Kernels of digraphs with finitely many ends

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Abstract

According to Richardson's theorem, every digraph G without directed odd cycles that is either (a) locally finite or (b) rayless has a kernel (an independent subset K with an incoming edge from every vertex in $G - K$). We generalize this theorem showing that a digraph without directed odd cycles has a kernel when (a) for each vertex, there is a finite set separating it from all rays, or (b) each ray contains at most finitely many vertices dominating it (having an infinite fan to the ray) and the digraph has finitely many ends. The restriction to finitely many ends in (b) can be weakened, admitting infinitely many ends with a specific structure, but the possibility of dropping it remains a conjecture.

Keywords: digraph kernel, infinite digraph, end of a digraph.

1 Introduction

A *kernel* of a digraph is an independent subset K of vertices with an incoming edge from every vertex $v \notin K$. The problem whether there exists a kernel for a given digraph is difficult. It is NP-complete for finite digraphs, Σ^1_1-complete for recursive ones and, in general, equivalent to consistency of theories in infinitary propositional logic [2, 1]. One can therefore hardly expect any simple characterization and most cases only specify sufficient conditions for kernel existence. The fundamental result, due to Richardson, is the following theorem from [8]. In this paper, "graph" means digraph unless stated otherwise, and all related terms like cycle, path, etc. refer to their directed versions. A *ray* is an infinite outgoing simple path.

Theorem 1.1 *A graph without odd cycles has a kernel if (a) each vertex has finite outdegree or (b) the graph has no rays.*

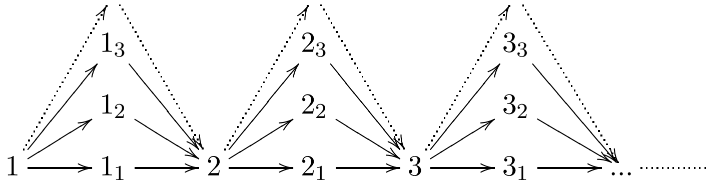
In particular, each finite graph without odd cycles has a kernel. For infinite graphs, one excludes rays or vertices with infinite outdegree, but these are very restrictive conditions. Few results, weakening these conditions for infinite graphs, identify specific classes possessing kernels but do not suggest any common pattern preventing their existence [3, 5, 6, 9]. The recurring example of an infinite graph without a kernel (nor odd cycle) is the countably infinite, acyclic tournament without a winner, $\langle \omega, < \rangle$. Motivated by its multiple variants, we propose Conjecture 1.2 below, using the following notions. A vertex v *dominates* a ray R if there exist infinitely many disjoint, except for v, paths from v to R. A graph is *safe* if it has no odd cycles nor any ray containing infinitely many vertices dominating it, and it is *kernel-perfect*, KP, if every induced subgraph has a kernel.

Conjecture 1.2 *Every safe graph is kernel-perfect.*

The paper proves this conjecture for graphs with finitely many ends and for some classes with infinitely many ends, where an *end* of a graph is the subgraph induced by all vertices with a path to some specific ray of the graph. This notion is coarser than that from [12], so graphs with finitely many ends, as defined there, have also finitely many ends in our sense, providing a special case of our main result:

Theorem 3.19 *A safe graph with finitely many ends is kernel-perfect.*

Unlike Richardson's result, Theorem 3.19 covers many graphs without odd cycles that have both rays and vertices with infinite outdegrees. For instance, in the graph from (1.3), every vertex $n \in \omega$ branches to infinitely many vertices $\{n_i \mid i \in \omega\}$, all with an edge (or a path) to the following vertex $n+1$. Uncountably many rays and infinite outdegree of each vertex $n \in \omega$ notwithstanding, each vertex n and n_i is separated from tails of all rays by vertex m, for each $m > n$. Thus, no vertex dominates any ray and the graph, having no odd cycles and only one end, is KP.



(1.3)

To sketch the proof of Theorem 3.19, some notation and definitions are needed. The sets of vertices and edges of a graph G are denoted by V_G and A_G, so $G = \langle V_G, A_G \rangle$. We use the following notation:

A^*_G – the reflexive transitive closure of A_G;

$A^-_G = \{(y,x) \in V_G \times V_G \mid (x,y) \in A_G\}$ – the converse of A_G;

$A^{\overset{*}{-}}_G$ – the reflexive transitive closure of A^-_G;

$E(x) = \{y \in V_G \mid (x,y) \in E\}$, for $x \in V_G$ and $E \subseteq V_G \times V_G$;

$E(X) = \bigcup_{x \in X} E(x)$ and $E[X] = E(X) \cup X$, for $X \subseteq V_G$ and $E \subseteq V_G \times V_G$.

An end, determined by a ray R, is the subgraph induced by $A^{\overset{*}{-}}_G(V_R)$. The graph G in (1.3) has only one end, since $A^{\overset{*}{-}}_G(V_R) = A^{\overset{*}{-}}_G(V_Q)$ for each pair R and Q of rays. Denoting by $G[X]$ the subgraph of G induced by X, for $X \subseteq V_G$, an end determined by a ray R should be denoted by $G[A^{\overset{*}{-}}_G(V_R)]$, but writing occasionally X for $G[X]$ simplifies notation, hopefully, without creating any confusion. By $H \sqsubseteq G$, we denote that H is an induced subgraph of G. Set difference is denoted by $X \setminus Y$, while for a graph G and $X \subseteq V_G$, the induced subgraph $G[V_G \setminus X]$ is denoted by $G - X$.

A subset V_H of V_G (or a subgraph H of G) is *free in* G if $A_G(V_H) \subseteq V_H$. A *tail* of a graph G is a nonempty induced subgraph T, free in G and such that $G - V_T$ has no rays.

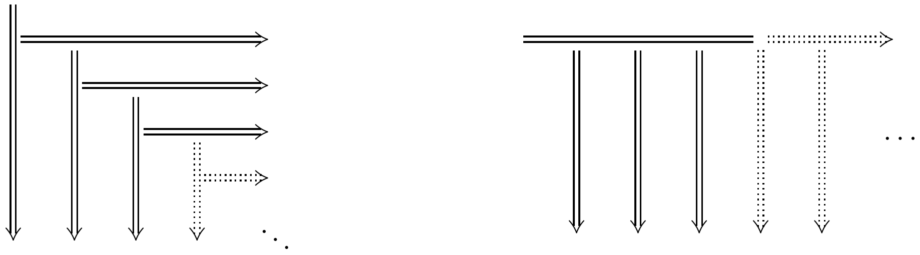
Theorem 3.19 follows from a more general result, Theorem 3.1, according to which a graph G is KP if there is a partition of its vertices, $V_G = \biguplus_{i \in I} V_{G_i}$, giving a KP induced subgraph $G[V_{G_i}]$ for each $i \in I$, and such that for each nonempty subset F of I, one subgraph $G[V_{G_k}]$, for some $k \in F$, is free in the subgraph induced by the union $\bigcup_{i \in F} V_{G_i}$.

Theorem 3.1 *A graph G is KP if there is a partition $V_G = \biguplus_{i \in I} V_{G_i}$ such that*

1. *for each $i \in I : G_i$ is KP, where $G_i = G[V_{G_i}]$, and*
2. *for each nonempty subset F of I, there is a $k \in F$ with G_k free in $G[\bigcup_{i \in F} V_{G_i}]$.*

This theorem enables a recursive construction of a kernel of G along free subgraphs, starting with a kernel of some G_0 free in the whole G and then, recursively, of a G_1 free in $G - V_{G_0}$, of a G_2 free in $G - (V_{G_0} \cup V_{G_1})$, etc.. Such kernels of free subgraphs can be combined into a kernel of the whole graph.

Theorem 3.19 follows by partitioning a safe graph with finitely many ends in the manner required by Theorem 3.1. More specifically, we apply the notion of a *flat* graph, namely, one where every tail of each ray has a path to every other ray, that is, $V_R \subseteq A^*_G(V_Q)$ and $V_Q \subseteq A^*_G(V_R)$ for every pair of rays R and Q. (A flat graph, like that in (1.3), has at most one end, but an end need not be flat.) The proof of Theorem 3.19 shows that (a) safe flat graphs are KP, and (b) vertices of a safe graph with finitely many ends can be partitioned so that the respective induced subgraphs are safe and flat, hence KP by (a), while the property (2) of Theorem 3.1 holds. Part (a), combined with Theorem 3.1, gives also kernel-perfectness of many graphs with infinitely many ends, for instance, safe graphs with countably many ends where each end is flat. Two examples are sketched in (1.4), with double arrows marking flat ends, which determine also the partition. (Vertices with infinite outdegree may occur anywhere as long as they do not violate safety or the end structure.)


(1.4)

Kernel-perfectness of safe flat graphs is shown by two cases. For rays Q and R satisfying $V_Q \subseteq A^*_G(V_R)$, a special situation occurs when every tail of Q reaches some vertex $r \in V_R$, that is, when $V_Q \subseteq A^*_G(r)$, denoted by $Q \;{}^{f}\!\preceq R$. If a flat graph contains a so related pair of rays, then it has a bipartite tail, which implies that it is KP.

Lemma 3.3 *A flat graph without odd cycles is KP if it has rays Q and R with $Q \;{}^{f}\!\preceq R$.*

The difficult part is the other case, which takes most of the proof.

Lemma 3.4 *A safe flat graph is KP if $Q \;{}^{f}\!\not\preceq R$ for each pair of rays Q and R.*

To show this, Definition 3.5 introduces finitary divisions, which allow to view a graph G as the limit $\bigcup_{i\in\omega} G_i$ of an ω-chain of rayless subgraphs with $G_i \subset G_{i+1} \subset G$ and where, for each $i \in \omega$, all paths leaving G_i intersect a finite subset of V_{G_i}. (Set operations/relations applied to graphs refer to their pointwise applications to the sets of vertices and edges.) By Corollary 3.9, a safe flat graph, containing no rays Q and R with $Q \;{}^{f}\!\preceq R$, has a tail $A^*_G(r)$ with a finitary division, for some vertex r. The proof of Lemma 3.4 is completed by showing that such a tail is KP, which follows by the last major result, Theorem 3.10.

Theorem 3.10 *A graph G with no odd cycles is KP if $G = A^*_G(v)$, for some $v \in V_G$, and G has a finitary division.*

The proof of this theorem uses compactness of Cantor space $\{\mathbf{1},\mathbf{0}\}^{V_G}$. Theorem 3.10 yields also Corollary 3.18 extending Richardson's Theorem 1.1.(a) to graphs without odd cycles where each vertex is finitely separable from tails of all rays. The graph in (1.3) exemplifies also this case, as does its generalization where each vertex $n \in \omega$, except 1, is replaced by finitely many vertices, each with edges from an arbitrary subset of $\{(n-1)_i \mid i \in \omega\}$ and to an arbitrary subset of $\{n_i \mid i \in \omega\}$. Such a graph can have uncountably many ends and is not covered by Theorem 3.1 or Theorem 3.19, but is KP by Corollary 3.18.

Section 2 introduces now the remaining notation, concepts and preliminary results, while Section 3 presents the proofs of the main statements.

2 Notation and preliminaries

Recall that graph-theoretical terms like graph, cycle, path, etc., refer to their directed versions. Paths are simple, and walks with repeated vertices are encountered only occasionally. We write $x \to y$ for $y \in A_G(x)$, $x \overset{*}{\to} y$ for $y \in A_G^*(x)$, and $\pi : x \overset{*}{\to} y$ for π being a path from x to y. When the terminal vertex of a path π and the initial vertex of a path ρ coincide, or there is an edge from the former to the latter, we denote by $\pi;\rho$ or $(\pi;\rho)$ the path π followed by the path ρ (which may be a walk). A walk π *intersects* X if $V_\pi \cap X \neq \varnothing$, *omits* X if $V_\pi \cap X = \varnothing$, and a walk $\pi : a \overset{*}{\to} b$ *crosses* X if $(V_\pi \setminus \{a,b\}) \cap X \neq \varnothing$.

