA(z_2).$$

The following formulas relating the inner products introduced above to each other, can be easily deduced from Green's identities. In all of the formulas, $\mathfrak{p},\mathfrak{q}\in\mathscr{D}(D)$.

$$(\mathfrak{p},\mathfrak{q}) = \frac{1}{4\pi}(-\Delta\mathfrak{p},\mathfrak{q})_{\mathcal{E}(D)}, \tag{1.41}$$

$$(\mathfrak{p},\mathfrak{q})_\nabla = (-\Delta\mathfrak{p},\mathfrak{q}), \tag{1.42}$$

$$(\mathfrak{p},\mathfrak{q})_\nabla = \frac{1}{4\pi}(\Delta\mathfrak{p},\Delta\mathfrak{q})_{\mathcal{E}(D)}. \tag{1.43}$$

The inner products induce the following norms on $\mathscr{D}(D)$.

$$\|\mathfrak{p}\| := \sqrt{(\mathfrak{p},\mathfrak{p})}, \quad \|\mathfrak{p}\|_\nabla := \sqrt{(\mathfrak{p},\mathfrak{p})_\nabla}, \quad \|\mathfrak{p}\|_{\mathcal{E}(D)} := \sqrt{(\mathfrak{p},\mathfrak{p})_{\mathcal{E}(D)}}.$$

The norm $\|\mathfrak{p}\|_\nabla$ is called the *Dirichlet energy* of $\mathfrak{p}$, and the norm $\|\mathfrak{p}\|_{\mathcal{E}(D)}$ is sometimes referred to as the *electrostatic potential energy* of $\mathfrak{p}$.

1.3.4 Definition of Gaussian free field

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. By a *random distribution* (or, more precisely, a *random Schwartz distribution*) on D we mean a measurable map $\Phi:\Omega\to\mathscr{D}'(D)$.

A random distribution Φ is called a *Gaussian free field* (*GFF*) on D if for every test function $\mathfrak{p}\in\mathscr{D}(D)$ the random variable $(\Phi,\mathfrak{p})$ is a centered Gaussian with variance $\|\mathfrak{p}\|^2_{\mathcal{E}(D)}$.

By polarization, it follows that the covariance satisfies

$$\mathrm{Cov}\left((\Phi, \mathfrak{p}), (\Phi, \mathfrak{q})\right) = (\mathfrak{p}, \mathfrak{q})_{\mathcal{E}(D)}.$$

Below we explain how one can construct Gaussian free fields explicitly.

Let $H(D)$ denote the Hilbert space completion of $\mathscr{D}(D)$ with respect to $(\,\cdot\,,\,\cdot\,)_\nabla$. The Hilbert space $H(D)$ is separable, and $H(D) \subset L^2(D, A) \subset L^1_{loc}(D) \subset \mathscr{D}'(D)$.

Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis in $H(D)$, and let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of independent standard Gaussian random variables. Consider the formal series

$$2\sqrt{\pi} \sum_{n=1}^{\infty} \alpha_n \, e_n. \tag{1.44}$$

The series (1.44) diverges almost surely, but nevertheless for any test function $\mathfrak{p} \in \mathscr{D}(D)$ the series

$$2\sqrt{\pi} \sum_{n=1}^{\infty} \alpha_n (\mathfrak{p}, e_n) \tag{1.45}$$

converges in $L^2(\Omega, \mathbb{P})$. In fact, by a result from the theory of Gaussian Hilbert spaces [Jan97, Example 1.25], the series (1.45) is a centered Gaussian with variance

$$\|2\sqrt{\pi} \sum_{n=1}^{\infty} \alpha_n \, (\mathfrak{p}, e_n)\|^2_{L^2(\Omega,\mathbb{P})} = 4\pi \sum_{n=1}^{\infty} (\mathfrak{p}, e_n)^2,$$

if the series on the right hand-side converges. But by (1.42), Parseval's identity and (1.43),

$$4\pi \sum_{n=1}^{\infty} (\mathfrak{p}, e_n)^2 = 4\pi \sum_{n=1}^{\infty} (-\Delta^{-1}\mathfrak{p}, e_n)^2_\nabla = 4\pi \|\Delta^{-1}\mathfrak{p}\|^2_\nabla = \|\mathfrak{p}\|^2_{\mathcal{E}(D)} < \infty,$$

which shows that (1.45) indeed converges for every $\mathfrak{p} \in \mathscr{D}(D)$, and that (1.44) is a Gaussian free field. Moreover, one can show that every Gaussian free field Φ on D can be represented as $\Phi = 2\pi \sum_{n=1}^\infty \alpha_n \, e_n$, where $\{e_n\}_{n=1}^\infty$ is an orthonormal basis for $H(D)$, and $\{\alpha_n\}_{n=1}^\infty$ is a sequence of standard Gaussian random variables.

