University of Bergen

Department of Physics and Technology
Particle Physics

Master Thesis

Renormalization of QED with a spontaneously broken gauge symmetry

By Ivar Mathias Grøvik
Supervisor: Jörn Kersten

December 14, 2020

Abstract

In this thesis we take an in-depth look at the process of renormalization. Important details such as regularization and renormalizability is also discussed. Renormalization of QED is used as a starting point to explore three schemes: on-shell, momentum subtraction and minimal subtraction. We then extend QED to include a complex scalar field that gives the photon mass through the Higgs mechanism, and apply the minimal subtraction scheme to find the renormalization constants and β_e and β_λ functions.

Acknowledgments

A huge thank you to my supervisor Jörn Kersten for being so patient with me, and being so helpful both in person and online, as we all have had to adjust to the situation out in the world.

I would like to also thank my girlfriend for encouraging me, and always being there for me. My friends also needs to be mentioned, thank you for making these years amazing. Last but not least my parents and sister, for always watching out for me, and pushing me to achieve my goals. None of this would be possible without any of you!

Contents

1 Introduction

In the search for extensions of the standard model, many exotic and interesting theories have been proposed. A hidden sector: particles and forces that remain hidden to us for one reason or another is a popular field of study. Particles that interact too weakly with standard model matter, or are so massive that their mediated force becomes too short range to detect are theorized in this sector. The simplest case would be a massive photon, interacting so weakly that it has avoided detection [1]. Nothing in the standard model prevents such a particle. In fact the Brout-Englert-Higgs (BEH)-mechanism provides the answer for how it might be created. The concept may be taken further; This hidden photon could mix with the massless photon, in a process known as kinetic mixing.

In contrast to these proposed extensions, another part of the standard model, renormalization, is quite well established at this point. This procedure provides the solution to the complication of divergences in loop integrals and allows us to extract observable quantities. A natural question is whether such a theory as kinetic mixing between a massless and a hidden photon can be renormalized, and what could be learned from this. The kinetic mixing parameter is a parameter that describes the strength of the interaction between these two photons. A β-function is a function that describes how a parameter changes with a change in renormalization scale. A classic example is the β-function of the electric charge e. This function determines the running coupling of e. Such a function should exist for the kinetic mixing parameter as well.

In this thesis we aim to explore the framework of renormalization and apply it to a photon that gains mass through the BEH mechanism. We explore this through QED and a complex scalar that acquires a non-zero vacuum expectation value. An in-depth discussion on the renormalization of QED is also included. In sections 2 and 3 we develop the tools necessary for renormalization, and in section 4 these tools are applied to QED. The minimal subtraction scheme is developed in section 5, along the Passarino-Veltman (P.V) functions. In section 6 we explore the renormalization of spontaneously broken QED. Topics in this section also include the R_ξ-gauge, which is useful for renormalization.

1.2 Conventions and tools

All Feynman diagrams are made using the *TikZ-Feynman* latex package[2]. For the heavier loop calculations we have used Mathematica, with the package *FeynCalc* ([3], [4], [5]). This allows for vastly more efficient calculations and is also a convenient way to double check results.

In the first part of this thesis (sections 4 and 5), Feynman gauge is exclusively used, meaning $\xi = 1$. This allows for easier to use propagators and simplifies results. This simplification is unable to be applied to the spontaneously broken theory, covered in section 6.

We use the covariant derivative

$$D^\mu = \partial^\mu - ieA^\mu.$$

For constructing Feynman rules for terms in the Lagrangian, the convention of simply multiplying by i has been chosen.

2 Regularization

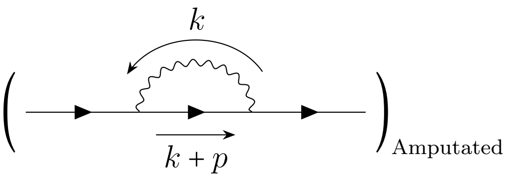
2.1 Justification

First order corrections to QED appear in the form of diagrams such as

(a) (b)

Figure 1: Examples of first order corrections to QED.

Corresponding to corrections to pair annihilation into pair creation, and Compton scattering, respectively. At each loop there is an integral over loop momenta k. These integrals are best studied without the external legs present, and from now on, we shall study the amputated diagrams. Figure 1(b) has the amputated amplitude



$$\equiv \Sigma(p). \tag{2.1.1}$$

The scalar Σ function has the form

$$\Sigma(p) =(-ie_0)^2 \int \frac{d^4k}{(2\pi)^4}\gamma_\mu \frac{i(\not{k}+\not{p}+m_0)}{(k+p)^2-m_0^2+i\varepsilon}\gamma_\nu i\frac{-g^{\mu\nu}}{k^2-\lambda^2+i\varepsilon}. \tag{2.1.2}$$

A standard method of calculating loop integrals is a Wick rotation, which is a change of variables defined as [6, p. 17]

$$k^0 \equiv ik_E^4 \ \ , \vec{k} = \vec{k}_E$$
$$\rightarrow k^2 = -k_E^2 \equiv -\vec{k}_E^{\,2} - (k_E^4)^2.$$

As an example we can consider the integral

$$\begin{aligned}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2+i\varepsilon)^2} &= \frac{i}{8\pi^2}\int_0^\infty \frac{k_E^3}{(k_E^2+m^2)^2}dk_E \\ &= \frac{i}{8\pi^2}\cdot\frac{1}{2}\left(\frac{m^2}{m^2+k_E^2}+\ln m^2+\ln\left(1+\frac{k_E^2}{m^2}\right)\right)\Bigg|_0^\infty,\end{aligned} \tag{2.1.3}$$

where the divergence is now made clear. In order to obtain this form, it is helpful to use Feynman parametrization, summarized in appendix B.2. This is just a convenient way of rewriting the denominator. Applying this to (2.1.2) would result in [7, p. 21]

$$\Sigma(p) \sim \ln l_E\Big|_0^\infty. \tag{2.1.4}$$

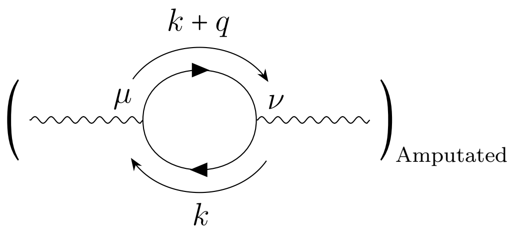
Hence the fermion self-energy is logarithmically divergent. The process of regularization aims to rewrite loop expressions into a limit of a convergent integral [8, p. 104]. Making the divergences explicit in this way allows us to subtract them, rendering the renormalized theory finite, this next step is the subject of section 3.

2.2 Cut-off

Perhaps the most intuitive method of regularization is introducing an upper limit Λ for the integration region, to remove the divergent high momentum region [8, p. 106], [9]. For the fermion self-energy

$$\Sigma(p) \sim \ln \Lambda, \tag{2.2.1}$$

where it is understood that Λ tends to infinity. As long as this limit is not taken, the amplitude remains finite. This method has the problem of breaking gauge symmetry. The vacuum polarization amplitude is defined as



$$\equiv \Pi_{\mu\nu}(q), \tag{2.2.2}$$

where, with a UV-cutoff, the loop amplitude has the form

$$\Pi_{\mu\nu} = -(ie_0)^2 \int_0^{\Lambda} \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma_\mu \frac{i(\not{k} + m_0)}{k^2 - m_0^2} \gamma_\nu \frac{i(\not{k} + \not{q} + m_0)}{(k+q)^2 - m_0^2} \right]. \tag{2.2.3}$$

Going through a normal loop calculation with a Wick rotation would result in a leading term [10, p. 248].

$$\Pi_{\mu\nu} \propto e^2 \Lambda^2 g_{\mu\nu}, \tag{2.2.4}$$

and no $q_\mu q_\nu$ term, resulting in an infinite photon mass, and ruining gauge invariance. This is a common problem for cut-off regularization, and it is not suitable for gauge theories.

2.3 Pauli-Villars regulator

Pauli-Villars regularization [11] is based on the concept of introducing ghost particles that cancel physical particle loop momenta at high energies. For each particle in the theory, the propagator is modified by [12, p. 12]

$$\frac{i}{p^2 - m^2} \to \frac{i}{p^2 - m^2} - \frac{i}{p^2 - \Lambda^2} = \frac{i(m^2 - \Lambda^2)}{(p^2 - m^2)(p^2 - \Lambda^2)}.$$

The modified propagator approaches the original as $M \to \infty$. The behavior of the integral is improved at large energies due to the higher power of k [6, p. 18]. Pauli-Villars ghosts either have the opposing sign in the kinetic term, for example the ghost photon in QED would have $+\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$, or the opposing statistic, such as fermion ghosts being bosonic

[13, p. 832]. The divergent integral (2.1.3) can be made finite with this. Adding a fictitious particle, the integral becomes

$$\int \frac{d^4k}{(2\pi)^4}\left(\frac{1}{(k^2-m^2+i\varepsilon)^2}-\frac{1}{(k^2-\Lambda^2+i\varepsilon)^2}\right)=\frac{i}{16\pi^2}\ln\frac{\Lambda^2}{m^2}.$$

QED can be regularized using this method, as it preserves gauge invariance in both QED and QCD [8, p. 107]. However, the method is not simple as several sets of Pauli-Villars fermions are required in order to regularize the vacuum polarization (4.3.3) [10, p. 248].

2.4 Dimensional regularization

The final method we will discuss, and the one which will be used from this point, is dimensional regularization [14]. Dimensional regularization directly modifies the space-time dimension d, inspired by the observation that by reducing the number of multiple integrals, an integral could be made convergent. In this way, all symmetries are preserved [8, p. 108]. The space-time dimension d is redefined as[1]

$$d = 4 - \epsilon. \tag{2.4.1}$$

In the limit $\epsilon \to 0$, the original theory is restored. The space-time dimension d is treated as a continuous variable, and in this sense, ϵ acts as a regulator [12, p. 14]. In this new dimension, standard formulae for Wick rotations are modified [13, p. 825], one such example is

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-\Delta)^n}=i\frac{(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}. \tag{2.4.2}$$

Properties of the Γ-function are given in appendix B.4. Most importantly, it can be expanded around zero and at negative values

$$\Gamma(\epsilon)=\frac{1}{\epsilon}-\gamma_E+O(\epsilon). \tag{2.4.3}$$

The divergent term $\sim \frac{1}{\epsilon}$ is now isolated from the finite terms.

A redefinition of the space-time dimension d has the consequence of changing the dimensions of fields and parameters. The Lagrangian density still has mass dimension d, in order to keep the action dimensionless [13, p. 828]. The fields and parameter therefore have the mass dimension [15]

$$[\psi]=\frac{d-1}{2}=\frac{3}{2}-\frac{\epsilon}{2},\ \ [A^\mu]=\frac{d-2}{2}=1-\frac{\epsilon}{2},\ \ [m]=1,\ \ [e]=\frac{4-d}{2}=\frac{\epsilon}{2}.$$

To have a dimensionless coupling constant, the dimension can be extracted

$$e \to \mu^{\frac{\epsilon}{2}}e, \tag{2.4.4}$$

where μ is a auxiliary mass scale with mass dimension 1, and e is dimensionless [13].

[1]Some authors use the definition $d = 4 - 2\epsilon$, this has the advantage of getting rid of factors of 2 in some results.

Propagators in QED will always have a factor of e^2. The auxiliary mass scale has a dependency on ϵ, which in turn will slightly modify the result of an expansion around $\epsilon = 0$. Integrals which involve parameters such as e are slightly modified [13, p. 828]. Looking at (2.4.2) for $n = 2$, this is modified by a factor μ^ϵ

$$\int \frac{d^dk}{(2\pi)^d}\frac{\mu^\epsilon}{(k^2-\Delta)^2} = \frac{i}{(4\pi)^{d/2}}\Gamma(2-\frac{d}{2})\left(\frac{\mu^2}{\Delta}\right)^{\frac{\epsilon}{2}} \tag{2.4.5}$$

Expanding around $\epsilon = 0$ yields

$$\begin{aligned} i\frac{\mu^\epsilon}{(4\pi)^{d/2}}\Gamma(2-\frac{d}{2})\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} &= \frac{i}{(4\pi)^2}\Gamma(\frac{\epsilon}{2})\left(\frac{4\pi\mu^2}{\Delta}\right)^{\frac{\epsilon}{2}} \\ &= \frac{i}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma_E + \ln 4\pi + \ln\mu^2 - \ln\Delta + O(\epsilon)\right). \end{aligned}$$

Similar analysis on the integrals gives the same result. The pole is unaffected, and the scale μ is absorbed into the finite terms. Other useful d-dimensional integrals are given in appendix B.3.

As a final note, altering the space-time dimension has a second effect, namely altering the Clifford algebra of the γ-matrices. This is summarized in appendix B.1. There is an issue on how to handle the γ_5-matrix, however as we do not encounter γ_5 in this work, we choose to skip over this.

All the necessary tools are now in place. We present a shortened version of the regularization of the fermion self-energy (4.3.1) using dimensional regularization. The full derivation can be found in appendix B.5.1. Note that the full calculation is done in renormalized theory, meaning we use e instead of e_0. We will discuss this in more detail in section 4.

$$\Sigma(p) = (-ie_0)^2\int\frac{d^4k}{(2\pi)^4}\gamma_\mu\frac{i(\not{k}+\not{p}+m_0)}{(k+p)^2-m_0^2}\gamma_\nu\frac{-ig^{\mu\nu}}{k^2-\lambda^2}. \tag{2.4.6}$$

The mass λ is a fictitious photon mass, which is to be taken to the limit $\lambda \to 0$. This is a regulator for infrared divergence, which is another type of divergence that occur at low energies. In QED this does not violate gauge invariance [16, p, 265]. Using the Feynman parametrization (B.2.1a) some terms cancel. Completing the square, substituting $l = k + xp$ and ignoring linear terms in l, which will vanish, yields

$$\Sigma(p) = -e_0^2\int\frac{d^4l}{(2\pi)^4}\int_0^1 dx\frac{\gamma_\mu\left((1-x)\not{p}+m_0\right)\gamma^\mu}{(l^2-\Delta)^2}, \tag{2.4.7}$$

where $\Delta = m_0^2x + (x-1)(p^2x-\lambda^2)$. These steps can be found in other regularization procedures as well. Since we want to use dimensional regularization now, the dimension is redefined to $d = 4-\epsilon$, and using the modified Clifford algebra (B.1.4)

$$\Sigma(p) = -\mu^\epsilon e_0^2\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{(2-d)(1-x)\not{p}+dm_0}{(l^2-\Delta)^2}. \tag{2.4.8}$$

Using the integral (B.3.1a) and $d = 4 - \epsilon$ we arrive at the final result

$$\Sigma(p) = -i\frac{e_0^2}{(4\pi)^2}\int_0^1 \Big((\epsilon - 2)(1-x)\not{p} + (4-\epsilon)m\Big)\,\Gamma(\frac{\epsilon}{2})\left(\frac{4\pi\mu^2}{\Delta}\right)^{\frac{\epsilon}{2}} dx, \tag{2.4.9}$$

The limit $\epsilon \to 0$ now restores the divergent amplitude; however the divergence is now isolated in the term $\frac{\epsilon}{2}$ the expansion of Γ gives. The rest of the terms are finite.

3 Renormalization

3.1 Concept of renormalization

Regularization allows us to rewrite loop integrals into more manageable forms dependent on a regulator. Taking the limit of this regulator once again causes the amplitude to diverge. Renormalization is the process of isolating the divergences to unphysical quantities, removing divergences from physically measurable quantities [17].

Loop corrections arising from higher order perturbation theory alters the parameters of the theory. Original parameters are called bare parameters, denoted as e_0. These are unphysical, and can be divergent without issues [18, p. 188]. Bare parameters are related to their finite counterpart by one of two simple definitions.

$$e_0 \equiv Z_e \cdot e. \tag{3.1.1}$$

This is known as *multiplicative renormalization*; the other definition is named *additive renormalization* and is given by

$$e_0 \equiv e + \delta e. \tag{3.1.2}$$

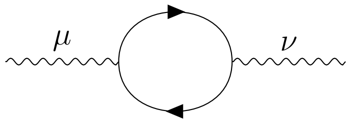
Z_e and δe are known as renormalization constants. They are divergent, allowing the renormalized quantity e to be finite.

The counter-term method consists of using the split of bare quantity into renormalization constant and renormalized quantity to construct a counter-term Lagrangian. These counter-terms contain the renormalization constants, and absorb terms from the divergent amplitudes, rendering observable quantities finite in the end. A renormalization scheme is a prescription of what the counter-terms absorb, which leads to a scheme dependence for the renormalization constants. This will be shown more in-depth in section 4.

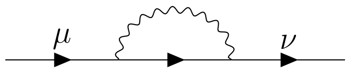
3.2 Renormalizability

A renormalizable theory means a theory whose divergences can be removed by a finite number of renormalization constants and interaction parameters. This has to apply to all orders of perturbation theory, thus in order to determine if a theory is renormalizable, we first need to know what diagrams are divergent, and what interactions are renormalizable [8].

To determine what amplitudes are divergent, the method of power-counting is often used. As an example, consider the diagrams



$$\sim d^4k\frac{1}{k^4}\cdot k^2,$$



$$\sim d^4k\frac{1}{k^4}k.$$

We can define s to be the resulting momenta power. Generally, the diagram is power-divergent for $s > 0$, logarithmically divergent for $s = 0$ and finite for $s < 0$. The number s is called the *superficial degree of divergence* and can, to 1-loop order, be generalized in dimension d as [8, p. 126]

$$s = d \cdot l + \Sigma_v \delta_v - 2n_b - n_f, \tag{3.2.1}$$

where l is the number of loops, δ_v is the number of momentum factors at vertex v and n_b, n_f are the number of boson and fermion internal lines respectively.

The next question is what interactions are renormalizable. To this end, we need to rewrite (3.2.1) in terms of external fields and define the *index of divergence* of the interaction.

$$\begin{aligned} s &= \sum_i r_i n_i - \frac{d-2}{2}N_B - \frac{d-1}{2}N_F + d \\ r_i &= \frac{d-2}{2}b_i + \frac{d-1}{2}f_i + \delta_i - d, \end{aligned} \tag{3.2.2}$$

where δ_i is the number of number of space-time derivatives, n is the number of vertices corresponding to $\mathscr{L}_I$, b_i, f_i are the number of boson or fermion fields in $\mathscr{L}_I$, and N_b, N_f are the number of external boson or fermion fields.

The index of divergence r has a key role in determining renormalizability. The value of r_i is only dependent on the interaction term $\mathscr{L}_I$. An interaction can be grouped into three categories based on these values [8, p. 131].

- $r > 0$: the theory is non-renormalizable, as for higher orders, an unlimited amount of new divergences appear, which cannot be removed by a finite number of renormalization coefficients.
- $r = 0$: the theory is renormalizable, the types of divergences are finite, and there is a chance that they can be removed by a finite number of coefficients.
- $r < 0$: the theory is super renormalizable, the number of divergent diagrams also becomes finite.

For QED in four dimensions, where the only interaction term is

$$\mathscr{L}_I = e\gamma_\mu\bar{\psi}A^\mu\psi,$$

the index of divergence takes the value

$$r = b + \frac{3}{2}f + \delta - 4 = 1 + 3 - 4 = 0.$$

QED is thereby renormalizable.

4 Renormalization of QED

In this section we summarize the renormalization of QED in two schemes, the on-shell scheme and momentum subtraction scheme. Renormalization constants Z_i can be determined in two ways, using Green functions or the counter-term method [19, p. 119]. Here we use the counter-term method to determine the renormalization constants in the momentum subtraction scheme. Furthermore, the β-function of QED is derived in the momentum-subtraction scheme.

4.1 Counter-terms

The counter-term method is based on using the relation between bare and renormalized variables to obtain counter-terms. These terms give Feynman rules containing the constants Z_i, which then can be determined when combined with an renormalization scheme. The bare QED Lagrangian is given by

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu,0}F_0^{\mu\nu} + \bar{\psi}_0(i\not{\partial} - m_0)\psi_0 - e_0 A_0^{\mu}\bar{\psi}_0\gamma_\mu\psi_0 \tag{4.1.1}$$

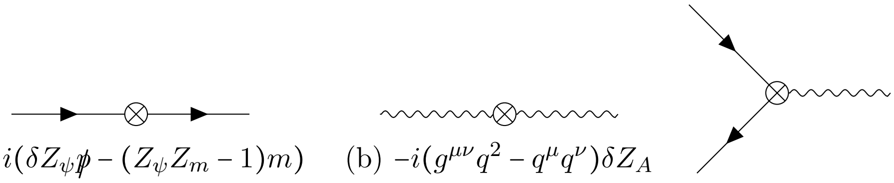
All possible counter-terms are already contained inside, and thus *multiplicative renormalization* can be used [18, p. 200]. Fields can be multiplicatively renormalized without any complications; The masses require more consideration. In a massless theory, multiplicative renormalization of the mass $m_0 = Z_m m$ would result in the bare mass always being equal to zero since m, the observable mass, is zero. Authors often use additive renormalization for masses, to relieve any potential problems. For QED this is not an issue, and the mass is renormalized multiplicatively.

$$\psi_0 = Z_\psi^{1/2}\psi,\ A_0^\mu = Z_A^{1/2}A^\mu,\ e_0 = Z_e\mu^{\frac{\epsilon}{2}}e,\ m_0 = Z_m m. \tag{4.1.2}$$

In the spirit of perturbation theory, the renormalization constants can be rewritten as $Z_\psi = 1 + \delta Z_\psi$.

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - \mu^{\frac{\epsilon}{2}}eA^\mu\bar{\psi}\gamma_\mu\psi \\ &-\frac{1}{4}\delta Z_A F_{\mu\nu}F^{\mu\nu} + \delta Z_\psi\bar{\psi}i\not{\partial}\psi - (Z_\psi Z_m - 1)\bar{\psi}m\psi - (Z_e Z_A^{1/2}Z_\psi - 1)\mu^{\frac{\epsilon}{2}}eA^\mu\bar{\psi}\gamma_\mu\psi.\end{aligned} \tag{4.1.3}$$

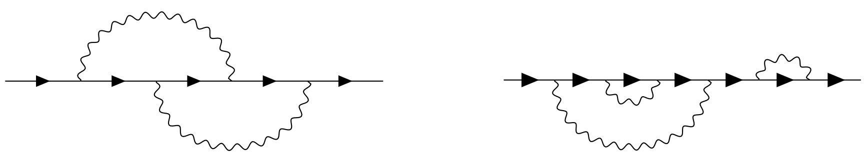
This splits the Lagrangian into two parts, the familiar, but now renormalized QED Lagrangian $\mathscr{L}_R$, and counter-terms $\mathscr{L}_{\text{CT}}$. Amplitudes are now calculated using $\mathscr{L}_R$, and from $\mathscr{L}_{\text{CT}}$ one obtains the explicit forms of the counter-terms [10, p. 332].



(a) $i(\delta Z_\psi \not{p} - (Z_\psi Z_m - 1)m)$ (b) $-i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta Z_A$ (c) $-(Z_e Z_A^{1/2}Z_\psi - 1)\mu^{\frac{\epsilon}{2}}e\gamma_\mu$

Figure 2: The counter-terms of QED.

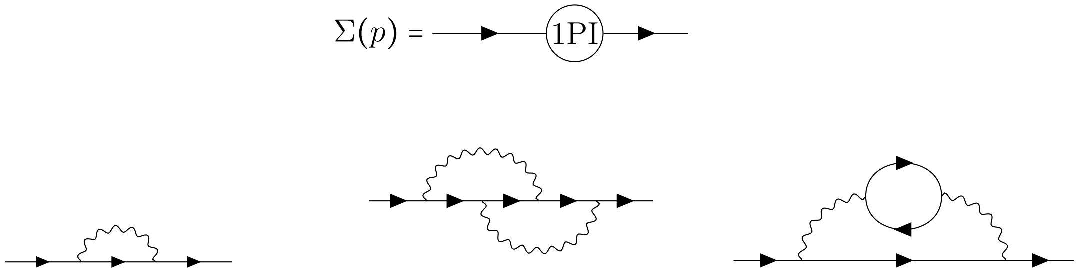
In order to properly write down the renormalized propagators and vertex function, an additional definition is needed; *1-Particle irreducible diagrams*. These diagrams have the property that they have no lines which can be removed or cut in order to construct two new viable diagrams. As an example [10, p. 219]



(a) Irreducible diagram. (b) Reducible diagram.

Figure 3: Irreducible and reducible diagram examples.

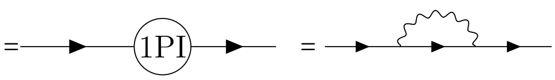
Reducible diagrams can always be deconstructed into subdiagrams. Irreducible diagrams represent the fundamental loop diagrams of perturbation theory, and to each order there several irreducible diagrams. Therefore 1-PI diagrams allow for easier grouping of the divergent loop diagrams of any order of α. Following [10], they are denoted Σ, Π and Γ for the fermion, photon and vertex, respectively.

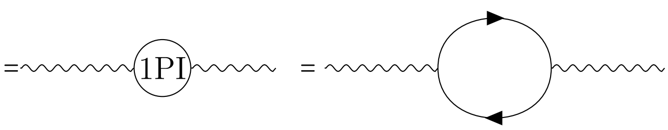


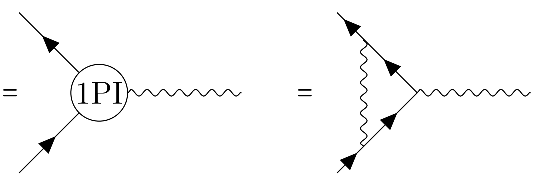
(a) (b) (c)

Figure 4: Notation for 1-PI diagrams, and some contributions.

Note that if one restricts perturbation expansion to the one loop order, these definitions simplify to

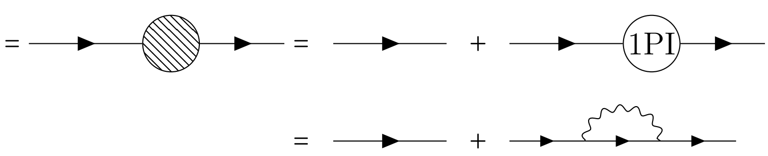
$\Sigma(p) =$  (4.1.4)

$\Pi_{\mu\nu}(q) =$  (4.1.5)

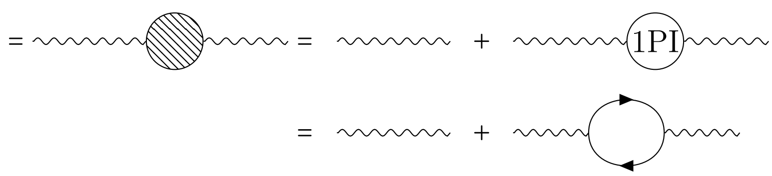
$\Gamma_\mu(p, q) =$  (4.1.6)

Using this new notation, the full propagators and QED vertex can be diagrammatically deconstructed as [10, p. 330]

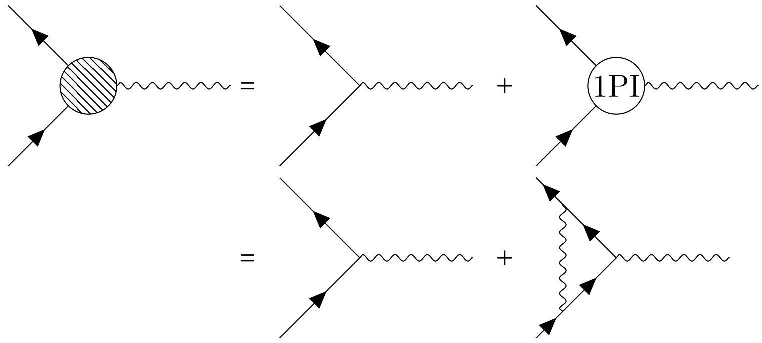
$$\frac{iZ_\psi}{\not{p} - m} =$$



$$-\frac{iZ_A g_{\mu\nu}}{q^2} =$$

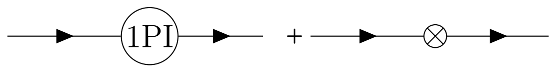


$$-ie\gamma_\mu + \Gamma_\mu =$$

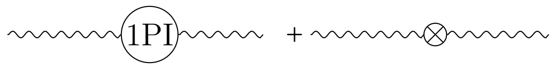


To make the shift into renormalized perturbation theory, the 1-Particle-Irreducible expressions (4.1.4) and (4.1.5), and the vertex function (4.1.6) are redefined, to include counter-terms.

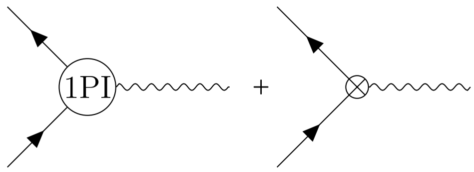
$$\Sigma(p) = \tag{4.1.7}$$



$$\Pi_{\mu\nu}(q) = \tag{4.1.8}$$



$$\Gamma_\mu(p,q) = \tag{4.1.9}$$



4.2 Renormalization schemes

In order to properly determine the renormalization constants Z_i, one needs to specify conditions on the renormalized propagators and vertex functions. This is equivalent with choosing a renormalization scheme. A key difference of schemes is the way they treat the finite terms. Renormalization constants varies depending on the chosen scheme. Observables do not, however, and any scheme must produce the same values for observable quantities.

4.2.1 On-shell scheme

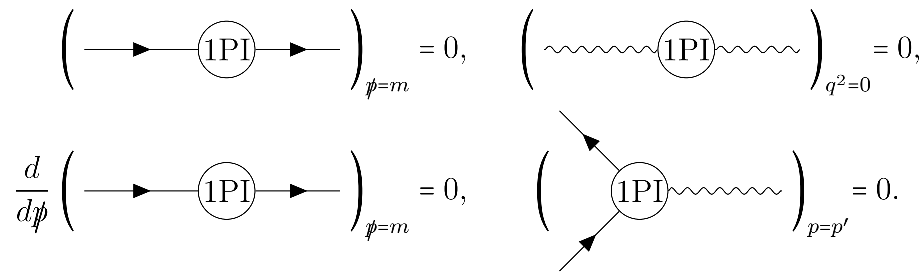
The on-shell scheme for QED is defined by associating the divergent poles with the physical mass of the fermions. This means that the renormalized mass m is the actual physical mass which can be observed through experiments. This relates the renormalization process nicely to the physical world, but it is a luxury few theories provide. Expanding $\Sigma(p)$ around $\not{p} = m$ shows it is ill-defined, possessing an infrared divergence as well as ultraviolet. This infrared divergence can be remedied by introducing a fictitious photon mass λ. This normally breaks the gauge-invariance of the theory. QED is, on the other hand, special in that the gauge invariance of the calculation is not affected [16, p. 265].