A *ray* is an infinite, outgoing, simple path. For a ray R and $1 \leq i \in \omega$, R_i denotes the i-th vertex of R. A *ray R crosses* a set X if $(V_R \setminus \{R_1\}) \cap X \neq \varnothing$. A ray R has the associated total ordering $<_R$, given by $R_i <_R R_j$ if $i < j$. By $R^{[v}$ we denote the tail of the ray R from vertex v, $R^{[v} = \{x \in V_R \mid v \leq_R x\}$, by $R^{v]}$ its initial segment up to v, $R^{v]} = \{x \in V_R \mid x \leq_R v\}$, and $R^{v)} = \{x \in V_R \mid x <_R v\}$. For a finite subset X of V_G, by $\max_R(X)$ we denote the $<_R$-maximal vertex in $V_R \cap X$, if it exists, and the first vertex of R if $V_R \cap X = \varnothing$. The set of rays in a graph G is denoted by $\vec{G}$, and the set of rays starting at $x \in V_G$ by $\vec{x}$. The subgraph of G induced by all its rays, $G[\bigcup_{R\in\vec{G}} A_G^*(V_R)]$, is denoted by $G[\vec{G}]$.

The set of strong components (with at least two vertices) in a graph G is denoted by $SC(G)$. The subset $ter(G)$ of *terminal* (strong) components is $\{X \in SC(G) \mid A_G(X) = X\}$.

A *kernel* (introduced as a *solution* in [10]) of a graph G is a subset K of V_G such that $A_G^-(K) = V_G \setminus K$. A subset X of V_G *absorbs* a subset Y of V_G if $A_G^-(X) \supseteq Y$. Thus, K is a kernel of G if it satisfies two conditions:

$A_G^-(K) \subseteq V_G \setminus K$, that is, K is independent, and

$A_G^-(K) \supseteq V_G \setminus K$, that is, K absorbs its complement.

The set of kernels of G is denoted by $sol(G)$, and G is *solvable* when $sol(G) \neq \varnothing$. By the second inclusion, only the empty graph $\langle\varnothing,\varnothing\rangle$ has $\varnothing$ as kernel. Equivalently, a subset K of vertices is a kernel of G if

$\forall x \in V_G : \big(x \in K \Leftrightarrow A_G(x) \cap K = \varnothing\big)$,

which can be expressed as an assignment $\alpha \in \mathbf{2}^{V_G}$, where $\mathbf{2} = \{\mathbf{1},\mathbf{0}\}$, subject to the condition

$\forall x \in V_G : \alpha(x) = \prod_{y\in A_G(x)}(\mathbf{1} - \alpha(y))$. [1]

An assignment α is *correct at a vertex v* if $\alpha(v)$ satisfies this equation, and it is *correct on a subset X* of V_G if it is correct at every $x \in X$.

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[1] This condition determines models of the propositional theory $\{x \Leftrightarrow \bigwedge_{y\in A_G(x)} \neg y \mid x \in V_G\}$, which is actually a normal form for propositional theories. A model $\alpha \in \mathbf{2}^{V_G}$ of such a theory determines the kernel of G given by $\alpha^{\mathbf{1}} = \{v \in V_G \mid \alpha(v) = \mathbf{1}\}$. Thus kernel existence and logical consistency are equivalent problems, also for infinitary logic. Relations and applications to logic are investigated in [1, 11].

2.1 Some basic facts

Given a graph G, we define $sinks(G) = \{x \in V_G \mid A_G(x) = \varnothing\}$. All sinks of a graph are contained in each of its kernels, forcing their predecessors $A^-_G(sinks(G))$ out of every kernel. Such an inducing from sinks continues until it reaches a sinkless residuum G°, which has a kernel if and only if G has it [1]. The process is captured by the construction in Figure 2.1, which removes repeatedly sinks and their predecessors, forming successive subgraphs G_i. At each stage, the sinks of G_i form the set $\sigma^{\mathbf{1}}_i$ and are assigned $\mathbf{1}$, while their predecessors form $\sigma^{\mathbf{0}}_i$ and are assigned $\mathbf{0}$. The so induced partial assignment $\overline{\sigma}$ is defined by ordinal recursion and is correct on its whole domain, $dom(\overline{\sigma})$.

$$
\begin{array}{l}
V_0 = V_G \\
G_i = G[V_i] \\
\sigma^{\mathbf{1}}_i = sinks(G_i) \\
\sigma^{\mathbf{0}}_i = A^-_G(\sigma^{\mathbf{1}}_i) \cap V_i \\
V_{i+1} = V_i \setminus (\sigma^{\mathbf{1}}_i \cup \sigma^{\mathbf{0}}_i) \quad \text{and} \quad V_\lambda = \bigcap_{i<\lambda} V_i \text{ for limit } \lambda \\
V^\circ = \bigcap_i V_i \text{ and } G^\circ = G[V^\circ] \text{ is an induced (sinkless) subgraph} \\
\sigma^{\mathbf{v}} = \bigcup_i \sigma^{\mathbf{v}}_i, \text{ for } \mathbf{v} \in \mathbf{2}
\end{array}
$$

Figure 2.1: The induced assignment is $\overline{\sigma} = (\sigma^{\mathbf{0}} \times \{\mathbf{0}\}) \cup (\sigma^{\mathbf{1}} \times \{\mathbf{1}\})$.

Theorem 2.2 ([1]) *For every graph* $G : sol(G) = \{\alpha \cup \overline{\sigma} \mid \alpha \in sol(G^\circ)\}$.

We can also induce from a given assignment α to a subset H of V_G, obtaining its unique extension $\overline{\alpha}$ to a subset of $V_G \setminus H$. The process above is then run on the subgraph $G - H$, starting with

$$
\begin{array}{l}
\alpha^{\mathbf{1}}_0 = \{x \in H \mid \alpha(x) = \mathbf{1}\} \cup (sinks(G) \setminus H) \\
\alpha^{\mathbf{0}}_0 = \{x \in H \mid \alpha(x) = \mathbf{0}\} \cup (A^-_G(\alpha^{\mathbf{1}}_0) \setminus H).
\end{array}
\tag{2.3}
$$

The so induced $\overline{\alpha}$ is correct on $dom(\overline{\alpha}) \setminus H$. In particular, every assignment α to the sinks of a KP graph G can be extended to an assignment correct on $V_G \setminus sinks(G)$ by inducing $\overline{\alpha}$ and, if the remaining G° is nonempty, adding its arbitrary solution, which exists since G is KP. For a rayless DAG (acyclic digraph), such a (relative) solution is induced uniquely [10, 1].

Solutions must respect the induced values. Two solutions, coinciding on a set H, coincide also on the part induced from their restrictions to H. As the observation below shows, if $\alpha_1|_H = \beta = \alpha_2|_H$ and both α_1 and α_2 are correct on $dom(\overline{\beta})$, then $\alpha_1|_{dom(\overline{\beta})} = \alpha_2|_{dom(\overline{\beta})}$. This follows by uniqueness of the inducing from Figure 2.1, which assigns only values forced by the prior assignment.

Observation 2.4 *Given a graph* G, $\alpha \in \mathbf{2}^{V_G}$ *and* $H \subseteq V_G$, *let* $\beta = \alpha|_H$. *If* α *is correct on* $dom(\overline{\beta}) \setminus H$, *then* $\alpha|_{dom(\overline{\beta})} = \overline{\beta}$.

PROOF. Given $\beta^{\mathbf{v}} = \{x \in H \mid \beta(x) = \mathbf{v}\}$, for $\mathbf{v} \in \mathbf{2}$, all $x \in A^-_G(\beta^{\mathbf{1}})$ must obtain value $\mathbf{0}$ under any correct assignment, in particular, $\alpha(x) = \mathbf{0} = \overline{\beta}(x)$. Similarly, all $y \in V_G$ satisfying $A_G(y) \subseteq \beta^{\mathbf{0}}$ must obtain value $\mathbf{1}$ under any correct assignment, in particular, $\alpha(y) = \mathbf{1} = \overline{\beta}(y)$. The claim follows by obvious induction. □

For α, H and β as in Observation 2.4, if α is correct on $V_G \setminus H$, then it is a *solution to G relative to* β. The set of such solutions is denoted by $solr(G, \beta)$.

We often apply inducing implicitly, using the following observation where case (b) allows to ignore sinks and terminal components also without inducing all their consequences.

Observation 2.5 *(a) A graph G is KP if and only if it has a free induced subgraph T such that both T and $G - V_T$ are KP.*

(b) A graph G without odd cycles is KP if and only if $G[\vec{G}]$ is KP.

In (a), the implication to the left follows since every induced subgraph H of G can be solved by solving first $H[V_H \cap V_T]$, inducing values from this solution to $V_H \setminus V_T$ – since T is free in G, there are no edges in H from $V_H \cap V_T$ to $V_H \setminus V_T$ – and then solving the remaining part. The implication to the left of (b) follows from (a), since the induced subgraph of G not reaching any ray, $G - V_{G[\vec{G}]}$, is free in G and, being rayless and having no odd cycles, is KP by Theorem 1.1.

A *fan* from a vertex v to a set of vertices X is a set of paths starting at v, terminating at X without crossing it (having only the terminal vertex in common with X), and being disjoint except for the common source v. A fan to a subgraph H is a fan to V_H. A vertex v *dominates* a ray R if v has an infinite fan to R. The set of vertices dominating a ray R is denoted by $dmi(R)$.

- A *ray* R is *safe* if $V_R \cap dmi(R)$ is finite.
- A *graph* is *safe* if it has no odd cycles and no unsafe (not safe) rays.

The fundamental example of an unsafe – and unsolvable – DAG is $\langle \omega, < \rangle$. The mere absence of its subdivision is not sufficient for solvability of DAGs, as shown by the unsolvable graph in Figure 2.6, with edges $b_i \to c_i, c_i \to a_{i+1}$ and $a_i \to b_j$, for all $i \in \omega$ and $j \geq i$.

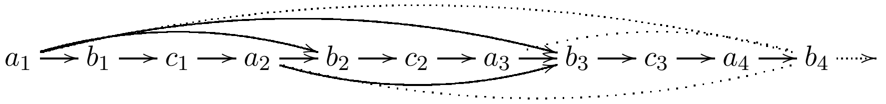

Figure 2.6: An unsolvable DAG without a subdivision of $\langle \omega, < \rangle$.

Fact 2.7 below characterizes safety using the following notion of finite separability. Let Q, F and R range over subsets of V_G in any graph G.

- *F separates Q from R* if the subgraph $G - F$ has no path from $Q \setminus F$ to $R \setminus F$, that is, if each path in G from Q to R intersects F.
- An *infinite Q* is *finitely separable from R* if there is a finite F separating Q from R.
- A *vertex q* is *finitely separable from R* if some finite F, not containing q, separates $\{q\}$ from $R \setminus \{q\}$.

Finite separability from R is trivial when R is finite – typically, it is not. Separation from a subgraph R refers to separation from V_R.

A vertex q dominates a ray R if and only if q is not finitely separable from R. If each path from q to R intersects a finite $F \not\ni q$, then there is no infinite fan from q to R. Conversely, if no finite set not containing q separates q from R, then each finite fan C from q to R can be extended with an additional path, disjoint (except for q) from all paths in C.

