ON WEAK SOLUTIONS AND A CONVERGENT NUMERICAL SCHEME FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

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ABSTRACT. In this paper, the three-dimensional compressible Navier-Stokes equations are considered on a periodic domain. We propose a semi-discrete numerical scheme and derive *a priori* bounds that ensures that the resulting system of ordinary differential equations is solvable for any $h > 0$. An *a posteriori* examination that density remain uniformly bounded away from 0 will establish that a subsequence of the numerical solutions converges to a weak solution of the compressible Navier-Stokes equations.

1. BACKGROUND

The compressible Navier-Stokes equations have received considerable attention and yet strong well-posedness results are lacking. For the incompressible counterpart, weak solutions were proven to exist in [Ler34] and for the isentropic compressible equations in [Lio98]. For the full compressible equations a particular form of weak solutions were derived in [Fei04]. In [FN12] these solutions were shown to satisfy a so-called "weak-strong uniqueness". The latter implies that as long as a weak solution has sufficient regularity, i.e., it is a strong solution, it is unique. The weak solutions they derive satisfy the usual continuity and momentum equations weakly. However, the energy equation is replaced with an entropy inequality and the constraint that the total energy is conserved. If the solutions, derived in [Fei04] and [FN12], are sufficiently smooth, they satisfy the energy equation in the usual sense. However, when not smooth, these weak solutions may be different from weak solutions satisfying the standard continuity, momentum, and energy equations weakly, whose existence is subject to investigation in this paper.

Establishing existence of solutions is of paramount importance for numerical simulations. The information provided from well-posedness results helps in the design of effective numerical schemes. Without such knowledge, any and all simulations are uncertain. There is no way to tell whether or not the solution produced by a numerical scheme is an approximation of the true solution. Of great importance to numerical simulations is also robustness, in the sense that the scheme *always* produces an approximation for reasonably bounded data. Given a numerical solution, it is possible to examine if it lies within the physical range of applicability of the model.

In this paper we consider the compressible Navier-Stokes equations in three space dimensions on a periodic domain. The system includes a, non-zero but possibly very small, bulk viscosity. (This condition can likely be weakened, which will be discussed later.) We propose a finite difference scheme and derive appropriate a priori estimates and we show that the discrete scheme is solvable on arbitrary fine grids producing a sequence of solutions. Furthermore, the a priori estimates ensure convergence (i.e. existence) of weak solutions, if the density remains uniformly bounded away from 0. Although we can not establish a weak solution in the presence

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of vacuum regions, we stress that the compressible Navier-Stokes equations are not a valid model when approaching vacuum as the continuum hypothesis breaks down.

2. A priori estimates and weak solutions

We consider the three-dimensional (3-D) problem on the domain $\Omega \times [0, \mathcal{T}]$, where $\Omega = [0, 1]^3$ is the unit cube with Cartesian coordinates, $\mathbf{x} = (x, y, z)$, and periodic boundary conditions. $\mathcal{T}$ is an arbitrary but finite time. Let $\mathbf{v}$ denote the velocity vector with components v_i, $i = 1, 2, 3$. Morevover, let ρ denote the density, p the pressure, $\mathbb{S}$ the viscous stress tensor, E the total energy, e the specific internal energy, T the temperature, κ the heat conductivity, μ the dynamic viscosity, and η the bulk viscosity. c_p and c_v denote the specific heats at constant pressure or volume.

The compressible Navier-Stokes equations take the form,

$$\partial_t \rho + div_{\mathbf{x}}(\rho \mathbf{v}) = 0$$

(1)
$$\partial_t(\rho \mathbf{v}) + div_x(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p = div_{\mathbf{x}} \mathbb{S}$$

$$\partial_t(E) + div_{\mathbf{x}}(E\mathbf{v} + p\mathbf{v}) = div_{\mathbf{x}} \mathbb{S}\mathbf{v} + div_{\mathbf{x}}(\kappa \nabla_{\mathbf{x}} T)$$

$$p = \rho R T \quad \text{ideal gas law}$$

where $E = \frac{1}{2}\rho|\mathbf{v}|^2 + \rho e$, $e = c_v T$, $\rho e = \frac{p}{\gamma - 1}$. Furthermore, $\gamma = c_p/c_v$ and R is the gas constant. The stress tensor is given by: $(\mathbb{S})_{ij} = \tau_{ij} = -\frac{2}{3}\mu v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i}) + \eta v_{k,k}\delta_{ij}$. The equations (1) are stated on the so-called conservative form and we refer to the variables, $\mathbf{u} = (\rho, \rho \mathbf{v}^T, E)^T$ as conservative. We consider a constant dynamic viscosity $\mu = \mu_0 > 0$ and heat conductivity $\kappa = \kappa_0 > 0$.

In Section 2, we assume that the solution satisfies $\rho, T \geq 0$, when deriving the a priori estimates.

2.1. **Entropy estimate.** Let $U = -\rho S$ be the entropy function, where $S = \ln(\frac{p}{\rho^\gamma})$ is the specific entropy. (For air, $\gamma = 7/5$ but generally $1 < \gamma < 5/3$.) Associated with the entropy are the entropy fluxes $F_i = -\rho v_i S$, $i = 1, 2, 3$. For an entropy, $U_{\mathbf{uu}}$ is symmetric positive definite. (For the Navier-Stokes equations this is the case if $\rho, T > 0$.) Furthermore, $\mathbf{q}^T = U_{\mathbf{u}}$ are the entropy variables.

The Navier-Stokes equations can be recast as

(2)
$$U_{\mathbf{q}}\mathbf{q}_t + A_i(\mathbf{q})\mathbf{q}_{,i} = (K_{ij}\mathbf{q}_{,j})_{,i}$$

where we have used tensor notation. (Einstein's summation rule and $,j$ signifying derivative with respect to coordinate x_j.) Since U is an entropy, A_i are symmetric (See [Moc80]), and

$$K = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix}$$

is symmetric and positive semi-definite. (See [HFM86]). Contracting the Navier-Stokes equations from the left by the entropy variables, and integrating over the periodic domain Ω, we arrive at the so-called entropy estimate.

(3)
$$\int_\Omega U_t \, d\mathbf{x} + \int_\Omega \mathbf{q}_{,i}^T K_{ij} \mathbf{q}_{,j} \, d\mathbf{x} = 0$$

This is a (global) entropy inequality and a bound on $U(t)$ implies that $\mathbf{u} \in C(0, \mathcal{T}; L^2(\Omega))$. (See [Daf00] and also [Svä15]).

Remark In the non-periodic case, we would also need to handle boundary terms appearing in (3). In the case of wall boundary conditions, entropy stability was established in [PCN14]. (See also [SO14].)

To obtain bounds on the solution, the initial data must be appropriately bounded.

Assumption 2.1. *Assume that the initial data are provided in the following spaces:*

$$\mathbf{u}(0, \mathbf{x}) \in (L^2(\Omega))^5, \quad \log(T(0, \mathbf{x})) \in H^1(\Omega),$$
$$\mathbf{v}(0, \mathbf{x}) \in (H^1(\Omega))^3, \quad \rho(0, \mathbf{x}), T(0, \mathbf{x}) > 0.$$

We end this section by summarizing the a priori estimates that can be inferred from the entropy estimate (3). (These results are standard and we omit detailed derivations.)

Theorem 2.2. *Assume that the initial conditions are given as in Assumption 2.1. Furthermore, we assume that $\rho(\mathbf{x}, t) > 0, T(\mathbf{x}, t) > 0, \ t \in [0, \mathcal{T}], \mathbf{x} \in \Omega$. Then solutions $\mathbf{u}$ of (1), satisfy*

$$\mathbf{u}(t) \in (L^2(\Omega))^5, \quad p, \rho|\mathbf{v}|^2 \in L^2(\Omega), \quad \log(T) \in L^2(0, \mathcal{T}; H^1(\Omega)).$$

Proof. $\mathbf{u} \in L^2(\Omega)$ follows from the entropy bound and an argument laid out in [Daf00]. (See also [SO14], [Svä15] and the proof of Proposition 3.4 below.) The bounds on p and $\rho|\mathbf{v}|^2$ follows from the L^2 bound on E and by positivity. (Note that positivity of p follows from the gas law and positivity of ρ and T.) The estimate on $\log(T)$ follows from the bound on the viscous terms in (3) and Poincare's inequality. $\qquad\square$

2.2. **Kinetic energy.** The kinetic energy is defined as $K_e = \frac{1}{2}\rho|\mathbf{v}|^2 = \frac{1}{2}\rho(v_1^2 + v_2^2 + v_3^2)$. It satisfies the following a priori estimate:

$$(4) \qquad \int_0^{\mathcal{T}} \int_\Omega (K_e)_t + cv_{i,j}v_{i,j} \, d\mathbf{x} \, dt \leq C,$$

where c and C are two a priori determined constants. We proceed and derive this estimate.