In the on-shell scheme, the conditions imposed on the 1-PI diagrams can be summarized as follows [10, p. 332]

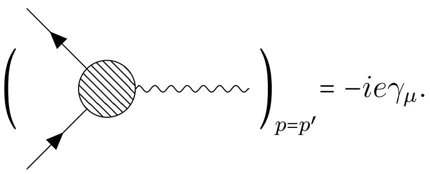
$$\Sigma(\not{p} = m) = 0, \qquad (4.2.1a) \qquad\qquad \Pi(q^2 = 0) = 0, \qquad (4.2.1c)$$

$$\frac{d}{d\not{p}}\Sigma(\not{p})|_{\not{p}=m} = 0, \qquad (4.2.1b) \qquad\qquad \Gamma_\mu(p' - p = 0) = 0. \qquad (4.2.1d)$$

Or diagrammatically as



Note that the condition on the vertex can equivalently be written



It is important to keep in mind the new definition of these diagrams, meaning the inclusion of the counter-terms. Using these conditions, one can obtain expressions for the renormalization constants. Using the explicit forms of the counter-terms, seen in figure 2.

$$\Sigma(\not{p} = m) + i(\delta Z_\psi \not{p} - (Z_\psi Z_m - 1)m) = 0, \qquad (4.2.2a)$$

$$\frac{d}{d\not{p}}(\Sigma_2(\not{p} = m) + i(\delta Z_\psi \not{p} - (Z_\psi Z_m - 1)m)) = 0, \qquad (4.2.2b)$$

$$(g_{\mu\nu}q^2 - q_\mu q_\nu)\Pi(q^2 = 0) - i(g_{\mu\nu}q^2 - q_\mu q_\nu)\delta Z_A = 0, \qquad (4.2.2c)$$

$$\Gamma_\mu(p' = p) - i(Z_e Z_A^{1/2} Z_\psi - 1)\mu^{\frac{\epsilon}{2}} e\gamma_\mu = 0. \qquad (4.2.2d)$$

Solving these leads to the renormalization constants

$$Z_\psi(1 + Z_m)m = -\Sigma(m) \qquad (4.2.3a) \qquad Z_A = 1 + \Pi(0) \qquad (4.2.3c)$$

$$Z_\psi = 1 - \frac{d}{d\not{p}}\Sigma(m) \qquad (4.2.3b) \qquad (Z_e Z_A^{1/2} Z_\psi - 1)\mu^{\frac{\epsilon}{2}} e\gamma_\mu = \Gamma_\mu(p' = p) \qquad (4.2.3d)$$

All four renormalization constants are now determined, the renormalization of QED is complete; the divergences which occur at 1 loop order are properly canceled by the counter-terms.

Using these expressions, or alternatively the *Ward Identity* of QED [18, p. 201], it can be shown that for all orders of perturbation theory, the following holds [10, p. 334]

$$Z_e = Z_A^{-1/2}. \qquad (4.2.4)$$

4.2.2 Momentum subtraction scheme

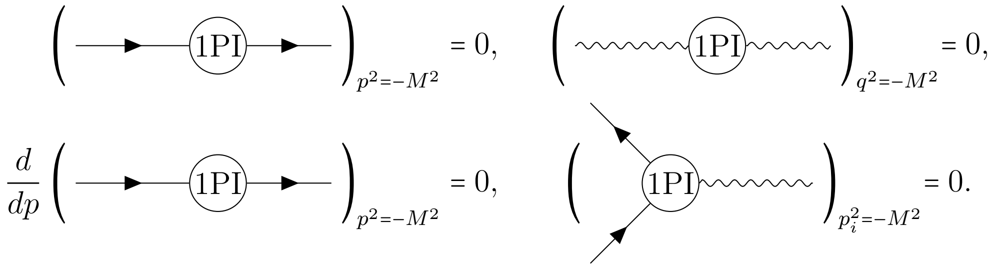
As mentioned earlier, multiplicative renormalization for the mass is fine in QED, since $m \neq 0$, and in a massless theory, one would have to use additive renormalization for the mass. There are, however, more steps required if one were to use a massless theory. In massless QED, the on-shell scheme would be vague, at least in the form discussed in the previous section. Equations (4.2.2) would not make much sense, and there are potential infrared divergences to worry about.

For QED, the on-shell scheme is enough to renormalize the theory, however there are some caveats. While this scheme immediately provides an intuitive understanding of the mechanism behind renormalization, several interesting phenomena, for example the *β-function* are more intricate to calculate. This can be seen by the generic form of the β-function[15, p. 13]

$$\beta(e) = \mu\frac{de}{d\mu} = -\epsilon e(\mu).$$

In taking the limit $\epsilon \to 0$ and restoring regular QED, the renormalized coupling constant becomes a scale independent constant. The on-shell scheme still has a β-function [20] but we will not focus on this issue here.

A convenient way around this is the closely related *Momentum subtraction scheme*, in which the counter-terms cancel at an arbitrary *renormalization scale M*. In regard to the β-function this means the renormalized parameters are no longer the physical parameters $e \neq e_{physical}$, and they can inhibit a scale dependence.



$$\Sigma(p)_{p^2=-M^2} = 0, \qquad (4.2.5a) \qquad\qquad \Pi(q)|_{q^2=-M^2} = 0, \qquad (4.2.5c)$$

$$\frac{d}{d\not{p}}\Sigma(\not{p})|_{p^2=-M^2} = 0, \qquad (4.2.5b) \qquad\qquad \Gamma_\mu(p_i)|_{p_i^2=-M^2} = 0. \qquad (4.2.5d)$$

In the same way as in the on-shell scheme, including the counter-terms leads to equations for the renormalization constants

$$\begin{aligned}
\Sigma(p)|_{p^2=-M^2} + i(\delta Z_\psi \not{p} - (Z_\psi Z_m - 1)m) &= 0, \\
\frac{d}{d\not{p}}\big(\Sigma(p)|_{p^2=-M^2} + i(\delta Z_\psi \not{p} - (Z_\psi Z_m - 1)m)\big) &= 0, \\
(g_{\mu\nu}q^2 - q_\mu q_\nu)\Pi(q^2)|_{q^2=-M^2} - i(g_{\mu\nu}q^2 - q_\mu q_\nu)\delta Z_A &= 0, \\
\Gamma_\mu(p_i)|_{p_i^2=-M^2} - i(Z_e Z_A^{1/2} Z_\psi - 1)\mu^{\frac{\epsilon}{2}} e\gamma_\mu &= 0.
\end{aligned}$$

This leads to the following expressions for the renormalization constants

$$(\delta_\psi \not{p} - (Z_\psi Z_m - 1)m) = \Sigma(p)|_{p^2=-M^2} \qquad (4.2.6a)$$

$$Z_\psi = 1 - \frac{d}{d\not{p}}\Sigma(p)|_{p^2=-M^2} \qquad (4.2.6b)$$

$$Z_A = 1 + \Pi(q)|_{q^2=-M^2} \qquad (4.2.6c)$$

$$(Z_e Z_A^{1/2} Z_\psi - 1)\mu^{\frac{\epsilon}{2}} e\gamma_\mu = \Gamma_\mu(p)|_{p^2=-M^2}, \qquad (4.2.6d)$$

which concludes the renormalization in the momentum subtraction scheme.

4.3 Amplitude calculations

Now that the conditions on each of the 1-PI diagrams and vertex function has been set, it is time to find explicit expressions for the loop amplitudes. Regularization was covered in section 2, and a shortened version of the fermion self-energy regularization using dimensional regularization was shown for the bare theory. The split of the Lagrangian into a renormalized Lagrangian and a counter-term Lagrangian (4.1.3) means that the calculated amplitudes are now functions of the renormalized parameters, e and m in this case, instead of the bare parameters.

4.3.1 Electron self-energy

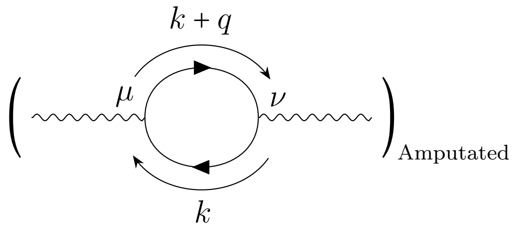
For the example of fermion self-energy already shown in section 2, nothing changes except for the swapping $e_0 \to e$, and we simply list the result here

$$\Sigma(p) = -i\frac{e^2}{(4\pi)^2}\int_0^1 \big((\epsilon-2)(1-x)\not{p} + (4-\epsilon)m\big)\,\Gamma(\frac{\epsilon}{2})\left(\frac{4\pi\mu^2}{\Delta}\right)^{\frac{\epsilon}{2}} dx, \tag{4.3.1}$$

where $\Delta = m^2x + (x-1)(p^2x - \lambda^2)$.

4.3.2 Vacuum polarization

This section and the following are shortened versions of the full calculations shown in appendices B.5.2 and B.5.3.



$$\equiv \Pi_{\mu\nu}(q). \tag{4.3.2}$$

The second rank tensor $\Pi_{\mu\nu}$ has the form

$$\Pi_{\mu\nu} = -(ie)^2\int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma_\mu\frac{i(\not{k}+m)}{k^2-m^2}\gamma_\nu\frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2}\right]. \tag{4.3.3}$$

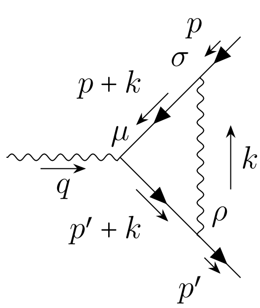
Using the trace relations (B.1.6), odd numbered γ-matrix terms vanish. Feynman parametrization, completing the square and substituting $l = k + qx$ gives

$$\Pi_{\mu\nu} = -4e^2\int_0^1 dx\int\frac{d^4l}{(2\pi)^4}\frac{2l_\mu l_\nu - 2x(1-x)q_\mu q_\nu - g_{\mu\nu}l^2 + g_{\mu\nu}(m^2+q^2x(1-x))}{(l^2-\Delta)^2}, \tag{4.3.4}$$

where $\Delta = m^2 - q^2x(1-x)$ and terms linear in l have been removed. Since there are terms proportional to both $g_{\mu\nu}l^2$ and $l_\mu l_\nu$, and some with no l-dependency, there are three different d-dimensional integrals after generalizing to $d = 4-\epsilon$ dimensions. These can be grouped together using (B.3.2) and (B.4.2)

$$\Pi_{\mu\nu} = -\,i(g_{\mu\nu}q^2 - q_\mu q_\nu)\frac{e^2}{4\pi^2}\int_0^1 (2x(1-x))\,\Gamma(\frac{\epsilon}{2})\left(\frac{4\pi\mu^2}{\Delta}\right)^{\frac{\epsilon}{2}} dx. \tag{4.3.5}$$

4.3.3 Vertex correction



$$\equiv \Gamma_\mu(p,q), \tag{4.3.6}$$

where $\Gamma_\mu(p,p')$ is calculated using the Feynman rules as usual

$$\Gamma_\mu(p,q) =(-ie)^3 \int \frac{d^4k}{(2\pi)^4}\gamma_\rho i\frac{-g^{\rho\sigma}}{k^2-\lambda^2}\frac{i(\not{p}+\not{k}+m)}{(k+p)^2-m^2}\gamma_\mu\frac{i(\not{k}+\not{q}+\not{p}+m)}{(k+p+q)^2-m^2}\gamma_\sigma$$

Though somewhat more complicated, this can be evaluated in the same way. Once again using Feynman parametrization, completing the square and substituting $l = k + pxy - py + p + qxy$

$$\Gamma_\mu(p,q) = -2e^3\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\cdot ydy$$
$$\cdot\frac{\gamma^\sigma(\not{p}+\not{l}-\not{p}xy+\not{p}y-\not{p}-\not{q}xy+m)\gamma_\mu(\not{l}-\not{p}xy+\not{p}y-\not{p}-\not{q}xy+\not{q}+\not{p}+m)\gamma_\sigma}{[l^2-\Delta]^3},$$

where

$$\Delta = y\left(p^2(x-1)((x-1)y+1)+2pq(x-1)xy+q^2x(xy-1)-\lambda^2(x-1)\right)-m^2(-xy+y-1).$$

Following [16, p. 254], Γ_μ is split into two parts, one proportional to l^2 and one independent of l. Generalizing to $d = 4-\epsilon$ and evaluating the integral for Γ^1_μ yields

$$\Gamma^1_\mu = -\frac{i}{2}\frac{e^3}{(4\pi)^2}\mu^{\frac{\epsilon}{2}}\int_0^1 dx\cdot ydy(4-4\epsilon+\epsilon^2)\gamma_\mu\Gamma(\frac{\epsilon}{2})\Big(\frac{4\pi\mu^2}{\Delta}\Big)^{\frac{\epsilon}{2}}, \tag{4.3.7}$$

The terms not proportional to l take the form

$$\Gamma^2_\mu =i\frac{e^3}{(4\pi)^2}\mu^{\frac{\epsilon}{2}}\int_0^1 dx\cdot ydy\gamma^\sigma(m-\not{p}xy+\not{p}y-\not{q}xy)\gamma_\mu(m-\not{p}xy+\not{p}y-\not{q}xy+\not{q})\gamma_\sigma\frac{\Gamma(1-\frac{\epsilon}{2})}{\Delta^{1-\frac{\epsilon}{2}}}\left(4\pi\mu^2\right)^{\frac{\epsilon}{2}}. \tag{4.3.8}$$

The total amplitude for the vertex correction is the sum of these two expressions

$$\Gamma_\mu = \Gamma^1_\mu + \Gamma^2_\mu \tag{4.3.9}$$

The reason for this split is that when $\epsilon \to 0$, only Γ_1 is divergent, and is the expression of most interest. Γ^2_μ does however provide a correction to the anomalous magnetic moment of the fermion [16, p. 269], but for our purposes is not particularly important.

4.4 Determination of Z_i in the momentum subtraction scheme

Following [10], we go back to the two first conditions in the momentum subtraction scheme (4.2.6), with the calculated fermion self-energy amplitude (4.3.1). These two equations determine the photon renormalization constant and the mass renormalization constant.

$$(\delta_\psi\not{p}-(Z_\psi Z_m-1)m) = -i\frac{e^2}{(4\pi)^2}\int_0^1\left((\epsilon-2)(1-x)\not{p}+(4-\epsilon)m\right)\Gamma(\frac{\epsilon}{2})\left(\frac{4\pi\mu^2}{\Delta_1}\right)^{\frac{\epsilon}{2}}dx\,\Big|_{p^2=-M^2},$$
$$Z_\psi =1+i\frac{e^2}{(4\pi)^2}\int_0^1\Gamma(\frac{\epsilon}{2})\left(\frac{4\pi\mu^2}{\Delta_1}\right)^{\frac{\epsilon}{2}}$$
$$\cdot\left((\epsilon-2)(1-x)-\frac{\epsilon}{2}\left((\epsilon-2)(1-x)\not{p}+(4-\epsilon)m\right)\cdot\frac{2\not{p}(x-1)x}{\Delta_1}\right)dx\,\Big|_{p^2=-M^2}. \tag{4.4.1}$$

The first equation gives, after a Taylor expansion.

$$
\begin{aligned}
Z_m =& 1 + i\frac{e^2}{(4\pi)^2 m}\int_0^1 dx \Bigg(m(x-5)\epsilon - 2m(x-9) - 5\not{p}(x-1)\epsilon + 10\not{p}(x-1) \\
&+ \frac{2\not{p}(x-1)x\epsilon(m-\not{p})(2m+x-1)}{\Delta_1}\Bigg)\Gamma(\frac{\epsilon}{2})\left(\frac{4\pi\mu^2}{\Delta_1}\right)^{\frac{\epsilon}{2}}\Bigg|_{p^2=-M^2},
\end{aligned}
\tag{4.4.2}
$$

where $\Delta_1 = m^2 x + (x-1)(p^2 x - \lambda^2)$

The third condition, along with (4.2.6c), gives the photon renormalization constant

$$
Z_A = 1 - \frac{e^2}{(4\pi)^2}\int_0^1 \Gamma(2-\frac{d}{2})\left(\frac{4\pi\mu^2}{\Delta_2}\right)^{\frac{\epsilon}{2}} \cdot 8x(1-x)dx\Bigg|_{q^2=-M^2}, \tag{4.4.3}
$$

where $\Delta_2 = m^2 - x(1-x)q^2$.

Combining this with the result from the Ward identity (4.2.4), determines the last renormalization constant

$$
Z_e = 1 + \frac{e^2}{(4\pi)^2}\int_0^1 \Gamma(2-\frac{d}{2})\left(\frac{4\pi\mu^2}{\Delta_2}\right)^{\frac{\epsilon}{2}} \cdot 4x(1-x)dx\Bigg|_{p^2=-M^2}. \tag{4.4.4}
$$

Ignoring terms which goes to zero as $\epsilon \to 0$ yields the results, summarized in table 1. To ease notation, it needs to be stressed that any momentum invariant is equal to $-M^2$.

	Constant	1-loop expression
Fermions	Z_ψ	$1 + i\frac{e^2}{(4\pi)^2}\int_0^1 \Gamma(\frac{\epsilon}{2})\left(\frac{4\pi\mu^2}{\Delta_1}\right)^{\frac{\epsilon}{2}} \cdot (1-x)\left(\epsilon - 2 - \left(4m - 2\not{p}\right)\cdot \frac{2\not{p}(x-1)x}{\Delta_1}\right)dx$
Gauge boson	Z_A	$1 - \frac{e^2}{(4\pi)^2}\int_0^1 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{\epsilon}{2}}}\left(\frac{\mu^2}{\Delta_2}\right)^{\frac{\epsilon}{2}} \cdot 8x(1-x)dx$
Parameter	Z_e	$1 + \frac{e^2}{(4\pi)^2}\int_0^1 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{\epsilon}{2}}}\left(\frac{\mu^2}{\Delta_2}\right)^{\frac{\epsilon}{2}} \cdot 4x(1-x)dx$

Table 1: Momentum subtraction scheme renormalization constants for QED at the 1-loop order in Feynman gauge.

4.5 β-function

In order to calculate β-functions, which determine how the gauge coupling of the interaction λ changes with a variation in the energy scale M, the *Callan-Symanzik equation* is needed [10, p. 411]

$$
[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)]G^{(n)}(\{x_i\}; M, \lambda) = 0, \tag{4.5.1}
$$

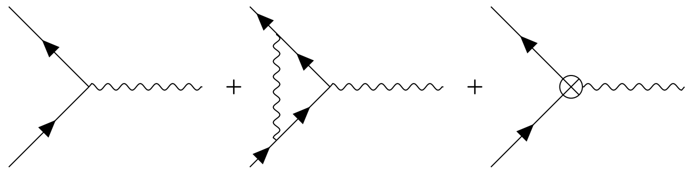
Or, the equivalent expression for QED

$$
[M\frac{\partial}{\partial M} + \beta(e)\frac{\partial}{\partial e} + n\gamma_2(e) + m\gamma_3(e)]G^{(n,m)}(\{x_i\}; M, e) = 0 \tag{4.5.2}
$$

This equation stems from the fact that observables are independent of the chosen renormalization scheme [13, p. 417]. It states that any shift in renormalization scale $M \to$

$M + dM$ is compensated by a shift in the functions β and γ [10].

In order to explicitly calculate the β-function, one has to use the 2- and 3-point *Green functions*, which has the diagrammatic forms

$G^{(3)} =$  (4.5.3)

The Green functions involve loops of external legs and counter-terms for these. Rather than delving into this, we quote the result for the QED β-function [10, p. 416], [21, p. 8] adopted to the defined renormalization constants (4.1.2).

$$\beta(e) = eM\frac{\partial}{\partial M}(-(Z_e Z_A^{1/2} Z_\psi - 1) + \delta Z_\psi + \frac{1}{2}\delta Z_A).$$

Using the established identity $Z_e = Z_A^{-1/2}$ (4.2.4), the β-function reduces to

$$\beta(e) = eM\frac{\partial}{\partial M}(\frac{1}{2}\delta Z_A). \tag{4.5.4}$$

Normally, the three constants (Z_ψ, Z_e, Z_A) would be needed in order to calculate the β-function. QED, which is an Abelian group, provided a simplification due to the Ward identity, and as such, only the photon field renormalization coefficient Z_A is needed. Now that the β-function is expressed in terms of a renormalization constant, a scheme needs to be chosen in order to express the constant explicitly. As mentioned earlier, the on-shell scheme poses a complication when calculating the β-function. Therefore, the momentum subtraction scheme (4.2.6) is used.

As discussed earlier, in the momentum subtraction scheme renormalization is performed for space-like momenta $q^2 = -M^2$ [18, p. 229]. The β-function is a high energy phenomenon. We calculated Z_A in the previous section

$$Z_A = 1 - \frac{e^2}{(4\pi)^2}\int_0^1 \Gamma(2-\frac{d}{2})\left(\frac{4\pi\mu^2}{m^2 - x(1-x)q^2}\right)^{\frac{\epsilon}{2}} \cdot 8x(1-x)dx\,\Bigg|_{q^2=-M^2}. \tag{4.5.5}$$

For high energies the mass is negligible, and the denominator can be set to M^2 [10, p. 527]. This allows for easy integration of the Feynman parameter, and the photon renormalization coefficient becomes

$$Z_A \to 1 - \frac{e^2}{(4\pi)^2}\frac{4}{3}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{\epsilon}{2}}}\left(\frac{\mu^2}{M^2}\right)^{\frac{\epsilon}{2}}. \tag{4.5.6}$$

From this, the *β-function*, with the use of (4.5.4), becomes

$$\begin{aligned}
\beta(e) &= eM\frac{\partial}{\partial M}(\frac{1}{2}\delta_A) \\
&= eM\frac{\partial}{\partial M}\left(\frac{1}{2}\left\{-\frac{e^2}{12\pi^2}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{\epsilon}{2}}}\left(\frac{\mu^2}{M^2}\right)^{\frac{\epsilon}{2}}\right\}\right) \\
&= eM\left\{-\frac{1}{2}\left(\frac{-\epsilon}{M^{\epsilon+1}}\right)\frac{e^2}{12\pi^2}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{\epsilon}{2}}}\mu^\epsilon\right\} \\
&= \epsilon\frac{e^3}{24\pi^2}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{\epsilon}{2}}}\left(\frac{\mu^2}{M^2}\right)^{\frac{\epsilon}{2}}.
\end{aligned}$$

Expanding around $\epsilon = 0$

$$\beta(e) = \epsilon \frac{e^3}{24\pi^2} \left\{ \frac{2}{\epsilon} - \gamma_E + \ln 4\pi + \ln \mu^2 - \ln \Delta + O(\epsilon) \right\}.$$

Finally, ignoring terms which vanish for $\epsilon \to 0$

$$\beta(e) = \frac{e^3}{12\pi^2}. \tag{4.5.7}$$

5 Minimal subtraction scheme

Different schemes mean different conditions set on the propagators and vertex function, leading to various finite terms. The renormalization constants calculated in the momentum subtraction scheme, shown in table 1, consist of multiple terms after an expansion around $\epsilon = 0$. All but one of these terms are finite. Yet in the schemes discussed previously, the finite terms must be accounted for. The idea behind the Minimal subtraction scheme is that only the divergent terms of the amplitudes are subtracted by the counterterms. In this section we will perform the renormalization of QED with the minimal subtraction scheme. A helpful tool in extracting the divergent terms of Feynman amplitudes is Passarino-Veltman (P.V) functions [22]. The P.V functions are standardized loop momenta integrals, with their divergent terms tabulated. Many loop integrals can therefore simply be defined in terms of these functions, and the divergent term can be extracted. These functions are also integrated into FeynCalc, which we will use in section 6. An overview of the P.V functions and an application to the divergent QED loop amplitudes are given in section 5.2.

5.1 Definition of the minimal subtraction scheme

As mentioned, in minimal subtraction only divergent terms are subtracted by the counterterms. For some divergent amplitude Σ this means that all finite terms are ignored.

$$\Sigma \propto \left(\frac{2}{\epsilon} + \ln 4\pi + \ln \frac{\mu^2}{\Delta} + \text{other finite terms} \right) \overset{\text{MS}}{=} \frac{2}{\epsilon}. \tag{5.1.1}$$

Naturally, this affects the renormalization constants Z_i as well. By removing the μ^2 term in (5.1.1), the explicit μ-dependence is removed for the constants. Of course, there is a μ-dependence, otherwise the renormalization group equations such as the β-function would not make much sense. However, the dependence is only implicit through the renormalized charge e, which has a scale dependence [19, p. 118]. We have already seen this dependence in (4.1.2). Additionally, the renormalization constants have no mass dependence. All the constants have the form [23]

$$Z_i = 1 + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} Z_{i,k}(e) \tag{5.1.2}$$

The β-function can easily be found in the MS scheme by using the generic form

$$\mu \frac{de}{d\mu} \equiv \beta(e), \tag{5.1.3}$$

and the property that the renormalization constants only have a μ-dependency through e. Following [24, p. 222] and [13, p. 424], the renormalized charge is

$$e_0 = \mu^{\epsilon/2} Z_e e. \tag{5.1.4}$$

Using the result $Z_e = Z_A^{-1/2}$, the definition of β-function becomes

$$\beta(e) = \mu\frac{de}{d\mu} = \mu\frac{d}{\partial\mu}(\mu^{-\epsilon/2} Z_A^{1/2} e_0).$$

One of the core concepts of the renormalization group equations is the fact that bare parameters are independent of the scale μ. Using this fact and some chain rule yields

$$\begin{aligned}\beta(e) &= -\frac{\epsilon}{2} e_0 \mu^{-\epsilon/2} Z_A^{1/2} + \mu^{-\epsilon/2+1} e_0 \frac{dZ_A^{1/2}\big(e(\mu)\big)}{d\mu} \\ &= -\frac{\epsilon}{2} e_0 \mu^{-\epsilon/2} Z_A^{1/2} + \mu^{-\epsilon/2+1} e_0 \frac{1}{2} Z_A^{-1/2} \frac{dZ_A}{d\mu} \\ &= -\frac{\epsilon}{2} e + \mu \frac{e}{2Z_A}\frac{dZ_A}{d\mu}.\end{aligned} \tag{5.1.5}$$

5.2 Passarino-Veltman functions

The P.V functions are an extension of dimensional regularization. In section 2 the loop integral (2.4.6) was discussed

$$\Sigma(p) \propto e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-\lambda^2)[(k+p)^2-m^2]}, \tag{5.2.1}$$

for simplicity, the numerator has been set to 1. After a Feynman parametrization and a substitution $l = k + px$ we found

$$\Sigma(p) \propto e^2 \int \frac{d^4l}{(2\pi)^4} \int_0^1 dx \frac{1}{(l^2-\Delta)^2}, \tag{5.2.2}$$

After generalizing the space-time dimension $d = 4 - \epsilon$

$$\Sigma(p) \propto e^2 \int_0^1 dx \int \frac{d^dl}{(2\pi)^d} \frac{\mu^\epsilon}{(l^2-\Delta)^2}, \tag{5.2.3}$$

the following result was obtained

$$\Sigma(p) \propto i\frac{e^2}{(4\pi)^2} \int_0^1 \Gamma(\frac{\epsilon}{2}) \left(\frac{4\pi\mu^2}{\Delta}\right)^{\frac{\epsilon}{2}} dx. \tag{5.2.4}$$

The expansion around $\epsilon = 0$ yielded the terms

$$\Gamma(\frac{\epsilon}{2}) \left(\frac{4\pi\mu^2}{\Delta}\right)^{\frac{\epsilon}{2}} = \frac{2}{\epsilon} - \gamma_E + \ln 4\pi + \ln\mu^2 - \ln\Delta + O(\epsilon).$$

If we generalize the space-time dimension $d = 4-\epsilon$ immediately in (5.2.1), one can simply write

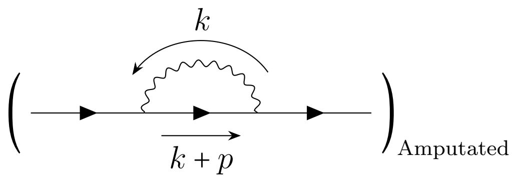
$$\int \frac{d^dk}{(2\pi)^d} \frac{\mu^\epsilon}{(k^2-\lambda^2)[(k+p)^2-m^2]} = \frac{i}{(4\pi)^2}\left(\frac{2}{\epsilon} + \text{Finite terms}\right). \tag{5.2.5}$$

The divergent term in the loop integral can now be easily extracted. (5.2.1) is one of the two-point functions, these are denoted with the letter B. This corresponds to the number of particles in the loop integral. The P.V functions can be found in appendix C.

