BOOLEAN NEGATION AND NON-CONSERVATIVITY III
THE ACKERMANN CONSTANT

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Abstract: It is known that many relevant logics can be conservatively extended by the truth-constant known as the *Ackermann constant*. It is also known that many relevant logics can be conservatively extended by Boolean negation. This essay, however, shows that a range of relevant logics *with* the Ackermann constant can not be conservatively extended by a Boolean negation.

Keywords: *Ackermann constant, Boolean negation, non-conservative extension, relevant logics*

1. Introduction

Acting on inspiration from Johansson's way of introducing negation for the minimal calculus, Willhelm Ackermann introduced the negative truth-constant $\curlywedge$ in [1] in order to define modal operators. Ackermann suggested reading $\curlywedge$ as "the absurd", and defined the modal operators of impossibility, necessity and possibility as, respectively $A \to \curlywedge$, $\sim A \to \curlywedge$ and $\sim(A \to \curlywedge)$. Anderson and Belnap showed in [2] that the addition of $\curlywedge$ is conservative and that these modal notions could in fact expressed without a truth-constant which prompted them to prefer a truth-constant-less formulation of their modified version of Ackermann's logic, namely the logic **E**, which turned out to be theorem-wise identical to Ackermann's logic.

The tradition of relevant logic has for the most part followed Anderson and Belnap in viewing the Ackermann constant as a constant of convenience; only to be added conservatively in cases where it simplifies presentations or proofs. Even though the addition is conservative in many cases, and even though the addition of other logical notions such as Boolean negation is also conservative in many cases, one cannot always put these fact together and conclude that the addition of Boolean negation to a logic extended by the Ackermann constant is itself conservative. This feature was first observed by Giambrone and Meyer ([9]) who noted that even though C**R**—the Boolean extension of **R**—is a conservative extension of **R**, and $\mathbf{R^t}$—**R** extended by the *positive* Ackermann constant **t** (basically $\sim\curlywedge$)—is a

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conservative extension of **R**, $C\mathbf{R}^\mathbf{t}$ is not a conservative extension of $\mathbf{R}^\mathbf{t}$. In this case it is provable that $C\mathbf{R}^\mathbf{t}$ *is* a conservative extension of $C\mathbf{R}$, and so the non-conservativeness of $C\mathbf{R}^\mathbf{t}$ over $\mathbf{R}^\mathbf{t}$ is only to be located in the **t**-involving formulas. Giambrone and Meyer further claim that the contraction-less logic $\mathbf{TW}^\mathbf{t}$ also fails to be conservatively extended by Boolean negation, although **TW** itself is. I will show that their proof of their $\mathbf{TW}^\mathbf{t}$-claim is wrong due to a confusion of two intuitive readings of **t** as either the conjunction of every *actual* truth or the conjunction of every *logical* truth. This paper studies Giambrone and Meyer's phenomenon in more depth and shows that in fact quite many relevant logics endowed with the Ackermann constant fail to be conservatively extended by Boolean negation.

The rest of this essay is divided into six sections. In sect. 2 I show how the various relevant logics are pieced together and give some basic definitions and lemmas. The two main interpretations of the Ackermann constant are given and their difference discussed. These two interpretations give rise to two interesting classes of relevant logics: the **t**-*distinctive* relevant logics for which the rule $A \vdash \mathbf{t} \to A$ is derivable, and the *semi*-**t**-*distinctive* relevant logics for which the rule is *admissible*.

Sect. 3 then shows that no semi-**t**-distinctive sublogic of **R** which validates excluded middle is conservatively extended by Boolean negation. It follows from this that none of the three "classical" relevant logics **T**, **E** and **R** strengthened by the Ackermann constant are conservatively extended by Boolean negation.

Sect. 4 and sect. 5 are on paracomplete relevant logics—relevant logics without excluded middle. Sect. 4 considers paracomplete **t**-distinctive relevant logics. It is shown that no such logic with reasoning by cases is conservatively extended by Boolean negation. Sect. 5 raises the question of whether this also carries over to paracomplete logics which are either merely semi-**t**-distinctive or for which reasoning by cases is not available. The question is, however, left unsettled, although it is shown that Giambrone and Meyer's claim in [9] to have settled this in the negative in the case of the semi-**t**-distinctive version of **TW** is incorrect. Sect. 6 then finally summarizes.

This is the third and last in a series of essays on Boolean negation and non-conservativeness pertaining to relevant logics. The first essay, [15], dealt with modal relevant logics, whereas the second essay, [16], dealt with the question whether the variable sharing property is always preserved when extending a logic with Boolean negation. Together the three essays paint a picture of relevant logics being quite often non-conservatively extended by Boolean negation. It should therefore be noted that many relevant logics in fact are conservatively extended by Boolean negation. Neither of the three papers make any effort to survey such proofs, however. The interested reader should consult [6], [9], [13] and [17].

2. Definitions and common lemmas

This section first gives some definitions of various relevant logics—see Tab. 1— as well as other notions and lemmas that will be useful throughout the paper. The notions of semi-**t**-distinctive and **t**-distinctive logics are introduced and discussed. As a start, then, all proofs in this essay will be standard Hilbert-style proofs:

Definition 1 (Hilbert proof)**.** *A Hilbert proof of a formula A from a set of formulas Γ in the logic* **L** *is defined to be a finite list $A_1, \ldots, A_n$ such that $A_n = A$ and every $A_{i \leq n}$ is either a member of Γ, a logical axiom of* **L***, or there is a set $\Delta \subseteq \{A_j | j < i\}$ such that $\Delta \vdash A_i$ is an instance of a rule of* **L***. The existential claim that there is such a proof is is written $\Gamma \vdash_{\mathbf{L}} A$.*

BB	A1–A5, R1–R7	**BBX**	**BB** +A12^b
B	**BB** +A6, +A7, −R5, −R6	**BBI**	**BB** +A12
DW	**B** +A8, −R7		
TW	**DW** +A9, +A10, −R3, −R4	**T**	**TW** +A12, +A13
EW	**TW** +R8	**E**	**T** +A14, +A15
RW	**TW** +A11	**R**	**T** +A11
$\mathbf{L}^{\mathbf{t}_1}$	**L** +**t**1, +**t**2.1	$\mathbf{L}^{\mathbf{t}_3}$	**L** +**t**1, +**t**2.3
$\mathbf{L}^{\mathbf{t}_2}$	$\mathbf{L}^{\mathbf{t}_1}$ +**t**2.2	$\mathbf{L}^{\mathbf{t}_4}$	**L** +**t**■, +**t**2.1, +**t**2.2
$C\mathbf{L}$	**L** +B1–B2		

Table 1. Definitions of various relevant logics

Definition 2 (Defined connectives)**.**

$$A \leftrightarrow B =_{df} (A \rightarrow B) \wedge (B \rightarrow A)$$
$$\Box C =_{df} (C \rightarrow C) \rightarrow C$$
$$\blacksquare C =_{df} \mathbf{t} \rightarrow C$$
$$\mathbf{f} =_{df} {\sim}\mathbf{t}$$
$$A \supset B =_{df} {\sim}A \vee B \qquad \textit{De Morgan material implication}$$
$$A \sqsupset B =_{df} \neg A \vee B \qquad \textit{Boolean material implication}$$

