WEAK SOLUTIONS AND CONVERGENT NUMERICAL SCHEMES OF BRENNER-NAVIER-STOKES EQUATIONS

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ABSTRACT. Lately, there has been some interest in modifications of the compressible Navier-Stokes equations to include diffusion of mass. In this paper, we investigate possible ways to add mass diffusion to the 1-D Navier-Stokes equations without violating the basic entropy inequality. As a result, we recover a general form of Brenner's modification of the Navier-Stokes equations. We consider Brenner's system along with another modification where the viscous terms collapse to a Laplacian diffusion. For each of the two modifications, we derive a priori estimates for the PDE, sufficiently strong to admit a weak solution; we propose a numerical scheme and demonstrate that it satisfies the same a priori estimates. For both modifications, we then demonstrate that the numerical schemes generate solutions that converge to a weak solution (up to a subsequence) as the grid is refined.

1. Conservation laws

Consider the system of conservation laws in one space dimension:

$$
\begin{aligned}
&u_t + f(u)_x = 0, \quad x \in \Omega, \quad 0 \le t \le \mathcal{T} \\
&u(x,0) = u^0(x),
\end{aligned} \tag{1}
$$

Here $u = (u_1, ..., u_n)^\top$ is the vector of unknowns and the fluxes $f = (f_1, f_2, \cdots, f_n)^\top$ are Lipschitz continuous functions of u. Ω is a bounded domain in one dimension (1-D). (We take $\Omega = (0,1)$.) The system is also subject to appropriate boundary conditions. $\mathcal{T}$ is an arbitrary finite time. $u^0(x)$ is a suitably bounded initial datum.

Conservation laws are often endowed with entropies. Entropy is a useful tool to obtain a priori bounds on the solution and sometimes infer uniqueness. We will briefly introduce the concept. Let (U, F) denote an entropy and entropy flux (for short, entropy pair). By definition $U_u^T f_u = F_u$, and $U_u = w^T$ is termed the entropy variables. Using the entropy variables, (1) can be rewritten as

$$
u_w w_t + g(w)_x = 0, \quad x \in \Omega \tag{2}
$$

where u_w is symmetric and positive definite and g_w is symmetric. (See [Moc80]).

Often the conservation law is considered to be a model of an associated viscous equation,

$$
\begin{aligned}
&u_t + f(u)_x = (G(u)u_x)_x, \quad x \in \Omega \\
&u(x,0) = u^0(x),
\end{aligned} \tag{3}
$$

where $G(u)$ is a matrix. The regularization $(G(u)u_x)$ is conservative and we will refer to (3) as being conservative. Using the entropy variables, (3) can be stated as

$$
u_w w_t + g(w)_x = (\tilde{G}(w)w_x)_x, \quad x \in \Omega. \tag{4}
$$

We require that $\tilde{G}$ is symmetric and positive semi-definite. (This property ensures that entropy is diffused.) Note that $\tilde{G}w_x = Gu_x = F^V$ where F^V is commonly known as the viscous flux.

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Often, solutions of conservation laws are interpreted in a weak (or averaged) sense. The weak form of (3) is obtained by multiplying the equation by a test function and integrating by parts.

Definition 1.1. *A locally integrable function u is defined as a* weak solution *of (3), if it satisfies the following integral identity for all compactly supported test functions $\varphi \in C^\infty(\Omega \times [0, \mathcal{T}))$*

$$\int_0^{\mathcal{T}} \int_\Omega (\varphi_t u + \varphi_x f(u) - \varphi_x(G u_x))\, dx\, dt + \int_\Omega \varphi(x,0) u^0(x)\, dx = 0. \tag{5}$$

Remark In the case of Ω being periodic, we employ periodic test functions in space.

1.1. **The compressible Navier-Stokes equations.** In this work, we focus on the special case of gas dynamics. An inviscid gas is governed by the Euler equations, which is a set of conservation laws (1) and in 1-D they take the form,

$$\begin{aligned} u_t + f(u)_x &= 0 \quad x \in \Omega, \quad 0 \le t \le \mathcal{T} \\ u &= (\rho, m, E)^T \\ f(u) &= (m, \rho q^2 + p, (E+p)q)^T \\ p &= (\gamma - 1)(E - \frac{1}{2}\rho q^2). \end{aligned}$$

ρ, q, p and E are the density, velocity, pressure and total energy of a gas. The momentum is denoted as $m = \rho q$ and γ is the ratio of the specific heats. The system is closed using the gas law $p = \rho R T$, where R is the gas constant and T the temperature. In the analysis, we will need the thermodynamic relations, $\gamma = c_p/c_v$ and $R = c_p - c_v$, where c_p and c_v are the specific heat capacities at constant pressure and volume, respectively.

The standard Navier-Stokes equations take the form (3) and are obtained by adding a diffusive flux to the Euler equations.

$$\begin{aligned} u_t + f(u)_x &= (f^{NS})_x \\ f^{NS} &= (0, \frac{4}{3}\mu q_x, \frac{4}{3}\mu q q_x + k T_x)^T \end{aligned} \tag{6}$$

where $\mu > 0$ is the first diffusion coefficient. (We have made the standard assumption that the second diffusion coefficient $\lambda = -2\mu/3$.) $k > 0$ is the thermal diffusivity. These equations are referred to as the Navier-Stokes(-Fourier) (NSF) equations which is the standard set of equations used in compressible viscous fluid dynamics.

In this study, we assume that the domain $\Omega = (0,1)$, i.e., it is bounded. The system (6) is subject to suitably bounded initial condition $u(x,0) = u^0(x)$ and we require that $p(x,0) > 0$ and $\rho(x,0) > 0$.

Furthermore, the system (6) must be augmented by appropriate boundary conditions. This is a topic in its own right and for simplicity we only consider thermally insulated wall boundary conditions.

$$q = 0|_{\partial\Omega}, \quad T_x = 0|_{\partial\Omega}. \tag{7}$$

Remark We will use boundary conditions when deriving a priori bounds. However, when considering numerical approximations, we will limit the analysis to the periodic case for simplicity. Demonstrating that it is possible to obtain bounds for the PDE with boundary conditions makes a good case for doing the same with the numerical scheme in the future.

1.2. **Background.** The standard compressible Navier-Stokes model has been studied extensively and still no general well-posedness results have been obtained. The literature on this subject is vast and we mention a only few results here. In [Ler34], the existence of weak solutions of the *incompressible* equations was proved and in [Lio98] for *isentropic compressible* fluids.

The lack of well-posedness results for the compressible equations hampers the design of effective numerical schemes. More importantly, it leaves doubts on any numerical results since the lack of knowledge of solutions precludes any convergence proofs. This uncertainty is not merely a mathematical nuisance. It affects engineering applications since there is no way of knowing if the numerical solution is in the vicinity of the true solution whose existence is assumed.

Many of the difficulties in proving existence stem from the incomplete parabolic structure of the Navier-Stokes equations. In other words, from the fact that the continuity equation lacks a diffusion term. In particular, this complicates proofs of positivity. Moreover, it is not clear that the Euler equations is the limiting case of the Navier-Stokes equations as the diffusion coefficients vanish. This is evident when considering boundaries. For the Euler equations it is commonplace to specify ingoing characteristic waves. This is a linearly well-posed procedure. As is perturbing the in-going characteristics with the viscous flux for the Navier-Stokes equations. This works for all boundaries but *subsonic* outflows. (See [SCN07]). Hence, the vanishing viscosity limit of the Navier-Stokes equations will not converge to the Euler solution in the vicinity of a subsonic outflow. With mass diffusion, this problem would disappear. (At least in the linear analysis.) (See also [MS11] for a similar study.)

Positivity and boundary conditions are two areas that would be much easier to treat if a diffusion term is added to the continuity equation. Mathematical arguments can indicate difficulties with a particular model, but it requires physical arguments and corroboration with experimental data before discarding one model in favor of another. Based on thermodynamical arguments, Brenner ([Bre05a, Bre05b]) has suggested that mass indeed is subject to a diffusion process. Brenner argues that in Newton's viscosity law, the volume velocity $\mathbf{u_v}$ should be used instead of the mass velocity $\mathbf{u_m}$. The latter is the velocity appearing everywhere in the conventional Navier-Stokes equations. The relation between the two velocities is, $\mathbf{u_v} = \mathbf{u_m} + \alpha_v \frac{1}{\rho}\nabla\rho$, where α_v is the volume diffusivity. This change of view introuduces mass diffusion to the Navier-Stokes equations. Support for this argument is found in [Ött05] on non-equilibrium processes. Brenner suggested a theoretical value of α_v, but its value is open for investigation.

The validity of Brenner-Navier-Stokes equations was carefully studied in [GR07] for a well-known shock tube problem for which there is exerimental data available. Specically they compared the standard equations with Brenner's modified set. They pointed out a major difficulty with such a comparison. Namely, the correct temperature dependence of the viscosity coefficient is uncertain, which in turn can give signicantly different results for the Navier-Stokes equations. (We remark that this implies that there is not one unviersally accepted model that is the Navier-Stokes equations.) Nevertheless, they made well-motivated choices for the diffusion coefficients and compared the standard Naver-Stokes with Brenner's. Their conclusion was that Brenner's system more accurately captured the strong shocks in their tests. They remarked that results of Brenner's system was similar to results of more elaborate models like Burnett's equations, without their stability problems equations. However, they concluded that more validation is needed.

The purpose of this work is ***not*** to demonstrate that the equations augmented with a mass diffusion supersede the traditional Navier-Stokes equations when it comes to modeling physics. That question is to date open. However, we remark that many studies in physics are concerned with the question whether or not mass diffusion can extend the range of applicability of the model, for instance to rarefied gases. We would argue that it would be a major step forward if the new models have the same physical range of applicability as the conventional Navier-Stokes, but in addition have strong well-posedness results and convergent numerical schemes. In fact, numerical convergence is for all practical purposes a premise for validating the model, since numerical solutions is the only feasible way to generate solutions that can be compared with experimental data.

A mathematical study of Navier-Stokes-Brenner (NSB) is found in the excellent article [FV09]. With certain choices of diffusion coefficients, the three-dimensional equations were proven to admit weak solutions, a very important result. However, these existence proofs are not easy to mimic with a practically useful numerical scheme.

The purpose of the current study is to mathematically study the properties of the Navier-Stokes system augmented with mass diffusion. In particular, the possibilities to design convergent numerical schemes. Instead of immediately analyzing Brenner's system, we begin by investigating in what way mass diffusion can be added. To this end, we take the entropy principle to be fundamental and require that modifications satisfy the usual entropy inequality. This condition gives us a family of modifications, which differ from Brenner's system only in the choice of mass diffusion coefficients. However, apart from Brenner's system, we note that a Laplacian diffusion model follows easily from the modified system. A Laplacian diffusion is a common way to stabilize numerical schemes for flow equations. Hence, we consider both the Laplacian diffusion model and the standard Brenner model, propose numerical schemes, and prove convergence to weak solutions.

Finally we remark that the schemes proposed in this work are not of high-order accuracy, which would be more effective in practice. Recently there has been efforts towards non-linearly stable high-order schemes in e.g. [FC13, Svä12]. However, proving convergence for high-order accurate schemes approximating the Navier-Stokes-Brenner system is more challenging and we put that task on the list of future work.

1.3. **Outline of paper.** As discussed above, our viewpoint is mathematical rather than physical. In Section 2, we begin by deriving the form of mass diffusion that admits local entropy inequalities and global entropy estimates. Not surprisingly, the system we obtain turns out to be a general form of Brenner's system. With particular relations between the three diffusion coefficients (mass, velocity and heat) the viscous terms can be cast as a Laplacian.

In Section 3, we consider the model with Laplacian diffusion. We derive a priori estimates for the equations, propose a numerical scheme and derive a priori bounds to ensure convergence to a weak solution.

In Section 4, we consider the form coinciding with Brenner's system. We choose a particular set of diffusion coefficients and derive a priori bounds for the system of equations. We propose a numerical scheme and demonstrate convergence to a weak solution. This set of diffusion coefficients differ from the set analyzed in [FV09]. Hence, these results are novel and complements the earlier results.

Although, we have limited the analysis to 1-D to reduce notation, we have deliberately avoided 1-D specific properties such as 1-D Sobolev embeddings. Hence, our results should be extendable to 3-D but we postpone that to a future paper. Furthermore, we have limited the analytical tools to those that have a counterpart in numerical analysis. This to be able to mimic the estimates for the numerical schemes.

We do not present any numerical computations, since that will immediately take us into the realm of selecting values for the diffusion coefficients and a discussion of physics. That said, the schemes have been tested in practice and we discuss this in the Conclusions.

In Appendix I.1 we define the function spaces and norms we will use.

2. Navier-Stokes equations with mass diffusion

We begin by deriving the entropy inequality for the standard Navier-Stokes-Fourier equations. Then we proceed by deriving the form of mass diffusion that can be added in an entropy consistent manner.

2.1. **Entropy inequality.** An entropy pair, as defined above, will symmetrize the Euler system but not necessarily the Navier-Stokes system. It turns out, [HFM86], that only one entropy symmetrizes the Navier-Stokes equations. Namely, $(U, F) = (-\rho S, -\rho q S)$ where $S = \ln(\frac{p}{\rho^\gamma})$ is the specific entropy. The corresponding entropy variables are

$$U_u = w^T = (-(S-\gamma) - \frac{q^2}{2c_vT}, \frac{q}{c_vT}, -\frac{1}{c_vT}). \tag{8}$$

(Although well known, we include the derivation of the entropy variables for the Navier-Stokes equations in Appendix I.)

These entropy variables symmetrize the Navier-Stokes system, such that (6) turns into

$$u(w)_t + g(w)_x = (C(w)w_x)_x \tag{9}$$

where u_w, g_w and C are symmetric matrices. The first row and column of C are 0 since no diffusion is added to the ρ equation. (c.f eqn (6).) u_w is positive definite if the entropy U is strictly convex, which in turn is the case if $\rho, T > 0$. Furthermore, C is positive semi-definite if the diffusion coefficients and ρ, p are positive.

Using (9) it is possible to obtain a priori L^2 bounds on the solution. Multiply (9) by w^T and integrate over the domain Ω. (Recall that $\Omega = (0, 1)$.)

$$\int_0^1 w^T u(w)_t\,dx + \int_0^1 w^T g(w)_x\,dx = \int_0^1 w^T (Cw_x)_x\,dx,$$
$$\int_0^1 U_t\,dx + \int_0^1 F_x\,dx = \int_0^1 w^T (Cw_x)_x\,dx,$$
$$\int_0^1 U_t\,dx + (F - w^T C w_x)|_0^1 + \int_0^1 w_x^T (Cw_x)\,dx \le 0.$$

Integration in time $[0, \mathcal{T}]$ and the use of the boundary conditions (7), give an upper bound on $U(\mathcal{T})$. Using the convexity of U we can obtain bounds on u in L^2. The argument is found in [Daf00] and we repeat it here for convenience.

Define a new entropy $\bar{U} = U - U'(u_0)^T(u - u_0)$ where u_0 is a constant (non-zero) state. (We choose $q = 0$, $\rho = \rho_0 > 0$ and $p = p_0 > 0$ and since it is a constant state $T_x = 0$, which is necessary for the boundary terms to vanish.) Similarly define $\bar{F}' = \bar{U}'f'$. Since this is an affine change of the entropy, this entropy satisfies an entropy inequality if U does.

To derive the entropy estimate, we note that $\bar{w}^T = \bar{U}' = U' - U'(u_0)$ and $\bar{w}_x = w_x$. Hence, upon multiplication from left by $\bar{w}^T$ and integrating in space, we get,

$$\int_0^1 \bar{U}_t\,dx \le -\int_0^1 w_x^T C w_x\,dx. \tag{10}$$

The diffusion terms are unaffected by the change of entropy and the boundary terms vanish in this case too.

Next, we Taylor expand U around u_0

$$U(u) = U(u_0) + U'(u_0)^T(u - u_0) + (u - u_0)^T U''(\theta)(u - u_0)$$

for some state θ. Since the density of our reference state $\rho_0 > 0$, we have $\rho(\theta) > 0$ even if the density of the solution only satisfies $\rho \geq 0$. (An analogical argument gives $T(\theta) > 0$.) Hence, $U''_{min} \geq c > 0$, where U''_{min} is the minimal eigenvalue of $U''(\theta, t)$ for all $\theta(x)$, $x \in [0, 1]$.

Furthermore, using $\bar{U} = U - U'(u_0)^T(u - u_0) = (u - u_0)^T U''(u - u_0)$ in (10), we obtain

$$\int_0^1 \bar{U}(u(\cdot, \mathcal{T}))\, dx \leq -\int_0^{\mathcal{T}} \int_0^1 w_x^T C w_x \, dx\, dt + \int_0^1 \bar{U}(u(\cdot, 0))\, dx$$

(11)

$$\int_0^1 (u - u_0)^T U''(\theta(\mathcal{T}))(u - u_0)\, dx + \int_0^{\mathcal{T}} \int_0^1 w_x^T C w_x \, dx\, dt \leq \int_0^1 \bar{U}(u(\cdot, 0))\, dx$$

From the bound on $\int_0^1 (u - u_0)^T U''(u - u_0)\, dx \leq \mathcal{C}$ we deduce that $U''_{min} \int_0^1 (u - u_0)^T(u - u_0)\, dx \leq \mathcal{C}$. Since

$$\begin{aligned} u^T u = (u - u_0 + u_0)^T(u - u_0 + u_0) = \\ (u - u_0)^T(u - u_0) + u_0^T u_0 + 2(u - u_0)^T u_0 \leq \\ 2(u - u_0)^T(u - u_0) + 2u_0^T u_0 \end{aligned}$$

we have that

$$\int_0^1 u^T u \, dx \leq 2\frac{\mathcal{C}}{U''_{min}} + 2\int_0^1 u_0^T u_0 \, dx \tag{12}$$

From (12) we obtain, $u \in C([0, \mathcal{T}]; L^2(\Omega)^3)$.

Remark Throughout this paper, we use $\mathcal{C}$ to denote an a priori known and bounded constant. Typically, $\mathcal{C}$ will contain bounds of initial and boundary data. We may incorporate new factors or terms into the constant without changing the notation.