Lastly, we define a symbol in order to signify that all finite terms are ignored. For the expression above we can write

$$\int \frac{d^dk}{(2\pi)^d} \frac{\mu^\epsilon}{(k^2-\lambda^2)[(k+p)^2-m^2]} \overset{\text{Div}}{=} \frac{i}{(4\pi)^2}\frac{2}{\epsilon}. \tag{5.2.6}$$

5.2.1 Electron self-energy



$$= \Sigma(p),$$

$$\Sigma(p) = (-ie)^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\gamma_\mu \frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}\gamma_\nu \frac{-ig^{\mu\nu}}{k^2-\lambda^2}.$$

The d-dimensional contraction identities B.1.4 gives

$$\begin{aligned}\Sigma(p) &= -e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{(2-d)(\not{k}+\not{p})+dm}{(k^2-\lambda^2)[(k+p)^2-m^2]} \\ &= -e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{(2-d)\not{k}+(2-d)\not{p}+dm}{(k^2-\lambda^2)[(k+p)^2-m^2]}.\end{aligned}$$

The two point P.V functions (C.1.2) are now easily applied

$$\Sigma(p) = -i\frac{e^2}{(4\pi)^2}\left((2-d)\gamma_\mu B^\mu + \left((2-d)\not{p}+dm\right)B_0\right),$$

where the arguments for $B^\mu(p^2,\lambda^2,m^2)$ and $B_0(p^2,\lambda^2,m^2)$ are hidden. Using the tensor decompositions (C.2.1)

$$\Sigma(p) = -i\frac{e^2}{(4\pi)^2}\left[(2-d)\frac{\not{p}}{2p^2}\Big(A_0(\lambda^2)-A_0(m^2)-(p^2+\lambda^2-m^2)B_0\Big) + \left((2-d)\not{p}+dm\right)B_0\right].$$

By removing terms which contain λ^2, which vanish in the limit $\lambda \to 0$, subsequently also $A(\lambda)$ [18, p. 222], and using $d = 4-\epsilon$

$$\Sigma(p) = -i\frac{e^2}{(4\pi)^2}\left[\not{p}(\epsilon-2)\frac{-A_0(m^2)-(p^2-m^2)B_0(p^2,m,\lambda)}{2p^2} + \left((\epsilon-2)\not{p}+dm\right)B_0\right]. \tag{5.2.7}$$

Then using the divergent part of A_0 and B_0 from table 5

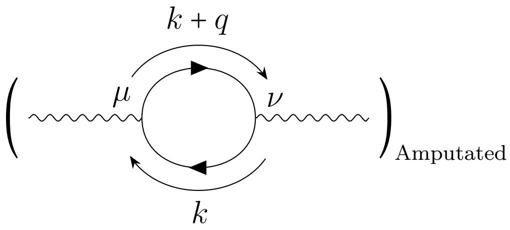
$$\begin{aligned}(\Sigma)_{\text{div}} &= -i\frac{e^2}{(4\pi)^2}\left[\not{p}(\epsilon-2)\frac{-\frac{2}{\epsilon}m^2-(p^2-m^2)\frac{2}{\epsilon}}{2p^2} + \left((\epsilon-2)\not{p}+dm\right)\frac{2}{\epsilon}\right] \\ &= -i\frac{e^2}{(4\pi)^2}\left[\not{p}(\epsilon-2)\left(-\frac{1}{\epsilon}\right) + \left((\epsilon-2)\not{p}+dm\right)\frac{2}{\epsilon}\right] \\ &= -i\frac{e^2}{(4\pi)^2}\left[\not{p}(\epsilon-2)\frac{1}{\epsilon} + (4-\epsilon)m\frac{2}{\epsilon}\right],\end{aligned}$$

Finally, keeping only the terms which are divergent

$$
\begin{aligned}
(\Sigma)_{\text{div}} &= -i\frac{e^2}{(4\pi)^2}\left[\,-2\not{p}\frac{1}{\epsilon} + 8m\frac{1}{\epsilon}\,\right] \\
&= i\frac{e^2}{(4\pi)^2}(\not{p} - 4m)\frac{2}{\epsilon}.
\end{aligned} \tag{5.2.8}
$$

5.2.2 Vacuum polarization

Remembering the previous diagram



$$= \Pi_{\mu\nu}(q)$$

$$\Pi_{\mu\nu} = -(-ie)^2\mu^{\epsilon}\int\frac{d^dk}{(2\pi)^d}\text{Tr}\left[\,\gamma_\mu\frac{i(\not{k}+m)}{k^2-m^2}\gamma_\nu\frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2}\,\right]. \tag{5.2.9}$$

Keeping the d-dimensional trace relations (B.1.6) in mind, we can immediately ignore terms that has an odd number of γ-matrices

$$\Pi_{\mu\nu} = -e^2\mu^{\epsilon}\int\frac{d^dk}{(2\pi)^d}\text{Tr}\left[\,\frac{\gamma_\mu\not{k}\gamma_\nu(\not{k}+\not{q}) + \gamma_\mu\gamma_\nu m^2}{(k^2-m^2)[(k+q)^2-m^2]}\,\right]. \tag{5.2.10}$$

The other trace relations gives

$$\Pi_{\mu\nu} = -4e^2\mu^{\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{2k_\mu k_\nu + k_\mu q_\nu + k_\nu q_\mu - g_{\mu\nu}g_{\rho\sigma}k^\rho k^\sigma - g_{\mu\nu}g_{\rho\sigma}k^\rho q^\sigma + g_{\mu\nu}m^2}{(k^2-m^2)[(k+q)^2-m^2]}. \tag{5.2.11}$$

Transforming into P.V functions with the use of (C.1.2)

$$\Pi_{\mu\nu} = -4i\frac{e^2}{(4\pi)^2}\left(2B_{\mu\nu} + q_\nu B_\mu + q_\mu B_\nu - g_{\mu\nu}g_{\rho\sigma}B^{\rho\sigma} - g_{\mu\nu}g_{\rho\sigma}q^\sigma B^\rho + g_{\mu\nu}m^2B_0\right), \tag{5.2.12}$$

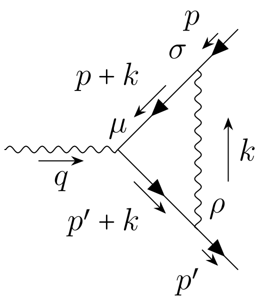
where the arguments of the P.V functions are $B_{\mu\nu}(q^2, m^2, m^2)$. Using table 5 to extract the divergent terms

$$
\begin{aligned}
\Pi_{\mu\nu} \overset{\text{Div}}{=} &- 4i\frac{e^2}{(4\pi)^2}\Bigg(2\left[\,g_{\mu\nu}\left(-\frac{1}{6\epsilon}\left(q^2-6m^2\right)\right) + q_\mu q_\nu\frac{2}{3\epsilon}\,\right] - q_\mu q_\nu\frac{2}{\epsilon} \\
&- g_{\mu\nu}g_{\rho\sigma}\left[\,g^{\rho\sigma}\left(-\frac{1}{6\epsilon}\left(q^2-6m^2\right)\right) + q^\rho q^\sigma\frac{2}{3\epsilon}\,\right] + g_{\mu\nu}g_{\rho\sigma}q^\sigma q^\rho\frac{1}{\epsilon} + g_{\mu\nu}m^2\frac{2}{\epsilon}\Bigg).
\end{aligned} \tag{5.2.13}
$$

Simplifying with the d-dimensional algebra, found in appendix B.1 and ignoring finite terms causes the mass terms to cancel. After some more algebra we get

$$
\begin{aligned}
\Pi_{\mu\nu} &= -4i\frac{e^2}{(4\pi)^2}\left(g_{\mu\nu}q^2\frac{2}{3\epsilon} - q_\mu q_\nu\frac{2}{3\epsilon}\right) \\
&= -i\frac{e^2}{6\pi^2}\left(g_{\mu\nu}q^2 - q_\mu q_\nu\right)\frac{1}{\epsilon}.
\end{aligned} \tag{5.2.14}
$$

5.2.3 Vertex correction



$$= \Gamma_\mu(p,q), \tag{5.2.15}$$

$$\Gamma_\mu(p,q) =(-ie)^3\mu^{\frac{3}{2}\epsilon}\int\frac{d^dk}{(2\pi)^d}\gamma_\rho i\frac{-g^{\rho\sigma}}{k^2-\lambda^2}\frac{i(\not{p}+\not{k}+m)}{(k+p)^2-m^2}\gamma_\mu\frac{i(\not{k}+\not{q}+\not{p}+m)}{(k+p+q)^2-m^2}\gamma_\sigma$$

Both C_0 and C_μ are finite integrals and does not contribute to the divergent amplitude. Therefore, the only term contributing to the divergent amplitude is the one proportional to $k^\lambda k^\sigma$.

$$\begin{aligned}\Big(\Gamma_\mu(p,q)\Big)_{\text{Div}} &= -\mu^{\frac{3}{2}\epsilon}e^3\int\frac{d^dk}{(2\pi)^d}\frac{\gamma^\delta\gamma_\lambda\gamma_\mu\gamma_\sigma\gamma_\delta k^\lambda k^\sigma}{(k^2-\lambda^2)[(k+p)^2-m^2][(k+p+q)^2-m^2]}\\ &= -i\mu^{\frac{\epsilon}{2}}\frac{e^3}{(4\pi)^2}(-2\gamma_\sigma\gamma_\mu\gamma_\lambda+(4-d)\gamma_\lambda\gamma_\mu\gamma_\sigma)C^{\lambda\sigma}(p^2,q^2,\lambda^2,m^2,m^2),\end{aligned}$$

using (B.1.4) and (C.1.3). Ignoring the finite second term, decomposing $C^{\lambda\sigma}$, and using that only C_{00} is divergent yields

$$\begin{aligned}\Big(\Gamma_\mu(p,q)\Big)_{\text{Div}} &= -i\mu^{\frac{\epsilon}{2}}\frac{e^3}{(4\pi)^2}(-2\gamma_\sigma\gamma_\mu\gamma_\lambda)g^{\lambda\sigma}C^{00}\\ &= -i\mu^{\frac{\epsilon}{2}}\frac{e^3}{(4\pi)^2}(-2(2-d)\gamma_\mu)\frac{1}{2\epsilon}\\ &= -i\mu^{\frac{\epsilon}{2}}\frac{e^3}{(4\pi)^2}\gamma_\mu\frac{2}{\epsilon}.\end{aligned}\tag{5.2.16}$$

5.3 Extracting UV-Divergent terms

For completeness sake, the calculations of the divergent terms using Feynman parametrization and the d-dimensional integrals are shown below. These follow from the amplitudes calculated in section 4.

Following the same structure, we start with the fermion self-energy (4.3.1)

$$\Sigma(p) = -i\frac{e^2}{(4\pi)^2}\int_0^1\Big((\epsilon-2)(1-x)\not{p}+(4-\epsilon)m\Big)\Gamma(\frac{\epsilon}{2})\left(\frac{4\pi\mu^2}{\Delta}\right)^{\frac{\epsilon}{2}}dx$$

Using (B.4.5)

$$\Sigma(p) = -i\frac{e^2}{(4\pi)^2}\int_0^1\Big((\epsilon-2)(1-x)\not{p}+(4-\epsilon)m\Big)\,dx\left\{\frac{2}{\epsilon}+\text{finite terms}\right\}$$

Some terms become finite when expanding, these are also ignored.

$$\Sigma(p) \stackrel{\text{Div}}{=} -i\frac{e^2}{(4\pi)^2}\int_0^1 \left(-2(1-x)\not{p}+4m\right)\frac{2}{\epsilon}dx$$

We are now free to evaluate the Feynman parameter integral, with the result

$$\Sigma(p) = -\frac{e^2}{(4\pi)^2}(4m-\not{p})\frac{2}{\epsilon}. \tag{5.3.1}$$

Moving on to the vacuum polarization (4.3.5)

$$\Pi_{\mu\nu}(q) = (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q). \tag{5.3.2}$$

This is evaluated in the same way

$$\begin{aligned}
(\Pi(q))_{\text{div}} &= \frac{-8e^2}{(4\pi)^{d/2}}\int_0^1 dx\, x(1-x)\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}} \\
&= \frac{-8e^2}{(4\pi)^2}\int_0^1 x(1-x)dx\left(\frac{2}{\epsilon}+\text{finite terms}\right) \\
&\stackrel{\text{Div}}{=} \frac{-8e^2}{(4\pi)^2}\left[\frac{1}{2}x^2-\frac{1}{3}x^3\right]_0^1\frac{2}{\epsilon} \\
&= -\frac{8}{3}\frac{e^2}{(4\pi)^2}\frac{1}{\epsilon},
\end{aligned} \tag{5.3.3}$$

with the use of (B.4.5). For the vertex function we start with (4.3.9) and look only for divergent terms.

$$\Gamma_\mu(p,q) = \Gamma^1_\mu(p,q) + \Gamma^2_\mu(p,q), \tag{5.3.4}$$

where the factors are given by (4.3.7) and (4.3.8) respectively

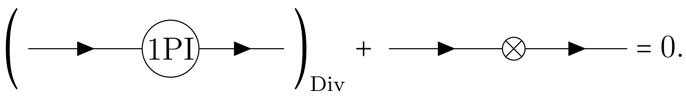
$$\begin{aligned}
\Gamma^1_\mu =& -\frac{i}{2}\mu^{\frac{\epsilon}{2}}\frac{e^3}{(4\pi)^2}\int_0^1 dx\cdot ydy(4-4\epsilon+\epsilon^2)\gamma_\mu\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}}, \\
\Gamma^2_\mu =& i\mu^{\frac{\epsilon}{2}}\frac{e^3}{(4\pi)^2}\int_0^1 dx\cdot ydy\gamma^\sigma(m-\not{p}xy+\not{p}y-\not{q}xy)\gamma_\mu(m-\not{p}xy+\not{p}y-\not{q}xy+\not{q})\gamma_\sigma\frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-\frac{d}{2}}}.
\end{aligned}$$

For $d = 4-\epsilon$ only Γ^1_μ is divergent, since $\Gamma(3-\frac{d}{2})$ is finite, and any potential infrared divergence in Δ has been remedied by the fictitious photon mass λ. In addition, since $\Delta^{2-\frac{d}{2}} \to 1$ as $\epsilon \to 0$, the Feynman parameter integrals can be evaluated without any complications.

$$\begin{aligned}
\left(\Gamma_\mu(p,q)\right)_{\text{Div}} =& -\frac{i}{4}\mu^{\frac{\epsilon}{2}}\frac{e^3}{(4\pi)^2}(4-4\epsilon+\epsilon^2)\gamma_\mu\frac{2}{\epsilon} \\
=& -i\mu^{\frac{\epsilon}{2}}\frac{e^3}{(4\pi)^2}\gamma_\mu\frac{2}{\epsilon}.
\end{aligned} \tag{5.3.5}$$

5.4 Renormalization constants

Following the procedure from [23], the appropriate counter-term is added to each of the 1-particle irreducible diagrams for the propagators and vertex function, and the result is required to be zero. Diagrammatically illustrated with



The only terms the counter-terms are absorbing are the divergent ones, unlike the on-shell or momentum subtraction schemes, where there would also be some finite terms. All of the renormalization constants be determined by using the divergent amplitudes and corresponding counter-terms. We start with the fermion self-energy (5.2.8) and the counter-term in figure 2(a).

$$i\frac{e^2}{(4\pi)^2}(\not{p} - 4m)\frac{2}{\epsilon} + i((Z_\psi - 1)\not{p} - (Z_\psi Z_m - 1)m) = 0. \tag{5.4.1}$$

p and m are independent variables; therefore, these are two equations that are fulfilled separately.

$$-\frac{e^2}{(4\pi)^2}\not{p}\frac{2}{\epsilon} = (Z_\psi - 1)\not{p}, \tag{5.4.2a}$$

$$-\frac{e^2}{(4\pi)^2}4m\frac{2}{\epsilon} = (Z_\psi Z_m - 1)m. \tag{5.4.2b}$$

The first equation readily gives

$$Z_\psi = 1 - \frac{e^2}{8\pi^2\epsilon}, \tag{5.4.3}$$

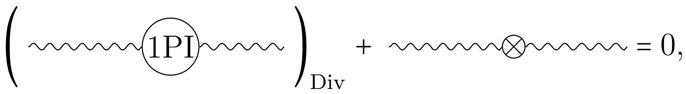
and the second leads to

$$Z_m = \frac{1 - 4\frac{e^2}{(4\pi)^2}\frac{2}{\epsilon}}{Z_\psi} = \frac{1 - 4\frac{e^2}{(4\pi)^2}\frac{2}{\epsilon}}{1 - \frac{e^2}{(4\pi)^2}\frac{2}{\epsilon}}. \tag{5.4.4}$$

We can use a Taylor expansion to simplify; the result at the 1-loop order is

$$Z_m = 1 - \frac{3e^2}{8\pi^2\epsilon}. \tag{5.4.5}$$

Similarly, the photon renormalization constant Z_A is determined from the divergent vacuum polarization amplitude (5.2.14) and the photon counter-term in figure 2(b)

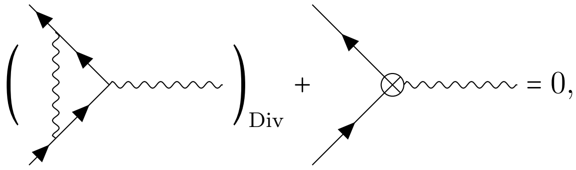


$$-i(g_{\mu\nu}q^2 - q_\mu q_\nu)\frac{e^2}{6\pi^2}\cdot\frac{1}{\epsilon} - i(g_{\mu\nu}q^2 - q_\mu q^\nu)\delta_A = 0, \tag{5.4.6}$$

which is solved with the use of $Z_i = 1 + \delta_i$

$$Z_A = 1 - \frac{e^2}{6\pi^2\epsilon}. \tag{5.4.7}$$

Finally, the divergent vertex amplitude (5.2.16) and the corresponding counter-term seen in figure 2(c) gives a relation between the other renormalization constants.



$$-i\frac{e^3}{(4\pi)^2}\mu^{\frac{\epsilon}{2}}\gamma_\mu\frac{2}{\epsilon} - i(Z_e Z_A^{1/2} Z_\psi - 1)e\mu^{\frac{\epsilon}{2}}\gamma_\mu = 0. \tag{5.4.8}$$

This can be solved for the coupling renormalization constant Z_e

$$\begin{aligned} Z_e &= \frac{1 - \frac{e^2}{(4\pi)^2}\frac{2}{\epsilon}}{Z_A^{1/2} Z_\psi} \\ &= 1 + \frac{e^2}{12\pi^2}\frac{1}{\epsilon}, \end{aligned} \tag{5.4.9}$$

where we have used (5.4.3) and a Taylor expansion. This result provides a confirmation of the previously established identity $Z_e = Z_A^{-\frac{1}{2}}$ (4.2.4). Therefore, for the purpose of calculating the β-function (5.1.5), the only needed renormalization constant is Z_A.

	Constant	1-loop expression
Fermions	Z_ψ	$1 - \frac{e^2}{8\pi^2\epsilon}$
	Z_m	$1 - \frac{3e^2}{8\pi^2\epsilon}$
Gauge boson	Z_A	$1 - \frac{e^2}{6\pi^2\epsilon}$
Parameter	Z_e	$1 + \frac{e^2}{12\pi^2\epsilon}$

Table 2: Renormalization constants for QED at the 1-loop order in Feynman gauge.

Now that all the constants have been determined, we can move on to the β-function. Following [24] again, we use the derived expression for the β-function (5.1.5), and the photon renormalization constant (5.4.7).

$$\begin{aligned} \beta(e) &= -\frac{\epsilon}{2}e + \mu\frac{e}{2Z_A}\frac{dZ_A}{d\mu} \\ &= -\frac{\epsilon}{2}e + \mu\frac{e}{2(1 - \frac{(e(\mu))^2}{6\pi^2\epsilon})}\frac{d}{d\mu}\Big(1 - \frac{\big(e(\mu)\big)^2}{6\pi^2\epsilon}\Big) \\ &= -\frac{\epsilon}{2}e + \mu\frac{e}{2}\Big(1 + \frac{e^2}{6\pi^2\epsilon}\Big)\Big(-\frac{2e}{6\pi^2\epsilon}\cdot\frac{de}{d\mu}\Big). \end{aligned}$$

At the one-loop order we can ignore the term of order e^4. The remaining terms together with the generic definition of the β-function (5.1.3) gives

$$\beta(e) = -\frac{\epsilon}{2}e - \frac{e^2}{6\pi^2\epsilon}\beta(e).$$

This yields the equation

$$\beta(e) = -\frac{\frac{\epsilon}{2}e}{1+\frac{e^2}{6\pi^2\epsilon}},$$

which, to the first order, gives

$$\beta(e) = -\frac{\epsilon}{2}e + \frac{e^3}{12\pi^2} \overset{\epsilon\to 0}{=} \frac{e^3}{12\pi^2}. \tag{5.4.10}$$

In the last step the limit $\epsilon \to 0$ is taken.

6 Spontaneously broken QED

In the presence of a complex scalar that acquires a non-zero vacuum expectation value v, the gauge boson or bosons of a theory gains mass. This is the Brout-Englert Higgs mechanism and the gauge boson mass is proportional to v. This section aims to develop a spontaneously broken QED theory, check for renormalizability, and calculate all the renormalization constants. In addition we want to see how the β-function changes. Consequently, the structure is different from a standard renormalization procedure. Before proof of renormalizability, renormalization constants and counter-terms can be established, the Lagrangian must be modified.

The concept and procedure of spontaneous symmetry breaking are well understood, and therefore the discussion here is limited to a shortened version in the case of an Abelian $U(1)$ symmetry breaking, up until the point where one usually chooses a gauge

6.1 BEH mechanism

We start with a Lagrangian describing regular QED and a complex scalar field coupled to the photon.

$$\begin{aligned}\mathscr{L} =&\bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2\\ =&\mathscr{L}_{\text{QED}} + (D^\mu\phi)(D_\mu\phi) - V(\phi),\end{aligned} \tag{6.1.1}$$

where D^μ is the covariant derivative defined as $D^\mu = \partial^\mu - ieA^\mu$ and ϕ is the complex scalar field. For $\mu^2 < 0$, the potential $V(\phi)$ in minimized at some value $\phi_0 \equiv v$. We adopt the procedure described in [24, p. 246]. Parametrizing the scalar field ϕ, into a real part η and a complex part χ

$$\phi = \frac{\eta + i\chi}{\sqrt{2}}. \tag{6.1.2}$$

The η will now become the Higgs boson, and χ will become a Goldstone boson. We denote the full Lagrangian as

$$\mathscr{L} = \mathscr{L}_{\text{QED}} + \mathscr{L}_\phi, \tag{6.1.3}$$

where $\mathscr{L}_\phi$ is given by

$$\mathscr{L}_\phi = (D^\mu\phi)(D_\mu\phi) - V(\phi).$$

After some calculations, see appendix G.1.1 for the full derivation, $\mathscr{L}_\phi$ takes the form

$$\mathscr{L}_\phi = \frac{1}{2}\Big(\big(\partial_\mu\eta + eA^\mu\chi\big)^2 + \big(\partial_\mu\chi - eA^\mu\eta\big)^2\Big) + \frac{\mu^2}{2}\big(\eta^2+\chi^2\big) - \frac{\lambda}{4}\big(\eta^2+\chi^2\big)^2. \tag{6.1.4}$$

In order for the η-field to attain a non-zero vev, one can use the shift [24, p. 248]

$$\begin{aligned} \eta &\to v + \eta, \\ \chi &\to \chi. \end{aligned} \tag{6.1.5}$$

The QED Lagrangian $\mathscr{L}_{\text{QED}}$ is of course unchanged for this transformation, while $\mathscr{L}_\phi$ takes the form

$$\begin{aligned} \mathscr{L}_\phi =& \frac{1}{2}M_A^2A_\mu^2 - evA^\mu\partial_\mu\chi + \frac{1}{2}\big(\partial_\mu\eta\big)^2 + \frac{1}{2}\big(\partial_\mu\chi\big)^2 \\ &+ \partial_\mu\eta eA^\mu\chi - \partial_\mu\chi eA^\mu\eta + \frac{\big(eA^\mu\chi\big)^2}{2} + \frac{\big(eA^\mu\eta\big)^2}{2} + eA^\mu veA_\mu\eta \\ &+ \frac{\mu^2}{2}\big(v^2+2v\eta+\eta^2+\chi^2\big) - \frac{\lambda}{4}\big(v^2+2v\eta+\eta^2+\chi^2\big)^2, \end{aligned} \tag{6.1.6}$$

where the full calculation is given in appendix G.1.2. The first term in (6.1.6) corresponds to the now massive gauge boson. The same line also includes a term mixing the gauge boson and the Goldstone field and kinetic terms for the Higgs boson and Goldstone boson. The term mixing the gauge boson and Goldstone boson can be dealt with by choosing a appropriate gauge, this will be postponed until section 6.3. For now, we move on to examine the new interactions and contributions to the 1-particle irreducible diagrams for the Higgs boson and Goldstone boson.

6.2 Higgs and Goldstone boson diagrams

The terms on the second and third line of (6.1.6) are new interaction terms. By rewriting some of the terms we obtain

$$\begin{aligned} \mathscr{L}_I =& \ (\chi\partial_\mu\eta - \eta\partial_\mu\chi)eA^\mu + \frac{g_{\mu\nu}e^2}{2}A^\mu A^\nu\chi^2 + \frac{g_{\mu\nu}e^2}{2}A^\mu A^\nu\eta^2 + g_{\mu\nu}e^2vA^\mu A^\nu\eta \\ &+ \frac{\mu^2}{2}\big(v^2+2v\eta+\eta^2+\chi^2\big) - \frac{\lambda}{4}\big(v^2+2v\eta+\eta^2+\chi^2\big)^2. \end{aligned} \tag{6.2.1}$$

We start by looking at the terms on the second line. These can be simplified by expanding and using the identity $v^2 = \frac{\mu^2}{\lambda}$. Several terms cancel, leaving behind the terms

$$\mathscr{L}_I \supset \frac{\lambda v^4}{4} - \lambda v^2\eta^2 - \frac{\lambda}{4}\big(4v\eta^3 + 4v\eta\chi^2 + 2\eta^2\chi^2 + \eta^4 + \chi^4\big). \tag{6.2.2}$$

These terms are a contribution to the vacuum energy, a mass term for the Higgs boson and interaction terms. Explicitly, the mass of the Higgs boson is given by

$$M_H^2 = 2\lambda v^2. \tag{6.2.3}$$

Equation (6.2.2) contains self-interactions for the Higgs boson and the Goldstone boson, and a interaction between them. Combining this with the rest of the terms from (6.2.1) gives the final interaction Lagrangian.

$$\mathscr{L}_I = (\chi\partial_\mu\eta - \eta\partial_\mu\chi)eA^\mu + \frac{g_{\mu\nu}e^2}{2}A^\mu A^\nu\chi^2 + \frac{g_{\mu\nu}e^2}{2}A^\mu A^\nu\eta^2 + g_{\mu\nu}e^2 vA^\mu A^\nu\eta$$
$$- \frac{\lambda}{4}\big(4v\eta^3 + 4v\eta\chi^2 + 2\eta^2\chi^2 + \eta^4 + \chi^4\big). \quad (6.2.4)$$

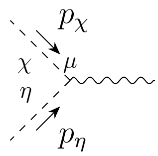
All but one of these new interactions are easily made into new Feynman diagrams and rules with the usual rule of taking the constants of each term, multiply with i, and account for symmetry factors. The first term of (6.2.4)

$$(\chi\partial_\mu\eta - \eta\partial_\mu\chi)eA^\mu,$$

leads to a momentum dependent vertex, which depend on how the particles flow in time [13]. The scalar fields are proportional to their creation and annihilation operators in the usual way

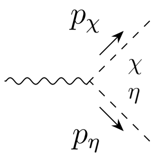
$$\eta \propto a_\eta(p)e^{-ipx} + a_\eta^\dagger(p)e^{ipx},$$
$$\chi \propto a_\chi(p)e^{-ipx} + a_\chi^\dagger(p)e^{ipx}.$$

For a vertex where both scalars are destroyed there is an overall factor of $-i$, with inwards momenta



$$= -ie(p_\eta - p_\chi).$$

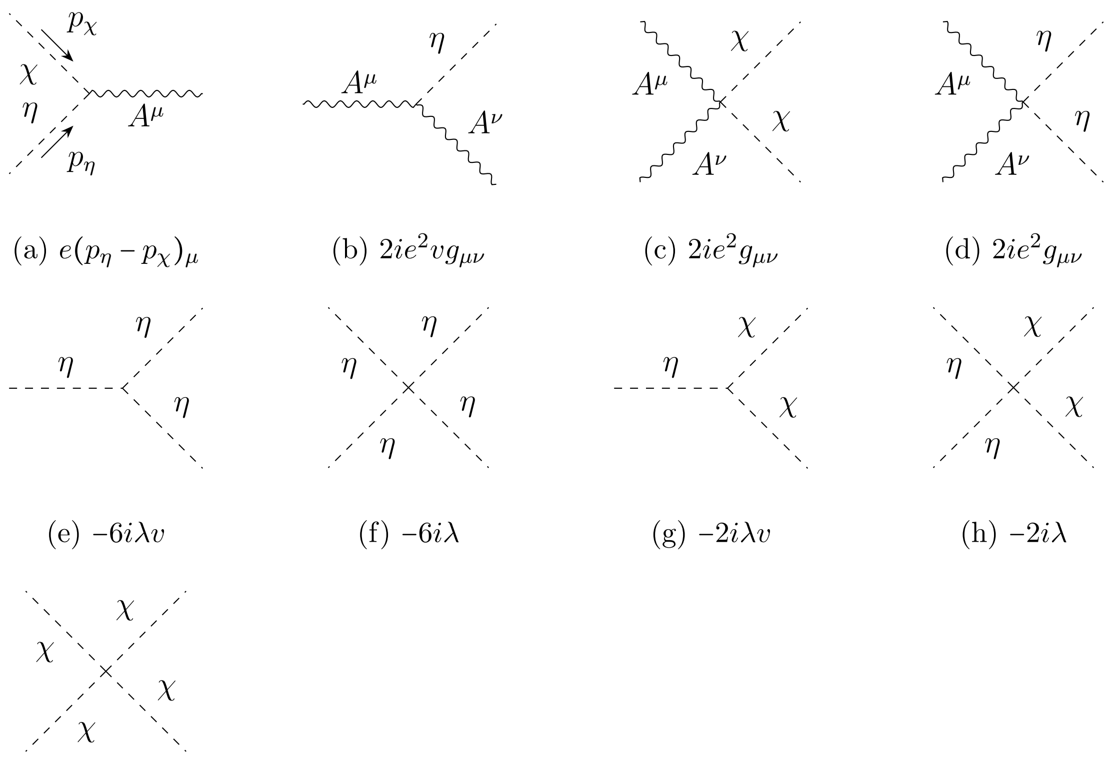
Meanwhile, a vertex where both scalars are created has a factor of i, with outgoing momenta



$$= ie(p_\eta - p_\chi).$$

Defining the momenta to always be pointing inwards accounts for both vertices and we have the Feynman rule shown in figure 5(a). This is the only real Feynman rule, due to the derivative which gave a factor of i. Fortunately, atleast at the one-loop level this does not cause any complications; in propagators this vertex appears in pairs. In section 6.7.2 we show that the divergent amplitudes of vertex corrections also has two vertices of this type.

As mentioned, the rest of the Feynman rules for the terms in (6.2.4) are obtained by adding a factor i and accounting for symmetry. All diagrams with repeating fields receive a numerical factor due to the symmetry in the S-matrix element, given by $\prod_c n_c!$ where n_c is the power of the repeating field [25]. The result is listed in figure 5.



(a) $e(p_\eta - p_\chi)_\mu$ (b) $2ie^2 v g_{\mu\nu}$ (c) $2ie^2 g_{\mu\nu}$ (d) $2ie^2 g_{\mu\nu}$

(e) $-6i\lambda v$ (f) $-6i\lambda$ (g) $-2i\lambda v$ (h) $-2i\lambda$

(i) $-6i\lambda$

Figure 5: Feynman rules for additional interactions.

In addition to the fermion, photon and their interaction from QED, there are now two new particles and nine new interactions. Fermions do not interact with these new particles. The photon is now massive and receives contributions to the divergent amplitude from new one-particle irreducible diagrams. However, before we move on to renormalization, we should choose an appropriate gauge.

6.3 R_ξ-gauge

The R_ξ gauges are the set of gauges for which ξ is a possible, finite value. The Feynman gauge ($\xi = 1$) we have used in the previous sections is one of them. A general R_ξ-gauge should produce physical quantities regardless of the choice of ξ [10, p. 738]. As we saw in the Lagrangian (6.1.6) there is a unfamiliar term mixing the Goldstone boson and photon

$$\mathscr{L} \supset -evA^\mu \partial_\mu \chi. \tag{6.3.1}$$

In the R_ξ-gauge this term can be dealt with by choosing the gauge-fixing term such that this term cancels, up to irrelevant total derivatives [18, p. 585]. Following [24, pp. 247-249], this is done by modifying the gauge-fixing term in the Lagrangian

$$\mathscr{L}_{\text{gf}} = -\frac{1}{2\xi}G^2 = -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2,$$

to include the Goldstone boson χ

$$G \equiv \partial_\mu A^\mu - \xi ev\chi = 0. \tag{6.3.2}$$

In the Lagrangian this means that there are two additional terms

$$\mathscr{L}_{\text{gf}} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 + evA_\mu\partial^\mu\chi - \frac{1}{2}\xi(ev)^2\chi^2. \tag{6.3.3}$$

The Goldstone boson ξ has now acquired a gauge-dependent mass, which is reflected in the propagator, shown in section 6.4. This propagator is unphysical, since it depends on ξ. It must be canceled by a ghost, which can be introduced by including the terms

$$\mathscr{L}_c = -\bar{c}\left[\,\Box + \xi e^2 v(v+\eta)\right]c = \left(\partial^\mu\bar{c}\right)\left(\partial_\mu c\right) - \xi M_A^2\bar{c}c - e^2 v\eta\bar{c}c. \tag{6.3.4}$$

6.4 Propagators and vertices in R_ξ-gauge

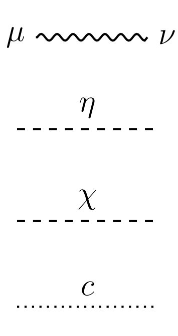
The full Lagrangian is now given by

$$\mathscr{L} = \mathscr{L}_{\text{QED}} + \mathscr{L}_\phi + \mathscr{L}_{\text{gf}} + \mathscr{L}_{\text{c}},$$

where $\mathscr{L}_\phi$, $\mathscr{L}_{\text{gf}}$, $\mathscr{L}_{\text{c}}$ are given by (6.1.6), (6.3.3) and (6.3.4) respectively. After some reordering and using the results from section 6.2 the full Lagrangian takes the form

$$\begin{aligned}\mathscr{L} = {} & \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M_A^2 A^2 \\ & + \frac{1}{2}\left[\left(\partial_\mu\eta\right)^2 - M_H^2\eta^2\right] + \frac{1}{2}\left(\partial_\mu\chi\right)^2 - \frac{1}{2}\xi(ev)^2\chi^2 \\ & + (\chi\partial_\mu\eta - \eta\partial_\mu\chi)eA^\mu + \frac{g_{\mu\nu}e^2}{2}A^\mu A^\nu\chi^2 + \frac{g_{\mu\nu}e^2}{2}A^\mu A^\nu\eta^2 + g_{\mu\nu}e^2 vA^\mu A^\nu\eta \\ & + \frac{\lambda v^4}{4} - \frac{\lambda}{4}\left(4v\eta^3 + 4v\eta\chi^2 + 2\eta^2\chi^2 + \eta^4 + \chi^4\right) \\ & - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 + \left(\partial^\mu\bar{c}\right)\left(\partial_\mu c\right) - \xi M_A^2\bar{c}c - \xi e^2 v\eta\bar{c}c.\end{aligned} \tag{6.4.1}$$

The propagators are directly obtained from the Lagrangian, and can be summarized as such [24, p. 249]



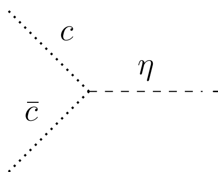
$$\rightarrow \frac{i}{q^2 - M_A^2 + i\epsilon}\left[-g^{\mu\nu} + (1-\xi)\frac{q^\mu q^\nu}{q^2 - \xi M_A^2}\right], \tag{6.4.2}$$

$$\rightarrow \frac{i}{p^2 - M_H^2 + i\epsilon}, \tag{6.4.3}$$

$$\rightarrow \frac{i}{p^2 - \xi M_A^2 + i\epsilon}, \tag{6.4.4}$$

$$\rightarrow \frac{i}{p^2 - \xi M_A^2 + i\epsilon}. \tag{6.4.5}$$

The last term in (6.4.1) determines how the ghost couple to the Higgs, and gives the diagram



$$\rightarrow -i\xi e^2 v \tag{6.4.6}$$

In the R_ξ-gauge, there is a new particle, a new interaction term and new 1-loop diagrams. Firstly, there are two new self-energy type diagrams, shown below.

