Algorithms for the Rainbow Vertex Coloring Problem on Graph Classes

Paloma T. Lima

Department of Informatics, University of Bergen, Norway
paloma.lima@uib.no

Erik Jan van Leeuwen

Department of Information and Computing Sciences, Utrecht University, The Netherlands
e.j.vanleeuwen@uu.nl

Marieke van der Wegen

Department of Information and Computing Sciences, Utrecht University, The Netherlands
Mathematical Institute, Utrecht University, The Netherlands
m.vanderwegen@uu.nl

Abstract

Given a vertex-colored graph, we say a path is a rainbow vertex path if all its internal vertices have distinct colors. The graph is rainbow vertex-connected if there is a rainbow vertex path between every pair of its vertices. In the Rainbow Vertex Coloring (RVC) problem we want to decide whether the vertices of a given graph can be colored with at most k colors so that the graph becomes rainbow vertex-connected. This problem is known to be NP-complete even in very restricted scenarios, and very few efficient algorithms are known for it. In this work, we give polynomial-time algorithms for RVC on permutation graphs, powers of trees and split strongly chordal graphs. The algorithm for the latter class also works for the strong variant of the problem, where the rainbow vertex paths between each vertex pair must be shortest paths. We complement the polynomial-time solvability results for split strongly chordal graphs by showing that, for any fixed $p \geq 3$ both variants of the problem become NP-complete when restricted to split $(S_3, \ldots, S_p)$-free graphs, where S_q denotes the q-sun graph.

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1 Introduction

Graph coloring is a classic problem within the field of structural and algorithmic graph theory that has been widely studied in many variants. An example is the rainbow coloring problem, which is an edge coloring problem [2, 11, 13]. One recent variant was defined by Krivelevich and Yuster [9] and has received significant attention: the *rainbow vertex coloring* problem. A vertex-colored graph is said to be *rainbow vertex-connected* if between any pair of its vertices, there is a path whose internal vertices are colored with distinct colors. Such a path is called a *rainbow path*. Note that this vertex coloring does not need to be a proper one; for instance, a complete graph is rainbow vertex-connected under the coloring that assigns the same color to every vertex. The Rainbow Vertex Coloring (RVC) problem takes as input a graph G and an integer k and asks whether G has a coloring with k colors under which it is rainbow vertex-connected. The *rainbow vertex connection number* of a graph G is the smallest number of colors needed in one such coloring and is denoted $\mathbf{rvc}(G)$. More recently, Li et al. [12] defined a stronger variant of this problem by requiring that the rainbow paths connecting the pairs of vertices are also shortest paths between those pairs. In this case we say the graph is *strong rainbow vertex-connected.* The analogous computational problem is called STRONG RAINBOW VERTEX COLORING (SRVC) and the corresponding parameter is denoted $\mathbf{srvc}(G)$.

Both the RVC and the SRVC problems are NP-complete for every $k \geq 2$ [4, 3, 5], and remain NP-complete even on bipartite graphs and split graphs [7]. Both problems are also NP-hard to approximate within a factor of $n^{1/3-\epsilon}$ for every $\epsilon > 0$, even when restricted to bipartite graphs and split graphs [7]. Contrasting these results, it was shown that RVC and SRVC are linear-time solvable on bipartite permutation graphs and block graphs [7], and on planar graphs for every fixed k [10]. In fact, if k is fixed, both problems are also solvable in linear time on graphs of constant treewidth and in cubic time on graphs of constant clique-width, as they can be expressed in monadic second order logic [5]. Furthermore, they are also solvable in linear time on graphs of vertex cover at most p, for any fixed p [5]. Finally, RVC is also known to be linear time solvable on interval graphs [7].

The above mentioned results on bipartite permutation graphs and interval graphs led Heggernes et al. [7] to formulate the following conjecture concerning diametral path graphs. A graph G is a *diametral path graph* if every induced subgraph H has a diametral path P that is dominating. Recall that a *diametral path* is a shortest path whose length is equal to the diameter, and that *dominating* means that every vertex either is in P, or is adjacent to a vertex in P.

▶ **Conjecture 1** (Heggernes et al. [7, Conjecture 15]). *Let G be a diametral path graph. Then* $\mathbf{rvc}(G) = \mathrm{diam}(G) - 1$.

In this context, it is interesting to remark that both bipartite permutation graphs and interval graphs are diametral path graphs, and that Heggernes et al. [7] showed that the conjecture is true for these graphs.

We remark that similar bounds were studied for the edge variant of the problem, in which we want to color the edges of a graph in such a way that every pair of vertices is connected by a path whose edges received pairwise distict colors (a rainbow path). For instance, it is known that AT-free graphs (a subclass of diametral path graphs) can be rainbow edge colored with $\mathrm{diam}(G) + 3$ colors. However, there are examples of such graphs G that need $\mathrm{diam}(G) + 2$ colors to be rainbow edge colored [16].

Our Results. [1] Our main contribution is to show that the above conjecture is true for permutation graphs.

▶ **Theorem A** (=Theorem 17). *If G is a permutation graph on n vertices, then* $\mathbf{rvc}(G) = \mathrm{diam}(G) - 1$ *and the corresponding rainbow vertex coloring can be found in $O(n^2)$ time.*

This generalizes the earlier result on bipartite permutation graphs [7]. The proof of our result follows from a thorough investigation of shortest paths in permutation graphs. We show that there are two special shortest paths that ensure that a rainbow vertex coloring with $\mathrm{diam}(G) - 1$ colors can be found.

We also investigate the rainbow vertex connection number of chordal graphs further. As the problem is NP-hard and hard to approximate on split graphs [7], the hope for polynomial-time solvability rests either within subclasses of split graphs or other chordal graphs that are not inclusion-wise related to split graphs (such as the previously studied interval graphs and block graphs [7]). We make progress in both directions. First, we show that the problem is polynomial-time solvable on split strongly chordal graphs.

[1] Statements marked with ♠ had their proofs omitted due to space constraints.