Since $(u - u_0)^T U''(\theta)(u - u_0) \geq 0$, we also obtain a bound on $\int w_x^T C w_x \, dx\, dt$ from (10) which is a measure by which the entropy is diffused. From (8), we have

$$w_x^T = (-S_x - \frac{q q_x}{c_v T} + \frac{q^2 T_x}{2 c_v T^2}, \frac{q_x}{c_v T} - \frac{q T_x}{c_v T^2}, \frac{T_x}{c_v T^2}).$$

and the entropy diffusion

$$\begin{aligned} w_x^T C w_x = w_x^T f^{NS} = w_x^T(0, \frac{4}{3}\mu q_x, \frac{4}{3}\mu q q_x + k T_x)^T = \\ (\frac{q_x}{c_v T} - \frac{q T_x}{c_v T^2})\frac{4}{3}\mu q_x + \frac{T_x}{c_v T^2}(\frac{4}{3}\mu q q_x + k T_x) = \\ \frac{4\mu q_x^2}{3 c_v T} + \frac{k T_x^2}{c_v T^2} \end{aligned} \tag{13}$$

We summarize the results and emphasize again that they are well known and found in the literature.

Proposition 2.1. *Let $u^0(x) \in (L^2(\Omega))^3$ and $\rho(x, 0) > 0, T(x, 0) > 0$. Assume that a solution to (6) obeys $\rho(x, t), T(x, t) \geq 0$ on $(0, \mathcal{T}]$. Then $u \in C([0, \mathcal{T}]; L^2(\Omega)^3)$ and*

$$\int_0^{\mathcal{T}} \int_0^1 \frac{4\mu q_x^2}{3 c_v T} + \frac{k T_x^2}{c_v T^2}\, dx\, dt < Constant. \tag{14}$$

Furthermore, $p \in C([0,\mathcal{T}]; L^2(\Omega))$ *and* $\rho q^2 \in C([0,\mathcal{T}]; L^2(\Omega))$.

Proof. We first comment on the positivity assumption. For these a priori estimates to hold it is enough to assume that ρ, T are non-negative. The key observation is that the state θ in (12) has positive density. It is a value in between u (where ρ, T are possibly 0) and u_0 where these states are bounded away from 0. Hence, U''_{min} is bounded away from 0 in (12).

The L^2 estimate of u and (14) follow from (11),(12) and (13). The statements that have not been shown are that $p, \rho q^2$ are in $L^2(\Omega)$. We note that $p, \rho \geq 0$ by assumption. Hence, $\frac{p}{\gamma-1} < \frac{p}{\gamma-1} + \frac{1}{2}\rho q^2$ and

$$\int_0^1 (\frac{p}{\gamma-1})^2 dx \leq \|E\|^2.$$

The L^2 bound on ρq^2 follows in the same way. □

The estimate (14) can be used to get bounds on q_x and T_x by choosing appropriate temperature dependence of μ and k. However, this comes at a cost since higher integrability of the temperature is needed to bound the $\mu q_x, \mu q q_x$ and kT_x terms in (6). The critical stumbling block is that the estimates rely on $\rho \geq 0$. For the Navier-Stokes-Fourier equations there is no proof that the density remains positive.

2.2. **Entropy Consistent Mass diffusion.** We will now begin with our program to add mass diffusion that does not violate the fundamental entropy principle. To this end, we modify (9) to,

$$u(w)_t + g(w)_x = (Cw_x)_x + (Dw_x)_x$$

where we have added the viscous flux Dw_x. We require that D is symmetric positive semi-definite to diffuse the entropy, and it should act on ρ. We remind of

$$w_x^T = (-S_x - \frac{qq_x}{c_vT} + \frac{q^2T_x}{2c_vT^2}, \frac{q_x}{c_vT} - \frac{qT_x}{c_vT^2}, \frac{T_x}{c_vT^2}).$$

The first row of Dw_x is

$$(Dw_x)_1 = d_{11}(w_1)_x + d_{12}(w_2)_x + d_{13}(w_2)_x = h_1(u)\rho_x \tag{15}$$

where the function $h_1(u) > 0$ is included to admit physical modeling of the diffusion coefficient.

To examine the expression (15), we need

$$S_x = (\log(\frac{p}{\rho^\gamma}))_x = \frac{\rho^\gamma}{p}(\frac{p}{\rho^\gamma})_x = \frac{\rho^\gamma}{p}(\frac{p_x}{\rho^\gamma} - \gamma\frac{p\rho_x}{\rho^{\gamma+1}}) = (\frac{p_x}{p} - \gamma\frac{\rho_x}{\rho}).$$

Hence,

$$(w_1)_x = -(\frac{p_x}{p} - \gamma\frac{\rho_x}{\rho}) - \frac{qq_x}{c_vT} + \frac{q^2T_x}{2c_vT^2}.$$

Furthermore, using the gas law $p = \rho RT$ we have

$$\frac{p_x}{p} = \frac{\rho_x RT}{\rho RT} + \frac{\rho RT_x}{\rho RT} = \frac{\rho_x}{\rho} + \frac{T_x}{T}$$

and

$$(w_1)_x = -(\frac{\rho_x}{\rho} + \frac{T_x}{T} - \gamma\frac{\rho_x}{\rho}) - \frac{qq_x}{c_vT} + \frac{q^2T_x}{2c_vT^2}.$$

Equipped with the above expressions, we choose $d_{11} = 1, d_{12} = q, d_{13} = c_v T + \frac{q^2}{2}$ and obtain

$$
\begin{aligned}
&(w_1)_x + q(w_2)_x + (c_v T + \frac{q^2}{2})(w_3)_x = \\
-(\frac{\rho_x}{\rho} + \frac{T_x}{T} - \gamma\frac{\rho_x}{\rho}) - \frac{q q_x}{c_v T} + \frac{q^2 T_x}{2 c_v T^2} &+ q(\frac{q_x}{c_v T} - \frac{q T_x}{c_v T^2}) + (c_v T + \frac{q^2}{2})\frac{T_x}{c_v T^2} = \\
&(\gamma - 1)\frac{\rho_x}{\rho}
\end{aligned}
$$

i.e., a mass diffusion. Note that $(\gamma - 1) > 0$. Up to a positive scaling this is the only way diffusion on ρ can be added to the first equation.

To determine D, we observe that the remaining two rows must be scaled versions of the first, or else, they will not result in diffusion on ρ. Moreover, D must be be symmetric and positive semi-definite to diffuse entropy. Hence, we are forced to choose

$$
D = \delta\frac{\rho}{\gamma - 1}\begin{pmatrix} 1 & q & \beta \\ q & q^2 & q\beta \\ \beta & q\beta & \beta^2 \end{pmatrix},
$$

where $\beta = c_v T + \frac{q^2}{2}$. $\delta > 0$ is a new diffusion coefficient. We allow it to be some function of u.

Denote $Cw_x = f^{NS}$ as the viscous flux in the standard Navier-Stokes equations and $f^{mod} = Dw_w$ is the new mass diffusion. We obtain the following set of equations on conservation form:

$$
u_t + f(u)_x = (f^{NS})_x + (f^{mod})_x \tag{16}
$$

$$
u = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix}, f(u) = \begin{pmatrix} m \\ \rho q^2 + P \\ (E + p)q \end{pmatrix}, f^{NS} = \begin{pmatrix} 0 \\ \frac{4}{3}\mu q_x \\ \frac{4}{3}\mu q q_x + kT_x \end{pmatrix}, f^{mod} = \begin{pmatrix} \delta\rho_x \\ \delta q\rho_x \\ \delta\beta\rho_x \end{pmatrix}
$$

$$
p = (\gamma - 1)(E - \frac{1}{2}\rho q^2).
$$

The system is by construction symmetrizable by the physical entropy $U = -\rho S$, just like the standard Navier-Stokes system.

Furthermore, we note that (16) is the Navier-Stokes-Brenner system. (See [Bre05b, FV09, GR07].) It is noted in [GR07], that Brenner suggest that $\delta = k/(c_p\rho)$ but they suggest that $\delta = constant$ is a more accurate model. In [FV09], δ is taken to be a constant. They then prove existence of weak solutions to the resulting system with $\mu = constant$ and k a third-order polynomial of T.

We will consider two systems. One differ from [FV09] only in the choice of diffusion coefficients δ, μ, k and will be analyzed in Section 4. The other one is derived in the next section.

3. Laplacian diffusion model

We consider f^{NS} and f^{mod}. Replace the dynamic viscosity with the kinematic counterpart. That is, we take $\mu = \nu\rho$.

$$
f^{mod} + f^{NS} = \begin{pmatrix} \delta\rho_x \\ \delta q\rho_x + \frac{4}{3}\nu\rho q_x \\ \delta(c_v T + \frac{q^2}{2})\rho_x + \frac{4}{3}\nu\rho\left(\frac{q^2}{2}\right)_x + kT_x \end{pmatrix}. \tag{17}
$$

We note that if $\delta = \frac{4}{3}\nu$, the second row forms a complete derivative. By the same token, we choose $k = \frac{4c_v\mu}{3} = \frac{4c_v\nu\rho}{3}$ and we end up with the viscous flux,

$$f^v = f^{mod} + f^{NS} = \frac{4\nu}{3}\begin{pmatrix} \rho_x \\ (\rho q)_x \\ (c_v T\rho)_x + \left(\frac{\rho q^2}{2}\right)_x \end{pmatrix} = \frac{4}{3}\nu u_x. \tag{18}$$

A few remarks on the choices made above. First, to replace the dynamic and kinematic viscosity is obviously the crucial choice to get complete derivatives. We will not attempt to motivate this physically but leaves validation of the model as a separate issue.

For the other choices something can be said. It is well known that k and μ are not independent. From a theoretical viewpoint it has been suggested that $\mu = \frac{3k}{4c_p}$ to a first approximation. (See e.g. [ČV94].) We then see that $\mu = \frac{3k}{4c_p}$ and $\delta = \frac{4}{3}\nu$ leads to $\delta = k/(c_p\rho)$ which is the choice suggested by Brenner. However, we use $\mu = \frac{3k}{4c_v}$ and $\delta = \frac{4}{3}\nu$ which leads to $\delta = k/(c_v\rho)$. (Not exactly the same, but strikingly similar.)

Remark Using a Laplacian diffusion is a common regularization used in computational fluid dynamics. In particular, when resolving shocks but also in the so-called ILES (Implicit Large Eddy Simulation) whereby turbulence is resolved using the internal diffusion of the scheme, commonly a Laplacian.

As mentioned above, $\delta = \delta_0 = \textit{constant}$ and $\delta \sim 1/\rho$ have been proposed in the literature. Here, we allow a blend of the two choices. Specifically

$$\nu = \nu_1\frac{1 + \nu_2\rho}{\rho} \tag{19}$$

where ν_1, ν_2 are positive constants.

Keeping in mind that (19) is our diffusion model we will drop all physical constants for notational convenience and analyze the system

$$\begin{aligned} u_t + f(u)_x &= (\nu u_x)_x, \quad x \in (0,1), \quad t \in (0, \mathcal{T}] \\ u(x,0) &= u^0(x) \\ \nu &= \frac{1+\rho}{\rho}. \end{aligned} \tag{20}$$

Assumption 3.1. *The initial datum $u^0(x) = u(x,0)$ is assumed to reside in the following spaces.*

- $U(u^0) = -\rho(x,0)S(x,0) \in L^1(\Omega)$.
- $u^0 \in L^2(\Omega)^3$.
- $\rho^{-1}(\cdot,0) \in L^1(\Omega)$.
- $\rho(x,0) > 0, p(x,0) > 0$ *and* $T(x,0) = (p/(R\rho))(x,0) > 0$.

We also assume that the initial datum satisfies the following boundary conditions.

$$\rho_x = 0, \quad T_x = 0, \quad q = 0 \tag{21}$$

The boundary conditions are the same that are used in [FV09]. The two latter are well-known and correspond to an insulated wall. The first one is required by the modification of the original system and corresponds to no mass flow through the wall.

Before, we proceed with derivation of various estimates, we present a Poincare inequality that will be a crucial tool. (We use a statement given in [FV09].)

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $B \subset \Omega$ be a measurable set such that $|B| \geq \mathrm{M} > 0$. Then*

$$\|v\|_{H^1(\Omega)} \leq c(M,\alpha)\left(\|\nabla_x v\|_{L^2(\Omega)} + \left(\int_B |v|^\alpha\right)^{1/\alpha}\right)$$

for any $v \in H^1(\Omega)$, where the constant $c = c(M,\alpha)$ depends solely on M and the parameter $\alpha > 0$.

3.1. **A priori estimates.** The goal in this section is to derive estimates for (20) that are strong enough to give meaning to a weak solution.

The first observation is that this system is not only symmetrizable with the Navier-Stokes entropy $U = -\rho S$ but with any entropy. Recall that $U_u = w^T$ are the entropy variables. Multiply (20) by w^T,

$$w^T u_t + w^T f(u)_x = w^T(\nu U_w w_x)_x$$

and note that f_w and U_w are symmetric and the latter positive definite thanks to U being an entropy. Integrating in space gives,

$$\int_\Omega U_t\,dx = -\int_\Omega \nu w_x U_w w_x\,dx. \tag{22}$$

The right-hand side is negative and we obtain the estimate. (The set of boundary conditions (21) ensures that all the boundary terms vanish.)

Specifically, using the entropy $U = -\rho S$, we calculate

$$\begin{aligned}
-w_x^T u_x =& -\left(-S_x - \frac{qq_x}{c_v T} + \frac{q^2 T_x}{2c_v T^2}\right)\rho_x - \left(\frac{q_x}{c_v T} - \frac{qT_x}{c_v T^2}\right)(\rho q)_x - \frac{T_x}{c_v T^2}E_x\\
=&(\frac{p_x}{p} - \gamma\frac{\rho_x}{\rho})\rho_x + \frac{qq_x\rho_x}{c_v T} - \frac{q^2 T_x}{2c_v T^2}\rho_x\\
&- \frac{q_x}{c_v T}(\rho_x q + \rho q_x) + \frac{qT_x}{c_v T^2}(\rho_x q + \rho q_x)\\
&- \frac{T_x}{c_v T^2}(\frac{\rho_x q^2}{2} + \rho q q_x + \frac{p_x}{\gamma - 1})\\
=&\rho_x(\frac{p_x}{p} - \gamma\frac{\rho_x}{\rho}) - c_v\frac{q_x}{T}\rho q_x - \frac{T_x}{c_v T^2}\frac{p_x}{(\gamma-1)}\\
=& -\gamma\frac{\rho_x^2}{\rho} - \frac{\rho}{c_v T}q_x^2 + \frac{p_x\rho_x}{p} - \frac{T_x p_x}{c_v T^2(\gamma-1)}\\
=& -\gamma\rho(\frac{\rho_x}{\rho})^2 - 2\frac{p_x T_x}{pT}\rho + \frac{p_x^2}{p^2}\rho - \frac{\rho}{c_v T}q_x^2\\
=& -\gamma\rho(\frac{\rho_x}{\rho})^2 + \rho(\frac{p_x}{p} - \frac{T_x}{T})^2 - \rho\frac{T_x^2}{T^2} - \frac{\rho}{c_v T}q_x^2\\
=& -(\gamma-1)\rho(\frac{\rho_x}{\rho})^2 - \rho\frac{T_x^2}{T^2} - \frac{\rho}{c_v T}q_x^2
\end{aligned} \tag{23}$$

We can now state the following proposition.

Proposition 3.3. *Let $u^0(x)$ satisfy Assumption 3.1. Assume that a solution of (20) obeys $\rho(x,t), T(x,t) \geq 0$ on $(0,\mathcal{T}] \times \Omega$. Then $u \in C([0,\mathcal{T}]; L^2(\Omega)^3)$ and*

$$\int_0^{\mathcal{T}}\int_0^1 \nu\left((\gamma-1)\rho(\frac{\rho_x}{\rho})^2 + \rho\frac{T_x^2}{T^2} + \frac{\rho}{T}q_x^2\right) < \mathcal{C} \tag{24}$$

Furthermore,

$$\frac{T_x}{T} \in L^2(0,\mathcal{T};L^2(\Omega)), \frac{q_x}{\sqrt{T}} \in L^2(0,\mathcal{T};L^2(\Omega)), \frac{\rho_x}{\rho} \in L^2(0,\mathcal{T};L^2(\Omega)),$$
$$p \in C([0,\mathcal{T}];L^2(\Omega)), \rho q^2 \in C([0,\mathcal{T}];L^2(\Omega)),$$

and $\rho > 0$ *a.e. in* $[0,\mathcal{T}]\times\Omega$.

Proof. The estimate (24) follows from (22) and (23) and the observation that the boundary conditions (21) cancel the boundary terms. With $\nu = \frac{1+\rho}{\rho}$, the estimates on T_x/T, ρ_x/ρ and $q_x/\sqrt{T}$ follow.

Furthermore, the bound on ρ_x/ρ is equivalent to $(\log\rho)_x \in L^2(0,\mathcal{T};L^2(\Omega))$. We use conservation of ρ,

$$\int_0^{\mathcal{T}}\int_0^1 (\rho_t + (\rho q)_x)\,dx\,dt = \int_0^{\mathcal{T}}\int_0^1 (\nu\rho_x)_x\,dx\,dt$$

leading to $\int_0^1 \rho(x,t)\,dx = \int_0^1 \rho(x,0)\,dx$ and since $\rho \geq 0$ it implies that $\rho(\cdot,t) \in L^1(\Omega)$. Hence, there is a non-zero subdomain B where $\log(\rho)$ is bounded in L^1. By the Poincare inequality, we have $(\log\rho) \in L^2(0,\mathcal{T};H^1(\Omega))$ and consequently $\rho > 0$ a.e.

