Theoretical Computer Science 815 (2020) 182–196

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

A complexity dichotomy for critical values of the b-chromatic number of graphs

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ARTICLE INFO

Article history:
Received 12 July 2019
Received in revised form 30 January 2020
Accepted 8 February 2020
Available online 12 February 2020
Communicated by L.M. Kirousis

Keywords:
b-coloring
b-chromatic number
Complexity dichotomy

ABSTRACT

A *b-coloring* of a graph G is a proper coloring of its vertices such that each color class contains a vertex that has at least one neighbor in all the other color classes. The b-Coloring problem asks whether a graph G has a b-coloring with k colors. The *b-chromatic number* of a graph G, denoted by $\chi_b(G)$, is the maximum number k such that G admits a b-coloring with k colors. We consider the complexity of the b-Coloring problem, whenever the value of k is close to one of two upper bounds on $\chi_b(G)$: The maximum degree $\Delta(G)$ plus one, and the m-degree, denoted by $m(G)$, which is defined as the maximum number i such that G has i vertices of degree at least $i-1$. We obtain a dichotomy result for all fixed $k \in \mathbb{N}$ when k is close to one of the two above mentioned upper bounds. Concretely, we show that if $k \in \{\Delta(G)+1-p, m(G)-p\}$, the problem is polynomial-time solvable whenever $p \in \{0,1\}$ and, even when $k=3$, it is NP-complete whenever $p \geq 2$. We furthermore consider parameterizations of the b-Coloring problem that involve the maximum degree $\Delta(G)$ of the input graph G and give two FPT-algorithms. First, we show that deciding whether a graph G has a b-coloring with $m(G)$ colors is FPT parameterized by $\Delta(G)$. Second, we show that b-Coloring is FPT parameterized by $\Delta(G)+\ell_k(G)$, where $\ell_k(G)$ denotes the number of vertices of degree at least k.

1. Introduction

Given a set of colors, a *proper coloring* of a graph is an assignment of a color to each of its vertices in such a way that no pair of adjacent vertices receive the same color. In the deeply studied Graph Coloring problem, we are given a graph and the question is to determine the smallest set of colors with which we can properly color the input graph. This problem is among Karp's famous list of 21 NP-complete problems [16] and since it often arises in practice, heuristics to solve it are deployed in a wide range of applications. A very natural such heuristic is the following. We greedily find a proper coloring of the graph, and then try to *suppress* any of its colors in the following way: say we want to suppress color c. If there is a vertex v that has received color c, and there is another color $c' \neq c$ that does not appear in the neighborhood of v, then we can safely recolor the vertex v with color c' without making the coloring improper. We terminate this process once we cannot suppress any color anymore.

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[1] Supported by the Bergen Research Foundation (BFS).
[2] Supported by Research Council of Norway via the project "CLASSIS", grant number 249994.

https://doi.org/10.1016/j.tcs.2020.02.007
0304-3975/

To predict the worst-case behavior of the above heuristic, Irving and Manlove defined the notions of a *b-coloring* and the *b-chromatic number* of a graph [14]. A b-coloring of a graph G is a proper coloring such that in every color class there is a vertex that has a neighbor in all of the remaining color classes, and the b-chromatic number of G, denoted by $\chi_b(G)$, is the maximum integer k such that G admits a b-coloring with k colors. We observe that in a b-coloring with k colors, there is no color that can be suppressed to obtain a proper coloring with $k-1$ colors, hence $\chi_b(G)$ describes the worst-case behavior of the previously described heuristic on the graph G. We consider the following two computational problems associated with b-colorings of graphs.

b-Coloring
Input: Graph G, integer k
Question: Does G admit a b-coloring with k colors?

b-Chromatic Number
Input: Graph G, integer k
Question: Is $\chi_b(G) \geq k$?

We would like to point out an important distinction from the 'standard' notion of proper colorings of graphs: If a graph G has a b-coloring with k colors, then this implies that $\chi_b(G) \geq k$. However, if $\chi_b(G) \geq k$ then we can in general not conclude that G has a b-coloring with k colors. A graph for which the latter implication holds as well is called *b-continuous*. This notion is mostly of structural interest, since the problem of determining if a graph is b-continuous is NP-complete even if an optimal proper coloring and a b-coloring with $\chi_b(G)$ colors are given [2].

Besides observing that $\chi_b(G) \leq \Delta(G) + 1$ where $\Delta(G)$ denotes the maximum degree of G, Irving and Manlove [14] defined the *m-degree* of G as the largest integer i such that G has i vertices of degree at least $i-1$. It follows that $\chi_b(G) \leq m(G)$; observe also that $m(G) \leq \Delta(G)+1$. Since the definition of the b-chromatic number originated in the analysis of the worst-case behavior of graph coloring heuristics, graphs whose b-chromatic numbers take on *critical* values, i.e. values that are close to these upper bounds, are of special interest. In particular, identifying them can be helpful in structural investigations concerning the performance of graph coloring heuristics.

In terms of computational complexity, Irving and Manlove showed that both b-Coloring and b-Chromatic Number are NP-complete [14] and Sampaio observed that b-Coloring is NP-complete even for every fixed integer $k \geq 3$ [19]. Panolan et al. [18] gave an exact exponential algorithm for b-Chromatic Number running in time $\mathcal{O}(3^n n^4 \log n)$ and an algorithm that solves b-Coloring in time $\mathcal{O}(\binom{n}{k} 2^{n-k} n^4 \log n)$. From the perspective of parameterized complexity [6,8], it has been shown that b-Chromatic Number is W[1]-hard parameterized by k [18] and that the dual problem of deciding whether $\chi_b(G) \geq n-k$, where n denotes the number of vertices in G, is FPT parameterized by k [13].

Since the above mentioned upper bounds $\Delta(G)+1$ and $m(G)$ on the b-chromatic number are trivial to compute, it is natural to ask whether there exist efficient algorithms that decide whether $\chi_b(G) = \Delta(G)+1$ or $\chi_b(G) = m(G)$. It turns out both these problems are NP-complete as well [12,14,17]. However, it is known that the problem of deciding whether a graph G admits a b-coloring with $k = \Delta(G)+1$ colors is FPT parameterized by k [18,19].

The Dichotomy Result. One of the main contributions of this paper is a complexity dichotomy of the b-Coloring problem for fixed k, whenever k is close to either $\Delta(G)+1$ or $m(G)$. In particular, for fixed $k \in \{\Delta(G)+1-p, m(G)-p\}$, we show that the problem is polynomial-time solvable when $p \in \{0,1\}$ and, even in the case $k=3$, NP-complete for all fixed $p \geq 2$. More specifically, we give XP time algorithms for the cases $k = m(G)$, $k = \Delta(G)$, and $k = m(G)-1$ which together with the FPT algorithm for the case $k = \Delta(G)+1$ [18,19] and the aforementioned NP-hardness result for $k=3$ completes the picture. We now formally state this result.

Theorem 1. *Let G be a graph, $p \in \mathbb{N}$ and $k \in \{\Delta(G)+1-p, m(G)-p\}$. The problem of deciding whether G has a b-coloring with k colors is*

(i) NP-*complete if k is part of the input and $p \in \{0,1\}$,*
(ii) NP-*complete if $k=3$ and $p \geq 2$, and*
(iii) polynomial-time solvable for any fixed positive k and $p \in \{0,1\}$.

Maximum Degree Parameterizations. The positive results in our dichotomy theorem provide XP-algorithms to decide whether a graph has a b-coloring with a number of colors that either *precisely meets* or is *one below* one of two upper bounds on the b-chromatic number, the parameter being the number of colors in each of the cases. Towards more 'flexible' (parameterized) tractability results, we consider parameterizations of the b-Coloring problem that involve the maximum degree $\Delta(G)$ of the input graph G, but ask for the existence of b-colorings with a number of colors that in general is different from $\Delta(G)+1$ or $\Delta(G)$.

First, as an addition to the result that in FPT time parameterized by $\Delta(G)$, one can decide whether G has a b-coloring with $\Delta(G)+1$ colors [18,19], we show that in the same parameterization we can decide in FPT time whether G has a b-coloring with $m(G)$ colors.

Theorem 2. *Let G be a graph. The problem of deciding whether G has a b-coloring with $m(G)$ colors is* FPT *parameterized by $\Delta(G)$.*

One of the crucially used facts in the algorithm of the previous theorem is that if we ask whether a graph G has a b-coloring with $k = m(G)$ colors, then the number of vertices of degree at least k is at most k. We generalize this setting and parameterize b-Coloring by the maximum degree plus the number of vertices of degree at least k. We show that this problem is FPT as well.

Theorem 3. *Let G be a graph. The problem of deciding whether G has a b-coloring with k colors is* FPT *parameterized by $\Delta(G)+\ell_k(G)$, where $\ell_k(G)$ denotes the number of vertices of degree at least k in G.*

We now argue that parameterizing by only one of the two invariants used in Theorem 3 is not sufficient to obtain efficient parameterized algorithms. From the result of Kratochvíl et al. [17], stating that b-Coloring is NP-complete for $k = \Delta(G)+1$, it follows that b-Coloring is NP-complete when $\Delta(G)$ is unbounded and $\ell_k(G)=0$. On the other hand, Theorem 1(ii) implies that b-Coloring is already NP-complete when $k=3$ and $\Delta(G)=4$. Together, this rules out the possibility of FPT- and even of XP-algorithms for parameterizations by one of the two parameters alone, unless P = NP. Parameterizations of graph coloring problems by the number of high degree vertices have previously been considered for vertex coloring [1] and edge coloring [10].

An extended abstract of this work appeared in the proceedings of MFCS 2019 [15].

Organization. The rest of the paper is organized as follows. After giving preliminary definitions in Section 2, we present the hardness results in Section 3, the algorithmic results of the dichotomy in Section 4, and the algorithms for the maximum degree parameterizations in Section 5. We conclude in Section 6.