We use the identity (with time or space derivative)

$$(5) \qquad \frac{1}{2}(v^2 m)_x + \frac{1}{2}m_x v^2 = (mv)_x v.$$

We contract the momentum equations with $\mathbf{v}$.

$$(6) \qquad \mathbf{v}^T \partial_t(\rho\mathbf{v}) + \mathbf{v}^T div_x(\rho\mathbf{v} \otimes \mathbf{v}) + \mathbf{v}^T \nabla_x p = \mathbf{v}^T div_x \mathbb{S}$$

Using (5) on the time derivative term, gives

$$\frac{1}{2}\partial_t(\rho|\mathbf{v}|^2)_t + \frac{1}{2}|\mathbf{v}|^2 \partial_t \rho + \mathbf{v}^T div_x(\rho\mathbf{v} \otimes \mathbf{v}) + \mathbf{v}^T \nabla_x p = \mathbf{v}^T div_x \mathbb{S}$$

With the relation,

$$\mathbf{v}^T div_x(\rho\mathbf{v} \otimes \mathbf{v}) = \frac{1}{2}div_x(\rho\mathbf{v}|\mathbf{v}|^2) + \frac{1}{2}|\mathbf{v}|^2 div_x(\rho\mathbf{v}),$$

(6) can be written as,

$$\frac{1}{2}\partial_t(\rho|\mathbf{v}|^2)_t + \frac{1}{2}|\mathbf{v}|^2(\partial_t \rho + div_x(\rho\mathbf{v})) + \frac{1}{2}div_x(\rho\mathbf{v}|\mathbf{v}|^2) + \mathbf{v}^T \nabla_x p = \mathbf{v}^T div_x \mathbb{S}.$$

Using the continuity equation and the chain rule on the pressure term, we get

$$\frac{1}{2}\partial_t(\rho|\mathbf{v}|^2)_t + \frac{1}{2}div_x(\mathbf{v}\rho|\mathbf{v}|^2) + div_x(p\mathbf{v}) = \mathbf{v}^T div_x \mathbb{S} + p \, div_x(\mathbf{v}).$$

We integrate in space and note that the divergence terms disappear thanks to periodicity.

$$(7) \qquad \int_\Omega (K_e)_t d\mathbf{x} = \int_\Omega (\mathbf{v}^T div_x \mathbb{S} + \frac{1}{\xi}p^2 + \xi(div_x(\mathbf{v}))^2)d\mathbf{x},$$

where $\xi > 0$ is a parameter. The pressure term on the right-hand side is bounded thanks to positivity and the L^2 estimate given in Theorem 2.2.

The last step in the derivation of (4) is to show that the $\mathbf{v}^T div_x \mathbb{S}$ term results in a bound on the gradients of the velocity. We write the stress tensor on component (tensor) form.

$$(8) \qquad (\mathbb{S})_{ij} = \tau_{ij} = -\frac{2}{3}\mu v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i}) + \eta v_{k,k}\delta_{ij}$$

The stress term

$$\mathbf{v}^T div_x(\mathbb{S}) = v_j \tau_{ij,i} = (v_j \tau_{ij})_{,i} - v_{j,i}\tau_{ij}.$$

Integrate in space and use the periodic boundary conditions.

$$-\int_\Omega \mathbf{v}^T div_x(\mathbb{S})d\mathbf{x} = -\int_\Omega v_{j,i}\tau_{ij}\,d\mathbf{x} =$$

$$\mu\int_\Omega \frac{2}{3}\left((u_x - v_y)^2 + (u_x - w_z)^2 + (v_y - w_z)^2\right)\,d\mathbf{x}$$

$$+\mu\int_\Omega (v_x + u_y)^2 + (u_z + w_x)^2 + (v_z + w_y)^2 d\mathbf{x}$$

$$+\int_\Omega \eta(u_x + v_y + w_z)^2\,d\mathbf{x} =$$

$$\int_\Omega \left(\frac{2\mu}{3} - \frac{\eta}{2}\right)\left((u_x - v_y)^2 + (u_x - w_z)^2 + (v_y - w_z)^2\right)\,d\mathbf{x}$$

$$+\int_\Omega \frac{\eta}{2}(u_x + v_y + w_z)^2\,d\mathbf{x}$$

$$+\mu\int_\Omega (v_x + u_y)^2 + (u_z + w_x)^2 + (v_z + w_y)^2\,d\mathbf{x}$$

$$(9) \qquad\qquad +\int_\Omega \eta\left(u_x^2 + v_y^2 + w_z^2\right)\,d\mathbf{x} \geq 0$$

The last two rows bound $\int_\Omega v_{i,j}v_{i,j}d\mathbf{x}$ via Korn's inequality. (See Appendix I.) By choosing $\xi \leq \eta/2$, we bound the last term of (7), and (4) follows.

Remark With the generalized Korn's inequality given in [Dai06], the estimate can be obtained with $\eta = 0$. However, that inequality need to be proven for periodic boundary conditions to be applicable here, and later also in a discrete setting.

Theorem 2.3. *Assume that the initial data are given as in Assumption 2.1 and ρ, T are positive for $t \in [0, \mathcal{T}]$. Then $v_i \in L^2(0, \mathcal{T}; H^1(\Omega))$, $i = 1, 2, 3$.*

Proof. By Theorem 2.2, we have $\rho\mathbf{v} \in C(0, \mathcal{T}; L^2(\Omega)^3)$. By positivity and conservation, we know that $\rho \geq constant > 0$ on a set, B_ρ, of non-zero measure. Hence, $\mathbf{v} \in C(0, \mathcal{T}; (L^2(B_\rho))^3)$. By (4) and Poincare's inequality, $\mathbf{v} \in L^2(0, \mathcal{T}; (H^1(\Omega))^3)$. $\qquad\square$

2.3. **Weak solutions.** A weak solution satisfies the equations in a distributional sense using periodic C^∞-test functions (denoted ϕ). That is, the equations (1) are multiplied by the test function and integrated. Derivatives are moved on to the test function by partial integration. (Boundary terms vanish thanks to periodicity.) This procedure reduces need of regularity of the solution. Instead L^1 integrability of the variables and fluxes is all that is needed to give meaning to the integrals appearing in the weak form of the equations. It is easy to see that Assumption 2.1 and the a priori estimates are sufficient (recalling that we consider a bounded domain) to bound the variables, inviscid fluxes and viscous fluxes in L^1. The only caveat is the temperature flux since we do not have a bound on temperature.

Note that

(10)
$$\int_{\mathbf{x}} \nabla \phi \kappa \nabla T d\mathbf{x} = -\int_{\mathbf{x}} \Delta \phi \kappa T d\mathbf{x}.$$

If $\rho \geq \epsilon$ where ϵ is a positive constant, we obtain from the gas law, $T \leq p/(R\epsilon) \in L^2(\Omega)$ which bounds the last integral.

3. THE NUMERICAL SCHEME

We discretize the domain with $N+1$ points in the x, y, z directions. That means $h = 1/N$ and $x_i = ih$, $y_j = jh$, and $z_k = kh$, $i, j, k = 0, .., N$. Let $\mathbf{u}_{ijk} = (\rho_{ijk}, m^1_{ijk}, m^2_{ijk}, m^3_{ijk}, E_{ijk})^T$ where the components are the numerical variables corresponding to density, momentum in the x-y-z-direction and total energy. We use u^k_{ijk} to denote the kth component of $\mathbf{u}_{ijk}$, e.g. $u^1_{ijk} = \rho_{ijk}$. All variables satisfy the same algebraic relations as their continuous counterparts. E.g. $E_{ijk} = \frac{p_{ijk}}{\gamma-1} + \frac{1}{2\rho_{ijk}}((m^1_{ijk})^2 + (m^2_{ijk})^2 + (m^3_{ijk})^2)$. To avoid cumbersome notation, we will use u_{ijk}, v_{ijk} and w_{ijk} to denote the velocity components. With a slight abuse of notation, we use D^x_- to denote the operator $D^x_- a_{ijk} = \frac{a_{ijk} - a_{i-1jk}}{h}$ irrespective if a is a scalar or a vector. If it is a vector, the operation is carried out on each component. We define $D^y_-, D^z_-, D^x_+, D^y_+, D^z_+$ analogously.

The periodic boundary conditions are enforeced through the following relations:

(11)
$$\mathbf{u}_{0jk} = \mathbf{u}_{N+1jk}, \quad \mathbf{u}_{i0k} = \mathbf{u}_{iN+1k}, \quad \mathbf{u}_{ij0} = \mathbf{u}_{ijN+1}.$$

Let

$$\mathbf{f}_{ijk} = (m^1_{ijk}, u_{ijk}m^1_{ijk} + p_{ijk}, u_{ijk}m^2_{ijk}, u_{ijk}m^3_{ijk}, u_{ijk}(E_{ijk} + p_{ijj}))^T,$$
$$\mathbf{g}_{ijk} = (m^2_{ijk}, v_{ijk}m^1_{ijk}, v_{ijk}m^2_{ijk} + p_{ijk}, v_{ijk}m^3_{ijk}, v_{ijk}(E_{ijk} + p_{ijk}))^T,$$
$$\mathbf{h}_{ijk} = (m^3_{ijk}, w_{ijk}m^1_{ijk}, w_{ijk}m^2_{ijk}, w_{ijk}m^3_{ijk} + p_{ijk}, v_{ijk}(E_{ijk} + p_{ijk}))^T,$$

be local inviscid flux vectors. The inviscid terms will be approximated by means of the local Lax-Friedrichs-type fluxes,

(12)
$$\mathbf{f}_{i+1/2jk} = \frac{\mathbf{f}_{i+1jk} + \mathbf{f}_{ijk}}{2} - \frac{\lambda_{i+1/2jk}}{2}(\mathbf{u}_{i+1,jk} - \mathbf{u}_{ijk}),$$
$$\mathbf{g}_{ij+1/2k} = \frac{\mathbf{g}_{ij+1k} + \mathbf{g}_{ijk}}{2} - \frac{\lambda_{ij+1/2k}}{2}(\mathbf{u}_{ij+1k} - \mathbf{u}_{ijk}),$$
$$\mathbf{h}_{ijk+1/2} = \frac{\mathbf{h}_{ijk+1} + \mathbf{h}_{ijk}}{2} - \frac{\lambda_{ijk+1/2}}{2}(\mathbf{u}_{ijk+1} - \mathbf{u}_{ijk}),$$

where $\lambda_{i+1/2,jk} = \lambda^{LF}_{i+1/2jk}$. $\lambda^{LF}_{i+1/2,jk}$ is almost the standard local Lax-Friedrichs diffusion. We define it as,

(13)
$$u^{LF}_{i+1/2jk} = \max(|u_{i+1jk}|, |u_{ijk}|),$$
$$c^{LF}_{i+1/2jk} = \max(c_{i+1jk}, c_{ijk}),$$
$$\lambda^{LF}_{i+1/2jk} = u^{LF}_{i+1/2jk} + c^{LF}_{i+1/2jk},$$

and similarly in the y, z-directions.

Remark This flux is slightly more diffusive than the entropy-stable local-Lax-Friedrichs flux, and hence also entropy stable. (See [Tad03].)