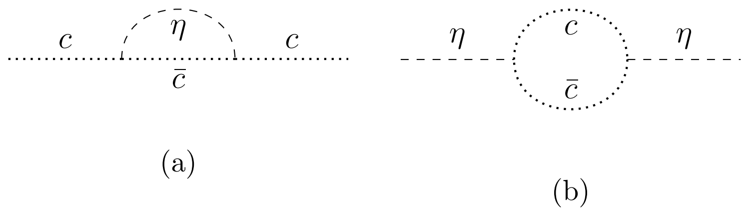


Figure 6: New self-energy type diagrams in R_ξ-gauge.

Secondly, the new vertex (6.4.6) has its own 1-loop diagrams.

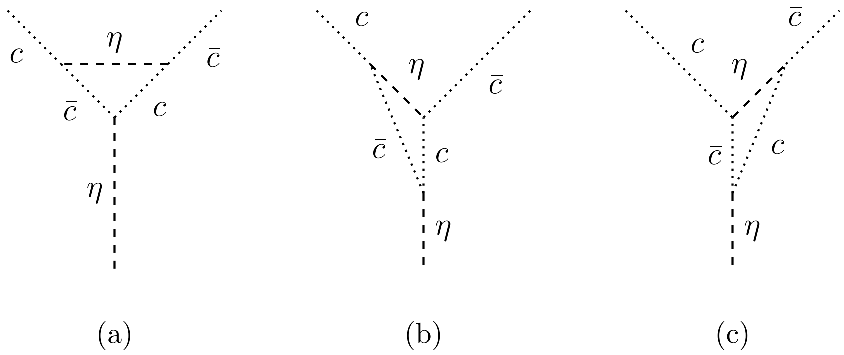


Figure 7: 1-loop contributions to the $\eta c\bar{c}$-vertex.

The Higgs boson propagator will therefore receive an additional contribution from figure 6(b). All the particles and interactions for spontanously broken QED in R_ξ-gauge have now been determined, and are contained in the full Lagrangian (6.4.1). We will first check the renormalizability of the theory. In sections 6.6 and 6.7 we will show the contributing diagrams and their divergent amplitude for each propagator and a couple of the vertices. After that we calculate the renormalization constants and the β-function.

6.5 Renormalizability

In order to check the renormalizability of this theory, we go back to the index of divergence (3.2.2), repeated here.

$$r_i = \frac{d-2}{2}b_i + \frac{d-1}{2}f_i + \delta_i - d.$$

The QED interaction term $e\gamma_\mu\bar{\psi}A^\mu\psi$ is unchanged, and so is the corresponding index of divergence; $r = 0$. It remains to determine all the other indices for the new interactions, summarized in table 3.

$\mathscr{L}_I$	r_i	$\mathscr{L}_I$	r_i
$e\chi\partial_\mu\eta A^\mu$	0	$e\eta\partial_\mu\chi A^\mu$	0
$g_{\mu\nu}\frac{e^2}{2}A^\mu A^\nu\chi^2$	0	$g_{\mu\nu}e^2 vA^\mu A^\nu\eta$	−1
$-\lambda v\eta^3$	−1	$-\lambda v\eta\chi^2$	−1
$-\frac{\lambda}{2}\eta^2\chi^2$	0	$-\frac{\lambda}{4}\eta^4$	0
$-\frac{\lambda}{4}\chi^4$	0	$-\xi e^2 v\eta c\bar{c}$	−1

Table 3: Indices of divergence for broken QED interactions.

Some of the interaction terms are super-renormalizable. The theory, however, is just renormalizable, since not all indices are less than 0. [8, p. 131].

Extracting the mass dimension of the parameters is necessary for calculating the β-functions. As before, this is done with dimensional analysis. The quartic coupling parameter λ can be obtained from one of many terms in the Lagrangian (6.4.1). We chose the quartic self-interaction term

$$\mathscr{L} \supset -\frac{\lambda}{4}\eta^4.$$

The mass dimension of η is the same as any scalar field

$$[\eta] = \frac{d-2}{2} = 1 - \frac{\epsilon}{2}.$$

In order for the term as a whole to have dimension $4-\epsilon$, the dimension of λ must be

$$[\lambda] = \epsilon. \tag{6.5.1}$$

For the vacuum expectation value parameter v, we use

$$\mathscr{L} \supset -\lambda v\eta^3.$$

The newly acquired value for $[\lambda]$ (6.5.1) determines the dimension for v

$$[v] = 1 - \frac{\epsilon}{2}, \tag{6.5.2}$$

the same as for any scalar field, such as η. This would of course have to be the case, as the field shift (6.1.5) would not make sense otherwise, but it serves as a simple check that the terms in the Lagrangian are correct.

Mass dimensions for the coupling constants e and λ are extracted, so that the parameters appearing in the amplitudes are dimensionless.

$$e \to \mu^{\frac{\epsilon}{2}}e, \tag{6.5.3}$$

$$\lambda \to \mu^{\epsilon}\lambda. \tag{6.5.4}$$

Although v appears as a parameter, the mass dimension is not extracted. As noted above, the mass dimension is equal to that of η, which has to hold both before and after renormalization.

The gauge-fixing parameter ξ is dimensionless. This can be seen by inspecting the mass term for the Goldstone boson

$$\mathscr{L} \supset \frac{1}{2}\xi M_A^2 \chi^2.$$

Renormalizability of the theory has now been proven and the necessary parameters have been made dimensionless. We are now ready to calculate the 1-loop diagrams.

6.6 Propagator amplitudes

In section 4 and 5 we used Feynman-gauge, where the photon propagator has the simple form

$$\Sigma \propto -i\frac{g^{\mu\nu}}{q^2 + i\varepsilon}. \tag{6.6.1}$$

This had the advantage of making generalization to P.V functions straightforward. In R_ξ-gauge the photon propagator (6.4.2) includes the mass the photon has acquired and is written in a gauge-invariant form. As a consequence, some amplitude Σ can how have the form

$$\Sigma = e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{(k^2 - M_A^2)(k^2 - \xi M_A^2)}.$$

In dimensional regularization these can be dealt with by splitting the denominator

$$\Sigma = e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\left(\frac{k^\mu k^\nu}{(\xi M_A^2 - M_A^2)(k^2 - \xi M_A^2)} - \frac{k^\mu k^\nu}{(\xi M_A^2 - M_A^2)(k^2 - M_A^2)}\right).$$

Evaluating these two integrals with the help of (B.3.1c) yields

$$\begin{aligned}\Sigma =& i\frac{e^2}{(4\pi)^2}\frac{g^{\mu\nu}}{2}\Gamma\left(-2 + \frac{\epsilon}{2}\right)\frac{(4\pi\mu^2)^{\frac{\epsilon}{2}}}{\xi M_A^2 - M_A^2}\left(\left(\frac{1}{\xi M_A^2}\right)^{-\frac{d}{2}} - \left(\frac{1}{M_A^2}\right)^{-\frac{d}{2}}\right)\\ =& i\frac{e^2}{(4\pi)^2}\frac{g^{\mu\nu}}{2}\Gamma\left(-2 + \frac{\epsilon}{2}\right)\frac{(4\pi\mu^2)^{\frac{\epsilon}{2}}}{\xi M_A^2 - M_A^2}\left(\frac{(\xi M_A^2)^2}{(\xi M_A^2)^{\frac{\epsilon}{2}}} - \frac{(M_A)^4}{(M_A)^\epsilon}\right)\end{aligned}$$

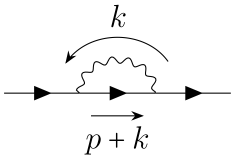
Expanding $\Gamma(-2 + \frac{\epsilon}{2})$ with the use of (B.4.7), and ignoring all higher order terms leaves

$$\Sigma \overset{\text{Div}}{=} i\frac{e^2}{(4\pi)^2}\frac{g^{\mu\nu}}{2\epsilon}\left(\xi M_A^2 + M_A^2\right). \tag{6.6.2}$$

From now on, we will use Mathematica for the more complex of the divergent amplitude calculations. We use the notation Σ^a to signify the sum of the divergent amplitudes for the propagator of particle a.

6.6.1 Fermion

As in the unbroken theory, there is only one contributing diagram at 1-loop level, however the amplitude is changed, due to the change in the photon propagator.



$$
\begin{aligned}
&=(-ie)^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\gamma_\mu \frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}\gamma_\nu \frac{i}{k^2-M_A^2}\left[-g^{\mu\nu}+(1-\xi)\frac{k^\mu k^\nu}{k^2-\xi M_A^2}\right] \\
&=e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\left[-\gamma_\mu \frac{(\not{k}+\not{p}+m)}{(k^2-M_A^2)[(k+p)^2-m^2]}\gamma^\mu \right. \\
&\quad \left. +(1-\xi)\frac{\gamma_\mu(\not{k}+\not{p}+m)\gamma_\nu k^\mu k^\nu}{(k^2-M_A^2)[(k+p)^2-m^2][k^2-\xi M_A^2]}\right].
\end{aligned}
$$

The first term in the brackets is the familiar term from the unbroken theory, calculated in section 5.2.1. These integrals are evaluated with the methods covered previously, with the result

$$
\begin{aligned}
\text{Amplitude} &\stackrel{\text{Div}}{=} i\frac{e^2}{(4\pi)^2}\left[2dm+(2-d)\not{p}+(1-\xi)\big(\not{p}+\not{p}-2m\big)\right]\frac{1}{d-4} \\
&= i\frac{e^2}{(4\pi)^2}\left[8m-2\not{p}+(1-\xi)\big(2\not{p}-2m\big)\right]\left(-\frac{1}{\epsilon}\right) \\
&= i\frac{e^2}{(4\pi)^2}\left[\not{p}-4m-(1-\xi)\big(\not{p}-m\big)\right]\frac{2}{\epsilon}, \qquad (6.6.3)
\end{aligned}
$$

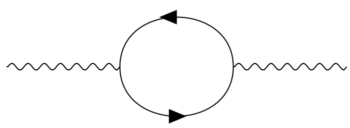
where we have used $d = 4-\epsilon$, and ignored finite terms.

There are no new couplings for the fermions, and therefore this is the only contributing diagram.

$$
\Sigma^\psi = i\frac{e^2}{(4\pi)^2}\left[\not{p}-4m-(1-\xi)\big(\not{p}-m\big)\right]\frac{2}{\epsilon}. \qquad (6.6.4)
$$

6.6.2 Photon

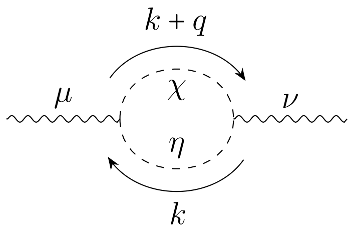
Two diagrams are particularly important, and their amplitude calculation is explicitly shown.



$$
= -(-ie)^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\mathrm{Tr}\left[\gamma_\mu \frac{i(\not{k}+m)}{k^2-m^2}\gamma_\nu \frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2}\right].
$$

This diagram is unchanged from the unbroken theory (5.2.14), so its amplitude is unchanged, and the result is simply listed here.

$$
\text{Amplitude} = -i\frac{e^2}{(4\pi)^2}\left(\frac{8}{3}\right)(g_{\mu\nu}q^2-q_\mu q_\nu)\frac{1}{\epsilon}. \qquad (6.6.5)
$$

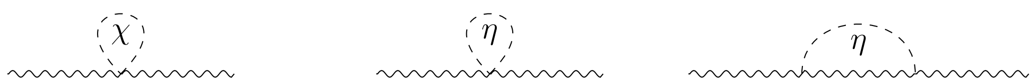


$$
\begin{aligned}
&= e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{[2k_\mu + q_\mu][2k_\nu + q_\nu]}{(k^2 - M_H^2)[(k+q)^2 - \xi M_A^2]}\\
&= e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{4k_\mu k_\nu + 2k_\mu q_\nu + 2k_\nu q_\mu + q_\mu q_\nu}{(k^2 - M_H^2)[(k+q)^2 - \xi M_A^2]}\\
&= e^2\frac{i}{(4\pi)^2}[4B_{\mu\nu} + 2q_\nu B_\mu + 2q_\mu B_\nu + q_\mu q_\nu B_0],
\end{aligned}
$$

where all the tensor integrals are functions of $q^2, M_H^2, \xi M_A^2$. Using the divergent parts of the integrals (C.2.1), the amplitude becomes

$$
\begin{aligned}
\Big(\text{Amplitude}\Big)_{\text{Div}} &= i\frac{e^2}{(4\pi)^2}[4\big(g_{\mu\nu}(-\frac{1}{6\epsilon}(q^2 - 3M_H^2 - 3\xi M_A^2)) + q_\mu q_\nu\frac{2}{3\epsilon}\big) - 2q_\mu q_\nu\frac{1}{\epsilon} - 2q_\mu q_\nu\frac{1}{\epsilon} + q_\mu q_\nu\frac{2}{\epsilon}]\\
&= i\frac{e^2}{(4\pi)^2}[4g_{\mu\nu}q^2\Big(-\frac{1}{6\epsilon}\Big) + 4q_\mu q_\nu\frac{2}{3\epsilon} - 2q_\mu q_\nu\frac{1}{\epsilon} + 2g_{\mu\nu}(M_H^2 + \xi M_A^2)\frac{1}{\epsilon}]\\
&= -i\frac{e^2}{(4\pi)^2}\Bigg[\frac{2}{3}(g_{\mu\nu}q^2 - q_\mu q_\nu) - 2g_{\mu\nu}(M_H^2 + \xi M_A^2)\Bigg]\frac{1}{\epsilon}. \qquad (6.6.6)
\end{aligned}
$$

These are singled out due to their amplitude. They both exhibit the tensor structure $g_{\mu\nu}q^2 - q_\mu q_\nu$, and therefore are the contributing factors to Z_A, as will be shown in section 6.9. The rest of the diagrams contributing to the photon propagator are shown in figure 8, and their calculations can be found in appendix E.2.1.



(a) (b) (c)

Figure 8: Other contributions to the photon propagator.

These do not exhibit the tensor structure and contribute only to δM_A^2. The sum of these diagrams is

$$
\begin{aligned}
\Sigma^A_{\mu\nu} = -i\frac{e^2}{(4\pi)^2}\frac{1}{\epsilon}\Bigg\{&\frac{8}{3}(g_{\mu\nu}q^2 - q_\mu q_\nu) + \frac{2}{3}(g_{\mu\nu}q^2 - q_\mu q_\nu) - 2g_{\mu\nu}(M_H^2 + \xi M_A^2)\\
&+ 4g_{\mu\nu}\xi M_A^2 + 4g_{\mu\nu}M_H^2 + 2M_A^2 g_{\mu\nu}(3+\xi)\Bigg\}.
\end{aligned}
$$

Reordering yields the result

$$
\Sigma^A_{\mu\nu} = -i\frac{e^2}{(4\pi)^2}\frac{1}{\epsilon}\Bigg\{\frac{10}{3}(g_{\mu\nu}q^2 - q_\mu q_\nu) + g_{\mu\nu}\big(2M_H^2 + 6M_A^2 + 4\xi M_A^2\big)\Bigg\}. \qquad (6.6.7)
$$

6.6.3 Higgs boson

All the contributions are summarized in figure 9, their calculation can be found in appendix E.2.2.

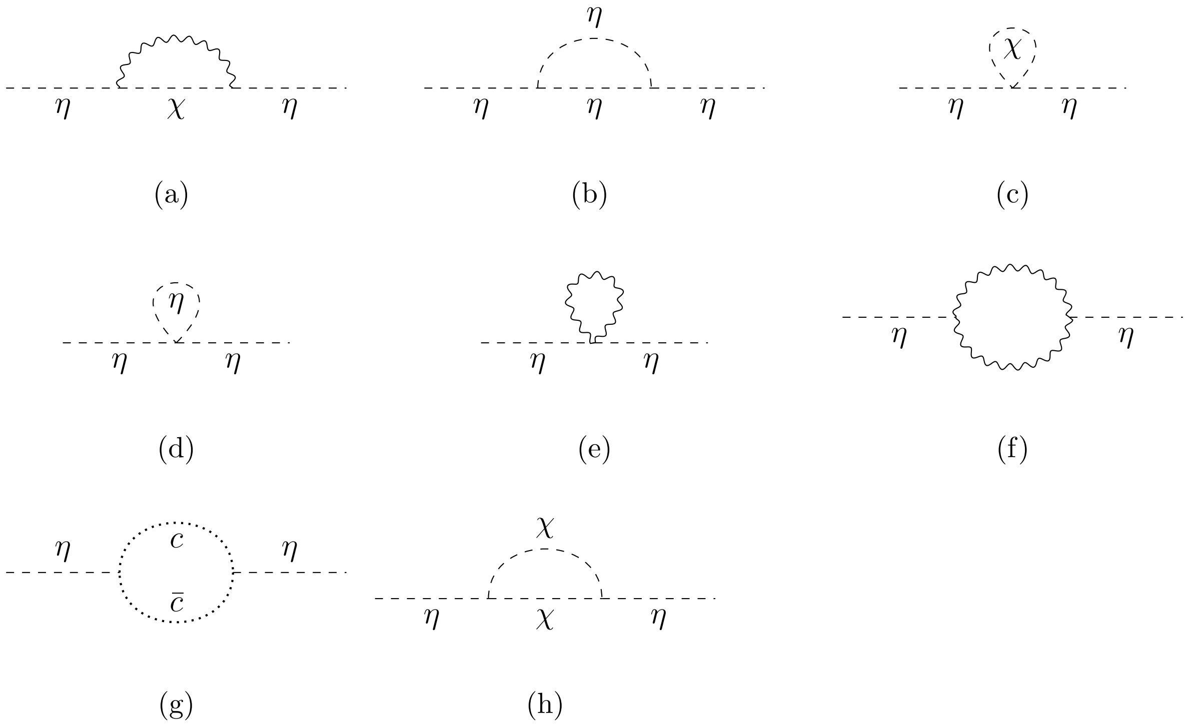


Figure 9: Contributions to the Higgs boson propagator.

The amplitudes sum up to

$$\begin{aligned}
\Sigma^\eta =& -i\frac{e^2\left(2\xi^2M_A^2-(\xi-3)p^2\right)}{8\pi^2\epsilon}+i\frac{e^2\left(\xi^2M_A^2+3M_A^2\right)}{4\pi^2\epsilon}\\
&+i\frac{e^2M_A^2\left(\xi^2+3\right)}{4\pi^2\epsilon}-i\frac{e^2M_A^2\xi^2}{8\pi^2\epsilon}+i\frac{\lambda\xi M_A^2}{4\pi^2\epsilon}+i\frac{3\lambda M_H^2}{4\pi^2\epsilon}+i\frac{5\lambda^2v^2}{2\pi^2\epsilon}\\
=& \, i\frac{\left(e^2(\xi-3)p^2+e^2\left(\xi^2M_A^2+12M_A^2\right)+2\lambda\left(\xi M_A^2+8M_H^2\right)\right)}{8\pi^2\epsilon}.
\end{aligned} \tag{6.6.8}$$

6.6.4 Goldstone boson

All the contributions are summarized in figure 10, their calculation can be found in appendix E.2.2.

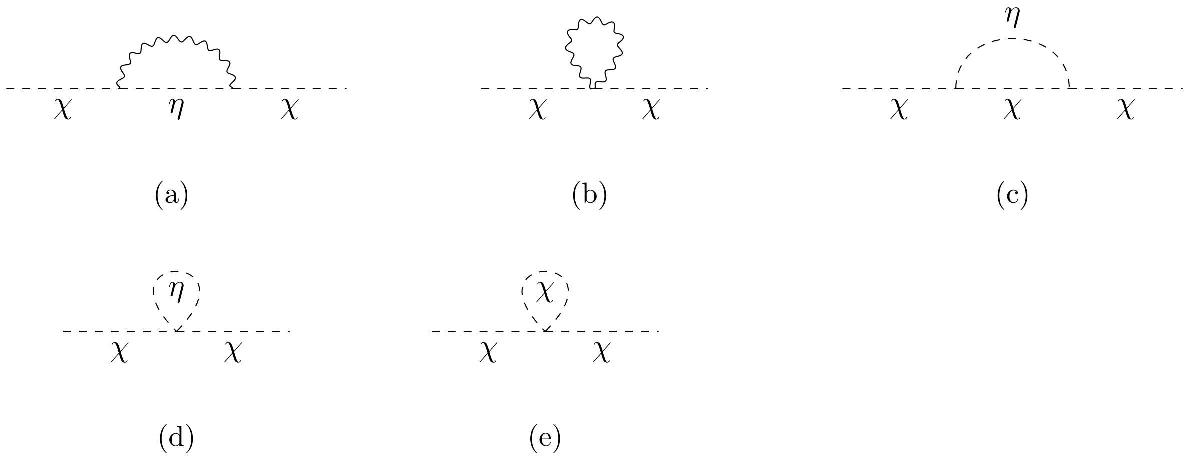


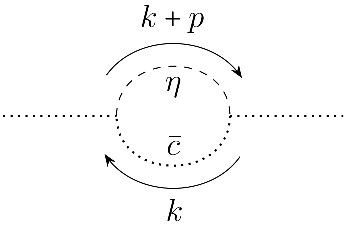
Figure 10: Contributions to the Goldstone boson propagator.

The amplitudes sum up to

$$\begin{aligned}\Sigma^{\chi} = &-\frac{ie^2\left(\xi\left(\xi M_A^2+M_H^2\right)-(\xi-3)p^2\right)}{8\pi^2\epsilon}+\frac{ie^2\left(\xi^2 M_A^2+3M_A^2\right)}{4\pi^2\epsilon}\\ &+\frac{3i\lambda\xi M_A^2}{4\pi^2\epsilon}+\frac{i\lambda M_H^2}{4\pi^2\epsilon}+\frac{i\lambda^2v^2}{2\pi^2\epsilon}\\ =&\frac{i\left(e^2(\xi-3)p^2+e^2\left(\xi(\xi M_A^2+M_H^2)+6M_A^2\right)+6\lambda\xi M_A^2+4\lambda M_H^2\right)}{8\pi^2\epsilon}.\end{aligned} \tag{6.6.9}$$

6.6.5 Ghost

At the one-loop order there is only one diagram contributing to the ghost propagator.



$$\begin{aligned}&= (-i\xi e^2 v)^2\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-\xi M_A^2}\frac{i}{\left[(k+p)^2-M_H^2\right]}\\ &= i\frac{(\xi e^2 v)^2}{(4\pi)^2}\mu^{\epsilon}B_0(p^2,\xi M_A^2,M_H^2).\\ &\overset{\text{Div}}{=} i\frac{e^2}{8\pi^2\epsilon}\xi^2 M_A^2.\end{aligned} \tag{6.6.10}$$

Notice the factor of μ^{ϵ} here. The coupling constants e in this equation are the dimensionless ones, as defined when we extracted the mass dimension in (6.5.3). In section 6.1 the photon mass was found to be

$$M_A^2 = (ev)^2,$$

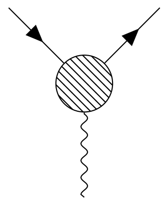
where the coupling constant e had its original mass dimension. This is necessary for the photon mass to have mass dimension 1. In expressions such as (6.6.10) the mass dimension μ^{ϵ} is therefore absorbed back into the mass. Since this was the only contribution to the ghost propagator, the sum of divergent amplitudes is simply given by

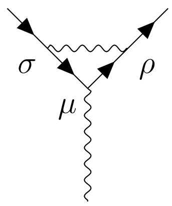
$$\Sigma^{c} = i\frac{e^2}{8\pi^2\epsilon}\xi^2 M_A^2. \tag{6.6.11}$$

6.7 Vertex amplitudes

We use the notation Γ^{abc} to signify the divergent amplitudes of each vertex. a, b and c represent the external particles in each vertex.

6.7.1 $\bar{\psi} A^{\mu} \psi$-vertex





$$=(-ie)^3\mu^{\frac{3}{2}\epsilon}\int\frac{d^dk}{(2\pi)^d}\gamma_\rho\frac{i(\not{q}+\not{k}+m)}{(q+k)^2-m^2}\gamma_\mu\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}\gamma_\sigma\frac{i}{k^2-M_A^2}\left[-g^{\rho\sigma}+(1-\xi)\frac{k^\rho k^\sigma}{k^2-\xi M_A^2}\right],$$

the first term corresponds to the regular expression from QED, calculated in section 5.2.3, with the result for the divergent term (5.2.16)

$$\Gamma_\mu^{\psi A^\mu \bar{\psi}} = -\mu^{\frac{\epsilon}{2}}\frac{e^3}{(4\pi)^2}\gamma_\mu\frac{2}{\epsilon}. \qquad (6.7.1)$$

The remaining term takes the form

$$\begin{aligned}
\text{Amplitude} =&(-ie)^3\mu^{\frac{3}{2}\epsilon}\int\frac{d^dk}{(2\pi)^d}\gamma_\rho\frac{i(\not{q}+\not{k}+m)}{(q+k)^2-m^2}\gamma_\mu\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}\gamma_\sigma\frac{i}{k^2-M_A^2}(1-\xi)\frac{k^\rho k^\sigma}{k^2-\xi M_A^2}\\
=&(1-\xi)e^3\mu^{\frac{3}{2}\epsilon}\int\frac{d^dk}{(2\pi)^d}\Big[\gamma_\rho\not{q}\gamma_\mu\not{p}\gamma_\sigma+\gamma_\rho\not{q}\gamma_\mu\not{k}\gamma_\sigma+\gamma_\rho\not{q}\gamma_\mu m\gamma_\sigma\\
&+\gamma_\rho\not{k}\gamma_\mu\not{p}\gamma_\sigma+\gamma_\rho\not{k}\gamma_\mu\not{k}\gamma_\sigma+\gamma_\rho\not{k}\gamma_\mu m\gamma_\sigma\\
&+\gamma_\rho m\gamma_\mu\not{p}\gamma_\sigma+\gamma_\rho m\gamma_\mu\not{k}\gamma_\sigma+\gamma_\rho m\gamma_\mu m\gamma_\sigma\Big]\frac{k^\rho k^\sigma}{[k^2-M_A^2][k^2-\xi M_A^2][(q+k)^2-m^2][(p+k)^2-m^2]}.
\end{aligned}$$

The tensor decomposition of this expression is lengthy. Luckily, there is only one term that is divergent, namely the 4-point P.V function, with the general form

$$\int\frac{d^dk}{(2\pi)^d}\frac{k^\delta k^\lambda k^\rho k^\sigma}{[k^2-M_A^2][k^2-\xi M_A^2][(q+k)^2-m^2][(p+k)^2-m^2]}\overset{\text{uv}}{=}-\frac{i(\pi^2g^{\delta\sigma}g^{\lambda\rho}+\pi^2g^{\delta\rho}g^{\lambda\sigma}+\pi^2g^{\delta\lambda}g^{\rho\sigma})}{12(d-4)}.$$

Thus, explicitly the term takes the form

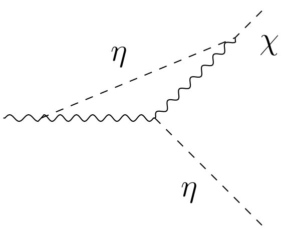
$$
\begin{aligned}
\Gamma_\mu^{\psi A^\mu \bar\psi} &= (1-\xi)e^3\mu^{\frac{3}{2}\epsilon}\int \frac{d^dk}{(2\pi)^d}\gamma_\rho \not{k}\gamma_\mu \not{k}\gamma_\sigma \frac{k^\rho k^\sigma}{[k^2-M_A^2][k^2-\xi M_A^2][(q+k)^2-m^2][(p+k)^2-m^2]}\\
&\overset{\text{Div}}{=} -(1-\xi)e^3\mu^{\frac{\epsilon}{2}}\frac{1}{(2\pi)^4}\gamma_\rho\gamma_\delta\gamma_\mu\gamma_\lambda\gamma_\sigma\left(\frac{i(\pi^2 g^{\delta\sigma}g^{\lambda\rho}+\pi^2 g^{\delta\rho}g^{\lambda\sigma}+\pi^2 g^{\delta\lambda}g^{\rho\sigma})}{12(d-4)}\right)\\
&= \frac{i}{(4\pi)^2}\mu^{\frac{\epsilon}{2}}(1-\xi)\frac{e^3}{12\epsilon}\big(\gamma_\rho\gamma^\sigma\gamma_\mu\gamma^\rho\gamma_\sigma+\gamma_\rho\gamma^\rho\gamma_\mu\gamma^\sigma\gamma_\sigma+\gamma_\rho\gamma^\lambda\gamma_\mu\gamma_\lambda\gamma^\sigma\big)\\
&= \frac{i}{(4\pi)^2}\mu^{\frac{\epsilon}{2}}(1-\xi)\frac{e^3}{12\epsilon}\big(-4\gamma_\mu-d^2\gamma_\mu+6d\gamma_\mu+d^2\gamma_\mu+4\gamma_\mu+d^2\gamma_\mu-4d\gamma_\mu\big)\\
&= \frac{i}{(4\pi)^2}\mu^{\frac{\epsilon}{2}}(1-\xi)\frac{e^3}{12\epsilon}\big((d+2)d\gamma_\mu\big), \qquad (6.7.2)
\end{aligned}
$$

using the contraction identities (B.1.5). The entire divergent expression is obtained by including the other divergent term (6.7.1)

$$
\begin{aligned}
\Gamma_\mu^{\psi A^\mu \bar\psi} &= -\frac{e^3}{(4\pi)^{d/2}}\mu^{\frac{\epsilon}{2}}\gamma_\mu\frac{2}{\epsilon}+\frac{i}{(4\pi)^2}\mu^{\frac{\epsilon}{2}}(1-\xi)\frac{e^3}{12\epsilon}\big((d+2)d\gamma_\mu\big)\\
&\overset{d\to4}{=} -i\mu^{\frac{\epsilon}{2}}\xi\frac{e^3}{(4\pi)^2}\gamma_\mu\frac{2}{\epsilon}. \qquad (6.7.3)
\end{aligned}
$$

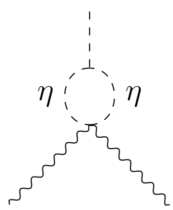
6.7.2 Divergence of three-point vertices

For the rest of the three-point vertices, power counting can be used to quickly filter out finite amplitudes in loops with three propagators. Referring to figure 5, the momentum-dependent vertex given first adds a power of integration momentum k, therefore some of the amplitudes may be divergent. As an example, imagine the diagram



$$\sim \int \frac{d^dk}{(2\pi)^d}k\frac{1}{k^2}\frac{1}{k^2}\frac{1}{k^2} = \text{UV-finite}, \qquad (6.7.4)$$

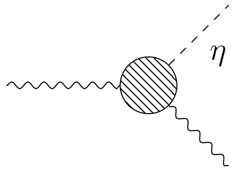
the second term in the photon propagator (6.4.2) does not change the powers of k in either the numerator or denominator, and the analysis is unchanged. From this we see that the only divergent diagrams are the ones containing two vertices of the type from figure ??. In addition, there are diagrams with two particles in the loop



$$, \qquad (6.7.5)$$

where we have slightly adjusted the external lines for readability. These are all divergent, and need to be considered as well. To obtain the renormalization constants not all of the vertices are needed, and we show only those that are used later.