Fact 2.7 *A graph G without odd cycles is unsafe if and only if it has a ray R which is not finitely separable from the set $dmi(R)$ of vertices dominating R.*

PROOF. The implication to the right is obvious, since an unsafe ray gives a required R. For the opposite, assume a ray R as specified, that is, such that for every finite subset F of V_G there is a path omitting F from R to some vertex dominating R. To exclude the trivial case, let R contain only finitely many of such dominating vertices, that is, $V_R \cap dmi(R)$ is finite. Recall that for a finite subset X of V_G, $\max_R(X)$ denotes the $<_R$-maximal vertex in $V_R \cap X$, if it exists, and the first vertex of R if $V_R \cap X = \varnothing$.

We construct a ray Y containing infinitely many vertices $d_h \in Y, h \in \omega$, dominating it. It starts with a path $(\beta_1;\gamma_1) : s_1 \xrightarrow{*} r_1$, where s_1 is the first vertex of R and

β_1 : $s_1 \xrightarrow{*} d_1$ is a path to an arbitrary vertex $d_1 \in dmi(R)$; we set $D_1 = \{d_1\}$;

γ_1 : $d_1 \xrightarrow{*} r_1$ is a path with $V_{\gamma_1} \cap V_{\beta_1} = \{d_1\}$, where $r_1 \in V_R$ is any (e.g., $<_R$-minimal) vertex with $\max_R(V_{\beta_1}) <_R r_1$, reachable from d_1 by a path sharing only d_1 with β_1. Such an r_1 and path γ_1 exist because d_1, dominating R, is not separated from R by the finite set $V_{\beta_1} \setminus \{d_1\}$.

Given $Y^{r_1]} = \beta_1;\gamma_1$, we append successively paths $\alpha_i : r_{i-1} \xrightarrow{*} s_i$, $\beta_i : s_i \xrightarrow{*} d_i$ and $\gamma_i : d_i \xrightarrow{*} r_i$, for $1 < i < \omega$, as explained below.

Suppose that we have already constructed a path $Y^{r_{i-1}]} : s_1 \xrightarrow{*} r_{i-1}$, for some $i > 1$. It contains $i-1$ chosen vertices, $\{d_1, ..., d_{i-1}\} \subseteq dmi(R) \cap Y^{r_{i-1}]}$, denoted by D_{i-1}, which will also dominate the constructed ray Y. When $i > 2$, each $d_h \in D_{i-1}$, for $1 \le h < i-1$, has a fan of $i-1-h$ paths to $Y^{r_{i-1}]}$, denoted by $F^{i-1}_h = \{\phi^k_h \mid h < k \le i-1\}$. (In Figure 2.8, dotted arrows mark the paths $\phi^k_h \in F^i_h$ for $1 \le h < k \le i \le 4$.) We let F^{i-1} denote the union of all such fans constructed along $Y^{r_{i-1}]}$, that is, $F^{i-1} = \bigcup_{h<i} F^{i-1}_h$.

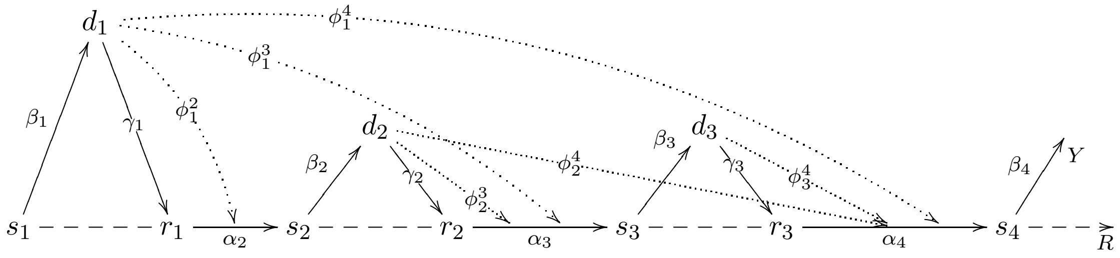

Figure 2.8:

Given a path $Y^{r_{i-1}]} : s_1 \xrightarrow{*} r_{i-1}$, for some $i > 1$, we append three following paths.

α_i : $r_{i-1} \xrightarrow{*} s_i$ is a segment of R with $s_i \in V_R$ satisfying $\max_R(Y^{r-1]} \cup F^{i-1}) <_R s_i$ and such that each $d_h \in D_{i-1}$ has a path ϕ^i_h to $V_{\alpha_i} \setminus \{s_i\}$ which is disjoint (except for its initial vertex d_h) from $Y^{r_{i-1}]} \cup F^{i-1}_h$. Since each $d_h \in D_{i-1}$ has an infinite fan to R, there is a path ϕ^i_h from d_h to R omitting the finite set $(Y^{r_{i-1}]} \cup F^{i-1}) \setminus \{d_h\}$. For each $d_h \in D_{i-1}$, we extend its fan F^{i-1}_h setting $F^i_h = F^{i-1}_h \cup \{\phi^i_h\}$.

β_i : $s_i \xrightarrow{*} d_i$. Using the Axiom of Choice, AC, we choose a new vertex $d_i \in dmi(R) \setminus D_{i-1}$ reachable from s_i by a path β_i omitting the construction up to now, so that $V_{\beta_i} \cap (Y^{s_i)} \cup F^i) = \varnothing$. Such d_i and β_i exist because R is not separated from $dmi(R)$ by the finite set $Y^{s_i)} \cup F^i$, while $R^{[s_i} \cap (Y^{s_i)} \cup F^i) = \varnothing$. We set $D_i = D_{i-1} \cup \{d_i\}$.

γ_i : $d_i \xrightarrow{*} r_i$. We find (e.g., $<_R$-minimal) $r_i \in V_R$ with $\max_R(Y^{d_i]} \cup F^i) <_R r_i$ and a path γ_i omitting vertices used so far, that is, $V_{\gamma_i} \cap (Y^{d_i)} \cup F^i) = \varnothing$. Such r_i and γ_i exist since d_i, dominating R, has a path to R omitting the finite set $Y^{d_i)} \cup F^i$. We set $F^i_i = \varnothing$.

The construction continues with $Y^{r_i]}$ and, in the ω-limit, yields the ray $Y = \bigcup_{i\in\omega} Y^{r_i]}$, where every $d_h \in \bigcup_{i\in\omega} D_i \subseteq V_Y$ dominates Y by the infinite fan $\{\phi^i_h \mid h < i\}$. $\square$

2.2 Ends of digraphs

The notion of an end of a digraph, as the subgraph induced by $A^{\overset{*}{-}}_G(V_R)$ for any ray $R \in \vec{G}$, can be given a different description, involving and leading to other relevant concepts.[2]

Two rays in a graph G are *equivalent*, $R \simeq Q$, if they determine the same end, $A^{\overset{*}{-}}_G(V_R) = A^{\overset{*}{-}}_G(V_Q)$. This is actually the largest equivalence contained in the quasiorder given by:

- $Q \preceq R$ if $V_Q \subseteq A^{\overset{*}{-}}_G(V_R)$, that is, if each tail of Q has a path to R.

The end $A^{\overset{*}{-}}_G(R)$ coincides with the subgraph induced by (the vertices on the rays belonging to) the equivalence class $[R]$ of R, given by $\{Q \in \vec{G} \mid Q \preceq R \wedge R \preceq Q\}$. This formulation relates our notion to that from [12], where an end is the equivalence class of rays $[R]^{\omega}$ with respect to the largest equivalence $\overset{\omega}{\simeq}$ contained in the quasiorder defined by:

- $Q \ {}^{\omega}\!\preceq R$ if there are infinitely many disjoint paths from Q to R.

Obviously, ${}^{\omega}\!\preceq \ \subseteq \ \preceq$ and for every ray $R : [R]^{\omega} \subseteq [R]$. The two notions are different when paths from (each tail of) Q to R are not disjoint so that, in addition to $V_Q \subseteq A^{\overset{*}{-}}_G(V_R)$, the rays Q and R stand also in the stronger relation ${}^{f}\!\preceq$, given by:

- $Q \ {}^{f}\!\preceq R$ if $V_Q \subseteq A^{\overset{*}{-}}_G(r)$ for some $r \in V_R$.

In general, ${}^{f}\!\preceq \ \not\subseteq \ {}^{\omega}\!\preceq$, while $\preceq \ = \ {}^{f}\!\preceq \cup {}^{\omega}\!\preceq$. Figure 2.9 illustrates the essentials.

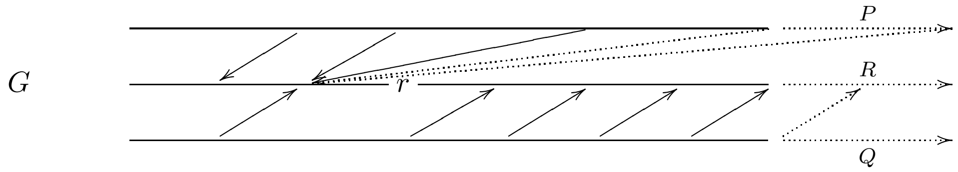

Figure 2.9: $P \ {}^{f}\!\prec R$, $P \ {}^{\omega}\!\not\preceq R$, $Q \ {}^{f}\!\not\preceq R$, $Q \ {}^{\omega}\!\preceq R$ and $R \not\preceq P, R \not\preceq Q$.

Unlike in undirected graphs, an end of a digraph can be a subgraph of another end; $Q \preceq R$ may reflect strict inclusion $A^{\overset{*}{-}}_G(V_Q) \subset A^{\overset{*}{-}}_G(V_R)$. The number of ends refers to distinct (not necessarily disjoint) such subgraphs. The graph in Figure 2.9 has three ends (of either kind): $A^{\overset{*}{-}}_G(V_P) = V_P, A^{\overset{*}{-}}_G(V_Q) = V_Q$, and $A^{\overset{*}{-}}_G(V_R) = V_G$, even though $A^{\overset{*}{-}}_G(V_Q) \cup A^{\overset{*}{-}}_G(V_P) \subset A^{\overset{*}{-}}_G(V_R)$.

A related difference from undirected graphs, exemplified by Q and R in Figure 2.9, is that (rays from) two ends of a digraph need not be finitely separable. For our purposes, a weaker separation property has to suffice, amounting to the fact that a vertex v dominating a ray Q dominates also every ray R with $Q \ {}^{\omega}\!\preceq R$.

Fact 2.10 *For each graph G, $v \in V_G$, finite subset F of V_G, and $Q, R \in \vec{G}$ satisfying $Q \ {}^{\omega}\!\preceq R$: if F separates v from some tail of R, then F separates v from some tail of Q.*

PROOF. Suppose that F does not separate v from any tail of Q and $v \notin F$. Let $q \in V_Q$ be such that $\max_Q(F) <_Q q$, and π be any path from v to the tail $Q^{[q}$ omitting F. Let γ be a path from q along Q until a $q' \in V_Q$, from which there is a path ρ to R omitting F. Such q' and ρ exist since $\max_Q(F) <_Q q \leq_Q q'$ and $Q \ {}^{\omega}\!\preceq R$. The concatenation $\pi;\gamma;\rho$ gives a path from v to a tail of R omitting F. $\square$

The equivalences on rays are used also for defining the following notion. For $\overset{\bullet}{\simeq} \in \{\overset{\omega}{\simeq}, \simeq\}$, a subset Z of V_G (or subgraph $G[Z]$) is $\overset{\bullet}{\simeq}$-*flat* if Z contains only $\overset{\bullet}{\simeq}$-equivalent rays. For

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[2] We consider only ends induced by rays because ends induced by inverse rays (infinite simple incoming paths, where $\forall i \in \omega : R_{i+1} \in A^{-}_G(R_i)$) are not significant here; Theorem 1.1 implies that a graph without odd cycles is solvable if and only if any of its tails is.

instance, the graph G in Figure 2.9 is neither $\simeq$-flat nor $\overset{\omega}{\simeq}$-flat, while the graph from (1.3) is both. When an end $A^{*}_{G}(V_R)$ is not flat, that is, when $\exists Q \in \vec{G} : A^{*}_{G}(V_Q) \subset A^{*}_{G}(V_R)$, it may be relevant to distinguish between the "whole end" induced by $A^{*}_{G}(V_R)$ and its "proper part" induced by $A^{*}_{G}(V_R) \setminus A^{*}_{G}(V_Q)$.