(A1)	$A \to A$	identity
(A2)	$A \to A \vee B$ and $B \to A \vee B$	$\vee$-introduction
(A3)	$A \wedge B \to A$ and $A \wedge B \to B$	$\wedge$-elimination
(A4)	$\sim\sim A \to A$	double negation elimination
(A5)	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$	distribution
(A6)	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$	strong lattice $\wedge$
(A7)	$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$	strong lattice $\vee$
(A8)	$(A \to \sim B) \to (B \to \sim A)$	contraposition axiom
(A9)	$(A \to B) \to ((B \to C) \to (A \to C))$	suffixing axiom
(A10)	$(A \to B) \to ((C \to A) \to (C \to B))$	prefixing axiom
(A11)	$A \to ((A \to B) \to B)$	assertion axiom
(A12$^\flat$)	$A \vee \sim A$	excluded middle
(A12)	$(A \to \sim A) \to \sim A$	reductio
(A13)	$(A \to (A \to B)) \to (A \to B)$	contraction axiom
(A14)	$((A \to A) \to B) \to B$	1. **E**-distinctive axiom
(A15)	$\Box A \wedge \Box B \to \Box(A \wedge B)$	2. **E**-distinctive axiom
(**t**1)	**t**	
(**t**$\blacksquare$)	$\blacksquare A \to A$	
(**t**2.1)	$\blacksquare(A \to A)$	
(**t**2.2)	$\blacksquare(A \vee \sim A)$	
(**t**2.3)	$A \vdash \blacksquare A$	
(B1)	$A \wedge \neg A \to B$	Boolean explosion axiom
(B2)	$A \to B \vee \neg B$	Boolean excl. middle axiom
(R1)	$A, B \vdash A \wedge B$	adjunction
(R2)	$A, A \to B \vdash B$	modus ponens
(R3)	$A \to B \vdash (B \to C) \to (A \to C)$	suffixing rule
(R4)	$A \to B \vdash (C \to A) \to (C \to B)$	prefixing rule
(R5)	$A \to B, A \to C \vdash A \to B \wedge C$	lattice $\wedge$-rule
(R6)	$A \to C, B \to C \vdash A \vee B \to C$	lattice $\vee$-rule
(R7)	$A \to \sim B \vdash B \to \sim A$	contraposition rule
(R8)	$A \to (B \to C), B \vdash A \to C$	δ
(R9)	$A, \sim A \vee B \vdash B$	γ

Definition 3 (*d*isjunctive extension)**.** $\mathbf{L}^d$*—the* disjunctive extension of **L***— is got from* **L** *by adding the disjunctive version of every primitive rule of* **L***, that is, if* $A_1, \ldots, A_n \vdash B$ *is such a primitive rule, then* $\mathbf{L}^d$ *has* $A_1 \vee C, \ldots, A_n \vee C \vdash B \vee C$ *as an additional the primitive rule.*

Not all relevant logics validate the meta-rule of *reasoning by cases*:

$$\text{(RbC)} \quad \frac{A \vdash C \qquad B \vdash C}{A \vee B \vdash C}.$$

However, an easy induction will suffice for showing that this meta-rule holds in any of the relevant logics dealt with in this paper if and only if the disjunctive versions of every primitive rule is derivable, that is, if and only if $\mathbf{L} = \mathbf{L}^d$. The sole purpose of introducing disjunctive rules in this paper is to make sure reasoning by cases holds and for that reason I'll sometimes state that some disjunctive version of a logic is got by adding reasoning by cases as a primitive rule.

Definition 4 (Admissible rule)**.** *A rule* $\Gamma \vdash A$ *is an* admissible rule *in* **L** *if it is the case that* $\varnothing \vdash_{\mathbf{L}} A$ *when* $\varnothing \vdash_{\mathbf{L}} B$ *for all* $B \in \Gamma$*.*

Definition 5 (Boolean extension)**.** $C\mathbf{L}$ *is called the* Boolean extension *of* **L***.*

Definition 6 (Conservative Extension)**.** *If* $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *are logics over, respectively, languages* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *such that* $\mathcal{L}_1 \subseteq \mathcal{L}_2$*, then* $\mathbf{L_2}$ *conservatively extends* $\mathbf{L_1}$ *if* $\varnothing \vdash_{\mathbf{L}_1} A$ *for every* $\mathcal{L}_1$*-formula* A *such that* $\varnothing \vdash_{\mathbf{L}_2} A$*.*

Definition 7 (**t**-distinctive logic)**.** *A logic is called* **t**-distinctive *if* **t**2.3 *is derivable.*

Definition 8 (Semi-**t**-distinctive logic)**.** *A logic is called* semi-**t**-distinctive *if it is the case that* $\varnothing \vdash_{\mathbf{L}} A$ *if and only if* $\varnothing \vdash_{\mathbf{L}} \blacksquare A$*.*

Thus a logic is semi-**t**-distinctive if **t**2.3 is an *admissible* rule in it.

Before we delve into the non-conservative, let me say something about truth-constants in relevant logics. There are two truth-constants which are often added to relevant logics: the intensional Ackermann constant **t**, and the extensional Church constant $\top$. The Church constant is always axiomatized by the single axiom $A \to \top$. Neither of these constants are definable in relevant logics. In every logic $C\mathbf{L}$, however, $\top$ can simply be defined as $A \vee \neg A$ for some A. This is not the case with the Ackermann constant which remains undefinable even in $C\mathbf{R}$. Note that *the Ackermann constant* is ambiguous. **t** is sometimes glossed as the conjunction of every logical truth/theorem, and sometimes as the conjunction of every actual truth. The first gives rise to a *modal* interpretation of the conditional in the sense that $\mathbf{t} \to A$ can be interpreted as "necessarily A". The latter gloss on **t** obviously does not invite such an interpretation. These interpretations are warranted by the fact, as we shall see, that $\blacksquare A$ is a theorem if and only if A is a theorem of any $\mathbf{L}^{\mathbf{t}_1}$-logic in which neither reasoning by cases nor excluded middle are primitive logical principles and that the same is true for any $\mathbf{L}^{\mathbf{t}_2}$ logic in which reasoning by cases is not a primitive principle, but excluded middle is. The rule $\blacksquare A \vdash A$ is derivable in all such logics. However, the converse rule, $A \vdash \blacksquare A$, is generally not derivable in $\mathbf{L}^{\mathbf{t}_1}$- and $\mathbf{L}^{\mathbf{t}_2}$-logics. The only logics in Table 1 for which this holds are the extensions of $\mathbf{EW}^{\mathbf{t}_1}$ since such logics validate Ackermann's δ rule which is easily seen to suffice for deriving **t**2.3: $\mathbf{t} \to (A \to A), A \vdash \mathbf{t} \to A$.