2. Preliminaries

We use the following notation: For $k \in \mathbb{N}$, $[k] := \{1,\ldots,k\}$. For a function $f\colon X \to Y$ and $X' \subseteq X$, we denote by $f|_{X'}$ the restriction of f to X' and by $f(X')$ the set $\{f(x) \mid x \in X'\}$. For a set X and an integer n, we denote by $\binom{X}{n}$ the set of all size-n subsets of X.

Graphs. Throughout the paper a graph G with vertex set $V(G)$ and edge set $E(G) \subseteq \binom{V(G)}{2}$ is finite and simple. We often denote an edge $\{u,v\} \in E(G)$ by the shorthand uv. For graphs G and H we denote by $H \subseteq G$ that H is a subgraph of G, i.e. $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We often use the notation $n := |V(G)|$. For a vertex $v \in V(G)$, we denote by $N_G(v)$ the *open neighborhood* of v in G, i.e. $N_G(v) = \{w \in V(G) \mid vw \in E(G)\}$, and by $N_G[v]$ the *closed neighborhood* of v in G, i.e. $N_G[v] := \{v\} \cup N_G(v)$. For a set of vertices $X \subseteq V(G)$, we let $N_G(X) := \bigcup_{v \in X} N_G(v) \setminus X$ and $N_G[X] := X \cup N_G(X)$. When G is clear from the context, we abbreviate 'N_G' to 'N'. The *degree* of a vertex $v \in V(G)$ is the size of its open neighborhood, and we denote it by $\deg_G(v) := |N_G(v)|$ or simply by $\deg(v)$ if G is clear from the context. For an integer k, we denote by $\ell_k(G)$ the number of vertices of degree at least k in G. A graph is *k-regular* if all its vertices have degree k.

For a vertex set $X \subseteq V(G)$, we denote by $G[X]$ the subgraph *induced* by X, i.e. $G[X] := (X, E(G) \cap \binom{X}{2})$. We furthermore let $G - X := G[V(G) \setminus X]$ be the subgraph of G obtained from removing the vertices in X and for a single vertex $x \in V(G)$, we use the shorthand '$G - x$' for '$G - \{x\}$'.

A graph G is said to be *connected* if for any 2-partition (X,Y) of $V(G)$, there is an edge $xy \in E(G)$ such that $x \in X$ and $y \in Y$, and *disconnected* otherwise. A *connected component* of a graph G is a maximal connected subgraph of G. A *path* is a connected graph of maximum degree two, having precisely two vertices of degree one, called its *endpoints*. The *length* of a path is its number of edges. Given a graph G and two vertices u and v, the *distance* between u and v, denoted by $dist_G(u,v)$ (or simply $dist(u,v)$ if G is clear from the context), is the length of the shortest path in G that has u and v as endpoints.

A graph G is a *complete graph* if every pair of vertices of G is adjacent. A set $C \subseteq V(G)$ is a *clique* if $G[C]$ is a complete graph. A set $S \subseteq V(G)$ is an *independent set* if $G[S]$ has no edges. A graph G is a *bipartite graph* if its vertex set can be partitioned into two independent sets. A bipartite graph with bipartition (A,B) is a *complete bipartite graph* if all pairs consisting of one vertex from A and one vertex from B are adjacent, and with $a = |A|$ and $b = |B|$, we denote it by $K_{a,b}$. A *star* is the graph $K_{1,b}$, with $b \geq 2$, and we call *center* the unique vertex of degree b and *leaves* the vertices of degree one.

Colorings. Given a graph G, a map $\gamma : V(G) \to [k]$ is called a *coloring of G with k colors*. If for every pair of adjacent vertices, $uv \in E(G)$, we have that $\gamma(u) \neq \gamma(v)$, then the coloring γ is called *proper*. For $i \in [k]$, we call the set of vertices $u \in V(G)$ such that $\gamma(u) = i$ the *color class i*. If for all $i \in [k]$, there exists a vertex $x_i \in V(G)$ such that

(i) $\gamma(x_i) = i$, and
(ii) for each $j \in [k] \setminus \{i\}$, there is a neighbor $y \in N_G(x_i)$ of x_i such that $\gamma(y) = j$,

then γ is called a *b-coloring* of G. For $i \in [k]$, we call a vertex x_i satisfying the above two conditions a *b-vertex for color i*.

Parameterized Complexity. Let Σ be an alphabet. A *parameterized problem* is a set $\Pi \subseteq \Sigma^* \times \mathbb{N}$. A parameterized problem Π is said to be *fixed-parameter tractable*, or contained in the complexity class FPT, if there exists an algorithm that for each $(x,k) \in \Sigma^* \times \mathbb{N}$ decides whether $(x,k) \in \Pi$ in time $f(k) \cdot |x|^c$ for some computable function f and fixed integer $c \in \mathbb{N}$. A parameterized problem Π is said to be contained in the complexity class XP if there is an algorithm that for all $(x,k) \in \Sigma^* \times \mathbb{N}$ decides whether $(x,k) \in \Pi$ in time $f(k) \cdot |x|^{g(k)}$ for some computable functions f and g.

A *kernelization algorithm* for a parameterized problem $\Pi \subseteq \Sigma^* \times \mathbb{N}$ is a polynomial-time algorithm that takes as input an instance $(x,k) \in \Sigma^* \times \mathbb{N}$ and either correctly decides whether $(x,k) \in \Pi$ or outputs an instance $(x',k') \in \Sigma^* \times \mathbb{N}$ with $|x'| + k' \le f(k)$ for some computable function f for which $(x,k) \in \Pi$ if and only if $(x',k') \in \Pi$. We say that Π *admits a kernel* if there is a kernelization algorithm for Π.

3. Hardness results

In this section we prove the hardness results of our complexity dichotomy. First, we show that b-Chromatic Number and b-Coloring are NP-complete for $k = m(G) - 1 = \Delta(G)$, based on a reduction due to Havet et al. [12] who showed NP-completeness for the case $k = m(G)$.

Theorem 3.1. *b-Chromatic Number and b-Coloring are NP-complete, even when $k = m(G) - 1 = \Delta(G)$.*

Proof. As in the proof of Havet et al. [12], the reduction is from the NP-complete problem 3-Edge Coloring of 3-regular graphs, which takes as input a 3-regular graph G and asks whether the edges of G can be properly colored with three colors.

Given an instance G of 3-Edge Coloring, an instance H of b-Chromatic Number and b-Coloring is constructed as follows. The graph H has one vertex for each vertex of G, that we denote by $v_1, \dots, v_n$, one vertex for each edge, that we denote by $u_1, \dots, u_m$ and a set of $4n + 13$ vertices that we denote by S. The edge set of H is such that $H[\{v_1, \dots, v_n\}]$ is a clique, $H[S]$ is the disjoint union of one copy of the complete bipartite graph $K_{n,n+3}$ and two copies of $K_{2,n+3}$ and $v_i u_j$ is an edge if the edge corresponding to u_j is incident to the vertex corresponding to v_i in G. The constructed graph H is such that $\Delta(H) = n + 3$ and H has $n + 4$ vertices of degree $n + 3$, which implies that $m(H) = n + 4$. The difference to the construction used in [12] is that instead of the three complete bipartite graphs mentioned above, the authors use three copies of the star $K_{1,n+2}$.

Claim 3.1.1. A connected component of H that is a complete bipartite graph can contain b-vertices of at most one color in any b-coloring of H with at least $n + 3$ colors.

Proof. Consider a component of H that induces a $K_{i,n+3}$, with $i \in \{2, n\}$. In any b-coloring with $k \ge n + 3$ colors, only the vertices of degree at least $n + 2$, so in this case the vertices of degree $n + 3$ can be b-vertices in H. If x is a b-vertex for a given color, then the remaining $k - 1$ colors appear on the vertices of $N(x)$. We conclude that any other vertex of degree $n + 3$ of this component will be assigned the same color as x. □

We prove that H has a b-coloring with $k = n + 3$ colors if and only if G is a Yes-instance for 3-Edge Coloring by using the same steps as in the proof of Theorem 3 of [12] and with the additional use of Claim 3.1.1. This proves the NP-completeness of b-Coloring when $k = m(G) - 1 = \Delta(G)$. Furthermore, we prove that $\chi_b(H) \ge n + 3$ if and only if H has a b-coloring with $n + 3$ colors. This yields the analogous result for b-Chromatic Number.

First, assume that G is a Yes-instance for 3-Edge Coloring. Let $\gamma_E : E(G) \to [3]$ be a proper 3-edge coloring for G. We construct a b-coloring γ_H for H in the following way. For each $1 \le i \le |E(G)|$, $\gamma_H(u_i) = \gamma_E(e_i)$ and each $1 \le j \le n$, we let $\gamma_H(v_j) = j + 3$. Note that since γ_E is a 3-edge coloring for G, the vertices $v_1, \dots, v_n$ in H are b-vertices for the colors $4, \dots, n + 3$: Any vertex in G is incident with 3 edges since G is 3-regular, and since γ_E is proper, each such edge receives a different color. Hence, for any vertex v_i, the colors $\{1, 2, 3\}$ appear on $N_H(v_i) \cap \{u_1, \dots, u_{|E(G)|}\}$. Now we can color the rest of the graph H in such a way that each connected component that is a complete bipartite graph contains a b-vertex for one of the three remaining colors.

Now we consider the other direction. We start by observing that Claim 3.1.1 implies that H does not admit a b-coloring with $n + 4 = m(H) = \Delta(H) + 1$ colors, since the set of vertices of degree $n + 3$ can contain b-vertices for at most three colors in any such a b-coloring. This implies that $\chi_b(H) \ge m(H) - 1 = \Delta(H)$ if and only if H has a b-coloring with $m(H) - 1 = \Delta(H)$ colors.