6.7.3 $\Gamma_{\mu\nu}^{A^\mu A^\nu \eta}$



(6.7.6)

All the divergent loops containing three propagators are listed in figure 11. For the calculation of each individual diagram see appendix E.3.1.

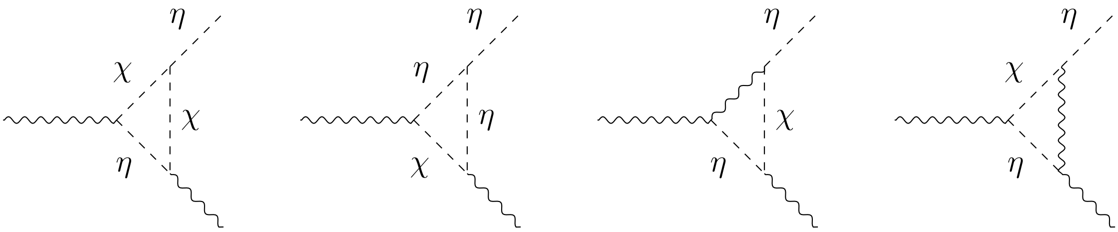


Figure 11: Divergent contributions to (6.7.6) with three propagators.

The sum of the contributions in figure 11 is

$$\left(\Gamma_{\mu\nu}^{A^\mu A^\nu \eta}\right)_1 = i\mu^\epsilon \frac{e^4 \xi v g_{\mu\nu}}{4\pi^2 \epsilon} + i\mu^\epsilon \frac{e^2 \lambda v g_{\mu\nu}}{\pi^2 \epsilon}. \tag{6.7.7}$$

As mentioned, there are several contributing diagrams with two propagators in the loop. These are shown in figure **??**

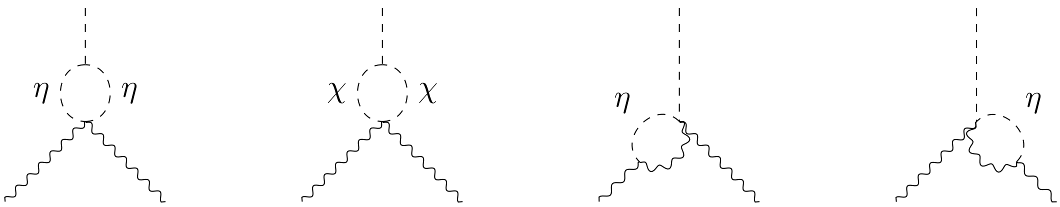


Figure 12: Two-point loop contributions to (6.7.6)

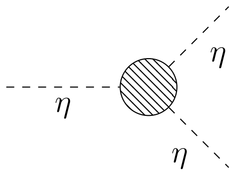
These diagrams have the combined amplitude

$$\left(\Gamma_{\mu\nu}^{A^\mu A^\nu \eta}\right)_2 = -i\mu^\epsilon \frac{e^2 \lambda v g_{\mu\nu}}{\pi^2 \epsilon} - i\mu^\epsilon \frac{e^4 (\xi + 3) v g_{\mu\nu}}{4\pi^2 \epsilon}. \tag{6.7.8}$$

Adding (6.7.7) and (6.7.8) causes some terms to cancel, leaving

$$\begin{aligned} \Gamma_{\mu\nu}^{A^\mu A^\nu \eta} &= i\mu^\epsilon \frac{e^4 \xi v g_{\mu\nu}}{4\pi^2 \epsilon} - i\mu^\epsilon \frac{e^4 (\xi + 3) v g_{\mu\nu}}{4\pi^2 \epsilon} \\ &= -i\mu^\epsilon \frac{3 e^4 v g_{\mu\nu}}{4\pi^2 \epsilon}. \end{aligned} \tag{6.7.9}$$

6.7.4 $\Gamma^{\eta\eta\eta}$



(6.7.10)

Following the same structure as above, all the divergent loop contributions with three propagators in the loop are listed in figure 13. Loop contributions with two propagators in the loop are shown in figure 14. For the calculation of each individual diagram see appendix E.3.2.

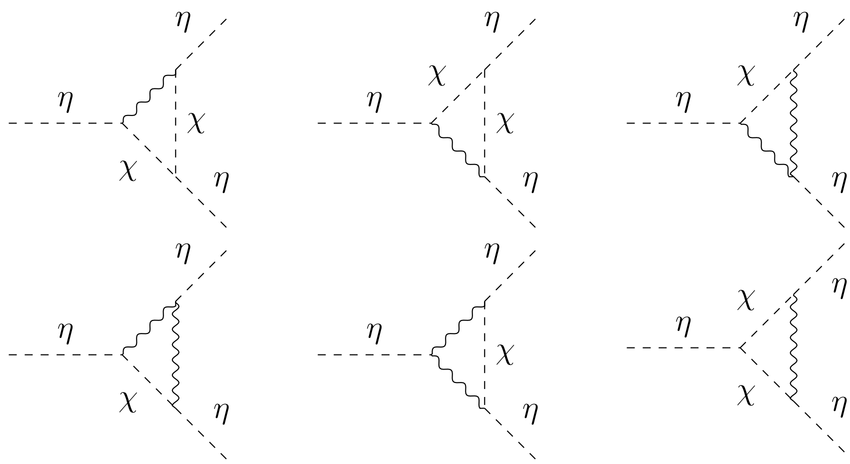


Figure 13: Three-point divergent contributions to (6.7.10)

The three-point loops have the combined divergent amplitude

$$(\Gamma^{\eta\eta\eta})_1 = -i\mu^{\epsilon}\frac{3e^4\xi^2 v}{4\pi^2\epsilon} - i\mu^{\epsilon}\frac{3e^2\lambda\xi v}{4\pi^2\epsilon}. \tag{6.7.11}$$

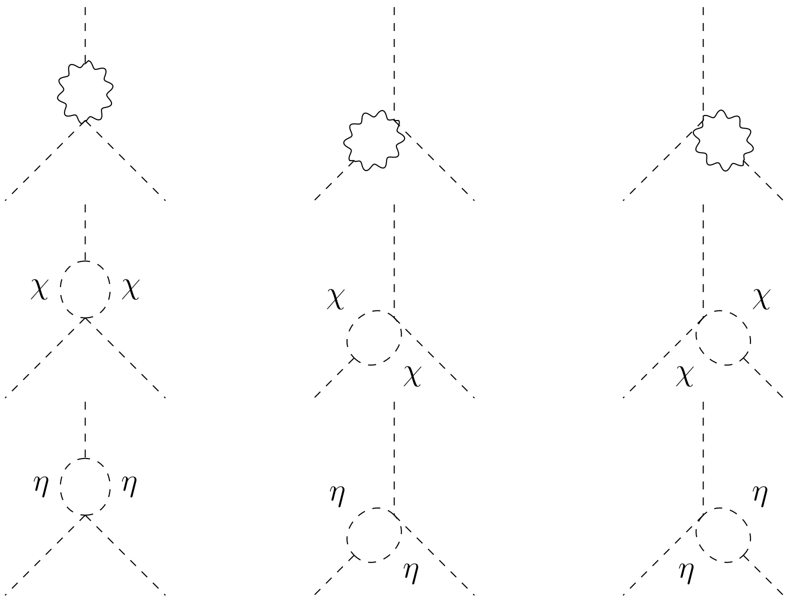


Figure 14: Two-point loop contributions to (6.7.10)

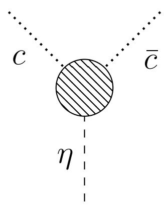
The two-point loops shown above have the combined divergent amplitude

$$\left(\Gamma^{\eta\eta\eta}\right)_2 = i\mu^\epsilon \frac{3e^4\left(\xi^2+3\right)v}{4\pi^2\epsilon} + i\mu^\epsilon \frac{15\lambda^2 v}{2\pi^2\epsilon}. \tag{6.7.12}$$

Once again adding the two contributions (6.7.11) and (6.7.12)

$$\begin{aligned}\Gamma^{\eta\eta\eta} &= -i\mu^\epsilon \frac{3e^4\xi^2 v}{4\pi^2\epsilon} + i\mu^\epsilon \frac{3e^4\left(\xi^2+3\right)v}{4\pi^2\epsilon} - i\mu^\epsilon \frac{3e^2\lambda\xi v}{4\pi^2\epsilon} + i\mu^\epsilon \frac{15\lambda^2 v}{2\pi^2\epsilon} \\ &= i\mu^\epsilon \frac{3v\left(3e^4 - e^2\lambda\xi + 10\lambda^2\right)}{4\pi^2\epsilon}. \end{aligned} \tag{6.7.13}$$

6.7.5 $\Gamma^{\eta c\bar{c}}$



(6.7.14)

This vertex is special in the sense that all contributions are finite. They are listed in figure 15.

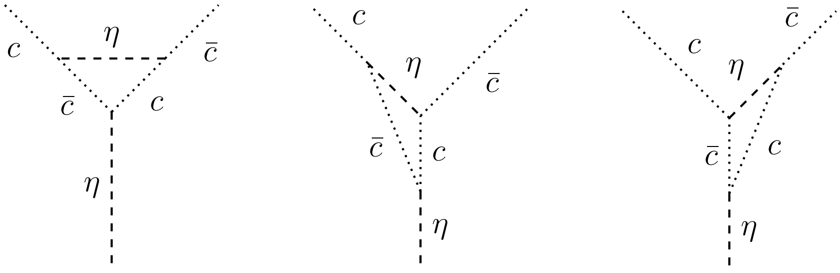
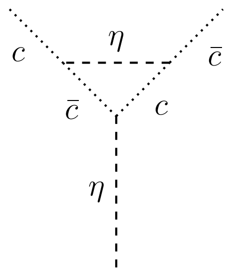


Figure 15: Finite contributions to (**??**)

As an example, we consider the first diagram



$$=\big(-i\xi e^2 v\big)^3 \int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-M_H^2}\frac{i}{(k+q)^2-\xi M_A^2}\frac{i}{(k+p)^2-\xi M_A^2}$$
$$=\frac{\big(\xi e^2 v\big)^3}{(4\pi)^2}C_0(q^2,p^2,M_H^2,\xi MA^2).$$

All the diagrams in figure 15 give the same result for the divergent part, and the sum is 0

$$\Gamma^{\eta c\bar{c}} = 0. \tag{6.7.15}$$

6.8 Rescaling and counter-terms

The full Lagrangian (6.4.1) adequately describes the interactions between all the necessary particles, and we able to move forward with the standard renormalization procedure. The renormalization constants for the fields, electric charge and fermion mass are defined in the same way as previously (section 4)

$$\begin{aligned}
A_0 =& Z_A^{1/2}A = \big(1+\frac{1}{2}\delta Z_A\big)A,\\
\eta_0 =& Z_\eta^{1/2}\eta = \big(1+\frac{1}{2}\delta Z_\eta\big)\eta,\\
\psi_0 =& Z_\psi^{1/2}\psi = \big(1+\frac{1}{2}\delta Z_\psi\big)\psi,\\
e_0 =& Z_e\mu^{\frac{\epsilon}{2}}e = \big(1+\delta Z_e\big)e,\\
m_{e,0} =& Z_m m,
\end{aligned} \tag{6.8.1}$$

In the previous constructions of counter-terms (section 4.1) multiplicative renormalization of the fermion mass was used. In the current theory, which consists of both fermion fields, and multiple scalar fields (η, χ, c), additive renormalization of the masses is useful [26].

$$\begin{aligned}
M_{A,0}^2 =& Z_A^{-1}\big(M_A^2+\delta M_A^2\big),\\
M_{H,0}^2 =& Z_\eta^{-1}\big(M_H^2+\delta M_H^2\big).
\end{aligned} \tag{6.8.2}$$

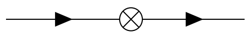
It is worth noting that this would be sufficient in order to render physical S-matrix elements finite [18, p. 658]. For a complete renormalization, the field renormalization of the Goldstone boson and ghost field, and the renormalization of the *gauge fixing parameter* ξ are needed

$$\begin{aligned}
\chi_0 =& Z_\chi^{1/2}\chi = \big(1+\frac{1}{2}\delta Z_\chi\big)\chi,\\
c_0 =& Z_c^{1/2}c = \big(1+\frac{1}{2}\delta Z_c\big)c,\\
\xi_0 =& Z_\xi\xi.
\end{aligned} \tag{6.8.3}$$

In addition, renormalization of the fundamental parameters of the theory (λ, μ, v) is needed. There exist methods that do not renormalize the vev [18, p. 580], however this approach yields non-finite vertex functions, and is not used here.

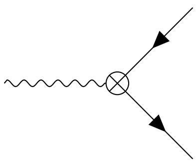
$$\begin{aligned} v_0 &= Z_v v, \\ \mu_0 &= Z_\mu \mu, \\ \lambda_0 &= Z_\lambda \mu^\epsilon \lambda. \end{aligned} \tag{6.8.4}$$

Counter-terms are constructed in the same was as before. Interactions between fermions and the photon are unchanged, therefore the fermion propagator counter-term



$$= i\big((Z_\psi - 1)\not{p} - (Z_\psi Z_m - 1)m\big), \tag{6.8.5}$$

and interaction counter-term



$$= -\, i\mu^{\frac{\epsilon}{2}}(Z_e Z_A^{1/2} Z_\psi - 1)e\gamma_\mu \tag{6.8.6}$$

are unchanged from the unbroken theory. The photon counter-term is modified, since it is now a massive particle

$$\begin{aligned} -\frac{1}{4}F_{\mu\nu,0}F_0^{\mu\nu} + \frac{1}{2}M_{A,0}^2 A_0^2 &= -\frac{1}{4}Z_A F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M_{A,0}^2 Z_A A^2 \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M_A^2 A^2 - \frac{1}{4}\delta Z_A F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\delta M_A^2 A^2. \end{aligned}$$

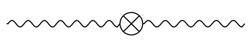
The kintetic terms $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ are equal to $-\frac{1}{2}A_\mu\big(-\partial^2 g^{\mu\nu} + \partial^\mu\partial^\nu\big)A_\nu$ by integration of parts [10] p. 331. The only terms relevant for the counter-term are

$$\begin{aligned} -\frac{1}{4}\delta Z_A F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\delta M_A^2 A^2 &= -\frac{1}{2}\delta Z_A A^\mu\big(-\partial^2 g_{\mu\nu} + \partial_\mu\partial_\nu\big)A^\nu + \frac{1}{2}\delta M_A^2 g_{\mu\nu}A^\mu A^\nu \\ &= -\frac{1}{2}A^\mu\Big(\big(-\partial^2 g_{\mu\nu} + \partial_\mu\partial_\nu\big)\delta Z_A - g_{\mu\nu}\delta M_A^2\Big)A^\nu \end{aligned}$$

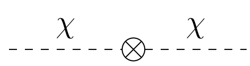
This gives the photon counter-term

$$= -i\Big(\big(q^2 g_{\mu\nu} - q_\mu q_\nu\big)\delta Z_A + g_{\mu\nu}\delta M_A^2\Big) \tag{6.8.7}$$

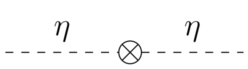
Counter-terms for the Higgs boson, Goldstone Boson and Ghost are constructed likewise. The derivation can be found in appendix E.1, and the results are listed in figure 16.



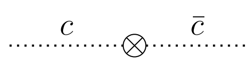
(a) $-i\Big(\big(q^2 g_{\mu\nu} - q_\mu q_\nu\big)\delta Z_A + g_{\mu\nu}\delta M_A^2\Big)$



(b) $i\Big(\delta Z_\chi p_\chi^2 - \big(Z_\xi Z_A^{-1} Z_\chi - 1\big)\xi M_A^2 - Z_\xi Z_A^{-1}\delta M_A^2 \xi\Big)$



(c) $i\big(\delta Z_\eta p_\eta^2 - \delta M_H^2\big)$



(d) $-i\Big(\frac{1}{2}\delta Z_c p_c^2 + \big(Z_\xi Z_c^{1/2} Z_A^{-1} - 1\big)\xi M_A^2 + Z_\xi Z_c^{1/2} Z_A^{-1}\xi\delta M_A^2\Big)$

Figure 16: Propagator counter-terms.

Now that the counter-terms for the propagators and fermion interaction are defined, we simply list the remaining counter-terms from the Lagrangian. These are the counter-terms for the interactions shown in figure 5. They come naturally from the Lagrangian (6.4.1) and are listed in figure 17.

$$\begin{aligned}
i\big(p_\eta - p_\chi\big)_\mu e_0\eta_0 A_0^\mu\chi_0 &= \; i\big(p_\eta - p_\chi\big)_\mu\big[e\eta A^\mu\chi + \big(Z_e Z_\eta^{1/2} Z_A^{1/2} Z_\chi^{1/2} - 1\big)e\eta A^\mu\chi\big],\\
\frac{1}{2}g_{\mu\nu}e_0^2 A_0^\mu A_0^\nu \chi_0^2 &= \; \frac{1}{2}g_{\mu\nu}e^2 A^\mu A^\nu\chi^2 + \frac{1}{2}\big(Z_e^2 Z_A Z_\chi - 1\big)g_{\mu\nu}e^2 A^\mu A^\nu\chi^2,\\
g_{\mu\nu}e_0^2 v_0 A_0^\mu A_0^\nu\eta_0 &= \; g_{\mu\nu}e^2 v A^\mu A^\nu\eta + \big(Z_e^2 Z_v Z_A Z_\eta^{1/2} - 1\big)g_{\mu\nu}e^2 v A^\mu A^\nu\eta,\\
-\lambda v\eta_0^3 &= -\lambda v\eta^3 - \big(Z_\lambda Z_v Z_\eta^{3/2} - 1\big)\lambda v\eta^3,\\
-\lambda v\eta_0\chi_0^2 &= -\lambda v\eta\chi^2 - \big(Z_\lambda Z_v Z_\eta^{1/2} Z_\chi - 1\big)\lambda v\eta\chi^2,\\
-\frac{\lambda}{2}\eta_0^2\chi_0^2 &= -\frac{\lambda}{2}\eta^2\chi^2 - \big(Z_\lambda Z_\eta Z_\chi - 1\big)\frac{\lambda}{2}\eta^2\chi^2,\\
-\frac{\lambda}{4}\eta_0^4 &= -\frac{\lambda}{4}\eta^4 - \big(Z_\lambda Z_\eta^2 - 1\big)\frac{\lambda}{4}\eta^4,\\
-\frac{\lambda}{4}\chi_0^4 &= -\frac{\lambda}{4}\chi^4 - \big(Z_\lambda Z_\chi^2 - 1\big)\frac{\lambda}{4}\chi^4.\\
-e_0^2 v_0\eta_0\bar{c}c_0 &= -e^2 v\eta\bar{c}c - \big(Z_e^2 Z_v Z_\eta^{1/2} Z_c^{1/2} - 1\big)e^2 v\eta\bar{c}c
\end{aligned}$$

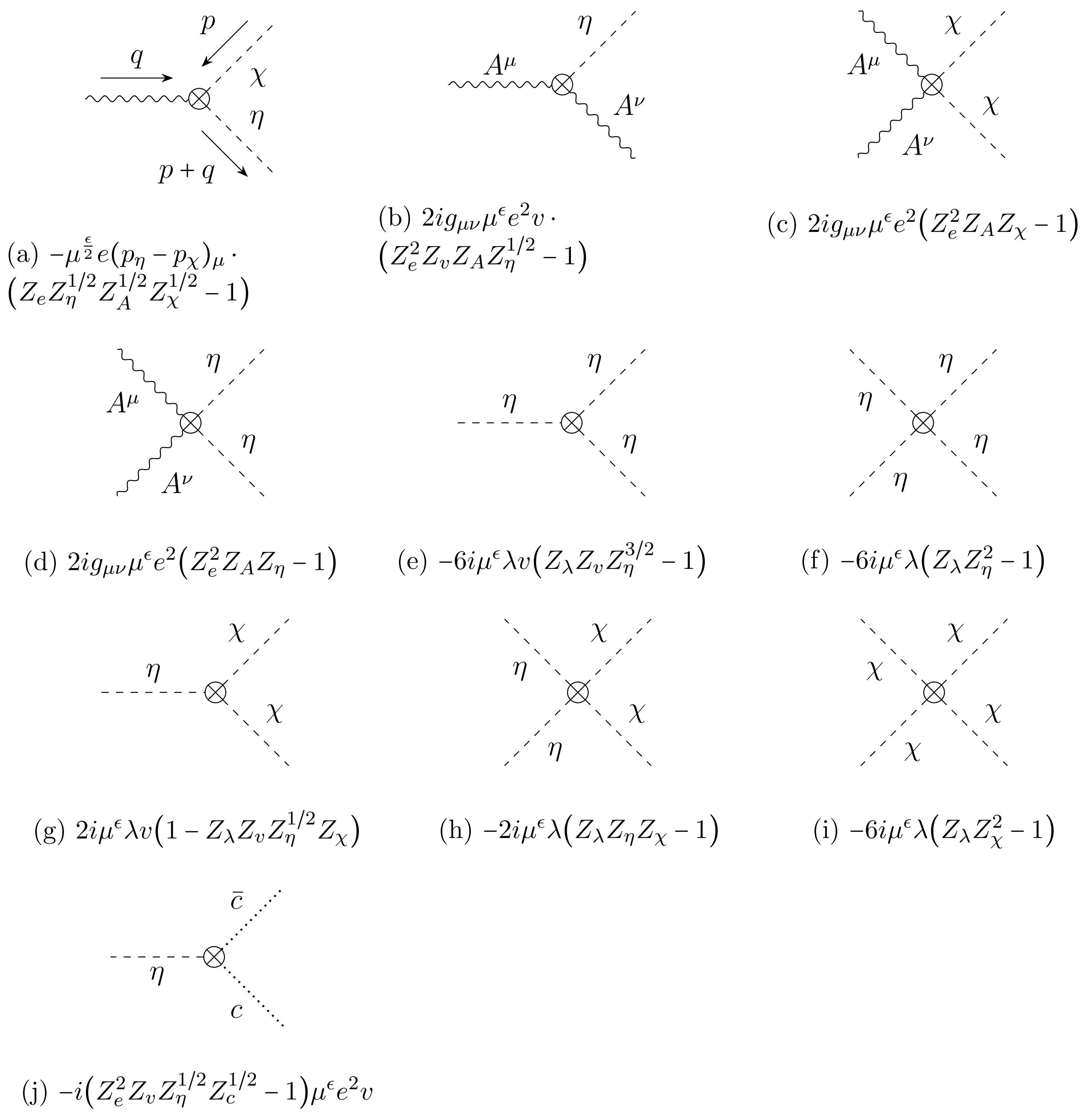


Figure 17: Feynman rules for the additional counter-terms.

6.9 Determination of renormalization constants

Z_ψ, Z_m

Although the fermion counter-term is the same as in the unbroken theory, the fermion and mass renormalization constants (5.4.3, 5.4.5) are changed due to changed fermion self-energy term (6.6.4)

$$i\frac{e^2}{(4\pi)^2}\left[\not{p} - 4m - (1-\xi)\big(\not{p} - m\big)\right]\frac{2}{\epsilon} + i((Z_\psi - 1)\not{p} - (Z_\psi Z_m - 1)m) = 0.$$

In order to isolate for the two coefficients, some reordering is convenient

$$i\frac{e^2}{(4\pi)^2}\left[\xi\not{p} - 3m - \xi m\right]\frac{2}{\epsilon} + i((Z_\psi - 1)\not{p} - (Z_\psi Z_m - 1)m) = 0.$$

This leads to the two independent equations

$$i\frac{e^2}{(4\pi)^2}\xi\not{p}\frac{2}{\epsilon} = -i\Big((Z_\psi - 1)\not{p}\Big),$$
$$-i\frac{e^2}{(4\pi)^2}(3+\xi)m\frac{2}{\epsilon} = i(Z_\psi Z_m - 1)m.$$

The first equation readily gives the first fermion renormalization coefficient

$$Z_\psi = 1 - \xi\frac{e^2}{(4\pi)^2}\frac{2}{\epsilon}, \tag{6.9.1}$$

and the second one can be rewritten as

$$Z_m = \frac{1-(3+\xi)\frac{e^2}{(4\pi)^2}\frac{2}{\epsilon}}{Z_\psi} = \frac{1-(3+\xi)\frac{e^2}{(4\pi)^2}\frac{2}{\epsilon}}{1-\xi\frac{e^2}{(4\pi)^2}\frac{2}{\epsilon}}.$$

Taylor expanding leads to the result

$$Z_m = 1 - \frac{3e^2}{8\pi^2}\frac{1}{\epsilon}, \tag{6.9.2}$$

where higher order terms have been omitted since they are not relevant at 1-loop order. It is interesting to note that while the fermion renormalization constant Z_ψ has a gauge dependence, the mass renormalization constant Z_m has not, at least not at the 1-loop level.

$Z_A, \delta M_A^2$

The only change in wave function constants from the unbroken theory, apart from the new ones, is Z_A, which now obeys a modified version of (5.4.6), due to all the added contributions from new diagrams (section 6.6.2). The divergent contributions to the photon propagator can be summarized by equation (6.6.7)

$$\Sigma^A_{\mu\nu} = -i\frac{e^2}{(4\pi)^2}\frac{1}{\epsilon}\left\{\frac{10}{3}(g_{\mu\nu}q^2 - q_\mu q_\nu) + g_{\mu\nu}\big(2M_H^2 + 6M_A^2 + 4\xi M_A^2\big)\right\}.$$

Adding the photon counter-term 16(a) and requiring the result to be zero

$$\begin{aligned}&-i\Big((g_{\mu\nu}q^2 - q_\mu q_\nu)\delta Z_A + g_{\mu\nu}\delta M_A^2\Big)\\ &-i\frac{e^2}{(4\pi)^2}\frac{1}{\epsilon}\left\{\frac{10}{3}(g_{\mu\nu}q^2 - q_\mu q_\nu) + g_{\mu\nu}\big(2M_H^2 + 6M_A^2 + 4\xi M_A^2\big)\right\} = 0.\end{aligned}$$

The same observation as for the fermion self-energy can be made, q and the different mass terms are independent quantities, and must evolve independently

$$\delta Z_A = -\frac{10}{3}\frac{e^2}{(4\pi)^2}\frac{1}{\epsilon},$$
$$\delta M_A^2 = -\frac{e^2}{(4\pi)^2}\big(2M_H^2 + 4\xi M_A^2 + 6M_A^2\big)\frac{1}{\epsilon},$$

from which follows

$$Z_A = 1 - \frac{10}{3}\frac{e^2}{(4\pi)^2}\frac{1}{\epsilon}. \tag{6.9.3}$$

Z_e

The divergent amplitude of the vertex function (6.7.3) gives a relation between the other constants

$$-i\xi\frac{e^3}{(4\pi)^2}\gamma_\mu\frac{2}{\epsilon} - i(Z_eZ_A^{1/2}Z_\psi - 1)e\gamma_\mu = 0,$$

$$Z_eZ_A^{1/2}Z_\psi = 1 - \xi\frac{e^2}{(4\pi)^2}\frac{2}{\epsilon} \tag{6.9.4}$$

Using equation (6.9.4) is the straightforward way to calculate the charge renormalization constant Z_e

$$Z_e = \frac{1 - \xi\frac{e^2}{(4\pi)^2}\frac{2}{\epsilon}}{Z_\psi Z_A^{1/2}} = Z_A^{-1/2},$$

and an expression for the 1-loop level can be found with Taylor expansion

$$Z_e = 1 + \frac{5e^2}{48\pi^2}\frac{1}{\epsilon}. \tag{6.9.5}$$

We note that the relation $Z_e = Z_A^{-1/2}$ derived from the Ward identity in regular QED still holds. Earlier in this section, and in section 6.6, there are multiple examples[2] of gauge-dependent amplitudes and constants that return to their unbroken values if the gauge parameter ξ is equal to 1 (Feynman gauge). The charge renormalization constant Z_e does not share this property[3], which will be reflected in the β-function.

Continuing the same way, we solve for the rest of the renormalization constants. Some of these are straightforward, mainly Z_η, Z_χ and δM_H^2, while others require us to solve more complicated equations. The calculations are split into small sections for readability.

$Z_\eta, \delta M_H^2$

These follow easily from a similar calculation to that of the photon. The counter-term is given in figure 16(c), with the divergent amplitude (6.6.8).

$$i\left(\delta Z_\eta p^2 - \delta M_H^2\right) + i\frac{(e^2(\xi-3)p^2 + e^2\left(\xi^2M_A^2 + 12M_A^2\right) + 2\lambda\left(\xi M_A^2 + 8M_H^2\right))}{8\pi^2\epsilon} = 0$$

Solving for the two variables yields the two constants

$$Z_\eta = 1 - \frac{e^2}{8\pi^2\epsilon}(\xi - 3), \tag{6.9.6}$$

$$\delta M_H^2 = \frac{(e^2\left(\xi^2M_A^2 + 12M_A^2\right) + 2\lambda\left(\xi M_A^2 + 8M_H^2\right))}{8\pi^2\epsilon}. \tag{6.9.7}$$

[2]See for example equations (6.6.4) or (6.9.1).

[3]See equation (5.4.9) for the unbroken case.

Z_χ

The Goldstone boson counter-term, found in figure 16(b), does not have the convenient $Z_i - Z_{i,m}$ structure seen in so far in propagator counter-terms. There is however, only one term dependent on p. The divergent amplitude (6.6.9) has a similar structure to that of the Higgs boson, with two p-dependent terms.

$$\begin{aligned} &i\Big(\delta Z_\chi p_\chi^2 - \big(Z_\xi Z_A^{-1} Z_\chi - 1\big)\xi M_A^2 - Z_\xi Z_A^{-1}\delta M_A^2\xi\Big) \\ &+ \frac{i\left(e^2(\xi-3)p^2 + e^2\left(\xi(\xi M_A^2 + M_H^2) + 6M_A^2\right) + 6\lambda\xi M_A^2 + 4\lambda M_H^2\right)}{8\pi^2\epsilon} = 0 \end{aligned} \tag{6.9.8}$$

As before the terms proportional to p^2 must evolve separately from the rest

$$\begin{aligned} i\delta Z_\chi p^2 &= -i\frac{e^2}{8\pi^2\epsilon}(\xi-3)p^2, \\ Z_\chi &= 1 - \frac{e^2}{8\pi^2\epsilon}(\xi-3). \end{aligned} \tag{6.9.9}$$

The rest of the constants are more complicated in form, as they also exhibit a λ-dependence in the same way as the higher order e-dependence, for which we have previously used a Taylor series to simplify to the 1-loop order. For these last constants there is therefore used a Taylor series in both e and λ.