Next, we turn to the diffusive fluxes which are divided in viscous and heat contributions. In the x-direction, we write

$$\mathbf{F}_{i+1/2jk} = \mathbf{F}^v_{i+1/2jk} + \mathbf{F}^T_{i+1/2jk}.$$

We will use the following short-hand notation, $D_+^x u_{ijk} = (\mathfrak{u}_x)_{ijk}, D_+^y u_{ijk} = (\mathfrak{u}_y)_{ijk}$ etc. Define,

$$(\tau_{xx})_{ijk} = \mu\left(\frac{4}{3}(\mathfrak{u}_x)_{ijk} - \frac{2}{3}((\mathfrak{v}_y)_{ijk} + (\mathfrak{w}_z)_{ijk})\right) + \eta((\mathfrak{u}_x)_{ijk} + (\mathfrak{v}_y)_{ijk} + (\mathfrak{w}_z)_{ijk}),$$

$$(\tau_{xy})_{ijk} = (\tau_{yx})_{ijk} = \mu\left((\mathfrak{u}_y)_{ijk} + (\mathfrak{v}_x)_{ijk}\right),$$

$$(\tau_{xz})_{ijk} = (\tau_{zx})_{ijk} = \mu\left((\mathfrak{u}_z)_{ijk} + (\mathfrak{w}_x)_{ijk}\right),$$

$$(\tau_{yy})_{ijk} = \mu\left(\frac{4}{3}(\mathfrak{v}_y)_{ijk} - \frac{2}{3}((\mathfrak{u}_x)_{ijk} + (\mathfrak{w}_z)_{ijk})\right) + \eta((\mathfrak{u}_x)_{ijk} + (\mathfrak{v}_y)_{ijk} + (\mathfrak{w}_z)_{ijk}),$$

$$(\tau_{yz})_{ijk} = (\tau_{zy})_{ijk} = \mu\left((\mathfrak{v}_z)_{ijk} + (\mathfrak{w}_y)_{ijk}\right),$$

$$(\tau_{zz})_{ijk} = \mu\left(\frac{4}{3}(\mathfrak{w}_z)_{ijk} - \frac{2}{3}((\mathfrak{u}_x)_{ijk} + (\mathfrak{v}_y)_{ijk})\right) + \eta((\mathfrak{u}_x)_{ijk} + (\mathfrak{v}_y)_{ijk} + (\mathfrak{w}_z)_{ijk}).$$

Then the viscous fluxes (excluding the heat flux) are discretized as

$$\mathbf{F}_{i+1/2,jk}^v = \begin{pmatrix} 0 \\ (\tau_{xx})_{ijk} \\ (\tau_{xy})_{ijk} \\ (\tau_{xz})_{ijk} \\ u_{i+1,jk}(\tau_{xx})_{ijk} + v_{i+1,jk}(\tau_{xy})_{ijk} + w_{ijk+1}(\tau_{xz})_{ijk} \end{pmatrix},$$

$$\mathbf{G}_{i,j+1/2k}^v = \begin{pmatrix} 0 \\ (\tau_{xy})_{ijk} \\ (\tau_{yy})_{ijk} \\ (\tau_{yz})_{ijk} \\ u_{i,j+1k}(\tau_{xy})_{ijk} + v_{i,j+1k}(\tau_{yy})_{ijk} + w_{i,j+1,k}(\tau_{yz})_{ijk} \end{pmatrix},$$

$$\mathbf{H}_{i,j,k+1/2}^v = \begin{pmatrix} 0 \\ (\tau_{zx})_{ijk} \\ (\tau_{zy})_{ijk} \\ (\tau_{zz})_{ijk} \\ u_{ijk+1}(\tau_{zx})_{ijk} + v_{ijk+1}(\tau_{zy})_{ijk} + w_{ijk+1}(\tau_{zz})_{ijk} \end{pmatrix}.$$

The heat fluxes are defined as

$$
\begin{aligned}
\mathbf{F}_{i+1/2jk}^T &= \left(0,0,0,\kappa D_+^x T_{ijk}\right)^T, \\
\mathbf{G}_{ij+1/2k}^T &= \left(0,0,0,\kappa D_+^y T_{ijk}\right)^T, \\
\mathbf{H}_{ijk+1/2}^T &= \left(0,0,0,\kappa D_+^z T_{ijk}\right)^T.
\end{aligned}
$$

(14)

Finally, the scheme approximating (1) takes the form.

$$(\mathbf{u}_{ijk})_t + D_-^x \mathbf{f}_{i+1/2jk} + D_-^y \mathbf{g}_{ij+1/2k} + D_-^z \mathbf{h}_{ijk+1/2} =$$
$$D_-^x \mathbf{F}_{i+1/2jk} + D_-^y \mathbf{G}_{ij+1/2k} + D_-^z \mathbf{H}_{ijk+1/2}, \qquad 0 \le i,j,k \le N.$$

(15)

Remark It is well-known that the inviscid fluxes are first-order accurate for smooth solutions. Furthermore, it is elementary to show that the proposed approximation of the viscous terms is also first-order accurate. Hence, the scheme is consistent.

3.1. **The discrete entropy estimate.** In the analysis below, we will need the entropy variables.

$$U_{\mathbf{u}} = \mathbf{q} = \frac{1}{c_v}\left(c_v(\gamma - S) - \frac{u^2 + v^2 + w^2}{T}, \frac{u}{T}, \frac{v}{T}, \frac{w}{T}, \frac{-1}{T}\right)$$

(16)

We denote the corresponding discrete variables as $\mathbf{q}_{ijk}$. Using the following discrete chain rule,

$$\frac{(a_{i+1}b_{i+1} - a_i b_i)}{h} = a_{i+1}\frac{b_{i+1} - b_i}{h} + b_i\frac{a_{i+1} - a_i}{h},$$

we can calculate

$$D_+^x \mathbf{q}_{ijk} = \frac{1}{c_v}\begin{pmatrix} c_v D_+^x q_{ijk}^1 \\ D_+^x \frac{u_{ijk}}{T_{ijk}} \\ D_+^x \frac{v_{ijk}}{T_{ijk}} \\ D_+^x \frac{w_{ijk}}{T_{ijk}} \\ D_+^x \frac{-1}{T_{ijk}} \end{pmatrix} = \frac{1}{c_v}\begin{pmatrix} c_v D_+^x q_{ijk}^1 \\ \left(\frac{1}{T}\right)_{ijk} D_+^x u_{ijk} + u_{i+1jk} D_+^x \frac{1}{T_{ijk}} \\ \left(\frac{1}{T}\right)_{ijk} D_+^x v_{ijk} + v_{i+1jk} D_+^x \frac{1}{T_{ijk}} \\ \left(\frac{1}{T}\right)_{ijk} D_+^x w_{ijk} + w_{i+1jk} D_+^x \frac{1}{T_{ijk}} \\ D_+^x \frac{-1}{T_{ijk}} \end{pmatrix}.$$

Remark $D_+^x q^1$ will not affect the subsequent calculation and we omit its precise form here.

Lemma 3.1. *If T_{ijk} is non-negative, then*

$$(D_+^x \mathbf{q}_{ijk})^T \mathbf{F}_{i+1/2jk}^v + (D_+^y \mathbf{q}_{ijk})^T \mathbf{G}_{ij+1/2k}^v + (D_+^z \mathbf{q}_{ijk})^T \mathbf{H}_{ijk+1/2}^v \geq 0.$$

Proof.

$$(D_+^x \mathbf{q}_{ijk})^T \mathbf{F}_{i+1/2jk}^v + (D_+^y \mathbf{q}_{ijk})^T \mathbf{G}_{ij+1/2k}^v + (D_+^z \mathbf{q}_{ijk})^T \mathbf{H}_{ijk+1/2}^v =$$

$$\left(\left(\frac{1}{T}\right)_{ijk} D_+^x u_{ijk} + u_{i+1jk} D_+^x \frac{1}{T_{ijk}}\right)(\tau_{xx})_{ijk}$$

$$+\left(\left(\frac{1}{T}\right)_{ijk} D_+^x v_{ijk} + v_{i+1jk} D_+^x \frac{1}{T_{ijk}}\right)(\tau_{xy})_{ijk}$$

$$+\left(\left(\frac{1}{T}\right)_{ijk} D_+^x w_{ijk} + w_{i+1jk} D_+^x \frac{1}{T_{ijk}}\right)(\tau_{xz})_{ijk}$$

$$-D_+^x \frac{1}{T_{ijk}}\left(u_{i+1jk}(\tau_{xx})_{ijk} + v_{i+1jk}(\tau_{xy})_{ijk} + w_{i+1jk}(\tau_{xz})_{ijk}\right)$$

$$+\left(\left(\frac{1}{T}\right)_{ijk} D_+^y u_{ijk} + u_{ij+1k} D_+^y \frac{1}{T_{ijk}}\right)(\tau_{yx})_{ijk}$$

$$+\left(\left(\frac{1}{T}\right)_{ijk} D_+^y v_{ijk} + v_{ij+1k} D_+^y \frac{1}{T_{ijk}}\right)(\tau_{yy})_{ijk}$$

$$+\left(\left(\frac{1}{T}\right)_{ijk} D_+^y w_{ijk} + w_{ij+1k} D_+^y \frac{1}{T_{ijk}}\right)(\tau_{yz})_{ijk}$$

$$-D_+^y \frac{1}{T_{ijk}}\left(u_{ij+1k}(\tau_{xy})_{ijk} + v_{ij+1k}(\tau_{yy})_{ijk} + w_{ij+1k}(\tau_{yz})_{ijk}\right)$$

$$+\left(\left(\frac{1}{T}\right)_{ijk} D_+^z u_{ijk} + u_{ijk+1} D_+^z \frac{1}{T_{ijk}}\right)(\tau_{zx})_{ijk}$$

$$+\left(\left(\frac{1}{T}\right)_{ijk} D_+^z v_{ijk} + v_{ijk+1} D_+^z \frac{1}{T_{ijk}}\right)(\tau_{zy})_{ijk}$$

$$+\left(\left(\frac{1}{T}\right)_{ijk} D_+^z w_{ijk} + w_{ijk+1} D_+^z \frac{1}{T_{ijk}}\right)(\tau_{zz})_{ijk}$$

$$-D_+^z \frac{1}{T_{ijk}}\left(u_{ijk+1}(\tau_{zx})_{ijk} + v_{ijk+1}(\tau_{zy})_{ijk} + w_{ijk+1}(\tau_{zz})_{ijk}\right)$$