1.3.5 Gaussian free field on a subdomain

Let $B \subset D$ be a simply connected subdomain of D. The space $H(B)$ is a closed subspace of $H(D)$, and we denote its orthogonal complement by $\mathrm{Harm}(B)$, so that $H(D)$ can be decomposed orthogonally as follows

$$H(D) = H(B) \oplus \mathrm{Harm}(B).$$

We denote by $P_{H(B)}$ and $P_{\mathrm{Harm}(B)}$ the orthogonal projections onto the corresponding closed subspaces.

By definition

$$\begin{aligned} f \in \mathrm{Harm}(B) &\Leftrightarrow (f,g)_\nabla = 0 \text{ for all } g \in H(B) \\ &\Leftrightarrow (f,\Delta g)_{L^2(B,A)} = 0 \text{ for all } g \in H(B). \end{aligned}$$

Functions satisfying the last condition are called *weakly harmonic* on B (or harmonic in the sense of distributional derivatives), but by Weyl's lemma [Wey40], weak harmonicity is equivalent to harmonicity. Hence, $\mathrm{Harm}(B)$ consists precisely of those elements of $H(D)$ that are harmonic on B.

Let Φ_B be a Gaussian free field on B. It may be written as

$$\Phi_B := 2\sqrt{\pi}\sum_{n=1}^{\infty} \alpha_n\, f_n$$

where $\{f_n\}_{n=1}^\infty$ is an orthonormal basis of $H(B)$, and $\{\alpha_n\}_{n=1}^\infty$ is a sequence of independent standard Gaussian variables. It turns out that Φ_B may be treated as a random distribution on the whole of D, such that for a test function $\mathfrak{p} \in \mathscr{D}(D)$, $(\Phi_B, \mathfrak{p})$ is a centered Gaussian with variance $\|\mathfrak{p}\|^2_{\mathcal{E}(B)}$.

Indeed, for a $\mathfrak{p} \in \mathscr{D}(D)$,

$$\begin{aligned} (\Phi_B, \mathfrak{p}) &= 2\sqrt{\pi}\sum_{n=1}^{\infty} \alpha_n\,(f_n, \mathfrak{p}) \\ &= 2\sqrt{\pi}\sum_{n=1}^{\infty} \alpha_n\,(f_n, -\Delta^{-1}\mathfrak{p})_\nabla \\ &= 2\sqrt{\pi}\sum_{n=1}^{\infty} \alpha_n\,(f_n, P_{H(B)}(-\Delta^{-1}\mathfrak{p}))_\nabla, \end{aligned}$$

which shows that $(\Phi_B, \mathfrak{p})$ is a centered Gaussian with variance

$$\begin{aligned} 4\pi\|P_{H(B)}(-\Delta^{-1}\mathfrak{p})\|^2_\nabla &= \|\Delta P_{H(B)}(-\Delta^{-1}\mathfrak{p})\|^2_{\mathcal{E}(B)} \\ &= \|\Delta P_{H(B)}(-\Delta^{-1}\mathfrak{p}) + \Delta P_{\mathrm{Harm}(B)}(-\Delta^{-1}\mathfrak{p})\|^2_{\mathcal{E}(B)} = \|\mathfrak{p}\|^2_{\mathcal{E}(B)}. \end{aligned}$$

1.3.6 Modifications of GFF

Every harmonic function h in D is locally integrable, and hence, $h \in \mathscr{D}'(D)$. We can define the *GFF with mean h* as the sum of distributions $\hat{\Phi}_D = \Phi_D + h$. We also refer to such distribution as *modifications* of GFF.

By Fatou's theorem (see [Hof88, Chapter 3]), every bounded harmonic function can be represented by the Poisson integral of a bounded measurable function h_∂ on ∂D. In this case we can call $\hat{\Phi}_D$ the Gaussian free field with Dirichlet boundary condition h_∂. If h is only bounded from above or from below, then the Herglotz representation theorem implies that it can be represented as the Poisson integral of a signed measure μ_h on ∂D.

Let $B \subset D$ be a simply connected domain, and let the area $A(D\setminus B)$ be zero. Let $\hat{\Phi}_B = \Phi_B + h$ be a modification of GFF on B, and let $\mathfrak{p} \in \mathscr{D}(D)$. As we have seen in the previous section, Φ_B defines a distribution on D. Even though h is only defined on B, the inner product

$$(h, \mathfrak{p})_{L^2(D,A)} = \int_D h(z)\,\mathfrak{p}(z)\,dA(z)$$

is still well-defined, hence $\hat{\Phi}_B$ defines a random Schwartz distribution on the whole of D.