The algebraic interpretation of **t** in **t**-distinctive logics is as the least designated element. A distinctive **t** needs to be interpreted as this element, whereas a non-distinctive **t** only needs to be interpreted as the least designated element assigned to any formula $A \to A$ in the case of **t**2.1 and $A \vee {\sim}A$ in case **t**2.2 is added. This, as we shall see, need not be the least designated element. Both these two interpretations of **t**—as the conjunction of all logical truths/theorems and the conjunction of all actual truths—and these two semantic interpretations are sometimes run together; this is especially tempting when one only cares about logical theorems.[1] However, I will show in the next section that this leads to error. The essential difference in the case of logical theorems is whether **t**2.3 applies to assumptions for reasoning by cases or not: if **t**2.3 is a mere *admissible* rule, then it will not in general be applicable in sub-derivations using reasoning by cases. If **t**2.3 is derivable, however, then one may freely use it also in such sub-derivations.

Lemma 1. $A \to B \vdash_{BB} C \wedge A \to C \wedge B$

Proof. Left for the reader. □

Lemma 2. $A \to B \vdash_{BB} C \vee A \to C \vee B$

Proof. Left for the reader. □

Lemma 3 (Boolean facts)**.** *The following are all theorems and derivable rules of C***BB***:*

$$\begin{array}{ll} (BF1) & \neg\neg A \leftrightarrow A \\ (BF2) & \neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B) \\ (BF7) & A \to B \vdash \neg B \to \neg A \end{array}$$

Proof. Left for the reader. □

Lemma 4. $A \to B \vdash_{\mathbf{L}^{\mathbf{t}_{i\leq 4}}} \blacksquare(A \to B)$

Proof.

(1)	$A \to B$	assumption
(2)	$(A \to A) \to (A \to B)$	1, R4
(3)	$\mathbf{t} \to (A \to A)$	**t**2.1
(4)	$\blacksquare(A \to B)$	2, 3, transitivity + def. of $\blacksquare$

□

Lemma 5 (Ackermann's lemma)**.** *If* **L** *is any explicitly named logic different than* **BBX** *in Table 1, then* $\vdash_{\mathbf{L}^{\mathbf{t}_{i\leq 4}}} A$ *if and only if* $\vdash_{\mathbf{L}^{\mathbf{t}_{i\leq 4}}} \blacksquare A$.[2]

Proof. The *if* part is trivial since $\vdash_{\mathbf{L}^{\mathbf{t}_{i\leq 4}}} \mathbf{t}$. The *only if* part is an induction on proofs. First consider the axioms. All such, except for **t**1, are →-formulas. Lem. 4, together with the fact that $\mathbf{t} \to \mathbf{t}$ is an instance of axiom A1, then, ensures that all axioms are accounted for. Now assume that it holds for the premises of a rule. If the rule in question is R3–R8, then the conclusion C

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[1] A prime example is found in the appendix to Routley's *Exploring Meinong's Jungle and Beyond*:

> `But t - which is interpreted as the conjunction of all theorems, and semantically marks out the class of logically regular worlds - can be added conservatively to DLQ through the two-way rule: A ⇔ t → A.` ([20, p. 923])

A world is *regular* if it validates all the logical theorems of the given logic. DLQ is first-order version of the logic **DW** plus the axioms A12 and $(A \to B) \wedge (B \to C) \to (A \to C)$.

[2] This was first proven by Ackermann in [1, p. 125].

of the rule is a provable →-formula, and so Lem. 4 again ensures that $\blacksquare C$ is also provable. For R1, assume that $\blacksquare A$ and $\blacksquare B$ are provable. By R1 itself it follows that $\blacksquare A \wedge \blacksquare B$ is provable, from which $\blacksquare(A \wedge B)$ follows using R5. For R2, assume that $\blacksquare A$ and $\blacksquare(A \to B)$ are provable, that is, $\mathbf{t} \to A$ and $\mathbf{t} \to (A \to B)$ are both provable. By R3 it follows that $(A \to B) \to (\mathbf{t} \to B)$ is provable, and so the transitivity of → suffices for ensuring that $\blacksquare\blacksquare B$ is also provable, and so using **t**1 and R2 one can then derive $\blacksquare B$. □

Lemma 6. $\mathbf{BBX}^{\mathbf{t}_2}$ *is semi-***t***-distinctive, i.e.* $\vdash_{\mathbf{BBX}^{\mathbf{t}_2}} A$ *if and only if* $\vdash_{\mathbf{BBX}^{\mathbf{t}_2}} \blacksquare A$.

Proof. Similar to Lem. 5; **t**2.2 is needed to ensure that $\blacksquare(A \vee {\sim}A)$. □

Corollary 1. *If the rule* $A \vdash \blacksquare A$ *is admissible in* **L** *and* **L**′ *is got by adding axioms the main connective of which is* →*, or rules for which the conclusion has* → *as the main connective, then* $A \vdash \blacksquare A$ *is admissible in* **L**′ *as well. In particular,* $A \vdash \blacksquare A$ *is admissible in* C**L** *if it is admissible in* **L***.*

Proof. Immediate from Lem. 4 □

3. Results on logics with excluded middle

Lemma 7. $A \vdash_{C\mathbf{BBX}} {\sim}\neg A$

Proof.

(1)	$A \wedge \neg A \to {\sim}\neg A$	B2
(2)	$A \wedge {\sim}\neg A \to {\sim}\neg A$	A3
(3)	$(A \wedge \neg A) \vee (A \wedge {\sim}\neg A) \to {\sim}\neg A$	1, 2, R6
(4)	$\neg A \vee {\sim}\neg A$	A12$^\flat$
(5)	A	assumption
(6)	$A \wedge (\neg A \vee {\sim}\neg A)$	4, 6, R1
(7)	$(A \wedge \neg A) \vee (A \wedge {\sim}\neg A)$	6, A5
(8)	${\sim}\neg A$	3, 7, R1

□

Corollary 2. $\vdash_{C\mathbf{BBX}^{\mathbf{t}_2}} \mathbf{t} \to {\sim}\neg\mathbf{t}$

Proof. Axiom **t**1 yields **t** as a logical theorem from which Lem. 7 yields ${\sim}\neg\mathbf{t}$. $\mathbf{t} \to {\sim}\neg\mathbf{t}$ now follows since $A \vdash \blacksquare A$ is admissible in $C\mathbf{BBX}^{\mathbf{t}_2}$ (Lem. 6 and Cor. 1). □

Lemma 8. $\vdash_{C\mathbf{BBX}^{\mathbf{t}_2}} A \to \mathbf{t} \vee \mathbf{f}$[3]

[3]Meyer showed in [12, p. 324] that $\top \leftrightarrow \mathbf{t} \vee \mathbf{f}$ is a theorem of $C\mathbf{R}^{\mathbf{t}_1\top}$ and noted that this is not a provable formula in $\mathbf{R}^{\mathbf{t}_1\top}$. He also showed that $\top \leftrightarrow ((\mathbf{f} \to \mathbf{t}) \to \mathbf{t})$ is a theorem of $C\mathbf{R}^{\mathbf{t}_1\top}$. This latter formula, however fails to be a theorem of weaker logics such as $C\mathbf{E}^{\mathbf{t}_1\top}$ and $C\mathbf{RWX}^{\mathbf{t}_1\top}$.