A number of terms cancel.

$$(D_+^x \mathbf{q}_{ijk})^T \mathbf{F}_{i+1/2jk}^v + (D_+^y \mathbf{q}_{ijk})^T \mathbf{G}_{ij+1/2k}^v + (D_+^z \mathbf{q}_{ijk})^T \mathbf{H}_{ijk+1/2}^v =$$

$$\left(\left(\frac{1}{T} \right)_{ijk} D_+^x u_{ijk} \right)(\tau_{xx})_{ijk} + \left(\left(\frac{1}{T} \right)_{ijk} D_+^x v_{ijk} \right)(\tau_{xy})_{ijk} + \left(\left(\frac{1}{T} \right)_{ijk} D_+^x w_{ijk} \right)(\tau_{xz})_{ijk}$$

$$+ \left(\left(\frac{1}{T} \right)_{ijk} D_+^y u_{ijk} \right)(\tau_{xy})_{ijk} + \left(\left(\frac{1}{T} \right)_{ijk} D_+^y v_{ijk} \right)(\tau_{yy})_{ijk} + \left(\left(\frac{1}{T} \right)_{ijk} D_+^y w_{ijk} \right)(\tau_{zy})_{ijk}$$

$$+ \left(\left(\frac{1}{T} \right)_{ijk} D_+^z u_{ijk} \right)(\tau_{xz})_{ijk} + \left(\left(\frac{1}{T} \right)_{ijk} D_+^z v_{ijk} \right)(\tau_{yz})_{ijk} + \left(\left(\frac{1}{T} \right)_{ijk} D_+^z w_{ijk} \right)(\tau_{zz})_{ijk}$$

Factoring out μ/T, we have

$$(D_+^x \mathbf{q}_{ijk})^T \mathbf{F}_{i+1/2jk}^v + (D_+^y \mathbf{q}_{ijk})^T \mathbf{G}_{ij+1/2k}^v + (D_+^z \mathbf{q}_{ijk})^T \mathbf{H}_{ijk+1/2}^v =$$
$$\left(\frac{\mu}{T} \right)_{ijk} (\{\mathfrak{u}, \mathfrak{v}, \mathfrak{w}\}_{x,y,z})_{ijk},$$

where

$$(\{\mathfrak{u}, \mathfrak{v}, \mathfrak{w}\}_{x,y,z})_{ijk} = \frac{1}{\mu} (D_+^x u_{ijk}(\tau_{xx})_{ijk} + D_+^x v_{ijk}(\tau_{xy})_{ijk} + D_+^x w_{ijk}(\tau_{xz})_{ijk}$$
$$+ D_+^y u_{ijk}(\tau_{yx})_{ijk} + D_+^y v_{ijk}(\tau_{yy})_{ijk} + D_+^y w_{ijk}(\tau_{yz})_{ijk}$$
$$+ D_+^z u_{ijk}(\tau_{zx})_{ijk} + D_+^z v_{ijk}(\tau_{zy})_{ijk} + D_+^z w_{ijk}(\tau_{zz})_{ijk}) =$$
$$(\mathfrak{u}_x)_{ijk} \left(\frac{4}{3}(\mathfrak{u}_x)_{ijk} - \frac{2}{3}((\mathfrak{v}_y)_{ijk} + (\mathfrak{w}_z)_{ijk}) \right)$$
$$+ (\mathfrak{v}_x)_{ijk}((\mathfrak{u}_y)_{ijk} + (\mathfrak{v}_x)_{ijk})$$
$$+ (\mathfrak{w}_x)_{ijk}((\mathfrak{u}_z)_{ijk} + (\mathfrak{w}_x)_{ijk})$$
$$+ (\mathfrak{u}_y)_{ijk}((\mathfrak{u}_y)_{ijk} + (\mathfrak{v}_x)_{ijk})$$
$$+ (\mathfrak{v}_y)_{ijk} \left(\frac{4}{3}(\mathfrak{v}_y)_{ijk} - \frac{2}{3}((\mathfrak{u}_x)_{ijk} + (\mathfrak{w}_z)_{ijk}) \right)$$
$$+ (\mathfrak{w}_y)_{ijk}((\mathfrak{v}_z)_{ijk} + (\mathfrak{w}_y)_{ijk})$$
$$+ (\mathfrak{u}_z)_{ijk}((\mathfrak{u}_z)_{ijk} + (\mathfrak{w}_x)_{ijk})$$
$$+ (\mathfrak{v}_z)_{ijk}((\mathfrak{v}_z)_{ijk} + (\mathfrak{w}_y)_{ijk})$$
$$+ (\mathfrak{w}_z)_{ijk} \left(\frac{4}{3}(\mathfrak{w}_z)_{ijk} - \frac{2}{3}((\mathfrak{u}_x)_{ijk} + (\mathfrak{v}_y)_{ijk}) \right)$$
$$+ \frac{\eta}{\mu}((\mathfrak{u}_x)_{ijk} + (\mathfrak{v}_y)_{ijk} + (\mathfrak{w}_z)_{ijk})^2.$$

Since $(\mu/T)_{ijk}$ is non-negative, we are left to show non-negativity of $(\{\mathfrak{u}, \mathfrak{v}, \mathfrak{w}\}_{x,y,z})_{ijk}$. (We suppress the common index ijk.)

$$
(\{\mathfrak{u}, \mathfrak{v}, \mathfrak{w}\}_{x,y,z}) = \frac{4}{3}(\mathfrak{u}_x)^2 - \frac{2}{3}(\mathfrak{v}_y \mathfrak{u}_x + \mathfrak{w}_z \mathfrak{u}_x) + \mathfrak{v}_x \mathfrak{u}_y + \mathfrak{v}_x \mathfrak{v}_x + \mathfrak{u}_z \mathfrak{w}_x + \mathfrak{w}_x \mathfrak{w}_x
$$

$$
+\mathfrak{u}_y \mathfrak{u}_y + \mathfrak{v}_x \mathfrak{u}_y + \frac{4}{3}(\mathfrak{v}_y)^2 - \frac{2}{3}(\mathfrak{u}_x \mathfrak{v}_y + \mathfrak{w}_z \mathfrak{v}_y) + \mathfrak{v}_z \mathfrak{w}_y + \mathfrak{w}_y \mathfrak{w}_y
$$

$$
+\mathfrak{u}_z \mathfrak{u}_z + \mathfrak{w}_x \mathfrak{u}_z + \mathfrak{v}_z \mathfrak{v}_z + \mathfrak{w}_y \mathfrak{v}_z + \frac{4}{3}\mathfrak{w}_z^2 - \frac{2}{3}(\mathfrak{u}_x \mathfrak{w}_z + \mathfrak{v}_y \mathfrak{w}_z)
$$

$$
+\frac{\eta}{\mu}(\mathfrak{u}_x + \mathfrak{v}_y + \mathfrak{w}_z)^2 =
$$

$$
\frac{2}{3}\left((\mathfrak{u}_x - \mathfrak{v}_y)^2 + (\mathfrak{u}_x - \mathfrak{w}_z)^2 + (\mathfrak{v}_y - \mathfrak{w}_z)^2\right)
$$

$$
+(\mathfrak{v}_x + \mathfrak{u}_y)^2 + (\mathfrak{u}_z + \mathfrak{w}_x)^2 + (\mathfrak{v}_z + \mathfrak{w}_y)^2
$$

(17)
$$
+\frac{\eta}{\mu}(\mathfrak{u}_x + \mathfrak{v}_y + \mathfrak{w}_z)^2 \geq 0
$$

$\square$

Lemma 3.2. *If $T_{ijk}, T_{i+1jk}, T_{ij+1k}, T_{ijk+1}$ are non-negative, then the following relations hold,*

$$
(D_+^x \mathbf{q}_{ijk})^T \mathbf{F}_{i+1/2jk}^T \geq 0, \quad (D_+^y \mathbf{q}_{ijk})^T \mathbf{G}_{ij+1/2k}^T \geq 0, \quad (D_+^z \mathbf{q}_{ijk})^T \mathbf{H}_{ijk+1/2}^T \geq 0.
$$

Proof. The heat fluxes are defined as

$$
\mathbf{F}_{i+1/2jk} = (0, 0, 0, \kappa D_+^x T_{ijk})^T.
$$

Suppressing the jk indices, we have $D_+^x T_i = T_{i+1} T_i D_+^x \frac{-1}{T_i}$. Then

(18)
$$
c_v (D_+^x \mathbf{q}_{ijk})^T \mathbf{F}_{i+1/2jk}^T = \kappa T_{ijk} T_{i+1jk} (D_+^x \frac{-1}{T_{ijk}})^2 \geq 0.
$$

The other statements are proved analogously. $\square$

We will use the notation $L^2(\Omega_N)$ to denote the discrete L^2(-equivalent when $h \to 0$) space. It is equipped with the norm, $\|u^h\|_2^2 = \sum_{ijk=0}^N h^3 u_{ijk}^2$ where u^h denotes the entire vector of (in this case x-velocity) values u_{ijk}. (The superscript h distinguishes the discrete from the corresponding continuous variable.)