1.3.7 Pullbacks of distributions and conformal invariance of GFF

Let $\phi : D_1 \to D_2$ be a conformal isomorphism, and let $\Psi \in \mathscr{D}'(D_2)$. The *pullback* $\Psi \circ \phi$ of Ψ with respect to ϕ is the distribution on D_1 defined by the formula

$$(\Psi \circ \phi, \mathfrak{p}) = (\Psi, |(\phi^{-1})'|^2\, \mathfrak{p}(\phi^{-1})), \quad \mathfrak{p} \in C_0^\infty(D_1).$$

If Ψ is an $L^1_{loc}(D_2)$ function, then this definition corresponds to the change of variables in the formula

$$\int_{D_1} \Psi(\phi(z))\, \mathfrak{p}(z)\, dA(z) = \int_{D_2} \Psi(w)\, |(\phi^{-1}(w))'|^2\, \mathfrak{p}(\phi^{-1}(w))\, dA(w).$$

The Gaussian free field is *conformally invariant*, meaning that the pullback $\Phi_{D_2} \circ \phi$ of a Gaussian free field on D_2 has the same law as GFF in D_1.

1.3.8 SLE and GFF

One of the possible formulations of the conformal field theory (CFT) is from the point of view of random fields, in particular, the Gaussian free field. The mathematical aspects of this formulations are well-understood, see [KM13].

The fact that the Gaussian free field is closely related to SLE_4 was first realized in [SS05], and since then a lot of progress has been made in this direction. Connections between GFF and SLE_κ for other values of κ have also been established [Dub09, She10].

Below we describe the simplest example of connection between SLE and GFF (chordal SLE_4), see the details in [SS09], [She10] and [SS13].

Let $\Phi_{\mathbb{H}}$ a Gaussian free field in the upper half-plane, and consider the following modification of $\Phi_{\mathbb{H}}$:

$$\hat{\Phi}_{\mathbb{H}} = \Phi_{\mathbb{H}} - \sqrt{2}\, \arg z.$$

In fact, $\hat{\Phi}_{\mathbb{H}}$ is a Gaussian free field with the boundary condition $\hat{\Phi}_{\mathbb{H}} = 0$ on the positive real axis, and $\hat{\Phi}_{\mathbb{H}} = \pi$ on the negative real axis.

Let B_t be a Brownian motion independent of $\hat{\Phi}_{\mathbb{H}}$, and let $\{w_t\}_{t\geq 0}$ be the corresponding chordal SLE_4 stochastic flow, so that

$$dw_t(z) = \frac{2}{w_t(z)}\, dt - 2\, dB_t, \quad w_0(z) = z, \quad z \in \mathbb{H}.$$

The interesting connection between the modification $\hat{\Phi}_{\mathbb{H}}$ and the flow of chordal SLE_4 is that for any deterministic time $T > 0$, the random fields $\hat{\Phi}_{\mathbb{H}}$ and $\hat{\Phi}_{\mathbb{H}} \circ w_T$ have the same law.

In Paper D we explain this relationship in detail and investigate whether similar result may be true for other slit holomorphic stochastic flows.

Chapter 2

Main Results

In this chapter we give an overview of the new results introduced in this thesis. The structure is as follows: Sections 2.1-2.4 are summaries of each of the research papers included in the thesis, and in Section 2.5 we outline possible directions for further research.

2.1 Non-slit and singular solutions to the Löwner equation

Let us return to backward radial Löwner evolution families $\{\phi_{s,t}\}_{0\le s\le t<+\infty}$ satisfying

$$\begin{cases} \frac{\partial}{\partial t}\phi_{s,t}(z) = -\phi_{s,t}(z)\,\frac{e^{iu(t)}+\phi_{s,t}(z)}{e^{iu(t)}-\phi_{s,t}(z)}, \\ \phi_{s,s}(z) = z, \quad z\in\mathbb{D}, \end{cases} \tag{2.1}$$

and the corresponding backward Löwner chains $\{f_t\}_{t\ge 0}$, which satisfy

$$\frac{\partial}{\partial t} f_t(z) = z f_t'(z)\frac{e^{i\,u(t)}+z}{e^{i\,u(t)}-z}, \quad f_0(z) = \lim_{t\to\infty} e^t\phi_{0,t}(z). \tag{2.2}$$

Recall that the evolution family corresponding to the Löwner chain $\{f_t\}_{t\ge0}$ is given by $\phi_{s,t} = f_t^{-1}\circ f_s$ (see (1.22)).

Suppose that the driving function is such that $\{f_t\}_{t\ge0}$ is generated by a *Jordan arc* (i.e., a homeomorphic image of $[0,+\infty)$). Then it is easy to see that $\phi_{s,t}(\mathbb{D})$ in this case is a *slit disk*, meaning that $\mathbb{D}\setminus\phi_{s,t}(\mathbb{D})$ is a Jordan arc γ.

Löwner proved that for any slit disk $\mathbb{D}\setminus\gamma$ containing 0, it is possible to find a driving function $u(t)$ generating $\mathbb{D}\setminus\gamma$. In other words, it is possible to find $u(t)$, such that for the evolution family solving the Löwner ODE (2.1), the equality $\phi_{0,t_0}(\mathbb{D}) = \mathbb{D}\setminus\gamma$ holds for some $t_0>0$.