Proof.

(1)	$\neg\mathbf{t} \to \neg\mathbf{t} \lor \mathbf{f}$	A2
(2)	$\neg(\neg\mathbf{t} \lor \mathbf{f}) \to \neg\neg\mathbf{t}$	2, Lem. 3 (BF7)
(3)	$\neg\mathbf{t} \land \neg(\neg\mathbf{t} \lor \mathbf{f}) \to \neg\mathbf{t} \land \neg\neg\mathbf{t}$	3, Lem. 1
(4)	$\neg\mathbf{t} \land \neg\neg\mathbf{t} \to \neg A$	B1
(5)	$\neg\mathbf{t} \land \neg(\neg\mathbf{t} \lor \mathbf{f}) \to \neg A$	3, 4, transitivity
(6)	$\mathbf{t} \to {\sim}\neg\mathbf{t}$	Cor. 2
(7)	$\mathbf{t} \to ({\sim}\neg\mathbf{t} \land \mathbf{t})$	6, fiddling
(8)	$(\neg\mathbf{t} \lor \mathbf{f}) \to \mathbf{f}$	7, R7 + De Morgan-fiddle
(9)	$\neg\mathbf{f} \to \neg(\neg\mathbf{t} \lor \mathbf{f})$	8, Lem. 3 (BF7)
(10)	$\neg\mathbf{t} \land \neg\mathbf{f} \to \neg\mathbf{t} \land \neg(\neg\mathbf{t} \lor \mathbf{f})$	9, Lem. 1
(11)	$\neg\mathbf{t} \land \neg\mathbf{f} \to \neg A$	5, 10, transitivity
(12)	$\neg(\mathbf{t} \lor \mathbf{f}) \to \neg\mathbf{t} \land \neg\mathbf{f}$	Lem. 3 (BF2)
(13)	$\neg(\mathbf{t} \lor \mathbf{f}) \to \neg A$	11, 12, transitivity
(14)	$A \to \neg\neg(\mathbf{t} \lor \mathbf{f})$	13, Lem. 3 (BF7)
(15)	$A \to \mathbf{t} \lor \mathbf{f}$	14, Lem. 3 (BF1), transitivity

□

$\mathbf{BBX}^{\mathbf{t}_2}$ is a sublogic of the three-valued logic $\mathbf{RM}_3^{\mathbf{t}_1}(= \mathbf{RM}_3^{d\mathbf{t}_3} = \mathbf{RM}_3^{d\mathbf{t}_4})$, which is the logic of the algebra displayed in Fig. 1.[4] That algebra, like others in this essay, will have its partial order displayed. Conjunction and disjunction are to be interpreted as infimum and supremum over this ordering. Alongside there will be a matrix which shows how the conditional and the negation(s) are to be interpreted. A subset $\mathcal{T}$ of the algebra—a *filter* to be precise—is selected as the set of *designated elements*. A rule holds in an algebra just in case the conclusion is assigned a value in $\mathcal{T}$ when all its premises are. The Ackermann constant **t** is assigned some element in $\mathcal{T}$—the least such if the logic is **t**-distinctive or the least such value assigned to any theorem if it is not, but is semi-**t**-distinctive. I also list how to interpret the relevant formulas so as to make the model a counter-model to the intended formula.[5]

Theorem 1. *C***L** *is not a conservative extension of* **L** *for any logic* $\mathbf{BBX}^{\mathbf{t}_2} \leqslant \mathbf{L} \leqslant \mathbf{RM}_3^{\mathbf{t}_1}$.

Proof. This follows from Lem. 8 together with the fact that $A \to \mathbf{t} \lor \mathbf{f}$ is not valid in the semantics for $\mathbf{RM}_3^{\mathbf{t}_1}$. □

Corollary 3. $C\mathbf{BBI}^{\mathbf{t}_{i\leq 4}}$, $C\mathbf{T}^{\mathbf{t}_{i\leq 4}}$, $C\mathbf{E}^{\mathbf{t}_{i\leq 4}}$, *and* $C\mathbf{R}^{\mathbf{t}_{i\leq 4}}$ *are not conservative extensions of their Boolean-free counterparts.*

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[4] $\mathbf{RM}_3^{\mathbf{t}_1}$ is got from $\mathbf{R}^{\mathbf{t}_1}$ by adding the axioms $A \to (A \to A)$ and $A \lor (A \to B)$. See Brady's paper [3] for soundness and completeness proofs.

[5] All models depicted in this paper have been found with the help of `MaGIC`—an acronym for *Matrix Generator for Implication Connectives*—which is an open source computer program created by John K. Slaney ([22]). I have made heavy use of both it as well as William McCune's theorem prover/model generator package *Prover9/Mace4* ([10]) in arriving at the results reported in this essay.

$\mathcal{T} = \{1, 2\}$
$[\![\mathbf{t}]\!] = 1$
$[\![A]\!] = 2$

$$0 \rightarrow 1 \rightarrow 2$$

$\rightarrow$	0	1	2	$\sim$
0	2	2	2	2
1	0	1	2	1
2	0	0	2	0

FIGURE 1. $\mathbf{RM}_3^{\mathbf{t}_1}$ counter-model to $A \rightarrow \mathbf{t} \vee \mathbf{f}$

Thus virtually non of the standard semi-**t**-distinctive paraconsistent relevant logics with excluded middle can be conservatively extended by Boolean negation. Note that the presence of **t** here is vital as for instance both C**T** and C**R** are conservative extensions of, respectively, **T** and **R** (cf. [9] and [13]).

4. RESULTS ON **t**-DISTINCTIVE PARACOMPLETE LOGICS

The previous section showed that quite a few logics with excluded middle fail to be conservatively extended by Boolean negation. How, then, about *paracomplete* logics—logics without excluded middle? As the following two proofs show, such logics also often fail to be conservatively extended by Boolean negation provided the Ackermann constant is present. The first proof regards logics for which γ is a derivable rule, whereas the latter relies instead on reasoning by cases being available.