The estimates on p and ρq^2 follows in the same way as for Proposition 2.1. $\square$

3.1.1. *Kinetic energy.* To derive the kinetic energy we use the continuity and momentum equations of (20).

$$\rho_t + (\rho q)_x = (\nu\rho_x)_x,$$
$$(\rho q)_t + (\rho q^2 + p)_x = (\nu(\rho q)_x)_x.$$

We will use the following identities.

$$q(\rho q)_t = \frac{1}{2}(\rho q^2)_t + \frac{\rho_t q^2}{2}, \quad q(\rho q^2)_x = \frac{1}{2}(\rho q^3)_x + (\rho q)_x\frac{q^2}{2}.$$

Multiplying the momentum equation by q yields

$$q(\rho q)_t + q(\rho q^2 + p)_x = q(\nu(\rho q)_x)_x$$
$$\frac{1}{2}(\rho q^2)_t + \frac{\rho_t q^2}{2} + \frac{1}{2}(\rho q^3)_x + (\rho q)_x\frac{q^2}{2} + qp_x = q(\nu(\rho q)_x)_x$$
$$\frac{1}{2}(\rho q^2)_t + \frac{q^2}{2}(\rho_t + (\rho q)_x) + \frac{1}{2}(\rho q^3)_x + qp_x = q(\nu(\rho q)_x)_x$$

Use the continuity equation.

$$\frac{1}{2}(\rho q^2)_t + \frac{q^2}{2}(\nu\rho_x)_x + \frac{1}{2}(\rho q^3)_x + (pq)_x - pq_x = q(\nu(\rho q)_x)_x$$

Integrating by parts.

$$\int_0^1 \frac{1}{2}(\rho q^2)_t dx + \frac{1}{2}(\rho q^3)|_0^1 + (pq)|_0^1 - \int_0^1 pq_x\,dx =$$
$$q\nu(\rho q_x + \rho_x q)|_0^1 - \frac{q^2}{2}(\nu\rho_x)|_0^1 + \int_0^1 \left(-q_x\nu(\rho q_x + \rho_x q) + \left(\frac{q^2}{2}\right)_x(\nu\rho_x)\right)dx$$

Use the boundary conditions (21).

$$\int_0^1 \frac{1}{2}(\rho q^2)_t dx - \int_0^1 pq_x\,dx = \int_0^1 -q_x\nu(\rho q_x + \rho_x q) + qq_x\nu\rho_x\,dx$$
$$\int_0^1 \frac{1}{2}(\rho q^2)_t dx = \int_x pq_x\,dx - \int_0^1 \nu\rho q_x^2\,dx.$$

Use Cauchy-Schwarz

$$\int_0^1 pq_x\, dx \le \|p\|\|q_x\| \le \frac{1}{\eta}\|p\|^2 + \eta\|q_x\|^2$$

and since $\nu = \frac{1+\rho}{\rho}$ we obtain

$$\int_0^1 \frac{1}{2}(\rho q^2)_t dx \le \frac{1}{\eta}\|p\|^2 + \eta\|q_x\|^2 - \int_0^1 (1+\rho)q_x^2\, dx.$$

By choosing $\eta < 1$, we get

$$\int_0^1 \frac{1}{2}(\rho q^2)(x,\mathcal{T})\, dx \le -\int_0^1\int_0^{\mathcal{T}} (1-\eta)q_x^2 dx\, dt - \int_0^{\mathcal{T}}\int_0^1 \rho q_x^2 dx\, dt + \frac{1}{\eta}\|p\|^2. \tag{25}$$

Recall that $p \in C([0,\mathcal{T}];L^2(\Omega))$. Assuming that ρ is non-negative, we have proved $\rho q^2 \in C([0,\mathcal{T}];L^2(\Omega))$ (which we already knew from Proposition 3.3). In addition, we have a now obtained a bound on q_x.

Proposition 3.4. *Let the initial datum u^0 satisfy Assumption 3.1. Then solutions of (20) with boundary conditions (21) satisfy*

$$q \in L^2(0,\mathcal{T};H^1(\Omega))$$

Proof. The estimate (25) gives $q_x \in L^2(0,\mathcal{T};L^2(\Omega))$. To prove that $q \in L^2(0,\mathcal{T};L^2(\Omega))$, we intend to use the Poincare inequality.

By conservation of ρ, $\int_0^1 \rho(x,\mathcal{T})\, dx = \int_0^1 \rho_0(x)\, dx$ and since $\rho(x,t) \ge 0$ this implies that $\rho(x,t) \ge c > 0$ on a set B of non-zero measure. (c is a constant.) Let $\|\cdot\|_{2,B}$ denote the local L^2-norm on B. Consequently, $c\|q\|_{2,B} \le \|\rho q\|_{2,B} \le \mathcal{C}\|\rho q\|_2$. Hence, we conclude that $q \in L^2(0,\mathcal{T};L^2(\Omega))$. □

3.1.2. *Estimates of density and momentum.*

Proposition 3.5. *Let the initial datum u^0 satisfy Assumption 3.1. Then solutions of (20), with boundary conditions (21) satisfy*

$$\rho \in C([0,\mathcal{T}];L^2(\Omega)) \cap L^2(0,\mathcal{T};H^1(\Omega)).$$

Proof. The first statement, $\rho \in C([0,T];L^2(\Omega))$, is already known from Proposition 3.3.

To prove the second statement (and the first again), we use the energy method on the first equation of (20).

$$\rho_t + (\rho q)_x = (\frac{\rho+1}{\rho}\rho_x)_x$$

We multiply by ρ and integrate.

$$\int_0^{\mathcal{T}} \frac{1}{2}\|\rho\|_t^2\, dt + \int_0^{\mathcal{T}}\int_0^1 \rho(\rho q)_x\, dx\, dt = \int_0^{\mathcal{T}}\int_0^1 \rho(\frac{\rho+1}{\rho}\rho_x)_x\, dx\, dt$$

Integrating by parts yields,

$$\int_0^{\mathcal{T}} \frac{1}{2}\|\rho\|_t^2\, dt - \int_0^{\mathcal{T}}\int_0^1 \rho_x\rho q\, dx\, dt + \int_0^{\mathcal{T}} \rho^2 q|_0^1 dt = -\int_0^{\mathcal{T}}\int_0^1 \frac{\rho+1}{\rho}\rho_x^2\, dx\, dt + \int_0^{\mathcal{T}} \rho(\frac{\rho+1}{\rho}\rho_x)dt|_0^1.$$

We apply the boundary conditions and estimate the convective term

$$\int_0^{\mathcal{T}} \frac{1}{2}\|\rho\|_t^2\, dt \le \int_0^{\mathcal{T}} (\eta\|\rho_x\|^2 + \frac{1}{\eta}\|\rho q\|^2)\, dt - \int_0^{\mathcal{T}} \|\rho_x\|^2\, dt - \int_0^{\mathcal{T}}\int_0^1 \frac{\rho_x^2}{\rho}\, dx\, dt$$

By choosing $0 < \eta < 1$ and using that $\rho q \in C([0,\mathcal{T}];L^2(\Omega))$ from Proposition 3.3, the estimate follows. □

Proposition 3.6. *Let the initial datum u^0 satisfy Assumption 3.1. Then solutions of (20), with boundary conditions (21) satisfy*

$$\rho q \in C([0,\mathcal{T}];L^2(\Omega)) \cap L^2([0,\mathcal{T}];H^1(\Omega)).$$

Proof. The proof is completely analogous to that of Proposition 3.5. The energy method is applied to the momentum equation.

$$\int_0^{\mathcal{T}} \frac{1}{2}\|\rho q\|_t^2\,dt + \int_0^{\mathcal{T}}\int_0^1 \rho q(\rho q^2+p)_x\,dx\,dt = \int_0^{\mathcal{T}}\int_0^1 \rho q(\frac{\rho+1}{\rho}(\rho q)_x)_x\,dx\,dt$$

The partial integrations are the same from here. In particular, the second integral above is integrated by parts, moved to the right-hand side and estimated in the same way. The bound follows since $(\rho q^2+p)(\cdot,t) \in L^2(\Omega)$ which follows from Proposition 3.3. □

3.1.3. *Positivity.* We return to positivity of the thermodynamic variables. We have already established that $\rho > 0$ a.e. in Proposition 3.3 but we need a stronger positivity result on the density.

Proposition 3.7. *Let the initial datum u^0 satisfy Assumption 3.1. Then the solution of (20) satisfies $\frac{1}{\rho} = L \in C([0,\mathcal{T}];L^1(\Omega)) \cap L^2(0,\mathcal{T};H^1(\Omega))$ and the viscosity coefficient $\nu \in C([0,\mathcal{T}];L^1(\Omega)) \cap L^2(0,\mathcal{T};H^1(\Omega))$.*

Proof. Consider the continuity equation,

$$\rho_t + (\rho q)_x = \left(\frac{\rho+1}{\rho}\rho_x\right)_x \tag{26}$$

Introduce,

$$L = \frac{1}{\rho}, \quad L_x = \frac{-1}{\rho^2}\rho_x \quad L_t = \frac{-1}{\rho^2}\rho_t$$

Multiply (26) by $-1/\rho^2$.

$$\begin{aligned}
-\frac{1}{\rho^2}\rho_t - \frac{1}{\rho^2}(\rho_x q + \rho q_x) &= -\frac{1}{\rho^2}\left(\frac{\rho+1}{\rho}\rho_x\right)_x \\
L_t + L_x q - L q_x &= \left(-\frac{1}{\rho^2}\frac{\rho+1}{\rho}\rho_x\right)_x + \left(\frac{1}{\rho^2}\right)_x\left(\frac{\rho+1}{\rho}\right)\rho_x
\end{aligned}$$

Introduce $A(x,t) = \left(-\frac{1}{\rho^2}\frac{\rho+1}{\rho}\rho_x\right)$ and note that $A(0,t) = A(1,t) = 0$ thanks to the boundary conditions (21). We obtain,

$$L_t + L_x q - L q_x = A_x - \left(\frac{2\rho_x^2}{\rho^3}\right)\left(\frac{\rho+1}{\rho}\right)$$

Integrate in space and note that $L \geq 0$ thanks to $\rho \geq 0$. Hence, the integral $\int_0^1 L\,dx = \|L\|_1$ (the L^1-norm) such that

$$\begin{aligned}
(\|L\|_1)_t + \int_0^1 L_x q - L q_x\,dx &= \int_0^1 A_x\,dx - \int_0^1 2L_x^2(\rho+1)\,dx \\
(\|L\|_1)_t + \int_0^1 ((Lq)_x - 2L_x q)\,dx &= -\int_0^1 2L_x^2(\rho+1)\,dx \\
(\|L\|_1)_t &= 2\int_0^1 L_x q\,dx - \int_0^1 2L_x^2(\rho+1)\,dx
\end{aligned}$$

$$(\|L\|_1)_t \leq \|L_x\|_2^2 + \|q\|_2^2 - 2\|L_x^2\|_2 - \int_0^1 2\rho L_x^2\, dx$$
$$(\|L\|_1)_t \leq -\|L_x\|_2^2 + \|q\|_2^2 - \int_0^1 2\rho L_x^2\, dx$$

By positivity of ρ, the bound on q (Prop. 3.4), we obtain $L \in C([0,\mathcal{T}]; L^1(\Omega))$. In addition, we also obtain $L_x \in L^2(0,\mathcal{T}; L^2(\Omega))$. By the Poincare inequality, we get $L \in L^2(0,\mathcal{T}; H^1(\Omega))$. (Once again, we have established that $\rho > 0$ a.e.)

Since $\nu = 1 + \frac{1}{\rho} = 1 + L$, ν is bounded in the same space.

□

We can also infer positivity on the other two thermodynamic variables.

Proposition 3.8. *Solutions of (20) satisfy $T \in L^1(0,\mathcal{T}; L^1(\Omega))$ and $p, T > 0$ a.e.*

Proof. $\|T\|_1 = \|\frac{p}{R\rho}\|_1 \leq \frac{1}{R}\|p\|_2\|L\|_2 \leq \frac{1}{R}(\|p\|_2^2 + \|L\|_2^2)$. Since, $p, L \in L^2(0,\mathcal{T}; L^2(\Omega))$ we obtain the desired result.

From Prop. 3.3, we have an L^2 bound on $(\log T)_x$. The L^1 bound on the temperature implies that there is a non-zero subset on which $\log T$ is bounded in L^1. By Poincare we have, $\log T \in L^2(0,\mathcal{T}; H^1(\Omega))$ and hence $T > 0$ a.e. The gas law gives $p > 0$. □

Remark Positivity of p could have been obtained from the minimum entropy principle [Tad86], which holds since the equations (20) satisfy an entropy inequality for any entropy. Then $\exp(S_{min})\rho^\gamma = p > 0$. However, the argument above holds also for the next model we consider below.

3.1.4. *Formal estimates for a weak solution.* We now return to the notion of weak solution.

Definition 3.9. *Let $\rho^0, (\rho q)^0$ and E^0 be the components of the initial datum $u^0(x)$. A locally integrable function u, is a* weak solution *of (20) if it satisfies*

(27)
$$\int_0^1 \varphi\rho^0\, dx + \int_0^{\mathcal{T}}\int_0^1 \varphi_t \rho\, dx\, dt + \int_0^{\mathcal{T}}\int_0^1 \varphi_x \rho q\, dx\, dt - \int_0^{\mathcal{T}}\int_0^1 \varphi_x \nu \rho_x\, dx\, dt = 0$$

$$\int_0^1 \varphi(\rho q)^0\, dx + \int_0^{\mathcal{T}}\int_0^1 \varphi_t(\rho q)\, dx\, dt$$
$$+ \int_0^{\mathcal{T}}\int_0^1 \varphi_x(\rho q^2 + p)\, dx\, dt - \int_0^{\mathcal{T}}\int_0^1 \varphi_x \nu(\rho q)_x\, dx\, dt = 0 \tag{28}$$

$$\int_0^1 \varphi E^0\, dx + \int_0^{\mathcal{T}}\int_0^1 \varphi_t E\, dx\, dt$$
$$+ \int_0^{\mathcal{T}}\int_0^1 \varphi_x(q(E+p))\, dx\, dt + \int_0^{\mathcal{T}}\int_0^1 (\varphi_{xx}\nu E + \phi_x \nu_x E)\, dx\, dt = 0 \tag{29}$$

for every compactly supported test function φ on $\Omega \times [0,\mathcal{T})$.

Remark The first two equations are equivalent to Definition 1.1. The diffusion term in the energy equation is partially integrated twice since we do not have a bound on the temperature gradient.

Using the a priori estimates, we will show that Definition 3.9 has meaning in the sense that the integrals are bounded. (We have yet to construct approximate solutions so this is just a formal consideration.)

First consider (27). By assumption, ρ^0 is bounded in $L^2(\Omega)$. For the others to be bounded, we need $\rho, \rho q$ and $\nu\rho_x = \frac{\rho_x}{\rho} + \rho_x$ to be at least in $L^1(0,\mathcal{T};L^1(\Omega))$, which follows from Proposition 3.3 and Proposition 3.5.

Next, we consider (28). As before, $(\rho q)^0$ is bounded in L^2. Furthermore, $\rho q \in L^2$ by Proposition 3.3. Next consider $\rho q^2 + p$. Since E, p and ρq^2 are positive, we have

$$2E = \frac{2p}{\gamma - 1} + \rho q^2 > p + \rho q^2.$$

Hence, $\|(\rho q^2+p)\|_2 \leq 2\|E\|_2$ for $t \in [0,\mathcal{T}]$ and consequently $\rho q^2+p \in L^1(0,\mathcal{T};L^1(\Omega))$ which is what we minimally require.

Finally, $\int_0^1 \|\nu(\rho q)_x\|_1\,dt \leq \int_0^{\mathcal{T}} \|\nu\|_2\|(\rho q)_x\|_2\,dt \leq \int_0^{\mathcal{T}} (\|\nu\|_2^2 + \|(\rho q)_x\|_2^2)\,dt$. Both ν and ρq are bounded in $L^2(0,\mathcal{T};H^1(0,1))$. (Proposition 3.7 and Proposition 3.6).

Finally, we consider (29). We have $E^0 \in L^2$ and $E \in C([0,\mathcal{T}];L^2(\Omega))$. The non-trivial terms are $q(E+p) = \frac{1}{2}\rho q^3 + \frac{\gamma}{\gamma-1}qp$ and the diffusion terms. The bound on qp in $L^1(0,\mathcal{T};L^1(\Omega))$ follows from Cauchy-Schwarz and $q,p \in L^2(0,\mathcal{T};L^2(\Omega))$ by Proposition 3.3 and Proposition 3.4.

Next, we consider ρq^3.

$$\int_0^{\mathcal{T}}\int_0^1 |\rho q^3|\,dx\,dt \leq \int_0^{\mathcal{T}}\int_0^1 |\rho q^2|^2 + q^2\,dx\,dt \leq \int_0^{\mathcal{T}} 4\|E\|^2 + \|q\|^2\,dt$$

Again, we use Proposition 3.3 and Proposition 3.4 to conclude that the right-hand side is bounded.

Lastly, we turn to the diffusion terms. These integrals are bounded since $E(\cdot,t) \in L^2(\Omega)$ by Proposition 3.3 and $\nu \in L^2(0,\mathcal{T};H^1(0,1))$ by Proposition 3.7, implying that $(\nu+\nu_x)E \in L^1(0,\mathcal{T};L^1(\Omega))$.

So far, we have shown that the a priori estimates at hand are sufficiently strong to give meaning to a weak solution. Next, we will construct a sequence of approximate solutions.

3.2. **Introduction to numerical schemes for conservation laws.** We discretize the domain, Ω using $N+2$ grids points $x_i = ih$, $i = 0...N+1$ and the grid spacing $h > 0$. With a unit domain size, $hN = 1$. The resulting discrete space is denoted Ω_N. At each grid point we associate a numerical solution variable, e.g. u_i at x_i. We use a similar notation for all variables, i.e., ρ_i, q_i, etc at grid point x_i. We enforce periodicity by demanding that $u_0 = u_N$ and $u_1 = u_{N+1}$.

When deriving estimates, we will use the notational convention that v_i is the value of a vector at x_i and v the entire vector with components v_i, $i = 1...N$. For instance, $v \in L^2(\Omega_N)$ is a vector bounded in the norm $\|v\|_2^2 = \sum_{i=1}^N hv_i^2$. Similarly, if v_i is itself a vector (like the solution vector u), the norm is, $\|v\|_2^2 = \sum_{j=1}^n \sum_{i=1}^N h(v_i^j)^2$ where n is the number of components. (Here, $n = 3$.)

To write schemes compactly, we will use the operators

$$D_+u_i = \frac{u_{i+1} - u_i}{h}, \quad D_-u_i = \frac{u_i - u_{i-1}}{h}, \quad (\Delta u)_{i+1/2} = u_{i+1} - u_i.$$

Furthermore, the discrete Sobolev space $H^1(\Omega_N)$, is endowed with the norm $\|v\|_{H^1} = \|v\|_2 + \|D_+v\|_2$. Here $\|D_+v\|_2^2 = \sum_{i=1}^N h(D_+v_i)^2$ and we have used the notational convention that $D_+v = (D_+v_1, D_+v_2, ...D_+v_N)^T$.(Note also that it is equivalent to use $\|D_-v\|_2$ in the definition of the $H^1(\Omega_N)$ norm.)

3.2.1. *Entropy stable schemes.* A crucial part in the previous analysis was to obtain entropy estimates. For hyperbolic conservation laws, the procedure to derive entropy inequalities and entropy estimates have been mimicked by so called *entropy stable schemes.* We will give a short description and in the subsequent sections use one such scheme to approximate the inviscid flux of the Navier-Stokes equations. For a detailed treatise on entropy stability, we refer to [Tad03].