Z_ξ

The rest of the terms from (6.9.8) must obey

$$\xi M_A^2\left(\frac{Z_\xi Z_\chi}{Z_A} - 1\right) + \frac{\delta M_A^2 \xi Z_\xi}{Z_A} = \frac{i\left(e^2\left(\xi(\xi M_A^2 + M_H^2) + 6M_A^2\right) + 6\lambda\xi M_A^2 + 4\lambda M_H^2\right)}{8\pi^2\epsilon}.$$

This can then be solved for another coefficient, allowing another to be determined.

$$\begin{aligned} Z_\xi &= \frac{Z_A\left(e^2\xi^2 M_A^2 + 6e^2 M_A^2 - e^2 M_H^2\xi + 6\lambda\xi M_A^2 + 8\pi^2\xi M_A^2\epsilon + 4\lambda M_H^2\right)}{8\pi^2\epsilon\left(\delta M_A^2\xi + \xi M_A^2 Z_\chi\right)} \\ &= \frac{\left(24\pi^2\epsilon - 5e^2\right)\left(e^2\left(M_A^2\left(\xi^2+6\right) - M_H^2\xi\right) + 2M_A^2\xi\left(3\lambda + 4\pi^2\epsilon\right) + 4\lambda M_H^2\right)}{24\pi^2\xi\epsilon\left(8\pi^2 M_A^2\epsilon - e^2\left(3M_A^2\xi + M_H^2\right)\right)}. \end{aligned}$$

Taking the Taylor series in both e and λ, as discussed above, yields

$$\begin{aligned} Z_\xi =&1 + \frac{\lambda\left(6M_A^2\xi + 4M_H^2\right)}{8\pi^2 M_A^2\xi\epsilon} \\ &+ e^2\left(\frac{12\xi^2 - 5\xi + 18}{24\pi^2\epsilon\xi} + \frac{\left(27\xi^2 M_A^4 - 15\xi M_A^4 - 10M_H^2 M_A^2 + 27M_H^2\xi M_A^2 + 6M_H^4\right)\lambda}{96M_A^4\pi^4\epsilon^2\xi}\right). \end{aligned}$$

Terms which contain both λ and e^2 are associated with higher order diagrams; in order to construct a diagram with this factor, one would need to go to at least 2-loop order. These terms can therefore be ignored.

$$Z_\xi =1 + \frac{6\lambda}{8\pi^2\epsilon} + \frac{4\lambda M_H^2}{8\pi^2 M_A^2\xi\epsilon} + e^2\left(\frac{12\xi^2 - 5\xi + 18}{24\pi^2\epsilon\xi}\right). \tag{6.9.10}$$

In order to determine the rest of the parameter renormalization constants (Z_v, Z_λ), and the ghost constant Z_c, we use several three-point vertices, though the choice of which ones is quite arbitrary, and one could choose others. The charge renormalization constant is universal due to the Slavnov-Taylor identity [8]. This applies to the other parameter renormalization constants as well, no matter how a constant is calculated, it will be the same.

Z_v

Now that universality of renormalization constants has been established, and we are free to choose a convenient counter-term. The $A^\mu A^\nu \eta$-vertex counter-term in figure 17(b) with the divergent amplitude (6.7.9) is a straightforward choice.

$$-i\mu^\epsilon \frac{3e^4 v g_{\mu\nu}}{4\pi^2\epsilon} + 2i\mu^\epsilon e^2 v g_{\mu\nu} \left(Z_A Z_e^2 \sqrt{Z_\eta} Z_v - 1 \right) = 0,$$

which gives the relation

$$\begin{aligned} Z_v =& \frac{3e^2 + 8\pi^2\epsilon}{8\pi^2 Z_A Z_e^2 \sqrt{Z_\eta}\epsilon} \\ =& \frac{13824\pi^5\epsilon^2 \left(3e^2 + 8\pi^2\epsilon\right)}{\left(24\pi^2\epsilon - 5e^2\right)\left(5e^2 + 48\pi^2\epsilon\right)^2 \sqrt{4\pi^2 - \frac{e^2(\xi-3)}{2\epsilon}}}. \end{aligned}$$

Once again using the Taylor series

$$Z_v =1 + \frac{e^2}{16\pi^2\epsilon}\left(\xi + 3\right). \tag{6.9.11}$$

Z_λ

For the last parameter constant Z_λ, we chose the triple Higgs self-interaction vertex, found in figure 17(e). The divergent amplitude is given by (6.7.13).

$$i\mu^\epsilon \frac{3v\left(3e^4 - e^2\lambda\xi + 10\lambda^2\right)}{4\pi^2\epsilon} - 6\mu^\epsilon i\lambda v \left(Z_\eta^{3/2} Z_\lambda Z_v - 1 \right) = 0,$$

which gives the relation

$$\begin{aligned} Z_\lambda =& \frac{3e^4 - e^2\lambda\xi + 2\lambda\left(5\lambda + 4\pi^2\epsilon\right)}{8\pi^2\lambda Z_\eta^{3/2} Z_v\epsilon} \\ =& \frac{\left(24\pi^2\epsilon - 5e^2\right)\left(5e^2 + 48\pi^2\epsilon\right)^2\left(3e^4 - e^2\lambda\xi + 2\lambda\left(5\lambda + 4\pi^2\epsilon\right)\right)}{6912\pi^4\lambda\epsilon^2\left(3e^2 + 8\pi^2\epsilon\right)\left(8\pi^2\epsilon - e^2(\xi - 3)\right)}, \end{aligned}$$

Taylor expanding as usual gives

$$Z_\lambda =1 + \frac{5\lambda}{4\pi^2\epsilon} - \frac{e^2}{4\pi^2\epsilon}. \tag{6.9.12}$$

Z_c

The only missing constant is Z_c, which can be found from the ghost propagator counter-term in figure 16(d) together with the amplitude (6.6.11). This leads to a rather complicated equation. A simple workaround is the $\eta c\bar{c}$-vertex. From section 6.7.5 we know that all contributions are finite. Therefore the counter-term in figure 17(j) gives the relation

$$-i\left(Z_e^2 Z_v Z_\eta^{1/2} Z_c^{1/2} - 1\right)e^2 v = 0$$

$$Z_c = \frac{1}{Z_e^4 Z_v^2 Z_\eta} = \frac{\left(5e^2 - 24\pi^2\epsilon\right)^2}{9\left(3e^2 + 8\pi^2\epsilon\right)^2}.$$

A Taylor expansion in e yields

$$Z_c = 1 - \frac{7e^2}{6\left(\pi^2\epsilon\right)}. \tag{6.9.13}$$

All of the constants have now been determined and are summarized in table 4.

	Constant	1-loop expression
Fermions	Z_ψ	$1 - \xi\frac{e^2}{8\pi^2\epsilon}$
	Z_m	$1 - \xi\frac{3e^2}{8\pi^2\epsilon}$
Gauge boson	Z_A	$1 - \frac{10}{3}\frac{e^2}{16\pi^2\epsilon}$
	δM_A^2	$-\frac{e^2}{8\pi^2\epsilon}\left(M_H^2 + 2\xi M_A^2 + 3M_A^2\right)$
Higgs Boson	Z_η	$1 - \frac{e^2}{8\pi^2\epsilon}(\xi - 3)$
	δM_H^2	$\frac{\left(e^2\left(\xi^2 M_A^2 + 12M_A^2\right) + 2\lambda\left(\xi M_A^2 + 8M_H^2\right)\right)}{8\pi^2\epsilon}$
Goldstone boson	Z_χ	$1 - \frac{e^2}{8\pi^2\epsilon}(\xi - 3)$
Ghost	Z_c	$1 - \frac{7e^2}{6(\pi^2\epsilon)}$
Parameters	Z_e	$1 + \frac{5e^2}{48\pi^2}\frac{1}{\epsilon}$
	Z_v	$1 + \frac{e^2}{16\pi^2\epsilon}(\xi + 3)$
	Z_λ	$1 + \frac{5\lambda}{4\pi^2\epsilon} - \frac{e^2}{4\pi^2\epsilon}$
	Z_ξ	$1 + \frac{6\lambda}{8\pi^2\epsilon} + \frac{4\lambda M_H^2}{8\pi^2 M_A^2\xi\epsilon} + e^2\left(\frac{12\xi^2 - 5\xi + 18}{24\pi^2\epsilon\xi}\right)$

Table 4: Renormalization constants for broken QED at the 1-loop order.

6.10 β-functions

As noted in the last section, $Z_e = Z_A^{-1/2}$ still holds, and we able to still use (5.1.5) to calculate β_e

$$\begin{aligned}\beta_e(e) &= -\frac{\epsilon}{2}e + \mu\frac{e}{2Z_A}\frac{\partial Z_A}{\partial\mu} \\ &= -\frac{\epsilon}{2}e - \frac{5}{24}\cdot\frac{e^2}{Z_A}\mu\frac{\partial e}{\partial\mu}\end{aligned}$$

Solving this equation yields

$$\beta_e(e) = -\frac{e\epsilon}{2\left(\frac{5e^2}{24\pi^2\epsilon Z_A}+1\right)},$$

and we get the result by Taylor expanding

$$\beta_e(e) = -\frac{e}{2}\epsilon + \frac{5e^3}{48\pi^2} \overset{\epsilon\to 0}{=} \frac{5e^3}{48\pi^2}. \tag{6.10.1}$$

The other coupling constant λ has a β-function as well. The derivation is similar to $\beta(e)$, which now is denoted β_e.

$$\begin{aligned}
\beta_\lambda =& \mu\frac{d\lambda}{d\mu}, \quad \lambda = \lambda_0\mu^{-\epsilon}Z_\lambda^{-1} \\
=& \lambda_0\mu\Big(-\epsilon\mu^{-\epsilon-1}Z_\lambda^{-1} + \mu^{-\epsilon}\frac{\partial Z_\lambda^{-1}}{\partial\mu}\Big) \\
=& \lambda_0\mu\Big(-\epsilon\mu^{-\epsilon-1}Z_\lambda^{-1} + \mu^{-\epsilon}\frac{\partial Z_\lambda^{-1}}{\partial Z_\lambda}\frac{\partial Z_\lambda}{\partial\mu}\Big) \\
=& -\epsilon\lambda - \mu\frac{\lambda}{Z_\lambda}\frac{dZ_\lambda}{d\mu} \\
=& -\epsilon\lambda - \mu\frac{\lambda}{Z_\lambda}\Big(\frac{dZ_\lambda}{de}\frac{de}{d\mu} + \frac{dZ_\lambda}{d\lambda}\frac{d\lambda}{d\mu}\Big) \\
=& -\epsilon\lambda - \frac{\lambda}{Z_\lambda}\Big(-\frac{2e}{4\pi^2\epsilon}\beta_e + \frac{5}{4\pi^2\epsilon}\beta_\lambda\Big)
\end{aligned}$$

solving this for β_λ yields

$$\beta_\lambda = \frac{-\epsilon\lambda + \frac{\lambda}{Z_\lambda}\left(\frac{e}{4\pi^2\epsilon}\right)\beta_e}{1 + \frac{\lambda}{Z_\lambda}\frac{5}{4\pi^2\epsilon}},$$

which, with a Taylor expansion in both e and λ becomes

$$\beta_\lambda(\lambda, e) = -\epsilon\lambda + \frac{5\lambda^2}{4\pi^2} - \frac{e^2\lambda}{4\pi^2} \overset{\epsilon\to 0}{=} \frac{\lambda}{4\pi^2}\left(5\lambda - e^2\right). \tag{6.10.2}$$

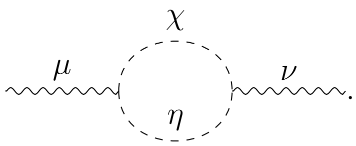
We have finally arrived at the β_λ function. In the R_ξ-gauge gauge independence for physical quantities is guaranteed if the corresponding β-function is gauge-independent [27]. The calculated β_λ function is gauge-independent as it has no dependency on the gauge-fixing parameter ξ. We can therefore conclude that λ is a gauge-independent quantity.

6.10.1 The Higgs contribution to the β_e-function

The β_e-function provides an interesting method of comparing spontaneously broken QED with the unbroken theory. The β_e-function is a function of Z_e only, which in turn depends on Z_ψ and Z_A. The change in Z_e, compared to the unbroken coefficient, comes solely from the extra contributions to Z_A. Looking back at (6.6.6), one can see that the only term modifying Z_A is

$$-i\frac{e^2}{(4\pi)^2}\frac{2}{3}\left(g_{\mu\nu}q^2 - q_\mu q_\nu\right)\frac{1}{\epsilon},$$

while the rest of the terms contribute to δM_A^2. This term was a part of the amplitude from the diagram



This can be used in order to calculate the contribution the Higgs boson has on the β_e-function.

$$i\big(g_{\mu\nu}q^2 - q_\mu q_\nu\big)\delta Z_A = -i\big(g_{\mu\nu}q^2 - q_\mu q_\nu\big)\frac{e^2}{(4\pi)^2}\left(\frac{2}{3}\right)\frac{1}{\epsilon},$$

with the result

$$\big(Z_A\big)_{\text{Higgs}} = 1 - \frac{2}{3}\frac{e^2}{(4\pi)^2}\frac{1}{\epsilon}.$$

Using this in (6.9.4) gives Z_e:

$$\big(Z_e\big)_{\text{Higgs}} = 1 + \frac{e^2}{48\pi^2}\frac{1}{\epsilon}, \tag{6.10.3}$$

which, using (6.10.1), yields the β-function contribution from the Higgs

$$\beta_e^{\text{Higgs}}(e) = \frac{e^3}{48\pi^2}. \tag{6.10.4}$$

Removing this contribution from the β-function also restores the QED β-function

$$\beta_e^{\text{QED}} + \beta_e^{\text{Higgs}} = \frac{5e^3}{48\pi^2},$$
$$\beta_e^{\text{QED}} = \frac{e^3}{12\pi^2},$$

as expected. The result of this is that adding a scalar particle to the theory of QED increases the value of the β-function.

6.10.2 β-function from Dynkin indices and particle content

Utilizing the renormalization group equations, it is possible to derive a form for the β-function that depends on different parameters of the groups and representations within the theory [28]. To the one-loop order

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}\kappa S_2(F) - \frac{1}{6}S_2(S) + \frac{2\kappa}{(4\pi)^2}Y_4(F)\right], \tag{6.10.5}$$

where C_2 is the quadratic Casimir of the gauge group G , $S_2(F), S_2(S)$ are the Dynkin indices for the fermion and scalar representations, the factor κ is equal to $\frac{1}{2}$ or 1 for two component fermions and four component fermions respectively and $Y_4(F)$ is an invariant defined from the Yukawa couplings, which can be ignored in the current theory.

For QED with a complex scalar, both the fermions and complex scalar transform under a fundamental representation of an Abelian $U(1)$ gauge group, this means that $C_2(G) = 0$ [29, p. 5]. The Dynkin indices for an irreducible representation has the general form

$$\mathrm{Tr}_R\left(T^A T^B\right) = S_2(R)\delta^{AB}, \tag{6.10.6}$$

which reduces to $S_2 = 1$ for both the fermions and the scalar, since there is only one generator for the $U(1)$ transformation.

$$\begin{aligned} \beta_e(e) &= -\frac{e^3}{(4\pi)^2}\left[-\frac{4}{3} - \frac{1}{3}\right] \\ &= \frac{5e^3}{48\pi^2}. \end{aligned} \tag{6.10.7}$$

From this comparison we can draw the conclusion that for that the β_e-function of a spontaneously broken theory is the same as in the unbroken theory. This is not a surprising result. At high energies, masses are negligible. The β_e-function, which is sensitive to UV-divergences, is therefore unaffected by whether the photon has a mass.

7 Conclusions and future work

In this thesis, we have discussed renormalization, the techniques within, and whether a theory can be renormalized. We have looked at different renormalization schemes for QED and investigated the β-function in the on-shell scheme and momentum subtraction scheme. The renormalization constants and β-function were calculated in the momentum subtraction scheme. After this we looked at the very useful minimal subtraction scheme, along with the P.V functions and applied the scheme to QED. We then looked at spontaneously broken QED. First we determined all the interactions between the real scalars and photon. We then used an R_ξ-gauge to eliminate a complicating term and in the process introduced a gauge-dependent mass for the Goldstone boson. To cancel this unphysical degree of freedom a Faddeev-Popov ghost was used. Interactions between all particles, including the unphysical Goldstone boson and ghost were studied. We tested the renormalizability of such a theory, and applied minimal subtraction to find all the renormalization constants and β-functions. It was shown that the β_e-function has the expected form based on particle content, and that it behaves as one would expect. We also found the β_λ function. Both of the derived β-functions had the expected property of being gauge-independent. This is important as is it an essential condition for guaranteeing that observables are gauge-independent.

Moving forward, further analysis on the β_λ-function would be a first priority. The β_e-function has told us much about the behavior of QED at high energies and investigating the behavior of the β_λ could potentially tell us more about the spontaneously broken QED theory. Finding the β-functions for other parameters could be interesting as well.

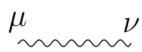
ξ is dimensionless and v has an unusual mass dimension that would be an intriguing complication to handle. This would also help to check if the procedure is consistent at every level. Continuing this analysis into more exotic theories incorporating kinetic mixing would be the next logical step after this. The renormalization of the kinetic mixing parameter would be of particular interest. Finding a β-function for such a parameter could be invaluable to help us understand the phenomenon.

Appendices

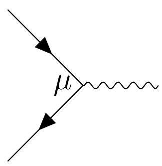
A QED Feynman rules

In the Feynman gauge $\xi = 1$, the Feynman rules are

$$\rightarrow i\frac{\not{p} + m}{p^2 - m^2 + i\epsilon}$$



$$\rightarrow -i\frac{g^{\mu\nu}}{q^2 + i\epsilon}$$



$$\rightarrow -ie\gamma_\mu$$

B Dimensional regularization

B.1 D-Dimensional Clifford algebra

The d-dimensional Clifford algebra satisfies the same basic anti-commutation as the 4-dimensional

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{B.1.1}$$

And, the metric is symmetric in the same way, and lowers and raises indices in the usual way

$$g^{\mu\nu} = g^{\nu\mu} \tag{B.1.2a}$$
$$g^{\mu\sigma}g_\sigma{}^\nu = g^{\mu\nu} \tag{B.1.2b}$$

However, the trace is updated, to reflect that the metric is now in a d-dimensional vector space

$$g_{\mu\nu}g^{\mu\nu} = d \tag{B.1.3}$$

And therefore the contractions of gamma matrices are also modified

$$\gamma_\mu\gamma^\mu = d \tag{B.1.4a}$$
$$\gamma_\nu\gamma^\mu\gamma^\nu = (2-d)\gamma^\mu \tag{B.1.4b}$$
$$\gamma_\lambda\gamma^\mu\gamma^\nu\gamma^\lambda = 4g^{\mu\nu} - (4-d)\gamma^\mu\gamma^\nu \tag{B.1.4c}$$
$$\gamma_\sigma\gamma^\mu\gamma^\nu\gamma^\delta\gamma^\sigma = -2\gamma^\delta\gamma^\nu\gamma^\mu + (4-d)\gamma^\mu\gamma^\nu\gamma^\delta \tag{B.1.4d}$$

These relations can be used to construct many new identities, a few that will be useful are

$$\gamma^\mu\gamma_\nu\gamma^\sigma\gamma_\mu\gamma^\nu = -4\gamma^\sigma - d^2\gamma^\sigma + 6d\gamma^\sigma, \quad \text{(B.1.5a)}$$
$$\gamma^\mu\gamma_\mu\gamma^\sigma\gamma_\nu\gamma^\nu = d^2\gamma^\sigma, \quad \text{(B.1.5b)}$$
$$\gamma^\mu\gamma_\nu\gamma^\sigma\gamma^\nu\gamma_\mu = 4\gamma^\sigma + d^2\gamma^\sigma - 4d\gamma^\sigma. \quad \text{(B.1.5c)}$$

Trace relations are unchanged, so long as they do not involve γ_5 [6, p. 78].

$$\mathrm{Tr}(\mathbb{1}) = 4, \quad \text{(B.1.6a)}$$
$$\mathrm{Tr}(\text{odd number of } \gamma\text{'s}) = 0, \quad \text{(B.1.6b)}$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}, \quad \text{(B.1.6c)}$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \quad \text{(B.1.6d)}$$

B.2 Feynman parametrization

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax + (1-x)B]^2}. \quad \text{(B.2.1a)}$$
$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^1 dy \frac{y}{[xyA + (1-y)xB + (1-y)C]^3}. \quad \text{(B.2.1b)}$$
$$\frac{1}{A_1A_2\cdots A_n} = \int_0^1 dx_1\cdots dx_n\Big(\Sigma x_i - 1\Big)\frac{(n-1)!}{[x_1A_1 + x_2A_2\cdots x_nA_n]^n}. \quad \text{(B.2.1c)}$$

B.3 D-Dimensional integrals

$$\int \frac{d^dk}{(2\pi)^d}\frac{\mu^\epsilon}{(k^2-\Delta)^n} = i\frac{(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}\mu^\epsilon, \quad \text{(B.3.1a)}$$
$$\int \frac{d^dk}{(2\pi)^d}\frac{k^2}{(k^2-\Delta)^n}\mu^\epsilon = -i\frac{d}{2}\frac{(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-1-\frac{d}{2})}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-1-\frac{d}{2}}\mu^\epsilon, \quad \text{(B.3.1b)}$$
$$\int \frac{d^dk}{(2\pi)^d}\frac{k_\mu k_\nu}{(k^2-\Delta)^n}\mu^\epsilon = -\frac{i}{2}g_{\mu\nu}\frac{(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-1-\frac{d}{2})}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-1-\frac{d}{2}}\mu^\epsilon. \quad \text{(B.3.1c)}$$

There are amplitudes in which we encounter terms of both $l_\mu l_\nu$ and l^2, in which case there is a convenient way of transforming one into the other

$$\int \frac{d^dk}{(2\pi)^d}k_\mu k_\nu f(k^2) = \frac{1}{d}g_{\mu\nu}\int \frac{d^d}{(2\pi)^d}k^2 f(k^2) \quad \text{(B.3.2)}$$

B.4 Gamma functions

The above Γ-function is defined as [6, p. 82]

$$\Gamma(z) = \int_0^\infty dt t^{z-1}e^{-t}, \quad \text{(B.4.1)}$$

and it satisfies

$$\Gamma(x+1) = x\Gamma(x). \tag{B.4.2}$$

For positive values of z this function is finite, some useful values are

$$\Gamma(1) = \Gamma(2) = 1, \tag{B.4.3}$$
$$\Gamma(3) = 2. \tag{B.4.4}$$

At negative values or zero, the function diverges, and can be expanded around the pole

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon), \tag{B.4.5}$$
$$\Gamma(\frac{\epsilon}{2}) = \frac{2}{\epsilon} - \gamma_E + O(\epsilon), \tag{B.4.6}$$
$$\Gamma\left(-n + \frac{\epsilon}{2}\right) = \frac{(-1)^n}{n!}\left[\frac{2}{\epsilon} + \psi(n+1) + O(\epsilon)\right], \tag{B.4.7}$$

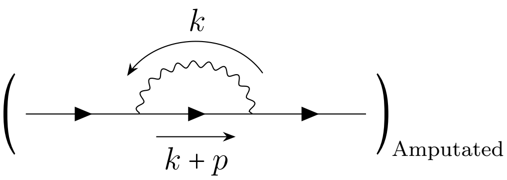
for positive ϵ. ψ satisfies

$$\psi(1) = -\gamma_E, \tag{B.4.8}$$
$$\psi(z+1) = \psi(z) + \frac{1}{z}, \tag{B.4.9}$$

where $\gamma_E = 0.5772...$ is Euler's constant.

B.5 Regularization of QED

B.5.1 Fermion self-energy



$$\equiv \Sigma(p). \tag{B.5.1}$$

The scalar Σ function has the form

$$\Sigma(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\mu \frac{i(\not{k} + \not{p} + m)}{(k+p)^2 - m^2} \gamma_\nu \frac{-ig^{\mu\nu}}{k^2 - \lambda^2}.$$

Using the Feynman parametrization (B.2.1a)

$$\begin{aligned}\Sigma(p) &= -e^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{\gamma_\mu(\not{k} + \not{p} + m)\gamma^\mu}{[((k+p)^2 - m^2)\,x + (1-x)(k^2 - \lambda^2)]^2} \\ &= -e^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{\gamma_\mu(\not{k} + \not{p} + m)\gamma^\mu}{[(k^2 + 2pk + p^2 - m^2)x + k^2 - \lambda^2 - x(k^2 - \lambda^2)]^2}.\end{aligned}$$

The terms k^2x cancel, this makes completing the square in the denominator straight forward

$$\Sigma(p) = -e^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{\gamma_\mu(\not{k} + \not{p} + m)\gamma^\mu}{[(k+px)^2 + p^2x - m^2x - \lambda^2 + \lambda^2x - \frac{(2px)^2}{4}]^2},$$

and making the substitution $l = k + xp$

$$\Sigma(p) = -e^2 \int \frac{d^4l}{(2\pi)^4} \int_0^1 dx \frac{\gamma_\mu(\not{l} - x\not{p} + \not{p} + m)\gamma^\mu}{[l^2 + p^2x - m^2x - \lambda^2 + \lambda^2x - p^2x^2]^2}.$$

The term linear in l vanishes, and the following is obtained for the divergent integral

$$\Sigma(p) = -e^2 \int \frac{d^4l}{(2\pi)^4} \int_0^1 dx \frac{\gamma_\mu\big((1-x)\not{p} + m\big)\gamma^\mu}{(l^2 - \Delta)^2},$$

where $\Delta = m^2x + (x-1)(p^2x - \lambda^2)$. Generalizing to $d = 4 - \epsilon$ dimensions, and using the contraction identities (B.1.4)

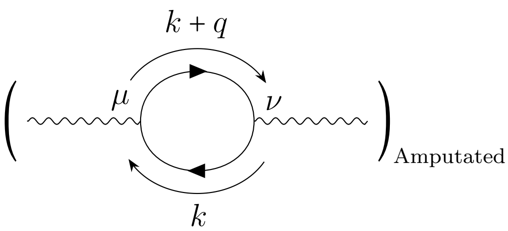
$$\Sigma(p) = -\mu^\epsilon e^2 \int_0^1 dx \int \frac{d^dl}{(2\pi)^d} \frac{(2-d)(1-x)\not{p} + dm}{(l^2 - \Delta)^2}$$

Using the integral (B.3.1a)

$$\begin{aligned}\Sigma(p) &= -e^2 \int_0^1 dx \big((2-d)(1-x)\not{p} + dm\big)\left(\frac{i}{(4\pi)^{d/2}}\Gamma(\frac{\epsilon}{2})\left(\frac{\mu^2}{\Delta}\right)^{2-d/2}\right)\\ &= -i\frac{e^2}{(4\pi)^2}\int_0^1 \big((\epsilon-2)(1-x)\not{p} + (4-\epsilon)m\big)\Gamma(\frac{\epsilon}{2})\left(\frac{\mu^2}{4\pi\Delta}\right)^{\frac{\epsilon}{2}} dx, \qquad \text{(B.5.2)}\end{aligned}$$

where $d = 4 - \epsilon$ has been used in the last step.

