Hitting Topological Minor Models in Planar Graphs is Fixed Parameter Tractable

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Abstract

For a finite collection of graphs $\mathcal{F}$, the $\mathcal{F}$-TM-Deletion problem has as input an n-vertex graph G and an integer k and asks whether there exists a set $S \subseteq V(G)$ with $|S| \leq k$ such that $G \setminus S$ does not contain any of the graphs in $\mathcal{F}$ as a topological minor. We prove that for every such $\mathcal{F}$, $\mathcal{F}$-TM-Deletion is fixed parameter tractable on planar graphs. In particular, we provide an $f(h,k) \cdot n^2$ algorithm where h is an upper bound to the vertices of the graphs in $\mathcal{F}$.

Keywords: Topological minors, irrelevant vertex technique, treewidth, vertex deletion problems

1 Introduction

1.1 The $\mathcal{P}$-deletion problem and its variants

In general, a $\mathcal{P}$-deletion problem is determined by some graph class $\mathcal{P}$ and asks, given an n-vertex graph G and an integer k, whether G can be transformed to a graph in $\mathcal{P}$ after the deletion of k vertices. In other words, the class $\mathcal{P}$ represents some desired property that we want to impose to the input graph after deleting k vertices. This is a general graph modification problem with great expressive power as it encompasses many problems, depending on the choice of the property $\mathcal{P}$. Unfortunately for most instantiations of $\mathcal{P}$, this problem is not expected to admit a polynomial time algorithm. Lewis and Yannakakis showed in [23] that for any non-trivial and hereditary graph class $\mathcal{P}$, the $\mathcal{P}$-vertex deletion problem is NP-complete. Given this hardness result, an attractive alternative is to consider the standard parameterized version of the problem, called p-$\mathcal{P}$-deletion where the parameter is the number k of vertex deletions. In this case the challenge is to investigate for which instantiations of $\mathcal{P}$, p-$\mathcal{P}$-deletion is fixed parameter tractable (or, in short, is FPT), i.e., it can be solved by an $O_k(n^c)$-time algorithm[1] (or FPT-algorithm), for some constant c. There is a long line of research on this general question. In many cases, this concerns particular properties and possible optimizations of the contribution of k in the function hidden in the "O_k" notation (see e.g. [5]). However, it is interesting to notice that FPT-algorithms exist for general families of properties. In this direction the more general (and compact) results concern properties $\mathcal{P}$ that can be characterized by the exclusion of some finite set $\mathcal{F}$ of graphs (i.e., of size bounded by some constant h) with respect to some partial ordering relation $\leq$. We define

$$\mathcal{P}_{\mathcal{F},\leq} = \{G \mid \forall H \in \mathcal{F} : H \not\leq G\}$$

and ask whether p-$\mathcal{P}_{\mathcal{F},\leq}$-deletion is FPT. Let us now consider the general status of this problem for the main known instantiations of the partial ordering relation $\leq$.

(1) $\leq$ is the *contraction*[2] relation: then there are graphs H such that $\mathcal{P}_{\{H\},\leq}$-deletion is NP-complete even for the case where $k = 0$. For instance one may take H to be the path on 4 vertices, as indicated in [7]. Using the terminology of fixed parameter complexity, this implies that there are choices of $\mathcal{F}$ such that p-$\mathcal{P}_{\mathcal{F},\leq}$-deletion is para-NP-complete.

(2) $\leq$ is the *induced minor*[3] relation: as in the previous case there are choices of $\mathcal{F}$ such that p-$\mathcal{P}_{\mathcal{F},\leq}$-deletion is para-NP-complete. For instance, one may consider $\mathcal{F}$ to contain the graph in [12, Theorem 4.3].

(3) $\leq$ is the *subgraph* or the *induced subgraph* relation: because of the result of Cai in [8], p-$\mathcal{P}_{\mathcal{F},\leq}$-deletion

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[1] Let $(x_1, \ldots, x_l) \in \mathbb{N}^l$ and $\chi, \psi : \mathbb{N} \to \mathbb{N}$. We use the notation $\chi(n) = O_{x_1,\ldots,x_l}(\psi(n))$ to denote that there exists a computable function $\phi : \mathbb{N}^l \to \mathbb{N}$ such that $\chi(n) = O(\phi(x_1, \ldots, x_l) \cdot \psi(n))$.

[2] A graph G is a *contraction* of a graph G' if G can be obtained from G by applying edge contractions.

[3] A graph G is an *induced minor* of a graph G' if G can be obtained from some contraction of G' after removing vertices.

is FPT, for every $\mathcal{F}$. In particular, the result in [8] implies an $O(h^k n^{h+1})$-time algorithm for both these problems. However, if instead we parameterize $\mathcal{P}_{\mathcal{F},\leq}$-DELETION by h, then there are instantiations of $\mathcal{F}$ for which the problem is W[1]-hard even for $k = 0$: just take $\mathcal{F} = \{K_h\}$ to generate the p-CLIQUE problem.

(4) $\leq$ is the *minor*[4] relation: again p-$\mathcal{P}_{\mathcal{F},\leq}$-DELETION is FPT, for every $\mathcal{F}$. To see this, observe that, for every k, the set of YES-instances of this problem is closed under taking of minors. On the other hand, Robertson and Seymour [27] proved that graphs are well-quasi-ordered with respect to the minor relation. These two facts together imply that there is a finite set $\mathcal{B}_k$ (whose size depends on k and h) such that (G, k) is a YES-instance if and only if G contains no graph in $\mathcal{B}_k$ as a minor. As minor checking for a graph on c vertices can be done in $O_c(n^3)$-steps [26], we derive the existence of an $O_{k,h}(n^3)$-step algorithm.

1.2 Our contribution. Interestingly, we are not aware of other partial ordering relations where p-$\mathcal{P}_{\mathcal{F},\leq}$-DELETION is FPT, for every $\mathcal{F}$. Among the possible candidates, the most relevant one is the topological minor relation, denoted by $\preceq$: a graph H is a *topological minor* of a graph G if G contains as a subgraph some subdivision[5] of H.

In this paper we make a first step on the study of the p-$\mathcal{P}_{\mathcal{F},\preceq}$-DELETION problem, also called $\mathcal{F}$-TM-DELETION, and we conjecture that it is FPT. Unfortunately, there are no known meta-algorithmic results, similar to those of the case of minors, that permit a straightforward resolution of this conjecture, as graphs are not well-quasi-ordered under topological minors. On the positive side, there is an algorithm that checks topological minor containment in $O_h(n^3)$-time [15] and this result would be a special case of our conjecture for the case where $k = 0$. In this paper we prove that this conjecture is true, when we are restricted to planar graphs. Moreover, we develop results and techniques that may serve as the base of its full resolution.

Given a finite set $\mathcal{F}$ of graphs, we use $h(\mathcal{F})$ for the maximum size of a graph in $\mathcal{F}$. We also write $\mathcal{F} \not\preceq G$ to denote the fact that none of the graphs in $\mathcal{F}$ is a topological minor of G. We define $\mathbf{p}_{\mathcal{F}}(G) = \min\{k \mid \exists S \subseteq V(G) : |S| \leq k \wedge \mathcal{F} \not\preceq G \setminus S\}$. The main result of this paper is the following:

THEOREM 1.1. *There exists an algorithm that given a finite set of graphs $\mathcal{F}$, a $k \in \mathbb{N}$, and a planar graph G, outputs whether $\mathbf{p}_{\mathcal{F}}(G) \leq k$ in $O_{h,k}(n^2)$ steps, where $h = h(\mathcal{F})$.*

We stress that the algorithm of Theorem 1.1 can be straightforwardly modified so to output a set S of size $\leq k$ that intersects all models of the graphs in $\mathcal{F}$.

1.3 High level description of our algorithm Our main approach towards proving Theorem 1.1 is the application of the so-called *irrelevant vertex technique*. This technique was introduced for the first time by Roberston and Seymour in [26] for the design of an FPT-algorithm for the DISJOINT PATHS problem, parameterized by the number of terminals. Subsequently, its was applied, in diverse ways, for the design of FPT-algorithms for several graph-theoretical problems and is now considered as a powerful technique of parameterized algorithm design [1, 11, 13, 14, 16–21, 24, 25]. We also refer to [10, Chapter 7] for a high-level overview of the irrelevant vertex technique. The general algorithmic paradigm of the irrelevant vertex technique takes advantage of some structural characteristic of the input graph in order to detect, in FPT-time, some vertex, called *irrelevant*, whose removal from G generates an equivalent instance of the problem. By recursing on the produced equivalent instance we end up with a graph where the structural parameter is bounded (by some function of k), a fact that permits the resolution of the problem with other techniques – typically by dynamic programming. In most of the times, this structural parameter is treewidth (see §2 for the formal definition) and this is the one that we use in this paper. Towards proving Theorem 1.1, the application of the irrelevant vertex technique is based on the following theorem.

THEOREM 1.2. *There exists a function $f_1 : \mathbb{N} \to \mathbb{N}$, and an algorithm with the following specifications:*

Find_Irrelevant_Vertex(k, h, G)
Input: $k, h \in \mathbb{N}_{\geq 0}$ *and an n-vertex planar graph G*
Output:

1. *an (irrelevant) vertex $v \in V(G)$ such that, for every graph class $\mathcal{F}$ where $h(\mathcal{F}) \leq h$, it holds that $\mathbf{p}_{\mathcal{F}}(G) \leq k \iff \mathbf{p}_{\mathcal{F}}(G \setminus v) \leq k$ or*
2. *a tree decomposition of G of width at most $f_1(h) \cdot k$.*

Moreover, this algorithm runs in $O_{k,h}(n)$ steps.

After applying the algorithm of Theorem 1.2 at most n times, the problem is reduced to instances of bounded treewidth. As topological minor containment can be expressed by a MSOL formula and vertex deletion to some MSOL definable property can also be expressed in MSOL, it follows from the Theorem of Courcelle [9] (see also [2, 6, 30]) that the problem for reduced

[4] A graph G is an *minor* of a graph G' is G is the contraction of some subgraph of G'.

[5] A graph G is a *subdivision* of a graph G' if G can be obtained from G' if we replace its edges by paths with the same endpoints.

instances can be solved in $O_{k,h}(n)$ steps. Theorem 1.1 follows. The version of the algorithm that outputs a certificate of the solution follows again from the version of the Theorem of Courcelle that returns such a certificate, if exists.

In the rest of this section we give an outline on how Theorem 1.2 is proved. All combinatorial concepts used in this description are presented in an intuitive way; formal definitions can be found in §2. Given a tuple of variables $\mathbf{x} = (x_1, \ldots, x_q)$ by the term $\mathbf{x}$-*big/small* we refer to a quantity that is lower/upper bounded by some (unbounded) function of $\mathbf{x}$. Alternatively, we use the term $\mathbf{x}$-*many/few* that is defined analogously. We work on some embedding of G in the plane.

Walls and annuli. An important combinatorial object is the one of a r-*wall* , as the one in Figure 1, that can be seen as the union of r horizontal paths intersected by r vertical paths. The layers of a wall W are defined as indicated in Figure 1.

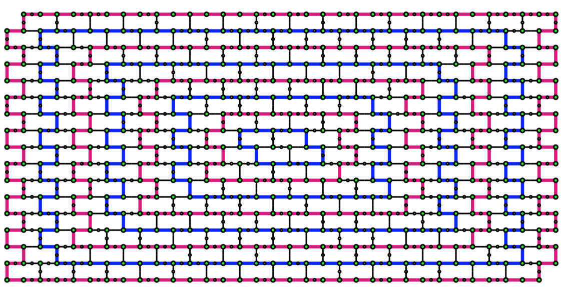

Figure 1: A 17-wall and its 8 layers.

We call the outermost layer *perimeter* of the wall W. Using a result of [13] we know that if the treewidth of a planar graph is (k, h)-big, then G contains a (k, h)-big wall such that the subgraph of G, called *the compass* of W, inside the closed disk defined by the perimeter of W has (k, h)-small treewidth (see Proposition 2.1). This additional property will permit us to answer queries expressed by MSOL sentences on subgraphs of the compass of W.

The next step is to detect some more structure in the wall W that is intuitively depicted in the left side of Figure 2. We first distinguish the collection $\mathcal{C}$ of the (k, h)-many outmost layers, drawn in yellow, and then we consider in the rest of W a packing of (k, h)-many (h)-big walls, drawn in green. This is done in Lemma 2.1.

We now work on the "annulus" of the (k, h)-many outer layers of W. For this, it is convenient to see those cycles as "crossed" by a collection $\mathcal{P}$ of disjoint paths (that are monotone subpaths of the horizontal/vertical paths of W) called *rails*. We call this system of cycles

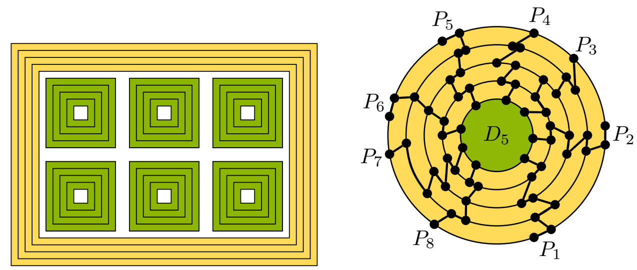

Figure 2: Left: The partition of a wall into a yellow annulus and several green subwalls. Right: An example of a $(5, 8)$-railed annulus depicted in yellow; its inner disk D_5 is depicted in green.

and rails *railed annulus*, denoted by $\mathcal{A} = (\mathcal{C}, \mathcal{P})$. (See the right side of Figure 2 for an example of a railed annulus with 5 cycles and 8 rails).

Taming topological minor models. Notice that if H is a topological minor of a graph G, then this is materialized by a pair (M, T) where M is a subgraph of G and T is a set of vertices of M, called *branches*, such that all vertices of $V(M)\setminus T$ have degree 2. We say that (M, T) is a *topological minor model* of H in G if a graph isomorphic to H is created after dissolving in M all vertices in $V(M) \setminus T$ (which means deleting every such vertex and making its two neighbors adjacent). For simplicity, assume that $\mathcal{F} = \{H\}$ and recall that $\mathbf{p}_{\mathcal{F}}(G) \leq k$ if there is a set $S \subseteq V(G), |S| \leq k$, called from now on *solution set*, that intersects all topological minor models of H in G.

Our next aim is to analyze how topological minor models of H may cross the cycles and the rails of a railed annulus $\mathcal{A} = (\mathcal{C}, \mathcal{P})$. For this we dedicate §5 to the proof of a general theorem stating that if the branches of (M, T) are situated outside the annulus and the annulus is (h)-big then it is possible to find an alternative "rail-tamed" model (M', T') of G, whose intersection with the "middle cycle" of $\mathcal{A}$ consists only of (h)-few rail vertices. We refer to this theorem as the *"model taming theorem"* (Theorem 2.1). As it has independent combinatorial interest, we present it in a slightly more general form that will appear useful on further algorithmic applications. The proof of this theorem is technical and it is based on the so-called *unique linkage theorem* by Robertson and Seymour in [28, 29] (also appeared in an alternative form as the *unique-linkage* theorem in [22]).

Representations of topological minor models. Using the model-taming theorem, we can pick a (h)-small collection $\mathcal{P}'$ of the rails of $\mathcal{A}$ for which the following holds: for every topological minor model (M, T) of H that crosses $\mathcal{A}$, there is a disk Δ bounded by some cycle C of $\mathcal{A}$ and a "tamed" (through $\mathcal{P}'$) version (M', T') of (M, T) that *represents* (M, T) in the sense that a set of vertices that are "not so close" to C, intersects $M \cap \Delta$ iff the same set intersects $M' \cap \Delta$. From now on we refer to the instantiations of $M' \cap \Delta$ as the *inner tamed models* of $\mathcal{A}$ and we can see them as models representing the "inner part" of all annulus-crossing models.