A hyperbolic system of conservation laws is generally stated as,

$$u_t + f_x = 0. \tag{30}$$

As discussed above, an entropy solution satisfies $U(u)_t + F(u)_x \leq 0$ where (U, F) is the entropy pair. Consequently, an estimate $\int_\Omega U(u)_t\, dx \leq 0$ is obtained. (See (22) above.)

The idea of entropy stable schemes is to approximate the non-diffusive equation (30) and add a numerical diffusion, that vanishes as $h \to 0$ (i.e., $N \to \infty$) but at the same time ensure that a discrete entropy inequality is satisfied by the numerical solution. The following generic semi-discrete form is analyzed in [Tad03],

$$(u_j)_t + \frac{\tilde{f}_{j+1/2} - \tilde{f}_{j-1/2}}{h} = 0, \tag{31}$$

where $\tilde{f}_{j+1/2} = \frac{f(u_j)+f(u_{j+1})}{2} - \frac{Q_{j+1/2}}{2}(u_{j+1} - u_j)$. $Q_{i+1/2}$ is the numerical diffusion matrix. We let $w_j = w(u_j)$, denote the entropy variables. A discrete entropy flux is defined as,

$$\tilde{F}_{j+1/2} = \frac{1}{2}(w_{j+1} + w_j)^T \tilde{f}_{j+1/2} - \frac{\Psi_{j+1} + \Psi_j}{2}$$

where $\Psi = w^T f - F$ is the entropy potential. Provided that the scheme is sufficiently diffusive, i.e., the matrix $Q_{j+1/2}$ is large enough, one can derive a discrete version of a local entropy inequality.

$$(U_j)_t + \frac{F_{j+1/2} - F_{j-1/2}}{h} \leq 0. \tag{32}$$

It was shown in [Tad03] that choices of $Q_{j+1/2}$, including Lax-Friedrichs, Rusanov and entropy fixed Roe, lead to a discrete entropy inequality (32). Summing in space over the domain Ω_N yields the global estimate

$$\sum_{j=1}^{N} h(U_j)_t \leq 0. \tag{33}$$

(Recall that we use periodic boundary conditions. For physical boundary conditions and entropy stable schemes, see e.g. [SO14].) Assuming positivity, the estimate (33) of U_i leads to $u \in C([0, \mathcal{T}]; L^2(\Omega_N)^3)$ by the same argument as in Section 2. (Note that the estimate of $\|u\|_2$ is obtained even in the absence of physical diffusion.)

The local Lax-Friedrichs is obtained by choosing $Q_{j+1/2} = \lambda^{LF}_{j+1/2} I$ where I is the 3×3 identity matrix and $\lambda^{LF}_{j+1/2} = \max(|q_j| + c_j, |q_{j+1}| + c_{j+1})$ where q_j, c_j are velocity and sound speed at x_j. We will utilize the entropy stability property of the local Lax-Friedrichs scheme below but it turns out that we will need a somewhat larger artificial diffusion to obtain other estimates. To define the diffusion coefficient below, we will use the following notation:

$$\begin{aligned}
q^{LF}_{j+1/2} &= \max(|q_j|, |q_{j+1}|),\\
c^{LF}_{j+1/2} &= \max(c_j, c_{j+1}),\\
\lambda^{LF}_{j+1/2} &= q^{LF}_{j+1/2} + c^{LF}_{j+1/2},\\
\tilde{\lambda}_{j+1/2} &= \max(\lambda^{LF}_{j+1/2}, 2q^{LF}_{j+1/2}).
\end{aligned}$$

We end this section with a few remarks. Firstly, time is not discretized and we will analyze a semi-discrete scheme. For generalizations to fully discrete schemes, we refer to [Tad03] and [LMR02]. Using a high-order strong-stability preserving Runge-Kutta scheme will introduce a small amount of extra diffusion. However, this is an effort to construct weak solutions and to this effect a semi-discrete scheme will do. However, we must show that the ODE system is well-posed and we will return that below.

Secondly, it is well known that any version of Lax-Friedrichs scheme is very diffusive and only first-order accurate. For practical simulations this will be less than optimal, but once again we emphasize that provable convergence properties is the goal. We mention that there have been several efforts to derive high-order accurate entropy stable schemes, see [SM09, Svä12, FMT12, FC13]. However, entropy stability is necessary but not sufficient to prove convergence.

3.3. **The numerical scheme for the Laplacian diffusion model.** We will propose a semi-discrete numerical scheme that satisfies the corresponding discrete estimates that were derived in the previous section. For every N, we will have a system of ODEs and we will demonstrate that it is solvable up to a time $\mathcal{T}$. Hence, as the grid is refined (i.e $N \to \infty$ and $h \to 0$), we obtain a sequence of approximate solutions. Thanks to the a priori bounds, we will be able to extract a subsequence that will converge to a weak solution.

To simplify and reduce notation we will limit the analysis to the periodic case. For a numerical scheme it is a non-trivial task to impose boundary conditions while maintaining the necessary a priori bounds and we postpone that task to a future study.

We will need the following averages:

$$\begin{aligned}
\phi_{i+1/2} &= \frac{\phi_i + \phi_{i+1}}{2} \quad \text{arithmetic average} \\
\bar{\phi}_{i+1/2} &= \frac{2}{(\phi_{i+1}^{-1} + \phi_i^{-1})} \quad \text{harmonic average} \\
\bar{\phi}_{i+1/2}^{-1} &= (\phi_{i+1}^{-1} + \phi_i^{-1})/2 \quad \text{inverse harmonic average}
\end{aligned}$$

The scheme approximating (20) is given below.

$$\begin{aligned}
(\rho_i)_t + D_- \tilde{f}^1_{i+1/2} &= D_- \tilde{f}^{V,1}_{i+1/2} \\
((\rho q)_i)_t + D_- \tilde{f}^2_{i+1/2} &= D_- \tilde{f}^{V,2}_{i+1/2} \\
(E_i)_t + D_- \tilde{f}^3_{i+1/2} &= D_- \tilde{f}^{V,3}_{i+1/2}
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
\tilde{f}^1_{i+1/2} &= \frac{f^\rho_{i+1} + f^\rho_i}{2} - \frac{\lambda_{i+1/2}}{2}(u^1_{i+1} - u^1_i) \\
\tilde{f}^2_{i+1/2} &= \frac{f^m_{i+1} + f^m_i}{2} - \frac{\lambda_{i+1/2}}{2}(u^2_{i+1} - u^3_i) \\
\tilde{f}^3_{i+1/2} &= \frac{f^E_{i+1} + f^E_i}{2} - \frac{\lambda_{i+1/2}}{2}(u^3_{i+1} - u^3_i)
\end{aligned} \tag{35}$$

and

$$f^\rho_i = (\rho q)_i = m_i, \quad f^m_i = p_i + \rho_i q_i^2, \quad f^E_i = q_i(E_i + p_i).$$

The artificial diffusion is given by,

$$\lambda_{i+1/2} = (1+\epsilon)\tilde{\lambda}_{i+1/2}, \quad \epsilon > 0.$$

Remark As mentioned above, for entropy stability it is sufficient with $\lambda_{i+1/2} = \lambda^{LF}_{i+1/2}$, but in the supersonic case, we will need more diffusion. ϵ is a small positive parameter that is included to secure a bound on the artificial diffusion term.

The viscous flux is approximated by,

$$\begin{pmatrix} \tilde{f}^{V,1} \\ \tilde{f}^{V,2} \\ \tilde{f}^{V,3} \end{pmatrix}_{i+1/2} = \nu_{i+1/2} \begin{pmatrix} D_+\rho_i \\ D_+(\rho q)_i \\ D_+E_i \end{pmatrix},$$

where

$$\nu_{i+1/2} = 1 + \bar{\rho}^{-1}_{i+1/2}$$

3.3.1. *Auxiliary results.* We will need the following algebraic relations

$$\frac{a_{i+1}b_{i+1} - a_i b_i}{h} = \frac{a_{i+1} + a_i}{2}\frac{b_{i+1} - b_i}{h} + \frac{b_{i+1} + b_i}{2}\frac{a_{i+1} - a_i}{h} \tag{36}$$

and

$$\frac{1}{2}(a_{i+1}b_{i+1} + a_i b_i) = \frac{a_{i+1} + a_i}{2}\frac{b_{i+1} + b_i}{2} + \frac{a_{i+1} - a_i}{2}\frac{b_{i+1} - b_i}{2} = a_{i+1/2}b_{i+1/2} + \frac{1}{4}(\Delta a)_{i+1/2}(\Delta b)_{i+1/2}$$

For the positive quantity ρ we have:

$$\begin{aligned} \rho_{i+1/2} - \bar{\rho}_{i+1/2} = \frac{1}{2}(\rho_{i+1} + \rho_i) - \frac{2\rho_{i+1}\rho_i}{\rho_{i+1} + \rho_i} &= \\ \frac{(\rho_{i+1} + \rho_i)^2 - 4\rho_{i+1}\rho_i}{2(\rho_{i+1} + \rho_i)} = \frac{(\rho_{i+1} - \rho_i)^2}{2(\rho_{i+1} + \rho_i)} &= \\ \frac{1}{2}\frac{\rho_{i+1} - \rho_i}{\rho_{i+1} + \rho_i}(\rho_{i+1} - \rho_i) &= \\ \frac{c(\rho)}{2}(\rho_{i+1} - \rho_i) & \end{aligned} \tag{37}$$

In this case, when $\rho_i > 0$, we have $0 < c(\rho) < 1$.

We will also need a discrete version of the Poincare inequality.

Lemma 3.10. *Let Ω denote a bounded periodic domain with length L. Let Ω_N denote the discretized bounded periodic domain with N cells each of length h such that $Nh = L$. Introduce a subset $B_M \in \Omega_N$ with length $l = Mh > 0$ independent of h. Let u be a grid function. The p-norm on B_M is defined as $\|u\|_{p,B} = (\sum_{i\in B_M} u^p h)^{1/p}$.*

If $\|D_+u\|_2 \leq \mathcal{C}$ on Ω_N and $\|u\|_{p,B} \leq \mathbb{K}$, then

$$\|u\|_2 \leq \mathcal{C}_1\|u\|^2_{p,B} + \mathcal{C}_2\|D_+u\|^2_2.$$

is bounded. ($\mathcal{C}_{1,2}$ are constants.)

Proof. By $\|u\|_{p,B} \leq \mathbb{K}$, we know that even as $h \to 0$ only a set of points of vanishing measure (as number of points times h) may become unbounded. Pick a bounded point u_m at x_m in B_M. We further assume that $u_m^2 = \min_{i\in B_M}(u_i^2)$.

By periodicity, we may shift the indices and assume that the boundary point x_0 in Ω_N is the location of the minimal point, i.e., $u_0 = u_N$. That is $u_0^2 \leq \|u\|^2_{p,B}$.

Hence,

$$\|u\|_2^2 \leq \sum_{i=0}^{N-1} u_i^2 h = \sum_{i=0}^{N-1} (D_+ x_i) u_i^2 h =$$
$$-x_0 u_0^2 + x_N u_N^2 - \sum_{i=0}^{N-1} x_i 2 u_{i+1/2} D_+ u_i h \leq$$
$$L\|u\|_{p,B}^2 + 4L\|u\|_2 \|D_+ u_i\|_2 \leq$$
$$L\|u\|_{p,B}^2 + 4L(\eta\|u\|_2^2 + \frac{1}{\eta}\|D_+ u_i\|_2^2)$$

where we have used that $\|u_{i+1/2}\|^2 \leq 2\|u_i\|^2$. By choosing $0 < \eta < 1/4L$, we obtain the desired estimate. □

3.3.2. *A priori estimates.*

Assumption 3.11. *The discrete initial condition is $u^0(x_i)$, where $u^0(x)$ is the initial function associated with (20) which satisfies Assumption 3.1.*

Let $\tilde{u}^0(x)$ be the piecewise constant periodic function on $[0,1]$ where $\tilde{u}^0(x_i) = u^0(x_i)$ on $x \in (x_{i+1/2}, x_{i+1/2}]$ (with the obvious adjustments of notation at the periodic boundaries).

Proposition 3.12. *Let the discrete initial datum u^0 satisfy Assumption 3.11. Assume that $\rho_i(t), T_i(t) \geq 0$ for all $i = 1...N$ and $t = (0, \mathcal{T}]$. Then solutions of (34) satisfy $u \in C([0,\mathcal{T}]; L^2(\Omega_N)^3)$ and $p, \rho q^2 \in C([0,\mathcal{T}]; L^2(\Omega_N))$.*

The proposition is analogous to Proposition 3.3. However, we do not explicitly specify the diffusion on the entropy caused by the right-hand side as we do not need it.

Proof. We begin by deriving an entropy estimate. The scheme (34) can be written on vector form (as in (31)) as

$$(u_j)_t + \frac{\tilde{f}_{j+1/2} - \tilde{f}_{j+1/2}}{h} = \frac{\tilde{f}^V_{j+1/2} - \tilde{f}^V_{j+1/2}}{h}$$

where $\tilde{f}_{j+1/2}$ is the entropy stable flux defined in (35). Multiplying by w_j^T and summing lead to

$$\sum_{j=1}^N h(U_j)_t \leq -\sum_{j=1}^N hD_+\tilde{F}_{j-1/2} + \sum_{j=1}^N hw_j^T \frac{\tilde{f}^V_{j+1/2} - \tilde{f}^V_{j-1/2}}{h} + \frac{\epsilon}{2}\sum_{j=1}^N hw_j^T hD_-\tilde{\lambda}_{j+1/2} D_+ u_j,$$

where part of the artificial diffusion term, $\tilde{\lambda}_{j+1/2}$, is used to construct $\tilde{F}_{j+1/2}$ and the remainder, $\epsilon\tilde{\lambda}_{j+1/2}$, sits on the right-hand side. We sum by parts,

$$\sum_{j=1}^N (U_j)_t \leq -\sum_{j=1}^N h(D_+ w_j)^T \tilde{f}^V_{j+1/2} - \frac{\epsilon}{2}\sum_{j=1}^N hD_+ w_j^T h\tilde{\lambda}_{j+1/2} D_+ u_j$$

To demonstrate that the right-hand side is negative, we use the following construction (see [Tad03]). Define the straight line

$$w_{i+1/2}(\xi) = \frac{w_i + w_{i+1}}{2} + \xi(w_{i+1} - w_i), \quad -1/2 \leq \xi \leq 1/2.$$

Then

$$(u_{i+1} - u_i) = \int_{\xi=-1/2}^{1/2} \frac{d}{d\xi} u(w_{i+1/2}(\xi)) d\xi = \int_{\xi=-1/2}^{1/2} u_w(w_{i+1/2}(\xi)) d\xi (w_{i+1} - w_i)$$

and denote $(\tilde{u}_w)_{i+1/2} = \int_{\xi=-1/2}^{1/2} u_w(w_{i+1/2}(\xi))d\xi$ which is a positive semi-definite matrix under the assumption that $\rho \geq 0$. Hence,

$$-\sum_{j=1}^{N} h(D_+w_j)^T \tilde{f}^V_{j+1/2} = -\sum_{j=1}^{N} h(D_+w_j)^T(\nu_{j+1/2}D_+u_j) =$$
$$-\sum_{j=1}^{N} h(D_+w_j)^T(\nu_{j+1/2}(\tilde{u}_w)_{j+1/2}D_+w_j)$$

Using the same trick on the artificial diffusion term, we end up with

$$\sum_{j=1}^{N}(U_j)_t \leq -\sum_{j=1}^{N} h(D_+w_j)^T(\nu_{j+1/2}(\tilde{u}_w)_{j+1/2}D_+w_j) - \frac{\epsilon}{2}\sum_{j=1}^{N} h^2(D_+w_j)^T\tilde{\lambda}_{j+1/2}(\tilde{u}_w)_{j+1/2}(D_+w_j) \tag{38}$$

We obtain $\sum_i (U_j)_t \leq 0$. An L^2 bound on u_i is obtained by the same reasoning as in (10) and on. The estimates on $p, \rho q^2$ follow in the same way as for the continuous problem by observing that $p, \rho q^2$ are both positive quantities that can be bounded by $E \in L^2(\Omega_N)$. □

3.3.3. *Estimates of density and momentum.*

Proposition 3.13. *Let the initial datum u^0 satisfy Assumption 3.11. Assume that $\rho_i(t) \geq 0$ for $t \in (0, \mathcal{T}]$. Then $\rho \in C([0,T]; L^2(\Omega_N)) \cap L^2(0,\mathcal{T}; H^1(\Omega_N))$ and $\rho q \in C([0,\mathcal{T}]; L^2(\Omega_N)) \cap L^2(0,\mathcal{T}; H^1(\Omega_N)$.*

Proof. The proof is the analog of the proof of Proposition 3.5. Namely, we will derive an energy bound from the continuity equation.

$$(\rho_i)_t + \frac{D_+ + D_-}{2}(\rho q)_i = hD_-\frac{\lambda_{i+1/2}}{2}D_+\rho_i + D_-\nu_{i+1/2}D_+\rho_i$$

Multiply by $h\rho_i$ and sum.

$$\frac{1}{2}\|\rho_i\|_t + \sum_{i=1}^{N} h\rho_i\frac{(D_- + D_+)}{2}(\rho q)_i = -\sum_{i=1}^{N}\frac{\lambda_{i+1/2}}{2}(hD_+\rho_i)^2 - \sum_{i=1}^{N} h\nu_{i+1/2}(D_+\rho_i)^2$$
$$\frac{1}{2}\|\rho_i\|_t - \sum_{i=1}^{N} h\frac{(D_- + D_+)\rho_i}{2}(\rho q)_i \leq \sum_{i=1}^{N} -h\frac{1}{2}(\nu_{i+1/2}(D_+\rho_i)^2 + \nu_{i-1/2}(D_-\rho_i)^2)$$

Recall that $\nu_{i+1/2} = 1 + \bar{\rho}^{-1}_{i+1/2}$.

$$\frac{1}{2}\|\rho_i\|_t \leq \frac{1}{2}(\|(D_-\rho_i)\| + \|D_+\rho_i\|)\|\rho q\| - \frac{1}{2}(\|D_+\rho_i\|^2 + \|D_-\rho_i\|^2)$$
$$\frac{1}{2}\|\rho_i\|_t \leq \frac{1}{2}(\eta\|(D_-\rho_i)\|^2 + \eta\|D_+\rho_i\|^2 + 2\frac{1}{\eta}\|\rho q\|^2) - \frac{1}{2}(\|D_+\rho_i\|^2 + \|D_-\rho_i\|^2)$$

By proposition 3.12 we have a bound on $\|\rho q\|$. We choose $\eta < 1$ Hence, we get the desired bounds on $\int_0^T \|D_+\rho_i\|^2\,dt$ and $\int_0^T \|D_-\rho_i\|^2\,dt$. (The two bounds are equal on a periodic domain.)