If the driving function $u(t)$ is such that the corresponding Löwner chain is generated by a curve with self-intersections or by a closed curve in $\hat{\mathbb{C}}$, then $\phi_{s,t}(\mathbb{D})$ is in general not a slit disk. Löwner knew that such driving functions exist, and an explicit example was given later by Kufarev in [Kuf47].

We use the geometric considerations above to construct a family of non-slit solutions to (2.1), for which the driving functions $u(t)$ are Lip(1/2) and their norms $\|u(t)\|_{1/2_{loc}}$ take all values in the interval $(4, \infty)$. This should be compared with the results cited in Section 1.2.6. Similar results for the chordal case had been earlier constructed in [KNK04].

In second part of the paper we study the chordal equation, and prove, in particular, that the driving term of an analytic orthogonal slit is Lip(1/2), and $\|u(t)\|_{1/2_{loc}}= 0$. This result was later generalized in [WD13].

2.2 Löwner evolution driven by a stochastic boundary point

The classical Löwner equation played a crucial role in the study of two-dimensional conformally invariant stochastic processes. After the general Löwner theory had been formulated, a natural question was whether the general theory can have similar stochastic applications.

We suggest that the first step in this direction may be analyzing the evolution families corresponding to the Herglotz vector field

$$V(t, z) = (z - \tau(t))(\overline{\tau(t)}\, z - 1)\, p(z, t),$$

where $\tau(t) = e^{i\,k\,B_t}$, $k \geq 0$, i.e., it is a Brownian motion on the unit circle.

In order to acquire geometric intuition we first work with the simpler deterministic counterpart $\tau(t) = e^{i\,k\,B_t}$, $k \in \mathbb{R}$, and study the backward evolution family

$$\begin{cases} \frac{\partial}{\partial t}\phi_t(z) = \frac{(\tau(t)-\phi_t(z))^2}{\tau(t)}\, p(\phi_t(z), t), \\ \phi_0(z) = z, \end{cases} \qquad z \in \mathbb{D}. \tag{2.3}$$

First, we note that if the Herglotz function can be written in the special form $p(z, t) = \tilde{p}(z/\tau(t))$, where $\tilde{p}(z) : \mathbb{D} \to \mathbb{C}$ is a holomorphic function with non-negative real part, then the change of variables $\psi_t(z) = \frac{\phi_t(z)}{\tau(t)}$ leads to the initial-value problem

$$\begin{cases} \frac{\partial}{\partial t}\psi_t(z) = (\psi_t(z) - 1)^2\; \tilde{p}(\psi_t(z)) - i\,k\,\psi_t(z), \\ \psi_0(z) = z. \end{cases}$$

In other words, $\{\psi_t\}_{t\geq 0}$ is a holomorphic semiflow, and we can use the well-developed theory of holomorphic semiflows to investigate $\{\psi_t\}_{t\geq 0}$ and $\{\phi_t\}_{t\geq 0}$.

We prove that $\{\phi_t\}_{t\geq 0} \subset \mathrm{Aut}(\mathbb{D})$ if and only if $p(z, t) = \tilde{p}(z/\tau(t))$ with

$$\tilde{p}(z) = A\,\frac{1 + z}{1 - z} + B\,i, \quad A,\, B \in \mathbb{R}.$$

The type of the corresponding semiflow of automorphisms ψ_t depends on the radial velocity k of the attracting point $\tau(t)$: it is

- hyperbolic, if $2\,(-\operatorname{Im}\tilde{p}(0) - |\tilde{p}(0)|) < k < 2\,(-\operatorname{Im}\tilde{p}(0) + |\tilde{p}(0)|)$,
- parabolic, if $2\,(-\operatorname{Im}\tilde{p}(0) - |\tilde{p}(0)|) < k < 2\,(-\operatorname{Im}\tilde{p}(0) + |\tilde{p}(0)|)$,
- elliptic, if $k < 2\,(-\operatorname{Im}\tilde{p}(0) - |\tilde{p}(0)|)$, or $k > 2\,(-\operatorname{Im}\tilde{p}(0) + |\tilde{p}(0)|)$.

In the case of elliptic automorphisms, the trajectories $\phi_t(z)$ are curves similar to the beautiful rose (rhodonea) curves. The trajectories are closed if and only if $\frac{k}{\sqrt{-4\,A^2+4\,B\,k+k^2}}$ is a rational number.

The semigroup $\{\psi_t\}_{t\geq 0}$ becomes elliptic for sufficiently high radial velocities of the attracting point also in the general case of holomorphic endomorphisms, no matter what Herglotz function $\tilde{p}(z)$ we choose.