▶ **Theorem B** (=Theorem 20). *If G is a split strongly chordal graph with ℓ cut vertices, then* $\mathbf{rvc}(G) = \mathbf{srvc}(G) = \max\{\mathrm{diam}(G) - 1, \ell\}$.

In order to obtain the above result, we exploit an interesting structural property of split strongly chordal graphs. Namely, if G is a split strongly chordal graph with clique K and independent set S, there exists a spanning tree of $G[K]$ such that the neighborhood of each vertex of S induces a subtree of this tree.

Second, we show that RVC remains polynomial-time solvable on powers of trees. This proof is based on a case analysis, depending on whether the diameter of the tree is a multiple of the power and how many long branches the tree has. In some cases $\mathrm{diam}(G) - 1$ many colors are enough to rainbow vertex color these graphs, but surprisingly this is not always true. There are graphs in this graph class that actually require $\mathrm{diam}(G)$ colors in order to be rainbow vertex colored. We provide a complete characterization of such graphs, as well as a polynomial time algorithm to optimally rainbow vertex color any power of tree.

▶ **Theorem C** (=Theorem 26). *If G is a power of a tree, then* $\mathbf{rvc}(G) \in \{\mathrm{diam}(G) - 1, \mathrm{diam}(G)\}$, *and the corresponding optimal rainbow vertex coloring can be found in time that is linear in the size of G.*

2 Preliminaries

Whenever we write graph, we will mean a finite undirected simple graph. We assume throughout that all graphs are connected and have at least four vertices.

Let $G = (V, E)$ be a graph. For two vertices $u, v \in V$, we use $u \sim v$ to denote that u and v are adjacent. For a vertex $v \in V$, we write $d_G(v)$ for its degree. For a subgraph H of G, we write V_H for the set of vertices of H. Specifically, for a path P in G, we write V_P for the vertices of P. If $X \subseteq V$, then by $G[X]$ we denote the subgraph of G induced by X, that is, $G[X] = (X, E \cap (X \times X))$. We use $N(v) = \{u \in V \mid u \sim v\}$ and $N[v] = N(v) \cup \{v\}$.

The length of a path P equals the number of edges of P. The distance $d_G(u, v)$ is the length of a shortest u, v-path in G. If the graph G is clear from the context, we simply write $d(u, v)$. The *diameter* $\mathrm{diam}(G)$ of G is the length of the longest shortest path between two vertices in G, that is, $\mathrm{diam}(G) = \max\{d_G(u, v) \mid u, v \in V\}$. A *center* of a graph G is a vertex c such that $\max\{d_G(c, v) \mid v \in V\}$ is minimum among all vertices of G. Note that a graph can have multiple centers and that a tree can have at most two.

A graph G is a *permutation graph* if it is an intersection graph of line segments between two parallel lines (see Figure 1). The set of line segments that induce the permutation graph is called an *intersection model*. Alternatively, if G has n vertices, then there is a permutation σ of $\{1, \dots, n\}$ such that vertex i and vertex j with $i < j$ are adjacent in G if and only if j comes before i in σ.

A graph G is a *chordal graph* if every cycle $C = \{c_1, \dots, c_\ell\}$ on $\ell \geq 4$ vertices has a chord, meaning an edge between two non-consecutive vertices of the cycle.

A graph G is a *split graph* if V_G can be split into two sets, K and S, such that K induces a clique in G and S induces an independent set in G.

For any $k \geq 3$, we denote by S_k the k-sun on $2k$ vertices, that is, a graph with a clique $c_1, \dots, c_k$ on k vertices and an independent set $v_1, \dots, v_k$ of k vertices such that v_i is adjacent to c_i and c_{i+1} for every $1 \leq i < k$ and v_k is adjacent to c_k and c_1. A graph G is a *strongly chordal graph* if it is chordal and every even cycle C has a chord uv such that the distance in C between u and v is odd. The strongly chordal graphs are exactly the chordal graphs which have no induced subgraphs isomorphic to a k-sun for any $k \geq 3$.

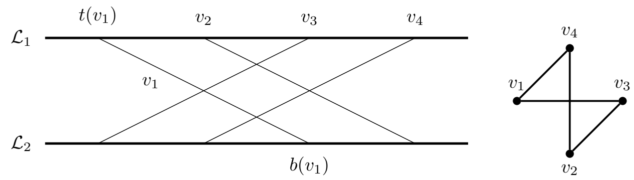

Figure 1 An intersection model and the corresponding permutation graph.

The *k-th power* of a graph G for $k \geq 1$, denoted by G^k, is the graph on the same vertex set of G where $u \sim v$ in G^k if and only if there is a path of length at most k from u to v in G. In particular, $G^1 = G$. If G is a tree, then G^k is a chordal graph for any $k \geq 1$.

Finally we make the following observation about diameter two graphs, which follows from the fact that we can color all the vertices of G with the same color.

▶ **Observation 1.** *If* $\operatorname{diam}(G) \leq 2$*, then* $\mathbf{srvc}(G) = \mathbf{rvc}(G) = 1$.

3 Permutation graphs

In this section, we consider rainbow colorings on permutation graphs. Let G be a permutation graph. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two parallel lines in the plane and for each $v \in V_G$, let s_v be the segment associated to v in the intersection model (see Figure 1). We denote by $t(v)$ the extreme of s_v in $\mathcal{L}_1$, that is $t(v) = s_v \cap \mathcal{L}_1$, and we refer to $t(v)$ as the top end point of s_v. By $b(v)$ we denote the extreme of s_v in $\mathcal{L}_2$, the bottom end point of s_v. Throughout, we assume that an intersection model is given; otherwise, one can be computed in linear time [14].

Whenever we write "u intersects v" for two vertices u and v, we mean s_u intersects s_v. For two vertices u and v, with $u \neq v$, there are several options for u, v in the intersection model. If

$$\begin{aligned}
&t(u) < t(v) \text{ and } b(u) > b(v), \text{ then } u \sim v,\\
&t(u) > t(v) \text{ and } b(u) < b(v), \text{ then } u \sim v,\\
&t(u) < t(v) \text{ and } b(u) < b(v), \text{ then we say "}u \text{ is left of } v\text{" and write } u \prec v,\\
&t(u) > t(v) \text{ and } b(u) > b(v), \text{ then we say "}u \text{ is right of } v\text{" and write } u \succ v.
\end{aligned}$$

We use the notation $u \preceq v$ if $t(u) \leq t(v)$ and $b(u) \leq b(v)$. Note that "$\prec$" is a partial ordering on the vertices of the graph and $u \not\succ v$ does not imply $u \preceq v$.