The following observation about flat graphs will be used a couple of places.

Fact 2.11 *In a $\simeq$-flat (or $\overset{\omega}{\simeq}$-flat) graph G, $A^{*}_{G}(v)$ is a tail of G, for every $v \in V_G$ with $\vec{v} \neq \varnothing$.*

Proof. $A^{*}_{G}(v)$ is a nonempty free induced subgraph of G. If it is not a tail of G, then there is some ray R in $G - A^{*}_{G}(v)$, so $V_Q \not\subseteq A^{*}_{G}(V_R)$ for any $Q \in \vec{v}$, that is, $R \not\simeq Q$ (and $R \overset{\omega}{\not\simeq} Q$). □

2.3 Semikernels

A *semikernel* of G is a subset L of V_G which is independent, $A^{-}_{G}(L) \subseteq V_G \setminus L$, and absorbs its out-neighbors, $A_G(L) \subseteq A^{-}_{G}(L)$ [7]. The set of all semikernels in G is denoted by $SK(G)$.

For instance, $\sigma^{\mathbf{1}}$ obtained by inducing in Figure 2.1 is a semikernel. Every kernel of a graph is also a semikernel of that graph, while a semikernel L is a kernel of the subgraph induced by $A^{-}_{G}[L]$, where $A^{-}_{G}[L] = A^{-}_{G}(L) \cup L$. A kernel of an induced subgraph H of G need not be a semikernel of G, but a kernel of a free induced subgraph is, as the following fact implies, since $sol(H) \subseteq SK(H)$.

Fact 2.12 *If $H \sqsubseteq G$ and H is free in G, then $SK(H) \subseteq SK(G)$.*

Proof. Let L be an arbitrary semikernel of H.

1. $A^{-}_{G}(L) \cap V_H = A^{-}_{H}(L) \subseteq V_H \setminus L$: since $H \sqsubseteq G$ and $L \in SK(H)$;
2. $A^{-}_{G}(L) \setminus V_H \subseteq V_G \setminus L$: since $L \subseteq V_H$;
3. $A^{-}_{G}(L) \subseteq (V_H \setminus L) \cup (V_G \setminus L) = V_G \setminus L$: by 1., 2., and $V_H \subseteq V_G$;
4. $A_G(L) \cap V_H = A_H(L) \subseteq A^{-}_{H}(L)$: since $H \sqsubseteq G$ and $L \in SK(H)$;
5. $A_G(L) \setminus V_H \subseteq A_G(V_H) \setminus V_H = \varnothing$: since $L \subseteq V_H$ and H is free in G, that is, $A_G(V_H) \subseteq V_H$;
6. $A_G(L) = (A_G(L) \cap V_H) \cup (A_G(L) \setminus V_H) \subseteq A^{-}_{H}(L) \subseteq A^{-}_{G}(L)$: by 4., 5., and $H \sqsubseteq G$;

Thus, $A_G(L) \subseteq A^{-}_{G}(L) \subseteq V_G \setminus L$ by 6. and 3., so $L \in SK(G)$. □

Semikernels are useful for proving (un)solvability, mainly, thanks to the following result.

Theorem 2.13 ([7]) *A graph G is KP if and only if every nonempty induced subgraph H of G has a nonempty semikernel.*

This theorem follows also from the iterative construction of kernels below, which generalizes a technique from [4] to infinite graphs. Also the inducing process in Figure 2.1 specializes it, by dropping point 5 and, in point 2, taking at each stage $L_i = sinks(G_i)$.

Definition 2.14 ([4]) *A* solver *for a graph G is a sequence of induced subgraphs and semikernels $\langle G_i, L_i \rangle_{1 \leq i \leq \kappa}$, for some ordinal κ, such that:*

1. *$G_1 = G$,*
2. *$L_i \in SK(G_i)$ for $1 \leq i \leq \kappa$,*
3. *$G_{i+1} = G_i - A^{-}_{G}[L_i]$,*
4. *$G_\lambda = G[\bigcap_{i<\lambda} V_{G_i}]$ – for limit ordinals λ,*
5. *$L_\kappa \in sol(G_\kappa)$.*

Theorem 2.15 ([4]) *A graph has a kernel if and only if it has a solver.*

PROOF. $\Rightarrow$) If $K \in sol(G)$, then $\langle G, K\rangle$ is a solver for G.

$\Leftarrow$) Let $\langle G_i, L_i\rangle_{1\leq i\leq\kappa}$ be a solver for G and $K = \bigcup_{1\leq i\leq\kappa} L_i$. We show that

(a) K is independent, $A^-_G(K) \subseteq V_G \setminus K$, and

(b) K absorbs its complement, $A^-_G(K) \supseteq V_G \setminus K$.

To show (a) suppose, toward a contradiction, $y \in A^-_G(x)$ for some $x, y \in K$. Since every semikernel is independent and K is the union of all semikernels L_i, x and y belong to different ones, say $x \in L_i$, $y \in L_j$. If $i < j$, then $y \in A^-_G[L_i]$ and, by Definition 2.14, $y \notin V_{G_j}$, so $y \notin L_j$. If $j < i$, then $x \in A_G(y) \subseteq A^-_G[L_j]$ since L_j is a semikernel and, by Definition 2.14, $x \notin V_{G_i}$ so $x \notin L_i$.

(b) follows because if there is some $x \in V_G \setminus A^-_G[K]$, then $x \notin A^-_G[L_i]$ for $1 \leq i \leq \kappa$. In particular, $x \in V_{G_\kappa} \setminus A^-_G[L_\kappa]$, contradicting the fact that $L_\kappa \in sol(G_\kappa)$. $\square$

In particular, if every nonempty induced subgraph has a nonempty semikernel, then one can easily form a solver. This gives, for instance, the following corollary.

Corollary 2.16 *Every bipartite graph G is KP.*

This follows because for a nonempty induced subgraph H of G, $sinks(H) \subseteq SK(H)$, while if $sinks(H) = \varnothing$, then vertices at even distances from a fixed vertex of H form a semikernel.

Corollary 2.17 *A graph G is KP if and only if $G[A^*_G(x)]$ is KP, for every $x \in V_G$.*

PROOF. The implication to the right is obvious. To show the opposite, we start with $i = 1$ and $G_1 = G$. Given G_i, we choose some $x_i \in V_{G_i}$, let L_i be a kernel of $G[A^*_{G_i}(x_i)]$ and $G_{i+1} = G_i - A^-_G[L_i]$. A kernel L_i of $G[A^*_{G_i}(x_i)]$ exists because $G[A^*_{G_i}(x_i)]$ is an induced subgraph of $G[A^*_G(x_i)]$, which is KP. By Fact 2.12, L_i is a semikernel of G_i, since $G[A^*_{G_i}(x_i)]$ is also free in G_i. For the limit ordinals λ, we set $G_\lambda = G[\bigcap_{i<\lambda} V_{G_i}]$. Using AC for the successive choices of x_i and kernels of $G[A^*_{G_i}(x_i)]$, eventually, we reach $V_{G_\kappa} = \varnothing$, obtaining a solver $\langle G_i, L_i\rangle_{i\leq\kappa}$. Thus G has a kernel by Theorem 2.15. The graph G is KP because kernel-perfectness of all $G[A^*_G(x)]$ is inherited by all $H[A^*_H(x)]$, whenever $H \sqsubseteq G$.[3] $\square$

3 The main result

The main result, Theorem 3.19, specializes the following general statement.

Theorem 3.1 *A graph G is KP if there is a partition $V_G = \biguplus_{i\in I} V_{G_i}$ such that*

1. *for each $i \in I : G_i$ is KP, where $G_i = G[V_{G_i}]$, and*
2. *for each nonempty subset F of I, there is a $k \in F$ with G_k free in $G[\bigcup_{i\in F} V_{G_i}]$.*

PROOF. Let H be an arbitrary nonempty induced subgraph of G. By Theorem 2.13, it suffices to show that H has a nonempty semikernel. Let

$F = \{i \in I \mid V_{G_i} \cap V_H \neq \varnothing\}$,
$V_{H_i} = V_H \cap V_{G_i}$, for each $i \in F$, and
$H_i = G[V_{H_i}]$, for each $i \in F$.

---

[3] A property $P(\_)$ is *inherited by* – or is *hereditary in* – induced subgraphs if $P(G)$ and $H \sqsubseteq G$ imply $P(H)$. This holds above if we let $P(G)$ denote that $G[A^*_G(v)]$ is KP for every $v \in V_G$. A hereditary property, implying solvability, implies also kernel-perfectness.

We show first that $H[V_{H_k}]$ is free in H. Let $K = G[\bigcup_{i \in F} V_{G_i}]$. The three facts – $H \sqsubseteq G$, $V_{H_k} \subseteq V_H \subseteq V_K$ for $k \in F$ given by 2., and $K \sqsubseteq G$ – give the respective equalities:

$A_H(V_{H_k}) = A_G(V_{H_k}) \cap V_H = A_G(V_{H_k}) \cap V_K \cap V_H = A_K(V_{H_k}) \cap V_H$.

This gives the first equality below, while the following inclusions and the final equality follow because $V_{H_k} = V_H \cap V_{G_k}$ and because G_k is free in K, $A_K(V_{G_k}) \subseteq V_{G_k}$:

$A_H(V_{H_k}) = A_K(V_{H_k}) \cap V_H \subseteq A_K(V_{G_k}) \cap V_H \subseteq V_{G_k} \cap V_H = V_{H_k}$.

Thus $A_H(V_{H_k}) \subseteq V_{H_k}$, that is, $H[V_{H_k}]$ is free in H.

Now, $H[V_{H_k}] = G_k[V_{H_k}]$ because $G_k[V_{H_k}] \sqsubseteq G_k \sqsubseteq G$, $H \sqsubseteq G$ and $V_{H_k} \subseteq V_H$. (Generally, if $X \sqsubseteq G$, $Y \sqsubseteq G$ and $V_X \subseteq V_Y$, then $X = Y[V_X]$ because $V_X = V_{Y[V_X]}$ by definition, while $A_X = A_{Y[V_X]}$ follows by verifying for every $x \in V_X : A_X(x) = A_G(x) \cap V_X = A_G(x) \cap V_Y \cap V_X = A_{Y[V_X]}(x)$. Setting $X = G_k[V_{H_k}]$ and $Y = H$ yields $G_k[V_{H_k}] = H[V_{H_k}]$.)

Thus $H[V_{H_k}]$, being the induced subgraph $G_k[V_{H_k}]$, has a kernel because G_k is KP. Since it is nonempty and free in H, its kernel is a nonempty semikernel of H by Fact 2.12. □

We apply this theorem partitioning a safe graph with finitely many ends into $\simeq$-flat sets, which are shown to be KP by two cases: when a $\simeq$-flat set contains a pair of rays with $Q \,{}^f\!\preceq P$, Lemma 3.3, and when it does not, Lemma 3.4.