Reducing the solution space. The next step is to compute, for every cycle C of $\mathcal{A}$, a set S_C of at most (k, h)-many vertices intersecting each possible inner tamed model of $\mathcal{A}$ (it is possibe that S_C is an empty set) This computation can be done in $O_{k,h}(n)$-time as this question can be expressed in MSOL and concerns subgraphs of the compass of W that has (k, h)-small treewidth. Let Δ_{in} be the disk bounded by the innermost cycle of $\mathcal{C}$ (cycle C_5 in Figure 2). We then compute $S_{\text{in}} = \Delta_{\text{in}} \cap (\bigcup_{C \in \mathcal{C}} S_C)$ and observe that S_{in} has (k, h)-small size. Based on the fact that the inner tamed models represent the inner part of all models crossing $\mathcal{A}$ and the fact that all these models are intersected by subsets of at most k vertices whose restriction in Δ_{in} is in S_{in}, we prove that if $G \setminus S$ does not contain any topological minor model of H, then we can replace $S \cap \Delta_{\text{in}}$ by vertices of S_{in} to obtain a new solution that is not larger than S (Lemma 3.1). This is an important restriction of the solution space of the problem in what concerns its intersection with Δ_{in}. As the (h)-big sub-walls packed inside Δ_{in} are (k, h)-many, there is a sub-wall whose compass can be avoided by all possible solution sets. In the above, H can might any graph on h vertices, however its is more convenient to think about some specific planar graph H in $\mathcal{F}$.

Finding an irrelevant vertex. We now fix our attention to the solution-free compass of some (h)-big subwall of W. Once again, we see this wall as a railed annulus $\mathcal{A}'$ and use the model taming theorem in order to represent all ways topological minor models of H can "invade" the compass of W by tamed topological models going through the rails of $\mathcal{A}'$. This in turn permit us to detect a vertex v of the solution-free compass of W such that if a solution set S intersects a topological minor model that contains v, then it should also intersect some representation of it that avoids v, therefore v is irrelevant (Lemma 3.2).

2 Definitions and preliminaries

We denote by $\mathbb{N}$ the set of all non-negative integers. Given an $n \in \mathbb{N}$, we denote by $\mathbb{N}_{\geq n}$ the set containing all integers equal or greater than n. Given two integers x and y we define by $[x, y] = \{x, x+1, \ldots, y-1, y\}$. Given an $n \in \mathbb{N}_{\geq 1}$, we also define $[n] = \{1, \ldots, n\}$. Let U be a set, $r \in \mathbb{N}_{\geq 1}$, and $\mathcal{A} = [A_1, \ldots, A_r] \subseteq (2^U)^r$, $\mathcal{B} = [B_1, \ldots, B_r] \subseteq (2^U)^r$. We say that $\mathcal{A} \subseteq \mathcal{B}$ if for all $i \in [r]$, $A_i \subseteq B_i$. Also, if $S \subseteq U$ we denote $\mathcal{A} \cap S = [A_1 \cap S, \ldots, A_r \cap S]$.

2.1 Basic concepts on Graphs All graphs in this paper are undirected, finite, and they do not have loops or multiple edges. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs, then we denote $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ and $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. Also, given a graph G and a set $S \subseteq V(G)$, we denote by $G \setminus S$ the graph obtained if we remove from G the vertices in S, along with their incident edges. Given a graph G, we say that the pair (A, B) is a *separation* of G if $A \cup B = V(G)$ and there is no edge in G with one endpoint in $A \setminus B$ and the other in $B \setminus A$. A *path* (*cycle*) in a graph G is a connected subgraph with all vertices of degree at most (exactly) 2. A path is *trivial* if it has only one vertex and is empty if it is the empty graph (i.e., the graph with empty vertex set).

Partially disk-embedded graphs. A *closed disk* (resp. *open disk*) Δ is a set homeomorphic to the set $\{(x, y) \mid x^2 + y^2 \leq 1\}$ (resp. $\{(x, y) \mid x^2 + y^2 < 1\}$). A *disk* of Δ is a closed or an open disk of Δ. We use $\mathbf{bor}(\Delta)$ to denote the boundary of Δ and $\mathsf{int}(\Delta)$ to denote the open disk $\Delta \setminus \mathbf{bor}(\Delta)$. When we embed a graph G in the plane or in a disk, we treat G as a set of points. This permits us to make set operations operations between graphs and sets of points. We say that a graph G is *partially disk-embedded in some closed disk* Δ, if there is some subgraph K of G that is embedded in Δ such that $\mathbf{bor}(\Delta)$ is a cycle of K and $(V(G) \cap \Delta, V(G) \setminus \mathsf{int}(\Delta))$ is a separation of G. From now on, we use the term *partially* Δ*-embedded graph* G to denote that a graph G is partially disk-embedded in some closed disk Δ. We also call the graph K *compass* of the partially Δ-embedded graph G and we always assume that we accompany a partially Δ-embedded graph G together with an embedding of its compass in Δ that is the set $G \cap \Delta$.

Let G be a partially Δ-embedded graph and let $\mathcal{C} = [C_1, \ldots, C_r]$, $r \geq 2$, be a collection of vertex-disjoint cycles of the compass of G. We say that the sequence $\mathcal{C}$ is a Δ*-nested sequence of cycles* of G if every C_i is the boundary of an open disk D_i of Δ such that $\Delta \supseteq D_1 \supseteq \cdots \supseteq D_r$. From now on, each Δ-nested sequence $\mathcal{C}$ will be accompanied with the sequence $[D_1, \ldots, D_r]$ of the corresponding open disks as well as the sequence $[\overline{D}_1, \ldots, \overline{D}_r]$ of their closures. Given $x, y \in [r]$ where $x \leq y$, we call the set $\overline{D}_x \setminus D_y$ (x, y)*-annulus* of $\mathcal{C}$ and we denote it by $\mathsf{ann}(\mathcal{C}, x, y)$. Finally, we say that $\mathsf{ann}(\mathcal{C}, 1, r)$ is the *annulus* of $\mathcal{C}$ and we denote it by $\mathsf{ann}(\mathcal{C})$.

Grids and Walls. Let $k, r \in \mathbb{N}$. The $(k \times r)$-*grid* is the Cartesian product of two paths on k and r vertices, respectively. An *elementary r-wall*, for some odd $r \geq 3$, is the graph obtained from a $(2r \times r)$-grid with vertices (x, y), $x \in [2r] \times [r]$, after the removal of the "vertical" edges $\{(x, y), (x, y + 1)\}$ for odd $x + y$, and then the removal of all vertices of degree one. Notice that, as $r \geq 3$, an elementary r-wall is a planar graph that has a unique (up to topological isomorphism) embedding in the plane such that all its finite faces are incident to exactly six edges. The *perimeter* of an elementary r-wall is the cycle bounding its infinite face. Given an elementary wall $\overline{W}$, a *vertical path* of $\overline{W}$ is one whose vertices, in ordering of appearance, are $(i, 1), (i, 2), (i + 1, 2), (i + 1, 3), (i, 3), (i, 4), (i + 1, 4), (i + 1, 5), (i, 5), \ldots, (i, r - 2), (i, r - 1), (i + 1, r - 1), (i + 1, r)$, for some $i \in \{1, 3, \ldots, 2r - 1\}$. Also a *horizontal path* of $\overline{W}$ is one whose vertices, in ordering of appearance, are $(1, j), (2, j), \ldots, (2r, j)$, for some $j \in [2, r - 1]$, or $(1, 1), (2, 1), \ldots, (2r - 1, 1)$ or $(2, r), (2, r), \ldots, (2r, r)$. (see Figure 1).

An *r-wall* is any graph W obtained from an elementary r-wall $\overline{W}$ after subdividing edges. We call the vertices that were added after the subdivision operations *subdivision vertices*. The *perimeter* of W is the cycle of W whose non-subdivision vertices are the vertices of the perimeter of $\overline{W}$. Also, a vertical (resp. horizontal) path of W is a subdivided vertical (resp. horizontal) path of $\overline{W}$. An *r'-subwall* W' of a wall W is any r'-wall that is a subgraph of W and whose horizontal/vertical paths are subpaths of the horizontal/vertical paths of W.

A subgraph W of a graph G is called a *wall* of G if W is an r-wall for some odd $r \geq 3$ and we refer to r as the *height* of the wall W. The *layers* of an r-wall W are recursively defined as follows. The first layer of W is its perimeter. For $i = 2, \ldots, (r-1)/2$, the i-th layer of W is the $(i - 1)$-th layer of the subwall W' obtained from W after removing from W its perimeter and all occurring vertices of degree one. Notice that each $(2r + 1)$-wall has r layers.

Treewidth. A *tree decomposition* of a graph G is a pair (T, χ) where T is a tree and $\chi : V(T) \to 2^{V(G)}$ such that

1. $\bigcup_{t \in V(T)} \chi(t) = V(G)$;
2. for every edge e of G there is a $t \in V(T)$ such that $\chi(t)$ contains both endpoints of e and
3. for every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

The *width* of (T, χ) is defined as

$$\mathbf{w}(T, \chi) := \max\left\{ |\chi(t)| - 1 \mid t \in V(T)\right\}.$$

The *treewidth of G* is defined as $\mathbf{tw}(G) := \min\left\{\mathbf{w}(T, \chi) \;\middle|\; (T, \chi) \text{ is a tree decomposition of } G\right\}$.

The following result from [13] intuitively states that given a $q \in \mathbb{N}$ and a graph G with "big" enough treewidth, we can find a q-wall of G whose compass has "small" enough treewidth.

PROPOSITION 2.1. ([13]) *There exists a constant c_1 and an algorithm with the following specifications:*

Find_Wall(G, q)
Input: a planar graph G and a $q \in \mathbb{N}$.
Output:

1. *A q-wall W of G whose compass has treewidth at most $c_1 \cdot q$ or*
2. *A tree decomposition of G of width at most $c_1 \cdot q$.*

Moreover, this algorithm runs in $O_q(n)$ steps.

2.2 Railed annuli

Let $r \in \mathbb{N}_{\geq 3}$ and $q \in \mathbb{N}_{\geq 3}$. Assume also that r is an odd number. An (r, q)-*railed annulus* of a partially Δ-embedded graph G is a pair $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ where $\mathcal{C} = [C_1, \ldots, C_r]$ is a Δ-nested collection of cycles of G and $\mathcal{P} = [P_1, \ldots, P_q]$ is a collection of pairwise vertex-disjoint paths in G such that

- For every $j \in [q]$, $P_j \subseteq \mathsf{ann}(\mathcal{C})$.
- For every $(i, j) \in [r] \times [q]$, $C_i \cap P_j$ is a non-empty path, that we denote $P_{i,j}$.

We refer to the paths of $\mathcal{P}$ as the *rails* of $\mathcal{A}$ and to the cycles of $\mathcal{C}$ as the *cycles* of $\mathcal{A}$.

Let $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ be an (r, q)-railed annulus of a partially Δ-embedded graph G. We call $\overline{D}_r$ (resp. $\overline{D}_1$) the *inner (resp. outer) disk* of $\mathcal{A}$. We also extend the notion of an annulus and we say that the *annulus* of $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ is the annulus of $\mathcal{C}$.

We now prove the following lemma which intuitively states that there is an algorithm that given a "big enough" wall, outputs a collection of railed annuli whose number and size will be useful in the proof of Theorem 1.2.

LEMMA 2.1. *There exists a function $f_2 : \mathbb{N}^3 \to \mathbb{N}$ and an algorithm with the following specifications:*

Find_Collection_of_Annuli(x, y, z, Δ, G, W)
Input: two odd integers $x, y \in \mathbb{N}_{\geq 3}$, an integer $z \in \mathbb{N}$, a partially Δ-embedded graph G and a q-wall W of the compass of G whose perimeter is the boundary of Δ and such that $q \geq f_2(x, y, z)$.
Output: a collection $\mathfrak{A} = \{\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_z\}$ of railed annuli of the compass of G such that

- $\mathcal{A}_0$ *is a* (x,x)*-railed annulus whose outer disk is* Δ *and whose inner disk is* Δ',
- *for* $i \in [z]$, $\mathcal{A}_i$ *is a* (y,y)*-railed annulus of* $G \cap \Delta'$, *and*
- *for every* $i,j \in [z]$ *where* $i \neq j$, *the outer disk of* $\mathcal{A}_i$ *and the outer disk of* $\mathcal{A}_j$ *are disjoint.*

Moreover, this algorithm runs in $O(n)$ *steps and* $f_2(x,y,z) = O(x + y\sqrt{z})$.

Proof. Let $y' := y + \lceil (y-2)/4 \rceil$ and assume that y' is an odd integer (otherwise, make it odd by adding 1) and let $f_2(x,y,z) = x + \max\{\lceil (x-2)/4 \rceil, \lceil \sqrt{z}/2 \rceil \cdot y'\} + 1$. We argue that the following holds:

Claim: Let $p \in \mathbb{Z}_{\geq 3}$. If H is an h-wall of G, where $h \geq p + \lceil (p-2)/4 \rceil$, then H contains a (p,p)-railed annulus $\mathcal{A} = (\mathcal{C}, \mathcal{P})$, where $\mathcal{C} = [C_1, \ldots, C_p]$ and for every $i \in [p]$, C_i is the i-th layer of H.

Proof of Claim: Let H be an h-wall of G, where $h \geq p + \lceil (p-2)/4 \rceil$. We define the Δ-nested collection $\mathcal{C} = [C_1, \ldots, C_p]$ of cycles of G, where, for every $i \in [p]$, C_i is the i-th layer of H. Let $\hat{\mathcal{P}}$ be the collection of the vertical and horizontal paths of H that intersect C_p. Observe that for every $i \in [p-1]$, every path in $\hat{\mathcal{P}}$ also intersects C_i and that $\hat{\mathcal{P}} \cap \mathsf{ann}(\mathcal{C})$ is a collection of pairwise-vertex disjoint paths of G. Also, notice that since $h - p \geq \lceil (p-2)/4 \rceil$ then $\hat{\mathcal{P}} \cap \mathsf{ann}(\mathcal{C})$ contains at least p paths. Let $\mathcal{P} := [P_1, \ldots, P_p]$ be a subset of $\hat{\mathcal{P}} \cap \mathsf{ann}(\mathcal{C})$. Then, $\mathcal{P}$ is a collection of pairwise vertex-disjoint paths of G and it holds that for every $j \in [p]$, $P_j \subseteq \mathsf{ann}(\mathcal{C})$ and for every $(i,j) \in [p] \times [p]$, $C_i \cap P_j$ is a non-empty path. Therefore, H contains a (p,p)-railed annulus $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ of G and the claim follows.

Following the claim above, for $H := W$, $h := q$, and $p := x$, since $q \geq x + \lceil (x-2)/4 \rceil$, we deduce the existence of a (x,x)-railed annulus A_0 whose inner disk is $\overline{D}_x$ and whose outer disk is $\overline{D}_1$ - that is Δ. Observe that since $q - x \geq \lceil \sqrt{z}/2 \rceil \cdot y' + 1$, then there exists an r-wall $\hat{W}$ of G for some odd $r \in \mathbb{Z}_{\geq 3}$ such that $r \geq \lceil \sqrt{z}/2 \rceil \cdot y'$ and $\hat{W} \subseteq G \cap \overline{D}_x$.