For each pair $u, v \in V(G)$, Mondal et al. [15] define two u-v paths, one of which is shortest. Define a path $X_{u,v}$ as follows. If $u \sim v$, $X_{u,v}$ will be u, v. Otherwise, assume without loss of generality that $u \prec v$. Start with $x_1 = u$. Of all vertices x that intersect u with $t(x) > t(u)$, let x_2 be the one with largest $t(x_2)$. If there is no vertex x that intersects u with $t(x) > t(u)$, we say that the path $X_{u,v}$ does not exist. Otherwise, define x_i, with $i \geq 3$, as follows. If x_{i-1} is incident to v, set $x_i = v$ and end the path. Otherwise, if i is even (resp. odd), let x_i be the vertex that intersects x_{i-1} where $t(x_i)$ (resp. $b(x_i)$) is largest.

Notice that it cannot be that $x_{i-2} = x_i$, or G would not be connected.

Analogously, we define the path $Y_{u,v}$. This path starts with $y_1 = u$. If u intersects v, set $y_2 = v$ and end the path. Otherwise, let y_2 be the vertex that intersects u with largest $b(y_2)$, if $b(y_2) > b(u)$ (otherwise the path $Y_{u,v}$ does not exist).

If y_{i-1} intersects v, set $y_i = v$ and end the path. Otherwise, the next vertex y_i is the vertex that intersects y_{i-1} with largest $b(y_i)$ (resp. $t(y_i)$) if i is even (resp. odd). Notice that it cannot be that $y_{i-2} = y_i$, or G would not be connected.

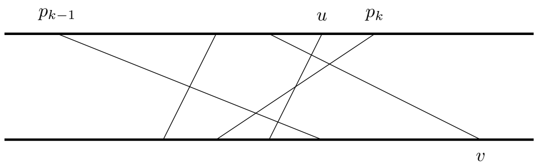

Figure 2 If a vertex v in layer L_{k+1} intersect a vertex u in layer L_k, then it also intersects p_k. See Lemma 7.

The paths we just defined satisfy the following property. Let $z_1, z_2, z_3, \ldots, z_a$ be a path. For all $2 \leq i \leq a$,

$$t(z_i) > t(z_{i-1}) \text{ and } b(z_i) < b(z_{i-1}) \text{ if } i \text{ is even,} \tag{1}$$
$$t(z_i) < t(z_{i-1}) \text{ and } b(z_i) > b(z_{i-1}) \text{ if } i \text{ is odd,} \tag{2}$$

or, for all $2 \leq i \leq a$,

$$t(z_i) < t(z_{i-1}) \text{ and } b(z_i) > b(z_{i-1}) \text{ if } i \text{ is even,} \tag{3}$$
$$t(z_i) > t(z_{i-1}) \text{ and } b(z_i) < b(z_{i-1}) \text{ if } i \text{ is odd.} \tag{4}$$

Note that Equations (1) and (2) hold for $X_{u,v}$, by definition, and that Equations (3) and (4) hold for $Y_{u,v}$, by definition.

▶ **Lemma 2** (♠, Mondal et al. [15]). *$X_{u,v}$ or $Y_{u,v}$ is a shortest u,v-path.*

We define two special paths P and Q. For P, let p_1 be the vertex such that $t(p_1)$ is smallest among all vertices of G. Perform the same process as in the construction of $X_{p_1,\cdot}$: for $i \geq 2$, if i is even (resp. odd), let p_i be the vertex that intersects p_{i-1} where $t(p_i)$ (resp. $b(p_i)$) is largest. Let P denote the resulting path and let p_d denote the last vertex of P. Observe that $P = X_{p_1,p_d}$.

For Q, let q_1 be the vertex such that $b(q_1)$ is smallest among all vertices of G. Perform the same process as in the construction of $Y_{q_1,\cdot}$: for $i \geq 2$, if i is even (resp. odd), let q_i be the vertex that intersects q_{i-1} where $b(q_i)$ (resp. $t(q_i)$) is largest. Let Q denote the resulting path and let $q_{d'}$ denote the last vertex of Q. Observe that $Q = Y_{q_1,q_{d'}}$.

▶ **Corollary 3** (♠). *P is a shortest p_1, p_d-path and Q is a shortest $q_1, q_{d'}$-path.*

We will prove some more useful properties about the paths P and Q.

▶ **Lemma 4** (♠). *Let v_t, resp. v_b, be the segment that has the rightmost top, resp. bottom, end point. Then $p_d = v_t$ and $p_{d-1} = v_b$ if d is even, and vice versa if d is odd. Furthermore, we have $q_{d'} = v_b$ and $q_{d'-1} = v_t$ if d' is even, and vice versa if d' is odd.*

▶ **Lemma 5** (♠). *The sets $V_P \setminus \{p_d\}$ and $V_Q \setminus \{q_{d'}\}$ are dominating sets of G.*

▶ **Lemma 6** (♠). *It holds that $d = \mathrm{diam}(G)$ or $d = \mathrm{diam}(G) + 1$, and $d' = \mathrm{diam}(G)$ or $d' = \mathrm{diam}(G) + 1$.*

Now we start a breadth-first search from p_1. Call the layers $L_1, L_2, \ldots, L_r$. Since P is a shortest path, it follows that $p_i \in L_i$ for every i. Thus $r \geq d$. Since $V_P \setminus \{p_d\}$ is a dominating set, we conclude that $r = d$, thus the layers of the breadth-first search are $L_1, L_2, \ldots, L_d$. We also start a breadth-first search in q_1, and call the layers $M_1, M_2, \ldots, M_{d'}$. Again, we have that $q_i \in M_i$ for every i. A nice property of the path P is that every vertex p_i is adjacent to all vertices in the next layer L_{i+1}.