Assumption 3.3. *The initial data are projections of the initial data given in Assumption 2.1 onto the grid. That is $\mathbf{u}^h(0)_{ijk} = \mathbf{u}(0, \mathbf{x}_{ijk})$. Hence, the discrete initial data reside in the equivalent discrete spaces.*

Proposition 3.4. *Assume that inital data are given as in Assumption 3.3. Assume that $T^h(t), \rho^h(t) \geq 0$, $t \in [0, \mathcal{T}]$, then the scheme (15) is entropy stable and its solutions satisfy $\mathbf{u}^h \in C(0, \mathcal{T}; (L^2(\Omega_N))^5)$ and $p^h, (\rho(u^2 + v^2 + w^2))^h \in C(0, \mathcal{T}; L^2(\Omega_N))$. Furthermore, $\log(T^h) \in L^2(0, \mathcal{T}; H^1(\Omega_N))$.*

Proof. Multiplying the scheme by $\mathbf{q}_{ijk}^T h^3$ and summing in space, lead to,

$$
\sum_{i,j,k=0}^N h^3 (U_{ijk})_t + \sum_{i,j,k=0}^N h^3 \mathbf{q}_{ijk}^T (D_-^x \mathbf{f}_{i+1/2jk} + D_-^y \mathbf{g}_{ij+1/2k} + D_-^z \mathbf{h}_{i,j,k+1/2}) =
$$

$$
\sum_{i,j,k=0}^N h^3 \mathbf{q}_{ijk}^T (D_-^x \mathbf{F}_{i+1/2jk} + D_-^y \mathbf{G}_{ij+1/2k} + D_-^z \mathbf{H}_{ijk+1/2}).
$$

Sum by parts and use periodicity and the entropy stability property of the inviscid fluxes.

$$\sum_{i,j,k=0}^{N} h^3(U_{ijk})_t \leq$$

(19)
$$-\sum_{i,j,k=0}^{N-1} h^3 \left((D_+^x \mathbf{q}_{ijk})^T \mathbf{F}_{i+1/2jk} + (D_+^y \mathbf{q}_{ijk})^T \mathbf{G}_{ij+1/2k} + (D_+^z \mathbf{q}_{ijk})^T \mathbf{H}_{ijk+1/2} \right).$$

Thanks to Lemmas 3.1 and 3.2, we obtain

$$\sum_{i,j,k=0}^{N} h^3(U_{ijk})_t \leq 0.$$

To obtain an L^2 bound on the variables, we repeat the calculation for the entropy $\bar{U} = U - U(\mathbf{u}_0) - U'(\mathbf{u}_0)^T(\mathbf{u} - \mathbf{u}_0)$ where $\mathbf{u}_0$ is a constant state. (This is an affine change, which ensures that $\bar{U}$ is an entropy.) We choose the constant state as $\mathbf{u}_0^1 = \rho_0 > 0$, $\mathbf{u}_0^{2,3,4} = 0$ and $\mathbf{u}_0^5 = E_0 > 0$. This corresponds to a state at rest with constant density, temperature and pressure.

The entropy $\bar{U}$ satisifes the analog estimate (19). We can recast this as

$$\frac{1}{2} \sum_{i,j,k=0}^{N} \{h^3(\mathbf{u} - \mathbf{u}_0)^T U''(\theta(\mathcal{T})))(\mathbf{u} - \mathbf{u}_0)\}_{ijk}$$

$$+ \int_0^{\mathcal{T}} \sum_{i,j,k=0}^{N} h^3 \left((D_+^x \mathbf{q}_{ijk})^T \mathbf{F}_{i+1/2jk} + (D_+^y \mathbf{q}_{ijk})^T \mathbf{G}_{ij+1/2k} \right.$$

$$\left. + (D_+^z \mathbf{q}_{ijk})^T \mathbf{H}_{ijk+1/2} \right) dt \leq$$

(20)
$$\sum_{i,j,k=0}^{N} h^3 \bar{U}(\mathbf{u}^h(0)).$$

Observe that $U''(\theta(t))$, for $t \in [0, \mathcal{T}]$, is symmetric positive definite, since $\theta(t)$ is an intermediate state between $\mathbf{u}$ and $\mathbf{u}_0$. Hence, the thermodynamic variables of $\theta(t)$ are positive and bounded away from 0 since we have assumed that $\rho_{ijk} \geq 0, T_{ijk} \geq 0$. Hence, we obtain an L^2 bound on $\mathbf{u}$ (continuosly in time). (This argument was given in [Daf00] and also presented in detail in [Svä15].)

Turning to the estimate of $\log(T^h)$, we will use the estimate of the viscous terms. We begin with positivity of temperature. From the estimate (20) and (18), we have an estimate on

(21)
$$\int_0^T \sum_{ijk=0}^{N-1} h^3 T_{ijk} T_{i+1jk} (D_+^x \frac{-1}{T_{ijk}})^2 + \sum_{ijk=0}^{N-1} h^3 T_{ijk} T_{ij+1k} (D_+^y \frac{-1}{T_{ijk}})^2$$

$$+ \sum_{ijk=0}^{N-1} h^3 T_{ijk} T_{ijk+1} (D_+^z \frac{-1}{T_{ijk}})^2 \, dt \leq \mathcal{C}.$$

For $\xi_1, \xi_2 > 0$, the log average is $\xi_{log} = (\xi_1 - \xi_2)/(\log(\xi_1) - \log(x_2))$. The geometric average is $\xi_{geo} = \sqrt{\xi_1 \xi_2}$ and $\xi_{geo} \leq \xi_{log}$. Since

$$D_+ \log(T_i) = \frac{1}{T_{log,i+1/2}} D_+ T_i = \frac{1}{T_{log,i+1/2}} T_{i+1} T_i D_+ \frac{-1}{T_i},$$

and $T_{i+1}T_i/T_{log,i+1/2} \leq T_{geo,i+1/2}$, the bound (21) implies an estimate of $D_+ \log(T^h)$. By Poincare's inequality (noting that T^h has to be bounded and non-zero on a finite subset) we obtain $\log(T^h) \in L^2(0,\mathcal{T};L^2(\Omega_N))$. (See Appendix II.) Therefore, $T^h > 0$ a.e. as $h \to 0$ and specifically it implies a lower positive bound on T^h for every fixed h.

$\square$

Note that thanks to the estimate on $\log(T^h)$, it is only necessary to assume positivity of the density. Then positivity of the temperature follows. Furthermore, it ensure that for every fixed h, there is an upper bound on the temperature. (Of course, not uniformly as $h \to 0$.)

3.2. Kinetic energy.

Proposition 3.5. *Let the initial data satisfy Assumption 3.3 and $\rho_{ijk} \geq 0$ for $t \in [0,\mathcal{T}]$. Then a semi-discrete solution of (15) satisfies*

$$u^h, v^h, w^h \in L^2(0,\mathcal{T};H^1(\Omega_N)).$$

Proof. Multiply x,y,z-momentum equations by $u_{ijk}, v_{ijk}, w_{ijk}$, respectively and sum the resulting equations.

$$u_{ijk}(\rho_{ijk}u_{ijk})_t + u_{ijk}D_-^x\left(\mathbf{f}_{i+1/2jk}^2\right) + u_{ijk}D_-^y\left(\mathbf{g}_{ij+1/2k}^2\right) + u_{ijk}D_-^z\left(\mathbf{h}_{ijk+1/2}^2\right)$$

$$+v_{ijk}(\rho_{ijk}v_{ijk})_t + v_{ijk}D_-^x\left(\mathbf{f}_{i+1/2jk}^3\right) + v_{ijk}D_-^y\left(\mathbf{g}_{ij+1/2k}^3\right) + v_{ijk}D_-^z\left(\mathbf{h}_{ijk+1/2}^3\right)$$

$$+w_{ijk}(\rho_{ijk}w_{ijk})_t + w_{ijk}D_-^x\left(\mathbf{f}_{i+1/2jk}^4\right) + w_{ijk}D_-^y\left(\mathbf{g}_{ij+1/2k}^4\right) + w_{ijk}D_-^z\left(\mathbf{h}_{ijk+1/2}^4\right) =$$

$$u_{ijk}D_-^x\mathbf{F}_{i+1/2jk}^2 + u_{ijk}D_-^y\mathbf{G}_{ij+1/2k}^2 + u_{ijk}D_-^z\mathbf{H}_{ijk+1/2}^2$$

$$+v_{ijk}D_-^x\mathbf{F}_{i+1/2jk}^3 + v_{ijk}D_-^y\mathbf{G}_{ij+1/2k}^3 + v_{ijk}D_-^z\mathbf{H}_{ijk+1/2}^3$$

$$+w_{ijk}D_-^x\mathbf{F}_{i+1/2jk}^4 + w_{ijk}D_-^y\mathbf{G}_{ij+1/2k}^4 + w_{ijk}D_-^z\mathbf{H}_{ijk+1/2}^4$$

$$(22) \hspace{8cm} (:= RHS_{ijk})$$

The last equality is a definition and will be used as a short-hand notation for the viscous terms. Next, we recast the temporal derivatives as

$$(23) \qquad u_{ijk}(\rho_{ijk}u_{ijk})_t = \left(\frac{1}{2}(\rho_{ijk}u_{ijk}^2)_t + (\rho_{ijk})_t\frac{u_{ijk}^2}{2}\right), \quad \text{etc.}$$

We introduce the kinetic energy $K_{ijk} = \frac{1}{2}\rho_{ijk}(u_{ijk}^2 + v_{ijk}^2 + w_{ijk}^2)$ and use the continuity equation to obtain.

$$(K_{ijk})_t + (-D_-^x\mathbf{f}_{i+1/2jk}^1 - D_-^y\mathbf{g}_{ij+1/2j}^1 - D_-^z\mathbf{h}_{ijk+1/2}^1)\frac{u_{ijk}^2 + v_{ijk}^2 + w_{ijk}^2}{2}$$

$$+u_{ijk}D_-^x\left(\mathbf{f}_{i+1/2jk}^2\right) + u_{ijk}D_-^y\left(\mathbf{g}_{ij+1/2k}^2\right) + u_{ijk}D_-^z\left(\mathbf{h}_{ijk+1/2}^2\right)$$

$$+v_{ijk}D_-^x\left(\mathbf{f}^3_{i+1/2jk}\right) + v_{ijk}D_-^y\left(\mathbf{g}_{ij+1/2k}^3\right) + v_{ijk}D_-^z\left(\mathbf{h}_{ijk+1/2}^3\right)$$

$$+w_{ijk}D_-^x\left(\mathbf{f}^4_{i+1/2jk}\right) + w_{ijk}D_-^y\left(\mathbf{g}_{ij+1/2k}^4\right) + w_{ijk}D_-^z\left(\mathbf{h}_{ijk+1/2}^4\right) = RHS_{ijk}$$