The proof for ρq is analogous. The key point is to show that f^m is bounded in L^2. This is accomplished by observing that $f^m_i = p_i + \rho_i q_i^2$ is positive and that both ρq^2 and p are bounded by E independently.

□

3.3.4. *Kinetic energy.*

Proposition 3.14. *Let the initial datum u^0 satisfy Assumption 3.11. Assume that $\rho_i(t) \geq 0$ for $t \in (0, \mathcal{T}]$. Then a semi-discrete solution of (34) satisfies*

$$q \in L^2(0, \mathcal{T}; H^1(\Omega_N)).$$

Proof. To simplify notation we write $\tilde{\nu}_{i+1/2} = \nu_{i+1/2} + h\frac{\lambda_{i+1/2}}{2}$. We also use the relation $q(\rho q)_t = \frac{1}{2}(\rho q^2)_t + \rho_t \frac{q^2}{2}$. Multiply the momentum equation by q_i and let the kinetic energy be $K_i = \frac{1}{2}\rho_i q_i^2$.

$$q_i(\rho_i q_i)_t + q_i D_- \left(q_{i+1/2} f^\rho_{i+1/2} + \frac{1}{4}(\Delta q)_{i+1/2}(\Delta f^\rho)_{i+1/2} + p_{i+1/2} \right) = q_i D_- \tilde{\nu}_{i+1/2} D_+(\rho q)_i$$

Rewrite the term with time-derivative and expand the right-hand side using the discrete Leibniz's rule (36).

$$\left(\frac{1}{2}(\rho_i q_i^2)_t + (\rho_i)_t \frac{q_i^2}{2} \right) + q_i D_- \left(q_{i+1/2} f^\rho_{i+1/2} + \frac{1}{4}(\Delta q)_{i+1/2}(\Delta f^\rho)_{i+1/2} + p_{i+1/2} \right) = q_i D_- \tilde{\nu}_{i+1/2} \left(\rho_{i+1/2} D_+ q_i + q_{i+1/2} D_+ \rho_i \right)$$

Use the continuity equation of (34).

$$\left(\frac{1}{2}(\rho_i q_i^2)_t + \left(-D_- f^\rho_{i+1/2} + D_- \tilde{\nu}_{i+1/2} D_+ \rho_i \right) \frac{q_i^2}{2} \right) + q_i D_- \left(q_{i+1/2} f^\rho_{i+1/2} + \frac{1}{4} h D_+ q_i h (D_+ f^\rho_i) + p_{i+1/2} \right) = q_i D_- \tilde{\nu}_{i+1/2} \left(\rho_{i+1/2} D_+ q_i + q_{i+1/2} D_+ \rho_i \right)$$

Multiply by h and sum over $i = 1...N$ to obtain the time evolution of the kinetic energy.

$$\sum_i h(K_i)_t + \sum_i h \left(-D_- f^\rho_{i+1/2} + D_-(\tilde{\nu}_{i+1/2}) D_+ \rho_i \right) \frac{q_i^2}{2} + \sum_i h q_i D_-(q_{i+1/2} f^\rho_{i+1/2}) + \sum_i h q_i D_-(\frac{1}{4} h D_+ q_i h (D_+ f^\rho_i) + p_{i+1/2}) = \sum_i h q_i D_- \tilde{\nu}_{i+1/2} \left(\rho_{i+1/2} D_+ q_i + q_{i+1/2} D_+ \rho_i \right)$$

and use a standard summation-by-parts rule.

$$\sum_i h(K_i)_t - \sum_i h \left(-f^\rho_{i+1/2} + (\tilde{\nu}_{i+1/2}) D_+ \rho_i \right) D_+ \frac{q_i^2}{2} - \sum_i h(D_+ q_i) q_{i+1/2} f^\rho_{i+1/2} - \sum_i h \frac{1}{4}(D_+ q_i) h (D_+ q_i) h (D_+ f^\rho_i) - \sum_i h (D_+ q_i) p_{i+1/2} = - \sum_i h (D_+ q_i) \tilde{\nu}_{i+1/2} \left(\rho_{i+1/2} D_+ q_i + q_{i+1/2} D_+ \rho_i \right)$$

Estimate the pressure term.

$$\sum_i h(K_i)_t - \sum_i h\left(-f^\rho_{i+1/2} + \tilde{\nu}_{i+1/2} D_+\rho_i\right) q_{i+1/2} D_+ q_i$$
$$- \sum_i (D_+q_i) q_{i+1/2} f^\rho_{i+1/2} - \sum_i h\frac{1}{4}(D_+q_i)^2 h^2 (D_+ f^\rho_i) \leq$$
$$\sum_i h\left(\eta (D_+q_i)^2 + \frac{1}{\eta} p^2_{i+1/2}\right) - \sum_i h(D_+q_i)\tilde{\nu}_{i+1/2}\left(\rho_{i+1/2} D_+ q_i + q_{i+1/2} D_+\rho_i\right)$$

Several terms cancel and we are left with,

$$\sum_i h(K_i)_t \leq -\sum_i h(\tilde{\nu}_{i+1/2}\rho_{i+1/2} - \eta)(D_+q_i)^2 + \frac{2}{\eta}\|p\|_2^2 + \sum_i h\frac{1}{4}(D_+q_i)^2 h^2 (D_+ f^\rho_i) \tag{39}$$

Recall that $\tilde{\nu}_{i+1/2} = \nu_{i+1/2} + h\lambda_{i+1/2}/2$. Furthermore, $\nu_{i+1/2}\rho_{i+1/2}$ is bounded from below by $1/2$ since

$$\nu_{i+1/2}\rho_{i+1/2} =$$
$$\rho_{i+1/2} + (\rho_{i+1/2}\bar{\rho}^{-1}_{i+1/2}) =$$
$$\rho_{i+1/2} + \frac{1}{4}\left(\frac{\rho_{i+1}}{\rho_i} + \frac{\rho_i}{\rho_{i+1}}\right) + \frac{1}{2}.$$

Hence, we choose $0 < \eta < 1/2$.

The last term of (39) contains a factor

$$\frac{h^2}{4}(D_+ f^\rho_i) = \sum_i \frac{h^2}{4}((D_+q_i)\rho_{i+1/2} + q_{i+1/2}(D_+\rho_i))$$

where

$$\frac{1}{4}h^2 q_{i+1/2}(D_+\rho_i) \leq \frac{h}{4} q^{LF}_{i+1/2}|(\rho_{i+1} - \rho_i)| \leq \frac{h}{4} q^{LF}_{i+1/2}(\rho_{i+1} + \rho_i) = \frac{h}{2} q^{LF}_{i+1/2}\rho_{i+1/2}$$

Similarly,

$$\frac{h^2}{4}\rho_{i+1/2}(D_+q_i) \leq \frac{h}{4}\rho_{i+1/2}(|q|_{i+1} + |q_i|) \leq \frac{h}{2}\rho_{i+1/2} q^{LF}_{i+1/2}$$

Using these relations in (39) yields

$$\sum_i h(K_i)_t \leq -\sum_i h(\frac{1}{2} - \eta)(D_+q_i)^2 + \frac{2}{\eta}\|p\|_2^2$$
$$+ \sum_i h(D_+q_i)^2 (h\rho_{i+1/2} q^{LF}_{i+1/2}) - \sum_i h\rho_{i+1/2}\frac{h\lambda_{i+1/2}}{2}(D_+q_i)^2$$

Since $\lambda_{i+1/2} \geq 2q^{LF}_{i+1/2}$, the right hand side is negative and we specifically obtain a bound on the velocity gradient.

$$\int_0^{\mathcal{T}} \sum_i (D_+q_i)^2 dt \leq \mathcal{C}.$$

By the discrete Poincare inequality (Lemma 3.10) we also obtain that $q \in L^2(0, \mathcal{T}; L^2(\Omega_N))$. $\square$

3.3.5. *Positivity.* Positivity is a much researched topic with respect to numerical schemes. The standard notion of positivity commonly associated with numerical schemes is that the time step for marching a PDE forward in time should never become vanishingly small in order to ensure positivity (at least not as long as the signal speeds remain bounded).

However, we will prove positivity in another sense. Namely, a global a priori statement that ρ, T, p remain positive in any finite time interval.

We introduce $L_i = \frac{1}{\rho_i}$ and prove the following proposition.

Proposition 3.15. *Let the initial datum u^0 satisfy Assumption 3.11. Assume that $\rho_i(t) \geq 0$ for $i = 1...N$ and $t \in (0, \mathcal{T}]$. Then, $L, \nu \in L^2(0, \mathcal{T}; H^1(\Omega_N))$ and $\rho_i > 0$ in $\Omega_N \times (0, \mathcal{T}]$.*

Proof. The scheme for the density is,

$$(\rho_i)_t + \frac{f^\rho_{i+1/2} - f^\rho_{i-1/2}}{h} = D_-(\nu_{i+1/2} + h\frac{\lambda_{i+1/2}}{2})D_+\rho_i$$

Multiply by $-h/\rho_i^2$ and sum.

(40)

$$\sum_i \left(-\frac{h}{\rho_i^2}(\rho_i)_t - \frac{h}{\rho_i^2}\frac{f^\rho_{i+1/2} - f^\rho_{i-1/2}}{h} \right) = -\sum_i \frac{h}{\rho_i^2} D_-(\nu_{i+1/2} + \frac{h\lambda_{i+1/2}}{2})D_+\rho_i$$

Note that

$$\frac{-1}{\rho_i^2}(\rho_i)_t = \left(\frac{1}{\rho_i}\right)_t = (L_i)_t.$$

Split $f^\rho_{i+1/2} = \rho_{i+1/2}q_{i+1/2} + \frac{1}{4}(\Delta\rho)_{i+1/2}(\Delta q)_{i+1/2}$ and sum (40) by parts,

$$\sum_i \left((L_i)_t - D_+\frac{-1}{\rho_i^2}(\rho_{i+1/2}q_{i+1/2} + \frac{1}{4}(\Delta\rho)_{i+1/2}(\Delta q)_{i+1/2}) \right) =$$

$$\sum_i hD_+(\frac{1}{\rho_i^2})(\nu_{i+1/2} + \frac{h\lambda_{i+1/2}}{2})D_+\rho_i \tag{41}$$

To further manipulate (41) we introduce

$$(L_x)_{i+1/2} = D_+(L_i) = D_+(\frac{1}{\rho_i}) = -\frac{(D_+\rho_i)}{\rho_{i+1}\rho_i}$$

and

$$(D_+\frac{-1}{\rho_i^2}) = \frac{\rho_{i+1} + \rho_i}{\rho_{i+1}^2\rho_i^2}D_+\rho_i = -2\bar{\rho}^{-1}_{i+1/2}D_+(\frac{1}{\rho_i})$$

We also use (37): $\rho_{i+1/2} = \bar{\rho}_{i+1/2} + \frac{c(\rho)}{2}\Delta\rho_{i+1/2}$.

Then (41) becomes

$$\sum_i h\left((L_i)_t + 2\bar{\rho}^{-1}_{i+1/2}D_+(\frac{1}{\rho_i})(\bar{\rho}_{i+1/2}q_{i+1/2} + \frac{1}{4}(\Delta\rho)_{i+1/2}(\Delta q)_{i+1/2} + \frac{c(\rho)}{2}\Delta\rho_{i+1/2}q_{i+1/2}) \right) =$$

$$\sum_i h2\bar{\rho}^{-1}_{i+1/2}D_+(\frac{1}{\rho_i})\left(\nu_{i+1/2} + \frac{h\lambda_{i+1/2}}{2}\right)D_+\rho_i$$

or

(42)

$$\sum_i h\left((L_i)_t + 2(L_x)_{i+1/2}q_{i+1/2} - \frac{\rho_{i+1} + \rho_i}{\rho_{i+1}^2\rho_i^2}D_+\rho_i(\frac{1}{4}(\Delta\rho)_{i+1/2}(\Delta q)_{i+1/2} + \frac{c(\rho)}{2}\Delta\rho_{i+1/2}q_{i+1/2}) \right) =$$

$$\sum_i h2\bar{\rho}^{-1}_{i+1/2}\nu_{i+1/2}D_+(\frac{1}{\rho_i})D_+\rho_i - \sum_i h\frac{\rho_{i+1} + \rho_i}{\rho_{i+1}^2\rho_i^2}D_+(\rho_i)(\frac{h\lambda_{i+1/2}}{2})D_+\rho_i$$

We manipulate the last term on the left-hand side and the last term on the right-hand side.

$$\begin{aligned}(D_+\rho_i)(\frac{1}{4}\Delta\rho_{i+1/2}\Delta q_{i+1/2} + \frac{c(\rho)}{2}\Delta\rho_{i+1/2}q_{i+1/2} - \frac{h\lambda_{i+1/2}}{2}D_+\rho_i) &= \\ (D_+\rho_i)^2(\frac{h}{4}\Delta q_{i+1/2} + \frac{h}{2}c(\rho)q_{i+1/2} - \frac{h\lambda_{i+1/2}}{2}) &\leq \\ (D_+\rho_i)^2(\frac{h}{2}q^{LF}_{i+1/2} + \frac{h}{2}q^{LF}_{i+1/2} - \frac{h\lambda_{i+1/2}}{2}) &= \\ (D_+\rho_i)^2(hq^{LF}_{i+1/2} - \frac{h\lambda_{i+1/2}}{2}) &\leq 0\end{aligned}$$

Hence, these terms are negative (when sitting on the right-hand side).

We are left with,

$$\begin{aligned}\sum_i \big((L_i)_t + 2(L_x)_{i+1/2}q_{i+1/2}\big) &\leq \\ \sum_i 2h\bar{\rho}^{-1}_{i+1/2}\nu_{i+1/2}D_+(\frac{1}{\rho_i})(-\rho_i\rho_{i+1}D_+(\frac{1}{\rho_i})) &= \\ -\sum_i 2h\bar{\rho}^{-1}_{i+1/2}\nu_{i+1/2}\rho_i\rho_{i+1}(L_x)^2_{i+1/2} &= \\ -\sum_i 2h\rho_{i+1/2}\nu_{i+1/2}(L_x)^2_{i+1/2}\end{aligned}$$

By Cauchy-Schwarz,

$$\sum_i (L_i)_t \leq \sum_i 2h\left(\eta(L_x)^2_{i+1/2} + \frac{1}{\eta}q^2_{i+1/2}\right) - \sum_i 2h\rho_{i+1/2}\nu_{i+1/2}(L_x)^2_{i+1/2} \tag{43}$$

Since, $\|q\|_2$ is bounded, it remains to show that

$$\sum_i 2h\eta(L_x)^2_{i+1/2} - \sum_i 2h\rho_{i+1/2}(\nu_{i+1/2})(L_x)^2_{i+1/2} < Constant$$

The estimate follows since

$$\rho_{i+1/2}\nu_{i+1/2} = \frac{\rho_{i+1}+\rho_i}{2}(1+\bar{\rho}^{-1}_{i+1/2}) = \frac{\rho_{i+1}+\rho_i}{2} + \frac{1}{2} + \frac{1}{4}\left(\frac{\rho_{i+1}}{\rho_i} + \frac{\rho_i}{\rho_{i+1}}\right)$$

is bounded from below by $1/2$.

Choose $0 < \eta < 1/2$. We obtain from (43) the estimate

$$\sum_i (L_i)_t \leq \frac{2}{\eta}\|q\|^2 - \sum_i 2h(\frac{1}{2} - \eta)(L_x)^2_{i+1/2}$$

We conclude that $L \in C([0,\mathcal{T}];L^1(\Omega_N))$ and $D_+(L) \in L^2(0,\mathcal{T};L^2(\Omega_N))$. By the Poincare inequality we get, $L \in L^2(0,\mathcal{T};H^1(\Omega_N))$. $\nu_{i+1/2} = 1 + L_i$ is of course bounded in the same space as L_i on a bounded domain.

□

Lemma 3.16. *Solutions of (34) satisfy $T \in L^1(0,\mathcal{T};L^1(\Omega_N))$, and $p_i, T_i > 0$.*

Proof. The L^1 estimate is obtained by the gas law and Cauchy-Schwarz. The positivity statement will be proven differently than for the equations themselves. Here, we utilize that the Laplacian diffuses any entropy. Hence, the minimum principle of the entropy, see [Tad86] is applicable.

That is $S_i = \log(p_i/\rho_i^\gamma)$ has a minimum. Consequently, $p_i = \exp(S_i)\rho_i^\gamma > 0$ (since $\rho_i > 0$). The gas law ensures that $T_i > 0$. □

Remark In [Tad86], the domain is not bounded and the result hinges on a maximum convective velocity in an integral limit in Lemma 3.1. In our case, the domain is bounded and periodic, and the assumption redundant.

Lemma 3.17. *With the same assumptions as in Prop. 3.12,* $\sqrt{h\lambda}D_+u \in L^2(0,\mathcal{T};L^2(\Omega_N)^3)$.

Proof. Define

$$u_{i+1/2}(\xi) = \frac{u_i+u_{i+1}}{2} + \xi(u_{i+1}-u_i), \quad -1/2 \le \xi \le 1/2.$$

Then

$$(w_{i+1}-w_i) = \int_{\xi=-1/2}^{1/2} \frac{d}{d\,\xi} w(u_{i+1/2}(\xi))d\xi = \int_{\xi=-1/2}^{1/2} w_u(u_{i+1/2}(\xi))d\xi(u_{i+1}-u_i)$$

Noting that $U_u = w^T$, we introduce $(\tilde{U}_{uu})_{i+1/2} = \int_{\xi=-1/2}^{1/2} w_u(w_{i+1/2}(\xi))d\xi$. U_{uu} is positive definite (see [Har83]), if $\rho > 0$, which is the case by Prop. 3.15. Hence, there exists a $\kappa > 0$ such that $\lambda_{min}(\tilde{U}_{uu}) \ge \kappa$ for all $t \in [0,T]$ and all $i = 1...N$. (λ_{min} denotes the minimal eigenvalue.)