Now, notice that $\hat{W}$ contains a collection $\mathcal{W} = \{W'_1, \ldots, W'_z\}$ of z y'-subwalls of W such that, for every $i,j \in [z], i \neq j$, $K(W'_i) \cap K(W'_j) = \emptyset$. Therefore, for every $i \in [z]$, applying again the claim above for $H := W'_i$, $h := y'$ and $p := y$, we deduce the existence of a (y,y)-railed annulus $\mathcal{A}_i$ of W'_i. Furthermore, for every $i,j \in [z], i \neq j$, recall that $K(W'_i) \cap K(W'_j) = \emptyset$ which implies that the outer disk of $\mathcal{A}_i$ and the outer disk of $\mathcal{A}_j$ are disjoint. The proof concludes by setting $\mathfrak{A} = \{\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_z\}$. ∎

2.3 Rerouting topological minors

We say that (M,T) is a tm*-pair* if M is a graph, $T \subseteq V(M)$, and all vertices in $V(M) \setminus T$ have degree two. We denote by $\mathsf{diss}(M,T)$ the graph obtained from M by dissolving all vertices in $V(M) \setminus T$. A tm*-pair* of a graph G is a tm-pair (M,T) where M is a subgraph of G. Given two graphs H and G, we say that a tm-pair (M,T) of G, is a *topological minor model of H in G* if H is isomorphic to $\mathsf{diss}(M,T)$. We call the vertices in T *branch* vertices of (M,T).

Topological minor models in railed annuli. Let G be a partially Δ-embedded graph, let H be a graph, $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ be a (r,q)-railed annulus of G. Let $r = 2t+1$. Let also $s \in [r]$ where $s = 2t'+1$. Given some $I \subseteq [q]$, we say that a topological minor model (M,T) of H in G is (s,I)*-confined in* $\mathcal{A}$ if

$$M \cap \mathsf{ann}(\mathcal{C}, t+1-t', t+1+t') \subseteq \bigcup_{i \in I} P_i.$$

Intuitively, the above definition demands that M traverses the "middle" (s,q)-annulus by intersecting it only at the rails of $\mathcal{A}$.

Our algorithms are strongly based on the following combinatorial result, whose proof is postponed to §5.

Theorem 2.1. (Model Taming) *There exist two functions* $f_3, f_4 : \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}$ *such that if*

- s *is a positive odd integer,*
- H *is a graph on* g *edges,*
- G *is a partially* Δ*-embedded graph,*
- $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ *is a* (r,q)*-railed annulus of* G, *where* $r \geq f_4(g) + 2 + s$ *and* $q \geq 5/2 \cdot f_3(g)$,
- (M,T) *is a topological minor model of* H *in* G *such that* $T \cap \mathsf{ann}(\mathcal{A}) = \emptyset$, *and*
- $I \subseteq [q]$ *where* $|I| > f_3(g)$,

then G *contains an topological minor model* $(\tilde{M}, \tilde{T})$ *of* H *in* G *such that*

1. $\tilde{T} = T$,
2. $\tilde{M}$ *is* (s,I)*-confined in* $\mathcal{A}$ *and*
3. $\tilde{M} \setminus \mathsf{ann}(\mathcal{A}) \subseteq M \setminus \mathsf{ann}(\mathcal{A})$.

Apart from being the combinatorial base of our results, the Model Taming Theorem will appear useful in other results using the irrelevant vertex technique (see [3]).

2.4 Boundaried graphs and folios Let $t \in \mathbb{N}$. A *t-boundaried graph* is a triple $\mathbf{G} = (G, B, \rho)$ where G is a graph, $B \subseteq V(G)$, $|B| \leq t$, and $\rho : B \to [t]$ is an injective function. We call B the *boundary* of $\mathbf{G}$ and we call the vertices of B *the boundary vertices* of $\mathbf{G}$. We also call G *the underlying graph* of $\mathbf{G}$. Moreover, we call $|B|$ the *boundary size* of $\mathbf{G}$. We say that the t-boundaried $\mathbf{G}' = (G', B', \rho')$ is a *subgraph* of $\mathbf{G}$ if G' is a subgraph of G, $B' = B \cap V(G')$, and $\rho' = \rho|_{B'}$. Also, for $S \subseteq V(G)$, we define $\mathbf{G} \setminus S$ to be the t-boundaried graph (G', B', ρ') where $G' = G \setminus S$, $B' = B \setminus S$ and $\rho' = \rho|_{B'}$. Two t-boundaried graphs $\mathbf{G}_1 = (G_1, B_1, \rho_1)$ and $\mathbf{G}_2 = (G_2, B_2, \rho_2)$ are isomorphic if G_1 is isomorphic to G_2 via a bijection $\phi : V(G_1) \to V(G_2)$ such that $\rho_1 = \rho_2 \circ \phi|_{B_1}$, i.e., the vertices of B_1 are mapped via ϕ to equally indexed vertices of B_2. A *boundaried graph* is any t-boundaried graph for some $t \in \mathbb{N}$.

We also define the *treewidth* of a boundaried graph $\mathbf{G} = (G, B, \rho)$, denoted by $\mathbf{tw}(\mathbf{G})$ as the minimum width of a tree decomposition (T, χ) of G for which there is some $t \in V(T)$ such that $B \subseteq \chi(t)$. Notice that the treewidth of a t-boundaried graph is always lower bounded by its boundary size.

Topological minors of boundaried graphs. If $\mathbf{M} = (M, B, \rho)$ is a boundaried graph and $T \subseteq V(M)$ with $B \subseteq T$, we call $(\mathbf{M}, T)$ a btm*-pair* and we define $\mathsf{diss}(\mathbf{M}, T) = (\mathsf{diss}(M, T), B, \rho)$ (notice that we consider all boundary vertices to be branch vertices, therefore we do not permit their dissolution). If $\mathbf{G} = (G, B, \rho)$ is a boundaried graph and (M, T) is a tm-pair of G where $B \subseteq T$, then we say that $(\mathbf{M}, T)$, where $\mathbf{M} = (M, B, \rho)$, is a btm*-pair* of $\mathbf{G} = (G, B, \rho)$. Let $\mathbf{G}_i = (G_i, B_i, \rho_i), i \in [2]$. We say that $\mathbf{G}_1$ is a *topological minor* of $\mathbf{G}_2$, denoted by $\mathbf{G}_1 \preceq \mathbf{G}_2$, if there is a btm-pair $(\mathbf{M}, T)$ of $\mathbf{G}_2$ such that $\mathsf{diss}(\mathbf{M}, T)$ is isomorphic to $\mathbf{G}_1$. We call $\mathsf{diss}(\mathbf{M}, T)$ the *representation* of the btm-pair $(\mathbf{M}, T)$ of $\mathbf{G}$.

Folios. Let $h, t \in \mathbb{N}$ where $h \geq t$. We denote by $\mathcal{B}_h^{(t)}$ the set of all (pairwise non-isomorphic) t-boundaried graphs with at most h vertices. We set the function $f_5 : \mathbb{N}^2 \to \mathbb{N}$ such that $f_5(t, h) = |\mathcal{B}_h^{(t)}|$. Given a t-boundaried graph $\mathbf{G}$ and an integer $h \in \mathbb{N}$, we define the *h-folio* of $\mathbf{G}$, denoted by $\mathcal{F}_h^{(t)}(\mathbf{G})$, as the set containing all t-boundaried graphs in $\mathcal{B}_h^{(t)}$ that are representations of the btm-pairs of $\mathbf{G}$. Notice that $|\mathcal{F}_h^{(t)}(\mathbf{G})| \leq f_5(t, h)$.

Given that topological minor containment can be expressed in Monadic Second Order logic, the next lemma follows from Courcelle's theorem.

Lemma 2.2. *There is an algorithm with the following specifications:*

Compute_Folio$(h, w, t, \mathbf{G})$
Input: $h, w, t \in \mathbb{N}$, where $h \geq t$ and a t-boundaried graph $\mathbf{G}$ of treewidth at most w.
Output: the set $\mathcal{F}_h^{(t)}(\mathbf{G})$.
Moreover, this algorithm runs in $O_{h,w}(n)$ steps.

3 The two main subroutines of the algorithm

In this section, we provide two main subroutines that will be useful in the proof of Theorem 1.2. From now on, functions f_3, f_4 will always denote the functions of Theorem 2.1.

Boundaried graphs in railed annuli. Let $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ be a (r, q)-railed annulus of a partially Δ-embedded graph G. We can see each path P_j in $\mathcal{P}$ as being oriented towards the "inner" part of Δ, i.e., starting from an endpoint of $P_{1,j}$ and finishing to an endpoint of $P_{r,j}$. For every $(i, j) \in [r] \times [q]$, we define $r_{i,j}$ as the first vertex of P_j that appears in $P_{i,j}$ while traversing P_j according to this orientation.

Given an $i \in [r]$ and a $t \in [q]$, we define the t-boundaried graph $\mathbf{G}_{i,t} = (G_i, R_{i,t}, \rho_{i,t})$ where $G_i = G \cap \overline{D}_i$, $R_{i,t} = \{r_{i,1} \ldots, r_{i,t}\}$ and, for $j \in [t]$, $\rho(r_{i,j}) = j$.

3.1 Reducing the solution space We now prove the following lemma that intuitively states that there is an algorithm that given a graph G and a "big enough" railed annulus $\mathcal{A}$ of G, it "reduces" the set of vertices that are candidates to the hitting set S.

Lemma 3.1. *There exists an algorithm with the following specifications:*

Reduce_Solution_Space$(k, h, g, \Delta, G, w, \mathcal{C}, \mathcal{P})$
Input: three integers $k, h, g \in \mathbb{N}_{\geq 0}$, a partially Δ-embedded graph G whose compass has treewidth $\leq w$ and an (r, q)-railed annulus $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ of G, where $r = (k+1)(h+1)(f_4(g)+3)$ and $q \geq 5/2 \cdot f_3(g)$.
Output: a set $R \subseteq D_r \cap V(G)$ such that

- *$|R| \leq f_5(f_3(g)+1, h+f_3(g)+1)^{h+1} \cdot k(k+1)$ and*
- *for every h-vertex and g-edge graph H and every $S \subseteq V(G)$, if $|S| \leq k$ and $H \not\preceq G \setminus S$, then there is some $S' \subseteq (V(G) \setminus D_r) \cup R$ such that $|S'| \leq k$ and $H \not\preceq G \setminus S'$.*

Moreover, this algorithm runs in $O_{h,w}(n)$ steps.

Proof. We set $\mu := f_4(g) + 3$ and $\lambda := f_3(g)$. Given an $i \in [k+1]$, we use notation A_i as a shortcut to $\mathsf{ann}(\mathcal{C}, (i-1)(h+1)\mu + 1, i(h+1)\mu)$ and for every $j \in [h+1]$ we define $B_{i,j} = \mathsf{ann}(\mathcal{C}, (i-1)(h+1)\mu + (j-1)\mu + 1, (i-1)(h+1)\mu + j\mu)$. Intuitively, we partition $\mathcal{C}$ in $k+1$ sets of consecutive cycles (i.e., the cycles of $A_i, i \in [k+1]$) and then, for every $i \in [k+1]$ we partition the set of cycles of A_i into $h+1$ sets of consecutive cycles (i.e., the cycles of $B_{i,j}$, $j \in [h+1]$). Notice that for every $i, j \in [k+1] \times [h+1]$, $|B_{i,j} \cap \mathcal{C}| = \mu$.

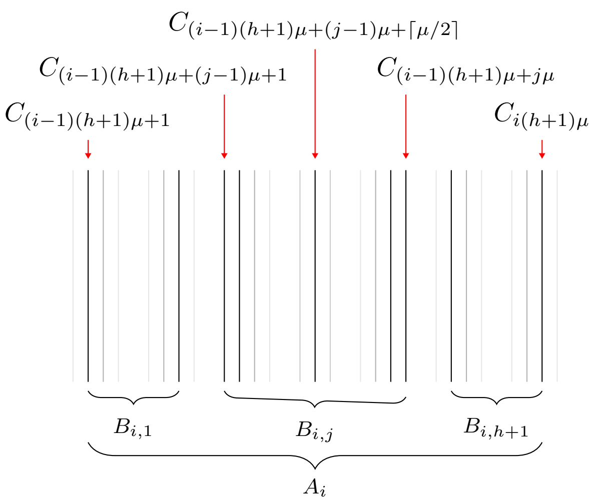

Figure 3: Visualization of the partition of the cycles of $\mathcal{A}$ in sets $A_i, i \in [k+1]$ and of the partition in sets $B_{i,j}, i, j \in [k+1] \times [h+1]$.

Also, we define for every $(i, j) \in [k+1] \times [h+1]$ the $(\lambda+1)$-boundaried graph

$$\hat{\mathbf{G}}_{i,j} = \mathbf{G}_{(i-1)(h+1)\mu+(j-1)\mu+\lceil\mu/2\rceil,\lambda+1}.$$

To get some intuition, notice that the boundary vertices of $\hat{\mathbf{G}}_{i,j}$ lie on the "middle" cycle of $B_{i,j}$ – see Figure 3. Let $i \in [k+1]$. For every collection $\mathfrak{F}_i := [\mathcal{F}_1, \ldots, \mathcal{F}_{h+1}] \in (\mathcal{B}^{(\lambda+1)}_{h+\lambda+1})^{h+1}$, let $S_{i,\mathfrak{F}_i}$ be the minimum-size subset of $V(G) \cap D_{(i-1)(h+1)\mu+1}$ of at most k vertices such that, for every $j \in [h+1]$, it holds that $\mathcal{F}^{(\lambda+1)}_{h+\lambda+1}(\hat{\mathbf{G}}_{i,j} \setminus S_{i,\mathfrak{F}_i}) \cap \mathcal{F}_j = \emptyset$. If such a set does not exist, then we set $S_{i,\mathfrak{F}_i} = \emptyset$. We define

$$R = \Big(\bigcup_{\substack{i\in[k+1] \\ \mathfrak{F}_i \in (\mathcal{B}^{(\lambda+1)}_{h+\lambda+1})^{h+1}}} S_{i,\mathfrak{F}_i}\Big) \cap D_r$$

Notice that as each $\hat{G}_{i,h+1}$, that is the underlying graph of $\hat{\mathbf{G}}_{i,h+1}$, is a subgraph of the compass of G, it has treewidth at most w. Moreover, the set $S_{i,\mathfrak{F}_i}$ can be expressed in MSOL and, again from Courcelle's theorem, each $S_{i,\mathfrak{F}_i}$, and therefore R as well, can be computed in $O_{k,g}(|G|)$ steps.

Let H be h-vertex graph and g-edge graph and $S \subseteq V(G)$ such that $|S| \leq k$ and $H \not\leq G \setminus S$. As $r = (k+1)(h+1)\mu$ and $|S| \leq k$, then by the pigeonhole principle there is some $\ell \in [k+1]$ such that $S \cap A_\ell = \emptyset$. (In case there are many such ℓ's, we take the minimum one.) Let $S_{\text{in}} = S \cap D_{\ell(h+1)\mu}$ and $S_{\text{out}} = S \setminus \overline{D}_{(\ell-1)(h+1)\mu+1}$. Let also $k_{\text{in}} := |S_{\text{in}}|$ and $k_{\text{out}} := |S_{\text{out}}|$ and keep in mind that $k_{\text{in}}+k_{\text{out}} = |S| \leq k$. Let $\mathcal{H}$ be the set of all topological minor models of H in G and notice that for every $(M, T) \in \mathcal{H}$, $S \cap V(M) \neq \emptyset$, i.e., S intersects at least one vertex of each graph in $\mathcal{H}$. Let $\mathcal{H}_\ell$ be the members of $\mathcal{H}$ that are intersected *only* by vertices in S_{in}.