Assume H has a b-coloring γ_H with $n + 3$ colors. Since by Claim 3.1.1 the set S contains b-vertices for at most three colors, we have that the vertices $v_1, \dots, v_n$ are b-vertices in this coloring. Moreover, since they induce a clique in H, they all have distinct colors. Assume, without loss of generality, that $\gamma_H(\{v_1, \dots, v_n\}) = \{4, \dots, n+3\}$. This implies that for each i, $\gamma_H(N(v_i) \cap \{u_1, \dots, u_{|E(G)|}\}) = \{1, 2, 3\}$. It follows that $\gamma_E \colon E(G) \to \mathbb{N}$, defined as $\gamma_E(e_i) = \gamma_H(u_i)$, for $i \in \{1, \dots, |E(G)|\}$, is a 3-edge coloring of G. We argue that γ_E is proper. Suppose for a contradiction that there exist adjacent edge e_i and e_j, sharing the endpoint v_s, such that $\gamma_E(e_i) = \gamma_E(e_j) = c$. Since $\deg_G(v_s) = 3$, and two of its incident edges received the same color c, we can conclude that at least one of the colors $\{1, 2, 3\}$ does not appear in the neighborhood of v_s in H, a contradiction with the fact that v_s is a b-vertex of its color in γ_H. □

The previous theorem, together with the result that b-Coloring is NP-complete when $k = \Delta(G) + 1$ [17] and when $k = m(G)$ [12], proves Theorem 1(i). We now turn to the proof of Theorem 1(ii), that is, we show that b-Coloring remains NP-complete for $k = 3$ if $k = \Delta(G) + 1 - p$ or $k = m(G) - p$ for any $p \geq 2$, based on a reduction due to Sampaio [19]. Note that the following proposition indeed proves Theorem 1(ii) as for fixed $p \geq 2$, we have that $3 \in \{\Delta(G) + 1 - p, m(G) - p\}$ if and only if $\Delta(G) = p + 2$ or $m(G) = p + 3$.

Proposition 3.2. *For every fixed integer $p \geq 2$, the problem of deciding whether a graph G has a b-coloring with 3 colors is* NP-*complete when $\Delta(G) = p + 2$ or $m(G) = p + 3$.*

Proof. Sampaio showed that the problem of deciding whether a graph G has a b-coloring with k colors is NP-complete for any fixed $k \in \mathbb{N}$ [19, Proposition 4.5.1]. For the case of $k = 3$, the reduction is from 3-Coloring on planar 4-regular graphs which is known to be NP-complete [11]. In this reduction, one takes the graph of the 3-Coloring instance and adds three stars with two leaves each to the graph which can serve as the b-vertices in the resulting instance of b-Coloring. Since this does not increase the maximum degree, we immediately have that the problem of deciding whether a graph of maximum degree 4 has a b-coloring with 3 colors is NP-complete. In other words, this proves NP-completeness of the question of whether a graph G with maximum degree $p + 2$ admits a b-coloring with 3 colors in the case $p = 2$. Furthermore, by adding more leaves to one of the stars and thereby increasing the maximum degree of the graph in the resulting instance, we have that for any fixed integer $p \geq 2$, it is NP-complete to decide whether a graph of maximum degree $\Delta(G) = p + 2$ has a b-coloring with three colors.

Towards the statement regarding $m(G)$, we first observe that for a 4-regular graph G on at least five vertices, we have that $m(G) = 5$. We observe that in any star with *at least* two leaves, the center vertex can be a b-vertex in a coloring with 3 colors. We construct a graph G' by adding five stars with four leaves each to G, and we again have that G has a 3-coloring if and only if G' has a b-coloring with 3 colors, showing that the problem of deciding whether a graph H with $m(H) = 5$ has a b-coloring with 3 colors, is NP-complete. In other words, it is NP-complete to decide if a graph H has a b-coloring with $m(H) = p + 3$ has a b-coloring with 3 colors in the case $p = 2$. Note that in this reduction, the center vertices of the stars can be regarded as the vertices determining the m-degree of the graph in the resulting instance of b-coloring with 3 colors, so we can extend this result in a similar way as above. That is, for any $p \geq 2$, given a 4-regular graph G, we can add $p + 3$ stars with $p + 2$ leaves each to G, implying that for the resulting graph G', $m(G') = p + 3$. Again, G has a 3-coloring if and only if G' has a b-coloring with 3 colors, implying the second statement of the proposition. □

We conclude this section by considering the complexity of the two problems on graphs with few vertices of high degree. Since b-Chromatic Number and b-Coloring are known to be NP-complete when $k = \Delta(G) + 1$ [17], we make the following observation which is of relevance to us since in Section 5.2, we show that b-Coloring is FPT parameterized by $\Delta(G) + \ell_k(G)$.

Observation 3.3. *b-Chromatic Number and b-Coloring are* NP-*complete on graphs with $\ell_k(G) = 0$, where k is the integer associated with the respective problem.*

4. Dichotomy algorithms

In this section we give the algorithms in our dichotomy result, proving Theorem 1(iii). We show that for fixed $k \in \mathbb{N}$, the problem of deciding whether a graph G admits a b-coloring with k colors is polynomial-time solvable when $k = m(G)$ (Section 4.2), when $k = \Delta(G)$ (Section 4.3), and when $k = m(G) - 1$ (Section 4.4), by providing XP-algorithms for each case. A natural way of solving the b-Coloring problem is to try to identify a set of k b-vertices, and for each vertex in the set a set of $k - 1$ neighbors that can be used to make a vertex a b-vertex for its color, and then extend the resulting coloring to the remainder of the graph. We guess all such sets and colorings, and show that the problem of deciding whether a given coloring can be extended to a proper coloring of the remainder of the graph is solvable in polynomial time in each of the above cases.

The strategy of identifying the set of b-vertices and subsets of their neighbors that make them b-vertices was (for instance) also used to give polynomial-time algorithms to compute the b-chromatic number of trees [14] and of graphs with large girth [4]. We capture it by defining the notion of a b-*precoloring* in the next subsection.

4.1. b-precolorings

All algorithms in this section are based on guessing a proper coloring of several vertices in the graph, for which we now introduce the necessary terminology and establish some preliminary results.

Definition 4.1 *(Precoloring).* Let G be a graph and $k \in \mathbb{N}$. A *precoloring* with k colors of a graph G is an assignment of colors to a subset of its vertices, i.e. for $X \subseteq V(G)$, it is a map $\gamma_X \colon X \to [k]$. We call γ_X *proper*, if it is a proper coloring of $G[X]$. We say that a coloring $\gamma \colon V(G) \to [k]$ *extends* γ_X, if $\gamma|_X = \gamma_X$.

We use the following notation. For two precolorings γ_X and γ_Y with $X \cap Y = \emptyset$, we denote by $\gamma_X \cup \gamma_Y$ the precoloring that colors the vertices in X according to γ_X and the vertices in Y according to Y, i.e. the precoloring $\gamma_{X\cup Y} := \gamma_X \cup \gamma_Y$ defined as:

$$\gamma_{X\cup Y}(v) = \begin{cases} \gamma_X(v), & \text{if } v \in X \\ \gamma_Y(v), & \text{if } v \in Y \end{cases} \quad \text{for all } v \in X \cup Y$$

Next, we define a special type of precoloring with the property that any proper coloring that extends it is a b-coloring of the graph.

Definition 4.2 *(b-precoloring).* Let G be a graph, $k \in \mathbb{N}$, $X \subseteq V(G)$ and γ_X a precoloring. We call γ_X a *b-precoloring with k colors* if γ_X is a b-coloring of $G[X]$. A b-precoloring γ_X is called *minimal* if for any $Y \subset X$, $\gamma_X|_Y$ is not a b-precoloring.

It is immediate that any b-coloring can be obtained by extending a minimal b-precoloring, a fact that we capture in the following observation.

Observation 4.3. *Let G be a graph, $k \in \mathbb{N}$, and γ a b-coloring of G with k colors. Then, there is a set $X \subseteq V(G)$ such that $\gamma|_X$ is a minimal b-precoloring.*

The next observation captures the structure of minimal b-precolorings with k colors. Roughly speaking, each such precoloring only colors a set of k b-vertices and for each b-vertex a set of $k-1$ of its neighbors that make that vertex the b-vertex of its color. We will use this property in the enumeration algorithm in this section to guarantee that we indeed enumerate all minimal b-precolorings with a given number of colors.

Observation 4.4. *Let γ_X be a minimal b-precoloring with k colors. Then, $X = B \cup Z$, where*

(i) $B = \{x_1, \ldots, x_k\}$ *and for* $i \in [k]$, $\gamma_X(x_i) = i$, *and*
(ii) $Z = \bigcup_{i\in[k]} Z_i$, *where* $Z_i \in \binom{N(x_i)}{k-1}$ *and* $\gamma_X(Z_i) = [k] \setminus \{i\}$.

We are now ready to give the enumeration algorithm for minimal b-precolorings.

Lemma 4.5. *Let G be a graph on n vertices and $k \in \mathbb{N}$. The number of minimal b-precolorings with k colors of G is at most*

$$\beta(k) := n^k \cdot \Delta^{k(k-1)} \cdot (k-1)!^k, \tag{1}$$

where $\Delta := \Delta(G)$ and they can be enumerated in time $\beta(k) \cdot k^{\mathcal{O}(1)}$.

Proof. By Observation 4.4, any minimal b-precoloring only colors a set of k b-vertices, and for each of them a size-$(k-1)$ subset of its neighbors that are colored bijectively with the remaining colors.

To guess all b-vertices in G, we enumerate all ordered vertex sets of size k, let $\{x_1, \ldots, x_k\}$ be such a set. Next, we enumerate all size-$(k-1)$ subsets of neighbors of each x_i that can make x_i the b-vertex of color i. Let $(Z_1, \ldots, Z_k)$ be a tuple of such sets of neighbors. Then we enumerate for each $i \in [k]$, all bijective colorings of $\pi_i \colon Z_i \to [k] \setminus \{i\}$ – these are precisely the colorings of Z_i that can make x_i the b-vertex for color i. Given such a tuple $(\pi_1, \ldots, \pi_k)$, we make sure that it is consistent: for each $i, j \in [k]$ and each vertex $v \in Z_i \cap Z_j$, we ensure that π_i and π_j assign v the same color, i.e. $\pi_i(v) = \pi_j(v)$; similarly, if $x_i \in Z_j$, then we ensure that $\pi_j(x_i) = i$ (recall that x_i is supposed to be the b-vertex of color i). If so, we construct a precoloring $\gamma_{B\cup Z}$ according to our choice of $B = \{x_1, \ldots, x_k\}$ and $(\pi_1, \ldots \pi_k)$ and if it is a minimal b-precoloring we output it. We give the details in Algorithm 1.