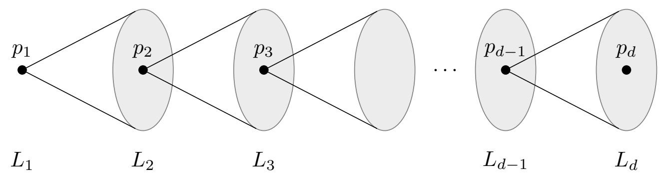

Figure 3 The layers of the BFS, the layers $L_1, L_2, \ldots, L_{d-1}$ are all assigned a different color. See Lemma 8.

▶ **Lemma 7.** *For every i, it holds that $L_{i+1} \subseteq N(p_i)$ and $M_{i+1} \subseteq N(q_i)$.*

Proof. We will prove a somewhat stronger result by induction, namely that $L_{i+1} \subseteq N(p_i)$ and if i is even (resp. odd) we have that for every u in L_{i+1}:

$$t(u) < t(p_i) \text{ and } b(u) > b(p_i) \text{ (resp. } t(u) > t(p_i) \text{ and } b(u) < b(p_i)). \tag{5}$$

We use a proof by induction. The first layer L_1 contains only p_1. It is clear that every vertex in the second layer L_2 is a neighbour of p_1. Moreover, by the definition of p_1, we have that $t(u) > t(p_1)$ for all $u \in L_2$, and thus $b(u) < b(p_1)$.

Suppose that $L_{i+1} \subseteq N(p_i)$ and Equation (5) holds for every $i < k$. Let v be a vertex in L_{k+1}. We know that v does not intersect p_{k-1}, otherwise v would be contained in L_k. So we know that $t(v) > t(p_{k-1})$ and $b(v) > b(p_{k-1})$. Since v is in layer L_{k+1}, v intersects u for some $u \in L_k$ (see Figure 2). So we either have that $t(v) < t(u)$ and $b(v) > b(u)$ or we have $t(v) > t(u)$ and $b(v) < b(u)$. If k is even (resp. odd), we have $t(v) < t(u)$ and $b(v) > b(u)$ (resp. $t(v) > t(u)$ and $b(v) < b(u)$), otherwise, by the induction hypothesis, v would intersect p_{k-1}. By the induction hypothesis u intersects p_{k-1}, so by the definition of p_k, we have that $t(p_k) \geq t(u)$ (resp. $b(p_k) \geq b(u)$). It follows that $t(v) < t(p_k)$ (resp. $b(v) < b(p_k)$). We know that $b(p_k) < b(p_{k-1})$ (resp. $t(p_k) < t(p_{k-1})$), thus $b(v) > b(p_k)$ (resp. $t(v) > t(p_k)$). We conclude that v intersects p_k, and $t(v) < t(p_k)$ and $b(v) > b(p_k)$ (resp. $t(v) > t(p_k)$ and $b(v) < b(p_k)$). So $L_{k+1} \subseteq N(p_k)$ for every k. The proof that $M_{k+1} \subseteq N(q_k)$ is analogous. ◀

For an illustration of the structure of G, see Figure 3. If $d = \mathrm{diam}(G)$ or $d' = \mathrm{diam}(G)$, we will color G layer by layer to obtain a rainbow coloring.

▶ **Lemma 8.** *If $d = \mathrm{diam}(G)$ or $d' = \mathrm{diam}(G)$, then $\mathbf{rvc}(G) = \mathrm{diam}(G) - 1$.*

Proof. Assume that $d = \mathrm{diam}(G)$. Consider the following coloring (see Figure 3): $c(v) = i$ if $v \in L_i$, $1 \leq i \leq d-1$, and $c(v) = 1$ otherwise. This coloring uses $d - 1 = \mathrm{diam}(G) - 1$ colors. We claim that it is a rainbow coloring. Let u and v be two vertices. Then $u \in L_i$, $v \in L_j$ for some i, j. Without loss of generality, assume that $i \leq j$. If $u = p_i$, then, by Lemma 7, the path $p_1, p_2, \ldots, p_{j-1}, v$ is a rainbow path. If $i > 1$, again by Lemma 7, the path $u, p_{i-1}, p_i, \ldots, p_{j-1}, v$ is a rainbow path. We conclude that $\mathbf{rvc}(G) = \mathrm{diam}(G) - 1$. The proof for $d' = \mathrm{diam}(G)$ is analogous. ◀

Consider the case where $d = d' = \mathrm{diam}(G) + 1$. In this case, we will still color the layers of a breadth-first search that starts at p_1, but we have to reuse the color of the first layer for layer L_{d-1}. We consider the coloring: $c(v) = i$ if $v \in L_i$, $1 \leq i \leq d-2$, $c(v) = 1$ if $v \in L_{d-1}$, and $c(v) = 2$ if $v \in L_d$. We will show that this is a rainbow coloring. For almost every u and v, we readily construct a rainbow path using path P.

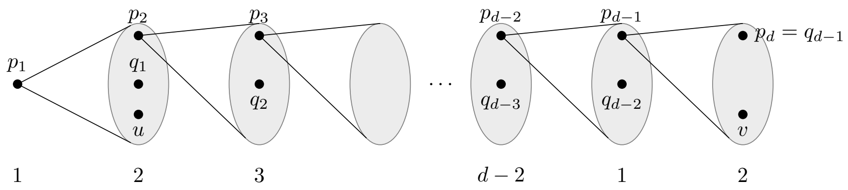

Figure 4 The numbers indicate the colors of the layers. If $u \sim q_1$ and $v \sim q_{d-1}$, the path $u, q_1, p_2, \ldots, q_{d-2}, v$ is a rainbow path. If $u \sim q_2$ and $v \nsim q_{d-2}$, the path $u, q_2, p_3, \ldots, q_{d-1}, v$ is a rainbow path. See Lemma 14.