Sum over all points in space. Next, we sum by parts and use a discrete chain rule on the square velocity terms. (That is, $D_+^x u_{i+1/2jk}^2 = 2u_{i+1/2jk}D_+^x u_{ijk}$, where

$u_{i+1/2jk} = (u_{i+1jk} + u_{ijk})/2.)$

$$\sum_{ijk=0}^{N} h^3 (K_{ijk})_t$$

$$+ \sum_{ijk=0}^{N} h^3 \left(\mathbf{f}^1_{i+1/2jk} \left(u_{i+1/2jk} D^x_+ u_{ijk} + v_{i+1/2jk} D^x_+ v_{ijk} + w_{i+1/2jk} D^x_+ w_{ijk} \right) \right.$$

$$+ \mathbf{g}^1_{ij+1/2k} \left(u_{ij+1/2k} D^y_+ u_{ijk} + v_{ij+1/2k} D^y_+ v_{ijk} + w_{ij+1/2k} D^y_+ w_{ijk} \right)$$

$$+ \mathbf{h}^1_{ijk+1/2} \left(u_{ijk+1/2} D^z_+ u_{ijk} + v_{ijk+1/2} D^z_+ v_{ijk} + w_{ijk+1/2} D^z_+ w_{ijk} \right)$$

$$- D^x_+ u_{ijk} \left(\mathbf{f}^2_{i+1/2jk} \right) - D^x_+ v_{ijk} \left(\mathbf{f}^3_{i+1/2jk} \right) - D^x_+ w_{ijk} \left(\mathbf{f}^4_{i+1/2jk} \right)$$

$$- D^y_+ u_{ijk} \left(\mathbf{g}^2_{ij+1/2k} \right) - D^y_+ v_{ijk} \left(\mathbf{g}^3_{ij+1/2k} \right) - D^y_+ w_{ijk} \left(\mathbf{g}^4_{ij+1/2k} \right)$$

$$\left. - D^z_+ u_{ijk} \left(\mathbf{h}^2_{ijk+1/2} \right) - D^z_+ v_{ijk} \left(\mathbf{h}^3_{ijk+1/2} \right) - D^z_+ w_{ijk} \left(\mathbf{h}^4_{ijk+1/2} \right) \right) =$$

$$+ \sum_{ijk=0}^{N} RHS_{ijk}$$

We have organized the flux terms in two 3×3-blocks. In the continous case, terms on the same position in the two blocks would cancel pairwise. In the discrete case, there will not be an exact cancellation. From each pair a rest term will emerge and we will show that this rest term is bounded by the artifical diffusion that is built into all fluxes.

Take the "$(1,1)$"-element

$$h^3 \sum_{ijk=0}^{N} \mathbf{f}^1_{i+1/2jk} u_{i+1/2jk} D^x_+ u_{ijk} - h^3 \sum_{ijk=0}^{N} D^x_+ u_{ijk} \left(\mathbf{f}^2_{i+1/2jk} \right).$$

The fluxes are defined as

$$\mathbf{f}^1_{i+1/2jk} = (\rho u)_{i+1/2jk} - \frac{\lambda_{i+1/2jk}}{2} (\rho_{i+1jk} - \rho_{ijk})$$

$$\mathbf{f}^2_{i+1/2jk} = (\rho u^2 + p)_{i+1/2jk} - \frac{\lambda_{i+1/2jk}}{2} ((\rho u)_{i+1jk} - (\rho u)_{ijk}).$$

In the expressions above, and the subsequent analysis, $i+1/2$ and $j+1/2$ indices of a variable indicates the arithmetic mean value between i and $i+1$. (Except in λ whose definition is given in (13).) Furthermore, we use the notation $\Delta a_{i+1/2} = a_{i+1} - a_i$. Rewriting the second one,

$$\mathbf{f}^2_{i+1/2jk} = u_{i+1/2jk} (\rho u)_{i+1/2jk} + \frac{1}{4} \Delta u_{i+1/2jk} \Delta (\rho u)_{i+1/2jk}$$

$$+ p_{i+1/2jk} - \frac{\lambda_{i+1/2jk}}{2} (\Delta(\rho u)_{i+1/2jk}),$$

leads to

$$h^3 \sum_{ijk=0}^{N} \mathbf{f}^1_{i+1/2jk} u_{i+1/2jk} D^x_+ u_{ijk} - h^3 \sum_{ijk=0}^{N} D^x_+ u_{ijk} \left(\mathbf{f}^2_{i+1/2jk} \right) =$$

$$h^3 \sum_{ijk=0}^{N} \left(-\frac{\lambda_{i+1/2jk}}{2} \Delta \rho_{i+1/2jk} \right) u_{i+1/2jk} D^x_+ u_{ijk}$$

$$- h^3 \sum_{ijk=0}^{N} \left(\frac{1}{4} \Delta u_{i+1/2jk} \Delta(\rho u)_{i+1/2jk} + p_{i+1/2jk} - \frac{\lambda_{i+1/2jk}}{2} \Delta(\rho u)_{i+1/2jk} \right) D^x_+ u_{ijk}.$$

Rewrite the second artificial diffusion term as,

$$(\rho u)_{i+1jk} - (\rho u)_{ijk} = \rho_{i+1/2jk}\Delta u_{i+1/2jk} + u_{i+1/2jk}\Delta\rho_{i+1/2jk}.$$

Then,

$$h^3 \sum_{ijk=0}^{N} \left(\mathbf{f}^1_{i+1/2jk}u_{i+1/2jk}D^x_+u_{ijk} - D^x_+u_{ijk}\left(\mathbf{f}^2_{i+1/2jk}\right)\right) =$$

$$-h^3 \sum_{ijk=0}^{N} \left(\frac{1}{4}\Delta u_{i+1/2jk}\Delta(\rho u)_{i+1/2jk} + p_{i+1/2jk} - \frac{\lambda_{i+1/2jk}}{2}\rho_{i+1/2jk}\Delta u_{i+1/2jk}\right) D^x_+u_{ijk} =$$

$$-h^3 \sum_{ijk=0}^{N} p_{i+1/2jk}D^x_+u_{ijk} - h^3 \sum_{ijk=0}^{N} \frac{1}{h}(\Delta u_{i+1/2jk})^2 \left(\frac{1}{4}\Delta(\rho u)_{i+1/2jk} - \frac{\lambda_{i+1/2jk}}{2}\rho_{i+1/2jk}\right).$$

Since $\lambda_{i+1/2jk} > |u_i|$ and $\lambda_{i+1/2jk} > |u_{i+1}|$, the last parenthesis must be negative (and consequently the whole term positive). Hence,

$$h^3 \sum_{ijk=0}^{N} \left(\mathbf{f}^1_{i+1/2jk}u_{i+1/2jk}D^x_+u_{ijk} - D^x_+u_{ijk}\left(\mathbf{f}^2_{i+1/2jk}\right)\right) > -h^3 \sum_{ijk=0}^{N} p_{i+1/2jk}D^x_+u_{ijk}.$$

To demonstrate that all terms can be treated in this way, we will also consider the "$(2,1)$"-term, where both u and v velocities appear.

$$h^3 \sum_{ijk=0}^{N} \left(\mathbf{g}^1_{ij+1/2k}u_{ij+1/2k}D^y_+u_{ijk} - D^y_+u_{ijk}\left(\mathbf{g}^2_{ij+1/2k}\right)\right)$$

The fluxes are defined as

$$\mathbf{g}^1_{ij+1/2jk} = (\rho v)_{ij+1/2k} - \frac{\lambda_{ij+1/2k}}{2}(\rho_{ij+1,jk} - \rho_{ijk}),$$

$$\mathbf{g}^2_{ij+1/2jk} = (\rho uv)_{ij+1/2k} - \frac{\lambda_{ij+1/2k}}{2}((\rho u)_{i,j+1,k} - (\rho u)_{ijk}).$$

Rewriting the second one,

$$\mathbf{g}^2_{ij+1/2jk} = u_{ij+1/2k}(\rho v)_{ij+1/2k} + \frac{1}{4}\Delta u_{ij+1/2k}\Delta(\rho v)_{ij+1/2k} - \frac{\lambda_{ij+1/2k}}{2}\Delta(\rho u)_{ij+1/2k}.$$

Inserting into the sums

$$h^3 \sum_{ijk=0}^{N} \left(\mathbf{g}^1_{ij+1/2k}u_{ij+1/2k}D^y_+u_{ijk} - D^y_+u_{ijk}\left(\mathbf{g}^2_{ij+1/2k}\right)\right) =$$

$$h^3 \sum_{ijk=0}^{N} (-\frac{\lambda_{ij+1/2k}}{2}\Delta\rho_{ij+1/2k})u_{ij+1/2k}D^y_+u_{ijk}$$

$$-h^3 \sum_{ijk=0}^{N} D^y_+u_{ijk}\left(\frac{1}{4}\Delta u_{ij+1/2k}\Delta(\rho v)_{ij+1/2k} - \frac{\lambda_{i,j+1/2k}}{2}\Delta(\rho u)_{ij+1/2k}\right) =$$

$$-h^3 \sum_{ijk=0}^{N} D^y_+u_{ijk}\left(\frac{1}{4}\Delta u_{ij+1/2k}\Delta(\rho v)_{ij+1/2k} - \frac{\lambda_{ij+1/2k}}{2}\rho_{ij+1/2}\Delta u_{ij+1/2k}\right).$$

Since $\lambda_{ij+1/2jk} > |v_{ijk}|$ and $\lambda_{ij+1/2k} > |v_{ij+1k}|$, the last parenthesis must be negative and we conclude,

$$(24) \qquad h^3 \sum_{ijk=0}^{N} \left(\mathbf{g}^1_{ij+1/2k}u_{ij+1/2k}D^y_+u_{ijk} - D^y_+u_{ijk}\left(\mathbf{g}^2_{ij+1/2k}\right)\right) > 0.$$

All the pairs will result in a positive rest term (and on the diagonal also a term including the pressure.)