In the stochastic case, Herglotz functions of the form $p(z,t) = \tilde{p}(z/\tau(t))$ also turn out to play an important role. The change of variables $w_t = \phi_t/e^{i\,k\,B_t}$ now leads to a time-homogeneous Itô diffusion

$$\begin{cases} dw_t = \left(-\frac{k^2}{2}\,w_t + (w_t-1)^2\tilde{p}(w_t)\right)dt - i\,k\,w_t\,dB_t, \\ w_0(z) = z. \end{cases} \tag{2.4}$$

This has two important implications. First, the Markov property of the stochastic flow $\{w_t\}_{t\geq 0}$ leads to a version of the domain Markov property for the domain family $\{\phi_t(\mathbb{D})\}_{t\geq 0}$. Second, one can use the rich machinery of stochastic calculus, such as Kolmogorov equations, to study the process.

We have thus found a new type of the general Löwner evolution (in addition to the already known radial, chordal and dipolar equations), that leads to a time-homogeneous diffusion equation in the stochastic case. Can we find more such types of the general Löwner evolution? Can we characterize them all? Can they have slit geometry similar to the classical SLE cases? If yes, will the random slits be similar to the classical SLE curves?

These questions are answered in the next paper.

2.3 General slit Löwner chains

Holomorphic stochastic semiflows

We start by considering the stochastic flow

$$\begin{cases} dw_t(z) = b(w_t(z))\,dt + \sigma(w_t(z)) \circ dB_t, \\ w_0(z) = z, \end{cases} \qquad z \in D, \tag{2.5}$$

in a simply connected hyperbolic domain D, and ask what conditions should be imposed on the functions b and σ, so that $w_t \in \operatorname{Hol}(D,D)$ for all $t \geq 0$.

It is known from the work of Kunita quoted in Section 1.1.2 that if the coefficients b and σ are complete holomorphic vector fields, then $w_t \in \operatorname{Aut}(D)$ for all $t \geq 0$.

We use the general Löwner theory to study the case when $b(z)$ is semicomplete. First we consider the auxiliary flow

$$\begin{cases} dH_t(z) = \sigma(H_t(z)) \circ dB_t, \\ H_0(z) = z, \end{cases} \quad z \in D, \tag{2.6}$$

of automorphisms of $\mathbb{D}$. In fact, $\{H_t\}_{t\geq 0}$ is related to the deterministic flow

$$\begin{cases} \frac{\partial}{\partial t} h_t(z) = \sigma(h_t(z)), \\ h_0(z) = z, \quad z \in D \end{cases} \tag{2.7}$$

by the formula $H_t = h_{B_t}$.

By (1.18) and (1.19), the composite flow $\{g_t\}_{t\geq 0}$, where $g_t := H_t^{-1} \circ w_t = h_{B_t}^{-1} \circ w_t$, satisfies

$$dg_t(z) = \left(h_{B_t}^{-1}{}_* b\right) (g_t(z))\, dt, \quad g_0(z) = z, \quad z \in D. \tag{2.8}$$

It is easy to check that $\left(h_{B_t}^{-1}{}_* b\right)$ is a Herglotz vector field of order ∞, implying that $\{g_t\}_{t\geq 0} \subset \mathrm{Hol}(D, D)$, which is equivalent to $\{w_t\}_{t\geq 0} \subset \mathrm{Hol}(D, D)$.

Thus, *if b is a semicomplete holomorphic vector field and σ is a complete holomorphic vector field, then the flow* (2.5) *consists of holomorphic endomorphisms of D.*

In the preceding paper we used mostly the backward (increasing) framework. In this paper, thanks to the appearance of the work [CDMG14], we were able to direct our attention towards the forward (decreasing) framework, the main tool of the SLE theory.

Using the forward Löwner theory, we study the solutions to

$$\begin{cases} dw_t(z) = -b(w_t(z))\, dt + \sigma(w_t(z)) \circ dB_t^k, \\ w_0(z) = z, \end{cases} \quad z \in D, \tag{2.9}$$

where b is a semicomplete holomorphic vector field and σ is a complete holomorphic field. We denote

$$D_t = \{z \in D : w_t(z) \text{ is defined at least up to time } t\},$$

and using the forward Löwner theory prove that $w_t : D_t \to D$ is a conformal isomorphism for all $t \geq 0$ with probability 1. Similarly to the case of (2.5), the crucial step here is to note that the composition $g_t = H_t^{-1} \circ w_t$ satisfies

$$dg_t(z) = -\left(h_{B_t}^{-1}{}_* b\right) (g_t(z))\, dt, \quad g_0(z) = z, \quad z \in D, \tag{2.10}$$

and hence $\{g_t\}_{t\geq 0}$ is a forward Löwner chain of order ∞.

Slit vector fields and Virasoro generators

The next question is what conditions should be imposed on the vector field b to ensure that the generated Löwner chains have slit geometry, as in the classical $SLEs$. From the geometrical point of view, the natural necessary condition is to require that the field b is tangent to the boundary of the canonical domain at all boundary points but one.