From (38), we have

$$\mathcal{C} \ge \int_0^{\mathcal{T}} \sum_{j=1}^N h^2 D_+ w_j^T \tilde{\lambda}_{i+1/2} D_- u_j \ge \kappa \int_0^T \sum_j h^2 \tilde{\lambda}_{j+1/2}(D_+u_j)^T(D_+u_j).$$

Hence, $\sqrt{h\tilde{\lambda}}D_+u \in L^2(0,\mathcal{T};L^2(\Omega_N)^3)$. Since $\lambda_{j+1/2} = (1+\epsilon)\tilde{\lambda}_{j+1/2}$, we get the desired estimate.

□

3.3.6. *Convergence to a weak solution.* So far, we have shown that the numerical scheme satisfies the analogous a priori estimates but strictly speaking we do not yet know if there are solutions to the numerical scheme.

The scheme can be compactly written as,

$$(u_i)_t = -D_+\tilde{f}_{i+1/2} + D_+\nu_{i+1/2}D_i u_i, \quad i = 1...N \tag{44}$$

where $u_i = (\rho_i, (\rho q)_i, E_i)^T$ and $\tilde{f}_{i+1/2}$ contains the three components of the flux. This is a system of $3 \times N$ ODEs. We write it more compactly as $\bar{u}_t = G(\bar{u})$ where $\bar{u}$ is a vector of all unknowns and G the vector of the right-hand side. The system (44) has a unique solution, up to $t = \mathcal{T}$, if:

(1) $G(\bar{u})$ is continuous.
(2) $|G(\bar{u})|$ is bounded.
(3) G is Lipschitz continuous.

(See [HNG93].) In our case, G is clearly continuous. For the other two conditions we need to use our a priori estimates. We know that $\bar{u}(t) \in L^2(\Omega)$ for all $t \in [0,\mathcal{T}]$.

We shall consider this L^2 estimate on a fixed grid, i.e., for a fixed N. Then $\max_{1\le i\le N}(|u_i|) < \frac{\mathcal{C}}{\sqrt{h}}$. We shall denote this as an l_N^∞ bound. Hence, $|G(\bar{u})|$ is bounded in l_N^∞. Similarly, from the a priori estimate of L, we deduce that $L_i \le \mathcal{C}/\sqrt{h}$ and hence $\rho_i \ge \sqrt{h}/\mathcal{C} > 0$.

Using that ρ_i is positive and bounded away from 0 and that $\bar{u} \in l_N^\infty$, it is easy to show that $\tilde{f}_{i+1/2}$ is Lipschitz continuous. Hence, $D_+\tilde{f}$ is Lipschitz continuous. (On a fixed grid.) The same holds for $D_+\nu D_-\bar{u}$. We conclude that for a fixed N, the ODE system has a unique solution up to $t = \mathcal{T}$. Hence, we can use the numerical scheme to generate a sequence of solutions $\bar{u}^N(t)$, $0 \le t \le \mathcal{T}$ that satisfies the a priori estimates.

Remark Note that we do not have uniform L^∞ bounds. This means that the limiting solution need not be in L^∞. Nevertheless, for any N the ODE system is solvable.

We will now show that we can extract a subsequence that converges to a weak solution. First, we note that under Assumption 3.11, the initial datum converges strongly in $L^2(\Omega)$. That is $\|\tilde{u}^0(x) - u^0(x)\|_2 \to 0$ as $h \to 0$ and consequently,

$$\sum_{j=1}^{N} h\varphi_j (u^0(x_j))^i = \int_0^1 \varphi(x)(\tilde{u}^0(x))^i dx \to \int_0^1 \varphi(x)(u^0(x))^i \, dx, \quad i = 1, 2, 3$$

We now state the first of two main results of this article.

Theorem 3.18. *Let the initial datum of the discrete scheme (34) satisfy Assumption 3.11. Then a subsequence of the solutions generated by (34) on ever finer grids ($h \to 0$), will converge, to a weak solution of (20) in the sense of Definition 3.9.*

Proof. The integrals of Def. 3.9 will be approximated by sums. We need to show that these sums are bounded independently of h. The we can pick a convergent subsequence that satisfies Def. 3.9.

Let φ denote the projection of a smooth periodic test function onto the grid.

The first equation of the scheme is:

$$(\rho_i)_t + D_- \tilde{f}^1_{i+1/2} = D_-(\nu_{i+1/2} D_+ \rho_i)$$

We multiply by $h\varphi_i$, sum and integrate in time.

$$\sum_i h\varphi_i \rho_i(0) + \int_0^{\mathcal{T}} \sum_i h(\varphi_i)_t(\rho_i) dt + \int_0^{\mathcal{T}} \sum_i h(D_+\varphi_i)\tilde{f}^1_{i+1/2} \, dt = \int_0^{\mathcal{T}} \sum_i hD_+\varphi_i(\nu_{i+1/2} D_+ \rho_i) \, dt$$

Clearly, the first two integrals/sums converge as $h \to 0$ since $\rho_i \in L^2$. We also need, at least an $L^1(0, \mathcal{T}; L^1(\Omega_N))$ bound on $\tilde{f}^1 = \frac{(\rho q)_{i+1} + (\rho q)_i}{2} + \frac{\lambda_{i+1/2}}{2}(\rho_{i+1} - \rho_i)$. We have an $L^2(\Omega_N)$-bound on ρq taking care of the first part of $\tilde{f}^1$.

The artificial diffusion term takes the weak form:

$$\int_0^T \sum_i h(D_+\varphi_i)(\lambda_{i+1/2} h D_+ \rho_i) =$$
$$\int_0^T \sum_i h(D_+\varphi_i) h^{1/2} (\sqrt{\lambda_{i+1/2}} \sqrt{\lambda_{i+1/2}} h^{1/2} D_+ \rho_i) \leq$$
$$\max_{i,t}(D_+\varphi_i(t)) \int_0^T \sqrt{h}(\|\sqrt{\lambda_{i+1/2}}\|^2 + \|\sqrt{h\lambda_{i+1/2}} D_+\rho_i\|^2) \leq$$

$\|\sqrt{\lambda}\|$ is bounded since it scales as $\sqrt{q}$ and $\sqrt{c}$, the latter which in turn is proportional to $T^{1/4}$. $T_i^{1/4}$ is bounded thanks to $T \in L^1(0, \mathcal{T}; L^1(\Omega_N))$. By Lemma 3.17, we have the necessary bound on $\sqrt{\lambda_{i+1/2} h} D_+ \rho_i$. Finally, we note that the terms vanish as $h \to 0$, as they should since they are not part of the PDE.

Next, we consider the right-hand side. We must bound:

$$\int_0^T \sum_i h|(\nu_{i+1/2} D_+ \rho_i)| dt$$

We have $D_+\rho_i \in L^2(0, \mathcal{T}; L^2(\Omega_N))$ and by Proposition 3.15 $\nu_{i+1/2} \in L^2(0, \mathcal{T}; H^1(\Omega_N))$ which bound the diffusive term.

The second/momentum equation is bounded in the same way. f_i^m is bounded in $C([0,\mathcal{T}];L^2(\Omega_N))$. The artificial and diffusive terms follow the same pattern as for the continuity equation.

The third/energy equation is somewhat different. Multiplying by φ, integrating in time and summing in space yield,

$$\sum_i h\varphi_i E_i(0) + \int_0^{\mathcal{T}} \sum_i h(\varphi_i)_t(E_i)dt + \int_0^{\mathcal{T}} \sum_i h(D_+\varphi_i)\tilde{f}^3_{i+1/2}\,dt = \\ -\int_0^{\mathcal{T}} \sum_i D_-((D_+\varphi_i)\nu_{i+1/2})E_i\,dt$$

The convective term $f^E_{i+1/2} = (q(E+p))_{i+1/2}$ is bounded in $L^1(0,\mathcal{T};L^1(\Omega_N))$ thanks to the $L^2(0,\mathcal{T};L^2(\Omega_N))$ bounds on q,E,p. (The artificial diffusion is bounded as described above.) The diffusive term on the right-hand side is bounded thanks to $\nu \in L^2(0,\mathcal{T};H^1(\Omega_N))$ and $E_i \in L^2(0,\mathcal{T};L^2(\Omega_N))$.

□

4. Navier-Stokes-Brenner equations

The previous model was easier to address than the general Navier-Stokes-Brenner system, (16), because of the simpler form of the diffusion terms. However, the key to the estimates was diffusion in the continuity equation that led to a sufficiently strong positivity result. This property is shared with Brenner's system.

For the Navier-Stokes-Brenner equations, we could adopt the same diffusion coefficient on the mass diffusion while keeping the standard Navier-Stokes terms. However, we will make one modification. Instead of choosing $\nu = \delta = 1 + 1/\rho$, we only consider $\delta = 1/\rho$. This results in weaker estimates on the density but, as will be evident below, we are still able to get sufficiently strong bounds to admit a weak solution. We simply want to show that a range of models work well mathematically.

We will consider the following system.

$$u_t + f(u)_x = (f^{NS})_x + (f^{mod})_x \tag{45}$$

$$u = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix}, f(u) = \begin{pmatrix} m \\ \rho q^2 + p \\ (E+p)q \end{pmatrix}, f^{NS} = \begin{pmatrix} 0 \\ \frac{4}{3}\mu q_x \\ \frac{4}{3}\mu q q_x + kT_x \end{pmatrix}, f^{mod} = \begin{pmatrix} \delta\rho_x \\ \delta q\rho_x \\ \delta\beta\rho_x \end{pmatrix}$$

$$p = (\gamma-1)(E - \frac{1}{2}\rho q^2).$$

Here, we take the diffusion coefficients to be

$$\delta = \frac{\delta_0}{\rho}, \quad \mu = \mu_0, \quad k = k_0$$

where δ_0, μ_0, k_0 are physical constants. With our choice of δ, we can write,

$$f^{mod} = \delta_0 \begin{pmatrix} 1 \\ q \\ \beta \end{pmatrix} (\log\rho)_x$$

Assumption 4.1. *The initial datum $u^0(x) = u(x,0)$ is assumed to reside in the following spaces:*

$$U(u^0) \in L^1(\Omega), \quad (u^0) \in L^2(\Omega)^3, \quad \rho^{-1}(\cdot,0) \in L^1(\Omega)$$
$$\rho(x,0) > 0, \quad p(x,0) > 0, \quad T(x,0) = (p/(R\rho))(x,0) > 0$$

We use the boundary conditions (21) and consider the bounded domain $\Omega = (0,1)$.

4.1. **A priori estimates.**

Proposition 4.2. *Let u^0 satisfy Assumption 4.1. Assume that $\rho(x,t), T(x,t) \geq 0$ on $\Omega \times (0, \mathcal{T}]$. Then $u = (\rho, \rho q, E) \in C([0,\mathcal{T}]; L^2(\Omega)^3)$ and*

$$\int_0^{\mathcal{T}} \int_0^1 \delta_0(\gamma-1)\frac{\rho_x^2}{\rho^2} + \frac{4\mu_0 q_x^2}{3c_v T} + \frac{k_0 T_x^2}{c_v T^2} dx\, dt < Constant. \tag{46}$$

Furthermore, $\rho \in L^2(0,\mathcal{T}; H^1(\Omega))$ and $1/\rho = L \in L^2(0,\mathcal{T}; H^1(\Omega)) \cap C([0,\mathcal{T}]; L^1(\Omega))$. $\rho(x,t) > 0$, $p(x,t) > 0$ and $T(x,t) > 0$ almost everywhere, and $T \in L^1(0,\mathcal{T}; L^1(\Omega))$. $p, \rho q^2 \in C([0,\mathcal{T}]; L^2(\Omega))$ and $q \in L^2(0,\mathcal{T}; H^1(\Omega))$.

Proof. All these estimates follow in the same way as in the previous sections. The entropy estimate gives $u \in C([0,\mathcal{T}]; L^2(\Omega)^3)$.

The diffusive terms result in (46) and were derived in (13). It is straightforward to calculate

$$w_x^T f^{mod} = \delta(\gamma-1)\frac{\rho_x^2}{\rho}$$

and with the choice $\delta = \delta_0/\rho$, the first term of (46) is recovered. From the L^2 estimate of u and positivity we get $p, \rho q^2 \in C([0,\mathcal{T}]; L^2(\Omega))$.

Conservation of ρ gives that $\rho \geq c > 0$ in a non-zero subset $B \subset \Omega$. The L^2-estimate of ρq along with $\rho > c$ gives $q \in L^2(B)$. In this case, the estimate of kinetic energy becomes

$$\int_0^1 \frac{1}{2}(\rho q^2)_t \leq (\eta - \frac{4}{3}\mu_0)\|q_x\|^2 + \frac{1}{\eta}\|p\|^2$$

and gives a bound on $\int_0^{\mathcal{T}} \mu_0 \|q_x^2\|\, dx\, dt$ by choosing $0 < \eta < \frac{4}{3}\mu_0$. The Poincare inequality gives $q \in L^2(0,\mathcal{T}; H^1(\Omega))$.

The estimate $L = 1/\rho$ is identical. It relies on $q \in L^2(0,\mathcal{T}; L^2(\Omega))$ which we have already established. The only difference is that previously $\nu = 1 + 1/\rho$ while here we only use a diffusion proportional to $1/\rho$. However, it is easily seen in the proof of Prop. 3.7 that it is only $1/\rho$ that plays a role. Hence, $L \in L^2(0,\mathcal{T}; H^1(\Omega))$.

$T \in L^1(0,\mathcal{T}; L^1(\Omega))$ follows from the gas law, Cauchy-Schwarz and $p, L \in L^2(0,\mathcal{T}; L^2(\Omega))$ as before.

The bound on L ensures $\rho > 0$ a.e. Similarly, the estimate of $(\log T)_x$ along with the L^1 estimate on T gives $T(x,t) > 0$.

□

The estimates admits a weak solution in the following sense.

Definition 4.3. *A locally integrable function $u = (\rho, \rho q, E)^T$ is a weak solution of (45), if, for every compactly supported test function $\varphi \in C^\infty(\Omega \times [0,\mathcal{T}))$, it satisfies*

(47)
$$\int_0^1 \varphi(x,0)\rho^0\, dx + \int_0^{\mathcal{T}} \left(\int_0^1 \varphi_t \rho\, dx + \int_0^1 \varphi_x \rho q\, dx - \int_0^1 \varphi_x(\delta_0(\log\rho)_x)\, dx \right) dt = 0$$

$$\begin{aligned} &\int_0^1 \varphi(x,0)(\rho q)^0\, dx + \int_0^{\mathcal{T}} \left(\int_0^1 \varphi_t(\rho q)\, dx + \int_0^1 \varphi_x(\rho q^2 + p)\, dx \right) dt \\ &\qquad - \int_0^{\mathcal{T}} \left(\int_0^1 \varphi_x(\delta_0 q(\log\rho)_x + \frac{4}{3}\mu q_x)\, dx \right) dt = 0 \end{aligned} \tag{48}$$

$$\int_0^1 \varphi(x,0)E^0\,dx + \int_0^{\mathcal{T}} \left(\int_0^1 \varphi_t E\,dx + \int_0^1 \varphi_x (q(E+p))\,dx \right) dt$$
$$- \int_0^{\mathcal{T}} \left(\int_0^1 \varphi_x (\delta_0 \beta (\log\rho)_x + \frac{4}{3}\mu q q_x) - \varphi_{xx} kT\,dx \right) dt = 0 \tag{49}$$

where $\beta = c_v T + \frac{q^2}{2}$.

Remark The definition coincides with Definition 1.1 apart from the temperature diffusion term in the last equation. It is partially integrated twice since we lack an estimate of T_x.

All the integrals are bounded thanks to the estimates of Proposition 4.2. We briefly outline the arguments. In the time derivative terms we have the solution variables which are in $C([0,\mathcal{T}];L^2(\Omega))$. The inviscid fluxes are bounded in $L^1(0,\mathcal{T};L^1(\Omega))$ thanks to $\rho q, p, E, q \in L^2(0,\mathcal{T};L^2(\Omega))$. As before, it is the viscous terms that require a closer look.

For (47) and (48) the bounds on the viscous terms, are readily obtained by observing that q, q_x and $(\log\rho)_x$ are all in $L^2(0,\mathcal{T};L^2(\Omega))$.

Lastly, we consider the viscous terms of (49). kT is bounded thanks to $T \in L^1(0,\mathcal{T};L^1(\Omega))$ and $k = k_0$ being a constant. Furthermore, qq_x is bounded in $L^1(0,\mathcal{T};L^1(\Omega))$ since $q \in L^2(0,\mathcal{T};H^1(\Omega))$.

Next, we consider $\beta\rho_x/\rho$. We have,

$$\beta\frac{\rho_x}{\rho} \le \mathcal{C}(p+\rho q^2)\frac{\rho_x}{\rho^2} = \mathcal{C}(p+\rho q^2)L_x$$

and we obtain the L^1 bound observing that $p, \rho q^2$ and L_x are bounded in $L^2(0,\mathcal{T};L^2(\Omega))$

4.2. **Numerical scheme for Brenner's model.** We will consider the following scheme approximating (45).

$$\begin{aligned}
(u_i^1)_t + D_-\tilde{f}^1_{i+1/2} &= D_- f^{mod,1}_{i+1/2} \\
(u_i^2)_t + D_-\tilde{f}^2_{i+1/2} &= D_- f^{mod,2}_{i+1/2} + D_- f^{NS,2}_{i+1/2} \\
(u_i^3)_t + D_-\tilde{f}^3_{i+1/2} &= D_- f^{mod,3}_{i+1/2} + D_- f^{NS,3}_{i+1/2}
\end{aligned} \tag{50}$$

where

$$\begin{aligned}
\tilde{f}^j_{i+1/2} &= \frac{f^{\alpha_j}_{i+1} + f^{\alpha_j}_i}{2} - \frac{\lambda_{i+1/2}}{2}(u^j_{i+1} - u^j_i), \\
\alpha_1 &= \rho,\ \alpha_2 = m,\ \alpha_3 = E \\
(f_i^\rho, f_i^m, f_i^E)^T &= ((\rho q)_i, (\rho q^2 + p)_i, (q(E+p))_i)^T, \\
(u_i^1, u_i^2, u_i^3)^T &= (\rho_i, (\rho q)_i, E_i)^T.
\end{aligned}$$

The artificial diffusion is chosen to be the same as for the previous model.

$$\lambda_{i+1/2} = (1+\epsilon)\tilde{\lambda}_{i+1/2}, \quad \epsilon > 0.$$

The viscous fluxes are approximated by,

$$\begin{pmatrix} f^{mod,1} \\ f^{mod,2} \\ f^{mod,3} \end{pmatrix}_{i+1/2} = \delta_0 \begin{pmatrix} 1 \\ q_{i+1/2} \\ \hat{\beta}_{i+1/2} \end{pmatrix} D_+ \log\rho_i,$$
$$\begin{pmatrix} f^{NS,1} \\ f^{NS,2} \\ f^{NS,3} \end{pmatrix}_{i+1/2} = \begin{pmatrix} 0 \\ \frac{4}{3}\mu_0 D_+ q_i \\ \frac{4}{3}\mu_0 q_{i+1/2} D_+ q_i + k_0 D_+ T_i \end{pmatrix}$$

where,

$$q_{i+1/2} = \frac{q_{i+1} + q_i}{2}$$
$$\hat{\beta}_{i+1/2} = \frac{q_{i+1}q_i}{2} + c_v \frac{T_{i+1}T_i}{\tilde{T}_{i+1/2}}$$

and

$$\tilde{T}_{i+1/2} = \frac{D_+ T_i}{D_+ \log(T_i)} \quad \text{log average}$$

(See [IR09] for a numerically well conditioned way to approximate the log average.) We will also need the identity

$$\frac{\log(T_{i+1}/T_i)}{\frac{T_{i+1}-T_i}{T_{i+1}T_i}} = \frac{T_{i+1}T_i}{\tilde{T}_{i+1/2}}$$

Furthermore, we define the viscous flux $f^V_{j+1/2} = f^{mod}_{j+1/2} + f^{NS}_{j+1/2}$. Finally, we state the assumptions on the initial condition.