We now show that the algorithm is correct.

Claim 4.5.1. A precoloring γ_X is a minimal b-precoloring with k colors if and only if Algorithm 1 returns it in line 8 in some iteration.

Input : A graph G, a positive integer k.
Output: All minimal b-precolorings with k colors of G
1 **foreach** $B \in \binom{V(G)}{k}$ *and every ordering* $x_1, \dots, x_k$ *of the elements of* B **do**
2 **foreach** $(Z_1, \dots, Z_k) \in \binom{N(x_1)}{k-1} \times \cdots \times \binom{N(x_k)}{k-1}$ **do**
3 **foreach** $(\pi_1, \dots, \pi_k)$,
(i) where for all $i \in [k]$, $\pi_i \colon Z_i \to [k] \setminus \{i\}$ *is a bijection,*
(ii) for all $i, j \in [k]$ *and all* $v \in Z_i \cap Z_j$, $\pi_i(v) = \pi_j(v)$, *and*
(iii) for all $i, j \in [k]$, *if* $x_i \in Z_j$, *then* $\pi_j(x_i) = i$
4 **do**
5 Let $Z := \cup_{i \in [k]} Z_i$ and $\gamma_{B \cup Z} \colon B \cup Z \to [k]$;
6 **for** $i \in [k]$ **do** $\gamma_{B \cup Z}(x_i) := i$;
7 **for** $i \in [k]$ *and* $v \in Z_i$ **do** $\gamma_{B \cup Z}(v) := \pi_i(v)$;
8 **if** $\gamma_{B \cup Z}$ *is a minimal b-precoloring* **then output** $\gamma_{B \cup Z}$ **and continue**;

Algorithm 1: Enumerating all minimal b-precolorings with k colors of a graph.

Proof. Suppose Algorithm 1 returns a precoloring γ_X. We first argue that γ_X is well-defined. Let $X = B \cup Z$ following the notation of Algorithm 1. It is immediate that $\gamma_X|_B$ is well-defined. For the remaining vertices $v \in Z$, we verify as condition (ii) in line 3 that whenever $v \in Z_i \cap Z_j$, $\pi_i(v) = \pi_j(v)$. Moreover, condition (iii) in line 3 ensures that whenever $x_i \in Z_j$, then $\pi_j(x_i) = i$. Hence if the tuple $(\pi_1, \dots, \pi_k)$ passes the check in line 3, then for each vertex $v \in Z$ there is precisely one color that γ_X assigns to v in line 7, and if $v \in Z_j \cap B$, then π_j assigns v a color that is consistent with line 6. We can conclude that γ_X is well-defined. By the check performed in line 8, we can conclude that γ_X is a minimal b-precoloring.

Now suppose that G contains a minimal b-precoloring γ_X. By Observation 4.4, X consists of an (ordered) set of b-vertices $B = \{x_1, \dots, x_k\}$ with $\gamma_X(x_i) = i$ for $i \in [k]$, and a set Z that contains, for each x_i, a set of $k-1$ neighbors $Z_i \subseteq Z$ such that $\gamma_X(Z_i) = [k] \setminus \{i\}$. Since Algorithm 1 enumerates all such possible sets in lines 1 and 2, we have that in some iteration, it guessed $B \cup Z$ as the set of vertices to color. Since the algorithm enumerates all combinations of possibilities of coloring the sets Z_i bijectively with colors $[k] \setminus \{i\}$ in line 3, it guessed a tuple of bijections $(\pi_1, \dots, \pi_k)$ from which we obtain γ_X. Clearly we have that in that case, $(\pi_1, \dots, \pi_k)$ passes the check in line 3 and by assumption, γ_X passes the check in line 8. □

It remains to argue its runtime. In line 1, there are $\binom{n}{k}$ choices for the set B and $k!$ choices for its orderings, in line 2, there are at most $\binom{\Delta}{k-1}^k$ choices of k-tuples of size-$(k-1)$ sets of neighbors, and in line 3, we enumerate $(k-1)!^k$ k-tuples of bijections of sets of size $k-1$. The remaining steps can be executed in time $k^{\mathcal{O}(1)}$: By construction, $|B \cup Z| \le k^2$, and every color has a b-vertex. It remains to verify whether the coloring $\gamma_{B \cup Z}$ is proper on $G[B \cup Z]$ to conclude that it is a b-precoloring. If so, we can verify minimality in polynomial time by simply trying for each vertex $x \in B \cup Z$, whether $\gamma_{B \cup Z \setminus \{x\}}$ is still a b-precoloring. If we can find such a vertex x, then $\gamma_{B \cup Z}$ is not minimal, otherwise it is. The total runtime amounts to

$$\binom{n}{k} k! \cdot \binom{\Delta}{k-1}^k \cdot (k-1)!^k \cdot k^{\mathcal{O}(1)} \le n^k \cdot \Delta^{k(k-1)} \cdot (k-1)!^k \cdot k^{\mathcal{O}(1)} = \beta(k) \cdot k^{\mathcal{O}(1)},$$

as claimed. The upper bound of $\beta(k)$ on the number of b-precolorings with k colors follows since the $k^{\mathcal{O}(1)}$ factor in the runtime only concerns the construction of the precolorings and the verification of whether they are indeed b-precolorings. □

4.2. Algorithm for $k = m(G)$

Our first application of Lemma 4.5 is to solve the b-Coloring problem in the case when $k = m(G)$ in time XP parameterized by k. It turns out that in this case, we are dealing with a Yes-instance as soon as we found a b-precoloring in the input graph that also colors all high-degree vertices (see Claim 4.6.1).

Theorem 4.6. *Let G be a graph. There is an algorithm that decides whether G has a b-coloring with $k = m(G)$ colors in time $n^{k^2} \cdot 2^{\mathcal{O}(k^2 \log k)}$.*

Proof. Let $D \subseteq V(G)$ denote the set of vertices in G that have degree at least k. Note that by the definition of $m(G)$, we have that $|D| \le k$.

Claim 4.6.1. G has a b-coloring with k colors if and only if G has a b-precoloring γ_X such that $D \subseteq X$ and there exists $S \subseteq D$ such that $\gamma_X|_{(X \setminus S)}$ is a minimal b-precoloring.

Proof. Suppose G has a b-precoloring γ_X satisfying the condition of the claim. By our choice of D, each vertex in $V(G) \setminus D$ has degree at most $k-1$. Hence we can greedily compute an extension γ of γ_X that is a proper coloring of G. By the definition of b-precoloring, we have that γ is a b-coloring of G.

Now suppose that G has a b-coloring γ with k colors. Let $B = \{x_1, \ldots, x_k\}$ be the set of b-vertices of γ and for each $i \in [k]$, let Z_i be a set of $k-1$ neighbors of x_i such that $\gamma(Z_i) = [k] \setminus \{i\}$. Let $Z := \cup_{i \in [k]} Z_i$. Then, $\gamma|_{B \cup Z}$ is a b-precoloring. Clearly, $\gamma|_{B \cup Z}$ contains a minimal b-precoloring on vertex set $W \subseteq B \cup Z$. Then, $\gamma|_{W \cup D}$ is a b-precoloring of G that satisfies the condition of the claim. □

The algorithm enumerates all minimal b-precolorings with k colors and for each such precoloring, it enumerates all colorings of the vertices D. If combining one such pair of precolorings gives a b-precoloring, it returns a greedy extension of it; otherwise it reports that there is no b-coloring with k colors, see Algorithm 2.

Input : A graph G
Output: A b-coloring with $m(G)$ colors if it exists, No otherwise.
foreach *minimal b-precoloring* $\gamma_X : X \to [k]$ **do**
  foreach *precoloring* $\gamma_{D \setminus X} : (D \setminus X) \to [k]$ **do**
    if $\gamma_{X \cup D} := \gamma_X \cup \gamma_{D \setminus X}$ *is proper* **then return** a greedy extension of $\gamma_{X \cup D}$;
return No;

Algorithm 2: Algorithm for b-Coloring with $k = m(G)$.

The correctness of the algorithm follows from the fact that it enumerates all precolorings that can satisfy Claim 4.6.1. We discuss its runtime. By Lemma 4.5, we can enumerate all minimal b-precolorings with k colors in time $\beta(k) \cdot k^{\mathcal{O}(1)}$. For each such minimal b-precoloring, we also enumerate all colorings of D. Since $|D| \le k$, this gives an additional factor of k^k to the runtime which (with $\Delta \le n$) then amounts to

$$\beta(k) \cdot k^k \cdot k^{\mathcal{O}(1)} = n^k \cdot \Delta^{k(k-1)} \cdot (k-1)!^k \cdot k^k \cdot k^{\mathcal{O}(1)} \le n^{k^2} \cdot k!^k \cdot k^{\mathcal{O}(1)} = n^{k^2} \cdot 2^{\mathcal{O}(k^2 \log k)},$$

as claimed. □

4.3. Algorithm for $k = \Delta(G)$

Next, we turn to the case when $k = \Delta(G)$. Here the strategy is to again enumerate all minimal b-precolorings, and then for each such precoloring we check whether it can be extended to the remainder of the graph. Formally, we use an algorithm for the following problem as a subroutine.

Precoloring Extension (PrExt)
Input: A graph G, an integer k, and a precoloring $\gamma_X : X \to [k]$ of a set $X \subseteq V(G)$
Question: Does G have a proper coloring with k colors extending γ_X?

Naturally, Precoloring Extension is a hard problem, since it includes Graph Coloring as the special case when $X = \emptyset$. However, when $\Delta(G) \le k-1$, then the problem is trivially solvable: we simply check if the precoloring at the input is proper and if so, we compute an extension of it greedily. Since each vertex has degree at most $k-1$, there is always at least one color available. The case when $k = \Delta(G)$ has also been shown to be solvable in polynomial time.