Theorem 2. *C**L** is a non-conservative extension for any logic between $\mathbf{BB}^{\mathbf{t}_3}[\gamma]$ and the four-valued logic $\mathbf{RM}_4$.*

Proof. The first part of the proof is to the effect that $\vdash_{C\mathbf{L}} \mathbf{t} \wedge \mathbf{f} \rightarrow A$ for any such logic:

(1)	$\mathbf{t}$	$\mathbf{t}1$
(2)	$\mathbf{t} \rightarrow \mathbf{f} \vee \neg\mathbf{f}$	B2
(3)	$\mathbf{f} \vee \neg\mathbf{f}$	1, 2, R2
(4)	$\neg\mathbf{f}$	1, 3, γ
(5)	$\mathbf{t} \rightarrow \neg\mathbf{f}$	4, $\mathbf{t}2.3$
(6)	$\mathbf{t} \wedge \mathbf{f} \rightarrow \mathbf{f} \wedge \neg\mathbf{f}$	5, R5 + fiddling
(7)	$\mathbf{f} \wedge \neg\mathbf{f} \rightarrow A$	B1
(8)	$\mathbf{t} \wedge \mathbf{f} \rightarrow A$	6, 7, transitivity

The last part is simply to note that $\mathbf{t} \wedge \mathbf{f} \rightarrow A$ fails in the $\mathbf{RM}_4$-algebra depicted in Fig. 2.[6] □

[6]See [7] for more information on **RM** and its finitely-valued extensions.

$\mathcal{T} = \{2, 3\}$
$[\![\mathbf{t}]\!] = 2$
$[\![A]\!] = 0$

$3 \uparrow 2 \uparrow 1 \uparrow 0$

$\rightarrow$	0	1	2	3	$\sim$
0	3	3	3	3	3
1	0	2	2	3	2
2	0	1	2	3	1
3	0	0	0	3	0

FIGURE 2. $\mathbf{RM}_4$-counter-model to $\mathbf{t} \wedge \mathbf{f} \rightarrow A$

Relevant logics are most often taken to be *paraconsistent* and so do not have γ as a derivable rule, although some logics do.[7] The next result concerns relevant logics without excluded middle as well as γ. The proof, however, relies on reasoning by cases being available, and unlike the other results which showed that either $A \rightarrow \mathbf{t} \vee \mathbf{f}$ or $\mathbf{t} \wedge \mathbf{f} \rightarrow A$ are derivable in the Boolean extensions, and therefore involve both the De Morgan negation $\sim$ as well as the Boolean one, the next result shows that $A \vee (A \wedge \mathbf{t} \rightarrow B)$ is derivable for arbitrary A's and B's. This, then, shows also that the non-conservativeness is in this case not due to the interaction between the two negations.

Lemma 9. $\vdash_{\mathbf{CBB}^{dt_3}} A \vee (A \wedge \mathbf{t} \rightarrow B)$

Proof.

(1)	$A \vee \neg A$	B2 + fiddling
(2)	$\quad A$	1, 1. assumption for RbC
(3)	$\quad A \vee (A \wedge \mathbf{t} \rightarrow B)$	2, fiddling
(4)	$\quad \neg A$	1, 2. assumption for RbC
(5)	$\quad \mathbf{t} \rightarrow \neg A$	4, R10
(6)	$\quad A \rightarrow \neg \mathbf{t}$	5, Lem. 3
(7)	$\quad A \wedge \mathbf{t} \rightarrow \mathbf{t} \wedge \neg \mathbf{t}$	6, fiddling
(8)	$\quad \mathbf{t} \wedge \neg \mathbf{t} \rightarrow B$	B1
(9)	$\quad A \wedge \mathbf{t} \rightarrow B$	7, 8, transitivity
(10)	$\quad A \vee (A \wedge \mathbf{t} \rightarrow B)$	9, fiddling
(11)	$A \vee (A \wedge \mathbf{t} \rightarrow B)$	1, 2–3, 4–10, RbC

□

Theorem 3. *No logic between the positive fragment of* $\mathbf{BB}^{dt_3}$ *and the three-valued Łukasiewicz logic* $\mathbf{Ł}_3$ is conservatively extended by Boolean negation.

Proof. This follows from Lem. 9 together with the fact that $\mathbf{t}$ is definable as $A \rightarrow A$ for an arbitrary A in $\mathbf{Ł}_3$, that reasoning by cases is derivable for $\mathbf{Ł}_3$

[7]Both Ackermann's Π', as well as the logic $\Pi'_{\mathbf{E}}$ presented in [14] are worth mentioning as exceptions. See the latter paper for a discussion of why relevant logics ended up being wrongly viewed as inherently paraconsistent.

([11, lem. 3.54]), and that $A \vee (A \wedge \mathbf{t} \to B)$ fails in the Ł_3-algebra depicted in Fig. 3.

$\mathcal{T} = \{2\}$
$[\![\mathbf{t}]\!] = 2$
$[\![A]\!] = 1$
$[\![B]\!] = 0$

$2 \uparrow 1 \uparrow 0$

$\to$	0	1	2	$\sim$
0	2	2	2	2
1	1	2	2	1
2	0	1	2	0

FIGURE 3. Ł_3 counter-model to $A \vee (A \wedge \mathbf{t} \to B)$

□

This, then, shows that none of the standard contraction-free relevant logics $\mathbf{B}^{dt_3}$, $\mathbf{DW}^{dt_3}$, $\mathbf{TW}^{dt_3}$, $\mathbf{EW}^{dt_1} = \mathbf{EW}^{dt_3}$ and $\mathbf{RW}^{dt_1} = \mathbf{RW}^{dt_3}$ are conservatively extended by Boolean negation.

Could one improve upon this result? Note first that $\mathbf{t}$ is definable as $A \to A$ for arbitrary A in Ł_3. This is essentially because Ł_3 validates the weakening axiom $A \to (B \to A)$; any logic which validates even the weakening rule $A \vdash B \to A$ will be able to define $\mathbf{t}$ in this way.[8] For relevant logics, however, $\mathbf{t}$ is not definable. The distinction between $\mathbf{t}$-distinctive and semi-$\mathbf{t}$-distinctive logics holds only for logics without the δ rule. With regards to $\mathbf{t}$, then, one may hope to improve the result by showing that also semi-$\mathbf{t}$-distinctive logics—sublogics of $\mathbf{TW}^{dt_1}$—are non-conservatively extended by Boolean negation.

The other path along which one might hope to do better is by eliminating the need to assume reasoning by cases as a primitive rule; one may here hope to find a proof of non-conservativeness which does not rely on RbC, or simply by showing that RbC, or at least sufficient parts of it, is in fact

[8]Restall showed in [17, thm. 20] that $\neg A \to {\sim}A$, and therefore $A \vee {\sim}A$ is provable in $C\mathbf{RWK}$—the Boolean extension of $\mathbf{RW}$ augmented by the weakening axiom—and that the extension is therefore non-conservative. It can be shown that this also holds for $\mathbf{DWK}$—$\mathbf{DW}$ augmented by the weakening axiom. To show this, first consider the following lemma that $\neg A \to (A \to B)$ is a theorem of $C\mathbf{DWK}$:

(1)	$\neg A \to (A \to \neg A)$	weakening axiom
(2)	$\neg A \to (A \to A)$	weakening axiom + fiddling
(3)	$\neg A \to (A \to A) \wedge (A \to \neg A)$	1, 2, R5
(4)	$\neg A \to (A \to (A \wedge \neg A))$	3, A6 + fiddling
(5)	$(A \to A \wedge \neg A) \to (A \to B)$	B1 + R4
(6)	$\neg A \to (A \to B)$	4, 5, transitivity