A precise formulation of this condition in $\mathbb{D}$ is that

$$\lim_{r\to 1} \operatorname{Re} b(re^{i\theta})\, re^{-i\theta} = 0 \tag{2.11}$$

for all $e^{i\theta} \in \partial\mathbb{D}$ except, perhaps, for one point $e^{i\theta_0}$. Without loss of generality we may assume $e^{i\theta_0} = 1$. We call such fields *slit vector fields in* $\mathbb{D}$. In the autonomous deterministic case such vector fields indeed generate semiflows having slit geometry.

We prove that b is a slit vector field in $\mathbb{D}$ if and only if

$$b(z) = \alpha - z\left(i\,\beta + \gamma\,\frac{1+z}{1-z}\right) - \overline{\alpha}\, z^2, \quad z \in \mathbb{D}, \tag{2.12}$$

for some $\alpha \in \mathbb{C}$, $\beta \in \mathbb{R}$ and $\gamma \geq 0$.

The vector fields

$$\ell_n^{\mathbb{H}}(z) := -z^{n+1}, \quad n \in \mathbb{Z},\, z \in \mathbb{H},$$

are called the *Virasoro generators* for the upper half-plane. Using the pushforward operation, the generators can be defined for any simply connected hyperbolic domain D by the formula $\ell_n^D := \phi_* \ell_n^{\mathbb{H}}$, where $\phi : \mathbb{H} \to D$ is a conformal isomorphism. This definition of ℓ_n^D depends on the choice of ϕ.

For the unit disk, we choose $\phi(z) = -\frac{z-2i}{z+2i}$, and get

$$\ell_n^{\mathbb{D}}(z) = -2^{n-1}\,(-i)^n\,(z-1)^{n+1}\,(z+1)^{-n+1}.$$

For the infinite strip $\mathbb{S} = \{z : 0 < \operatorname{Im} z < \pi\}$ with $\psi(z) = \operatorname{Log}\frac{2+z}{2-z}$ we get

$$\ell_n^{\mathbb{S}}(z) = \psi_*\,\ell_n^{\mathbb{H}}(z) = -2^n\,\sinh(z)\tanh^n\left(\frac{z}{2}\right).$$

An interesting fact is that the space of complete holomorphic vector fields in D is given by $\operatorname{span}_{\mathbb{R}}\{\ell_{-1}^D, \ell_0^D, \ell_1^D\}$. What is even more interesting for us, is that we can use $\ell_n^{\mathbb{D}}$, $n = -2, \ldots, 1$ to represent slit vector fields in $\mathbb{D}$. A vector field b is a slit vector field in $\mathbb{D}$ if and only if it can be represented in the form

$$b(z) = b_{-2}\,\ell_{-2}^{\mathbb{D}}(z) + b_{-1}\,\ell_{-1}^{\mathbb{D}}(z) + b_0\,\ell_0^{\mathbb{D}}(z) + b_1\,\ell_1^{\mathbb{D}}(z), \tag{2.13}$$

with $b_{-2} \geq 0$ and b_{-1}, b_0, $b_1 \in \mathbb{R}$.

We can then define slit vector fields in an arbitrary simply connected hyperbolic domain D by replacing $\ell_n^{\mathbb{D}} \mapsto \ell_n^D$ in (2.13).

Definition of slit Löwner chains

Let D be a simply connected hyperbolic domain D. Let σ be a complete vector field in D, and b be a slit vector field in D, so that

$$b(z) = b_{-2}\,\ell_{-2}^D(z)+b_{-1}\,\ell_{-1}^D(z)+b_0\,\ell_0^D(z)+b_1\,\ell_1^D(z), \quad b_{-2}\geq 0, b_{-1},\, b_0,\, b_1 \in \mathbb{R},$$

and

$$\sigma(z) = \sigma_{-1}\,\ell_{-1}^D(z) + \sigma_0\,\ell_0^D(z) + \sigma_1\,\ell_1, \quad \sigma_{-1}, \sigma_0, \sigma_1 \in \mathbb{R},\, \sigma_{-1} \neq 0.$$

Let $u_t : [0, +\infty) \to \mathbb{R}$ be a continuous function. The *slit Löwner chain driven by b, σ and u_t* is by definition the forward Löwner chain $\{f_t\}_{t\geq 0}$ corresponding to the Herglotz vector field

$$V(t,z) = \left(h_{u_t}^{-1}{}_*\, b\right)(z),$$

where $\{h_t\}_{t\geq 0}$ is the flow of automorphisms of D generated by σ.