Assumption 4.4. *The discrete initial datum is $u^0(x_i)$, where $u^0(x)$ is the initial function associated with (45) which satisfies Assumption 4.1.*

Let $\tilde{u}^0(x)$ be the piecewise constant periodic function on $[0, 1]$ where $\tilde{u}^0(x_i) = u^0(x_i)$ on $x \in (x_{i+1/2}, x_{i+1/2}]$ (with the obvious notational adjustments at the periodic boundaries).

4.2.1. *A priori estimates.* For the continuous equation, the derivations were very similar to the previous model but for the discrete scheme more work is required.

Proposition 4.5. *Assume that the initial conditions satisfy Assumption 4.4. Assume further that $\rho_i(t) \geq 0$ for all $i = 1...N$ and $t = (0, \mathcal{T}]$. Then solutions of (50) satisfy, $u \in C([0, \mathcal{T}]; L^2(\Omega_N)^3)$, and $p, \rho q^2 \in C([0, \mathcal{T}]; L^2(\Omega_N))$. Furthermore, $\log \rho \in L^2(0, \mathcal{T}; H^1(\Omega_N))$ and $\rho_i(t) > 0$, $t \in (0, \mathcal{T}]$ and $\sqrt{h\lambda} D_+ u \in L^2(0, \mathcal{T}; L^2(\Omega_N)^3)$*

Proof. The statements will follow from the discrete entropy estimate. The scheme (50) can be written on vector form as

$$(u_j)_t + \frac{f^{LF}_{j+1/2} - f^{LF}_{j+1/2}}{h} = \frac{f^V_{j+1/2} - f^V_{j+1/2}}{h} + D_-\left(\frac{h(\lambda_{i+1/2} - \lambda^{LF}_{j+1/2})}{2} D_+ u_j\right)$$

where $f^{LF} = \frac{f_{j+1}+f_j}{2} - \frac{h\lambda^{LF}}{2} D_+ u_j$ is the entropy stable Lax-Friedrichs flux. Multiplying by hw_j^T and summing lead to

$$\sum_{j=1}^N h(U_j)_t \leq \sum_{j=1}^N hw_j^T \frac{f^V_{j+1/2} - f^V_{j-1/2}}{h} + \sum_{j=1}^N hw_j^T D_-\left(\frac{h(\lambda_{i+1/2} - \lambda^{LF}_{j+1/2})}{2} D_+ u_j\right).$$

Hence, we must show that the left-hand side is negative to obtain a bound on U_j. To this end, we sum by parts,

$$\sum_{j=1}^N (U_j)_t \leq -\sum_{j=1}^N h(D_+ w_j)^T f^V_{j+1/2} - \sum_{j=1}^N (D_+ w_j)^T \frac{h(\lambda_{i+1/2} - \lambda^{LF}_{j+1/2})}{2} D_+ u_j \tag{51}$$

We calculate the two contributions from viscosity separately: $(w_{j+1} - w_j)^T f^V_{j+1/2} = (w_{j+1} - w_j)^T (f^{mod}_{j+1/2} + f^{NS}_{j+1/2})$ and recall that

$$w^T = \left(-(S - \gamma) - \frac{q^2}{2c_v T}, \frac{q}{c_v T}, -\frac{1}{c_v T}\right).$$

We begin by rewriting the mass diffusion,

$$
\begin{aligned}
&\frac{h}{\delta_0}D_+w_if^{mod}_{i+1/2}=\frac{1}{\delta_0}(w_{i+1}-w_i)^Tf^{mod}_{i+1/2}=\\
&\Big((w^1_{i+1}-w^1_i)+q_{i+1/2}(w^2_{i+1}-w^2_i)+\hat\beta_{i+1/2}(w^3_{i+1}-w^3_i)\Big)D_+\log\rho_i\\
&\left(-D_+(S_i)+D_+(\frac{q_i^2}{c_vT_i}))+q_{i+1/2}D_+\frac{q_i}{c_vT_i}-\hat\beta_{i+1/2}D_+\frac{1}{c_vT_i}\right)D_+\log\rho_i=\\
&\left(\left(-D_+\frac{q_i^2}{2c_vT_i}+q_{i+1/2}D_+\frac{q_i}{c_vT_i}-\hat\beta^q_{i+1/2}D_+\frac{1}{T_i}\right)\right.\\
&\left.+\left(-D_+(S_i)-\hat\beta^T_{i+1/2}D_+\frac{1}{c_vT_i}\right)\right)D_+\log\rho_i=\\
&(A+B)D_+\log\rho_i
\end{aligned}
\tag{52}
$$

where we have split $\hat\beta=\hat\beta^q+\hat\beta^T$ with $\hat\beta^q_{i+1/2}=\frac{q_{i+1}q_i}{2}$ and $\hat\beta^T_{i+1/2}=c_v\frac{\log(T_{i+1}/T_i)}{\frac{T_{i+1}-T_i}{T_{i+1}T_i}}$.

Next, we consider terms of (52) that contain the velocity, i.e. A,

$$
\begin{aligned}
&A=-D_+\frac{q_i^2}{2c_vT_i}+q_{i+1/2}D_+\frac{q_i}{T_i}-\hat\beta^q_{i+1/2}D_+\frac{1}{c_vT_i}=\\
&-\frac{q^2_{i+1}}{2c_vT_{i+1}}+\frac{q_i^2}{2c_vT_i}+\frac{q_{i+1}+q_i}{2}(\frac{q_{i+1}}{c_vT_{i+1}}-\frac{q_i}{c_vT_i})-\hat\beta^q_{i+1/2}(\frac{1}{c_vT_{i+1}}-\frac{1}{c_vT_i}).
\end{aligned}
$$

We use the relation

$$
\begin{aligned}
&\frac{q_{i+1}+q_i}{2}(\frac{q_{i+1}}{c_vT_{i+1}}-\frac{q_i}{c_vT_i})=\frac{1}{2}(\frac{q^2_{i+1}+q_{i+1}q_i}{c_vT_{i+1}}-\frac{q_i^2+q_iq_{i+1}}{c_vT_i})=\\
&\frac{1}{2}(\frac{q^2_{i+1}}{c_vT_{i+1}}-\frac{q_i^2}{c_vT_i})+\frac{q_iq_{i+1}}{2}(\frac{1}{c_vT_{i+1}}-\frac{1}{c_vT_i})
\end{aligned}
$$

and $\hat\beta^q_{i+1/2}=\frac{q_{i+1}q_i}{2}$, to obtain

$$
\begin{aligned}
hA=&-\frac{q^2_{i+1}}{2c_vT_{i+1}}+\frac{q_i^2}{2c_vT_i}+\frac{1}{2}(\frac{q^2_{i+1}}{c_vT_{i+1}}-\frac{q_i^2}{c_vT_i})+\frac{q_iq_{i+1}}{2}(\frac{1}{c_vT_{i+1}}-\frac{1}{c_vT_i})\\
&-\frac{q_{i+1}q_i}{2}(\frac{1}{c_vT_{i+1}}-\frac{1}{c_vT_i})=0
\end{aligned}
$$

Next, we turn to $B=\left(-D_+(S_i)-\hat\beta^T_{i+1/2}D_+\frac{1}{c_vT_i}\right)$. We use

$$S=\log(p\rho^{-\gamma})=\log(p)-\gamma\log(\rho)=\log(\rho RT)-\gamma\log(\rho)=(1-\gamma)\log(\rho)+\log(RT)$$

such that

$$S_{i+1}-S_i=(1-\gamma)(\log(\rho_{i+1})-\log(\rho_i))+\log(T_{i+1})-\log(T_i)$$

Using the expression on $\hat\beta^T_{i+1/2}$, we find

$$
\begin{aligned}
&B=\frac{1}{h}\left(-(S_{i+1}-S_i)-\hat\beta_T(\frac{1}{c_vT_{i+1}}-\frac{1}{c_vT_i})\right)=\\
&(\gamma-1)D_+\log(\rho_i)-D_+\log(T_i)-\frac{1}{hc_v}\hat\beta_T(\frac{1}{T_{i+1}}-\frac{1}{T_i})=\\
&(\gamma-1)D_+\log(\rho_i)
\end{aligned}
$$

So far, we have shown that,

$$(D_+w_i)^Tf^{mod}_{i+1/2}=\delta_0(A+B)D_+\log\rho_i=\delta_0(\gamma-1)(D_+\log\rho_i)^2. \tag{53}$$

Now we turn to the viscous and heat diffusion. (See also [TZ06].) We will need the following identity,

$$\frac{q_{i+1}}{T_{i+1}}-\frac{q_i}{T_i}=\frac{q_{i+1}+q_i}{2}(\frac{1}{T_{i+1}}-\frac{1}{T_i})+\frac{1}{2}(\frac{1}{T_{i+1}}+\frac{1}{T_{i+1}})(q_{i+1}-q_i).$$

The terms in the entropy estimate are:

$$(w_{j+1}-w_j)^T f^{NS}_{j+1/2}=(w_{j+1}-w_j)^T\begin{pmatrix}0\\ \frac{4}{3}\mu_0 D_+q_i\\ \frac{4}{3}\mu_0 q_{i+1/2}D_+q_i+k_0D_+T_i\end{pmatrix}. \tag{54}$$

From (54), we get

$$hD_+w_if^{NS}_{i+1/2}=(\frac{q_{i+1}}{c_vT_{i+1}}-\frac{q_i}{c_vT_i})\frac{4}{3}\mu_0\frac{(q_{i+1}-q_i)}{h}$$
$$-(\frac{1}{c_vT_{i+1}}-\frac{1}{c_vT_i})\left(\frac{4}{3}\mu_0q_{i+1/2}\frac{(q_{i+1}-q_i)}{h}+k_0\frac{(T_{i+1}-T_i)}{h}\right)$$

Using (36)

$$\begin{aligned}&hD_+w_if^{NS}_{i+1/2}=\\ &\frac{1}{c_v}\left(q_{i+1/2}(\frac{1}{T_{i+1}}-\frac{1}{T_i})+\frac{1}{2}(\frac{1}{T_{i+1}}+\frac{1}{T_{i+1}})(q_{i+1}-q_i)\right)\frac{4}{3}\mu_0\frac{(q_{i+1}-q_i)}{h}\\ &-\frac{1}{c_v}(\frac{1}{T_{i+1}}-\frac{1}{T_i})\left(\frac{4}{3}\mu_0q_{i+1/2}\frac{(q_{i+1}-q_i)}{h}+k_0\frac{(T_{i+1}-T_i)}{h}\right)=\\ &\frac{1}{c_v}\frac{4}{3}\mu_0\bar{T}^{-1}_{i+1/2}\frac{(q_{i+1}-q_i)^2}{h}-\frac{k_0}{c_v}(\frac{1}{T_{i+1}}-\frac{1}{T_i})\frac{(T_{i+1}-T_i)}{h}=\\ &\frac{4}{3}\mu_0\frac{\bar{T}^{-1}_{i+1/2}}{c_v}\frac{(q_{i+1}-q_i)^2}{h}+k_0\frac{1}{c_vT_iT_{i+1}}\frac{(T_{i+1}-T_i)^2}{h}\end{aligned} \tag{55}$$

where $\bar{T}^{-1}_{i+1/2}=\frac{1}{2}(\frac{1}{T_{i+1}}+\frac{1}{T_i})$.

Using (55), (53) in (51), we obtain the global entropy estimate

$$\begin{aligned}\sum_{j=1}^N(U_j)_t\le&-\sum_{j=1}^N h\left(\delta_0(\gamma-1)(D_+\log\rho_j)^2+\frac{4}{3}\mu_0\frac{\bar{T}^{-1}_{j+1/2}}{c_v}(D_+q_j)^2\right)\\ &-\sum_{j=1}^N hk_0\frac{(D_+T_j)^2}{c_vT_jT_{j+1}}-\sum_{j=1}^N h(D_+w_j)^T\frac{h(\lambda_{j+1/2}-\lambda^{LF}_{j+1/2})}{2}D_+u_j\end{aligned} \tag{56}$$

The L^2 bound on $\|u\|$ follows by the arguments resulting in (12). Then the bounds on ρq^2 and p follow from positivity. $\log\rho_i\in L^2(0,\mathcal{T};H^1(\Omega_N))$ follows from (56), and by observing that $\log\rho_i$ is bounded on a non-zero subset (due to conservation of ρ_i) and Lemma 3.10 (the discrete Poincare inequality).

The bound on $\log\rho_i$ implies that $\rho_i(t)>0$ a.e. (Here, a.e. means that as $h\to 0$ the measure of the cells associated non-positive densities is 0. For any finite N, $\rho_i(t)>0$ for all time.)

The artificial diffusion is bounded by the same argument as in the proof of Prop. 3.12 and the particular bound obtained as in Lemma 3.17. □

4.2.2. *Kinetic energy.*

Proposition 4.6. *Let the initial datum u^0 satisfy Assumption 4.4. Then the semi-discrete solution of (50) satisfies*

$$q\in L^2(0,\mathcal{T};H^1(\Omega_N)).$$

Proof. Let $K_i = \frac{1}{2}\rho_i q_i^2$ be the kinetic energy. Multiply the momentum equation by q_i and use the relation $q(\rho q)_t = \frac{1}{2}(\rho q^2)_t + \rho_t \frac{q^2}{2}$.

$$\left(\frac{1}{2}(\rho_i q_i^2)_t + (\rho_i)_t \frac{q_i^2}{2}\right) + q_i D_- \left(q_{i+1/2} f^\rho_{i+1/2} + \frac{1}{4}(\Delta q)_{i+1/2}(\Delta f^\rho)_{i+1/2} + p_{i+1/2}\right) =$$
$$q_i D_- \left(\frac{4}{3}\mu_0 D_+ q_i + \delta_0 q_{i+1/2} D_+ \log \rho_i\right) + q_i D_-(\frac{h\lambda_{i+1/2}}{2}(D_+(f_i^\rho))$$

using the continuity equation

$$\left(\frac{1}{2}(\rho_i q_i^2)_t + \left(-D_- f^\rho_{i+1/2} + \delta_0 D_- D_+ \log\rho_i + D_- \frac{h\lambda_{i+1/2}}{2} D_+ \rho_i\right)\frac{q_i^2}{2}\right)$$
$$+ q_i D_- \left(q_{i+1/2} f^\rho_{i+1/2} + \frac{1}{4} h D_+ q_i h(D_+ f_i^\rho) + p_{i+1/2}\right) =$$
$$q_i D_- \left(\frac{4\mu_0}{3} D_+ q_i + \delta_0 q_{i+1/2} D_+ \log\rho_i\right) + q_i D_-(\frac{h\lambda_{i+1/2}}{2}(D_+(f_i^\rho))$$

We sum over all $i = 1..N$ to obtain the kinetic energy

$$\sum_i h(K_i)_t + \sum_i h\left(-D_- f^\rho_{i+1/2} + \delta_0 D_- D_+ \log\rho_i + D_- \frac{h\lambda_{i+1/2}}{2} D_+\rho_i\right)\frac{q_i^2}{2}$$
$$+ \sum_i h q_i D_-(q_{i+1/2} f^\rho_{i+1/2}) + \sum_i h q_i D_-(\frac{1}{4} h D_+ q_i h(D_+ f_i^\rho) + p_{i+1/2}) =$$
$$\sum_i h q_i D_- \left(\frac{4\mu_0}{3} D_+ q_i + \delta_0 q_{i+1/2} D_+ \log\rho_i\right) + \sum_i h q_i D_-(\frac{h\lambda_{i+1/2}}{2}(D_+(f_i^\rho))$$

We have dropped the term $h\frac{1}{4}(D_+q_i)h(D_+q_i)h(D_+f_i^\rho)$ along with the artificial diffusion term, since we have already shown that the latter bound the former. (See proof of Prop 3.14.) Use a standard summation-by-parts rule.

$$\sum_i h(K_i)_t - \sum_i h\left(-f^\rho_{i+1/2} + \delta_0 D_+ \log\rho_i\right) D_+ \frac{q_i^2}{2}$$
$$- \sum_i h(D_+q_i) q_{i+1/2} f^\rho_{i+1/2} - h(D_+q_i) p_{i+1/2} \le$$
$$- \sum_i h(D_+q_i)\left(\frac{4\mu_0}{3} D_+ q_i + \delta_0 q_{i+1/2} D_+ \log\rho_i\right)$$

We estimate the pressure term

$$\sum_i h(K_i)_t - \sum_i h\left(-f^\rho_{i+1/2} + \delta_0 D_+\log\rho_i\right) q_{i+1/2} D_+ q_i$$
$$- \sum_i (D_+q_i) q_{i+1/2} f^\rho_{i+1/2} \le$$
$$\sum_i h\left(\eta(D_+q_i)^2 + \frac{1}{\eta} p^2_{i+1/2}\right) - \sum_i h(D_+q_i)\left(\frac{4\mu_0}{3} D_+ q_i + \delta_0 q_{i+1/2} D_+ \log\rho_i\right)$$

Several terms cancel and we are left with,

$$\sum_i h(K_i)_t \le -\sum_i h(\frac{4\mu_0}{3} - \eta)(D_+q_i)^2 + \frac{2}{\eta}\|p\|_2^2$$