If we first instantiate $\neg A$ for A and ${\sim}A$ for B, we get $\neg\neg A \to (\neg A \to {\sim}A)$. Now instantiate A for A and ${\sim}{\neg}A$ for B so that we get $\neg A \to (A \to {\sim}{\neg}A)$ and by using the contraposition axiom therefore $\neg A \to (\neg A \to {\sim}A)$. Putting these together using R6, we then get $\neg A \vee \neg\neg A \to (\neg A \to {\sim}A)$ and therefore that $\neg A \to {\sim}A$. Fiddling then yields $A \vee \neg A \to A \vee {\sim}A$ and therefore $A \vee {\sim}A$ since $A \vee \neg A$ is a theorem. Since excluded middle is not a theorem of Ł_3, it follows that $C\mathbf{L}$ is a non-conservative extension of $\mathbf{L}$ for any logic between $\mathbf{DWK}$ and Ł_3.

derivable. Specifically, then, is it possible to prove non-conservativity for any of the logics depicted in Fig. 4? Before ending this section I will show that $\mathbf{RW}^{\mathbf{t}_1}$ is in fact a $\mathbf{t}$-distinctive logic for which reasoning by cases holds, and so $C\mathbf{RW}^{\mathbf{t}_1}$ fails to be a conservative extension of $\mathbf{RW}^{\mathbf{t}_1}$. The next and final section delves into these problems for the other logics. Some issues will be clarified, but, regrettably, the question whether any of them are conservatively extended by Boolean negation will be left unsettled. Giambrone and Meyer claim in [9] that $\mathbf{TW}^{\mathbf{t}_1}$ fails to be conservatively extended by Boolean negation. I will show that Giambrone and Meyer's proof of this, however, is faulty.

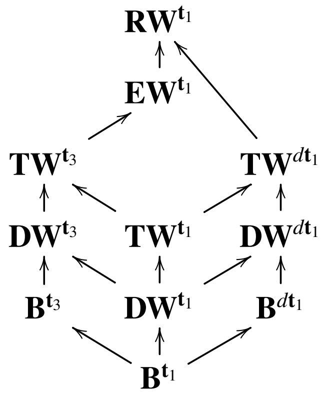


FIGURE 4.

Logics with only modus ponens and adjunction as primitive rules often have a deduction theorem. Such a deduction theorem is then often sufficient for showing that reasoning by cases holds for a logic. Brady showed that any axiomatic extension of **TW** by →-formulas has a deduction theorem ([5, thm. 9]) on the form

$$\Gamma \vdash_{\mathbf{L}} B \Longleftrightarrow \varnothing \vdash_{\mathbf{L}} A_1 \to (A_2 \to (\ldots \to (A_n \to B)\ldots))$$

where $n \geq 1$ and all the A_i's are members of, or conjunctions thereof, of the set $\Gamma \cup \{A \mid \varnothing \vdash_{\mathbf{L}} A\}$. Using this one can then show that reasoning by cases is derivable $\mathbf{RW}^{\mathbf{t}_1}$ and $C\mathbf{RW}^{\mathbf{t}_1}$.[9] Furthermore, $\vdash_{\mathbf{RW}^{\mathbf{t}_1}} A \to (\mathbf{t} \to A)$, and so $\mathbf{RW}^{\mathbf{t}_1}$ is $\mathbf{t}$-distinctive. Since $C\mathbf{RW}^{\mathbf{t}_1}$ is an axiomatic extension by →-formulas of **RW** we therefore get that:

Corollary 4. *$C\mathbf{RW}^{\mathbf{t}_1}$ is not a conservative extension of* $\mathbf{RW}^{\mathbf{t}_1}$.[10]

[9] Reasoning by cases for $\mathbf{RW}^{\mathbf{t}_1}$ is shown in [11, Lem. 3.54]. A strait forward proof that the disjunctive versions of the rules of $\mathbf{RW}^{\mathbf{t}_1}$ are derivable is given Restall in [18, thm. 5.2].

[10] This, and Cor. 3, contradicts Restall's claim in [19, cor. 13.17] that in fact both $C\mathbf{R}^{\mathbf{t}_1}$ and $C\mathbf{RW}^{\mathbf{t}_1}$ are conservative extensions of $\mathbf{R}^{\mathbf{t}_1}$ and $\mathbf{RW}^{\mathbf{t}_1}$. Restall simply overlooks the fact that a counter-model using a reduced frame extended to evaluate the Boolean negation will only have one point at which $\mathbf{t}$ is satisfied. However, if such a model is to be a counter-model to $\mathbf{t} \wedge \mathbf{f} \to A$, that point will have to satisfy both $\mathbf{t}$ and $\mathbf{f}$, but not A. However, since

«Parenthetical remark. Let me digress a bit to note an important consequence of having a deduction theorem for the concept of conservative extension. Def. 6 above defines a logic to conservatively extend another basically if the logical theorems in the original language is unaltered upon the extension. It doesn't require *non-logical theories* to be unaltered. In the case of Boolean negation, it would therefore be possible that $C\mathbf{L}$ to conservatively extend $\mathbf{L}$, yet there be formulas $\Gamma \cup \{A\}$ over the Boolean free language such that $\Gamma \vdash_{C\mathbf{L}} A$, yet $\Gamma \nvdash_{\mathbf{L}} A$. Let's make this precise:

Definition 9 (Strong Conservative Extension)**.** *If* $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *are logics over, respectively, languages* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *such that* $\mathcal{L}_1 \subseteq \mathcal{L}_2$*, then* $\mathbf{L_2}$ strongly conservatively extends $\mathbf{L_1}$ *if for every set of* $\mathcal{L}_1$*-formula* $\Gamma \cup \{A\}$ *such that* $\Gamma \vdash_{\mathbf{L}_2} A$*,* $\Gamma \vdash_{\mathbf{L}_1} A$*.*

For logics with a **t**-deduction theorem, however, strong conservative extension is entailed by mere conservative extension:

Lemma 10. *If a deduction theorem holds for* $\mathbf{L}$ *in the form*

$$\Gamma \vdash_{\mathbf{L}} B \Longleftrightarrow \varnothing \vdash_{\mathbf{L}} A_1 \to (\ldots \to (A_n \to B))$$

for some n where all A_i*'s are either members of* $\Gamma \cup \{\mathbf{t}\}$*, or conjunctions thereof, then it is strongly conservatively extended by a logic* $\mathbf{L}'$ *for which the same deduction theorem holds if it is conservatively extended by it.*