The next claim shows that there is a collection of cycles of $\mathcal{A}$ such that for each tm-pair $(M, T) \in \mathcal{H}_\ell$ there exists a cycle C of this collection and a tm-pair $(\tilde{M}, \tilde{T}) \in \mathcal{H}_\ell$ that is equivalent to (M, T) and is "tamed in C" in the sense that $M \cap C$ is a subgraph of the rails of $\mathcal{A}$.

Claim: For every $(M, T) \in \mathcal{H}_\ell$, there is an $j_M \in [h+1]$ and a topological minor model $(\tilde{M}, \tilde{T})$ in $\mathcal{H}_\ell$, such that $\tilde{M} \setminus A_\ell \subseteq M \setminus A_\ell$ and whose intersection with C_{y_M} is the union of the paths $\{P_{y_M,c^M_1}, \ldots, P_{y_M,c^M_{z_M}}\}$ where $y_M = (\ell-1)(h+1)\mu + (j_M-1)\mu + \lceil\mu/2\rceil$ and $\{c^M_1, \ldots, c^M_{z_M}\} \subseteq [\lambda+1]$ (see Figure 4).

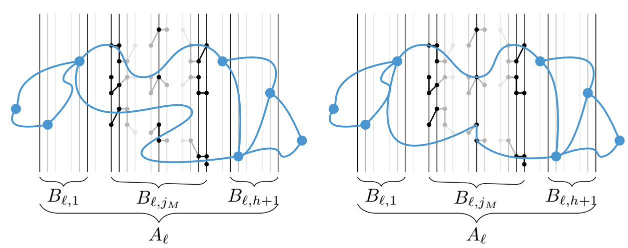

Figure 4: Visualization of the statement of the Claim. (M, T) is depicted in the left figure, while $(\tilde{M}, \tilde{T})$ is depicted in the right figure.

Proof of Claim: Let $(M, T) \in \mathcal{H}_\ell$ and notice that $S_{\text{in}} \cap V(M) \neq \emptyset$. As $|T| = h$, there is some $j_M \in [h+1]$ such that $T \cap B_{\ell,j_M} = \emptyset$ (if many such j_M's exist, take the minimum one). We use notation $\mathcal{A}^{(M)} = (\mathcal{C}^{(M)}, \mathcal{P}^{(M)})$ instead of $\mathcal{A} \cap B_{\ell,j_M}$. We can now apply Theorem 2.1 for $s = 1$, $\mathcal{A} := \mathcal{A}^{(M)}$, and $I = [\lambda+1]$ and obtain a topological minor model $(\tilde{M}, \tilde{T})$ of H in G such that $\tilde{T} = T$, $\tilde{M}$ is $(1, I)$-confined in $\mathcal{A}^{(M)}$ and $\tilde{M} \setminus B_{\ell,j_M} \subseteq M \setminus B_{\ell,j_M}$, which implies that $\tilde{M} \setminus A_\ell \subseteq M \setminus A_\ell$. Let $y_M = (\ell-1)(h+1)\mu + (i_M-1)\mu + \lceil\mu/2\rceil$. Notice that $(\tilde{M}, \tilde{T})$ is a topological minor model in $\mathcal{H}_\ell$ whose intersection with C_{y_M} is the union of some of the paths in $\{P_{y_M,1}, \ldots, P_{y_M,\lambda+1}\}$. Suppose that these paths are $\{P_{y_M,c^M_1}, \ldots, P_{y_M,c^M_{z_M}}\}$ where $\{c^M_1, \ldots, c^M_{z_M}\} \subseteq [\lambda+1]$. The claim follows.

Following the above claim, for every $(M, T) \in \mathcal{H}_\ell$ we define the $(\lambda+1)$-boundaried graph $\mathbf{G}_M = (G_M, B_M, \rho_M)$ where $G_M = (\tilde{M} \cap \overline{D}_{y_M}) \cup (B_M, \emptyset)$ (i.e. the graph $\tilde{M} \cap \overline{D}_{y_M}$ together with the vertices B_M), $B_M = \{r_{y_M,1}, \ldots, r_{y_M,\lambda+1}\}$, and for $d \in [\lambda+1]$,

$\rho_M(r_{y_M,d}) = d$.

We now define, for every $j \in [h+1]$, the set $\mathcal{F}_j = \{\mathsf{diss}(\mathbf{G}_M, T \cup B_M) \mid j_M = j \text{ and } (M,T) \in \mathcal{H}_\ell\}$ and we set $\mathfrak{F}_S = [\mathcal{F}_1, \ldots, \mathcal{F}_{h+1}]$.

Notice that S_{in} is a subset of $V(G) \cap D_{(\ell-1)(h+1)\mu+1}$ of k_{in} vertices such that, for every $j \in [h+1]$, it holds that $\mathcal{F}^{(\lambda+1)}_{h+\lambda+1}(\hat{\mathbf{G}}_{\ell,j} \setminus S_{\text{in}}) \cap \mathcal{F}_j = \emptyset$. Clearly, $|S_{\ell,\mathfrak{F}_S}| \leq |S_{\text{in}}|$. We now set $S' = S_{\ell,\mathfrak{F}_S} \cup S_{\text{out}}$. Observe that $S_{\ell,\mathfrak{F}_S} \cap D_r \subseteq R$ and therefore $S' \subseteq (V(G) \setminus D_r) \cup R$. Since $|S'| \leq k$, it remains to prove that $H \not\leq G \setminus S'$.

Suppose to the contrary that the graph $G \setminus S'$ contains some topological minor model (M,T) of H as a subgraph. Since $H \not\leq G \setminus S$, then it holds that (M,T) is intersected only by vertices in S_{in} - thus $(M,T) \in \mathcal{H}_\ell$. According to the claim above, there is an $j_M \in [h+1]$ and a topological minor model $(\tilde{M}, \tilde{T}) \in \mathcal{H}_\ell$ such that $\tilde{M} \setminus A_\ell \subseteq M \setminus A_\ell$ and whose intersection with C_{y_M} is the union of the paths $\{P_{y_M,c^M_1}, \ldots, P_{y_M,c^M_{z_M}}\}$ where $y_M = (\ell-1)(h+1)\mu + (i_M - 1)\mu + \lceil \mu/2 \rceil + 1$ and $\{c^M_1, \ldots, c^M_{z_M}\} \subseteq [\lambda+1]$. Therefore $\mathsf{diss}(\mathbf{G}_M, T \cup B_M) \in \mathcal{F}_{j_M}$, which contradicts the definition of $S_{\ell,\mathfrak{F}_S}$. □

3.2 Finding an irrelevant area

Before we proceed with the proof of the second result of this section we need some more definitions. Let $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ be an (r,q)-railed annulus of a partially Δ-embedded graph G.

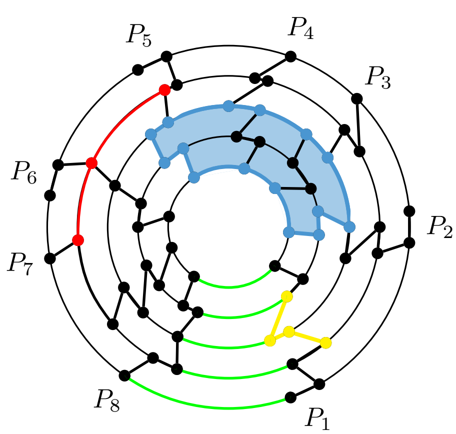

Figure 5: An example of a $(5,8)$-railed annulus $\mathcal{A}$, the set $F_{\mathcal{A}}$ (depicted in green), and the graphs $L_{2,5\to7}$ (depicted in red), $R_{2\to4,1}$ (depicted in yellow), and $\Delta_{3,5,2,5}$ (depicted in blue).

For every $i \in [r]$, we define $F^{(i)}_{\mathcal{A}}$ as the edge set of the unique $(P_{i,q}, P_{i,1})$-path that does not contain any vertex from P_2. We also set $F_{\mathcal{A}} = \bigcup_{i\in[r]} F^{(i)}$. Let $(i,j,j') \in [r] \times [q]^2$ where $j \neq j'$. We denote by $L_{i,j\to j'}$ the shortest path in C_i starting from a vertex of $P_{i,j}$ and finishing to a vertex of $P_{i,j'}$ and that does not contain any edge from $F_{\mathcal{A}}$. Let $(i,i',j) \in [r]^2 \times [q]$ where $i \neq i'$ (see Figure 5). We denote by $R_{i\to i',j}$ the shortest path in P_j starting from a vertex of $P_{i,j}$ and finishing to a vertex of $P_{i',j}$. Let $(i,i',j,j') \in [r]^2 \times [q]^2$ such that $i < i'$ and $j < j'$. We define $\Delta_{i,i',j,j'}$ as the closed disk bounded by the unique cycle in the graph

$$P_{i,j} \cup L_{i,j\to j'} \cup P_{i,j'} \cup R_{i\to i',j'} \cup \\ P_{i',j'} \cup L_{i',j'\to j} \cup P_{i',j} \cup R_{i'\to i,j}.$$

The next lemma intuitively states that there exists an algorithm that given a partially Δ-embedded graph G and a "big enough" railed annulus of G, then there exists a bidimensional area $\Delta' \subseteq \Delta$ such that for every hitting set S outside Δ, $\Delta' \cap V(G)$ is an irrelevant part of the instance.

Lemma 3.2. *There exists an algorithm with the following specifications:*

Find_irrelevant_area$(h, g, b, \Delta, G, w, \mathcal{C}, \mathcal{P})$
Input: three integers $h, g \in \mathbb{N}_{\geq 1}$ *and* $b \in \mathbb{N}_{\geq 2}$*, a partially* Δ*-embedded graph* G *whose compass has treewidth at most* w*, and an* (r,q)*-railed annulus* $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ *of* G*, where* $r = f_5(f_3(g)+1, h+f_3(g)+1) \cdot \big((h+1)(f_4(g)+3)+b+1\big)$ *and* $q = \max\{5/2 \cdot f_3(g), f_3(g)+b\}$.
Output: a closed disk $\Delta' \subseteq \Delta$ *such that*

- $\mathbf{tw}(G \cap \Delta') \geq b$ *and*
- *for every* h*-vertex and* g*-edge graph* H *and for every* $S \subseteq V(G) \setminus \Delta$*, if* $H \not\leq (G \setminus (\Delta' \cap V(G))) \setminus S$ *then* $H \not\leq G \setminus S$.

Moreover, this algorithm runs in $O_{h,w}(b \cdot |G|)$ *steps.*

Proof. Let $t := f_3(g)+1$, $\mu := f_4(g)+3$, $\ell := (h+1)\mu + b+1$. Using this notation we have that $r = f_5(t, h+t) \cdot \ell$.

We consider the t-boundaried graphs $\mathbf{G}_{i,t}$, $i \in [r]$. As the underlying graph of each $\mathbf{G}_{i,t}$ is a subgraph of the compass of G, we have that $\mathbf{tw}(\mathbf{G}_{i,t}) \leq w + t = O_h(w)$. For each $i \in [r]$, we call the algorithm **Compute_Folio**$(h+t, t, \mathbf{G}_{i,t}, w+t)$ and compute the $(h+t)$-folio of $\mathbf{G}_{i,t}$ which, from now on, we denote by $\mathcal{F}_i$. According to Lemma 2.2 $\mathcal{F}_i$, for all $i \in [r]$ can be computed in $O_{h,w}(r \cdot |G|) = O_{h,w}(b \cdot |G|)$ steps. We observe that if $1 \leq i \leq i' \leq r$, then $\mathcal{F}_{i'} \subseteq \mathcal{F}_i$. This, together with the fact that $|\mathcal{B}^{(t)}_h| = f_5(t, h+t)$, implies that there is an $i' \in [r-\ell+1]$ such that $\mathcal{F}_{i'} = \mathcal{F}_{i'+1} = \ldots = \mathcal{F}_{i'+\ell-1}$. We define

$$\Delta' = \Delta_{i'+\mu(h+1), i'+\ell-2, t+1, t+b}$$

and notice that $G \cap \Delta'$ contains a $(b \times b)$-grid as a minor, therefore $\mathbf{tw}(G \cap \Delta') \geq b$ (see Figure 6). Also keep in mind that Δ' does not intersect the cycle $C_{i'+\ell-1}$.

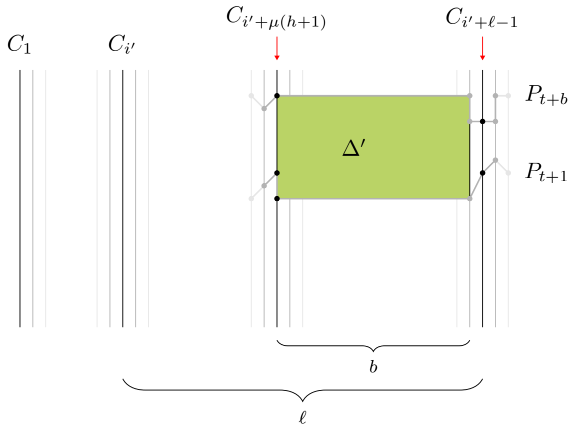

Figure 6: An example showing the disk Δ'.

Let now H be a h-vertex and g-edge graph and $S \subseteq V(G) \setminus \Delta$ such that $H \not\leq (G \setminus \Delta') \setminus S$. It remains to prove that $H \not\leq G \setminus S$. Suppose to the contrary that the graph $G \setminus S$ contains some topological minor model (M, T) of H as a subgraph. As $|T| = h$ and $\ell = \mu \cdot (h+1) + b + 1$ there is some $i'' \in [i', i' + \ell - \mu]$ such that $T \cap \mathsf{ann}(\mathcal{C}, i'', i'' + \mu - 1) = \emptyset$.

We consider the (μ, q)-railed annulus $\mathcal{A}' = (\mathcal{C}', \mathcal{P})$ of $G \setminus S$ where

- $\mathcal{C}' = [C'_1, \ldots, C'_\mu] := [C_{i''}, \ldots, C_{i''+\mu-1}]$ and
- $\mathcal{P}' = [P'_1, \ldots, P'_q] := [P_1 \cap \mathsf{ann}(\mathcal{A}'), \ldots, P_q \cap \mathsf{ann}(\mathcal{A}')]$. (See Figure 7.)

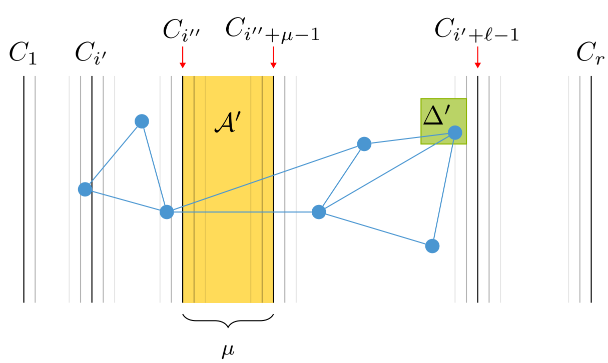

Figure 7: An example showing the (μ, q)-railed annulus $\mathcal{A}'$.

We are now in position to apply Theorem 2.1 for $s := 1$, H, $G := G \setminus S$, $\mathcal{A} := \mathcal{A}'$, $r := \mu$, M, and $I = [t]$. We deduce the existence of a topological minor model $(\tilde{M}, \tilde{T})$ of H in $G \setminus S$ such that $\tilde{T} = T$, $\tilde{M}$ is $(1, I)$-confined in $\mathcal{A}'$, and $\tilde{M} \setminus \mathsf{ann}(\mathcal{A}') \subseteq M \setminus \mathsf{ann}(\mathcal{A}')$ (see Figure 8).