Theorem 4.7 *(Thm. 3 in [5], see also [7]). There is an algorithm that solves* Precoloring Extension *in polynomial time whenever* $\Delta(G) \le k$.

Theorem 4.8. *There is an algorithm that decides whether a graph G has a b-coloring with* $\Delta(G)$ *colors in time* $n^{k+\mathcal{O}(1)} \cdot 2^{\mathcal{O}(k^2 \log k)}$.

Proof. The algorithm simply enumerates all minimal b-precolorings and then applies the algorithm for PrExt of Theorem 4.7. This algorithm can be applied with any precoloring of G since $k = \Delta(G)$. We give the details in Algorithm 3.

We now show that the algorithm is correct.

Claim 4.8.1. A graph G contains a b-coloring with $k = \Delta(G)$ colors if and only if Algorithm 3 returns a coloring γ.

Proof. Suppose Algorithm 3 returns a coloring γ. Since γ extends a b-precoloring γ_X with k colors, we can conclude that γ has a b-vertex for each color. By the correctness of the algorithm of Theorem 4.7, we can conclude that γ is a b-coloring with k colors.

Input : A graph G
Output: A b-coloring of G with $k = \Delta(G)$ colors if it exists, and No otherwise.
foreach *minimal b-precoloring γ_X of G* **do**
  Apply the algorithm for PrExt of Theorem 4.7 with input (G, k, γ_X);
  if *the algorithm found a proper coloring γ extending γ_X* **then return** γ;
return No;

Algorithm 3: Algorithm for b-Coloring with $k = \Delta(G)$.

Suppose G contains a b-coloring with k colors, say γ. By Observation 4.3, γ contains a minimal b-precoloring, say γ_X. Hence, Algorithm 1, guessed γ_X in some iteration. Furthermore, since γ is a proper coloring that extends γ_X, (G, k, γ_X) is a Yes-instance of PrExt, so the algorithm of Theorem 4.7 returned a b-coloring γ' that extends γ_X. □

It remains to argue the runtime. By Lemma 4.5, we can enumerate all b-precolorings in $\beta(k) \cdot k^{\mathcal{O}(1)}$ time and by Theorem 4.7, the algorithm for PrExt runs in time $n^{\mathcal{O}(1)}$. The total runtime is hence (with $\Delta(G) = k$)

$$\begin{aligned}\beta(k) \cdot k^{\mathcal{O}(1)} \cdot n^{\mathcal{O}(1)} &= n^k \cdot \Delta^{k(k-1)} \cdot (k-1)!^k \cdot n^{\mathcal{O}(1)} = n^{k+\mathcal{O}(1)} \cdot k^{k(k-1)} \cdot (k-1)!^k \\ &\leq n^{k+\mathcal{O}(1)} \cdot 2^{\mathcal{O}(k^2 \log k)},\end{aligned}$$

as claimed. □

4.4. Algorithm for $k = m(G) - 1$

Before we proceed to describe the algorithm for b-Coloring when $k = m(G) - 1$, we show that the algorithm of Theorem 4.7 can be used for a slightly more general case of Precoloring Extension, namely the case when all high-degree vertices in the input instance are precolored.

Lemma 4.9. *There is an algorithm that solves an instance (G, k, γ_X) of* Precoloring Extension *in polynomial time whenever $\max_{v \in V(G) \setminus X} \deg(v) \leq k$.*

Proof. First, we check whether γ_X is a proper coloring of $G[X]$ and if not, the answer is No. We create a new instance of Precoloring Extension $(G', k, \delta_{X'})$ as follows. For every vertex $x \in X$ and every vertex $y \in N_G(x) \setminus X$, we let x_y be a new vertex that is only adjacent to y. We denote the set of these newly introduced vertices by $X' := \{x_y \mid x \in X, y \in N_G(x) \setminus X\}$. We obtain G' from G as follows. Let $G'' = G - X$. Then, the vertex set of G' is $V(G') := V(G'') \cup X'$ and its edge set is $E(G') := E(G'') \cup \{x_y y \mid x_y \in X'\}$. Now, we define a precoloring $\delta_{X'} \colon X' \to [k]$ such that for $x_y \in X'$, $\delta_{X'}(x_y) := \gamma_X(x)$.

It is clear that (G, k, γ_X) is a Yes-instance of Precoloring Extension if and only if $(G', k, \delta_{X'})$ is a Yes-instance of Precoloring Extension. Furthermore, for every vertex $z \in X'$, $\deg_{G'}(z) = 1$ and for every vertex $v \in V(G') \setminus X'$, $\deg_{G'}(v) = \deg_G(v) \leq k$, so $\Delta(G') \leq k$. This means that we can solve the instance $(G', k, \gamma_{X'})$ in polynomial time using Theorem 4.7. □

Theorem 4.10. *There is an algorithm that decides whether a graph G has a b-coloring with $k = m(G) - 1$ colors in time $n^{k^2+\mathcal{O}(1)} \cdot 2^{k^2 \log k}$.*

Proof. Let D denote the set of vertices of degree at least $k+1$ in G. By the definition of $m(G)$, we have that $|D| \leq k+1$. We first enumerate all minimal b-precolorings of G and, for each such precoloring, we enumerate all precolorings of D. Then, given a b-precoloring γ_X with $D \subseteq X$, we have that every vertex in $V(G) \setminus X$ has degree at most k, so we can apply the algorithm of Lemma 4.9 to verify whether there is a proper coloring of G that extends γ_X. If so, we output that extension. If no such precoloring can be found, then we conclude that we are dealing with a No-instance. We give the details in Algorithm 4.

We now prove the correctness of the algorithm.

Claim 4.10.1. G has a b-coloring with $k = m(G) - 1$ colors if and only if Algorithm 4 returns a coloring γ.

Proof. Suppose Algorithm 4 returns a coloring γ. Then, γ is obtained from a minimal b-precoloring γ_X and a precoloring $\gamma_{D \setminus X}$, both with k colors, such that $\gamma_{X \cup D} = \gamma_X \cup \gamma_{D \setminus X}$ is proper. Furthermore, since all vertices in $V(G) \setminus D$ have degree at most k, the application of the algorithm of Lemma 4.9 returns a correct answer. Hence, γ is a proper coloring and since it is obtained by extending a b-precoloring, it is a b-coloring with k colors.

The forward direction can be proved as in Claim 4.8.1 using Observation 4.3 which states that every b-coloring can be obtained by extending a minimal b-precoloring. □

Input : A graph G
Output: A b-coloring of G with $k = m(G) - 1$ colors if it exists, and No otherwise.
1 **foreach** *minimal* b*-precoloring* γ_X *of* G **do**
2 **foreach** *precoloring* $\gamma_{D \setminus X} : (D \setminus X) \to [k]$ **do**
3 **if** $\gamma_{X \cup D} := \gamma_X \cup \gamma_{D \setminus X}$ *is proper* **then**
4 Apply the algorithm for PrExt of Lemma 4.9 with input (G, k, γ_X);
5 **if** *the algorithm found a proper coloring* γ *extending* $\gamma_{X \cup D}$ **then return** γ;
6 **return** No;

Algorithm 4: Algorithm for b-Coloring with $k = m(G) - 1$.

It remains to argue the runtime. In line 1, we enumerate $\beta(k)$ (see (1)) minimal b-precolorings in time $\beta(k) \cdot k^{\mathcal{O}(1)}$ using Lemma 4.5. For each such precoloring, we enumerate all precolorings of $D \setminus X$. Since $|D| \le k+1$, there are at most k^{k+1} such colorings. Finally, we run the algorithm for PrExt due to Lemma 4.9 which takes time $n^{\mathcal{O}(1)}$. The total runtime becomes

$$\beta(k) \cdot k^{\mathcal{O}(1)} \cdot k^{k+1} \cdot n^{\mathcal{O}(1)} = n^k \cdot \Delta^{k(k-1)} \cdot (k-1)!^k \cdot k^{k+1} \cdot n^{\mathcal{O}(1)} \le n^{k^2+\mathcal{O}(1)} \cdot 2^{k^2 \log k},$$

as claimed. □

5. Maximum degree parameterizations

In this section we consider parameterizations of b-Coloring that involve the maximum degree $\Delta(G)$ of the input graph G. In Section 5.1 we show that we can solve b-Coloring when $k = m(G)$ in time FPT parameterized by $\Delta(G)$ and in Section 5.2 we show that b-Coloring is FPT parameterized by $\Delta(G) + \ell_k(G)$.

Both algorithms presented in this section make use of the following reduction rule, which has already been applied in [18,19] to obtain the FPT algorithm for the problem of deciding whether a graph G has a b-coloring with $k = \Delta(G) + 1$ colors, parameterized by k.

Reduction Rule 5.1 *([18,19]).* Let (G, k) be an instance of b-Coloring. If there is a vertex $v \in V(G)$ such that every vertex in $N[v]$ has degree at most $k-2$, then reduce (G, k) to $(G - v, k)$.

5.1. FPT *algorithm for* $k = m(G)$ *parameterized by* $\Delta(G)$

Sampaio [19] and Panolan et al. [18] independently showed that parameterized by $\Delta(G)$, it can be decided in FPT time whether a graph G has a b-coloring with $\Delta(G)+1$ colors. In this section we show that in the same parameterization, it can be decided in FPT time whether a graph has a b-coloring with $m(G)$ colors.

Theorem 5.2 *(Theorem 2, restated). There is an algorithm that given a graph G on n vertices decides whether G has a b-coloring with $k = m(G)$ colors in time $2^{\mathcal{O}(k^4 \cdot \Delta)} + n^{\mathcal{O}(1)} < 2^{\mathcal{O}(\Delta^5)} + n^{\mathcal{O}(1)}$, where $\Delta := \Delta(G)$.*

Proof. We apply Reduction Rule 5.1 exhaustively to G and consider the following 3-partition (D, T, R) of $V(G)$, where D contains the vertices of degree at least k, T the vertices of degree precisely $k-1$ and R the remaining vertices, i.e. $R := V(G) \setminus (D \cup T)$. Since we applied Reduction Rule 5.1 exhaustively, we make

Observation 5.2.1. Every vertex in R has at least one neighbor in $D \cup T$.