Proof. Assume that $\mathbf{L}'$ conservatively extends $\mathbf{L}$ and that the requisite deduction theorem holds for them. Now assume that $\Gamma \vdash_{\mathbf{L}'} B$ where $\Gamma \cup \{B\}$ are in the language of $\mathbf{L}$. Since the deduction theorem holds, there are A_i's such that $\vdash_{\mathbf{L}'} A_1 \to (\ldots \to (A_n \to B))$ where also $A_1 \to (\ldots \to (A_n \to B))$ is a formula in the language of $\mathbf{L}$. Since $\mathbf{L}'$ conservatively extends $\mathbf{L}$, it now follows that $\vdash_{\mathbf{L}} A_1 \to (\ldots \to (A_n \to B))$. Since all A_i's are either members of $\Gamma \cup \{\mathbf{t}\}$, or conjunctions thereof, it follows that $\Gamma \vdash_{\mathbf{L}} A_i$ for all i's, and therefore that $\Gamma \vdash_{\mathbf{L}} B$ which ends the proof. □

Corollary 5. *Let* $\mathbf{L}$ *be any axiomatic* $\to$*-extension of* $\mathbf{TW}^{\mathbf{t}_1}$*. Then* $\mathbf{L}'$ *conservatively extends* $\mathbf{L}$ *if and only if it strongly conservatively extends* $\mathbf{L}$*.*

Proof. Looking at the proof of Lem. 5 it is easy to see that any such logic will be semi-**t**-distinctive. The result now follows from Brady's deduction theorem for $\mathbf{TW}^{\mathbf{t}_1}$. *End parenthetical.»* □

5. Semi-t-distinctive paracomplete logics without reasoning by cases

There are two main features that went into Lem. 9, namely reasoning by cases and the **t**-distinctive rule $A \vdash \mathbf{t} \to A$. As we have seen, reasoning by cases is provable of $\mathbf{RW}^{\mathbf{t}_1}$ and of the disjunctive logics $\mathbf{TW}^{d\mathbf{t}_1}$, $\mathbf{DW}^{d\mathbf{t}_1}$ and $\mathbf{B}^{d\mathbf{t}_1}$. It is not, however, provable for any of the other logics on display in Fig. 4:

γ is a derivable rule of $C\mathbf{R}$ ([15, thm. 4]), that point will be closed under γ, and therefore also satisfy A. Thus the flattening-technique appealed to in Restall's proof of his Boolean negation theorem ([19, thm. 13.16]) fails to work in the presence of **t**.

Lemma 11. *Reasoning by cases is not provable for any of the sublogics of* $\mathbf{EW}^{\mathbf{t}_1}$ *displayed in Fig. 4.*

Proof. That RbC does not hold for **TW** was shown in [15, cor. 9]. That proof is easily seen to also cover all the sublogics of $\mathbf{TW}^{\mathbf{t}_3}$ in Fig. 4, so we're left with showing that RbC is not true of $\mathbf{EW}^{\mathbf{t}_1}$ either.

Restall noted ([18, thm. 5.5] that the formula $\sim A \vee ((A \to B) \to B)$ is not a theorem of **E** augmented by the δ rule.[11] Evidently it would have been if reasoning by cases held true. Restall does not display a counter-model, but simply appeals to that MaGIC can be used to generate one. Like Restall, I will in this case simply leave it as a MaGICal exercise for the reader. Note, then, that such a counter-model to $\sim A \vee ((A \to B) \to B)$ is then also a **EW** counter-model to the inference $A \vee \sim A \vdash \sim A \vee ((A \to B) \to B)$ which would also have held if reasoning by cases held true of **EW**. Such a counter-model can also be generated which also validates all of $\mathbf{EW}^{\mathbf{t}_1}$, which therefore ends the proof. □

Thus the proof of Lem. 9 does not extend to any of the sublogics of $\mathbf{EW}^{\mathbf{t}_1}$ Fig. 4. Even the instance $\mathbf{f} \vee (\mathbf{f} \wedge \mathbf{t} \to A)$ fails to be a theorem of $\mathbf{EW}^{\mathbf{t}_1}$:

Lemma 12. $\mathbf{f} \vee (\mathbf{f} \wedge \mathbf{t} \to A)$ *is not a theorem of* $\mathbf{EW}^{\mathbf{t}_1}$

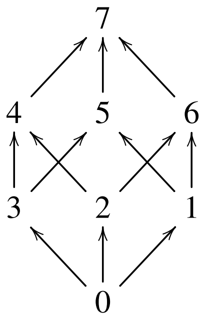
Proof. The model displayed in Fig 5 is a model for $C\mathbf{TW}^{d\mathbf{t}_1}$. However, by replacing its set of designated elements $\mathcal{T} = \{2, 4, 6, 7\}$ with the set $\mathcal{T}' = \{4, 7\}$ we get a model for $\mathbf{EW}^{\mathbf{t}_1}$. The same evaluation of **t** and A yields, then, a counter-model to $\mathbf{f} \vee (\mathbf{f} \wedge \mathbf{t} \to A)$ pertaining to $\mathbf{EW}^{\mathbf{t}_1}$. □

We are therefore left to investigate the disjunctive logics $\mathbf{TW}^{d\mathbf{t}_1}$, $\mathbf{DW}^{d\mathbf{t}_1}$ and $\mathbf{B}^{d\mathbf{t}_1}$. Even with RbC thus forced to hold, however, one will not be able to derive $A \vee (A \wedge \mathbf{t} \to B)$—even when A is the formula **f**—in either $C\mathbf{TW}^{\mathbf{t}_1}$ or $C\mathbf{TW}^{d\mathbf{t}_1}$ as the counter-model in Fig. 5 shows.

Lem. 9 utilized not only reasoning by cases, but also the rule $A \vdash \mathbf{t} \to A$ to get $\mathbf{t} \to \neg A$ from the RbC-assumption $\neg A$. This, it seems, will not be possible to do in mere semi-**t**-distinctive logics such as $C\mathbf{TW}^{d\mathbf{t}_1}$ since assumptions for reasoning by cases are not established logical theorems. In fact, the model in Fig. 5 falsifies precisely the needed inference in that $\neg\mathbf{f}$ is assigned the designated value 6, but $[\![\mathbf{t} \to \neg\mathbf{f}]\!] = [\![4 \to 6]\!] = 0$. As the matter stands, therefore, it seems that one needs *both* reasoning by cases as well as a **t**-distinctive Ackermann constant to obtain the sentence $\mathbf{f} \vee (\mathbf{f} \wedge \mathbf{t} \to B)$.

[11]Restall credits Brady's [4] for first noting this. The δ rule $A \to (B \to C), B \vdash A \to C$ is interderivable in all relevant logics with the *assertion rule* $A \vdash (A \to B) \to B$. Note that Restall—in [18], as well as in [17, p. 509] and [19, p. 305]—takes **E** to have the assertion rule as a derivable rule. δ is admissible in **E**, and so theorem-wise, this is harmless. However, it is not a derivable rule of **E** as that logic was formulated by Anderson and Belnap, and so the logics are not identical; reasoning by cases *does* hold for **E** but does not hold for $\mathbf{E}[\delta]$.