B.5.2 Vacuum polarization



$$\equiv \Pi_{\mu\nu}(q). \qquad \text{(B.5.3)}$$

The second rank tensor $\Pi_{\mu\nu}$ has the form

$$\Pi_{\mu\nu} = -(ie)^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma_\mu \frac{i(\not{k} + m)}{k^2 - m^2}\gamma_\nu \frac{i(\not{k} + \not{q} + m)}{(k+q)^2 - m^2}\right]. \qquad \text{(B.5.4)}$$

Taking the trace relations (B.1.6) into account, the terms with an odd number of gamma matrices can immediately be removed, leaving

$$\begin{aligned}\Pi_{\mu\nu} &= -e^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\frac{\gamma_\mu\gamma_\sigma\gamma_\nu\gamma_\delta k^\sigma k^\delta + \gamma_\mu\gamma_\sigma\gamma_\nu\gamma_\delta k^\sigma q^\delta + \gamma_\mu\gamma_\nu m^2}{(k^2 - m^2)[(k+q)^2 - m^2]}\right]\\ &= -e^2 \int \frac{d^4k}{(2\pi)^4} 4\frac{(g_{\mu\sigma}g_{\nu\delta} - g_{\mu\nu}g_{\sigma\delta} + g_{\mu\delta}g_{\sigma\nu})(k^\sigma k^\delta) + (g_{\mu\sigma}g_{\nu\delta} - g_{\mu\nu}g_{\sigma\delta} + g_{\mu\delta}g_{\sigma\nu})(k^\sigma q^\delta) + g_{\mu\nu}m^2}{(k^2 - m^2)[(k+q)^2 - m^2]}\\ &= -e^2 \int \frac{d^4k}{(2\pi)^4} 4\frac{k_\mu k_\nu - g_{\mu\nu}k^2 + k_\mu k_\nu + k_\mu q_\nu - g_{\mu\nu}k\cdot q + k_\nu q_\mu + g_{\mu\nu}m^2}{(k^2 - m^2)[(k+q)^2 - m^2]}\\ &= -e^2 \int \frac{d^4k}{(2\pi)^4} 4\frac{k_\mu(k_\nu + q_\nu) + k_\nu(k_\mu + q_\mu) - g_{\mu\nu}(k(k+q) - m^2)}{(k^2 - m^2)[(k+q)^2 - m^2]}.\end{aligned}$$

Using Feynman parametrization (B.2.1a)

$$\Pi_{\mu\nu} = -4e^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu(k_\nu + q_\nu) + k_\nu(k_\mu + q_\mu) - g_{\mu\nu}(k(k+q) - m^2)}{[((k+q)^2 - m^2)x + (1-x)(k^2 - m^2)]^2}$$

Completing the square, and substituting $l = k + qx$ yields

$$\Pi_{\mu\nu} = -4e^2 \int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \frac{2l_\mu l_\nu - 2x(1-x)q_\mu q_\nu - g_{\mu\nu}l^2 + g_{\mu\nu}(m^2 + q^2x(1-x))}{(l^2 - \Delta)^2}, \quad \text{(B.5.5)}$$

where $\Delta = m^2 - q^2x(1-x)$ and terms linear in l have been removed. Generalizing to $d = 4 - \epsilon$ dimensions, and using the transformation property (B.3.2)

$$\begin{aligned}
\Pi_{\mu\nu} =& -4e^2\mu^\epsilon \int_0^1 dx \Bigg[\int \frac{d^dl}{(2\pi)^d} \frac{-2x(1-x)q_\mu q_\nu + g_{\mu\nu}(m^2 + q^2x(1-x))}{(l^2 - \Delta)^2} \\
&+ \frac{2}{d} g_{\mu\nu} \int \frac{d^dl}{(2\pi)^d} \frac{l^2}{(l^2 - \Delta)^2} - g_{\mu\nu} \int \frac{d^dl}{(2\pi)^d} \frac{l^2}{(l^2 - \Delta)^2} \Bigg] \\
=& -4ie^2\mu^\epsilon \int_0^1 dx \Bigg[\frac{-2x(1-x)q_\mu q_\nu + g_{\mu\nu}(m^2 + q^2x(1-x))}{(4\pi)^{d/2}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta} \right)^{2-\frac{d}{2}} \\
&- \left(\frac{2}{d} - 1\right) g_{\mu\nu} \frac{d}{2} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(1 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta} \right)^{1-\frac{d}{2}} \Bigg],
\end{aligned}$$

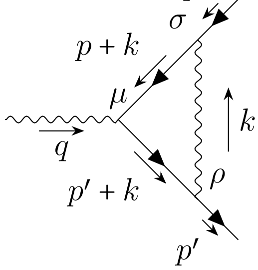
by changing the value of the argument of the second Γ-function with (B.4.2), this can be grouped together.

$$\begin{aligned}
\Pi_{\mu\nu} =& -4ie^2\mu^\epsilon \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^1 \frac{-2x(1-x)q_\mu q_\nu + g_{\mu\nu}(m^2 + q^2x(1-x)) - \Delta g_{\mu\nu}}{\Delta^{2-\frac{d}{2}}} dx \\
=& -4ie^2\mu^\epsilon \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^1 \frac{-2x(1-x)q_\mu q_\nu + 2q^2 g_{\mu\nu} x(1-x))}{\left(m^2 - q^2x(1-x)\right)^{2-\frac{d}{2}}} dx,
\end{aligned}$$

by using $\Delta = m^2 - q^2x(1-x)$. Lastly, it will be convenient to factor out the tensor structure.

$$\Pi_{\mu\nu} = -4i(g_{\mu\nu}q^2 - q_\mu q_\nu)e^2\mu^\epsilon \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^1 \frac{2x(1-x)}{\left(m^2 - q^2x(1-x)\right)^{2-\frac{d}{2}}} dx. \quad \text{(B.5.6)}$$

B.5.3 Vertex correction



At one-loop order, the vertex is replaced by

$$ie\gamma^{\mu} \to ie\Gamma^{\mu}(p,p'). \tag{B.5.7}$$

where $\Gamma_{\mu}(p,p')$ is calculated using the Feynman rules

$$\Gamma_{\mu}(p,p') = (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \gamma_{\rho} i \frac{-g^{\rho\sigma}}{k^2-\lambda^2} \frac{i(\not{p}+\not{k}+m)}{(k+p)^2-m^2} \gamma_{\mu} \frac{i(\not{k}+\not{q}+\not{p}+m)}{(k+p+q)^2-m^2} \gamma_{\sigma}$$

$$= -2e^3 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \cdot ydy \frac{\gamma^{\sigma}(\not{p}+\not{k}+m)\gamma_{\mu}(\not{k}+\not{q}+\not{p}+m)\gamma_{\sigma}}{[(1-x)y(k^2-\lambda^2)+xy((k+p+q)^2-m^2)+(1-y)((k+p)^2-m^2)]^3}$$

$$= -2e^3 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \cdot ydy$$

$$\frac{\gamma^{\sigma}(\not{p}+\not{k}+m)\gamma_{\mu}(\not{k}+\not{q}+\not{p}+m)\gamma_{\sigma}}{[k^2+2k(pxy-py+p+qxy)-m^2xy+m^2y-m^2+p^2xy-p^2y+p^2+2pqxy+q^2xy+\lambda^2xy-\lambda^2y]^3}$$

Defining $l = k + pxy - py + p + qxy$, this becomes

$$= -2e^3 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \cdot ydy \frac{\gamma^{\sigma}(\not{p}+\not{l}-\not{p}xy+\not{p}y-\not{p}-\not{q}xy+m)\gamma_{\mu}(\not{l}-\not{p}xy+\not{p}y-\not{p}-\not{q}xy+\not{q}+\not{p}+m)\gamma_{\sigma}}{[l^2-\Delta]^3},$$

where $\Delta = y\left(p^2(x-1)((x-1)y+1)+2pq(x-1)xy+q^2x(xy-1)-\lambda^2(x-1)\right)-m^2(-xy+y-1)$. Following [16, p. 254], Γ_{μ} is split into two parts, one proportional to l^2 and one independent of l.

$$\begin{aligned}
\Gamma^1_{\mu} &= -2e^3\mu^{\frac{3}{2}\epsilon} \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \cdot ydy \frac{\gamma^{\sigma}\not{l}\gamma_{\mu}\not{l}\gamma_{\sigma}}{[l^2-\Delta]^3} \\
&= -2e^3\mu^{\frac{3}{2}\epsilon} \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \cdot ydy \frac{(-2\gamma_{\lambda}\gamma_{\mu}\gamma_{\delta}+(4-d)\gamma_{\delta}\gamma_{\mu}\gamma_{\lambda})l^{\lambda}l^{\delta}}{[l^2-\Delta]^3} \\
&= -2\frac{e^3}{(4\pi)^2}\mu^{\frac{\epsilon}{2}} \int_0^1 dx \cdot ydy(-2\gamma_{\lambda}\gamma_{\mu}\gamma_{\delta}+(4-d)\gamma_{\delta}\gamma_{\mu}\gamma_{\lambda})\frac{i}{4}g^{\lambda\delta}\Gamma(2-\frac{d}{2})\Big(\frac{\mu^2}{\Delta}\Big)^{2-\frac{d}{2}} \\
&= -\frac{i}{2}\frac{e^3}{(4\pi)^2}\mu^{\frac{\epsilon}{2}} \int_0^1 dx \cdot ydy(-2(2-d)+(4-d)(2-d))\gamma_{\mu}\Gamma(2-\frac{d}{2})\Big(\frac{\mu^2}{\Delta}\Big)^{2-\frac{d}{2}} \\
&= -\frac{i}{2}\frac{e^3}{(4\pi)^2}\mu^{\frac{\epsilon}{2}} \int_0^1 dx \cdot ydy(4-4\epsilon+\epsilon^2)\gamma_{\mu}\Gamma(2-\frac{d}{2})\Big(\frac{\mu^2}{\Delta}\Big)^{2-\frac{d}{2}},
\end{aligned} \tag{B.5.8}$$

using (B.1.4) and (B.3.1c). The terms not proportional to l take the form

$$\begin{aligned}
\Gamma^2_{\mu} &= -2e^3\mu^{\frac{3}{2}\epsilon} \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \cdot ydy \frac{\gamma^{\sigma}(m-\not{p}xy+\not{p}y-\not{q}xy)\gamma_{\mu}(m-\not{p}xy+\not{p}y-\not{q}xy+\not{q})\gamma_{\sigma}}{[l^2-\Delta]^3} \\
&= -2e^3\mu^{\frac{3}{2}\epsilon} \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \cdot ydy \frac{\gamma^{\sigma}(m-\not{p}xy+\not{p}y-\not{q}xy)\gamma_{\mu}(m-\not{p}xy+\not{p}y-\not{q}xy+\not{q})\gamma_{\sigma}}{[l^2-\Delta]^3} \\
&= i\frac{e^3}{(4\pi)^2}\mu^{\frac{\epsilon}{2}} \int_0^1 dx \cdot ydy\gamma^{\sigma}(m-\not{p}xy+\not{p}y-\not{q}xy)\gamma_{\mu}(m-\not{p}xy+\not{p}y-\not{q}xy+\not{q})\gamma_{\sigma}\frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-\frac{d}{2}}}\mu^{\epsilon}.
\end{aligned} \tag{B.5.9}$$

The total amplitude for the vertex correction is the sum of these two expressions

$$\Gamma_{\mu} = \Gamma^1_{\mu} + \Gamma^2_{\mu} \tag{B.5.10}$$

C Passarino-Veltmann functions

C.1 Definition

The Passarino-Veltman integrals are readily tabulated, the following definitions is from [6].

The one point function A_0

$$A_0(m^2) = \frac{\mu^\epsilon}{i\pi^2}\int\frac{d^dk}{k^2-m^2} \tag{C.1.1a}$$

$$= m^2\left(\frac{2}{\epsilon} - ln\pi - \gamma_E + 1 + ln\frac{\mu^2}{m^2}\right) + O(\epsilon). \tag{C.1.1b}$$

The two point functions B

$$B_0 = \frac{\mu^\epsilon}{i\pi^2}\int d^dk\frac{1}{(k^2-m_1^2)[(k+p)^2-m_2^2]} \tag{C.1.2a}$$

$$= \frac{2}{\epsilon} - ln\pi - \gamma_E + ln\frac{\mu^2}{-p^2} - \int_0^1 ln[x(1-x) - (1-x)\frac{m_1^2}{p^2} - x\frac{m_2^2}{p^2}] + O(\epsilon),$$

$$B_\mu = \frac{\mu^\epsilon}{i\pi^2}\int d^dk\frac{k_\mu}{(k^2-m_1^2)[(k+p)^2-m_2^2]}, \tag{C.1.2b}$$

$$B_{\mu\nu} = \frac{\mu^\epsilon}{i\pi^2}\int d^dk\frac{k_\mu k_\nu}{(k^2-m_1^2)[(k+p)^2-m_2^2]}. \tag{C.1.2c}$$

And lastly, the triple point functions C

$$C_0 = \frac{\mu^\epsilon}{i\pi^2}\int d^dk\frac{1}{(k^2-m_1^2)[(k+p)^2-m_2^2][(q+k)^2-m_3^2]}, \tag{C.1.3a}$$

$$C_\mu = \frac{\mu^\epsilon}{i\pi^2}\int d^dk\frac{k_\mu}{(k^2-m_1^2)[(k+p)^2-m_2^2][(q+k)^2-m_3^2]}, \tag{C.1.3b}$$

$$C_{\mu\nu} = \frac{\mu^\epsilon}{i\pi^2}\int d^dk\frac{k_\mu k_\nu}{(k^2-m_1^2)[(k+p)^2-m_2^2][(q+k)^2-m_3^2]}, \tag{C.1.3c}$$

$$C_{\mu\nu\sigma} = \frac{\mu^\epsilon}{i\pi^2}\int d^dk\frac{k_\mu k_\nu k_\sigma}{(k^2-m_1^2)[(k+p)^2-m_2^2][(q+k)^2-m_3^2]}. \tag{C.1.3d}$$

C.2 Decompositions into scalar integrals

The decomposition of the tensor integrals are given by [18]

$$B^{\mu} = p^{\mu} B_1, \tag{C.2.1a}$$

$$B_1 = \frac{1}{2p^2}\Bigg(A_0(m_1) - A_0(m_2) - (p^2 + m_1^2 - m_2^2) B_0(p^2; m_1, m_2) \Bigg), \tag{C.2.1b}$$

$$B^{\mu\nu} = g^{\mu\nu} B_{00} + p^{\mu} p^{\nu} B_{11}, \tag{C.2.1c}$$

$$B_{00} = \frac{1}{2(d-1)}\Bigg(A_0(m_2) + 2m_1^2 B_0 + (p^2 + m_1^2 - m_2^2) B_1 \Bigg), \tag{C.2.1d}$$

$$B_{11} = \frac{1}{2(d-1)p^2}\Bigg(((d-2) A_0(m_2) - 2m_1^2 B_0 - d(p^2 + m_1^2 - m_2^2) B_1 \Bigg), \tag{C.2.1e}$$

$$\begin{aligned} B^{\mu\nu} =& \frac{1}{2}\Bigg[\frac{1}{d-1}\Bigg(g^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{p^2} \Bigg)\Big(A_0(m_2) + 2m_1^2 B_0 + (p^2 + m_1^2 - m_2^2) B_1 \Big), \\ &+ \frac{p^{\mu}p^{\nu}}{p^2}\Big(A_0(m_2) - (p^2 + m_1^2 - m_2^2) B_1 \Big) \Bigg]. \end{aligned} \tag{C.2.1f}$$

And the triple-point functions can be deconstructed as follows [6]

$$C_{\mu} = p_{\mu} C_1 + q_{\mu} C_2, \tag{C.2.2a}$$

$$C_{\mu\nu} = g_{\mu\nu} C_{00} + p_{\mu} p_{\nu} C_{11} + q_{\mu} q_{\nu} C_{22} + (p_{\mu} q_{\nu} + q_{\mu} p_{\nu}) C_{12}, \tag{C.2.2b}$$

where only C_{00} is divergent.
For the three point functions, the relations become trickier [22] appendix D, or [31] appendix B

C.3 Divergent terms of PV-functions

The following table summarizes the divergent terms for the P.V functions, we list the most commonly used, and a few that will be of use in our calculations.

Tensor integral	Divergent term
$A_0(m^2)$	$\frac{2}{\epsilon} m^2$
$B_0(p^2, m_1^2, m_2^2)$	$\frac{2}{\epsilon}$
B_1	$-\frac{1}{\epsilon}$
$B_{\mu\nu}$	$g_{\mu\nu}\Big(-\frac{1}{6\epsilon}(p^2 - 3m_1^2 - 3m_2^2) \Big) + p_{\mu} p_{\nu} \frac{2}{3\epsilon}$
$B_{\mu\nu\sigma}$	$-\frac{1}{12\epsilon}\Big(2m_1^2 (p^{\sigma} g^{\mu\nu} + p^{\nu} g^{\mu\sigma} + p^{\mu} g^{\nu\sigma})$ $+4m_2^2 (p^{\sigma} g^{\mu\nu} + p^{\nu} g^{\mu\sigma} + p^{\mu} g^{\nu\sigma})$ $-p^2 (p^{\sigma} g^{\mu\nu} + p^{\nu} g^{\mu\sigma} + p^{\mu} g^{\nu\sigma}) + 6p^{\mu} p^{\nu} p^{\sigma} \Big)$
$C_{\mu\nu}$	$g_{\mu\nu} \frac{1}{2\epsilon}$
$C_{\mu\nu\sigma}$	$-\frac{1}{6\epsilon}\left(2p^{\sigma} g^{\mu\nu} + 2p^{\nu} g^{\mu\sigma} + 2p^{\mu} g^{\nu\sigma} + q^{\sigma} g^{\mu\nu} + q^{\nu} g^{\mu\sigma} + q^{\mu} g^{\nu\sigma} \right)$

Table 5: Divergent terms of the PV-functions

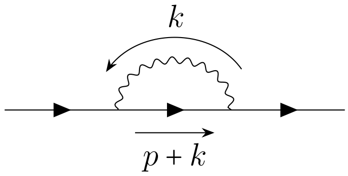
D Momentum assignment conventions for PV-functions

For ease of use of the PV-functions, some standardization of momentum assignments are in order. The three main categories of diagrams are self-energy, vacuum polarization and vertex corrections diagrams, which will use the following conventions, unless otherwise noted.

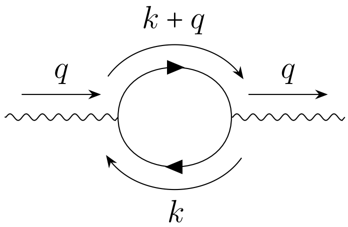
In addition, fermion and scalar momentum, photon momentum and momentum integrand are usually denoted p, q and k respectively.

D.1 Self-energy types

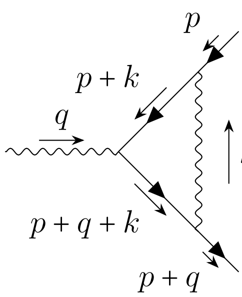
We use the following assignment. It obeys convention of aligning momentum with the direction of particle flow [13, p. 226] for fermions, and make the transformation into P.V functions simple.



D.2 Polarization types



D.3 Vertex correction types



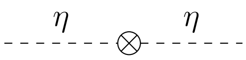
E Diagram calculations

E.1 Constructing the counter-terms of SSB QED

E.1.1 Higgs boson counter-term

$$
\begin{aligned}
\frac{1}{2}\left[(\partial_\mu \eta_0)^2 - 2\mu_0^2\eta_0^2\right] &= \frac{1}{2}\left[(\partial_\mu \eta_0)^2 - M_{H,0}^2\eta_0^2\right] \\
&= \frac{1}{2}\left[(\partial_\mu Z_\eta^{1/2}\eta)^2 - (M_H^2 + \delta M_H^2)\eta^2\right] \\
&= \frac{1}{2}\left[(\partial_\mu \eta)^2 - M_H^2\eta^2\right] + \frac{1}{2}\left[\delta Z_\eta(\partial_\mu \eta)^2 - \delta M_H^2\eta^2\right]
\end{aligned}
$$

The counter-term is then given by [24, p. 203]



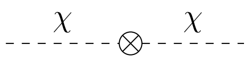
$$= i\left(\delta Z_\eta p_\eta^2 - \delta M_H^2\right) \tag{E.1.1}$$

E.1.2 Goldstone boson counter-term

$$
\begin{aligned}
\frac{1}{2}\left(\partial_\mu \chi_0\right)^2 - \frac{1}{2}\xi_0 M_{A,0}^2\chi_0^2 &= \frac{1}{2}\left(\partial_\mu \chi\right)^2 + \frac{1}{2}\delta Z_\chi\left(\partial_\mu \chi\right)^2 - \frac{1}{2}Z_\xi \xi Z_A^{-1}\left(M_A^2 + \delta M_A^2\right)Z_\chi\chi^2 \\
&= \frac{1}{2}\left(\partial_\mu \chi\right)^2 + \frac{1}{2}\delta Z_\chi\left(\partial_\mu \chi\right)^2 - \frac{1}{2}Z_\xi \xi Z_A^{-1}M_A^2 Z_\chi\chi^2 - \frac{1}{2}Z_\xi \xi Z_A^{-1}\delta M_A^2 Z_\chi\chi^2 \\
&= \frac{1}{2}\left(\partial_\mu \chi\right)^2 + \frac{1}{2}\delta Z_\chi\left(\partial_\mu \chi\right)^2 - \frac{1}{2}\xi M_A^2\chi^2 \\
&\quad - \frac{1}{2}\left(Z_\xi Z_A^{-1}Z_\chi - 1\right)\xi M_A^2\chi^2 - \frac{1}{2}Z_\xi \xi Z_A^{-1}\delta M_A^2 Z_\chi\chi^2
\end{aligned}
$$

The terms relevant for the counter-term take the form

$$\frac{1}{2}\Big(\delta Z_\chi(\partial_\mu \chi)^2 - \left(Z_\xi Z_A^{-2}Z_\chi - 1\right)\xi M_A^2\chi^2 - Z_\xi Z_A^{-1}\delta M_A^2\xi\chi^2\Big)$$



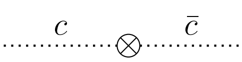
$$= i\Big(\delta Z_\chi p_\chi^2 - \left(Z_\xi Z_A^{-1}Z_\chi - 1\right)\xi M_A^2 - Z_\xi Z_A^{-1}\delta M_A^2\xi\Big) \tag{E.1.2}$$

E.1.3 Ghost counter-term

$$
\begin{aligned}
\left(\partial_\mu \bar{c}\right)\left(\partial^\mu c_0\right) - \xi_0 M_{A,0}^2\bar{c}c_0 &= \left(\partial_\mu \bar{c}\right)\left(\partial^\mu c\right) + \frac{1}{2}\delta Z_c\left(\partial_\mu \bar{c}\right)\left(\partial^\mu c\right) - Z_\xi Z_c^{1/2}Z_A^{-1}\xi\left(M_A^2 + \delta M_A^2\right)\bar{c}c \\
&= \left(\partial_\mu \bar{c}\right)\left(\partial^\mu c\right) + \frac{1}{2}\delta Z_c\left(\partial_\mu \bar{c}\right)\left(\partial^\mu c\right) - Z_\xi Z_c^{1/2}Z_A^{-1}\xi M_A^2\bar{c}c - Z_\xi Z_c^{1/2}Z_A^{-1}\xi\delta M_A^2\bar{c}c \\
&= \left(\partial_\mu \bar{c}\right)\left(\partial^\mu c\right) + \frac{1}{2}\delta Z_c\left(\partial_\mu \bar{c}\right)\left(\partial^\mu c\right) - \xi M_A^2\bar{c}c \\
&\quad - \left(Z_\xi Z_c^{1/2}Z_A^{-1} - 1\right)\xi M_A^2\bar{c}c - Z_\xi Z_c^{1/2}Z_A^{-1}\xi\delta M_A^2\bar{c}c
\end{aligned}
$$

The relevant terms take the form

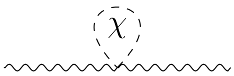
$$\bar{c}\Big(\frac{1}{2}\delta Z_c\left(\partial_\mu \bar{c}\right)\left(\partial^\mu c\right) - \left(Z_\xi Z_c^{1/2}Z_A^{-1} - 1\right)\xi M_A^2 - Z_\xi Z_c^{1/2}Z_A^{-1}\xi\delta M_A^2\Big)c$$



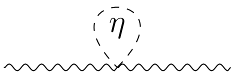
$$= i\Big(\frac{1}{2}\delta Z_c p_c^2 - \left(Z_\xi Z_c^{1/2}Z_A^{-1} - 1\right)\xi M_A^2 - Z_\xi Z_c^{1/2}Z_A^{-1}\xi\delta M_A^2\Big) \tag{E.1.3}$$

E.2 Spontaneously broken QED propagators

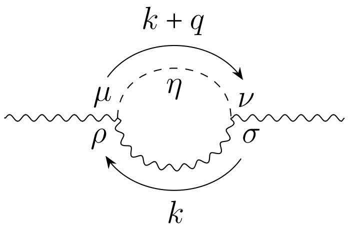
E.2.1 Photon propagator



$$\begin{aligned} &= 2ig_{\mu\nu}e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-\xi M_A^2} \\ &= -2ig_{\mu\nu}\frac{e^2}{(4\pi)^2}A_0(\xi M_A^2) \\ &\overset{\text{Div}}{=} -i\frac{e^2}{4\pi^2\epsilon}g_{\mu\nu}\xi M_A^2. \end{aligned} \tag{E.2.1}$$



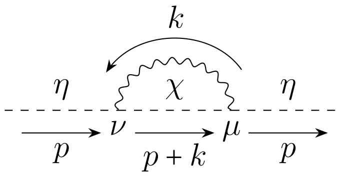
$$\begin{aligned} &= 2ig_{\mu\nu}e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-M_H^2} \\ &= -2g_{\mu\nu}i\frac{e^2}{(4\pi)^2}A_0(M_H^2) \\ &\overset{\text{Div}}{=} -i\frac{e^2}{4\pi^2\epsilon}g_{\mu\nu}M_H^2. \end{aligned} \tag{E.2.2}$$



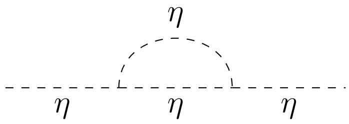
$$\begin{aligned} &= (2ie^2v)^2 g_{\mu\rho}g_{\sigma\nu}\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-M_A^2}\Big(-g^{\rho\sigma}+ \\ &\quad (1-\xi)\frac{k^\rho k^\sigma}{k^2-\xi M_A^2}\Big)\frac{i}{(k+q)^2-M_H^2}, \\ &\overset{\text{Div}}{=} -i\frac{e^4v^2}{8\pi^2\epsilon}\mu^\epsilon g_{\mu\nu}(3+\xi) \\ &= -i\frac{e^2}{8\pi^2\epsilon}M_A^2 g_{\mu\nu}(3+\xi), \end{aligned} \tag{E.2.3}$$

where we have used $M_A^2 = (ev)^2$.

E.2.2 Higgs boson

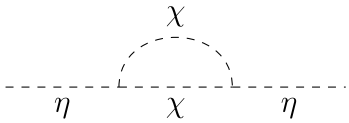


$$\begin{aligned} &= e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{i^2\big(-p-(k+p)\big)_\mu\big(p-(-(k+p))\big)_\nu}{(k^2-M_A^2)[(k+p)^2-\xi M_A^2]} \\ &\quad \cdot\left[-g^{\mu\nu}+(1-\xi)\frac{k^\mu k^\nu}{k^2-\xi M_A^2}\right] \\ &\overset{\text{Div}}{=} -i\frac{e^2}{(4\pi)^2}(2\xi M_A^2\cdot\xi-(\xi-3)p^2)\frac{2}{\epsilon}. \end{aligned} \tag{E.2.4}$$



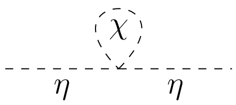
$$= \big(-6i\lambda v\big)^2 \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - M_H^2} \frac{i}{(k+p)^2 - M_H^2}$$
$$= i\frac{9(\lambda v)^2}{4\pi^2}\mu^\epsilon B_0(p^2, M_H^2, M_H^2)$$
$$\overset{\text{Div}}{=} i\frac{9(\lambda v)^2}{2\pi^2\epsilon}\mu^\epsilon$$
$$= i\frac{9\lambda}{8\pi^2\epsilon}M_H^2, \qquad \text{(E.2.5)}$$

where we have added a symmetry factor of $\frac{1}{2}$ in the last step.

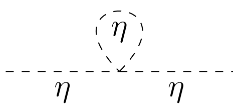


$$= \big(-2i\lambda v\big)^2 \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - \xi M_A^2} \frac{i}{(k+p)^2 - \xi M_A^2}$$
$$= i\frac{(\lambda v)^2}{4\pi^2}\mu^\epsilon B_0(p^2, \xi M_A^2, \xi M_A^2)$$
$$\overset{\text{Div}}{=} i\frac{(\lambda v)^2}{2\pi^2\epsilon}\mu^\epsilon$$
$$= i\frac{\lambda}{8\pi^2\epsilon}M_H^2, \qquad \text{(E.2.6)}$$

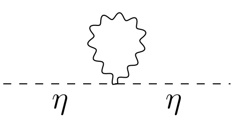
where we have added a symmetry factor of $\frac{1}{2}$ in the last step.



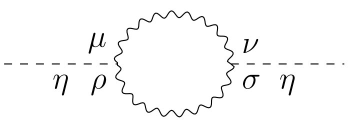
$$= (-2i\lambda)\mu^\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - \xi M_A^2}$$
$$= 2i\frac{\lambda}{(4\pi)^2}A_0(\xi M_A^2)$$
$$\overset{\text{Div}}{=} i\frac{\lambda}{4\pi^2\epsilon}\xi M_A^2. \qquad \text{(E.2.7)}$$



$$= (-6i\lambda)\mu^\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - M_H^2}$$
$$= 6i\frac{\lambda}{(4\pi)^2}A_0(M_H^2)$$
$$\overset{\text{Div}}{=} i\frac{3\lambda}{4\pi^2\epsilon}M_H^2. \qquad \text{(E.2.8)}$$

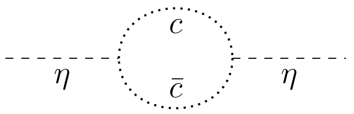


$$= \big(2ie^2 g_{\mu\nu}\big)\mu^\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - M_A^2}\left[-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2 - \xi M_A^2}\right]$$
$$\overset{\text{Div}}{=} i\frac{e^2}{4\pi^2\epsilon}M_A^2(\xi^2 + 3). \qquad \text{(E.2.9)}$$



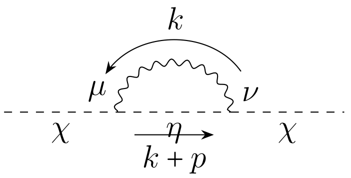
$$= \left(2ie^2v\right)^2\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}g_{\mu\rho}g_{\sigma\nu}\frac{i}{k^2-M_A^2}\left[-g^{\rho\sigma}+(1-\xi)\frac{k^\rho k^\sigma}{k^2-\xi M_A^2}\right]$$
$$\cdot\frac{i}{(k+p)^2-M_A^2}\left[-g^{\mu\nu}+(1-\xi)\frac{(k+p)^\mu(k+p)^\nu}{(k+p)^2-\xi M_A^2}\right]$$
$$\overset{\text{Div}}{=} i\frac{e^4v^2}{2\pi^2\epsilon}\mu^\epsilon(\xi^2+3)$$
$$= i\frac{e^2}{4\pi^2\epsilon}M_A^2(\xi^2+3), \tag{E.2.10}$$

where we have added a symmetry factor of $\frac{1}{2}$ in the last step.