We pick an inclusion-wise maximal set $B \subseteq D \cup T$ such that for each pair of distinct vertices $b_1, b_2 \in B$, we have that $dist(b_1, b_2) \ge 4$.

Case 1 ($|B \cap T| < k$).[3] We show that for any vertex in $u \in V(G) \setminus B$, there is a vertex $v \in B$ such that $dist(u, v) \le 4$. Suppose $u \in D \cup T$. Since we did not include u in B, it immediately follows that there is some $v \in B$ such that $dist(u, v) < 4$. Now suppose $u \in R$. By Observation 5.2.1, u has a neighbor w in $D \cup T$ and by the previous argument, there is a vertex $v \in B$ such that $dist(w, v) < 4$. We conclude that $dist(u, v) \le 4$. Using this observation, we now show that in this case, the number of vertices in G is polynomial in k and Δ.

Claim 5.2.2. If $|B \cap T| < k$, then $|V(G)| \le \mathcal{O}(k^4 \cdot \Delta)$.

[3] This case is almost identical to [18, Case II in the proof of Theorem 2].

Input : A graph G with $k = m(G)$ `// More generally, graph` G `with` $\ell_k(G) \leq k$
Output: A b-coloring with k colors of G if it exists, and No otherwise.
1 Apply Reduction Rule 5.1 exhaustively;
2 Let (D, T, R) be a partition of $V(G)$ such that for all $x \in D$, $\deg_G(x) \geq k$, for all $x \in T$, $\deg_G(x) = k - 1$, and $R = V(G) \setminus (D \cup T)$;
3 Let $B \subseteq D \cup T$ be a maximal set such that for distinct $b_1, b_2 \in B$, $dist(b_1, b_2) \geq 4$;
4 **if** $|B \cap T| < k$ **then** `// Case 1`
5 Solve the instance in time $2^{\mathcal{O}(k^4 \cdot \Delta)}$ using the b-Coloring algorithm [18];
6 **if** *the algorithm of [18] returned a b-coloring* γ **then return** γ;
7 **else return** No;
8 **else** `// Case 2, i.e.` $|B \cap T| \geq k$
9 Pick a size-k subset of $B \cap T$, say $B' := \{x_1, \ldots, x_k\}$;
10 Initialize a k-coloring $\gamma : V(G) \to [k]$;
11 For $i \in [k]$, let $\gamma(x_i) := i$;
12 Let γ color the vertices of D injectively such that γ remains proper on $G[B' \cup D]$;
13 For $i \in [k]$, let γ color $N(x_i) \cap D$ such that x_i is the b-vertex of color i;
14 Extend the coloring γ greedily to the remainder of G;
15 **return** γ;

Algorithm 5: An algorithm that either constructs a b-coloring of a graph G with $m(G)$ colors, or reports that there is none, and runs in FPT time parameterized by $\Delta(G)$.

Proof. Note that $(B \cup D, S_1, \ldots, S_4)$ constitutes a partition of $V(G)$, where S_i is the set of vertices of $V(G) \setminus (B \cup D)$ that are at distance exactly i from B. Since $|B \cap T| < k$ and $|D| \leq k$, we have that $|B \cup D| < 2k$, and therefore $|S_1| < 2k \cdot \Delta$. By the definition of $m(G)$, all the vertices in $S_1 \cup \ldots \cup S_4$ have degree at most $k - 1$. This implies that $|S_i| < (k-1)^{i-1} \cdot 2k \cdot \Delta$. We conclude that the number of vertices in G is at most $2k + 2k \cdot \Delta \cdot \sum_{i=1}^{4} (k-1)^{i-1} = \mathcal{O}(k^4 \cdot \Delta)$. $\square$

By Claim 5.2.2, we can solve the instance in Case 1 in time $2^{\mathcal{O}(k^4 \cdot \Delta)}$ using the algorithm of Panolan et al. [18].

Case 2 ($|B \cap T| \geq k$). Let $B' \subseteq B \cap T$ with $|B'| = k$ and denote this set by $B' = \{x_1, x_2, \ldots, x_k\}$. We show that we can construct a b-coloring $\gamma : V(G) \to [k]$ of G such that for $i \in [k]$, x_i is the b-vertex of color i. For $i \in [k]$, we let $\gamma(x_i) := i$. Next, we color the vertices in D. Recall that $|D| \leq k$, so we can color the vertices in D injectively with colors from $[k]$, ensuring that this will not create a conflict on any edge in $G[D]$. Furthermore, consider $i, j \in [k]$ with $i \neq j$. Since $dist(x_i, x_j) \geq 4$, we have that $N(x_i) \cap N(x_j) = \emptyset$. In particular, there is no vertex in D that has two or more neighbors in B'. To summarize, we can conclude that we can let γ color the vertices of D in such a way that:

(i) γ is injective on D, and
(ii) γ is a proper coloring of $G[B' \cup D]$.

These two items imply that for each x_i ($i \in [k]$), its neighbors $N(x_i) \cap D$ receive distinct colors which are also different from i. Let $\ell := |N(x_i) \cap D|$. It follows that we can let γ color the remaining $(k-1) - \ell$ neighbors of x_i in an arbitrary bijective manner with the $(k-1) - \ell$ colors that do not yet appear in the neighborhood of x_i.

After this process, x_i is a b-vertex for color i. We proceed in this way for all $i \in [k]$. Since for $i, j \in [k]$ with $i \neq j$ we have that $dist(x_i, x_j) \geq 4$, it follows that there are no edges between $N[x_i]$ and $N[x_j]$ in G. Hence, we did not introduce any coloring conflict in the previous step. Now, all vertices in G that have not yet received a color by γ have degree at most $k - 1$, so we can extend γ to a proper coloring of G in a greedy fashion.

We summarize the whole procedure in Algorithm 5. We now analyze its runtime. Clearly, exhaustively applying Reduction Rule 5.1 can be done in time $n^{\mathcal{O}(1)}$. As mentioned above, Case 1 can be solved in time $2^{\mathcal{O}(k^4 \cdot \Delta)}$. In Case 2, the coloring of $G[B' \cup D]$ can be found in time $\mathcal{O}(k^2)$, and extending the coloring to the remainder of G can be done in time $n^{\mathcal{O}(1)}$. The claimed bound follows. $\square$

We observe that Algorithm 5 in fact solves a more general case of the b-Coloring problem, a fact which we will use later in the proof of Theorem 5.5. By the definition of $m(G)$, $\ell_{m(G)} \leq m(G)$, and this is the only property of $m(G)$ that the algorithm relies on: it bounds the size of D by $|D| \leq m(G)$. This is crucially used in lines 5 and 12. Now, if we relax the condition of $k = m(G)$ to $\ell_k(G) \leq k$, we observe that the assumption $\ell_k(G) \leq k$ still guarantees that $|D| \leq k$. Hence, in line 5, the bound of $\mathcal{O}(k^4 \cdot \Delta)$ on $V(G)$ remains the same and in line 12 we can find a coloring that is injective on D as well.

Remark 5.3. *Algorithm 5 solves the problem of deciding whether G admits a b-coloring with k colors in time $2^{\mathcal{O}(k^4 \cdot \Delta)} + n^{\mathcal{O}(1)}$ whenever $\ell_k(G) \leq k$.*

Furthermore, in the proof of Theorem 5.2, we in fact provide a polynomial kernel for the problem: In Case 1, we have a kernelized instance on $\mathcal{O}(k^4 \cdot \Delta)$ vertices (see Claim 5.2.2) and in Case 2, we always have a Yes-instance.

Corollary 5.4. *The problem of deciding whether a graph G has a b-coloring with $k = m(G)$ colors admits a kernel on $\mathcal{O}(k^4 \cdot \Delta) = \mathcal{O}(\Delta^5)$ vertices.*

5.2. FPT algorithm parameterized by $\Delta(G) + \ell_k(G)$

The next parameterization of b-Coloring involving the maximum degree that we consider is by $\Delta(G) + \ell_k(G)$. We show that in this case, the problem is FPT. By Observation 3.3 we know that b-Coloring is NP-complete on graphs with $\ell_k(G) = 0$, and by Theorem 1, it is NP-complete even when $k = 3$ and $\Delta(G) = 4$. Hence, there is no FPT- nor XP-algorithm for a parameterization using only one of the two above mentioned parameters unless P = NP. Note that the algorithm we provide in this section can be used to solve the case of $k = m(G)$ for which we gave a separate algorithm in Section 5.1, see Algorithm 5. However, Algorithm 5 is much simpler than the algorithm presented in this section, and simply applying the following algorithm for the case $k = m(G)$ results in a runtime of $2^{\mathcal{O}(k^{k+3} \cdot \Delta)} + n^{\mathcal{O}(1)}$ which is far worse than the runtime of $2^{\mathcal{O}(k^4 \cdot \Delta)} + n^{\mathcal{O}(1)}$ of Theorem 5.2.

Theorem 5.5 *(Theorem 3, restated). There is an algorithm that given a graph G on n vertices decides whether G has a b-coloring with k colors in time $2^{\mathcal{O}(\ell \cdot \Delta \cdot \min\{\ell, \Delta\}^{\ell+2})} + n^{\mathcal{O}(1)}$, where $\Delta := \Delta(G)$ and $\ell := \ell_k(G)$.*

Proof. The overall strategy of the algorithm is similar to Algorithm 5. We can make the following assumptions. First, if $\ell \le k$, then we can apply Algorithm 5 directly to solve the instance at hand, see Remark 5.3. Hence we can assume that $k < \ell$. Furthermore, $k \le \Delta + 1$, otherwise we are dealing with a trivial No-instance; we have that $k \le \min\{\ell - 1, \Delta + 1\}$. Furthermore, we can assume that $k > 2$, otherwise the problem is trivially solvable in time polynomial in n.