Without loss of generality we may impose the normalization conditions

$$b_{-2} = 2, \quad \sigma_{-1} = 1,$$

and we call Löwner chains satisfying these conditions *normalized slit Löwner chains.*

Our general definition of normalized slit Löwner chains includes the classical radial, chordal and dipolar Löwner chains as special cases:

Löwner chain type	b	σ
Chordal	$2\ell_{-2}$	ℓ_{-1}
Radial	$2\ell_{-2} + \frac{1}{2}\ell_0$	$\ell_{-1} + \frac{1}{4}\ell_1$
Dipolar	$2\ell_{-2} - \frac{1}{2}\ell_0$	$\ell_{-1} - \frac{1}{4}\,\ell_1$

If $u_t = \sqrt{\kappa}B_t$, $\kappa \geq 0$, then, as we discussed earlier, the change of variables $w_t = h_{\sqrt{\kappa}B_t} \circ g_t$ leads to the diffusion

$$dw_t(z) = -b(w_t(z))\,dt + \sqrt{\kappa}\,\sigma(w_t(z)) \circ dB_t, \quad w_0(z) = z, \quad z \in D.$$

We call $\{w_t\}_{t\geq 0}$ the *slit holomorphic stochastic flow driven by b and σ.*

Properties of the hulls

At least for small times, it is possible to describe the hulls generated by an arbitrary slit Löwner chain in terms of a radial Löwner chain. The radial driving function is related to the driving function of the original chain by means of a differential equation.

In the deterministic case this relationship allows us to translate the well-known results about the properties of the driving function and the corresponding hulls to the case of general slit chains.

In the stochastic case this allows us to prove that the law of the hulls of a general slit holomorphic stochastic flow is locally absolutely continuous with respect to the radial SLE hulls with the same value of κ.

In particular, we prove that the hulls of a general slit holomorphic stochastic flow are almost surely generated by a curve. Analogously to the classical cases, the curve is simple for $\kappa \in [0, 4]$, has self-intersections for $\kappa \in (4, 8)$ and is space-filling for $\kappa \in [8, \infty)$.

Non-classical example: *ABP SLE*

If, combining the chordal and the radial cases, we choose $b = 2\ell_{-2}$ and $\sigma = \ell_{-1} + \frac{1}{4}\ell_1$, the equation for the forward Löwner chain in the unit disk is given by

$$\frac{\partial}{\partial t} f_t(z) = \frac{1}{4\, e^{iu(t)}} \frac{(e^{iu(t)} + f_t(z))^3}{e^{iu(t)} - f_t(z)}.$$

This is an example of a Löwner chain with an attracting boundary point (see Paper B). In the Berkson-Porta representation for the corresponding Herglotz vector field, $\tau(t) = -e^{iu(t)}$, $p(z) = \frac{1}{4}\frac{1+z}{1-z}$.

The process obtained by putting $u(t) = \sqrt{\kappa}\, B_t$, $\kappa \geq 0$, can be called provisionally *ABP SLE* (*ABP* standing for "attracting boundary point").

In contrast to chordal, radial and dipolar Löwner chains, the family of conformal maps $\{f_t\}_{t\geq 0}$ does not have common fixed points neither inside the canonical domain, nor on its boundary. This facts complicates the study of the process quite a bit. Simulations show that *SLE ABP* curves terminate at random points in the unit disk. Not much else, however, is known about their long-term behavior so far.

2.4 Slit holomorphic stochastic flows and Gaussian free field

Let $\{w_t\}_{t\geq 0}$ be a normalized slit holomorphic stochastic flow driven by b and σ, with $\kappa = 4$. For technical reasons, we prefer to work in the half-plane. The flow $\{w_t\}_{t\geq 0}$ then satisfies

$$dw_t(z) = -b(w_t(z))\, dt + 2\, \sigma(w_t(z)) \circ dB_t, \quad w_0(z) = z, \quad z \in \mathbb{H}.$$

Let $\Phi_{\mathbb{H}}$ denote a Gaussian free field in $\mathbb{H}$ independent of B_t. We address the following problem. For what b and σ is it possible to find a harmonic function h in $\mathbb{H}$, such that for the modified Gaussian free field $\hat{\Phi}_{\mathbb{H}} := \Phi_{\mathbb{H}} + h(z)$,

$$\text{the field } \hat{\Phi}_{\mathbb{H}} \circ w_T \text{ has the same law as } \hat{\Phi}_{\mathbb{H}} \text{ for any } T > 0? \tag{2.14}$$

This is a natural question in view of the works [Dub09, She10, SS13] where similar relationships of *GFF* with classical *SLEs* were discovered. In particular, this is a reasonable first step in constructing a conformal field theory for general slit holomorphic stochastic flows.

An important role in proving (2.14) is played by the equation

$$dG_{\mathbb{H}}(w_t(z_1), w_t(z_2)) = -\frac{1}{2}\langle h(w_t(z_1)), h(w_t(z_2))\rangle, \tag{2.15}$$

where $G_{\mathbb{H}}$ denotes the Green's function of $\mathbb{H}$, and $z_1, z_2 \in \mathbb{H}$, $z_1 \neq z_2$. If (2.15) holds, then we can adapt the techniques used in [SS13] to prove that (2.14) also holds. The only theoretical difficulty we face is that in [SS13] the uniform boundedness assumption of h was crucial, which is, unfortunately, too restrictive for us. We relax this assumption and require only that $h(w_t(z))$ is bounded by a deterministic continuous function not depending on t.