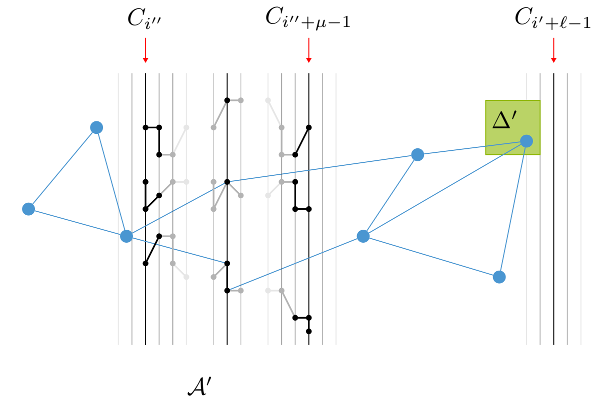

Figure 8: An example of $(\tilde{M}, \tilde{T})$, the result of applying Theorem 2.1 in the railed annulus $\mathcal{A}'$.

Let $y = i'' + \lfloor \mu/2 \rfloor$. Notice now that $\tilde{M} \cap C_y$ is the union of some of the paths in $\{P_{y,1}, \ldots, P_{y,t}\}$. Suppose that these paths are $\{P_{y,c_1}, \ldots, P_{y,c_\mu}\}$ where $\{c_1, \ldots, c_\mu\} \subseteq [t]$. We consider the boundaried graph $\mathbf{M}_y = (M_y, B_y, \rho_y)$ where $M_y = (\tilde{M} \cap \overline{D}_y) \cup (B_y, \emptyset)$ (i.e. the graph $\tilde{M} \cap \overline{D}_y$ together with the isolated vertices B_y), $B_y = \{r_{y,1}, \ldots, r_{y,t}\}$, and for every $d \in [t]$, $\rho_y(r_{y,d}) = d$. We also define $\hat{M}_y = \tilde{M} \setminus D_y \setminus \bigcup_{d \in [t]} (V(P_{y,c_d}) \setminus r_{y,c_d})$. Keep in mind that $\hat{M}_y$ does not intersect the disk Δ' (see Figure 9).

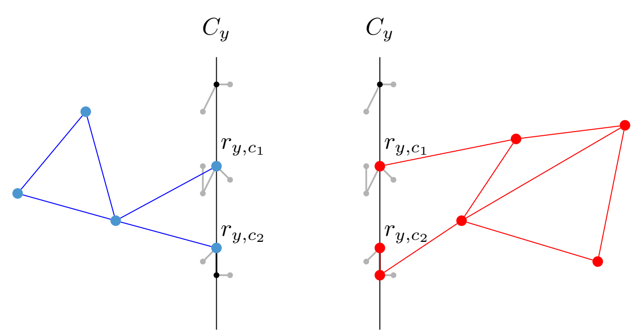

Figure 9: The graphs M_y (depicted in red) and $\hat{M}_y$ (depicted in blue).

Now consider the t-boundaried graph $\mathsf{diss}(\mathbf{M}_y, T \cup B_y)$ and notice that it is isomorphic to a member $\mathbf{F} \in \mathcal{F}_y$. We set $y' = i' + \ell - 1$. Recall that $\mathbf{F} \in \mathcal{F}_{y'}$, as $\mathcal{F}_y = \mathcal{F}_{y'}$. This means that $\mathbf{G}_{y'}$ contains as a subgraph a model $\mathbf{M}_{y'} = (M_{y'}, B_{y'}, \rho_{y''})$ of $\mathbf{F}$ where $B_{y'} = \{r_{y',1}, \ldots, r_{y',t}\}$, and for every $d \in [t]$, $\rho_y(r_{y',d}) = d$. Notice that $M_{y'}$ does not intersect Δ'. Let $\hat{M}_{y'}$ be the graph obtained from $M_{y'}$ after removing the vertices $r_{y',j}, j \in [t] \setminus \{c_1, \ldots, c_\mu\}$. For every $d \in [\mu]$, we define $P^*_d = P_{y,c_d} \cup R_{y \to y', c_d}$ and observe that none of the paths in $\mathcal{P}^* = \{P^*_d \mid d \in [\mu]\}$ intersects Δ'. Consider now the graph $M_0 := \hat{M}_y \cup \hat{M}_{y'} \cup \bigcup \mathcal{P}^*$ and observe that (M_0, T) is a topological minor model of H in $G \setminus S$ such that $V(M_0) \cap \Delta' = \emptyset$. Therefore $H \preceq (G \setminus \Delta') \setminus S$, a contradiction (see Figure 10).

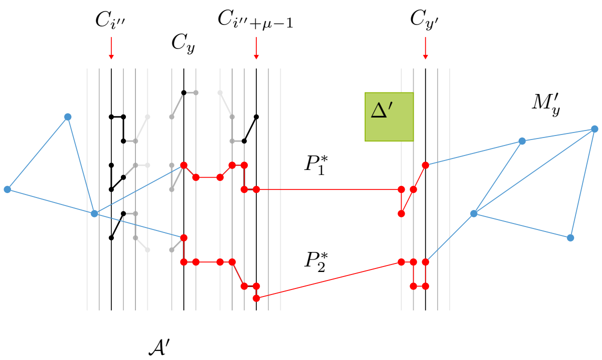

Figure 10: Visualization of the last part of the proof.

□

4 Proof of the main result

Now we have all necessary results in order to prove Theorem 1.2.

Proof. [Proof of Theorem 1.2] Let $g := \binom{h}{2}$, $\mu := f_4(g) + 3$, and $\lambda := f_3(g)$. Also, let

$$x := (k+1)(h+1)\mu,$$
$$y := f_5(\lambda+1, h+\lambda+1) \cdot ((h+1)\mu+1), \text{ and}$$
$$z := f_5(\lambda+1, h+\lambda+1)^{h+1} \cdot k(k+1) + 3.$$

For $q := f_2(x, y, z)$, we call the algorithm **Find_Wall**(G, q) of Proposition 2.1 which outputs either a q-wall W of G whose compass has treewidth at most $c_1 \cdot q$ or a tree decomposition of G of width at most $c_1 \cdot q$. We consider the first case.

Let Δ be the closed disk defined by the compass of W. Then, we call the algorithm **Find_Collection_of_Annuli**(x, y, z, Δ, G, W) of Lemma 2.1 which outputs a collection $\mathfrak{A} = \{\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_z\}$ of railed annuli of the compass of G such that

- $\mathcal{A}_0$ is a (x, x)-railed annulus whose outer disk is Δ and whose inner disk is Δ',
- for $i \in [z]$, $\mathcal{A}_i$ is a (y, y)-railed annulus of $G \cap \Delta'$, and
- for every $i, j \in [z]$ where $i \neq j$, the outer disk of $\mathcal{A}_i$ and the outer disk of $\mathcal{A}_j$ are disjoint.

Then, we call the algorithm **Reduce_Solution_Space**$(k, h, g, \Delta, G, w, \mathcal{C}, \mathcal{P})$ of Lemma 3.1 for $(\mathcal{C}, \mathcal{P}) := \mathcal{A}_0$ and $w := c_1 \cdot q$ which outputs a set $R \subseteq \Delta' \cap V(G)$ such that

- $|R| \leq f_5(\lambda+1, h+\lambda+1)^{h+1} \cdot k(k+1) = z - 1$ and
- for every graph H on at most h vertices and g edges and every $|S| \leq k$ and $H \not\preceq G \setminus S$, then there is some $S' \subseteq (V(G) \setminus \Delta') \cup R$ such that $|S'| \leq k$ and and $H \not\preceq G \setminus S'$.

Since $|R| < z$ then there exists a $j \in [z]$ such that $R \cap \mathsf{ann}(\mathcal{A}_j) = \emptyset$. Let $(\mathcal{C}^{(j)}, \mathcal{P}^{(j)}) := \mathcal{A}_j$. Now, for $b := 2$, the algorithm **Find_irrelevant_area**$(h, g, b, \Delta, G, w, \mathcal{C}^{(j)}, \mathcal{P}^{(j)})$ of Lemma 3.2 computes a closed disk $\Delta' \subseteq \Delta$ such that

- $\mathbf{tw}(G \cap \Delta') \geq b$, and
- for every graph H on at most h vertices and g edges and every $S \subseteq V(G) \setminus \Delta$, if $H \not\preceq (G \setminus (\Delta' \cap V(G))) \setminus S$ then $H \not\preceq G \setminus S$.

As the graphs in $\mathcal{F}$ have at most h vertices and $g = \binom{h}{2}$, we conclude that there exists a vertex $v \in V(G) \cap \Delta'$ such that $\mathbf{p}_{\mathcal{F}}(G) \leq k \iff \mathbf{p}_{\mathcal{F}}(G \setminus v) \leq k$. □

5 Proof of the Model Taming Theorem

This section is devoted to the proof of Theorem 2.1 that is the base of the correctness of both algorithmic results of the previous section.

5.1 Linkages in railled annuli A *linkage* in a graph G is a subgraph L of G whose connected components are all non-trivial paths. The *paths* of a linkage are its connected components and we denote them by $\mathcal{P}(L)$. The *size* of L is the number of its paths and is denoted by $|L|$. The *terminals* of a linkage L, denoted by $T(L)$, are the endpoints of the paths in $\mathcal{P}(L)$, and the *pattern* of L is the set

$$\{\{s,t\} \mid \mathcal{P}(L) \text{ contains some } (s,t)\text{-path}\}.$$

Two linkages L_1, L_2 of G are *equivalent* if they have the same pattern and we denote this fact by $L_1 \equiv L_2$.We say that a linkage L of a graph G is *vital* if $V(L) = V(G)$ and there is no other linkage of G that is equivalent to L.

Let G be a partially Δ-embedded graph, let $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ be a (r, q)-railed annulus of G and L be a linkage of G. If L is a linkage of a partially Δ-embedded graph, and $D \subseteq \Delta$, then we say that L is D-*avoiding* if $T(L) \cap D = \emptyset$. We also say that L is D-*free* if $D \cap L = \emptyset$. We also say that L is $\mathcal{A}$-*avoiding* if it is $\mathsf{ann}(\mathcal{C})$-avoiding (see Figure 11).

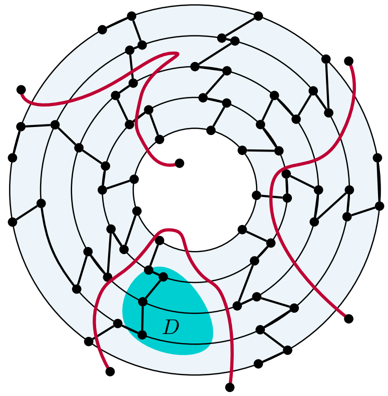

Figure 11: An example of a railed annulus $\mathcal{A}$, a closed disk D (depicted in blue) and a linkage L (depicted in red) that is D-free and $\mathcal{A}$-avoiding.

Let $r = 2t + 1$. Let also $s \in [r]$ where $s = 2t' + 1$. Given some $I \subseteq [q]$, we say that a linkage L is (s, I)-*confined in* $\mathcal{A}$ if

$$L \cap \mathsf{ann}(\mathcal{C}, t + 1 - t', t + 1 + t') \subseteq \bigcup_{i \in I} P_i.$$

Our purpose is to prove the following.

THEOREM 5.1. *There exist two functions* $f_3, f_4 : \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}$ *such that for every odd* $s \in \mathbb{N}_{\geq 1}$ *and every* $k \in \mathbb{N}_{\geq 0}$, *if* G *is a partially* Δ*-embedded graph,* $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ *is a* (r, q)*-railed annulus of* G, *where* $r \geq f_4(k) + s$ *and* $q \geq 5/2 \cdot f_3(k)$, L *is an* $\mathcal{A}$*-avoiding linkage of size at most* k, *and* $I \subsetneq [q]$, *where* $|I| > f_3(k)$, *then* G *contains a linkage* $\tilde{L}$ *where* $\tilde{L} \equiv L$, $\tilde{L}$ *is* $\mathcal{A}$*-avoiding,* $\tilde{L} \setminus \mathsf{ann}(\mathcal{C}) \subseteq L \setminus \mathsf{ann}(\mathcal{C})$, *and* $\tilde{L}$ *is* (s, I)*-confined in* $\mathcal{A}$.

We say that a function is *even* if its images are even numbers. We state the following result.

PROPOSITION 5.1. ([22, 28]) *There exists an even function* $f_3 : \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}$ *such that if* G *is a graph and* L *is a vital linkage of* G, *then* $\mathbf{tw}(G) \leq f_3(|L|)$.

In the above proposition, f_3 is a non-decreasing function that is important for the statement of many of the results of this paper. For this reason, for now on, f_3 will always denote the function of Proposition 5.1.

5.2 Taming a Linkage LB-pairs. Given a graph G, a *LB-pair* of G is a pair (L, B) where B is a subgraph of G with maximum degree 2 and L is a linkage of G. We define $\mathsf{cae}(L, B) = |E(L) \setminus E(B)|$ (i.e., the number of linkage edges that are not edges of B).

LEMMA 5.1. *Let* (L, B) *be an LB-pair of a* G. *If* $\mathbf{tw}(L \cup B) > f_3(|L|)$, *then* G *contains a linkage* L' *where*

1. $\mathsf{cae}(L', B) < \mathsf{cae}(L, B)$,
2. $L' \equiv L$,
3. $L' \subseteq L \cup B$.

Proof. Let $H = L \cup B$. From Proposition 5.1, L is not a vital linkage of H, therefore, H contains a linkage L' such that $L \neq L'$ and $L' \equiv L$. Notice that $E(L') \setminus E(B) \subseteq E(L) \setminus E(B)$. It remains to prove that this inclusion is proper.

Let $\{x, y\}$ be a member of the common pattern of L and L' such that the (x, y)-path P of L is different than the (x, y)-path P' of L'. Clearly, P and P', when oriented from x to y, have a common part P^*. Formally, this is the connected component of $P \cap P'$ that contains x. Let e be the $(m + 1)$th edge of P, starting from x, where m is the length of P^*. Notice that $e \in E(L) \setminus E(B)$, while $e \notin E(L') \setminus E(B)$.

We conclude that $E(L') \setminus E(B) \subsetneq E(L) \setminus E(B)$, therefore $|E(L') \setminus E(B)| < |E(L) \setminus E(B)|$, as required. ∎

Minimal linkages. Let G be a partially Δ-embedded graph, $\mathcal{C}$ be a Δ-nested cycle collection of G, $D \subseteq \Delta$, L be a $\mathsf{ann}(\mathcal{C})$-avoiding and D-free linkage of G. We say that a linkage L' of G is $(\mathcal{C}, D, L)$-*minimal* if, among all the $\mathsf{ann}(\mathcal{C})$-avoiding linkages of G that are equivalent to L and are subgraphs of $L \cup (\bigcup \mathcal{C} \setminus D)$, L' is one where the quantity $\mathsf{cae}(L', \bigcup \mathcal{C} \setminus D)$ is minimized.

LEMMA 5.2. *Let* G *be a partially* Δ*-embedded graph,* $\mathcal{C}$ *be a* Δ*-nested cycle collection of* G, $D \subseteq \Delta$, L *be an* $\mathsf{ann}(\mathcal{C})$*-avoiding and* D*-free linkage of* G, *and* L' *be a* $(\mathcal{C}, D, L)$*-minimal linkage of* G, *then* $\mathbf{tw}(L' \cup (\bigcup \mathcal{C} \setminus D)) \leq f_3(|L'|)$.