We consider a partition (D, T, R) of $V(G)$, where the vertices in D have degree at least k, the vertices in T have degree $k - 1$ and the vertices in R have degree less than $k - 1$. We assume that Reduction Rule 5.1 has been applied exhaustively, so Observation 5.2.1 holds, i.e. every vertex in R has at least one neighbor in $D \cup T$.

Now, we pick an inclusion-wise maximal set $B \subseteq D \cup T$ such that for each pair of distinct vertices $b_1, b_2 \in B$, $dist(b_1, b_2) \ge \ell + 3$.

Case 1 ($|B \cap T| < k$)**.** By the same argument given in Case 1 of the proof of Theorem 5.2, we have that any vertex in $T \cup R$ is at distance at most $\ell + 3$ from a vertex in B. We now give a bound on the number of vertices in G in terms of ℓ and Δ.

Claim 5.5.1. If $|B \cap T| < k$, then $|V(G)| = \mathcal{O}(\ell \cdot \Delta \cdot \min\{\ell, \Delta\}^{\ell+2})$.

Proof. The proof strategy is the same as in the proof of Claim 5.2.2. Note that $(B \cup D, S_1, \ldots, S_{\ell+3})$ constitutes a partition of $V(G)$, where S_i is the set of vertices of $V(G) \setminus (B \cup D)$ that are at distance exactly i from B. Since $|B \cap T| < k$ and $|D| \le \ell$, we have that $|B \cup D| < k + \ell$ and $|S_1| < \ell \cdot \Delta + k(k-1) = \mathcal{O}(\ell \cdot \Delta)$. By the definition of the set D, all the vertices in $S_1 \cup \ldots \cup S_{\ell+3}$ have degree at most $k - 1$. Thus, $|S_i| = (k-1) \cdot |S_{i-1}| = |S_1| \cdot (k-1)^{i-1}$ for all $i \in \{2, \ldots, \ell + 3\}$. We conclude that the number of vertices in G is at most

$$k + \ell + |S_1| \cdot \sum_{i=1}^{\ell+3} (k-1)^{i-1} = k + \ell + |S_1| \cdot \mathcal{O}((k-1)^{\ell+2}) = \mathcal{O}(\ell \cdot \Delta \cdot (k-1)^{\ell+2}),$$

where $(k-1) \le \min\{\ell - 2, \Delta\} \le \min\{\ell, \Delta\}$ and therefore $|V(G)| = \mathcal{O}(\ell \cdot \Delta \cdot \min\{\ell, \Delta\}^{\ell+2})$. $\square$

By the previous claim, we can solve the instance in time $2^{\mathcal{O}(\ell \cdot \Delta \cdot \min\{\ell, \Delta\}^{\ell+2})}$ in this case, using the exact exponential time algorithm for b-Coloring due to Panolan et al. [18].

Case 2 ($|B \cap T| \ge k$)**.** Let $B' \subseteq B \cap T$ be of size k and denote it by $B' := \{x_1, \ldots, x_k\}$. The strategy in this case is as follows: We compute a proper coloring of $G[D]$, and then modify it so that can be extended to a b-coloring of G. In this process we will be able to guarantee for each $i \in [k]$, that either x_i can be the b-vertex for color i, or we will have found another vertex in D that can serve as the b-vertex of color i. The difficulty here arises from the following situation: Suppose that in the coloring we computed for $G[D]$, a vertex x_i has two neighbors in D that received the same color. Then, x_i cannot be the b-vertex of color i in any extension of that coloring, since $\deg(x_i) = k - 1$, and $k - 1$ colors need to appear the neighborhood of x_i for it to be a b-vertex. However, recoloring a vertex in $N(x_i) \cap D$ might create a conflict in the coloring of $G[D]$. These potential conflicts can only appear in the connected component of $G[D \cup B']$ that contains x_i. We now show that each component of $G[D \cup B']$ can contain at most one such vertex, by our choice of the set B.

Claim 5.5.2. Let C be a connected component of $G[D \cup B']$. Then, C contains at most one vertex from B'.

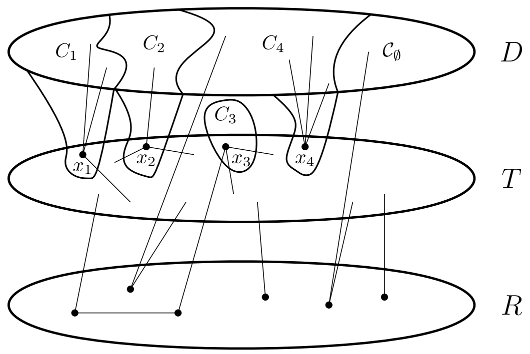

Fig. 1. Illustration of the structure of a graph G in the proof of Theorem 5.5 where $k = 4$. Here, $B' = \{x_1, \ldots, x_4\}$ and $C_1, \ldots, C_4$ are the components of $G[D \cup B']$ containing $x_1, \ldots, x_4$, respectively. Note that all vertices in T are of degree 3, all vertices in R of degree at most 2 and all vertices in R have a neighbor in $D \cup T$.

Proof. Let $Z := V(C) \cap B'$ and assume for the sake of a contradiction that $|Z| > 1$. Let $x_i, x_j \in Z$ be a pair of distinct vertices in Z such that $dist_{G[D\cup B']}(x_i, x_j)$ is minimized among all pairs of distinct vertices in Z. Hence, all vertices on the path from x_i to x_j in $G[D \cup B']$ are from D. Since $|D| = \ell$, we have that $dist_{G[D\cup B']}(x_i, x_j) \le \ell + 1$. However, we then have that $dist_G(x_i, x_j) \le dist_{G[D\cup B']}(x_i, x_j) \le \ell + 1$, a contradiction with the choice of B, by which we have that $dist_G(x_i, x_j) \ge \ell + 3$. $\square$

Throughout the following, for $i \in [k]$, we denote by C_i the connected component of $G[D \cup B']$ that contains x_i, and by ℓ_i the number of vertices of C_i, i.e. $\ell_i := |V(C_i)|$. By Claim 5.5.2, $C_i \neq C_j$, for all $i, j \in [k], i \neq j$. We now show that each neighbor of x_i has no neighbor in $D \cap N[B']$ outside of $V(C_i) \cup N[x_i]$.

Claim 5.5.3. Let $i \in [k]$, and $y \in N(x_i) \setminus D$. Then, $N_G[y] \cap (D \cup N[B']) \subseteq V(C_i) \cup N[x_i]$.

Proof. Suppose there is some $j \in [k] \setminus \{i\}$ such that y has a neighbor z in $V(C_j) \cup N[x_j]$. Since $dist_G(x_i, x_j) \ge \ell + 3 \ge 4$, z cannot be in $N[x_j]$, as it would imply that $dist_G(x_i, x_j) \le 3$. We can assume that $z \in V(C_j) \setminus N[x_j]$. Since $\left|V(C_j)\right| \le \ell$, there is a path of length at most ℓ between z and x_j; appending the edges $x_i y$ and yz to this path yields a path of length at most $\ell + 2$ between x_i and x_j, a contradiction with $dist_G(x_i, x_j) \ge \ell + 3$. $\square$

Let $\mathcal{C}_\emptyset$ be the set of connected components of $G[D \cup B']$ that do not contain any vertex from B'. We observe that any proper coloring of $G[D \cup B']$ can be obtained from independently coloring the vertices in $C_1, \ldots, C_k$, and $\mathcal{C}_\emptyset$. If for some $i \in [k]$, C_i is a trivial[4] component, then $N(x_i) \cap D = \emptyset$. Hence, we can assign x_i any color without creating any conflict with the remaining vertices in $G[D \cup B']$. On top of that, Claim 5.5.3 ensures that assigning a color to a neighbor of any x_i (that is not contained in D) cannot create a coloring conflict with any vertex in $D \cup N[B']$ that is not contained in $V(C_i) \cup N[x_i]$. We illustrate the structure of G in Fig. 1. Before we proceed with the proof of the next claim, we introduce some notation. For $X \subseteq V(G)$, a (pre-) coloring $\gamma\colon X \to [k]$, and $i, j \in [k]$, we denote by $\gamma_{i\leftrightarrow j}$ the (pre-) coloring obtained from γ by *switching* colors i and j, i.e. for $v \in X$ we let:

$$\gamma_{i\leftrightarrow j}(v) := \begin{cases} \gamma(v), & \text{if } \gamma(v) \notin \{i, j\} \\ i, & \text{if } \gamma(v) = j \\ j, & \text{if } \gamma(v) = i \end{cases}$$

It is immediate that γ is proper if and only if $\gamma_{i\leftrightarrow j}$ is proper.

Claim 5.5.4. Let $i \in [k]$ and let $\gamma\colon V(C_i) \to [k]$ be a proper coloring of C_i. Then, one can find in time $\mathcal{O}(k^2 \cdot \ell_i^2)$ a set $Y_i \subseteq N_G(x_i) \setminus D$ and a proper coloring $\delta\colon V(C_i) \cup Y_i \to [k]$ of $G[V(C_i) \cup Y_i]$ that has a b-vertex for color i.

Proof. We can assume that $\gamma(x_i) = i$, for if $\gamma(x_i) = j \neq i$, we can consider $\gamma_{i\leftrightarrow j}$ instead. The proof works in two stages. First, we show that within the claimed time bound we can find a proper coloring $\delta\colon V(C_i) \to [k]$ of C_i satisfying one of the following two conditions.

(i) There is a vertex in $V(C_i)$ different from x_i that is a b-vertex for color i in δ.
(ii) We have that $\delta(x_i) = i$ and δ is injective on $N_{C_i}[x_i]$.

[4] We call a connected component of a graph *trivial* if it contains only one vertex.