▶ **Lemma 9** (♠). *For the following u and v, there exists a rainbow path:*
1. *for $u = p_i$, and v arbitrary,*
2. *for $u \in L_i$ with $i \geq 3$, and $v \in L_j$ with $j \geq 3$,*
3. *for $u \in L_2$, and $v \notin L_d$,*
4. *for $u \in L_2$, $u \sim p_2$, and $v \in L_d$.*

There are some vertices u and v, for which we did not yet construct a rainbow path. The case that is left, is the following:

5. for $u \in L_2$, $u \nsim p_2$, and $v \in L_d$.

The path via P, $u, p_1, p_2, \ldots, p_{d-1}, v$, does not suffice in this case, since it uses p_1 and p_{d-1}, which are both colored with color 1. So this is not a rainbow path. For some cases we show that a similar path via Q is a rainbow path. For other cases, we show that $X_{u,v}$ or $Y_{u,v}$ is a rainbow path.

▶ **Lemma 10** (♠). *If $u \in L_2$ and $u \nsim p_2$, then $u \sim q_1$ or $u \sim q_2$.*

▶ **Lemma 11** (♠). *If $d = d' = \operatorname{diam}(G) + 1$, then $p_d = q_{d-1}$ and $p_{d-1} = q_d$.*

▶ **Corollary 12** (♠). *If $d = d' = \operatorname{diam}(G) + 1$, it holds that $q_i \in L_{i+1}$, for every $1 \leq i < d$.*

▶ **Lemma 13** (♠). *If $d = d' = \operatorname{diam}(G) + 1$, then for every $v \in L_d$, if $v \nsim q_{d-2}$, then $v \sim q_{d-1}$.*

Now we can prove for even more vertices u and v that there is a rainbow path from u to v, using path Q, see also Figure 4.

▶ **Lemma 14.** *For the following vertices u and v, there is a rainbow path:*
5a. *for $u \in L_2$, $u \nsim p_2$, $v \in L_d$, and $v \sim q_{d-2}$,*
5b. *for $u \in L_2$, $u \nsim p_2$, $v \in L_d$, and $v \nsim q_{d-2}$, $u \sim q_2$.*

Proof. 5a. By Lemma 10, we know that $u \sim q_1$ or $u \sim q_2$. By Corollary 12, we know that $q_1, q_2, \ldots, q_{d-2}$ are in layers $L_2, L_3, \ldots, L_{d-1}$, each vertex in a different layer. So, either $u, q_1, q_2, \ldots, q_{d-2}, v$ or $u, q_2, q_3, \ldots, q_{d-2}, v$ is a rainbow path.

5b. By Lemma 13, we know that $v \sim q_{d-1}$. By Corollary 12, we know that $q_1, q_2, \ldots, q_{d-2}$ are in layers $L_2, L_3, \ldots, L_{d-1}$, each vertex in a different layer. The path $u, q_2, q_3, \ldots, q_{d-1}, v$ is a rainbow path. ◀

Now there is still one case of vertices u and v for which we did not prove yet that there is a rainbow path. Namely:

5c. for $u \in L_2$, $u \nsim p_2$, $v \in L_d$, and $v \nsim q_{d-2}$, $u \nsim q_2$.

For this last case we can show that either $X_{u,v}$ or $Y_{u,v}$ is a rainbow path.

▶ **Lemma 15** (♠). *If $u \sim p_1$, $u \nsim p_2$ and $u \nsim q_2$, then $u \prec p_2$.*

▶ **Lemma 16.** *For u and v satisfying case 5c, there is a rainbow path.*

Proof. We distinguish two cases, based on Lemma 2: either $X_{u,v}$ is a shortest u, v-path or $Y_{u,v}$ is a shortest u, v-path.

Suppose that $X_{u,v}$ is a shortest u, v-path. Notice that $X_{u,v}$ has at least one vertex in every layer $L_2, L_3, \ldots, L_d$. Since $X_{u,v}$ has length at most $d-1$, there is at most one layer which contains two vertices of $X_{u,v}$. It is clear that $x_1 = u$ is in layer L_2. We will show that x_2 is in layer L_2 as well. By definition of x_2, we have $t(x_2) > t(u)$ and $b(x_2) < b(u)$. Since $u \sim p_1$ and p_1 has the leftmost top end, we know that $t(u) > t(p_1)$ and $b(u) < b(p_1)$. We conclude that $t(x_2) > t(p_1)$ and $b(x_2) < b(p_1)$, thus $x_2 \sim p_1$. So we see that x_1 and x_2 are both in layer L_2, so all internal vertices of $X_{u,v}$ are in different layers. So $X_{u,v}$ is a rainbow path.

Suppose that $Y_{u,v}$ is a shortest u, v-path. Write $Y_{u,v} = u, y_2, y_3, \ldots, y_{\alpha-1}, v$. Then $\alpha = d$ or $\alpha = d - 1$; note that $\alpha \leq \text{diam}(G) + 1 = d$ and that $d - 1 \leq \alpha$ because $Y_{u,v}$ contains a vertex from every layer. We prove by induction that $y_i \in L_{i+1}$ for all $2 \leq i \leq \alpha - 1$.

Since y_2 and p_1 both intersect u, by the definition of y_2, it follows that $b(y_2) \geq b(p_1)$. See Figure 5. If $y_2 = p_1$, then $y_{u,v} = u, p_1, p_2, \ldots, p_{d-1}, v$. Notice that the length of this path is $d = \text{diam}(G) + 1$. This yields a contradiction with the fact that $Y_{u,v}$ is a shortest u, v-path. Hence, $y_2 \neq p_1$, and $b(y_2) > b(p_1)$. Since p_1 is the vertex with the leftmost top end, we see that $t(p_1) < t(y_2)$. Hence $y_2 \succ p_1$. Since y_2 does not intersect p_1, it follows that $y_2 \notin L_2$.

By the definition of y_2, we know that $t(y_2) < t(u)$. By Lemma 15, it holds that $t(u) < t(p_2)$, thus $t(y_2) < t(p_2)$. Moreover, $b(y_2) > b(p_1) > b(p_2)$ (by Equation (1)). Hence, y_2 intersects p_2. We conclude that $y_2 \in L_3$.