Proof. Let $B = \bigcup \mathcal{C} \setminus D$ and observe that (L', B) is an LB-pair of G. Assume, towards a contradiction, that $\mathbf{tw}(L' \cup B) > f_3(|L'|)$. From Lemma 5.1, G contains a linkage L'' that is equivalent to L' where $\mathsf{cae}(L'', B) < \mathsf{cae}(L', B)$ and $L'' \subseteq L' \cup B$. This contradicts the choice of L' as a $(\mathcal{C}, D, L)$-minimal linkage of G. ∎

Streams, rivers, mountains, and valleys. Let G be a partially Δ-embedded graph, $\mathcal{C} = [C_1, \ldots, C_r]$ be Δ-nested cycle collection of G, and L be a $\mathsf{ann}(\mathcal{C})$-avoiding linkage of G. A $(\mathcal{C}, L)$-*stream* of G is a subpath of L that is a subset P of $\mathsf{ann}(\mathcal{C})$ and such that $V(P \cap C_1)$ consists of the one endpoint of P and $V(P \cap C_r)$ consists of the other. A *disjoint collection of* $(\mathcal{C}, L)$*-streams* of G is a collection $\mathcal{R}$ of $(\mathcal{C}, L)$-streams such that $\bigcup \mathcal{R}$ is a linkage of G. A $(\mathcal{C}, L)$-*river* of G is a $(\mathcal{C}, L)$-stream that is a subpath of a connected component of $L \cap \mathsf{ann}(\mathcal{C})$ that has one of its endpoints in C_1 and the other in C_r.

Notice that not each $(\mathcal{C}, L)$-stream of G is a $(\mathcal{C}, L)$-river and any collection of $(\mathcal{C}, L)$-rivers is a disjoint collection of $(\mathcal{C}, L)$-streams (see Figure 12).

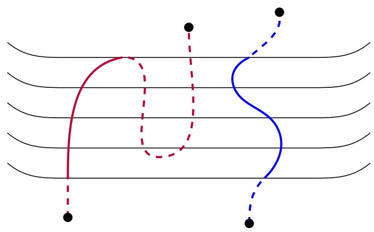

Figure 12: An example of a $(\mathcal{C}, L)$-stream (depicted in solid red) and a $(\mathcal{C}, L)$-river (depicted in solid blue).

Let $i \in [r]$ and $D \subseteq \Delta$. An $(\mathcal{C}, D, L)$*-mountain (resp.* $(\mathcal{C}, D, L)$-valley*) of* G *based on* C_i is a non-trivial subpath P of some path of L where

1. $P \subseteq \overline{D}_i$ (resp. $P \subseteq \Delta \setminus D_i$),
2. $P \cap D_r = \emptyset$ (resp. $P \cap (\Delta \setminus \overline{D}_1) = \emptyset$),
3. $P \cap C_i$ has two connected components, each containing exactly one of the endpoints of P,
4. if D' is the closure of the connected component of $D_i \setminus P$ (resp. $(\Delta \setminus \overline{D}_i) \setminus P$) that does not contain D_r (resp. $\Delta \setminus \overline{D}_1$), then $D' \cap T(L) = \emptyset$ and $D' \cap D = \emptyset$.

Clearly, in (4), D' is a closed disk. We call it, the *disk* of the $(\mathcal{C}, D, L)$-mountain (resp. valley) P and we denote it by $\mathsf{disk}(P)$.Notice that there is no $(\mathcal{C}, D, L)$-mountain based on C_r and there is no $(\mathcal{C}, D, L)$-valley based on C_1.

A $(\mathcal{C}, D, L)$*-mountain* (resp. $(\mathcal{C}, D, L)$*-valley*) of G is any $(\mathcal{C}, D, L)$-mountain (resp. $(\mathcal{C}, D, L)$-valley) of G based on some of the cycles of $\mathcal{C}$.

The *height* (resp. *depth*) of a $(\mathcal{C}, D, L)$*-mountain* (resp. $(\mathcal{C}, D, L)$*-valley*) P that is based on C_i is the maximum j such that C_{i+j-1} (resp. C_{i-j+1}) intersects P and, in both cases, we denote it by $\mathsf{dehe}(P)$. Moreover, the height (resp. depth) of P is at least 1 and at most r.

Notice that if a $(\mathcal{C}, L)$-stream P of G is a subpath of a $(\mathcal{C}, D, L)$-mountain P' or a $(\mathcal{C}, D, L)$-valley P' of G then $\mathsf{dehe}(P') = r$. Moreover, if a $(\mathcal{C}, L)$-stream P of G is not a subpath of some $(\mathcal{C}, D, L)$-mountain or some $(\mathcal{C}, D, L)$-valley of G, then P is a $(\mathcal{C}, L)$-river of G.

We say that a $(\mathcal{C}, D, L)$-mountain (resp. $(\mathcal{C}, D, L)$-valley) P based on C_i, is *tight* if $\mathsf{dehe}(P) = d \geq 2$ and there is a sequence $[P_2, \ldots, P_d]$ of $(\mathcal{C}, D, L)$-mountains (resp. $(\mathcal{C}, D, L)$-valleys) based on C_i such that

- $P = P_d$,

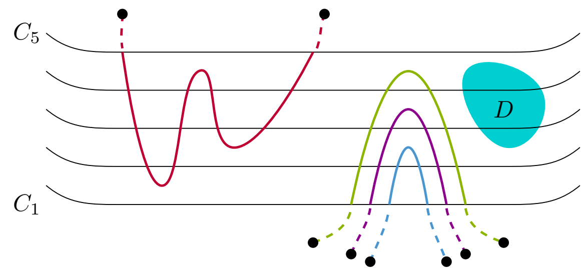

Figure 13: An example of a $(\mathcal{C}, D, L)$-valley (depicted in solid red), and some $(\mathcal{C}, D, L)$-mountains (depicted in solid colors). Notice that the $(\mathcal{C}, D, L)$-mountain depicted in green is tight.

- $\forall j \in [2, d]$, $\mathsf{dehe}(P_j) = j$, and
- $\forall j \in [2, d-1], P_j \subseteq \mathsf{disk}(P_{j+1})$ (see Figure 13).

Lemma 5.3. *Let G be a partially Δ-embedded graph, $\mathcal{C}$ be a Δ-nested cycle collection of G, $D \subseteq \Delta$, L be a $\mathsf{ann}(\mathcal{C})$-avoiding and D-free linkage of G. Let also L' be a $(\mathcal{C}, D, L)$-minimal linkage of G. Then all $(\mathcal{C}, D, L')$-mountains (resp. $(\mathcal{C}, D, L')$-valleys) of G are tight.*

Proof. Let $B = \bigcup \mathcal{C} \setminus D$. We present the proof for the case of $(\mathcal{C}, D, L')$-mountains as the case of $(\mathcal{C}, D, L')$-valleys is symmetric.

Claim: Let $i \in \mathbb{N}_{\geq 1}, j \in \mathbb{N}_{\geq 2}$. If P_j is a $(\mathcal{C}, D, L')$-mountain of G based on C_i such that $\mathsf{dehe}(P_j) = j$, then there exists a $(\mathcal{C}, D, L')$-mountain P' based on C_i such that $\mathsf{dehe}(P') = j - 1$ and $P' \subseteq \mathsf{disk}(P_j)$.

Proof of Claim: Suppose to the contrary that there does not exist a $(\mathcal{C}, D, L')$-mountain P' based on C_i such that $\mathsf{dehe}(P') = j - 1$ and $P' \subseteq \mathsf{disk}(P_j)$. Let $P_j^{(j-1)} = (P_j \setminus D_{i+(j-1)-1}) \cup (C_{i+(j-1)-1} \cap \mathsf{disk}(P_j))$ and notice that $\mathsf{dehe}(P_j^{(j-1)}) = j - 1$ (see Figure 14).

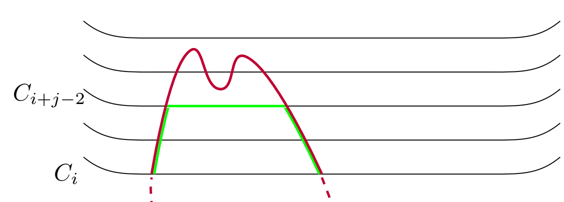

Figure 14: An example of a $(\mathcal{C}, D, L')$-mountain P_j of G based on C_i such that $\mathsf{dehe}(P_j) = j$ (depicted in red) and the $(\mathcal{C}, D, L')$-mountain $P_j^{(j-1)}$ (depicted in green).

Then G contains a linkage $L'' = (L' \setminus P_j) \cup (P_j^{(j-1)})$ that is equivalent to L where $\mathsf{cae}(L'', B) < \mathsf{cae}(L', B)$ and $L'' \subseteq L' \cup B$. This contradicts the choice of L' as a $(\mathcal{C}, D, L)$-minimal linkage of G. The claim follows.

Let P be a $(\mathcal{C}, D, L')$-mountain of G based on C_i such that $\mathsf{dehe}(P) = d \geq 2$. The fact that P is tight follows by recursively applying the Claim above. ∎

Orderings of streams. Let G be a partially Δ-embedded graph, $\mathcal{C}$ be a Δ-nested cycle collection of G, D be an open disk where $D \subseteq \mathsf{ann}(\mathcal{C})$, L be an $\mathsf{ann}(\mathcal{C})$-avoiding and D-free linkage of G.

If $\mathcal{Z}$ is a disjoint collection of $(\mathcal{C}, L)$-streams of G we define its *D-ordering* as follows: Consider the sequence $[Z_1, \ldots, Z_d]$ such that for each $i \in [d]$, one, say D_i, of the two connected components of $\mathsf{ann}(\mathcal{C}) \setminus (Z_i \cup Z_{i+1})$ does not intersect $\bigcup \mathcal{Z}$ (here Z_{d+1} denotes Z_1). Among all $(d-1!)$ such sequences we insist that $[Z_1, \ldots, Z_d]$ is the unique one where $D \subseteq D_q$ and that the order of $\mathcal{Z}$ is the counter-clockwise order that its elements appear around $\mathsf{ann}(\mathcal{C})$ (see Figure 15). We call $[Z_1, \ldots, Z_d]$ the *D-ordering* of $\mathcal{Z}$.

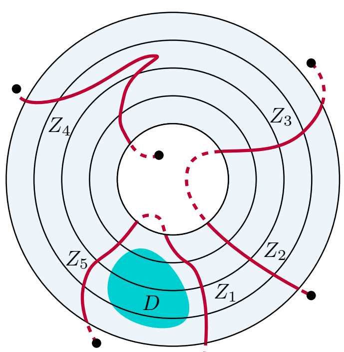

Figure 15: An example of an Δ-nested cycle collection $\mathcal{C}$, an open disk $D \subseteq \mathsf{ann}(\mathcal{C})$ (depicted in blue), a linkage L (depicted in red) that is D-free and $\mathsf{ann}(\mathcal{C})$-avoiding, a disjoint collection $\mathcal{Z}$ of $(\mathcal{C}, L)$-streams, and the D-ordering $[Z_1, \ldots, Z_5]$ of $\mathcal{Z}$.

Brambles. Given a graph G, we say that a subset S of $V(G)$ is *connected* if $G[S]$ is connected. Given $S_1, S_2 \subseteq V(G)$, we say that S_1 and S_2 touch if either $S_1 \cap S_2 \neq \emptyset$ or there is an edge $e \in E(G)$ where $e \cap S_1 \neq \emptyset$ and $e \cap S_2 \neq \emptyset$. A *bramble* in G is a collection $\mathcal{B}$ is pairwise touching connected subsets of $V(G)$. The *order* of a bramble $\mathcal{B}$ is the minimum number of vertices that intersect all of its elements.

Proposition 5.2. ([31]) *Let $k \in \mathbb{N}_{\geq 0}$. A graph G has a bramble of order $k+1$ if and only if $\mathbf{tw}(G) \geq k$.*

We now use the notions of ordering of streams and brambles to prove the following result.

Lemma 5.4. *Let G be a partially Δ-embedded graph, $\mathcal{C}$ be Δ-nested cycle collection of G, D be an open disk where $D \subseteq \mathsf{ann}(\mathcal{C})$, L be an $\mathsf{ann}(\mathcal{C})$-avoiding and D-free linkage of G, and $\mathcal{Z}$ be a disjoint collection of $(\mathcal{C}, L)$-streams of G. Then $\mathbf{tw}(L \cup (\bigcup \mathcal{C} \setminus D)) \geq \min\{|\mathcal{C}|, |\mathcal{Z}|\}$.*

Proof. Let $[Z_1, \ldots, Z_d]$ be the D-ordering of $\mathcal{Z}$ and let D' be the connected component of $\mathsf{ann}(\mathcal{C}) \setminus (Z_d \cup Z_1)$ that contains D. Let $r = \min\{|\mathcal{C}|, |\mathcal{Z}|\}$, and let $[Z_1, \ldots, Z_r]$ be the sequence consisting of the first r elements of the D-ordering of $\mathcal{Z}$. Let also $\mathcal{C}'$ be the sequence consisting of the first r elements of $\mathcal{C}$. Notice that there is a disjoint collection $\mathcal{Z}' = [Z'_1, \ldots, Z'_r]$ of $(\mathcal{C}', L)$-streams of G such that for each $i \in [r]$, $Z'_i \subseteq Z_i$.

We now set $\mathcal{B} = \mathcal{C}' \setminus D'$, denote $\mathcal{B} = [B_1, \ldots, B_r]$, and notice that both $\mathcal{B}$ and $\mathcal{Z}'$ are sequences of paths in G, such that both $\bigcup \mathcal{B}$ and $\bigcup \mathcal{Z}'$ are linkages of G. Consider now the graph $Q = \bigcup \mathcal{B} \cup \bigcup \mathcal{Z}'$ and notice that $C = B_1 \cup Z'_1 \cup B_r \cup Z'_r$ is a cycle of G.

As $Q \subseteq L \cup (\bigcup \mathcal{C} \setminus D)$, it remains to prove that $\mathbf{tw}(Q) \geq r$. For this, because of Proposition 5.2, it suffices to give a bramble of Q of order $r+1$. For each $(i,j) \in [2, r-1]^2$ we define $X^{(i,j)} = (B_i \cup Z'_j) \setminus V(C)$. It is easy to check that $\mathcal{X} = \{X^{(i,j)} \mid (i,j) \in [2, r-1]^2\}$ is a bramble of Q of order $\geq r-2$. Let also $X^{(1)} = Z_1 \setminus B_1$, $X^{(2)} = B_1$, and $X^{(3)} = Z'_r \cup B_r$. Notice that $\mathcal{X} \cup \{X^{(1)}, X^{(2)}, X^{(3)}\}$ is also a bramble of Q and its order is the order of $\mathcal{X}$ incremented by 3. Therefore Q contains a bramble of order at least $r+1$, as required (see Figure 16).

Figure 16: An example of the construction of a bramble of Q, where $|\mathcal{B}| = 5$ and $|\mathcal{Z}'| = 5$. Here, $X^{(2,2)}, X^{(3,3)}, X^{(4,4)}$ are depicted in red, green, and yellow, respectively, while $X^{(1)}, X^{(2)}, X^{(3)}$ are depicted in orange, brown, and blue, respectively.

∎

Lemma 5.5. *Let G be a partially Δ-embedded graph, $\mathcal{C}$ be a Δ-nested cycle collection of G, D be a connected subset of Δ, and L be a $\mathsf{ann}(\mathcal{C})$-avoiding and D-free linkage of G. If P is a tight $(\mathcal{C}, D, L)$-mountain (resp. $(\mathcal{C}, D, L)$-valley) of G, then $\mathbf{tw}(L \cup (\bigcup \mathcal{C} \setminus D)) \geq \frac{2}{3} \cdot \mathsf{dehe}(P)$.*

Proof. Let $d = \mathsf{dehe}(P)$. We examine the non-trivial case where $d \geq 3$. We present the proof for the case where P is a $(\mathcal{C}, D, L)$-mountain as the case where P is a $(\mathcal{C}, D, L)$-valley is symmetric.