We can now make an intermediate summary using the above results for the inviscid fluxes. We have

$$h^3 \sum_{ijk=0}^{N} \left((K_{ijk})_t - p_{i+1/2jk} D_+^x u_{ijk} - p_{ij+1/2k} D_+^y v_{ijk} - p_{ijk+1/2} D_+^z w_{ijk} \right) \leq \sum_{ijk=0}^{N} RHS_{ijk},$$

or

$$\sum_{ijk=0}^{N} (K_{ijk})_t \leq \sum_{ijk=0}^{N} RHS_{ijk} + \frac{1}{\xi_1} \|p_{i+1/2}^h\|_2^2 + \xi_1 \|D_+^x u^h\|_2^2$$

$$(25) \qquad \qquad + \frac{1}{\xi_2} \|p_{j+1/2}^h\|_2^2 + \xi_2 \|D_+^y v^h\|_2^2 + \frac{1}{\xi_3} \|p_{k+1/2}^h\|_2^2 + \xi_3 \|D_+^z w^h\|_2^2,$$

where $\xi_{1,2,3} > 0$ are parameters. The averaged index $(i+1/2, j+1/2, k+1/2)$ in the pressure is kept to signify that the variables are different. However, all the three L^2 norms can be estimate by the L^2 estimate of the pressure given in Proposition 3.4. To bound the velocity gradients we need to use the viscous terms in RHS.

The expression for RHS as defined in (22) is summed by parts.

$$\sum_{ijk=0}^{N} RHS_{ijk} = -h^3 \sum_{ijk=0}^{N} \Big(D_+^x u_{ijk} \mathbf{F}^2_{i+1/2,j,k} + D_+^y u_{ijk} \mathbf{G}^2_{i,j+1/2k} + D_+^z u_{ijk} \mathbf{H}^2_{ijk+1/2} $$

$$+ D_+^x v_{ijk} \mathbf{F^3}_{i+1/2jk} + D_+^y v_{ijk} \mathbf{G}^3_{ij+1/2k} + D_+^z v_{ijk} \mathbf{H}^3_{i,j,k+1/2}$$

$$+ D_+^x w_{ijk} \mathbf{F^4}_{i+1/2jk} + D_+^y w_{ijk} \mathbf{G}^4_{ij+1/2k} + D_+^z w_{ijk} \mathbf{H}^4_{ijk+1/2} \Big)$$

Then we employ the short-hand notation for all the differences and we drop the indices since all operators act on ijk.

$$\sum_{ijk=0}^{N} RHS_{ijk} = -h^3 \sum_{ijk=0}^{N} \mu \left(\mathfrak{u}_x (\frac{4}{3}\mathfrak{u}_x - \frac{2}{3}(\mathfrak{v}_y + \mathfrak{w}_z)) + \mathfrak{u}_y(\mathfrak{u}_y + \mathfrak{v}_x) + \mathfrak{u}_z(\mathfrak{w}_x + \mathfrak{u}_z) \right.$$

$$+ \mathfrak{v}_x(\mathfrak{u}_y + \mathfrak{v}_x) + \mathfrak{v}_y(\frac{4}{3}\mathfrak{v}_y - \frac{2}{3}(\mathfrak{u}_x + \mathfrak{w}_z)) + \mathfrak{v}_z(\mathfrak{v}_z + \mathfrak{w}_y)$$

$$\left. + \mathfrak{w}_x(\mathfrak{u}_z + \mathfrak{w}_x) + \mathfrak{w}_y(\mathfrak{v}_z + \mathfrak{w}_y) + \mathfrak{w}_z(\frac{4}{3}\mathfrak{w}_z - \frac{2}{3}(\mathfrak{u}_x + \mathfrak{v}_y)) \right)$$

$$+ \eta \left(\mathfrak{u}_x + \mathfrak{v}_y + \mathfrak{w}_z \right)^2$$

This expression is the same as appeared in the entropy estimate and leads to the expression (17), and can in turn be recast as in (9). The discrete analog of Korn's inequality will bound all the three differences of all velocity components. (See Appendix I for a proof.) By choosing $\xi_{1,2,3}$ sufficiently small, we obtain a bound on the velocity gradients.

Furthermore, the velocity components can be bounded on the set where $\rho_{ijk} \geq c > 0$ thanks to the L^2 estimate of the momentum components. (The set of density bounded away from 0 is non-zero and non-vanishing as $h \to 0$ due to conservation of mass.) We conclude that $u^h, v^h, w^h \in L^2(0, T; H^1(\Omega_N))$.

$\square$

3.3. **Positivity.** The estimates derived in the previous section relies on $\rho_{ijk}(t) \geq 0$. In the first step to ensure that $U_{uu}(\theta) > 0$, where $0 < \theta^1 \leq \rho_0$ if $\rho_{ijk} < \rho_0$. From the entropy estimate, that is Prop. 3.4 we get estimate $\log(T) \in L^2(0, \mathcal{T}; L^2(\Omega_n))$. Hence, $T_{ijk} > 0$ a.e. as long as the initial data satisfies positivity of temperature and ρ remains non-negative. Similarly, we need $\rho \geq 0$ to derive the estimate on

kinetic energy (Prop. 3.5) which leads to the estimates on the velocities. Therefore, we must show that the semi-discrete scheme implies $\rho \geq 0$. The approximation of the continuity equation is,

$$(\rho_{ijk})_t + D_-^x \left((m^1)_{i+1/2jk} - \frac{\lambda_{i+1/2jk}}{2}(\rho_{i+1jk} - \rho_{ijk}) \right)$$

$$+ D_-^y \left((m^2)_{ij+1/2k} - \frac{\lambda_{ij+1/2k}}{2}(\rho_{i,j+1,k} - \rho_{ijk}) \right)$$

$$+ D_-^z \left((m^3)_{1/2jk+1/2} - \frac{\lambda_{ijk+1/2}}{2}(\rho_{ijk+1} - \rho_{ijk}) \right) = 0.$$

Consider a solution up to a time τ, where $\rho(t) \geq 0$ for all $t \in (0, \tau]$. Hence all the theorems derived above are true on this time interval. Next, we will outline the argument keeping in mind that h is fixed for a particular approximation.

Assume that ρ_{ijk} is the global (positive) minimum. For simplicity we assume that it is a unique minimum in this neighborhood. (It is not difficult to generalize the argument by noting that a non-unique minimum, i.e., a number of neighboring points contain points whose neighbors are greater than the minimum.) Rewrite (26) by using a discrete Leibniz rule on the flux terms and use that ρ_{ijk} is less than its neighbors. (See [TX00]),

$$(\rho_{ijk})_t + \left(\frac{(u_{i+1ij} - u_{ijk})\rho_{ijk} - \rho_{ijk}(u_{ijk} - u_{i-1jk})}{2h} \right)$$

$$+ \left(\frac{(v_{ij+1j} - v_{ijk})\rho_{ijk} - \rho_{ijk}(v_{ijk} - v_{ij-1k})}{2h} \right)$$

$$+ \left(\frac{(w_{ijk+1} - w_{ijk})\rho_{ijk} - \rho_{ijk}(w_{ijk} - w_{ijk-1})}{2h} \right) \geq 0.$$

Since $\rho_{ijk} \geq 0$, the velocity gradients D_+u_{ijk}, D_+v_{ijk} and D_+w_{ijk} are all bounded in L^2. Furthemore h is fixed, the bounds on the velocity gradients imply an upper bound on $D_+^{x,y,z}u_{ijk}$ in all points. Hence, if we let $\rho_{ijk} \to 0$, the terms involving the velocity gradients, e.g.

$$\lim_{\rho_{ijk} \to 0^+} (u_{i+1ij} - u_{ijk})\rho_{ijk} = 0 \quad \text{(h fixed)}.$$

By making the same argument in the two remaining directions, we conclude that $(\rho_{ijk})_t \geq 0$, implying that $(\rho_{ijk}) \geq 0$ in a neighborhood of $t = \tau$. Hence, we can extend our a priori bounds beyond $t = \tau$, and repeat the argument till we reach any finite time $\mathcal{T}$ (for all $h > 0$).

We summarize the results of this section.

Lemma 3.6. *Let the initial data satisfy Assumption 3.3. Then a semi-discrete solution of (15) satisfies $\rho_{ijk}(t), T_{ijk}(t) \geq 0$ for all $t \in [0, \mathcal{T}]$.*

4. SOLVABILITY OF THE ODE SYSTEM

The semi-discrete system constitute a system of ODEs, $\mathbf{u}_t^h = \mathcal{F}(\mathbf{u}^h)$, where $\mathcal{F}(\mathbf{u}^h)$ symbolizes the spatial discretization of (15). We know that the a priori estimates can be extended to any finite time $\mathcal{T}$. From these estimates it is straightforward to show that for a given grid size h, the function $\mathcal{F}$ is Lipshitz continuous and hence there exists a unique solution on the arbitrary, but finite, interval $[0, \mathcal{T}]$.

Consequently, we can generate a sequence of solutions $\mathbf{u}^h$ satisfying the a priori bounds given in Proposition 3.4 and Prop. 3.5. This sequence also satisfies $\rho^h(t) \geq 0, T^h(t) \geq 0$ by Lemma 3.6.

The reason we stress the solvability of the scheme is because it provides numerical robustness. The scheme will always produce a solution up to any finite

time $\mathcal{T}$. However, to prove convergence we need a sufficiently strong estimate on temperature. We propose a condition that can be examined a posteriori.

Lemma 4.1. *If $\rho^h(t) \geq \epsilon > 0$, uniformly as $h \to 0$, then $T^h \in L^2(\Omega_N)$.*

Proof. By the gas law: $T^h \leq p^h/(R\epsilon) \in L^2(\Omega)$. $\hfill\square$

Remark We have chosen to use the condition $\rho^h(t) \geq \epsilon > 0$, since the Navier-Stokes equations break down for large Knudsen numbers, i.e., well before vacuum is reached. (Mathematically, we could just as well have chosen a condition like $T^h(t) \leq Constant$ on sets where $\rho^h(t) = 0$.)