The following fact is used in Lemma 3.3 and then in Definitions 3.14 and 3.15. A digraph without odd cycles need not be bipartite, for instance, $\begin{tikzcd}[cramped] a \arrow[r] \arrow[rr, bend right] & b \arrow[r] & c \end{tikzcd}$, but a strong component without odd cycles is (and its bipartition is unique).

Fact 3.2 *A strong component without odd cycles is bipartite.*

Proof. Vertices of such a strong component C can be partitioned by choosing any vertex $a \in V_C$, taking E to be all vertices reachable from a by a path of even length, and O all vertices reachable from a by a path of odd length. Since C is a strong component, $E \cup O = V_C$. To show $E \cap O = \varnothing$, suppose that there is some $b \in E \cap O$, that is, there is an even path $\pi_E : a \overset{*}{\to} b$ and an odd path $\pi_O : a \overset{*}{\to} b$. Since C is a strong component, there is also a path $\rho : b \overset{*}{\to} a$. There is therefore an odd closed walk: either $\rho; \pi_O$ if ρ is even, or $\rho; \pi_E$ if ρ is odd. Since every odd closed walk contains an odd cycle, while C has none, we conclude $E \cap O = \varnothing$. This implies also that both E and O are independent. Hence $\{E, O\}$ is a bipartition of C. □

A $\simeq$-flat graph, containing rays with $Q \,{}^f\!\preceq P$, contains a cycle. When none of its cycles is odd, we show that the graph has a bipartite tail which is KP by Corollary 2.16. The graph is then KP by Observation 2.5.(a), and this summarizes the proof of the following lemma.

Lemma 3.3 *A $\simeq$-flat graph G without odd cycles is KP if it has rays P and Q with $Q \,{}^f\!\preceq P$.*

Proof. Let Q and P be as in the statement of the lemma. By Observation 2.5.(b), we can assume $\vec{x} \neq \varnothing$ for all $x \in V_G$. Since G is $\simeq$-flat, this implies $V_G = A^*_G(V_R)$ for every $R \in \vec{G}$, in particular, $V_G = A^*_G(V_Q)$. For some $p_0 \in V_P : V_Q \subseteq A^*_G(p_0)$ and we consider $A^*_G(p_0)$. It is a tail of G by Fact 2.11. Also, $A^*_G(p_0)$ is a strong component of G because each two $s, t \in A^*_G(p_0)$ are connected by $(\alpha; \beta; \gamma) : s \overset{*}{\to} t$, combining the paths:

$\alpha : s \overset{*}{\to} q$, for some $q \in V_Q$, existing since $s \in V_G = A^*_G(V_Q)$;
$\beta : q \overset{*}{\to} p_0$, existing since $V_Q \subseteq A^*_G(p_0)$, and
$\gamma : p_0 \overset{*}{\to} t$, existing since $t \in A^*_G(p_0)$.

Thus $A^*_G(p_0)$ is a tail and a strong component of G. Having no odd cycles, it is bipartite by Fact 3.2 and KP by Corollary 2.16. Its complement $G - A^*_G(p_0)$, having no odd cycles nor rays, is KP by Theorem 1.1. By Observation 2.5.(a), the graph G is KP. □

The proof of the following lemma refers to further results because they take virtually the rest of the paper, stretching until Corollary 3.18.

Lemma 3.4 *A safe $\simeq$-flat graph G is KP if for all rays P and Q : $P \;{}^{f}\!\!\not\preceq Q$.*

Proof. Since G is $\simeq$-flat and contains no rays with $P \;{}^{f}\!\!\preceq Q$ (especially, no P with $P \;{}^{f}\!\!\preceq P$), G is actually $\overset{\omega}{\simeq}$-flat, that is, for each ray $R \in \vec{G}$: $[R] = [R]^{\omega} = \vec{G}$.

According to Corollary 3.9, such a safe $\overset{\omega}{\simeq}$-flat G, with no rays P and Q satisfying $P \;{}^{f}\!\!\preceq Q$, has a tail $A^*_G(r)$, for some $r \in V_G$, with a finitary division (Definition 3.5). This tail is KP by Theorem 3.10. Its complement $G - A^*_G(r)$ is rayless and has no odd cycles, so it is KP by Theorem 1.1. The lemma follows by Observation 2.5.(a). □

Constructions and arguments related to finitary divisions, to be defined now, are simplified by considering graphs with a *supersource*, namely, a vertex $v \in V_G$ such that $V_G = A^*_G(v)$.

Definition 3.5 A finitary division *of a graph G, with a chosen supersource v, is a sequence $\langle C_i \rangle_{i \in \omega}$ of finite disjoint subsets C_i of V_G, where $C_0 = \{v\}$ and C_{i+1}, for each i, is a minimal set separating C_i (every $c \in C_i$) from tails of all rays.*

Observation 3.6 *Sinks and finite terminal components are irrelevant for forming finitary divisions. A finitary division of G is a finitary division of its subgraph $G[\vec{G}]$, and vice versa.*

Fact 3.7 *If $\langle C_i \rangle_{i \in \omega}$ is a finitary division of G with $C_0 = \{v\}$, then every ray $R \in \vec{v}$ intersects every C_i for $i \in \omega$.*

Proof. If some $R \in \vec{v}$ omits some C_j, then letting j be the least such, $j > 0$ and C_j does not separate C_{j-1} from tails of all rays, violating Definition 3.5. □

Lemma 3.8 *For every graph G and $v \in V_G$, if every $x \in A^*_G(v)$ is finitely separable from tails of all rays, then $A^*_G(v)$ has a finitary division.*

Proof. Let $v \in V_G$ be arbitrary and, for every $x \in A^*_G(v)$, B_x be a minimal finite subset of V_G not containing x and separating x from tails of all rays. Let $C_0 = \{v\}$ and, given C_i, let

$C'_{i+1} = \bigcup_{y \in C_i} B_y$ and

$C''_{i+1} = C'_{i+1} \setminus \bigcup_{j \leq i} C_j$.

Since C''_{i+1} is finite and separates C_i from tails of all rays (as shown below), we can find (using AC) a minimal subset C_{i+1} of C''_{i+1} which still separates C_i from tails of all rays. All C_i are then disjoint and finite. We prove that the resulting $\langle C_i \rangle_{i \in \omega}$ is a finitary division of $A^*_G(v)$, showing by induction on i that C_{i+1} separates each C_j, for $j \leq i$, from tails of all rays.

The claim is obvious for $i = 0$, since then $C''_1 = C'_1 = B_v$ and B_v separates v from tails of all rays. Since C''_1 is finite, we can choose C_1 as described.

Given C_i, separating each C_j with $j < i$ from tails of all rays, C'_{i+1} separates C_i from tails of all rays by definition, so it separates each C_j, for $j \leq i$, from tails of all rays. We show that also C''_{i+1} separates C_i from tails of all rays. Suppose that some ray, intersecting C_i, intersects also C'_{i+1} at some $c \in \bigcup_{j \leq i} C_j$, and consider its tail

(*) $R \in \vec{c}_i$, for some $c_i \in C_i$ such that $V_R \cap C_i = \{c_i\}$.

Such a tail R and c_i exist for every ray intersecting C_i because C_i is finite and rays are acyclic. Since C'_{i+1} separates C_i from tails of all rays, R crosses C'_{i+1} but not $\bigcup_{j \leq i} C_j$. If it did, it would cross either C_i, contradicting (*), or $\bigcup_{j < i} C_j$, in which case, by the induction hypothesis for $j < i$, it would also cross C_i, again contradicting (*). Thus C''_{i+1} separates C_i from tails of all rays and, being finite, has a minimal subset C_{i+1} doing the same. Thus, $\langle C_i \rangle_{i\in\omega}$ is a finitary division of $A^*_G(v)$. □

Corollary 3.9 *A safe $\overset{\omega}{\simeq}$-flat graph G, having no rays with $P \,{}^{f}\!\!\preceq Q$, has a tail $A^*_G(r)$, for some $r \in V_G$, with a finitary division.*

PROOF. By Observation 3.6, we can assume the inequality $\vec{x} \neq \varnothing$ for all $x \in V_G$. Since G is $\overset{\omega}{\simeq}$-flat, fixing an arbitrary $R \in \vec{G}$ yields $V_G = A^*_G(V_R)$. Since G is safe, R is finitely separable from vertices dominating it by Fact 2.7. Hence, there is some $r \in V_R$ such that $A^*_G(r)$ contains no vertex dominating R. (If a vertex dominating R existed in $A^*_G(r)$ for every $r \in V_R$, then the paths from R to these vertices would all intersect a finite set F, since R is finitely separable from them. Consequently, for some $e \in F : V_R \subseteq A^*_G(e)$, that is, $R \,{}^{f}\!\!\preceq Q$ for each $Q \in \vec{e}$ – contradiction.) For every $x \in A^*_G(r)$, there is thus a finite set B_x not containing x and separating x from a tail of R. Since G is $\overset{\omega}{\simeq}$-flat, $Q \,{}^{\omega}\!\!\preceq R$ for every $Q \in \vec{G}$, so B_x separates x from tails of all rays $\vec{G}$, by Fact 2.10. By Lemma 3.8, $A^*_G(r)$ has then a finitary division, while by Fact 2.11, $A^*_G(r)$ is a tail of G. □

To complete the proof of Lemma 3.4, we need kernel-perfectness of $A^*_G(r)$ with a finitary division and no odd cycles, as in Corollary 3.9. The sketch of the proof of the following theorem, giving this fact, overviews the steps stretching until Corollary 3.18.

Theorem 3.10 *A graph with a supersource, a finitary division and no odd cycles is KP.*

PROOF SKETCH. Given a finitary division $\langle C_i \rangle_{i\in\omega}$ of a graph G with a supersource and no odd cycles, we cover G by ω many rayless subgraphs G_i such that $G_i \subseteq G_{i+1}$ and $C_i \subseteq sinks(G_i)$.

For each $i \in \omega$, given any assignment $\alpha_i \in \mathbf{2}^{C_i}$, we choose (using AC) a solution to G_i relative to this α_i. Since C_i is finite, and for each $\alpha_i \in \mathbf{2}^{C_i}$ one relative solution is chosen, the set $solr(G_i)$ of these selected relative solutions is finite.

We will also show that the choices are compatible in the sense that if β is selected for G_j, then $\beta|_{V_{G_i}}$ is selected for G_i, for every $i < j$. Thus, for every G_i we obtain a nonempty finite set $solr(G_i)$ of solutions with the property that $solr(G_j)|_{V_{G_i}} \subseteq solr(G_i)$ when $G_i \subseteq G_j$. Viewing solutions as elements of the product topology $\mathbf{2}^{V_G}$ and an appropriate extension $solr(G_i)^*$ of each $solr(G_i)$ as its closed subset, compactness of $\mathbf{2}^{V_G}$ yields a nonempty intersection $\bigcap_{i\in\omega} solr(G_i)^*$, containing solutions to G. Solvability gives kernel-perfectness because, as will be shown, a finitary division of G is inherited by subgraphs with supersources. □

Definition 3.11 *For a graph G with a chosen supersource v and a finitary division $\langle C_i \rangle_{i\in\omega}$, we set $G_0 = \langle \{v\}, \varnothing \rangle$ and, for every $i > 0$, define G_i as the subgraph consisting of the union of all walks from v that do not cross C_i.*

In general, G_i is not an induced subgraph of G. Paths terminating in C_i (without crossing it) belong to G_i, so $C_i \subseteq sinks(G_i)$. In the example in Figure 3.12, $C_1 = \{a_1, b_1, c_1, d_1, e_1\}$, $C_2 = \{a_2, b_2, c_2, d_2\}$, and edges A_{G_i} are marked by $j \leq i$.