We assume that P is based on C_i, for some $i \in [r]$. By the definition of tightness, there is a sequence $\mathcal{P} = [P_2, \ldots, P_d = P]$ of $(\mathcal{C}, D, L)$-mountains (resp. $(\mathcal{C}, D, L)$-valleys) based on C_i such that

- $P = P_d$,

- $\forall i \in [2,d]$, $\mathsf{dehe}(P_i) = i$, and
- $\forall i \in [2,d-1], P_i \subseteq \mathsf{disk}(P_{i+1})$.

For every $j \in [2,d]$, we denote $\mathcal{C}^{(j)} = [C_i, \ldots, C_{i+j-1}]$. Notice that for every $j \in [2,d]$, L is an $\mathsf{ann}(\mathcal{C}^{(j)})$-avoiding and D-free linkage of G.

Claim: For every $j \in [2,d-1]$ there exists a disjoint collection $\mathcal{Z}_j$ of $(\mathcal{C}^{(j)}, L)$-streams of G where $|\mathcal{Z}_j| \geq 2(d-j)+1$.

Proof of Claim: Let $j \in [2,d-1]$. Observe that for each $h \in [j+1,d]$ exactly two of the connected components of $\mathsf{ann}(C^{(j)}) \cap P_h$ are $(\mathcal{C}^{(j)}, L)$-rivers in G. This implies that there is a collection $\mathcal{R}_j$ of at least $2(d-j)$ $(\mathcal{C}^{(j)}, L)$-rivers in G. Recall that $\mathcal{R}_j$ is a disjoint collection of $(\mathcal{C}^{(j)}, L)$-streams of G. Observe also that we can pick some sub-path of $\mathsf{ann}(C^{(j)}) \cap P_j$ that has one endpoint in C_i and the other in C_{i+j-1}. As this path does not share vertices with any of the paths in $\mathcal{R}_j$ we can add it in $\mathcal{R}_j$ and obtain a disjoint collection $\mathcal{Z}_j$ of $(\mathcal{C}^{(j)}, L)$-streams of G where $|\mathcal{Z}_j| \geq 2(d-j)+1$. Claim follows (see Figure 17).

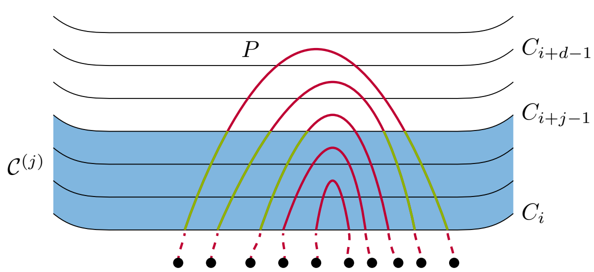

Figure 17: An example of a tight $(\mathcal{C}, D, L)$-mountain P based on C_i of height d and the respective sequence of $(\mathcal{C}, D, L)$-mountains based on C_i (depicted in red), an annulus $\mathcal{C}^{(j)}$ (depicted in blue), for some $j \in [2,d]$,and a disjoint collection $\mathcal{Z}_j$ (depicted in green) of $(\mathcal{C}^{(j)}, L)$-streams of G.

We now set $j' = \lfloor (2d+1)/3 \rfloor$ and observe that $2 \leq j' \leq d-1$. The above claim implies that there exists a disjoint collection $\mathcal{Z}_{j'}$ of $(\mathcal{C}^{(j')}, L)$-streams of G such that $|\mathcal{Z}_{j'}| \geq 2(d-j')+1 \geq j' = |\mathcal{C}^{(j')}|$. Therefore, we can apply Lemma 5.4 on $\mathcal{C}^{(j')}$ and deduce that $\mathbf{tw}(L \cup (\bigcup \mathcal{C}^{(j')} \setminus D)) \geq j'$. The Lemma follows as $L \cup (\bigcup \mathcal{C}^{(j')} \setminus D) \subseteq L \cup (\bigcup \mathcal{C} \setminus D)$ and $\lfloor (2d+1)/3 \rfloor \geq 2d/3$. $\square$

Lemma 5.6. *Let G be a partially Δ-embedded graph, $\mathcal{C}$ be a Δ-nested cycle collection of G, D be a connected subset of Δ, L be a $\mathsf{ann}(\mathcal{C})$-avoiding and D-free linkage of G, and L' be a $(\mathcal{C}, D, L)$-minimal linkage of G. Then all $(\mathcal{C}, D, L')$-mountains (resp. $(\mathcal{C}, D, L')$-valleys) of G have height (resp. depth) at most $\frac{3}{2} \cdot f_3(|L'|)$.*

Proof. We set $B = \bigcup \mathcal{C} \setminus D$. By Lemma 5.2, $\mathbf{tw}(L' \cup B) \leq f_3(|L'|)$. Let P be a $(\mathcal{C}, D, L')$-mountain (resp. $(\mathcal{C}, D, L')$-valley) of G based on C_i, for some $i \in [r-1]$ (resp. $i \in [2,r]$). From Lemma 5.3, P should be tight and, from Lemma 5.5, $\mathbf{tw}(L' \cup B) \geq \frac{2}{3} \cdot \mathsf{dehe}(P)$. Therefore, $\mathsf{dehe}(P) \leq \frac{3}{2} \cdot f_3(|L'|)$. $\square$

Lemma 5.7. *Let G be a partially Δ-embedded graph, $\mathcal{C} = [C_1, \ldots, C_r]$ be a Δ-nested cycle collection of G, and L be a $\overline{D}_1$-avoiding linkage. Then there is a linkage L' of G such that*

1. *L' is $\overline{D}_1$-avoiding,*
2. *$L' \equiv L$,*
3. *L' is $\overline{D}_{3m/2+1}$-free, where $m = f_3(|L'|)$.*

Proof. Let G^+ be the graph obtained if we take its disjoint union with a cycle $C_{r+1} \subseteq D_r$ and we set $\mathcal{C}^+ = [C_1, \ldots, C_r, C_{r+1}]$. Observe that L is an $\mathsf{ann}(\mathcal{C}^+)$-avoiding linkage of G^+. Let L' be a $(\mathcal{C}^+, \emptyset, L)$-minimal linkage of G^+.

As $L' \equiv L$, L' is a $\overline{D}_1$-avoiding linkage of both G and G^+. Therefore L' satisfies conditions (1) and (2). For condition (3), assume to the contrary that L' is a linkage of G that is intersecting $\overline{D}_{3m/2+1}$. As L' is a $\overline{D}_1$-avoiding linkage of G^+ we obtain that G^+ contains some $(\mathcal{C}, \emptyset, L')$-mountain P, based on C_1 where $\mathsf{dehe}(P) > 3m/2$. On the other side, as L is an $\mathsf{ann}(\mathcal{C}^+)$-avoiding linkage of G^+ we can apply Lemma 5.6, on G^+, $\mathcal{C}^+$, $\emptyset$, L, and L' and obtain that $\mathsf{dehe}(P) \leq 3m/2$, a contradiction. $\square$

Lemma 5.8. *Let G be a partially Δ-embedded graph, $\mathcal{C}$ be a Δ-nested cycle collection of G, $D \subseteq \Delta$, and L be an $\mathcal{A}$-avoiding and D-free linkage. If $|\mathcal{C}| > m = f_3(|L|)$, then G contains a linkage L' of G such that*

1. *$L' \equiv L$,*
2. *$L' \cap D = \emptyset$,*
3. *All $(\mathcal{C}, D, L')$-mountains of G have height at most $\frac{3}{2}m$,*
4. *All $(\mathcal{C}, D, L')$-valleys of G have depth at most $\frac{3}{2}m$, and*
5. *L' has at most m $\mathcal{A}$-rivers.*

Proof. Let L' be a $(\mathcal{C}, D, L)$-minimal linkage. (1) and (2) follow by the definition of a $(\mathcal{C}, D, L)$-minimal linkage. (3) and (4) follow directly from Lemma 5.6. To prove (5), assume that G contains a collection $\mathcal{Z}$ of $(\mathcal{C}, L')$-rivers where $|\mathcal{Z}| > m$. Recall that $\mathcal{Z}$ is a disjoint collection of $(\mathcal{C}, L')$-streams of G. From Lemma 5.4, $\mathbf{tw}(L' \cup (\bigcup \mathcal{C} \setminus D)) \geq \min\{|\mathcal{C}|, |\mathcal{Z}|\} > m$. We arrive at a contradiction as, from Lemma 5.2, $\mathbf{tw}(L' \cup (\bigcup \mathcal{C} \setminus D)) \leq m$. $\square$

5.3 Rerouting a linkage The following proposition is a direct consequence of [1, Lemma 7].

PROPOSITION 5.3. *Let k, k', d be integers such that $0 \leq d \leq k' \leq k$. Let Γ be a $(k \times k')$-grid and let $\{p_1^{\text{up}}, \ldots, p_\rho^{\text{up}}\}$ (resp. $\{p_1^{\text{down}}, \ldots, p_\rho^{\text{down}}\}$) be vertices of the higher (resp. lower) horizontal line arranged as they appear in it from left to right. Then the grid Γ contains ρ pairwise disjoint paths $P_1, \ldots, P_\rho$ such that, for every $h \in [\rho]$, the endpoints of P_h are p_h^{up} and p_h^{down}.*

Given two vertex disjoint paths P_1 and P_2 of G, we say that an (P_1, P_2)-path of G is a path that whose one endpoint is a vertex of P_1 the other endpoint is a vertex of H_2 and contains all edges of $P_1 \cup P_2$. We now prove the following:

LEMMA 5.9. *Let $r, q, s \in \mathbb{N}_{\geq 3}$, $b, d \in \mathbb{N}_{\geq 0}$, such that $r \geq s + 2b$ and $q \geq b + d$, where r and s are odd numbers. If G is a partially Δ-embedded graph, $\mathcal{A}$ is a (r, q)-railed annulus of G, $I \subseteq [q]$ where $|I| \geq d$, then there is a linkage K of G such that,*

(a) there is an ordering $\mathcal{P}(K) = [K_1, \ldots, K_d]$, where for $i \in [d]$, K_i is a $(P_{1,b+i}, P_{r,b+i})$-path of G.

(b) K is (s, I)-confined in $\mathcal{A}$.

Proof. Let $\mathcal{A} = (\mathcal{C}, \mathcal{P})$, let $t = \lfloor r/2 \rfloor$ and $t' = \lfloor s/2 \rfloor$. Also, let $\{i_1, \ldots, i_d\} \subseteq I$ such that $\forall j \in [d-1], i_j < i_{j+1}$. *Claim:* There is a collection of pairwise disjoint paths $\mathcal{P}^{\mathsf{down}} = \{P_1^{\mathsf{down}}, \ldots, P_d^{\mathsf{down}}\}$ such that, for every $h \in [d]$, P_h^{down} is a $(P_{1,b+h}, P_{b,i_h})$-path.

Proof of Claim: For $i \in [b], j \in [q]$ let $p_{i,j}$ be the vertex obtained after contracting all edges in $P_{i,j}$. We also define:

- $E_1 = \{e = \{p_{i,j}, p_{i,j+1}\} \mid e$ is the edge obtained after contracting all but one of the edges of $L_{i,j \to j+1}, i \in [b], j \in [q-1]\}$ and
- $E_2 = \{e = \{p_{i,j}, p_{i+1,j}\} \mid e$ is the edge obtained after contracting all but one of the edges of $R_{i \to i+1,j}, i \in [b-1], j \in [q]\}$.

Let H be the graph where $V(H) = \{p_{i,j} \mid (i,j) \in [b] \times [q]\}$ and $E(H) = E_1 \cup E_2$. Observe that H is isomorphic to a $(q \times b)$-grid (see Figure 18). For $h \in [d]$, let p_h^{low} (resp. p_h^{high}) be the vertex $p_{1,b+h}$ (resp. p_{b,i_h}).

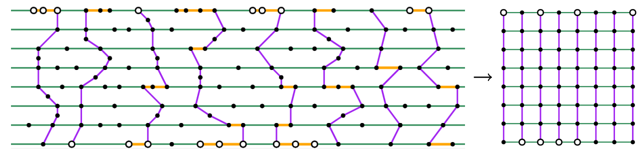

Figure 18: An example showing the construction of the graph H. For every $h \in [d]$, the resulting vertices p_h^{low} and p_h^{high} (corresponding to the vertices of the paths $P_{1,b+h}$ and P_{b,i_h}, respectively) are depicted in white.

Due to Proposition 5.3, H contains d pairwise disjoint paths $P_1, \ldots, P_d$ such that, for every $h \in [d]$, the endpoints of P_h are p_h^{low} and p_h^{high}. Therefore, if we substitute every vertex of each P_i with the the edges that where contracted in G in order to obtain it in H, we obtain the claimed result.

By applying the previous claim symmetrically, we can find a collection of pairwise disjoint paths $\mathcal{P}^{\mathsf{up}} = \{P_1^{\mathsf{up}}, \ldots, P_d^{\mathsf{up}}\}$ such that, for every $h \in [d]$, P_h^{up} is a $(P_{r-b,i_h}, P_{r,b+h})$ path.

Now, for every $h \in [d]$, let $P_h^{\mathsf{mid}} = \mathsf{ann}(\mathcal{C}, b, r-b) \cap P_{i_h}$ and let $K_h = P_h^{\mathsf{down}} \cup P_h^{\mathsf{mid}} \cup P_h^{\mathsf{up}}$. Since $r \geq s+2b$ and $s = 2t'+1$ then $\mathsf{ann}(\mathcal{C}, t-t', t+t') \subseteq \mathsf{ann}(\mathcal{C}, b, r-b)$ and therefore $K = \{K_1, \ldots, K_d\}$ is the desired linkage. This concludes the proof. ∎

Let $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ be an (r, q)-railed annulus of a partially Δ-embedded graph G. We set $z = \lfloor \min\{r, q\}/2 \rfloor$. For each $i \in [z]$, we define $C_i^{\mathcal{A}}$ as the unique cycle of the graph

$$L_{i,i \to q-i+1} \cup L_{r-i+1,i \to q-i+1} \cup R_{i \to r-i+1,i} \cup R_{i \to r-i+1,q-i+1}.$$

Notice that if $r, q \geq 5$, then $[C_1^{\mathcal{A}}, \ldots, C_z^{\mathcal{A}}]$ is a Δ-nested collection of cycles of G and we denote it by $\mathcal{C}_{\mathcal{A}}$ (see Figure 19).

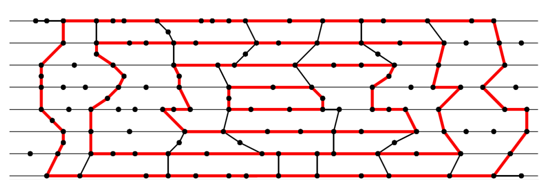

Figure 19: An example of an $(8, 8)$-railed annulus $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ and the sequence $C_{\mathcal{A}}$ (depicted in red).

We are now ready to prove Theorem 5.1.

Proof. [Proof of Theorem 5.1] Recall that f_3 is the function of Proposition 5.1. We define $f_4(k) := 3 \cdot (f_3(k))^2 + 6 \cdot f_3(k) + 2$. For simplicity, we use $m =$

$f_3(k)$. Let also $b = 3m/2$, and keep in mind that $r \geq f_4(k)+s = 3m^2+6m+2+s = 2(m+1)\cdot b+2+s+2b$ and that $|I| \geq m+1$.