After identifying flows that are closely related to each other, we arrive at six one-parameter families. Below we list those families together with the corresponding harmonic functions h. For simplicity, we always assume that the canonical domain is the upper half-plane.

1. Chordal SLE_4 driven by $2\,B_t - \alpha t$, $\alpha \in \mathbb{R}$ (i.e., Brownian motion *with drift* $-\alpha\, t$).

$$b(z) = -\frac{2}{z} - \alpha, \quad \sigma = -1, \quad h(z) = \frac{\alpha}{\sqrt{2}}\,\mathrm{Im}\, z - \sqrt{2}\arg z. \tag{2.16}$$

2.

$$b(z) = -\frac{2}{z} - \beta\, z,\, \beta \in \mathbb{R}, \quad \sigma = -1, \quad h(z) = -\sqrt{2}\arg z. \tag{2.17}$$

3. Dipolar SLE_4 driven by $2\,B_t - \alpha t$, $\alpha \in \mathbb{R}$.

$$\begin{aligned} b(z) &= -\frac{2}{z} - \alpha + \frac{z}{2} + \frac{\alpha}{4}\,z^2, \\ h(z) &= \frac{1-\alpha}{\sqrt{2}}\arg(2-z) - \sqrt{2}\arg z + \frac{1+\alpha}{\sqrt{2}}\arg(2+z). \end{aligned} \tag{2.18}$$

4.

$$\begin{aligned} b(z) &= -\frac{2}{z} + 1 - \left(\beta - \frac{1}{2}\right) z - \left(\frac{1}{4} - \frac{\beta}{2}\right) z^2,\, \beta \in \mathbb{R}, \\ h(z) &= \sqrt{2}\,\arg(2-z) - \sqrt{2}\arg z, \end{aligned} \tag{2.19}$$

5.

$$\begin{aligned} b(z) &= -\frac{2}{z} - 1 - \left(\beta - \frac{1}{2}\right) z - \left(\frac{\beta}{2} - \frac{1}{4}\right) z^2,\, \beta \in \mathbb{R}, \\ h(z) &= \sqrt{2}\arg(2+z) - \sqrt{2}\arg z, \end{aligned} \tag{2.20}$$

6. Radial SLE_4 driven by $2\,B_t - \alpha\, t$, $\alpha \in \mathbb{R}$.

$$\begin{aligned} b(z) &= -\frac{2}{z} - \alpha - \frac{z}{2} - \alpha\,\frac{z^2}{4}, \quad \alpha \in \mathbb{R}, \\ h(z) &= -\sqrt{2}\,\alpha\ \mathrm{Im}\arctan\frac{2}{z} - \sqrt{2}\,\arg z + \frac{1}{\sqrt{2}}\arg(4+z^2). \end{aligned} \tag{2.21}$$

The modifications of GFF for cases 1, 3, and 6 with $\alpha = 0$ (i.e., radial, dipolar and chordal SLE_4 *without drift*) were known previously. In the case 6, the function h is multivalued, which leads to multivalued modifications of GFF addressed in [KM12].

2.5 Further work

Below we formulate several questions that arose during the work on this thesis. They are quite interesting and may provide a basis for further research.

- How can we apply control theory techniques to general slit Löwner chains? What are the reachable sets of univalent functions corresponding to various combinations of b and σ? Can this have applications in the theory of univalent functions?

- Classical radial and chordal $SLEs$ were introduced in an attempt to describe analytically the scaling limits of several conformally invariant lattice models in statistical physics. Is it possible to define analogous lattice models that converge to non-classical slit holomorphic stochastic flows, e.g., $ABP\ SLE$?

- In general, $ABP\ SLE$ seems to be particularly difficult to work with. Any results about its properties, such as, e.g., long term behavior, are interesting.

- In Paper D, we found a way to connect the Gaussian free field with chordal, dipolar and radial $SLEs$ with drift, which is a natural first step in constructing the related conformal field theories. From the theoretical point of view, these cases are somewhat more challenging than the cases without drift, considered in [KM12, KM13, KT13], and constructing the CFT for these cases is of interest.

- In Paper D, we gave a classification of general slit holomorphic stochastic flows that may be related to the Gaussian free field with Dirichlet boundary conditions in such a way that the property (2.14) holds. Can we perform a similar classification for other boundary conditions, such as Neumann or Riemann-Hilbert (see [Kan13, IK13])?

- Another framework which allows to treat radial, chordal and dipolar SLE in a unified way was introduced by Schramm and Wilson in [SW05]. They extend the definition of the so-called $SLE(\kappa;\rho)$ evolution [LSW03] and show that in this case the classical $SLEs$ become special cases of $SLE(\kappa;\rho)$.

  In $SLE(\kappa;\rho)$ one uses the usual chordal Löwner equation, but employs a complicated stochastic process as a driving term. Our approach is the opposite: we consider various versions of the differential equation, but always use the Brownian motion as the driving term. A natural question is whether one of those approaches can be reduced to the other.

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