For this first step, we assume that C_i is a nontrivial component of $G[D \cup B']$, otherwise condition (ii) is vacuously satisfied. Suppose that neither of the two conditions holds. Let $j \in [k]$ with $j \neq i$ be a color that does not appear on any vertex in $N_{C_i}(x_i)$, i.e. there is no vertex $y \in N_{C_i}(x_i)$ such that $\gamma(y) = j$. Such a color must exist by the facts that γ is not injective on $N_{C_i}[x_i]$ and $\deg_G(x_i) = k - 1$. For each vertex $z \in V(C_i)$ with $\gamma(z) = j$, we do the following.

1) If $\gamma(N[z]) = [k]$, i.e. if all colors except j appear in the neighborhood of z, then z is a b-vertex for color j. We let $\delta := \gamma_{i \leftrightarrow j}$ and we are in case (i).
2) Otherwise, there is a color $j' \neq j$ that does not appear in the neighborhood of z. We update γ by setting $\gamma(z) := j'$, keeping the coloring γ proper.

If these two steps are executed for all vertices that γ colored j without ending up in case (i), then γ is a proper coloring of C_i with colors $[k] \setminus \{j\}$. If after these recoloring steps, γ remains non-injective on $N_{C_i}[x_i]$, then there are two vertices $y_1, y_2 \in N_{C_i}(x_i)$ that received the same color, i.e. $\gamma(y_1) = \gamma(y_2)$. Since no vertex in C_i received color j by γ, we can update $\gamma(y_2) := j$ without introducing a conflict.

We repeat this process until we either reached case (i) at some stage, or until the coloring γ is injective on $N_{C_i}[x_i]$, which means we are in case (ii). Since in each iteration, we increase the number of colors appearing in the neighborhood of x_i by one, we know that the latter condition is met after at most $\left|N_{C_i}(x_i)\right| \le k - 1$ iterations. This recoloring procedure terminates within time $\mathcal{O}(k \cdot |V(C_i)|^2) = \mathcal{O}(k \cdot \ell_i^2)$.

As any coloring δ satisfying case (i) yields the claim with $Y_i = \emptyset$, we can from now on assume that we are in case (ii), i.e. we have a coloring δ of C_i such that $\delta(x_i) = i$ and δ is injective on $N_{C_i}[x_i]$.

We proceed as follows. Let $Z = N_G(x_i) \setminus V(C_i)$ be the set of neighbors of x_i that are not contained in C_i, and initialize a set $Y_i := \emptyset$. We repeat the following steps, extending the coloring δ to one more vertex at a time, and adding it to Y_i, until the condition of the claim is met. We keep as an invariant that the coloring δ is proper and injective on $N_{C_i}[x_i] \cup Y_i$.

1) Let $z \in Z \setminus Y_i$, and let $j \in [k]$ be a color that does not appear on $N_{C_i}[x_i] \cup Y_i$ in δ. Let furthermore $w_1, \ldots, w_t$ be the neighbors of z in $V(C_i) \cup Y_i$ that δ colored j. (If there is no such vertex, we skip the next stage.) Note that for all $h \in [t]$, w_h is not a neighbor of x_i, since $\delta(w_h) = j$ and j is a color that does not appear in the neighborhood of x_i so far.
2) For $h = 1, \ldots, t$, we proceed as follows. If w_h is a b-vertex of color j in δ, then $\delta_{i \leftrightarrow j}$ yields the claim and we terminate this process. Otherwise, there is a color $j' \neq j$ that does not appear on any vertex in $N_G(w_h) \cap (V(C_i) \cup Y_i)$. We update $\delta(w_h) := j'$ without introducing a conflict, and repeat Stage 2 for vertex w_{h+1} (unless $h = t$).
3) If this stage is reached, then we modified δ in such a way that no neighbor of z in C_i has received color j. Hence, we let $\delta(z) := j$, add z to Y_i and continue with Stage 1 unless $Y_i = Z$. Note that δ remained injective on $N_{C_i}[x_i] \cup Y_i$, as color j has not yet appeared in the neighborhood of x_i.

If the above process is terminated in Stage 2, then we found a coloring satisfying the claim. Otherwise, the coloring δ is proper and injective on the neighborhood of x_i. Since $\deg_G(x_i) = k - 1$, this implies that x_i is a b-vertex for its color i in δ, so the target condition of the claim is satisfied as well. It can be verified that this process can be implemented to run in time $\mathcal{O}(k \cdot |V(C_i)| (|V(C_i)| + k)) = \mathcal{O}(k^2 \cdot \ell_i^2)$. $\square$

We now wrap up the treatment of this case. We compute a proper k-coloring γ of $G[D \cup B']$. We derive from γ another k-coloring δ of some induced subgraph of $G[D \cup N_G[B']]$ containing $D \cup B'$. For each $i \in [k]$, we do the following. With input $\gamma|_{V(C_i)}$ we compute a proper k-coloring δ_i of $G[V(C_i) \cup Y_i]$ using Claim 5.5.4, where Y_i is the set returned by its algorithm, and we let $\delta|_{V(C_i) \cup Y_i} := \delta_i$. Finally, we let $\delta|_{V(\mathcal{C}_\emptyset)} := \gamma|_{V(\mathcal{C}_\emptyset)}$. As for $i \neq j$, C_i and C_j are distinct connected components of $G[D \cup B']$ and by Claim 5.5.3, this construction is well-defined and there is no color conflict between any pair of vertices z_i, z_j where $z_i \in V(C_i) \cup Y_i$ and $z_j \in V(C_j) \cup Y_j$ for $i \neq j$. Since for each $i \in [k]$ we applied Claim 5.5.4, δ is a b-precoloring of G. All vertices in G that have not received a color so far (recall that δ colors all vertices in D) have degree at most $k - 1$, so we can extend the coloring δ greedily to the remainder of G.

It remains to argue the runtime of the algorithm. Applying Reduction Rule 5.1 exhaustively can be done in time $n^{\mathcal{O}(1)}$. As mentioned above, in Case 1 we can solve the instance in time $2^{\mathcal{O}(\ell \cdot \Delta \cdot \min\{\ell, \Delta\}^{\ell+2})}$. In Case 2, we can compute a proper k-coloring of $G[D \cup B']$ in time $\mathcal{O}(2^{\ell+k} \cdot \ell^{\mathcal{O}(1)})$ using standard methods [3]. Modifying this coloring to satisfy the conditions of Claim 5.5.4 for each $i \in [k]$ takes time at most $\mathcal{O}(\sum_{i=1}^{k} k^2 \cdot \ell_i^2) = \mathcal{O}(k^3 \cdot \sum_{i=1}^{k} \ell_i^2) = \mathcal{O}(k^3 \cdot \ell^2) = \mathcal{O}(\ell^5)$. Extending the coloring to the remainder of G can be done in time $n^{\mathcal{O}(1)}$, so the total runtime of the algorithm is

$$2^{\mathcal{O}(\ell \cdot \Delta \cdot \min\{\ell, \Delta\}^{\ell+2})} + 2^{\ell+k} \cdot \ell^{\mathcal{O}(1)} + \mathcal{O}(\ell^5) + n^{\mathcal{O}(1)} = 2^{\mathcal{O}(\ell \cdot \Delta \cdot \min\{\ell, \Delta\}^{\ell+2})} + n^{\mathcal{O}(1)},$$

as claimed. $\square$

Similar to above, we obtained a kernel for the problem. While this result does not provide a polynomial kernel for the parameterization $\Delta + \ell$, it does give a polynomial kernel if we consider the problem for *fixed* values of ℓ and parameter Δ.

Corollary 5.6. *The problem of deciding whether a graph G admits a b-coloring with k colors admits a kernel on $\mathcal{O}(\ell \cdot \Delta \cdot \min\{\ell, \Delta\}^{\ell+2})$ vertices, where $\Delta := \Delta(G)$ and $\ell := \ell_k(G)$.*

6. Conclusion

We have presented a complexity dichotomy for b-Coloring with respect to two upper bounds on the b-chromatic number, in the following sense: We have shown that given a graph G and for fixed $k \in \{\Delta(G)+1-p, m(G)-p\}$, it can be decided in polynomial time whether G has a b-coloring with k colors whenever $p \in \{0, 1\}$ and the problem remains NP-complete whenever $p \geq 2$, already for $k = 3$.

The most immediate question left open in this work is the parameterized complexity of the b-Coloring problem when $k \in \{m(G), \Delta(G), m(G) - 1\}$. In all of these cases, we have provided XP-algorithms, and it would be interesting to see whether these problems are FPT or W[1]-hard.

Open Problem 1. Let G be a graph and $k \in \{m(G), \Delta(G), m(G) - 1\}$. Is the problem of deciding whether a graph G has a b-coloring with k colors parameterized by k fixed-parameter tractable or W[1]-hard?

We showed that b-Coloring is FPT parameterized by $\Delta(G) + \ell_k(G)$, where $\ell_k(G)$ denotes the number of vertices of degree at least k in G, and this is optimal in the sense that there is no FPT nor XP algorithm for the problem parameterized by only one of the two invariants. It would be interesting to see if one could devise an FPT-algorithm for the parameterization that replaces the maximum degree by the number of colors.

Open Problem 2. Is b-Coloring parameterized by $k + \ell_k(G)$ fixed-parameter tractable?

Note that a positive answer to this question would also imply an FPT-algorithm for the question of whether a graph G has a b-coloring with $k = m(G)$ colors parameterized by k, partially answering Open Problem 1.

Recently, Effantin et al. [9] introduced the *relaxed b-chromatic number* of a graph G, $\chi_b^r(G)$, as the maximum b-chomatic number of any induced subgraph of G, i.e. $\chi_b^r(G) := \max_{X \subseteq V(G)} \chi_b(G[X])$. It is clear that $\chi_b(G) \leq \chi_b^r(G)$, so it would be interesting to see if for fixed k, the problem of deciding whether a graph G admits a b-coloring with k colors when the value of k is close to $\chi_b^r(G)$ admits a similar dichotomy as the ones we presented for the upper bounds $\Delta(G) + 1$ and $m(G)$ on $\chi_b(G)$.

Declaration of competing interest

None.

Acknowledgements

We would like to thank Fedor Fomin for useful advice with good timing and Petr Golovach for pointing out the reference [7].

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