Fact 3.13 *If $\langle C_i \rangle_{i\in\omega}$ is a finitary division of a graph G with a supersource v, then*

1. $G = \langle \bigcup_{i\in\omega} V_{G_i}, \bigcup_{i\in\omega} A_{G_i} \rangle$,

and the following facts hold for each subgraph G_i from Definition 3.11:

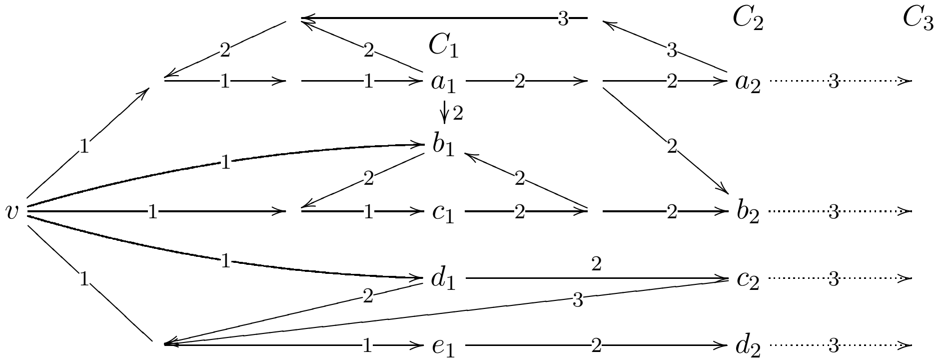

Figure 3.12: For $i \in \{1,2,3\}$, edges A_{G_i} are marked by $j \leq i$.

2. *for* $j > i : V_{G_i} \subseteq V_{G_j} \setminus C_j$,
3. *if* $\pi : x_i \overset{*}{\to} x_j$ *is a path in* G *with* $x_i \in V_{G_i}$ *and* $x_j \notin V_{G_i}$, *then* $V_\pi \cap C_i \neq \varnothing$,
4. $A_G(V_{G_i}) \subseteq V_{G_{i+1}}$,
5. G_i *is rayless.*

Proof. (1) Every $x \in V_G$ is reachable from v by some path π which, being finite, does not cross some C_i, so $x \in V_{G_i}$. For every edge $(x,y) \in A_G$, the finite set $V_\pi \cup \{y\}$ does not intersect some C_i. Hence the walk $\pi;(x,y)$ is in G_i and $(x,y) \in A_{G_i}$.

(2) This holds trivially for $i = 0$ since $V_{G_0} = \{v\} \subseteq V_{G_j} \setminus C_j$, for each $j > 0$. So let $j > i > 0$ and suppose $x \in V_{G_i} \setminus V_{G_j}$ for some x, that is, there is a path $\pi_0 : v \overset{*}{\to} x$ not crossing C_i (since $x \in V_{G_i}$), while every path $v \overset{*}{\to} x$ crosses C_j (which is the only reason why $x \notin V_{G_j}$). Letting $c_0 \neq x$ be such that $c_0 \in C_j \cap V_{\pi_0}$, we can write the path as $\pi_0 : v \overset{*}{\to} c_0 \overset{*}{\to} x$.

We claim that for every $c_j \in C_j$, there is a ray $R \in \vec{c}_j$ omitting C_i, for each $i < j$:

(*) $\forall c_j \in C_j\ \exists R \in \vec{c}_j\ \forall i < j : V_R \cap C_i = \varnothing$.

If some c_j contradicts this formula, then also $C_j \setminus \{c_j\}$ separates C_{j-1} from tails of all rays. This holds trivially if $\vec{c}_j = \varnothing$. Otherwise, every path $C_{j-1} \overset{*}{\to} c_j$ can be extended along any $R \in \vec{c}_j$. If every such R crosses C_i for some $i < j$, then it returns to C_j (since C_j separates C_i from tails of all rays) and crosses it again at some vertex in $C_j \setminus \{c_j\}$. This contradicts minimality of C_j as a member of the finitary division.

In particular, for $c_0 \in C_j \cap V_{\pi_0}$, there is such a ray $R \in \vec{c}_0$, omitting all C_i for $i < j$. Since π_0 does not cross C_i, its initial segment $\pi_0' : v \overset{*}{\to} c_0$ omits C_i. The walk $(\pi_0'; R)$ gives then a ray in $\vec{v}$ omitting C_i, which contradicts Fact 3.7 yielding $V_{G_i} \subseteq V_{G_j}$.

We show $V_{G_i} \cap C_j = \varnothing$. By Definition 3.5, $C_i \cap C_j = \varnothing$. If there is some $c_j \in C_j \cap (V_{G_i} \setminus C_i)$, then there is a path $\pi : v \overset{*}{\to} c_j$ omitting C_i. Since $c_j \in C_j$, some $R \in \vec{c}_j$ omits C_i by (*). The walk $(\pi; R)$ gives a ray in $\vec{v}$ omitting C_i. This contradicts Fact 3.7, yielding $V_{G_i} \subseteq V_{G_j} \setminus C_j$.

(3) If $j \leq i$, then $V_{G_j} \subseteq V_{G_i}$ by (2), hence if $x_j \in V_{G_j}$, then $x_j \in V_{G_i}$. So let $j > i$ and, toward a contradiction, suppose that for some $x_i \in V_{G_i}$ and $x_j \in V_{G_j} \setminus V_{G_i}$, there is a path $\pi : x_i \overset{*}{\to} x_j$ omitting C_i. In particular, $x_i \notin C_i$. Since $x_i \in V_{G_i}$, there is a path $\alpha : v \overset{*}{\to} x_i$ omitting C_i. This yields a walk $(\alpha; \pi) : v \overset{*}{\to} x_j$ omitting C_i, so $x_j \in V_{G_i}$ contradicting $x_j \in V_{G_j} \setminus V_{G_i}$.

(4) If there is an $x \in V_{G_i}$ and $y \in A_G(x) \setminus V_{G_{i+1}}$, then the 1-edge path (x,y) leaves $V_{G_{i+1}}$ ($V_{G_i} \subseteq V_{G_{i+1}}$ by (2)) and does not intersect C_{i+1} (since $C_{i+1} \subseteq V_{G_{i+1}}$), contradicting (3).

(5) If some G_i contains a ray R, then $V_R \cap C_j = \varnothing$, for $j > i$, because $V_{G_i} \cap C_j = \varnothing$ by (2), and $V_R \cap C_i = \varnothing$ because no edge going out of C_i belongs to A_{G_i}.

Let $\pi : v \overset{*}{\to} r$, for some $r \in V_R$, be any path omitting C_i (which exists since $V_R \subseteq V_{G_i} \setminus C_i$). By (3), π is contained in G_i. The walk $\pi; R^{[r}$ gives a ray in $\vec{v}$ contained in G_i, that is, omitting C_j, for $j \geq i$. This contradicts Fact 3.7. $\square$

This fact will justify a choice of solutions to G_i relatively to $\alpha \in \mathbf{2}^{C_i}$. More generally, we show how to select a solution to a rayless graph without odd cycles, relatively to (i) a fixed assignment to any subgraph (to be then specialized to C_i) and (ii) a given choice of one part from each strong component (all are bipartite by Fact 3.2). We first define the new concepts.

Definition 3.14 *For a strong component $X \in SC(G)$ in a graph G without odd cycles, let $\{L_X, R_X\}$ denote the bipartition of X. A* choice from $SC(G)$ *is a function λ selecting one part of the bipartition of each component, that is, $\forall X \in SC(G) : \lambda(X) \in \{L_X, R_X\}$.*

Given a choice λ from $SC(G)$ and a subgraph H of G, the induced choice $\lambda|_H$ *is defined, for each $Y \in SC(H)$, by $\lambda|_H(Y) = Y \cap \lambda(X)$, where $X \in SC(G)$ is the unique component of G such that $Y \subseteq X$.*

Given a rayless graph G without odd cycles, an assignment α to some $H \subseteq V_G$, and a choice λ from $SC(G)$, we generalize the concept of inducing from Figure 2.1 to a choice of a relative solution for G. It starts with α and keeps it unchanged on H, as in (2.3). (In the definition below, the initial modification of G to G_α ensures only that the first step induces the assignment α on H, represented by a semikernel of G_α.) From this initial point, the definition follows the induction process (Figure 2.1) with one difference: encountering a terminal component, its solution is chosen by the choice induced from λ. Although λ is thus a parameter to the construction, we drop it from the notation since it is applied only once for the whole graph and propagated to the subgraphs as the induced choice.

Definition 3.15 *Let G be a rayless graph without odd cycles, λ be a choice from $SC(G)$, $H \subseteq V_G$ and $\alpha \in \mathbf{2}^H$. The graph G_α is obtained by keeping G unchanged on $V_G \setminus H$, while*

– adding a new vertex w, $V_{G_\alpha} = V_G \uplus \{w\}$,

– for each $x \in H$ with $\alpha(x) = \mathbf{0}$, adding the edge (x, w), and

– for each $x \in H$ with $\alpha(x) = \mathbf{1}$, removing all edges out of x (making it a sink of G_α).

We proceed now recursively starting with $D_1 = G_\alpha$:

1. *If $sinks(D_i) \neq \varnothing$, then induce from them, as described in Figure 2.1, obtaining a semikernel L_i of D_i.*
2. *If $sinks(D_i) = \varnothing$, then let $T_i = \bigcup ter(D_i)$ be the subgraph induced by the terminal components and L_i its semikernel given by the induced choice $\lambda|_{D_i}$, that is, $L_i = \bigcup_{Y \in ter(D_i)} \lambda(X) \cap Y$, where $X \in SC(G)$ is unique such that $Y \subseteq X$.*
3. *Continue with $D_{i+1} = D_i - A^-_{D_i}[L_i]$, and for limit ordinals λ let $D_\lambda = G_\alpha[\bigcap_{i<\lambda} V_{D_i}]$.*

For κ being the least ordinal with $V_{D_\kappa} = \varnothing$, we define $L^\alpha = \bigcup_{i<\kappa} L_i$. For each $\alpha \in \mathbf{2}^H$:

$\epsilon(\alpha) = ((L^\alpha \setminus \{w\}) \times \{\mathbf{1}\}) \cup ((V_G \setminus L^\alpha) \times \{\mathbf{0}\})$.

Since G is rayless, it has sinks or terminal components, and the process starts with these, terminating with the empty graph in κ steps, for some ordinal κ with $|\kappa| \leq |V_G|$. The function ϵ is well-defined because in each encountered subgraph, sinks or the choice from terminal components determine its values uniquely. For each argument, this function yields a relative solution.

Fact 3.16 *For a rayless graph G without odd cycles, arbitrary $H \subseteq V_G$ and choice λ from $SC(G)$, the function ϵ from Definition 3.15 is such that $\forall \alpha \in \mathbf{2}^H : \epsilon(\alpha) \in solr(G, \alpha)$.*

PROOF. First, we show that $\epsilon(\alpha) \cup \{\langle w, \mathbf{1}\rangle\} \in sol(D_1)$, for any $\alpha \in \mathbf{2}^H$. In point 1 of Definition 3.15, L_n is a semikernel of D_n, being the result of inducing from $sinks(D_n)$, while in point 2, L_n is a kernel of T_n since T_n consists of mutually unreachable strong components $Y \in ter(D_n)$, each with the bipartition $\{\lambda(X) \cap Y, Y \setminus \lambda(X)\}$, where X is the unique strong component of G containing Y. Since T_n is a free induced subgraph of D_n, L_n is a semikernel of D_n by Fact 2.12. Thus, for the ordinal κ as in Definition 3.15, adding $G_\kappa = \langle \varnothing, \varnothing\rangle$ and $L_\kappa = \varnothing$, the sequence of subgraphs $\langle D_i\rangle_{i\leq\kappa}$ with the semikernels $\langle L_i\rangle_{i\leq\kappa}$ is a solver, yielding by (the proof of) Theorem 2.15 the kernel $L^\alpha = \bigcup_{i\leq\kappa} L_i$ for D_1, that is, for G_α.