$\mathcal{T} = \{2, 4, 6, 7\}$
$[\![\mathbf{t}]\!] = 4$
$[\![A]\!] = [\![\mathbf{f}]\!]$
$[\![B]\!] = 0$



$\to$	0	1	2	3	4	5	6	7	$\sim$	$\neg$
0	7	7	7	7	7	7	7	7	7	7
1	0	4	0	0	0	4	4	7	6	4
2	0	0	4	0	4	0	4	7	5	5
3	0	0	0	4	4	4	0	7	4	6
4	0	0	0	0	4	0	0	7	3	1
5	0	0	0	0	0	4	0	7	2	2
6	0	0	0	0	0	0	4	7	1	3
7	0	0	0	0	0	0	0	7	0	0

FIGURE 5. Counter-model to $\mathbf{f} \vee (\mathbf{f} \wedge \mathbf{t} \to B)$ for $C\mathbf{TW}^{d\mathbf{t}_1}$

The inspiration for this essay came from Giambrone and Meyer's essay [9] where theorem 8 and its preceding proof reads:[12]

> The following non-theorem of $\mathbf{R}^{\mathbf{t}_1}$ is valid in the $C\mathbf{TW}^{\mathbf{t}_1}$ semantics: $\mathbf{f} \vee (\mathbf{f} \wedge \mathbf{t} \to A)$. So:
>
> THEOREM 8. $C\mathbf{TW}^{\mathbf{t}_1}$, $C\mathbf{RW}^{\mathbf{t}_1}$, $C\mathbf{T}^{\mathbf{t}_1}$, and $C\mathbf{R}^{\mathbf{t}_1}$ are not conservative extensions of $\mathbf{TW}^{\mathbf{t}_1}$, $\mathbf{RW}^{\mathbf{t}_1}$, $\mathbf{T}^{\mathbf{t}_1}$, and $\mathbf{R}^{\mathbf{t}_1}$, respectively. ([9, p. 13])

First of all note that the latter three logics are covered by Cor. 4 and Cor. 3. That leaves their claim that $C\mathbf{TW}^{\mathbf{t}_1}$ is not a conservative extension of $\mathbf{TW}^{\mathbf{t}_1}$ which contradicts my claim that the model in Fig. 5 is a $C\mathbf{TW}^{\mathbf{t}_1}$-counter-model to $\mathbf{f} \vee (\mathbf{f} \wedge \mathbf{t} \to A)$.

Now Giambrone and Meyer demanded that the Boolean extension of **TW** should be got by adding B1, $A \to (B \to (C \vee \neg C))$ and the axiom $(A \to B) \sqsupset (A \sqsupset B)$. The latter axiom is interesting as it suffices for deriving reasoning by cases for any axiomatic extension of **TW**.[13] Thus reasoning by cases holds for Giambrone and Meyer's strengthened $C\mathbf{TW}^{\mathbf{t}_1}$.[14] However, the counter-model to $\mathbf{f} \vee (\mathbf{f} \wedge \mathbf{t} \to A)$ in Fig. 5 also validates these two stronger Boolean principles. Their claim, therefore, is wrong. Where, then, do they go astray? In order to show non-conservativeness, one needs to show that there is a sentence in the original language which is only derivable in the

[12]The citation is slightly modified to fit the current nomenclature. Note that there is a typesetting error in the essay as the displayed formula claimed to be a non-theorem of $\mathbf{R}^{\mathbf{t}_1}$ is in fact $\mathbf{t} \vee (\mathbf{f} \wedge \mathbf{t} \to A)$. The essay [9] is basically copied from §1.7 and §1.8 of Giambrone's PhD distertation ([8]), where the corresponding formula reads $\sim$t $\vee$ ($\sim$t & t $\to A$) ([8, thm. 1.8.5]). Meyer was, together with Routley, Giambrone's supervisor and §§1.7–1.8 is in [8, p. ii] acknowledged as the joint work of Giambrone and Meyer.

[13]See [15, §6.1] for a proof and a short discussion of whether $(A \to B) \sqsupset (A \sqsupset B)$ is too strong for logics like **TW**.

[14]The proof of [15, cor. 8] relies on there being no more primitive rules than adjunction and modus ponens. Note, then, that Giambrone and Meyer initially take the fusion connective to be part of the axiomatization of **TW** and its **t**- and $\neg$-extensions, although this is later retracted ([9, p. 11]). The proof of [15, cor. 8] would still hold, however, if the fusion rules $A \circ B \to C \dashv\vdash A \to (B \to C)$ are interpreted to be only logical theorem-preserving rules.

extended logic. The obvious lacuna in Giambrone and Meyer's proof is therefore the step from noting that $\mathbf{f} \vee (\mathbf{f} \wedge \mathbf{t} \to A)$ is valid in the semantics for $C\mathbf{TW}^{\mathbf{t}_1}$ they set forth—which it is—to the fact that $\mathbf{f} \vee (\mathbf{f} \wedge \mathbf{t} \to A)$ is derivable, i.e. they need a completeness proof for $C\mathbf{TW}^{\mathbf{t}_1}$. They never try to establish this, settling for claiming soundness. The stated semantic clause for **t** is in fact the semantic clause for the stronger **t**-distinctive truth constant, and so soundness does hold, but completeness will not be possible to prove for $C\mathbf{TW}^{\mathbf{t}_1}$ (nor for $C\mathbf{T}^{\mathbf{t}_1}$).[15] Their claim, therefore, is at best that $C^\sharp\mathbf{TW}^{\mathbf{t}_3}$—where $C^\sharp\mathbf{TW}^{\mathbf{t}_3}$ is got by adding the extra Boolean axiom $(A \to B) \sqsupset (A \sqsupset B)$ to $C\mathbf{TW}^{\mathbf{t}_3}$—is a non-conservative extension of $\mathbf{TW}^{\mathbf{t}_3}$. Since reasoning by cases is provable for $C^\sharp\mathbf{TW}^{\mathbf{t}_3}$ ([15, cor. 8]), this follows in fact from Thm. 3. For $\mathbf{TW}^{\mathbf{t}_1}$, $\mathbf{TW}^{d\mathbf{t}_1}$ and $\mathbf{TW}^{\mathbf{t}_3}$, however, it is still an open question whether adding B1 and B2 yields a conservative extension or not.

Of course, that $\mathbf{f} \vee (\mathbf{f} \wedge \mathbf{t} \to A)$ is not derivable in $C\mathbf{TW}^{d\mathbf{t}_1}$ or $C\mathbf{EW}^{\mathbf{t}_1}$ does not show that $C\mathbf{TW}^{d\mathbf{t}_1}$ and $C\mathbf{EW}^{\mathbf{t}_1}$ are conservative extensions of, respectively, $\mathbf{TW}^{d\mathbf{t}_1}$ and $\mathbf{EW}^{\mathbf{t}_1}$. Nor that they are not. I end this section, therefore, with the following open question:

Open Problem. *Are any of the logics different from* $\mathbf{RW}^{\mathbf{t}_1}$ *in Fig. 4 conservatively extended by Boolean negation?*

6. Summary

Many relevant logics can be conservatively extended by Boolean negation. This essay, however, shows that many such relevant logics fail to be conservatively extended by Boolean negation if the Ackermann constant is taken to be part of the logic.

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