Recall that $\mathcal{C}_{\mathcal{A}} = [C'_1,\ldots,C'_z]$, where $z = \lfloor \min\{q,r\}/2 \rfloor$, is a Δ-nested collection of cycles of G. For each $i \in [z]$, we denote by D'_i (resp. $\overline{D}'_i$) the open (closed) disk corresponding to C'_i. Let also $\breve{D} := D'_{b+1}$ and $D := \overline{D}'_{b+1}$. Keep in mind that $\overline{D}'_1 = \Delta_{1,r,1,q}$ and $D = \Delta_{b+1,r-b,b+1,q-b}$.

Observe now that L is a $\overline{D}'_1$-avoiding linkage. By applying Lemma 5.7 on G, $\mathcal{C}_{\mathcal{A}}$, L, and $\overline{D}'_1$, we obtain that G has a $\overline{D}'_1$-avoiding and D-free linkage L' such that $L' \equiv L$.

It is easy to verify that L' is $\mathcal{A}$-avoiding, $r \geq a > m$, $D \subseteq \mathsf{ann}(\mathcal{C})$, and $|L'| = |L| \leq k$. Therefore, we may apply Lemma 5.8 on k, G, $\mathcal{A}$, D, and L'. We obtain a D-free linkage L'' of G that is equivalent to L' (and therefore to L as well) and such that

(a) All $(\mathcal{C}, D, L'')$-mountains/valleys of G have height/depth at most b.

(b) L'' has at most m $\mathcal{A}$-rivers of G,

Let $\mathcal{Z} = [Z_1,\ldots,Z_d]$ be the D-ordering of the $\mathcal{A}$-rivers of L'' in G and keep in mind that, from (b), $d \leq m$.

For every $i \in [d]$, we define x_i^{down} (resp. x_i^{up}) as the vertex in the path $C_{(i+1)\cdot b+1} \setminus \breve{D}$ (resp. $C_{r-(i+1)\cdot b} \setminus \breve{D}$) that belongs in Z_i and has the minimum possible distance to the vertices of the path $P_{(i+1)b+1,q-b}$ (resp. $P_{r-(i+1)\cdot b,q-b}$). We also denote by Q_i^{down} (resp. Q_i^{up}) the path certifying this minimum distance.

For $i \in [d]$, let Z_i^{down} and Z_i^{up} be the two connected components of the graph obtained from Z_i if we remove the edges of its $(x_i^{\text{down}}, x_i^{\text{up}})$-subpath (see Figure 20). We choose Z_i^{down} (resp. Z_i^{up}) so that it intersects C_1 (resp. C_r).

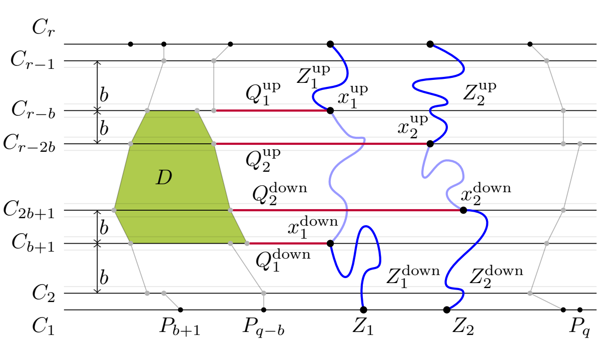

Figure 20: Visualization of an (r, q)-railed annulus and the notations introduced above.

Claim: For $i \in [d]$, Z_{i-1}^{down} (resp. Z_{i-1}^{up}) does not intersect Q_i^{down} (resp. Q_i^{up}) — where Z_0^{down} (resp. Z_0^{up}) denotes Z_q.

Proof of claim: If $Z_{i-1}^{\text{down}} \cap Q_i^{\text{down}} \neq \emptyset$ (resp. $Z_{i-1}^{\text{up}} \cap Q_i^{\text{up}} \neq \emptyset$) for some $i \in [d]$, then some of the connected components of $Z_{i-1}^{\text{down}} \cap \overline{D}_{i\cdot b+1}$ (resp. $Z_{i-1}^{\text{up}} \cap (\Delta \setminus D_{r-i\cdot b})$) whose endpoints are in $C_{i\cdot b+1}$ (resp. $C_{r-i\cdot b}$) should be a $(\mathcal{C}, D, L'')$-mountain (resp. $(\mathcal{C}, D, L'')$-valley) of G of height (resp. depth) $> b$, a contradiction to (a). Claim follows.

Because of the above claim, it follows that the paths $Q_i^{\text{down}} \cup Z_i^{\text{down}}$ (resp. $Q_i^{\text{up}} \cup Z_i^{\text{up}}$), $i \in [d]$ are pairwise vertex-disjoint $(Z_i \cap C_1, P_{(i+1)\cdot b+1,q-b})$-paths (resp. $(Z_i \cap C_r, P_{r-(i+1)\cdot b,q-b})$-paths) in G that do not intersect the open disk $\breve{D}$.

Let $w = (m+1)\cdot b+2$ and $w' = r-(m+1)\cdot b-1$. For $i \in [d]$, we now define (see Figure 21)

$$\begin{aligned}
Y_i^{\text{down}} &= \text{the } (P_{q-b}, P_{b+i})\text{-path} \\
&\quad \text{in } L_{(i+1)\cdot b+1,q-b\to b+i} \cup P_{(i+1)\cdot b+1,b+i} \cup \\
&\quad R_{(i+1)\cdot b+1\to w,b+i}, \\
Y_i^{\text{up}} &= \text{the } (P_{q-b}, P_{b+i})\text{-path} \\
&\quad \text{in } L_{r-(i+1)\cdot b,q-b\to b+i} \cup P_{r-(i+1)\cdot b+1,b+i} \cup \\
&\quad R_{r-(i+1)\cdot b\to w',b+i}.
\end{aligned}$$

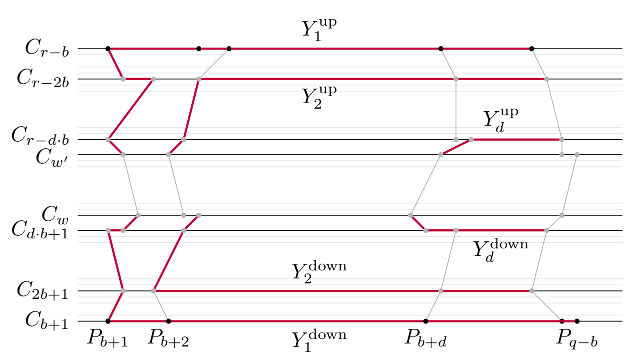

Figure 21: Visualization of the definition of Y_i^{up} and Y_i^{down}, $i \in [d]$.

By the definition of Y_i^{down} and Y_i^{up}, the graphs $X_i^{\text{down}} = Z_i^{\text{down}} \cup Q_i^{\text{down}} \cup Y_i^{\text{down}}$ and $X_i^{\text{up}} = Z_i^{\text{up}} \cup Q_i^{\text{up}} \cup Y_i^{\text{up}}$, $i \in [d]$, are pairwise vertex-disjoint paths. In particular, taking into account the definition of Y_i^{down} and Y_i^{up}, we have that

(5.1) $\quad X_i^{\text{down}}$ is a $(Z_i \cap C_1, P_{w,b+i})$-path and

(5.2) $\quad X_i^{\text{up}}$ is a $(Z_i \cap C_r, P_{w',b+i})$-path

Let $\Omega = \mathsf{ann}(\mathcal{C}, w, w')$ and $K' = \bigcup X_i^{\text{down}} \cup \bigcup X_i^{\text{up}}$. Observe that

(5.3) $\quad K' \cap \Omega = \{P_{w,b+i}, P_{w',b+1} \mid i \in [d]\}.$

Let $\bar{\mathcal{A}} = (\bar{\mathcal{C}}, \bar{\mathcal{P}})$, where $\bar{\mathcal{C}} = [C_w, \ldots, C_{w'}]$ and $\bar{\mathcal{P}} = \mathcal{P} \cap \Omega$.

Notice that $|\bar{\mathcal{C}}| = w' - w + 1 = r - 2(m+1) \cdot b - 2 \geq s + 2b$. Notice also that $d \leq |I|$, $I \subseteq [q]$. Finally, $b = 3/2m$ and $d \leq m$ imply that $d + b \leq 5/2m \leq q$. We can now apply Lemma 5.9 for $r, q, s, b, d, \bar{\mathcal{A}}$, and I and obtain a linkage K of $\bar{\mathcal{A}}$ satisfying properties (a), and (b) of Lemma 5.9.

From Property (a) we can write $\mathcal{P}(K) = [K_1, \ldots, K_d]$ and, using (5.3), we deduce that, for $i \in [d]$, K_i is a $(P_{w,b+i}, P_{w',b+i})$-path of G. This, together with (5.1), (5.2), and (5.3), implies that $K \cup K'$ is a linkage of G where $K \cup K' \subseteq \mathsf{ann}(\mathcal{C})$. From Property (b), K is (s, I)-confined in $\bar{\mathcal{A}}$, therefore, from (5.3), we get that $K \cup K'$ is (s, I)-confined in $\mathcal{A}$. Observe also that each of the d paths of $\mathcal{P}(K \cup K')$ is a $(Z_i \cap C_1, Z_i, \cap C_r)$-path of G for some $i \in [d]$. We define

$$\tilde{L} = (L \setminus A') \cup K \cup K'$$

where $A' = \mathsf{ann}(\mathcal{C}) \setminus (C_1 \cup C_r)$. By definition $\tilde{L}$ is a linkage of G where $\tilde{L} \equiv L$ and $\tilde{L} \setminus \mathsf{ann}(\mathcal{C}) \subseteq L \setminus \mathsf{ann}(\mathcal{C})$. Finally, as $K \cup K'$ is (s, I)-confined in $\mathcal{A}$, then $\tilde{L}$ (s, I)-confined in $\mathcal{A}$ as well. ∎

Now, since we proved Theorem 5.1, we will use it to in order to prove Theorem 2.1.

Proof. [Proof of Theorem 2.1] Let s be a positive odd integer, H be a graph on g edges, G be a partially Δ-embedded graph, $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ be a (r, q)-railed annulus of G, where $r \geq f_4(g) + 2 + s$ and $q \geq 5/2 \cdot f_3(g)$, (M, T) be a topological minor model of H in G such that $T \cap \mathsf{ann}(\mathcal{A}) = \emptyset$.

Let $\mathcal{A}' = ([C_2, \ldots, C_{r-1}], \mathcal{P} \cap \mathsf{ann}(\mathcal{C}, 2, r-1))$ and keep in mind that $\mathcal{A}'$ is a (r, q)-railed annulus of G, where $r \geq f_4(g) + s$ and $q \geq 5/2 \cdot f_3(g)$. Notice that it also holds that $T \cap \mathsf{ann}(\mathcal{A}') = \emptyset$ (see Figure 22).

Let $\tilde{M}^{(1)} = \tilde{M}[N_{\tilde{M}}[\tilde{T}]]$. Notice that all the connected components of $M \setminus T$ are paths of G. Let L be the linkage of $G \setminus T$ created if we take the union of all non-trivial connected components of $M \setminus T$. Observe that $\mathcal{P}(L)$ is the set of all paths of G connecting neighbors of branch vertices of M and consisting only of subdividing vertices of M. Also, notice that since $T \cap \mathsf{ann}(\mathcal{A}') = \emptyset$, then L is $\mathcal{A}'$-avoiding and there is an one-to-one correspondence of $\mathcal{P}(L)$ with $E(H)$ and thus $|L| \leq g$.

Let $I \subseteq [q]$, where $|I| > f_3(h)$. By applying Theorem 5.1 for $s, g, G, \mathcal{A}', L$, and I we obtain a linkage $\tilde{L}$ of G such that $\tilde{L} \equiv L$, $\tilde{L}$ is $\mathcal{A}'$-avoiding, $\tilde{L} \setminus \mathsf{ann}(\mathcal{A}') \subseteq L \setminus \mathsf{ann}(\mathcal{A}')$, and $\tilde{L}$ (s, I)-confined in $\mathcal{A}'$. We define

$$\tilde{M} = (M \setminus L) \cup \tilde{L}.$$

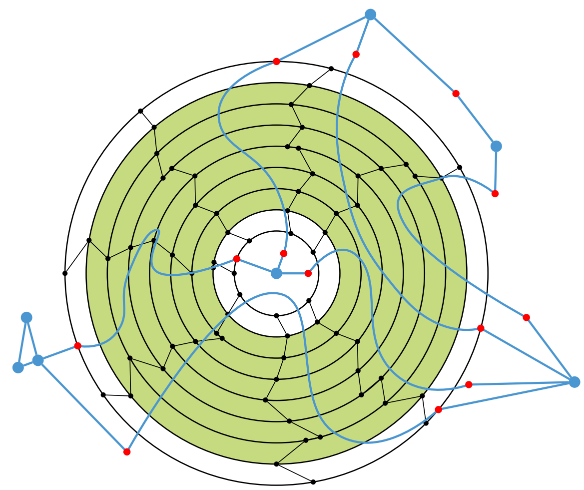

Figure 22: An example of a topological minor model (M, T) of H in G. Vertices of T are depicted in blue while the neighbors of vertices of T that are also subdividing vertices are depicted in red. Also, $\mathsf{ann}(\mathcal{A}')$ is depicted in green.

By definition, $(\tilde{M}, T)$ is a topological minor model of H in G. Also, since $L, \tilde{L} \subseteq \mathsf{ann}(\mathcal{A})$, then $\tilde{M} \setminus \mathsf{ann}(\mathcal{A}) \subseteq M \setminus \mathsf{ann}(\mathcal{A})$. Finally, as $\tilde{L}$ is (s, I)-confined in $\mathcal{A}'$ then $\tilde{M}$ is (s, I)-confined in $\mathcal{A}$ as well. ∎

6 Conclusions

In this paper we prove that $\mathcal{F}$-TM-Deletion is Fixed Parameter Tractable on planar graphs by designing an $O_{k,h}(n^2)$-time algorithm for his problem.

In this paper we did not make any effort to specify the contribution of the main parameter k in the complexity of the algorithm. However, we suggest that the contribution of k can be single-exponential, in particular $2^{O_h(k)} n^2 + O(n^3)$. This can be done if, instead of using Courcelle's theorem in the proofs of Theorem 1.2 and Lemma 2.2, we use the $2^{O_h(w)} n + O(n^3)$-time dynamic programming algorithm of [4] for computing $\mathbf{p}_{\mathcal{F}}(G)$ on planar graphs with treewidth at most w.

The remaining question is whether the same result can be derived for *all* graphs, as we conjectured in the introduction. Towards this, we chose to state all combinatorial theorems of this paper in more general forms. Based on them, a straightforward generalization is possible for the class of surface embeddable graphs, that is graphs with Euler genus at most γ. Indeed, the only piece of the proof that needs extension is the starting point of the proof, that is the algorithm of Proposition 2.1, that can easily be extended to work on graphs of Euler genus γ. Using this, we can directly derive a $O_{k,h,\gamma}(n^2)$-time algorithm for the version of the problem on surfaces (which can be further improved to one running in $2^{O_{h,\gamma}(k)}n + O(n^3)$-time, again using the results from [4]). It follows that with much more effort it is possible to extend the result to every class that excludes some fixed graph as a minor. However, for a complete resolution of our conjecture one has to deal with the case where the input graph contains a big clique minor. We believe that the techniques of the algorithm of [15] can be a good starting point in this direction. However, the technical challenges of such an extension are cumbersome.

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