Since for every $x \in V_G \setminus H : A_G(x) = A_{G_\alpha}(x)$, the obtained $\epsilon(\alpha)$ is correct in G at every $x \in V_G \setminus H$. The modification of G to G_α ensures that $\epsilon(\alpha)|_H = \alpha$, so $\epsilon(\alpha) \in solr(G, \alpha)$. □

Given a graph G with a supersource, a finitary division $\langle C_i\rangle_{i\in\omega}$ and no odd cycles, we apply this fact to the rayless graphs G_i from Definition 3.11, covering G according to Fact 3.13.1. Fixing a choice from $SC(G)$ and using the induced choice for each G_i, Definition 3.15 gives the function $\epsilon_i : \mathbf{2}^{C_i} \to \mathbf{2}^{V_{G_i}}$ providing a solution $\epsilon_i(\alpha_i)$ to G_i, relatively to every assignment $\alpha_i \in \mathbf{2}^{C_i}$. These relative solutions to different subgraphs G_i are compatible; restriction of a relative solution for G_j to G_i, for any $0 < i < j$, is a relative solution for G_i, as shown in the following lemma. More precisely, for each pair i, j with $0 < i < j$ and any $\alpha_j \in \mathbf{2}^{C_j}$,
$\epsilon_j(\alpha_j) \in solr(G_j, \alpha_j)$ by Fact 3.16. Let
$\sigma = \epsilon_j(\alpha_j)|_{C_i}$, and
$\beta = \epsilon_i(\sigma)$, where $\beta \in solr(G_i, \sigma)$ by Fact 3.16. Finally, let
$\gamma = \epsilon_j(\alpha_j)|_{V_{G_i}}$.
With this notation, $\beta|_{C_i} = \sigma = \gamma|_{C_i}$, and the following lemma gives the equality $\beta = \gamma$.

Lemma 3.17 *For a graph G with a supersource, a finitary division $\langle C_i\rangle_{i\in\omega}$ and no odd cycles,*
- *let λ be an arbitrary choice from $SC(G)$, and*
- *for each $i > 0$, let $\epsilon_i : \mathbf{2}^{C_i} \to \bigcup_{\alpha\in\mathbf{2}^{C_i}} solr(G_i, \alpha)$ be the function from Definition 3.15 (with the induced choice $\lambda_i = \lambda|_{G_i}$).*

For every i and j, with $1 \leq i < j \in \omega$, and every $\alpha_j \in \mathbf{2}^{C_j} : \epsilon_i(\epsilon_j(\alpha_j)|_{C_i}) = \epsilon_j(\alpha_j)|_{V_{G_i}}$.

PROOF. By Fact 3.13.5, each G_i is rayless, so we can use Definition 3.15 and Fact 3.16. With the notation introduced just before the lemma, both β and γ are correct on $V_{G_i} \setminus C_i$: β by Fact 3.16, while γ is correct on $V_{G_j} \setminus C_j$ by the same fact, and hence on $V_{G_i} \setminus C_i$, since $V_{G_i} \setminus C_i \subseteq V_{G_j} \setminus C_j$, by Fact 3.13.2.

The claim $\beta = \gamma$ follows by induction. Starting with $K_0 = C_i$ and $\sigma_0 = \sigma$, we extend in each step the induction hypothesis $\beta|_{K_n} = \sigma_n = \gamma|_{K_n}$ to some nonempty subset of vertices in the remaining subgraph $D_n = G_i - K_n$. Two cases depend on the result $\overline{\sigma}_n$ of inducing from σ_n to D_n.

i. $\sigma_n \neq \overline{\sigma}_n$, that is, $\sigma_n \subset \overline{\sigma}_n$.

Since $\beta|_{K_n} = \sigma_n = \gamma|_{K_n}$ and both β and γ are correct on D_n, Observation 2.4 yields $\beta|_{K'_n} = \overline{\sigma}_n = \gamma|_{K'_n}$, where $K'_n = dom(\overline{\sigma}_n) \cap V_{D_n} \neq \varnothing$. We continue with $\beta|_{K_{n+1}} = \gamma|_{K_{n+1}}$, where $K_{n+1} = K_n \cup K'_n$.

ii. $\sigma_n = \overline{\sigma}_n$, that is, no inducing from σ_n takes place; in particular, $sinks(D_n) = \varnothing$.

In this case, the rayless D_n has terminal components, denoted $ter(D_n)$, and we show that the assignments agree on the subgraph T_n consisting of all these components. Let $X \in ter(D_n)$

be arbitrary.

(a) The assignments agree on $A_{G_i}(X)\setminus X$ since they agree on K_n by the induction hypothesis, while $A_{G_i}(X)\setminus X \subseteq K_n$. This inclusion follows since $X \in \mathit{ter}(D_n)$ implies $A_{D_n}(X) = X$, while $A_{D_n}(X) = A_{G_i}(X) \cap V_{D_n}$, so $X = A_{G_i}(X) \cap V_{D_n}$ and hence $A_{G_i}(X) \setminus X \subseteq V_{G_i} \setminus V_{D_n} = K_n$. Since no inducing from σ_n to D_n takes place, all vertices in K_n with edges from D_n are assigned $\mathbf{0}$, that is, $\forall y \in A_{G_i}(V_{D_n}) \setminus V_{D_n} : \gamma(y) = \mathbf{0} = \beta(y)$. This holds, in particular, for all vertices in $A_{G_i}(X) \setminus X$.

(b) Since $X \in \mathit{SC}(D_n)$, for some $Y \in \mathit{SC}(G_j)$ and $Z \in \mathit{SC}(G) : X \subseteq Y \subseteq Z$.

- If $X = Y$, then by (a) and Definition 3.15.2, $\beta^{\mathbf{1}}|_X = \lambda_i(X) = \lambda(Z) \cap X = \lambda_j(X) = \gamma^{\mathbf{1}}|_X$.
- If $X \neq Y$, then, since no inducing occurs from σ_n to X, in particular, from $\sigma_n|_{Y \cap K_n}$ to X, we have $\gamma^{\mathbf{1}}|_X = \lambda_j(Y) \cap X = \lambda(Z) \cap X$, while $\lambda(Z) \cap X = \lambda_i(X) = \beta^{\mathbf{1}}|_X$, by Definition 3.15.2. Thus $\gamma^{\mathbf{1}}|_X = \beta^{\mathbf{1}}|_X$.

In either case, also $\beta^{\mathbf{0}}|_X = X \setminus \lambda_i(X) = \gamma^{\mathbf{0}}|_X$, so $\gamma|_X = \beta|_X$.

In a rayless and sinkless D_n, all $X \in \mathit{ter}(D_n)$, having no outgoing edges, are mutually unreachable, so this argument works simultaneously for all of them. Thus β and γ agree on T_n, and we continue with $\beta|_{K_{n+1}} = \gamma|_{K_{n+1}}$, where $K_{n+1} = K_n \cup V_{T_n}$. This completes point **ii** and the successor case.

Setting $K_\lambda = \bigcup_{i<\lambda} K_i$, in any limit λ, yields $\beta|_{K_\lambda} = \gamma|_{K_\lambda}$ because if not, then $\beta|_{K_i} \neq \gamma|_{K_i}$ for some $i < \lambda$. We continue with K_λ and this equality.

Eventually, $K_{\mu+1} = K_\mu$ for some ordinal μ with cardinality $|\mu| \leq |V_{G_i}|$. Then $K_\mu = V_{G_i}$, for suppose $K_{\mu+1} = K_\mu$ and some $x \in V_{G_i} \setminus K_\mu$. If there is a $y \in A_{G_i}(x) \cap K_\mu$ such that $\gamma(y) = \mathbf{1} = \beta(y)$, then x obtains the induced value $\mathbf{0}$ in step $\mu+1$, so $x \in K_{\mu+1}$, contradicting $K_{\mu+1} = K_\mu$. If there is no such y while $A_{G_i}(x) \subseteq K_\mu$, then $x \in \mathit{sinks}(D_\mu)$, so $x \in K_{\mu+1}$ by **i**, contradicting $K_{\mu+1} = K_\mu$. Hence, all $y \in A_{G_i}(x) \cap K_\mu$ are assigned $\mathbf{0}$ and $A_{G_i}(x) \not\subseteq K_\mu$. This implies that D_μ is sinkless. It is also nonempty (since $\exists x \in V_{G_i} \setminus K_\mu$) and rayless (since G_i is rayless), so $\mathit{ter}(D_\mu) \neq \varnothing$. This contradicts $K_{\mu+1} = K_\mu$ because $\mathit{ter}(D_\mu) \subseteq K_{\mu+1}$ by point **ii**. Thus, $K_\mu = V_{G_i}$ and $\beta = \beta|_{K_\mu} = \gamma|_{K_\mu} = \gamma$. □

We can now complete the proof of Theorem 3.10, according to which a graph with a supersource, a finitary division and no odd cycles is KP.

Proof of Theorem 3.10. (a) We show first solvability of such graphs. Let $G = A^*_G(v)$ and $\langle C_i \rangle_{i\in\omega}$ be a finitary division of G. Definition 3.11 gives an ω-chain of rayless subgraphs $G_1 \subset G_2 \subset ...$, such that $G = \langle \bigcup_{i\in\omega} V_{G_i}, \bigcup_{i\in\omega} A_{G_i} \rangle$ by Fact 3.13.1. Using AC, we obtain a choice λ from $\mathit{SC}(G)$ and, for each $i \in \omega$, we let function $\epsilon_i : \mathbf{2}^{C_i} \to \mathbf{2}^{V_{G_i}}$ be as in Definition 3.15. By Fact 3.16, $\epsilon_i(\alpha) \in \mathit{solr}(G_i, \alpha)$ for each $\alpha \in \mathbf{2}^{C_i}$. Let $\mathit{solr}(G_i)$ denote the set of such relative solutions:

- $\mathit{solr}(G_i) \ = \{\epsilon_i(\alpha) \mid \alpha \in \mathbf{2}^{C_i}\} \neq \varnothing$, and
- $\mathit{solr}(G_i)^* = \{\beta \in \mathbf{2}^{V_G} \mid \beta|_{V_{G_i}} \in \mathit{solr}(G_i)\}$.

Since each C_i is finite and since we have fixed a global choice λ from $\mathit{SC}(G)$, each $\mathit{solr}(G_i)$ is finite. Hence $\mathit{solr}(G_i)^*$ is a closed set in the product topology on $\mathbf{2}^{V_G}$ (with the discrete topology on $\mathbf{2}$). For every finite subset F of $\omega : \bigcap_{i\in F} \mathit{solr}(G_i)^* = \mathit{solr}(G_{\max F})^* \neq \varnothing$, since $\mathit{solr}(G_m)|_{V_{G_i}} \subseteq \mathit{solr}(G_i)$ for all $m, i \in F$ with $m \geq i$, by Lemma 3.17. Since $\mathbf{2}^{V_G}$ is a compact space, $\bigcap_{i\in\omega} \mathit{solr}(G_i)^* \neq \varnothing$. Finally, $\bigcap_{i\in\omega} \mathit{solr}(G_i)^* \subseteq \mathit{sol}(G)$. Take an arbitrary $\alpha \in \bigcap_{i\in\omega} \mathit{solr}(G_i)^*$ and $x \in V_G$. For some $i \in \omega : x \in V_{G_i}$, so $A_G(x) \subseteq V_{G_{i+1}}$, by Fact 3.13.4. Because $\alpha|_{V_{G_{i+1}}} \in \mathit{solr}(G_{i+1})$, the value $\alpha|_{V_{G_{i+1}}}(x)$ is correct. Thus $\alpha \in \mathit{sol}(G)$ since it is correct at every $x \in V_G$.