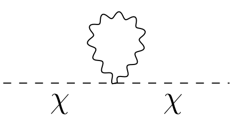


$$= -\left(-i\xi e^2v\right)^2\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-\xi M_A^2}\frac{i}{(k+p)^2-\xi M_A^2}$$
$$= -i\frac{e^2}{(4\pi)^2}\xi^2M_A^2\mu^\epsilon B_0(p^2,\xi M_A^2,\xi M_A^2)$$
$$\overset{\text{Div}}{=} -i\frac{e^2}{8\pi^2\epsilon}\mu^\epsilon\xi^2M_A^2. \tag{E.2.11}$$

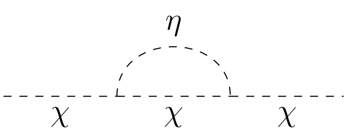
E.2.3 Goldstone boson



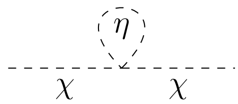
$$= -e^2\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\frac{(-k-2p)_\mu(k+2p)_\nu}{(k+q)^2-M_H^2}\cdot\frac{1}{k^2-M_A^2}\left[-g^{\mu\nu}+(1-\xi)\frac{k^\mu k^\nu}{k^2-\xi M_A^2}\right]$$
$$\overset{\text{Div}}{=} -i\frac{e^2}{8\pi^2\epsilon}\Big(\xi(\xi M_A^2+M_H^2)-(\xi-3)p^2\Big). \tag{E.2.12}$$



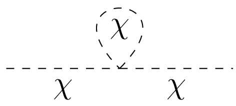
$$= \left(2ie^2g_{\mu\nu}\right)\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-M_A^2}\left[-g^{\mu\nu}+(1-\xi)\frac{k^\mu k^\nu}{k^2-\xi M_A^2}\right]$$
$$\overset{\text{Div}}{=} i\frac{e^2}{4\pi^2\epsilon}\left(\xi^2+3\right)M_A^2. \tag{E.2.13}$$



$$= \left(-2i\lambda v\right)^2\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-\xi M_A^2}\frac{i}{(k+p)^2-M_H^2}$$
$$= i\frac{(\lambda v)^2}{4\pi^2}\mu^\epsilon B_0(p^2,\xi M_A^2,M_H^2)$$
$$\overset{\text{Div}}{=} i\frac{(\lambda v)^2}{2\pi^2\epsilon}\mu^\epsilon. \tag{E.2.14}$$



$$= (-2i\lambda)\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2 - M_H^2}$$
$$= 2i\frac{\lambda}{(4\pi)^2}A_0(M_H^2)$$
$$\overset{\text{Div}}{=} i\frac{\lambda}{4\pi^2\epsilon}M_H^2. \tag{E.2.15}$$



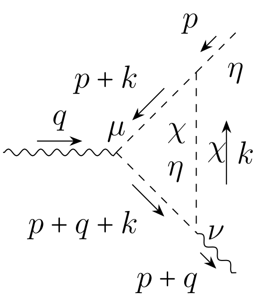
$$= (-6i\lambda)\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2 - \xi M_A^2}$$
$$= 6i\frac{\lambda}{(4\pi)^2}A_0(\xi M_A^2)$$
$$\overset{\text{Div}}{=} i\frac{3\lambda}{4\pi^2\epsilon}\xi M_A^2. \tag{E.2.16}$$

E.3 3-point vertices

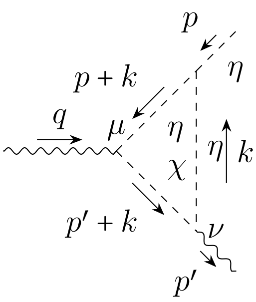
E.3.1 $\Gamma_{\mu\nu}^{A^\mu A^\nu \eta}$

To ease notation, we define

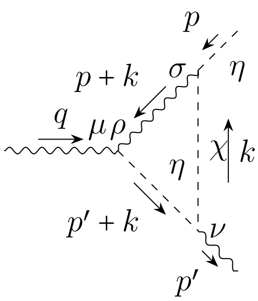
$$D^{\mu\nu}(k) = \left(\frac{(1-\xi)k^\mu k^\nu}{k^2 - \xi M_A^2} - g^{\mu\nu}\right).$$



$$= (-2ie^2\lambda v)\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^3(p+q+k+k)_\mu(-(p+q+k)-(p+k))_\nu}{(k^2-\xi M_A^2)[(p+k)^2-\xi M_A^2][(p'+k)^2-M_H^2]}$$
$$= (-2e^2\lambda v)\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{(p+q+2k)_\mu(-p-q-p-2k)_\nu}{(k^2-\xi M_A^2)[(p+k)^2-\xi M_A^2][(p'+k)^2-M_H^2]}$$
$$\overset{\text{Div}}{=} (-2e^2\lambda v)\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{-4k_\mu k_\nu}{(k^2-\xi M_A^2)[(p+k)^2-\xi M_A^2][(p'+k)^2-M_H^2]}$$
$$= 8i\frac{e^2\lambda v}{(4\pi)^2}\mu^\epsilon C_{\mu\nu}$$
$$\overset{\text{Div}}{=} i\frac{e^2\lambda v}{4\pi^2\epsilon}\mu^\epsilon g_{\mu\nu}. \tag{E.3.1}$$

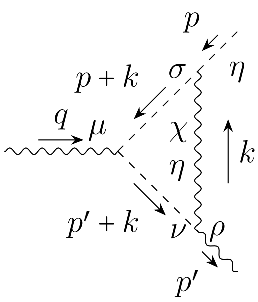


$$= (-6ie^2\lambda v)\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^3(2k+2p+q)_\mu(-2k-p-q)_\nu}{(k^2-M_H^2)[(k+p)^2-M_H^2][(k+p+q)^2-\xi M_A^2]}$$
$$\overset{\text{Div}}{=} i\frac{3e^2\lambda v}{4\pi^2\epsilon}\mu^\epsilon g_{\mu\nu}. \tag{E.3.2}$$



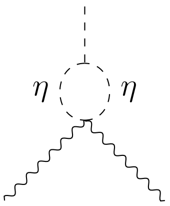
$$= (2ie^4 v g_{\mu\rho})\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^3(p-k)_\sigma(2k+p+q)_\nu D^{\rho\sigma}(k+p)}{(k^2-\xi M_A^2)[(k+p+q)^2-M_H^2][(k+p)-M_A^2]}$$

$$\overset{\text{Div}}{=} i\frac{e^4\xi v}{8\pi^2\epsilon}\mu^\epsilon g_{\mu\nu}. \tag{E.3.3}$$



$$= (2ie^4 v g_{\nu\rho})\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^3(k+2p)_\sigma(-2k-2p-q)_\mu D^{\rho\sigma}(k)}{(k^2-M_A^2)[(k+p)^2-\xi M_A^2][(k+p+q)^2-M_H^2]}$$

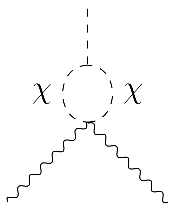
$$\overset{\text{Div}}{=} i\frac{e^4\xi v}{8\pi^2\epsilon}\mu^\epsilon g_{\mu\nu}. \tag{E.3.4}$$



$$= (-6i\lambda v)(2ie^2 g_{\mu\nu})\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^2}{(k^2-M_H^2)[(k+p)^2-M_H^2]}$$

$$\overset{\text{Div}}{=} -i\mu^\epsilon\frac{3e^2\lambda v g_{\mu\nu}}{4\pi^2\epsilon}, \tag{E.3.5}$$

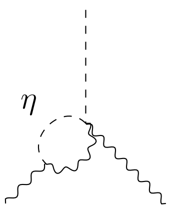
where we have added a symmetry factor of $\frac{1}{2}$ in the last step.



$$= (-2i\lambda v)(2ie^2 g_{\mu\nu})\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^2}{(k^2-\xi M_A^2)[(k+p)^2-\xi M_A^2]}$$

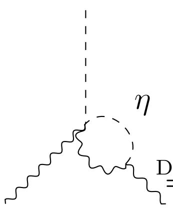
$$\overset{\text{Div}}{=} -i\mu^\epsilon\frac{e^2\lambda v g_{\mu\nu}}{4\pi^2\epsilon}, \tag{E.3.6}$$

where we have added a symmetry factor of $\frac{1}{2}$ in the last step.



$$= (2ie^2 g_{\rho\nu})(2ie^2 v g_{\mu\sigma})\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^2 D^{\rho\sigma}(k)}{(k^2-M_A^2)[(k+q)^2-M_H^2]}$$

$$\overset{\text{Div}}{=} -i\mu^\epsilon\frac{e^4(\xi+3)v g_{\mu\nu}}{8\pi^2\epsilon}. \tag{E.3.7}$$

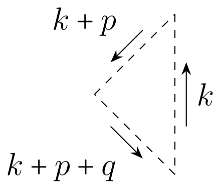


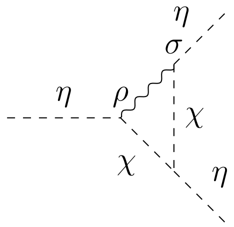
$$=(2ie^2 g_{\mu\rho}e^2)(2ie^2 v g_{\sigma\nu})\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^2 D^{\sigma\rho}(k)}{(k^2-M_A^2)[(k+p+q)^2-M_H^2]}$$

$$\overset{\text{Div}}{=} -i\mu^\epsilon\frac{e^4(\xi+3)v g_{\mu\nu}}{8\pi^2\epsilon}. \tag{E.3.8}$$

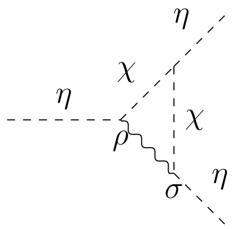
E.3.2 $\Gamma^{\eta\eta\eta}$

We define the flow of momenta in the loop here, to ease notation. This is of course arbitrary, however this definition gives straightforward applications to the P.V functions.

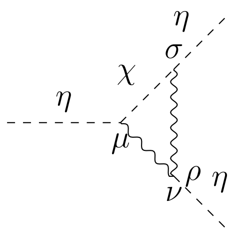




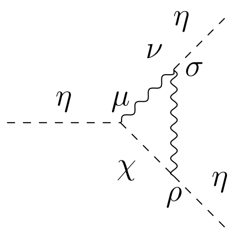
$$= (-2ie^2\lambda v)\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^3(p-k)_\sigma(k+p+2q)_\rho D^{\rho\sigma}(k+p)}{(k^2-\xi M_A^2)[(k+p)^2-M_A^2][(k+p+q)^2-\xi M_A^2]}$$
$$\overset{\text{Div}}{=} -i\frac{e^2\lambda\xi v}{4\pi^2\epsilon}\mu^\epsilon. \tag{E.3.9}$$



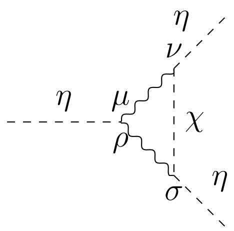
$$= (-2ie^2\lambda v)\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^3(-k-p+q)_\rho(k-p-q)_\sigma D^{\rho\sigma}(k+p+q)}{(k^2-\xi M_A^2)[(k+p)^2-\xi M_A^2][(k+p+q)^2-M_A^2]}$$
$$\overset{\text{Div}}{=} -i\frac{e^2\lambda\xi v}{4\pi^2\epsilon}\mu^\epsilon. \tag{E.3.10}$$



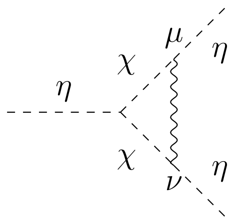
$$= (2ie^4vg_{\nu\rho})\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^3(k+2p)_\sigma(-k-p+q)_\mu D^{\rho\sigma}(k)D^{\mu\nu}(k+p+q)}{(k^2-M_A^2)[(k+p)^2-\xi M_A^2][(k+p+q)^2-M_A^2]}$$
$$\overset{\text{Div}}{=} -i\frac{e^4\xi^2 v}{4\pi^2\epsilon}\mu^\epsilon. \tag{E.3.11}$$



$$= (2ie^4vg_{\nu\sigma})\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^3(k+p+2q)_\mu(-k-2p-2q)_\rho D^{\rho\sigma}(k)D^{\mu\nu}(k+p)}{(k^2-M_A^2)[(k+p)^2-M_A^2][(k+p+q)^2-\xi M_A^2]}$$
$$\overset{\text{Div}}{=} -i\frac{e^4\xi^2 v}{4\pi^2\epsilon}\mu^\epsilon. \tag{E.3.12}$$

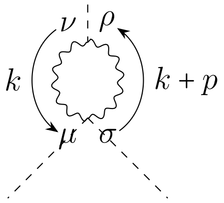


$$= (2ie^4vg_{\mu\rho})\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^3(p-k)_\nu(k-p-q)_\sigma D^{\mu\nu}(k+p)D^{\rho\sigma}(k+p+q)}{(k^2-\xi M_A^2)[(k+p)^2-M_A^2][(k+p+q)^2-M_A^2]}$$
$$\overset{\text{Div}}{=} -i\frac{e^4\xi^2 v}{4\pi^2\epsilon}\mu^\epsilon. \tag{E.3.13}$$

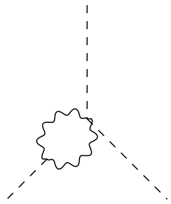


$$= (-2i\lambda v)e^2\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^2(2p+k)_\mu(-2p-2q-k)_\nu D^{\mu\nu}(k)}{(k^2-M_A^2)[(k+p)^2-\xi M_A^2][(k+p+q)^2-\xi M_A^2]}$$
$$\overset{\text{Div}}{=} -\mu^\epsilon\frac{ie^2\lambda\xi v}{4\pi^2\epsilon}. \tag{E.3.14}$$

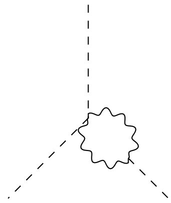
All of the diagrams below have a symmetry factor of $\frac{1}{2}$, which is added in the last step.



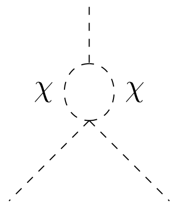
$$= (2ie^2 g_{\mu\sigma})(2ie^2 v g_{\nu\rho})\mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{i^2 D^{\mu\nu}(k) D^{\sigma\rho}(k+p)}{(k^2 - M_A^2)[(k+p)^2 - M_A^2]} \stackrel{\text{Div}}{=} i\mu^\epsilon \frac{e^4 v\,(\xi^2+3)}{4\pi^2\epsilon}. \tag{E.3.15}$$



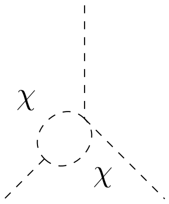
$$= (2ie^2 g_{\mu\sigma})(2ie^2 v g_{\nu\rho})\mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{i^2 D^{\mu\nu}(k) D^{\sigma\rho}(k+q)}{(k^2 - M_A^2)[(k+q)^2 - M_A^2]} \stackrel{\text{Div}}{=} i\mu^\epsilon \frac{e^4 v\,(\xi^2+3)}{4\pi^2\epsilon}. \tag{E.3.16}$$



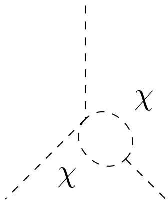
$$= (2ie^2 g_{\mu\sigma})(2ie^2 v g_{\nu\rho})\mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{i^2 D^{\mu\nu}(k) D^{\sigma\rho}(k+q+p)}{(k^2 - M_A^2)[(k+q+p)^2 - M_A^2]} \stackrel{\text{Div}}{=} i\mu^\epsilon \frac{e^4 v\,(\xi^2+3)}{4\pi^2\epsilon}. \tag{E.3.17}$$



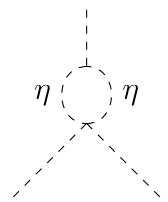
$$= (-2i\lambda v)(-2i\lambda)\mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{i^2}{(k^2 - \xi M_A^2)[(k+p)^2 - \xi M_A^2]} \stackrel{\text{Div}}{=} i\mu^\epsilon \frac{\lambda^2 v}{4\pi^2\epsilon}. \tag{E.3.18}$$



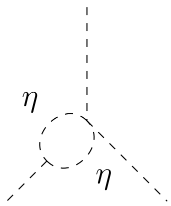
$$= (-2i\lambda v)(-2i\lambda)\mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{i^2}{(k^2 - \xi M_A^2)[(k+q)^2 - \xi M_A^2]} \stackrel{\text{Div}}{=} i\mu^\epsilon \frac{\lambda^2 v}{4\pi^2\epsilon}. \tag{E.3.19}$$



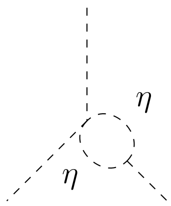
$$= (-2i\lambda v)(-2i\lambda)\mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{i^2}{(k^2 - \xi M_A^2)[(k+q+p)^2 - \xi M_A^2]} \stackrel{\text{Div}}{=} i\mu^\epsilon \frac{\lambda^2 v}{4\pi^2\epsilon}. \tag{E.3.20}$$



$$= (-6i\lambda v)(-6i\lambda)\mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{i^2}{(k^2 - M_H^2)[(k+p)^2 - M_H^2]} \stackrel{\text{Div}}{=} i\mu^\epsilon \frac{9\lambda^2 v}{4\pi^2\epsilon}. \tag{E.3.21}$$



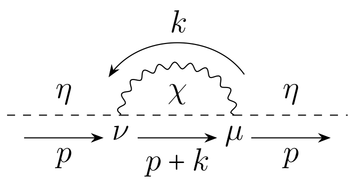
$$= (-6i\lambda v)(-6i\lambda)\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^2}{(k^2-M_H^2)[(k+q)^2-M_H^2]} \overset{\text{Div}}{=} i\mu^{\epsilon}\frac{9\lambda^2 v}{4\pi^2\epsilon}. \tag{E.3.22}$$



$$= (-6i\lambda v)(-6i\lambda)\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^2}{(k^2-M_H^2)[(k+q)^2-M_H^2]} \overset{\text{Div}}{=} i\mu^{\epsilon}\frac{9\lambda^2 v}{4\pi^2\epsilon}. \tag{E.3.23}$$

F Example of FeynCalc calculation

Here we show an example of a loop calculation done using Mathematica with the FeynCalc package. The example is the first diagram in appendix E.2.2.



$$= e^2\mu^{\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i^2\big(-p-(k+p)\big)_\mu\big(p-(-(k+p))\big)_\nu D^{\mu\nu}(k)}{(k^2-M_A^2)[(k+p)^2-\xi M_A^2]} \overset{\text{Div}}{=} -i\frac{e^2}{(4\pi)^2}(2\xi M_A^2\cdot\xi-(\xi-3)p^2)\frac{2}{\epsilon}. \tag{F.0.1}$$

```
In[385]:= amp = -(e^2) / (2*π)^4 * (FVD[-k - 2 p, μ] * FVD[k + 2 p, ν]) *
    FAD[{k, mA}, {k + p, m2}] *
    (-MTD[μ, ν] + (1 - ξ) * FVD[k, μ] * FVD[k, ν] * FAD[{k, m2}])
  Print["The full PV-decomp is"]
  TID[amp, k, UsePaVeBasis → True, ToPaVe → True]
  Print["With the UV-divergent part"]
  div = PaVeUVPart[TID[amp, k, UsePaVeBasis → True, ToPaVe → True], Dimension → 4 - ϵ]
  FullSimplify[div, {D == 4, m2^2 == ξ * mA^2}]
```

Figure 18: Example input in Mathematica.

Out[385]= $$-\frac{e^2\,(-k-2\,p)^{\mu}\,(k+2\,p)^{\nu}\left(\frac{(1-\xi)\,k^{\mu}\,k^{\nu}}{k^2-\text{m2}^2}-g^{\mu\,\nu}\right)}{16\,\pi^4\,(k^2-\text{mA}^2).((\,k+p\,)^2-\text{m2}^2)}$$

The full PV–decomp is

Out[387]= $$\frac{i\,e^2\,(1-\xi)\,(\text{m2}^2-p^2)^2\,\text{B}_0(p^2,\text{m2}^2,\text{m2}^2)}{16\,\pi^2\,(\text{m2}^2-\text{mA}^2)}-\frac{i\,e^2\,(\text{m2}^4\,(-\xi)+3\,\text{m2}^4-3\,\text{m2}^2\,\text{mA}^2+2\,\text{m2}^2\,\xi\,p^2+\text{mA}^4-2\,\text{mA}^2\,p^2-\xi\,p^4+p^4)\,\text{B}_0(p^2,\text{m2}^2,\text{mA}^2)}{16\,\pi^2\,(\text{m2}^2-\text{mA}^2)}+\frac{i\,e^2\,\text{A}_0(\text{m2}^2)\,(-2\,\text{m2}^2\,\xi+3\,\text{m2}^2-\text{mA}^2+\xi\,p^2-p^2)}{16\,\pi^2\,(\text{m2}^2-\text{mA}^2)}-\frac{i\,e^2\,\text{A}_0(\text{mA}^2)\,(\text{m2}^2\,(-\xi)+3\,\text{m2}^2-\text{mA}^2\,\xi-\text{mA}^2+\xi\,p^2-p^2)}{16\,\pi^2\,(\text{m2}^2-\text{mA}^2)}$$

With the UV–divergent part

Out[389]= $$-\frac{i\,(2\,e^2\,\text{m2}^2\,\xi+e^2\,(-\text{m2}^2)+e^2\,\text{mA}^2\,\xi-e^2\,\xi\,p^2+3\,e^2\,p^2)}{8\,\pi^2\,\epsilon}$$

Out[390]= $$-\frac{i\,e^2\,(2\,\text{m2}^2\,\xi-(\xi-3)\,p^2)}{8\,\pi^2\,\epsilon}$$

Figure 19: Corresponding output in Mathematica.

G Lengthy calculations

G.1 Finding the Spontaneously broken QED Lagrangian

G.1.1 Factorization

$$\begin{aligned}
\mathscr{L}_\phi =& \left(D_\mu\Big(\frac{\eta+i\chi}{\sqrt{2}}\Big) \right)^\dagger \left(D^\mu\Big(\frac{\eta+i\chi}{\sqrt{2}}\Big) \right) \\
&+ \mu^2\Big(\frac{\eta+i\chi}{\sqrt{2}}\Big)^\dagger \frac{\eta+i\chi}{\sqrt{2}} - \lambda \left(\Big(\frac{\eta+i\chi}{\sqrt{2}}\Big)^\dagger \Big(\frac{\eta+i\chi}{\sqrt{2}}\Big) \right)^2 \\
=& \left((\partial_\mu - ieA_\mu)\Big(\frac{\eta+i\chi}{\sqrt{2}}\Big) \right)^\dagger \left((\partial^\mu - ieA^\mu)\Big(\frac{\eta+i\chi}{\sqrt{2}}\Big) \right) \\
&+ \mu^2\Big(\frac{\eta+i\chi}{\sqrt{2}}\Big)^\dagger \frac{\eta+i\chi}{\sqrt{2}} - \lambda \left(\Big(\frac{\eta+i\chi}{\sqrt{2}}\Big)^\dagger \Big(\frac{\eta+i\chi}{\sqrt{2}}\Big) \right)^2 \\
=& \big(\partial_\mu + ieA_\mu\big)\Big(\frac{\eta-i\chi}{\sqrt{2}}\Big)\big(\partial^\mu - ieA^\mu\big)\Big(\frac{\eta+i\chi}{\sqrt{2}}\Big) \\
&+ \mu^2\Big(\frac{\eta-i\chi}{\sqrt{2}}\Big)\frac{\eta+i\chi}{\sqrt{2}} - \lambda \left(\Big(\frac{\eta-i\chi}{\sqrt{2}}\Big)\Big(\frac{\eta+i\chi}{\sqrt{2}}\Big) \right)^2 \\
=& \frac{1}{2}\big(\partial_\mu\eta - i\partial_\mu\chi + ieA_\mu\eta + eA_\mu\chi\big)\big(\partial^\mu\eta + i\partial^\mu\chi - ieA^\mu\eta + eA^\mu\chi\big) \\
&+ \frac{\mu^2}{2}\big(\eta^2+\chi^2\big) - \frac{\lambda}{4}\big(\eta^2+\chi^2\big)^2 \\
=& \frac{1}{2}\big(\partial_\mu\eta\partial^\mu\eta + i\partial_\mu\eta\partial^\mu\chi - \partial_\mu\eta ieA^\mu\eta + \partial_\mu\eta eA^\mu\chi \\
&- i\partial_\mu\chi\partial^\mu\eta + \partial_\mu\chi\partial^\mu\chi - \partial_\mu\chi eA^\mu\eta - i\partial_\mu\chi eA^\mu\chi \\
&+ ieA_\mu\eta\partial^\mu\eta - eA_\mu\eta\partial^\mu\chi + eA_\mu\eta eA^\mu\eta + ieA_\mu\eta eA^\mu\chi \\
&+ eA_\mu\chi\partial^\mu\eta + eA_\mu\chi i\partial\chi - eA_\mu\chi ieA^\mu\eta + eA_\mu\chi eA^\mu\chi\big) \\
&+ \frac{\mu^2}{2}\big(\eta^2+\chi^2\big) - \frac{\lambda}{4}\big(\eta^2+\chi^2\big)^2
\end{aligned}$$

The terms containing i cancel, leaving

$$\begin{aligned}
\mathscr{L}_\phi =& \frac{1}{2}\big(\partial_\mu\eta\partial^\mu\eta + \partial_\mu\eta eA^\mu\chi + \partial_\mu\chi\partial^\mu\chi - \partial_\mu\chi eA^\mu\eta \\
&- eA_\mu\eta\partial^\mu\chi + eA_\mu\eta eA^\mu\eta + eA_\mu\chi\partial^\mu\eta + eA_\mu\chi eA^\mu\chi\big) \\
&+ \frac{\mu^2}{2}\big(\eta^2+\chi^2\big) - \frac{\lambda}{4}\big(\eta^2+\chi^2\big)^2
\end{aligned}$$

By rearranging some terms, a factorization can be seen

$$\begin{aligned}
\mathscr{L}_\phi =& \frac{1}{2}\big(\partial_\mu\eta\partial^\mu\eta + \partial_\mu\eta eA^\mu\chi + eA_\mu\chi\partial^\mu\eta + eA_\mu\eta eA^\mu\eta \\
&+ \partial_\mu\chi\partial^\mu\chi - \partial_\mu\chi eA^\mu\eta - eA_\mu\eta\partial^\mu\chi + eA_\mu\chi eA^\mu\chi\big) \\
&+ \frac{\mu^2}{2}\big(\eta^2+\chi^2\big) - \frac{\lambda}{4}\big(\eta^2+\chi^2\big)^2 \\
=& \frac{1}{2}\Big(\big(\partial_\mu\eta + eA^\mu\chi\big)^2 + \big(\partial_\mu\chi - eA^\mu\eta\big)^2\Big) + \frac{\mu^2}{2}\big(\eta^2+\chi^2\big) - \frac{\lambda}{4}\big(\eta^2+\chi^2\big)^2 \qquad \text{(G.1.1)}
\end{aligned}$$

G.1.2 Field shift in Lagrangian

$$\begin{aligned}
\mathscr{L}_\phi =& \frac{1}{2}\Big(\big(\partial_\mu(v+\eta)+eA^\mu\chi\big)^2+\big(\partial_\mu\chi-eA^\mu(v+\eta)\big)^2\Big)\\
&+\frac{\mu^2}{2}\big((v+\eta)^2+\chi^2\big)-\frac{\lambda}{4}\big((v+\eta)^2+\chi^2\big)^2\\
=&\frac{1}{2}\Big(\big(\partial_\mu v+\partial_\mu\eta+eA^\mu\chi\big)^2+\big(\partial_\mu\chi-eA^\mu v-eA^\mu\eta\big)^2\Big)\\
&+\frac{\mu^2}{2}\big((v+\eta)^2+\chi^2\big)-\frac{\lambda}{4}\big((v+\eta)^2+\chi^2\big)^2
\end{aligned}$$

v represents the new non-zero potenial minimum, and thus it's derivate is 0; $\partial^\mu v = 0$

$$\begin{aligned}
\mathscr{L}_\phi =& \frac{1}{2}\Big(\big(\partial_\mu\eta+eA^\mu\chi\big)^2+\big(\partial_\mu\chi-eA^\mu v-eA^\mu\eta\big)^2\Big)\\
&+\frac{\mu^2}{2}\big((v+\eta)^2+\chi^2\big)-\frac{\lambda}{4}\big((v+\eta)^2+\chi^2\big)^2\\
=&\frac{1}{2}\Big(\big(\partial_\mu\eta\big)^2+2\partial_\mu\eta eA^\mu\chi+\big(eA^\mu\chi\big)^2\\
&+\big(\partial_\mu\chi\big)^2-\partial_\mu\chi eA^\mu v-\partial_\mu\chi eA^\mu\eta\\
&-eA^\mu v\partial_\mu\chi+\big(eA^\mu v\big)^2+eA^\mu veA^\mu\eta\\
&-eA^\mu\eta\partial_\mu\chi+eA^\mu\eta eA^\mu v+\big(-eA^\mu\eta\big)^2\Big)\\
&+\frac{\mu^2}{2}\big(v^2+2v\eta+\eta^2+\chi^2\big)-\frac{\lambda}{4}\big(v^2+2v\eta+\eta^2+\chi^2\big)^2\\
=&\frac{1}{2}M_A^2A_\mu^2-evA^\mu\partial_\mu\chi+\frac{1}{2}\big(\partial_\mu\eta\big)^2+\frac{1}{2}\big(\partial_\mu\chi\big)^2\\
&+\partial_\mu\eta eA^\mu\chi-\partial_\mu\chi eA^\mu\eta+\frac{\big(eA^\mu\chi\big)^2}{2}+\frac{\big(eA^\mu\eta\big)^2}{2}+eA^\mu veA^\mu\eta\\
&+\frac{\mu^2}{2}\big(v^2+2v\eta+\eta^2+\chi^2\big)-\frac{\lambda}{4}\big(v^2+2v\eta+\eta^2+\chi^2\big)^2
\end{aligned} \tag{G.1.2}$$

References

[1] J. Jaeckel, *A force beyond the standard model - status of the quest for hidden photons*, 2013, arXiv:1303.1821 [hep-ph].

[2] J. Ellis, *Tikz-feynman: feynman diagrams with tikz*, 2016, arXiv:1601.05437 [hep-ph].

[3] V. Shtabovenko, R. Mertig, and F. Orellana, "New developments in feyncalc 9.0", **207**, 432–444 (2016).

[4] V. Shtabovenko, R. Mertig, and F. Orellana, *Feyncalc 9.3: new features and improvements*, 2020, arXiv:2001.04407 [hep-ph].

[5] R. Mertig, M. Böhm, and A. Denner, "Feyn calc - computer-algebraic calculation of feynman amplitudes", Computer Physics Communications **64**, 345–359 (1991).

[6] J. Kersten, "Renormalization group evolution of neutrino masses", MA thesis (Technische Universitat Munchen, 2001).

[7] J. McGreevy, *Particles and fields*, Lecture notes in Quantum Field Theory at USCD, 2018, https://mcgreevy.physics.ucsd.edu/w18/ (visited on 11/28/2020).

[8] T. Muta, *Foundations of quantum chromodynamics*, 3rd ed., World Scientific Lecture Notes in Physics - Vol 78 (World Scientific Publishing, Singapore, 2010).

[9] J. M. Jauch and F. Rohrlich, *The theory of photons and electrons* (Springer Berlin Heidelberg, 1976) Chap. 2.4.

[10] M. Peskin and D. Schroeder, *An introduction to quantum field theory* (CRC Press, Boca Raton, 2018).

[11] W. Pauli and F. Villars, "On the invariant regularization in relativistic quantum theory", Reviews of Modern Physics **21**, 434–444 (1949).

[12] J. Collins, *Renormalization*, 1st ed. (Cambridge University Press, Cambridge, 1984).

[13] M. D. Schwartz, *Quantum field theory and the standard model*, 1st ed. (Cambridge University Press, Cambridge, 2014).

[14] G. '. Hooft and M. Veltman, "Combinatorics of gauge fields", Nuclear Physics B **50**, 318–353 (1972).

[15] M. Neubert, *Les houches lectures on renormalization theory and effective field theories*, 2019, arXiv:1901.06573 [hep-ph].

[16] P. Ramond, *Field theory : a modern primer*, 2nd ed., Frontiers in Physics Series, Vol 74 (Westview Press, Cambridge, 2001).

[17] L. F. Li, "Introduction to renormalization in field theory", 100 Years of Subatomic Physics, 465–491 (2013), http://dx.doi.org/10.1142/9789814425810_0017.

[18] M. Bohm, A. Denner, and H. Joos, *Gauge theories of the strong and electroweak interaction*, 3rd ed. (B.G Teubner, Stuttgart, 2001).

[19] A. J. Buras, *Gauge theory of weak decays* (Cambridge University Press, Cambridge, 2020).

[20] A. L. Kataev and S. A. Larin, *Analytical five-loop expressions for the renormalization group QED beta-function in different renormalization schemes*, 2012, arXiv:1205.2810 [hep-ph].

[21]G. Burdman, *The callan-symanzik equation*, Lecture notes on Quantum Field Theory, Chapter 3, 2020, http://fma.if.usp.br/~burdman/QFT2/qft2index.html (visited on 11/16/2020).

[22]G. Passarino and M. Veltman, "One-loop corrections for e^+e^- annihilation into $\mu^+\mu^-$ in the weinberg model", Nuclear Physics B **160**, 151–207 (1979).

[23]A. J. Buras, *Weak hamiltonian, cp violation and rare decays*, 1998, arXiv:hep-ph/9806471 [hep-ph].

[24]M. Kachelriess, *Quantum Fields: From the Hubble to the Planck Scale*, 1st ed. (Oxford University Press, Oxford, 2018).

[25]C. M. Becchi and G. Ridolfi, *An introduction to relativistic processes and the standard model of electroweak interactions*, 2nd ed. (Springer International Publishing, Switzerland, 2014) Chap. 8, p. 107.

[26]S. Pokorski, *Gauge Field Theories*, 2nd ed. (Cambridge University Press, Cambridge, 2003) Chap. 8, p. 107.

[27]N. Irges and F. Koutroulis, "Renormalization of the abelian–higgs model in the R_ξ and unitary gauges and the physicality of its scalar potential", Nuclear Physics B **924**, 178–278 (2017).

[28]M. E. Machacek and M. T. Vaughn, "Two-loop renormalization group equations in a general quantum field theory: (i). wave function renormalization", Nuclear Physics B **222**, 83–103 (1983).

[29]J. Roy, *Calculating -function coefficients of renormalization group equations*, 2019, arXiv:1907.10238 [hep-ph].

[30]T. Hahn, "Generating feynman diagrams and amplitudes with feynarts 3", Computer Physics Communications **140**, 418–431 (2001), http://dx.doi.org/10.1016/S0010-4655(01)00290-9.

[31]A. Denner and S. Dittmaier, *Reduction of one-loop tensor 5-point integrals*, 2002, arXiv:hep-ph/0212259 [hep-ph].