We choose $0 < \eta < \frac{4}{3}\mu_0$ to get the bound on q_x. By the discrete Poincare inequality we also obtain that $q \in L^2(0, \mathcal{T}; L^2(\Omega_N))$. □

4.2.3. *Positivity.* We introduce $L_i = 1/\rho_i$. Multiply the continuity equation by $-1/\rho_i^2$. The manipulations up to (42) are the same and we obtain

$$\sum_i h\left((L_i)_t + 2(L_x)_{i+1/2}q_{i+1/2}\right)$$
$$-\sum_i h\left(\frac{\rho_{i+1}+\rho_i}{\rho_{i+1}^2\rho_i^2}D_+\rho_i(\frac{1}{4}(\Delta\rho)_{i+1/2}(\Delta q)_{i+1/2} + \frac{1}{2}|\Delta\rho_{i+1/2}|q_{i+1/2})\right) =$$
$$\delta_0\sum_i h2\bar{\rho}_{i+1/2}^{-1}D_+(\frac{1}{\rho_i})D_+\log\rho_i - \sum_i h\frac{\rho_{i+1}+\rho_i}{\rho_{i+1}^2\rho_i^2}D_+(\rho_i)(\frac{h\lambda_{i+1/2}}{2})D_+\rho_i$$

The only term that differs is the diffusion term, now a function of $\log(\rho_i)$. The last term on the right-hand and on the left-hand side are dropped since they are negative (when sitting on the right-hand side). (The derivation of has already been carried out in a previous proof.) Hence,

$$\sum_i h\left((L_i)_t + 2(L_x)_{i+1/2}q_{i+1/2}\right) = \delta_0\sum_i h2\bar{\rho}_{i+1/2}^{-1}D_+(\frac{1}{\rho_i})D_+\log\rho_i$$

Estimating the indefinite term.

$$\sum(L_i)_t \leq \sum_i 2h\left(\eta((L_x)_{i+1/2})^2 + \frac{1}{\eta}q_{i+1/2}^2\right) + \delta_0\sum_i h2\bar{\rho}_{i+1/2}^{-1}D_+(\frac{1}{\rho_i})D_+\log\rho_i$$

and we recall that we have a bound on $\|q\|_2$. Let

$$\rho(\xi) = \frac{1}{2}(\rho_{i+1}+\rho_i) + \xi(\rho_{i+1}-\rho_i).$$

Then

$$D_+\log\rho_i = \frac{\log\rho(1/2) - \log\rho(-1/2)}{h} = \frac{1}{h}(\frac{d}{d\rho}\log\rho)|_{\xi=\theta}(\rho_{i+1}-\rho_i) \tag{57}$$

where $-1/2 < \theta < 1/2$. such that

$$h2\bar{\rho}_{i+1/2}^{-1}D_+(\frac{1}{\rho_i})D_+\log\rho_i = -h\frac{\rho_{i+1}+\rho_i}{\rho_{i+1}^2\rho_i^2}D_+\rho_i D_+\log\rho_i =$$
$$-h\frac{\rho_{i+1}+\rho_i}{\rho_{i+1}^2\rho_i^2}\frac{1}{\rho(\theta)}(D_+\rho_i)^2 = -h\frac{\rho_{i+1}+\rho_i}{\rho(\theta)}(L_x)_{i+1/2}$$

Note that $\rho(\theta)$ is an intermediate value between ρ_i and ρ_{i+1}. Hence, $\rho(\theta) = \alpha\rho_{i+1} + \beta\rho_i$ where $0 < \alpha, \beta < 1$ and

$$\frac{\rho_{i+1}+\rho_i}{\alpha\rho_{i+1}+\beta\rho_i} = 1 + \frac{(1-\alpha)\rho_{i+1} + (1-\beta)\rho_i}{\alpha\rho_{i+1}+\beta\rho_i}$$

where the last term is positive. Hence, we have a bound from below of $2\rho_{i+1/2}/\rho(\theta)$.

$$\sum(L_i)_t \leq \sum_i 2h\left(\eta((L_x)_{i+1/2})^2 + \frac{1}{\eta}q_{i+1/2}^2\right) + \sum_i h\delta_0(L_x)_{i+1/2}^2$$

and we obtain the bound by choosing $0 < \eta < \delta_0/2$. We summarize these estimates in the following proposition.

Proposition 4.7. *Let the initial datum u^0 satisfy Assumption 4.4. Then, $L \in L^2(0,\mathcal{T};H^1(\Omega_N))$.*

We also need positivity and estimates on the temperature.

Proposition 4.8. *Let the initial datum u^0 satisfy Assumption 4.4. Then $T_i(t) > 0$, $t \in (0,\mathcal{T}]$ and $T \in L^1(0,\mathcal{T};L^1(\Omega_N))$.*

Proof. From the estimate (56), we have the bound

$$\int_0^{\mathcal{T}} h\frac{(D_+T_i)^2}{T_iT_{i+1}}dt \le \mathcal{C}.$$

We use that $\tilde{T}_{i+1/2}D_+\log T_i = D_+T_i$ where $\tilde{T}_{i+1/2}$ is the log-average of T_i and T_{i+1}. (c.f (57)). We also note that $T^{geo}_{i+1/2} = \sqrt{T_iT_{i+1}}$ is the geometric average. Hence,

$$\int_0^{\mathcal{T}} h\left(\frac{\tilde{T}_{i+1/2}}{T^{geo}_{i+1/2}}D_+\log T_i\right)^2 dt \le \mathcal{C}.$$

Since, for positive numbers, $T^{geo} \le \tilde{T}$, we obtain a bound on $\log T \in L^2(0,\mathcal{T},L^2(\Omega_N))$. Hence, $T_i(t) > 0$. Next, we use the gas law $Lp = RT$, Cauchy-Schwarz and the L^2 bounds on L and p to get $T \in L^1(0,\mathcal{T};L^1(\Omega_N))$. □

4.2.4. *Weak solution.* By the same reasoning as for the previous scheme, we know that we can solve the numerical scheme as a system of ODEs on the domain $\Omega_N \times [0,\mathcal{T}]$ and obtain a sequence of approximate solutions as N increases. We shall now show that we can extract a subsequence that converges to a weak solution, which is the second of our two main results.

Theorem 4.9. *Let the initial datum u^0 of the discrete scheme (50) satisfy Assumption 4.4. Then a subsequence of the solutions generated by (50) on ever finer grids ($h \to 0$), will converge, to a weak solution of (45) in the sense of Definition 4.3.*

Proof. We multiply the three discrete equations by a smooth periodic and compactly supported test function. (Projected onto the grid as described for the previous model.)

$$\begin{aligned}\sum_i h\varphi_i(0)\rho_i^0 + \int_0^T\left(\sum_i h(\varphi_i)_t\rho_i + \sum h(D_+\varphi_i)\tilde{f}^1_{i+1/2}\right)dt\\ -\int_0^{\mathcal{T}}\left(\sum_i h(D_+\varphi_i)(\delta_0(D_+\log\rho_i))\right)dt = 0\end{aligned} \tag{58}$$

$$\begin{aligned}\sum_i h\varphi_i(0)(\rho q)_i^0 + \int_0^T\left(\sum_i h(\varphi_i)_t(\rho q)_i + \sum_i h(D_+\varphi_i)\tilde{f}^2_{i+1/2}\right)dt\\ -\int_0^{\mathcal{T}}\left(\sum_i h(D_+\varphi_i)(\delta_0 q_{i+1/2}(D_+\log\rho_i) + \frac{4}{3}\mu D_+q_i)\right)dt = 0\end{aligned} \tag{59}$$

$$\begin{aligned}\sum_i h\varphi_i(0)E_i^0 + \int_0^T\left(h\sum_i(\varphi_i)_tE_i + \sum_i h(D_+\varphi_i)\tilde{f}^3_{i+1/2}\right)dt\\ -\int_0^{\mathcal{T}}\left(\sum_i h(D_+\varphi_i)(\delta_0\hat{\beta}_{i+1/2}(D_+\log\rho_i) + \frac{4}{3}\mu q_{i+1/2}D_+q_i) - h(D_-D_+\varphi_i)kT_i\right)dt = 0\end{aligned} \tag{60}$$

The proof is essentially the same as that of Theorem 3.18. The inviscid fluxes, including the artificial viscosity, are the same and satisfy the same bounds. (Prop. 4.5 and Prop. 4.6.) What differs in the present model are the diffusion terms. We will present the argument for the bounds on the viscous terms in some detail.

In the continuity equation (58) the diffusion term is:

$$\int_0^T \sum \delta_0 D_+\varphi_i (D_+ \log \rho_i)\, dt$$

where $D_+ \log \rho_i \in L^2(0,\mathcal{T}; L^2(\Omega_N))$ by Prop. 4.5.

In the momentum equation (59), there are two terms:

$$\int_0^T \sum D_+\varphi_i (\delta_0 q_{i+1/2} D_+ \log \rho_i)\, dt$$

$$\int_0^T \sum D_+\varphi_i (\mu_0 D_+ q_i)\, dt$$

The first one is bounded in $L^1(0,\mathcal{T}; L^1(\Omega_N))$ by Cauchy-Schwarz and the L^2 bounds on q and $\log \rho$. (Propositions 4.5 and Prop. 4.6.) From Prop. 4.6, we also get the bound on $D_+ q_i$ for the second term.

Finally, we turn to the energy equation (60). Here the diffusive terms are

$$\int_0^{\mathcal{T}} \left(\sum_i h(D_+\varphi_i)(\delta_0 \hat{\beta}_{i+1/2}(D_+ \log \rho_i) + \frac{4}{3}\mu q_{i+1/2} D_+ q_i) - h(D_- D_+ \varphi_i) k T_i \right) dt$$

The bound on T_i (Prop. 4.8) bounds the last term. Proposition 4.6 and Cauchy-Schwarz bound the middle diffusion term. Finally, we must show that

$$\hat{\beta}_{i+1/2}(D_+ \log \rho_i)$$

is bounded in $L^1(0,\mathcal{T}; L^1(\Omega_N))$. We write

$$\sqrt{\rho_{i+1}\rho_i}\left(\frac{q_{i+1}q_i}{2} + c_v \frac{T_{i+1}T_i}{\tilde{T}_{i+1/2}}\right)\frac{D_+ \log \rho_i}{\sqrt{\rho_{i+1}\rho_i}}.$$

Furthermore,

$$\sqrt{\rho_{i+1}\rho_i}\, q_{i+1} q_i \leq \rho_{i+1} q_{i+1}^2 + \rho_i q_i^2 \in C([0,\mathcal{T}]; L^2(\Omega_N)) \tag{61}$$

and

(62)

$$\sqrt{\rho_{i+1}\rho_i}\frac{T_{i+1}T_i}{\tilde{T}_{i+1/2}} = \frac{1}{R}\sqrt{p_{i+1}p_i}\frac{T^{geo}_{i+1/2}}{\tilde{T}_{i+1/2}} \leq \sqrt{p_{i+1}p_i} \leq p_{i+1} + p_i \in C([0,T]; L^2(\Omega_N)).$$

Let $\tilde{\rho}_{i+1/2}$ denote the log-averaged state (and again use that the log-average is greater than the geometric mean). Then,

$$\frac{D_+ \log \rho_i}{\sqrt{\rho_{i+1}\rho_i}} = \frac{D_+\rho_i}{\tilde{\rho}_{i+1/2}\sqrt{\rho_{i+1}\rho_i}} \leq \frac{D_+\rho_i}{\rho_{i+1}\rho_i} = (L_x)_{i+1/2} \in L^2(0,\mathcal{T}, L^2(\Omega_N)). \tag{63}$$

Combining (61)-(63), yield the desired bound on the viscous terms.

□

5. Concluding remarks

We have derived the necessary a priori bounds to give meaning to weak solutions of two different forms of the Navier-Stokes equations augmented with mass diffusion. Furthermore, we have proposed numerical discretization schemes that satisfy the corresponding bounds and proved that as $h \to 0$, a subsequence of solutions converges to a weak solution of the equations. In particular, no 1-D specific estimates have been used to allow the extension of this analysis to 3-D.

Of particular importance is the estimate of ρ^{-1} that gave us strong positivity results, which is an advantage of these modified equations. Another advantage is the natural correspondence of boundary conditions between the viscous and inviscid case. This should make it easier to derive stable numerical boundary schemes than for the conventional Navier-Stokes. (The estimates of the continuous equations in this paper were derived with the boundary conditions (21).) In the case of wall boundary conditions, discrete entropy estimates are derived for the Navier-Stokes equations in the recent report [PCN14]. Their approach could be the starting point when deriving the estimates for the modified system with wall boundary conditions.

Furthermore, we have proved convergence to a weak solution but another important issue is whether or not this is unique. In the analysis, we deliberately avoided using 1-D specific embeddings to make a case for a generalization to multi-D. However, if 1-D Sobolev embeddings are used, the a priori estimates (of both the continuos and semi-discrete problem) would be much stronger. For instance, in 1-D the estimates in this article lead to $\rho \geq \epsilon > 0$ and $q \in L^2(0, \mathcal{T}; L^\infty(\Omega))$, which in turn can be used to obtain stronger estimates on all variables. This might be sufficient to prove uniqueness although that would have to be carefully analyzed. (See [SFN12] where they reduce the system analyzed in [FV09] to 1-D and prove uniqueness of strong solutions.)

Finally, there is the question of validity of this model. We do not wish to venture further into that discussion. Hence, we have not presented any numerical computations as they necessitate choices of all three diffusion coefficients. To validate the model in 1-D a shock tube problem is the obvious choice, which has already been thoroughly reported in [GR07]. (We summarized their results in Section 1.2.) We can summarily say that both our models display the positivity property that was proven, even when the artificial diffusion is turned off. (With Lax-Friedrichs diffusion the scheme is positive even for the Euler equations.) Testing the strong Argon shock ([GR07]), it is clear that both these models/schemes have similar properties as reported by Greenshields and Reese. In particular, by choosing the mass diffusion coefficient very small a closer match to the standard Navier-Stokes solution is obtained.

As already stated, more validation test cases must be run in order to draw any strong conclusions. In particular, well-known test cases in 2 and 3-D where experiemental data is avaialable, including vortex shedding around cylinders, and turbulent decay, would shed more light on these modified models. See also [Bre13] where experimental validation tests are proposed.

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APPENDIX

I. Detailed derivation of entropy variables

Although this is a well-known and admittedly trivial calculation, we include it since there are so many versions in the literature depending on different, not always well detailed, non-dimensionalizations. Here, we do not use any non-dimensionalizations. We use, $U = -\rho S$, and $S = \log(p/\rho^\gamma)$.

The internal energy is $e = \rho c_v T$ and the conservative total energy is $E = e + \frac{1}{2}\rho q^2$. With the above relations this is equivalent to $E = \frac{p}{\gamma-1} + \frac{1}{2}\rho q^2$ and hence,

$$p = (\gamma - 1)(E - \frac{1}{2\rho}(\rho q)^2)$$

Next, we calculate the entropy variables, $w^T = (U_\rho, U_{\rho q}, U_E)$, one at the time.

$$\begin{aligned} U_\rho &= (-\rho S)_\rho = -S - \rho S_\rho \\ S_\rho &= (\log(p/\rho^\gamma))_\rho = \frac{\rho^\gamma}{p}(\frac{p_\rho}{\rho^\gamma} - \gamma\frac{p}{\rho^{\gamma+1}}) \\ p_\rho &= (\gamma-1)\frac{1}{2}q^2 \end{aligned}$$

Substituting back,

$$\begin{aligned} U_\rho = -S - \rho\frac{\rho^\gamma}{p}\left(\frac{q^2}{2\rho^\gamma} - \gamma\frac{p}{\rho^{\gamma+1}}\right) = \\ -S - (\gamma-1)\frac{\rho}{p}\frac{1}{2}q^2 + \gamma = -S - \frac{(\gamma-1)}{RT}\frac{1}{2}q^2 + \gamma = \\ = -S - \frac{1}{c_v T}\frac{1}{2}q^2 + \gamma. \end{aligned}$$

$$\begin{aligned} U_{\rho q} &= -\rho\frac{\rho^\gamma}{p}(\frac{p}{\rho^\gamma})_{\rho q} = -\frac{\rho}{p}p_{\rho q}. \\ p_{\rho q} &= -(\rho q)\frac{1}{\rho}(\gamma-1) \end{aligned}$$

Hence,

$$U_{\rho q} = \frac{\rho}{p}q(\gamma-1) = \frac{q}{c_v T}$$

Finally and noting that $p_E = (\gamma - 1)$,

$$U_E = -\frac{\rho}{p}p_E = -(\gamma-1)\frac{1}{RT} = -\frac{1}{c_v T}$$

I.1. **Definitions of norms and spaces.** We denote the norm of the standard $L^p(\Omega)$ as

$$\|u\|_p = (\int_\Omega u^p)^{1/p}$$

and for the L^2 norm we usually drop the index and write $\|u\|$. As usual, H^1 denotes the space where both $\|u\|$ and $\|u_x\|$ are bounded. (In L^2, index is dropped.)

We also use the Bochner space

$$L^r(0, \mathcal{T}; L^s(\Omega))$$

which is equipped with the norm

$$\|u\|_{L^r(L^s)} = (\int_0^{\mathcal{T}} (\int_\Omega u^s)^{r/s}\,dx\,dt)^{1/r}$$

In particular we will use the following spaces and corresponding norms:

$$L^2(0,\mathcal{T};L^2(\Omega)), \quad \|u\|_{L^2(L^2)} = (\int_0^{\mathcal{T}} \int_\Omega u^2\,dx\,dt)^{1/2}$$

$$L^1(0,\mathcal{T};L^1(\Omega)), \quad \|u\|_{L^1(L^1)} = (\int_0^{\mathcal{T}} \int_\Omega |u|\,dx\,dt)$$

$$L^2(0,\mathcal{T};H^1(\Omega)), \quad \|u\|_{L^2(H^1)} = (\int_0^{\mathcal{T}} \|u\|_2^2 + \|u_x\|^2\,dt)^{1/2}$$

We also have,

$$C([0,\mathcal{T}];L^p(\Omega)), \quad \|u\|_{C(L^2)} = \sup_{t\in[0,\mathcal{T}]} \|u\|_p.$$

The semi-discrete counterparts of the spaces and norms are obtained by replacing the spatial norm with the corresponding discrete norms. E.g. $L^2(0,\mathcal{T};L^2(\Omega))$ has the norm

$$\|u\|_{L^2(L^2)} = (\int_0^{\mathcal{T}} \sum_i h u_i^2\,dt)^{1/2}.$$