With the estimate on temperature, we can now bound the artifical diffusion terms in the inviscid fluxes.

Lemma 4.2. *Under the assumptions of Prop. 3.4 and Lemma 4.1, $h\lambda_{ijk}D_+^{x,y,z}u_{ijk}^k \in L^1(0,\mathcal{T};L^1(\Omega_N)^5)$, $k = 1...5$.*

Proof. First, λ depends on velocity and the speed of sound, i.e., $\sqrt{T}$. All velocity compontents and $\sqrt{T}$ are bounded in $L^2(0,\mathcal{T};L^2(\Omega_N))$ by Prop. 3.5 and Lemma 4.1. Furthermore, $hD_+u_{ijk}^k$ are bounded thanks to $\mathbf{u}^h \in L^2(0,\mathcal{T};L^2(\Omega_N))$ by Prop. 3.4. The result follows by Cauchy-Schwarz. $\hfill\square$

In smooth regions the effect of the artificial diffusion will vanish as $h \to 0$.

4.1. **Weak solutions.** Multiplying (15) by test functions φ (projected onto the grid) it is straightforward to move the spatial differences onto the test function using summation by parts and periodicity. The time derivative is moved to the test function with integration by parts as usual. Hence, it suffices if the variables and fluxes are in $L^1(0,\mathcal{T},L^1(\Omega_N))$. Since $\mathbf{u}^h \in L^2$ and the domain is bounded, the variables are integrable L^1. The inviscid fluxes are given (essentially) as *velocity $\times$ conservative variable*. Using Cauchy-Schwarz and L^2 integrability of the velocities and conservative variables, we have the bound. Furthermore, the diffusive fluxes (F^v, G^v, H^v) are also integrable thanks to L^2 integrability of the velocities and their gradients. (In the energy equation we employ Cauchy-Schwarz.) The temperature flux is treated in the same way as described in Section 2.3. That is, the heat flux contains terms of the form

$$\int_0^T \sum_{ijk} h^3 \varphi_{ijk}D_-^x D_+^x \kappa T_{ijk}\, dt = \int_0^T \sum_{ijk} h^3 (D_+^x D_-^x \varphi_{ijk})\kappa T_{ijk}\, dt,$$

which is bounded in view of Lemma 4.1. Summability of the Lax-Friedrichs artificial diffusion follows from Lemma 4.2.

Theorem 4.3. *Assume that the initial data satisfy Assumption 3.3 and $\rho^h(t) \geq \epsilon > 0$ for all $t \in [0,\mathcal{T}]$. Then the scheme (15) generates a sequence of solutions $\mathbf{u}^h$ as $h \to 0$ on $\Omega_N \times [0,\mathcal{T}]$ for any finite time $\mathcal{T}$. Furthermore, a subsequence converges to a weak solution of (1).*

Proof. Under the given assumptions, all integrals are bounded and hence there is a convergent subsequence of $\mathbf{u}^h$ which satisfy our definition of a weak solution. $\hfill\square$

5. Conclusions

We have shown that weak solutions to the 3D Navier-Stokes equations exist on a periodic domain in space and a bounded interval in time. We needed two technical assumptions to achieve this.

The first assumption was that a non-zero, although possibly very small bulk viscosity must be present in the model. This assumption can be relaxed by proving a discrete and periodic equivalent of the Korn inequality given in [Dai06].

The second was that density remains bounded away from 0. This was necessary since we were not able to bound the temperature in vacuum regions. However, as discussed above, this condition is not necessary to generate the sequence of approximate solutions, so the condition can be examined *a posteriori*.

In summary, this work demonstrates the possibility to construct robust *numerical schemes that converges to weak solutions as long as vacuum states are avoided.*

Furthermore, we have used constant viscosity and heat conductivity coefficients. However, within the present framework it is easy to see that the choice,

$$\mu \in C^2[0,\infty),\, 0 < \mu_{min} \leq \mu(T) \leq \mu_{max},$$

leads to the same results. Moreover, with the assumption that the temperature remains bounded in vacuum regions we obtain that $T \in L^2$. Hence and without violating the proofs, we can choose

$$\kappa(T) = \kappa_0 + \sum_{i=1}^{m} \kappa_i T^{\alpha_i}, \quad 0 < \alpha_i \leq 1$$

where m is an integerer and κ_i constants.

Designing a convergent scheme with no-slip wall boundary conditions that satisfy the corresponding estimates, is work in progress. Other type of boundary conditions appear to be more challenging.

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APPENDIX

I. Korn's inequality

I.1. **Continuous space.** This is minor modification of a theorem and proof given
in [CPS07].

Assume that the vector-valued function u is periodic on the domain Ω. (In all
directions and all d components.) Let $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $|\nabla u|^2 = \sum_{i,j=1}^{d} |u_{i,j}|^2$

Theorem I.1. *Under the above assumptions, the following holds,*

$$(26) \qquad \int_\Omega |\nabla u|^2 d\mathbf{x} \leq 2 \int_\Omega e_{ij} e_{ij} d\mathbf{x}$$

Proof. We prove it for $u(\mathbf{x}) \in (C^\infty_{periodic})^d$. The general result will hold thanks to a
density argument. The derivation below utilizes repeated integration by parts and
the fact that boundary terms cancel thanks to periodicity.

$$2 \int_\Omega e_{ij} e_{ij} d\mathbf{x} = \frac{1}{2} \int_\Omega \sum_{i,j=1} (u_{i,j} + u_{j,i})^2 d\mathbf{x} =$$

$$\int_\Omega \sum_{i,j=1} (u_{i,j}^2 + u_{i,j} u_{j,i}) d\mathbf{x} = \int_\Omega |\nabla u|^2 d\mathbf{x} - \int_\Omega \sum_{i,j=1}^{d} u_j u_{j,ij} d\mathbf{x} =$$

$$\int_\Omega |\nabla u|^2 d\mathbf{x} + \int_\Omega \sum_{i,j=1}^{d} u_{i,i} u_{j,j} d\mathbf{x} =$$

$$\int_\Omega |\nabla u|^2 d\mathbf{x} + \int_\Omega (u_{1,1} + u_{2,2} + ... u_{d,d})^2 d\mathbf{x} \geq \int_\Omega |\nabla u|^2 d\mathbf{x}$$

Since $C^\infty_{periodic}$ is dense in $H^1_{periodic}$ the result follows.

$\square$

I.2. **Discrete space.** Under the corresponding assumptions as in the previous sec-
tion, i.e., boundedness of gradients and periodicity, the discrete counterpart of the
theorem holds. The proof is exactly the same with summation by parts in place of
integration by parts. Periodicity will cancel boundary terms in the same way.

II. Discrete Poincare Inequality

Theorem II.1. *Let $\|D^x_+ u\|$, $\|D^y_+ u\|$, $\|D^z_+ u\|$ be bounded. Let $B \subset \Omega$ be a non-
vanishing subset on which $u \in L^2(B)$. Then*

$$\|u\|_2^2 \leq C(\|D^x_+ u\|^2 + \|D^y_+ u\|^2 + \|D^z_+ u\|^2) + c\|u\|_{L^2(B)}^2.$$

Proof. We consider the periodic domain and to reduce notation we consider the two-dimensional case. Furthermore, we will say "almost every" to denote a set on which only a vanishingly small subset is excluded as $h \to 0$. For instance, a discrete L^2 bound will bound u almost everywhere, although for any finite h, the bound is "everywhere".

To reduce notation, we will assume that B is the rectangular set $x_1 \leq x \leq x_2$ and $y_1 \leq y \leq x_2$ wich translates to indices $N_1 \leq i \leq N_2$ and $M_1 \leq j \leq M_2$.

Remark Note that the values of $N_{1,2}$, $M_{1,2}$ change with h and the number of points in between them increase with smaller h. Note that the proof of course holds in the general case of non-consecutive points but it would entail complicated labelling of points.

In the set B we will find bounded points on every line in the x-direction and every line in the y-direction that crosses the set B. To further simplify notation, we relabel the points such that $N_1 = 0$ and $M_1 = 0$ are bounded. (This can be done thanks to periodicity but this is merely for convenient notation. The proof holds without periodicity as long as Ω is a bounded domain.)

$$\sum_{j=0}^{M_2} \sum_{i=0}^{N} h^2 u_{ij}^2 =$$

$$\sum_{j=0}^{M_2} \sum_{i=0}^{N} h^2 D_+^x x_{ij} u_{ij}^2 =$$

$$\sum_{j=0}^{M_2} \left(-x_{0j} u_{0j}^2 h + x_{M_1 j} u_{M_1 j}^2 h - \sum_{i=0}^{N} h^2 x_{ij} D_-^x u_{ij}^2 \right) =$$

$$(27) \qquad \sum_{j=0}^{M_2} \left(-x_{0j} u_{0j}^2 h + x_{M_1 j} u_{M_1 j}^2 h - \sum_{i=0}^{N} h^2 x_{ij} u_{i-1/2j} D_-^x u_{ij} \right)$$

For (almost every) $0 \leq j \leq M_2$, u_{0j} and $u_{M_2 j}$ are bounded and hence their sums are bounded. Consequently, the above expression implies that $\sum_{j=0}^{M_2} \sum_{i=0}^{N} h^2 u_{ij}^2$ is bounded by $\|u\|_B$ and $\|D_+^x u\|$.

Next, we define the set B' which contains all points with $0 \leq i \leq N$ and $0 \leq j \leq M_2$. (I.e. a strip along the x-axis and a subset of y values.) u is bounded (a.e.) on B' and in particular $u \in L^2(B')$. We can repeat the derivation above for with B' instead of B and summing by parts in the j index instead of i. This way, we can extend the estimate (now depending also on $\|D_+^y u\|$) to the entire domain Ω. (In 3D we repeat this process once more.)

$\square$