Suppose that for all $i < k$, for some $k > 2$, it holds that $y_i \succ p_{i-1}$ and $y_i \in L_{i+1}$. Now consider y_k. Suppose that k is even. By the induction hypothesis and Lemma 7, we know that $y_{k-1} \sim p_{k-1}$, since $y_{k-1} \in L_k$. By definition of y_k, it follows that $b(y_k) \geq b(p_{k-1})$. And by Equation (1), we know that $b(p_{k-1}) > b(p_k)$, thus $b(y_k) > b(p_k)$. Similarly, by the definition of p_k, we know that $t(p_k) \geq t(y_{k-1})$. And by Equation (3), we know that $t(y_k) < t(y_{k-1})$, hence $t(p_k) > t(y_k)$. We conclude that p_k intersects y_k. It follows that y_k is in layer $k - 1$, k or $k + 1$.

Notice that if $y_k \in L_{k-1}$, then the length of $Y_{u,v}$ is at least $d = \text{diam}(G) + 1$. This yields a contradiction with the fact that $Y_{u,v}$ is a shortest u, v-path. Thus $y_k \notin L_{k-1}$. Suppose that $y_k \in L_k$. Then y_k intersects p_{k-1}. We have seen that $b(y_k) \geq b(p_{k-1})$, thus $t(y_k) < t(p_{k-1})$. By Equation (2), we have $b(p_{k-1}) > b(p_{k-2})$, Thus $b(y_k) > b(p_{k-2})$. By Equation (2), we also have that $t(p_{k-1}) < t(p_{k-2})$, thus $t(y_k) < t(p_{k-2})$. It follows that $y_k \sim p_{k-2}$. This yields a contradiction with the assumption that $y_k \in L_k$. We conclude that $y_k \in L_{k+1}$.

The case for k odd is analogous. Since $y_i \in L_{i+1}$ for all internal vertices y_i of $Y_{u,v}$, we conclude that $Y_{u,v}$ is a rainbow path. ◀

▶ **Theorem 17** (=Theorem A). *For every n-vertex permutation graph G, it holds that $\mathbf{rvc}(G) = \text{diam}(G) - 1$. Moreover, we can compute an optimal rainbow vertex coloring in $O(n^2)$ time.*

Proof. By Lemma 6 we know that either $d = \text{diam}(G)$ or $d = \text{diam}(G) + 1$, and either $d' = \text{diam}(G)$ or $d' = \text{diam}(G) + 1$. If $d = \text{diam}(G)$ or if $d' = \text{diam}(G)$, we have seen a rainbow coloring of G with $\text{diam}(G) - 1$ colors in Lemma 8. If both d and d' equal $\text{diam}(G) + 1$, then we have seen a coloring of G with $\text{diam}(G) - 1$ colors. Lemmas 9, 14 and 16 show that this coloring is indeed a rainbow coloring. We conclude that $\mathbf{rvc}(G) = \text{diam}(G) - 1$.

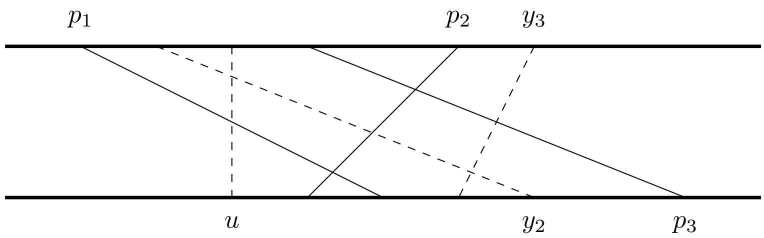

Figure 5 Vertex y_2 is in layer L_3. See Lemma 16.

Assume that we are given a permutation model of the graph and thus know the values $t(v)$ and $b(v)$ for each vertex $v \in V(G)$. Otherwise, a permutation model can be computed in linear time [14]. First, compute d and d'. Following the description of P and Q, this takes linear time. Computing the diameter of G takes $O(n^2)$ time using the algorithm of Mondal et al. [15]. The colorings given by Lemma 8 and before Lemma 9 can each be computed in linear time through a breadth-first search. By the preceding arguments, an optimal rainbow vertex coloring can be computed in $O(n^2)$ time. ◀

4 Split strongly chordal graphs

In this section, we show that RVC and SRVC are polynomial-time solvable on split strongly chordal graphs. We show this result is tight in the sense that both problems are NP-complete on split graphs if we forbid any finite family of suns.

In order to prove our next theorem we will use the following property of dually chordal graphs, a graph class that contains that of strongly chordal graphs [1].

▶ **Lemma 18.** (Brandstädt et al. [1]) *A graph G is dually chordal if and only if G has a spanning tree T such that all maximal cliques of G induce a subtree of T.*

We show a tree with a stronger property exists in split strongly chordal graphs.

▶ **Lemma 19 (♠).** *Let $G = (V, E)$ be a connected split strongly chordal graph, with $V = K \cup S$, where K is a clique and S is an independent set. Then G has a spanning tree T such that every maximal clique of G induces a subtree of T and every vertex of S is a leaf of T.*

▶ **Theorem 20** (=Theorem B)**.** *If G is a split strongly chordal graph with ℓ cut vertices, then* $\mathbf{rvc}(G) = \mathbf{srvc}(G) = \max\{\mathrm{diam}(G) - 1, \ell\}$.

Proof. Let $G = (V, E)$ be a split strongly chordal graph, with $V = K \cup S$, where K is a clique and S is an independent set. Note that if $\mathrm{diam}(G) \leq 2$, we can (strong) rainbow color G by assigning the same color to all the vertices. Notice that in this case $\ell \leq 1$, thus $\mathbf{rvc}(G) = \mathbf{srvc}(G) = \max\{\mathrm{diam}(G) - 1, \ell\}$.

Assume then that $\mathrm{diam}(G) = 3$ (recall that if G is a split graph, then $\mathrm{diam}(G) \leq 3$). By Lemma 19, G has a spanning tree T such that every maximal clique of G induces a subtree of T and every vertex of S is a leaf of T. Let $\mathcal{T}$ denote the subtree of T induced by the vertices of K, that is, the subtree of T obtained by the deletion of the leaves corresponding to vertices of S. Note that $\mathcal{T}$ is a tree with $V_{\mathcal{T}} = K$. We will now use the tree $\mathcal{T}$ to provide a (strong) rainbow coloring of G.