(b) To show that a graph G with a supersource, a finitary division $\langle C_i\rangle_{i\in\omega}$ and no odd cycles is KP, let H be an arbitrary nonempty induced subgraph of G. We show that H has a nonempty semikernel, which implies that G is KP by Theorem 2.13.

If H has a sink, then it is a semikernel of H, so assume $sinks(H) = \varnothing$. Let $h \in V_H$ be arbitrary and $k \in \omega$ be the least index for which $h \in V_{G_k}$, where indexing and G_k refer to the division of G into subgraphs according to Definition 3.11. Set $D_0 = \{h\}$ and, for each $i > k$, let $D'_{i-k} = C_i \cap A^*_H(h)$. Since C_{k+1} separates C_k from tails of all rays, and all paths leaving G_k intersect C_k by Fact 3.13.3, all rays in $\vec{h} \cap \vec{H}$ cross D'_1. Likewise, for $i > 1$, since C_{i+k} separates C_{i+k-1} from tails of all rays in $\vec{G}$, each D'_i separates D'_{i-1} from tails of all rays in $\vec{h} \cap \vec{H}$. Starting with $i = 1$ and using AC, we choose as D_i a minimal subset of D'_i separating D_{i-1} from tails of all rays, obtaining a finitary division $\langle D_i\rangle_{i\in\omega}$ of $A^*_H(h)$. By (a), the graph $A^*_H(h)$ has a kernel, which is nonempty since so is $A^*_H(h)$. Since $A^*_H(h)$ is also a free induced subgraph of H, this kernel is a nonempty semikernel of H by Fact 2.12. □

Local finiteness implies finite separability of each vertex from tails of all rays (implying safety, in the absence of odd cycles), so the following extends Richardson's Theorem 1.1.(a).

Corollary 3.18 *A graph G is KP if it has no odd cycles and satisfies any of the conditions:*

1. *$A^*_G(v)$ has a finitary division for every $v \in V_G$, or*
2. *every $v \in V_G$ is finitely separable from tails of all rays.*

Proof. 1. By Theorem 3.10, every $A^*_G(v)$ is KP, so the claim follows by Corollary 2.17.

2. For each $v \in V_G$, each $x \in A^*_G(v)$ is finitely separable from tails of all rays in $A^*_G(v)$, since x is finitely separable from tails of all rays in G. Each $A^*_G(v)$ has then a finitary division by Lemma 3.8, so G is KP by point 1. □

The proof of Theorem 3.10 concludes also the proof of Lemma 3.4: a safe $\simeq$-flat graph is KP if $P\ {}^{f}\!\not\precsim Q$ for every pair of rays. Lemma 3.4 together with Lemma 3.3 give kernel-perfectness of every safe $\simeq$-flat graph. This fact, together with Theorem 3.1, lead to the final result.

Theorem 3.19 *A safe graph G with finitely many $\simeq$-ends is KP.*

Proof. Each $\simeq$-flat subgraph of G is KP by Lemma 3.3 or Lemma 3.4, so the claim follows by Theorem 3.1 if we find a partition, $V_G = \biguplus_{i\in I} V_{G_i}$, such that (1) each G_i is $\simeq$-flat or rayless, and (2) each nonempty subset F of I has some $k \in F$ with G_k free in K, where $K = G[V_K]$ and $V_K = \bigcup_{i\in F} V_{G_i}$. By Observation 2.5.(b), we can assume

i. $\forall x \in V_G : \vec{x} \neq \varnothing$.

ii. Let $\{E_1, ..., E_n\}$ be a finite set of ends of G, and consider their strict partial order $\subset$. Let $E^{\downarrow}_j$ denote the set of strict subends of E_j, that is, $E^{\downarrow}_j = \{E_i \mid E_i \subset E_j\}$ and $F'_j = E_j \setminus \bigcup E^{\downarrow}_j$.
Define $F'_i \lhd' F'_j \Leftrightarrow E_i \subset E_j$, for $1 \leq i \leq n$ and $1 \leq j \leq n$. It follows that
(*) $F'_i \lhd' F'_j \Leftrightarrow F'_i \subset A^{\underline{*}}_G(F'_j)$ and $F'_i \lhd' F'_j \Rightarrow A^{\underline{*}}_G(F'_i) \cap F'_j = \varnothing$.
Distinct F'_i and F'_j may still intersect, when so do the $\subset$-unrelated ends E_i and E_j. We modify them obtaining disjoint sets as follows:
$F_0 = \bigcup_{1\leq i<j\leq n} F'_i \cap F'_j$, and
$F_i = F'_i \setminus F_0$ for $1 \leq i \leq n$.
We define $F_0 \ntriangleleft F_i$ for $0 \leq i \leq n$ and, for $i, j > 0$, apply (*) defining
(**) $F_i \lhd F_j \Leftrightarrow F_i \subset A^{\underline{*}}_G(F_j)$.

iii. The set $\{F_i \mid 0 \leq i \leq n\}$ partitions V_G. It covers V_G because every $x \in V_G$ belongs to some end by **i** and, since $\vec{\vec{G}}$ is finite, to some $\subset$-minimal end $E_j \setminus \bigcup E_j^{\downarrow} = F'_j$. So $V_G = \bigcup_{1 \leq j \leq n} F'_j$ but also $\bigcup_{1 \leq j \leq n} F'_j = \bigcup_{0 \leq j \leq n} F_j$. Distinct F_i, F_j are disjoint since $F_0 \cap F_i = \varnothing$ for all $i > 0$, while for $0 < i < j \leq n : F_i \cap F_j = (F'_i \setminus F_0) \cap (F'_j \setminus F_0) \subseteq (F'_i \cap F'_j) \setminus F_0 = \varnothing$.

iv. $G[F_0]$ is rayless. Toward a contradiction, suppose $V_R \subseteq F'_i \cap F'_j$, for some $F'_i \neq F'_j$ and ray R. Then $V_R \subseteq E_i \cap E_j$, giving an end $E_r = A^{*}_{G}(V_R)$ with $E_r \subseteq E_i \cap E_j$. If $E_i \subset E_j$ (or $E_j \subset E_i$), then $F'_i \lhd' F'_j$ (or $F'_j \lhd' F'_i$), hence $F'_i \cap F'_j = \varnothing$ by **ii**.(*), contradicting the existence of R. Since neither $E_i \subseteq E_j$ nor $E_j \subseteq E_i$, both $E_r \subset E_i$ and $E_r \subset E_j$, but then $V_R \cap F'_i = \varnothing$ by definition $F'_i = E_i \setminus \bigcup E_i^{\downarrow}$, so $V_R \not\subseteq F'_i \cap F'_j$.

Thus, $F'_i \cap F'_j$ contains no ray, for each pair i, j with $1 \leq i < j \leq n$. Since F_0 is the union of finitely many such intersections, if F_0 contains any ray R, then R contains infinitely many vertices from $F'_i \cap F'_j$, for some pair of distinct i, j, that is, $V_R \subseteq A^{*}_{G}(F'_i) \cap A^{*}_{G}(F'_j) \subseteq E_i \cap E_j$. Then $A^{*}_{G}(V_R) = E_r \subseteq E_i \cap E_j$, yielding a contradiction as in the previous paragraph.

Having no odd cycles and no rays, $G[F_0]$ is KP by Theorem 1.1.(b).

v. F_i is $\simeq$-flat for every $1 \leq i \leq n$. If R_i is a ray for which $E_i = A^{*}_{G}(V_{R_i})$ and Q a ray such that $V_Q \subseteq F_i$, then $A^{*}_{G}(V_Q) \subseteq A^{*}_{G}(F_i) \subseteq A^{*}_{G}(V_{R_i})$. If $A^{*}_{G}(V_Q) \neq A^{*}_{G}(V_{R_i})$, then $A^{*}_{G}(V_Q) \subset A^{*}_{G}(V_{R_i})$, so $A^{*}_{G}(V_Q) \in E_i^{\downarrow}$ and $A^{*}_{G}(V_Q) \cap F_i = \varnothing$ by **ii**.(**), contradicting $V_Q \subseteq F_i$. Hence $A^{*}_{G}(V_Q) = A^{*}_{G}(V_{R_i})$, that is, $Q \simeq R_i$. By Lemma 3.3 or Lemma 3.4, $G[F_i]$ is KP.

vi. Since the number n of ends is finite, if $\varnothing \neq S \subseteq \{F_0, ..., F_n\}$, then S has a $\lhd$-maximal element F_m. Setting $V_K = \bigcup_{i \in S} F_i$, $K = G[V_K]$, and $F = G[F_m]$, we claim that F is free in K. Toward a contradiction, suppose that for some $x \in F_m$, there is some $F_i \in S, F_i \neq F_m$, with some $y \in F_i \cap A_K(x)$. Note that $i \neq 0$ because $F_0 \cap A_G(F_j) = \varnothing$ for $1 \leq j \leq n$. Since $y \in F_i \subseteq E_i$ and $E_i = A^{*}_{G}(V_{R_i})$ for some ray R_i, the edge $x \to y$ implies also $x \in E_i$. Maximality of F_m in S gives two cases, each leading to a contradiction:

(a) If $F_i \lhd F_m$, then $E_i \in E_m^{\downarrow}$, so $x \notin F_m \subseteq E_m \setminus \bigcup E_m^{\downarrow}$.

(b) If $F_i \not\lhd F_m$, then let E_j be the $\subset$-minimal subend of E_i containing x. By its minimality, $x \in F'_j$. Then $x \in F'_j \cap F_m \subseteq F'_j \cap F'_m \subseteq F_0$, so $x \notin F_m = F'_m \setminus F_0$.

Hence, F_m is free in K.

The set $\{F_0, ..., F_n\}$ partitions thus V_G by **iii**, each of its nonempty subsets S has an element free in the subgraph $G[\bigcup S]$, **vi**, and each $G[F_i]$ is KP: for $i = 0$ by **iv**, while for $i > 0$ by **v**. The graph is thus KP by Theorem 3.1. □

Consequently, for any definition of a digraph minor admitting subgraphs and edge contractions along directed paths, a digraph is KP if it has finitely many $\simeq$-ends, no odd cycles and no $\langle \omega, < \rangle$-minor.

A special case is a graph with finitely many $\overset{\omega}{\simeq}$-ends, which are finer than $\simeq$-ends. On the other hand, a safe graph with infinitely many rays, each two satisfying $R_i \ {}^{f}\!\preceq R_j$ and $R_i \ {}^{\omega}\!\not\preceq R_j$, has infinitely many $\overset{\omega}{\simeq}$-ends but is KP by Theorem 3.19, having only one $\simeq$-end. Also, many safe graphs with infinitely many $\simeq$-ends can be shown KP by Theorem 3.1, Theorem 3.10 or Corollary 3.18, as exemplified by (1.4). Conjecture 1.2, that safety ensures solvability of arbitrary graphs, seems plausible but proving it remains an open problem.

Besides safety, parities of the involved paths play the obvious role. One can, for instance, admit arbitrary dominating vertices as long as subgraphs reachable from them have bipartite tails. More specific parity conditions on acyclic paths might therefore deserve closer attention.

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