▷ Claim 1 (♠). For every $x \in S$, $N(x)$ induces a subtree of $\mathcal{T}$.

▷ Claim 2. If G is 2-connected, then $\mathbf{srvc}(G) = \mathbf{rvc}(G) = \mathrm{diam}(G) - 1$.

Proof. Note that $\mathrm{diam}(G) - 1 = 2$. Color the vertices of K according to a proper 2-coloring of the vertices of $\mathcal{T}$, and give arbitrary colors to the vertices of S. Let ϕ be the coloring of G obtained in this way. Note that ϕ is indeed a (strong) rainbow coloring of G. To see this, let $u, v \in V$ be such that $d_G(u, v) = 3$. Since G is a split graph, we have that $u, v \in S$. Since G is 2-connected, $|N(u)| \geq 2$ and $|N(v)| \geq 2$. Moreover, since $N(u)$ and $N(v)$ induce subtrees of $\mathcal{T}$, we know that these two sets are not monochromatic under ϕ. Thus, there are $x \in N(u)$ and $y \in N(v)$ s.t. $\phi(x) \neq \phi(y)$, which shows $uxyv$ is a rainbow (shortest) path between u and v. ◁

We now consider the case in which G has cut vertices. Let $C \subset V$ be the set of cut vertices of G. Consider a proper 2-coloring ϕ of $\mathcal{T}$. If there exist $c_1, c_2 \in C$ such that $\phi(c_1) \neq \phi(c_2)$, then we can obtain a (strong) rainbow coloring for G with ℓ colors by assigning distinct colors in the set $\{3, \ldots, \ell\}$ to the remaining cut vertices of G. Note that with this coloring of $\mathcal{T}$, it holds that for every $w \in S$, if $|N(w)| > 1$, then $N(w)$ is not monochromatic under ϕ. Since all the cut vertices were assigned distinct colors, by the same argument used in the 2-connected case, this is indeed a (strong) rainbow coloring of G. Note that this reasoning also applies if $|C| = 1$, so from now on we may assume $|C| \geq 2$.

If all the vertices of C were assigned the same color, since ϕ was a proper 2-coloring of $\mathcal{T}$, we have that for every $x, y \in C$, $d_{\mathcal{T}}(x, y) \geq 2$. Let $c_1, c_2 \in C$ be two cut vertices such that the unique path connecting c_1 and c_2 in $\mathcal{T}$ contains no other vertex of C. Let z be the vertex adjacent to c_1 in this path. Note that $z \notin C$. We will consider the following coloring ϕ' of $\mathcal{T}$. Let $\phi'(c_1) = \phi'(z) = 1$. Now we extend ϕ' by considering a proper 2-coloring of the subtree of $\mathcal{T}$ rooted in c_1 (resp. z) that assigns color 1 to the vertex c_1 (resp. z). Note that now we have $\phi'(c_2) = 2$. Finally, assign distinct colors from $\{3, \ldots, \ell\}$ to the vertices of $C \setminus \{c_1, c_2\}$. To obtain a (strong) rainbow coloring of G, we color the vertices of K according to ϕ' and give arbitrary colors to the vertices of S.

▷ Claim 3 (♠). ϕ' is a (strong) rainbow coloring of G.

Since ϕ' uses ℓ colors, this concludes the proof. ◀

We now show that both RVC and SRVC are NP-complete if we only forbid a finite number of suns. In what follows, we make use of the same reduction of Heggernes et al. [7] for split graphs. Their reduction is from Hypergraph Coloring. In our case, we start with an instance of Graph Coloring restricted to $(C_3, \ldots, C_p)$-free graphs, a problem that was shown to be NP-complete by Král' et al [8] (see also [6]) for every fixed $k \geq 3$. We can see an input $G = (V, E)$ of Graph Coloring as a hypergraph in which every hyperedge has size two. We perform the same construction as Heggernes et al. [7], starting with an $(C_3, \ldots, C_p)$-free instance of Graph Coloring.

▶ **Theorem 21 (♠).** *For any fixed $p \geq 3$,* RVC *and* SRVC *are NP-complete on split $(S_3, \ldots, S_p)$-free graphs for any fixed $p \geq 3$.*

5 Powers of trees

In this section we study powers of trees. Let T be a tree, and z in the center of T. Let $e = zv$ be an edge that is incident to z, with v not in the center. When e is removed from the tree, the tree will fall apart in two parts, a *branch* is the part that does not contain z. If the center of T contains only one vertex, the number of branches equals the degree of z. We distinguish between squares and higher powers of trees. We first consider squares of trees.

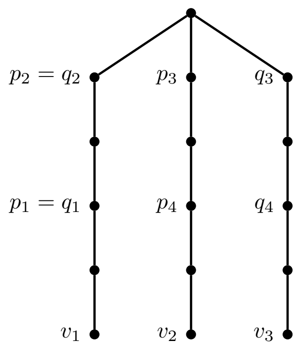

Figure 6 A graph T for which T^2 needs $\mathrm{diam}(T^2)$ colours, see Lemma 22.

Two trivial lower bounds for the rainbow coloring number of a graph G are the number of cut vertices in G and $\mathrm{diam}(G) - 1$. In squares of trees we found graphs that need more than $\mathrm{diam}(T^2) - 1$ colors. Notice that squares of trees are 2-connected, so there are no cut vertices.

▶ **Lemma 22.** *Let T be a tree such that the center of T consist of a single vertex z, T has diameter at least 6, and there are at least three branches from the center with maximum length. Then $\mathbf{srvc}(T^2) \geq \mathbf{rvc}(T^2) \geq \mathrm{diam}(T^2)$.*

Proof. Let v_1, v_2, and v_3 be three vertices with maximum distance to z in three different branches. We consider the case that $\mathrm{diam}(T^2)$ is odd. There is a unique shortest path $P = v_1, p_1, p_2, \ldots, p_k, v_2$ from v_1 to v_2 in T^2. Analogously, there is a unique shortest path $Q = v_1, q_1, q_2, \ldots, q_k, v_3$ from v_1 to v_3 in T^2. Notice that $q_1 = p_1$, $q_2 = p_2$, …, $q_j = p_j$, where $j = \lfloor \frac{\mathrm{diam}(T^2)}{2} \rfloor$. That is, P and Q use the same vertices in the branch of v_1. The unique shortest path R in T^2 from v_2 to v_3 is $v_2, p_k, \ldots, p_{j+1}, q_{j+1}, \ldots, q_k, v_3$. See Figure 6.

We give a proof by contradiction. Let c be a rainbow vertex coloring that uses at most $\mathrm{diam}(T^2) - 1$ colors. Notice that the paths P, Q, and R have length $\mathrm{diam}(T^2)$. Therefore, for each of these paths, all internal vertices are assigned different colors and all colors appear in the path. Since the first j vertices of the paths P and Q are equal, we see that the colors used for $p_{j+1}, \ldots, p_k$ are the same as the colors used for $q_{j+1}, \ldots, q_k$. Since $\mathrm{diam}(T) \geq 6$, $\{p_{j+1}, \ldots, p_k\}$ and $\{q_{j+1}, \ldots, q_k\}$ are non-empty. Hence, there is a color that appears twice in R, which yields a contradiction. We conclude that $\mathbf{rvc}(T^2) \geq \mathrm{diam}(T^2)$.

The case that $\mathrm{diam}(T^2)$ is even is analogous. ◀

The class of graphs described in the statement of Lemma 22 needs exactly $\mathrm{diam}(T^2)$ colors. We define layer i as the set of all vertices with distance $\lfloor \mathrm{diam}(T)/2 \rfloor - i$ to the center of T. For a vertex v, we write $l(v)$ for the layer that it is contained in, so $l(v) = \lfloor \mathrm{diam}(T)/2 \rfloor - d$, where d is the distance of v to the center of T. Our upper bounds all use a coloring by layer.

▶ **Lemma 23 (♠).** *Let T be a tree such that the center of T consist of a single vertex, T has diameter at least 6, and there are at least three branches from the center with maximum length. Then $\mathbf{rvc}(T^2) = \mathrm{diam}(T^2)$.*

In squares of trees this is the only example that needs more than $\mathrm{diam}(T^2) - 1$ colors. If tree T has $\mathrm{diam}(T) \leq 4$, then $\mathrm{diam}(T^2) \leq 2$ and thus $\mathbf{rvc}(T^2) = 1$ by Observation 1. We distinguish two cases for the remaining trees.

▶ **Lemma 24.** *Let T be a tree such that the center of T consist of a single vertex, T has diameter at least* 6*, and there are exactly two branches from the center with maximum length. Then* $\mathbf{rvc}(T^2) = \text{diam}(T^2) - 1$.

Proof sketch (♠). Let B_1 be one of the branches with maximum length. Let B_2 be all other branches, together with the center vertex. Suppose that $\text{diam}(T^2)$ is odd. Consider the following coloring c. We color B_2 per layer, using the color of layer 1 for the center as well. And we color B_1 similar, but with the colors of layer 1 and 2 swapped, and the colors of layers 3 and 4 swapped, etc. So, the colors used in the even layers of B_1 are exactly the colors of the odd layers of B_2 and vice versa. The number of colors used in this coloring equals $\text{diam}(T^2) - 1$.

Let u and v be two vertices of T. Suppose that u and v are both in B_i, for $i = 1, 2$, and assume that $l(u) \leq l(v)$. Use the even layer to go from u to the lowest common ancestor w and the odd layers to go from w to v. This is a rainbow path since every layer has a unique color, except for the center vertex. And if the center vertex is contained in this path, no vertex of layer 1 is an internal vertex.

If $u \in B_i$ and $v \in B_j$, with $i \neq j$, consider the following path. Use the even layers to go from u to the center and even layers to go from the center to v, but exclude the center itself from this path.

Suppose that $\text{diam}(T^2)$ is even. We slightly modify the coloring c: we color B_2 per layer, and use the color of layer 1 for the center as well. And we color B_1 similar, but with the colors of layer 2 and 3 swapped, and the colors of layers 4 and 5 swapped, etc. The paths constructed above are rainbow paths in this coloring as well. ◀

▶ **Lemma 25 (♠).** *Let T be a tree such that the center of T consist of two vertices and T has diameter at least* 5*. Then* $\mathbf{rvc}(T^2) = \text{diam}(T^2) - 1$.

We further consider higher powers of trees and generalize the above results for T^k for $k \geq 3$. Even though the corresponding statements are similar, we have to distinguish more cases in order to prove them. The proofs are deferred to the appendix. We then obtain the following theorem on all powers of trees.

▶ **Theorem 26** (=Theorem C, **♠**)**.** *If G is a power of a tree, then* $\mathbf{rvc}(G) \in \{\text{diam}(G) - 1, \text{diam}(G)\}$*, and the corresponding optimal rainbow vertex coloring can be found in time that is linear in the size of G.*

6 Conclusion and open problems

We provided polynomial-time algorithms to rainbow vertex color permutation graphs, powers of trees, and split strongly chordal graphs. The algorithm provided for the latter class also works for the strong variant of the problem.

An interesting question to be answered towards solving Conjecture 1 is whether RVC can be solved in polynomial time on AT-free graphs, i.e. graphs that do not contain an asteroidal triple. Conjecture 1 has been proved true for interval graphs [7] and, in this work, for permutation graphs, both of which are important subclasses of AT-free graphs.

Another direction of research within graph classes lies in determining the complexity of RVC and SRVC on strongly chordal graphs. Note that both powers of trees and split strongly chordal graphs form subclasses of strongly chordal graphs for which RVC is polynomial-time solvable, as we show in this work. Finally, note that every strongly chordal graph is also a chordal graph, and the problems are known to be NP-hard when restricted to chordal graphs.

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