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Admissibility conditions for Riemann data in shallow water theory

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Abstract: Consideration is given to the shallow-water equations, a hyperbolic system modeling the propagation of long waves at the surface of an incompressible inviscible fluid of constant depth. It is well known that the solution of the Riemann problem associated to this system may feature dry states for some configurations of the Riemann data. This article will discuss various scenarios in which the Riemann problem for the shallow water system arises in a physically reasonable sense. In particular, it will be shown that if certain physical assumptions on the disposition of the Riemann data are made, then dry states can be avoided in the solution of the Riemann problem.

Keywords: bores; cavitation; Hugoniot locus; Riemann problem; shallow-water equations.

1 Introduction

Many physical systems can be described by systems of conservation laws. Such systems are first-order quasilinear partial differential equations, and for one-dimensional problems they can be written in the general form

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0 \text{ for } (x, t) \in \mathbb{R} \times \mathbb{R}^+, \tag{1}$$

where $\mathbf{u}$ is a vector of unknowns, x is the one-dimensional spatial coordinate, and t is the time. The flux function $\mathbf{f}$ is a nonlinear vector function often satisfying certain mild assumptions, such as that the function be twice continuously differentiable, the flux Jacobian $\nabla\mathbf{f}$ have a full set of distinct eigenvalues and the wave families be either genuinely nonlinear or linearly degenerate [12, 17, 24].

Due to the special nonlinear structure of such systems, solutions naturally develop discontinuities in time, even if the original state of the system is given by a smooth function of x. Once a discontinuity has developed, solutions of the system need to be interpreted in a weak sense. If the solution is to include a jump between a left state $\mathbf{u}_L$ and a right state $\mathbf{u}_R$, the weak formulation leads to the well-known Rankine-Hugoniot condition

$$\sigma(\mathbf{u}_R - \mathbf{u}_L) = \mathbf{f}(\mathbf{u}_R) - \mathbf{f}(\mathbf{u}_L). \tag{2}$$

The Riemann problem distills the essence of the problem of singularity formation into a simple initial-value problem where the initial data have a prescribed discontinuity. For the system above, the Riemann problem would prescribe initial data of the form

$$\mathbf{u}(x, 0) = \begin{cases} \mathbf{u}_L & \textit{for } x < 0, \\ \mathbf{u}_R & \textit{for } x > 0, \end{cases} \tag{3}$$

where $\mathbf{u}_L$ and $\mathbf{u}_R$ are given left and right states. Under the conditions on $\mathbf{f}$ mentioned above, the Riemann problem can always be solved as long as the left and right states are close enough (see [12, 17, 19, 20]). However, if the left and right state are not close, then there is no general theory guaranteeing the existence of a solution to the Riemann problem (see [8]). Indeed, it can be shown explicitly, that there is no solution using the standard theory in some cases because the solution becomes unbounded [27]. On the other hand, for a large number of systems, solutions of the Riemann problem can be shown to exist by elementary methods.

In the present work, our focus is on the well-known shallow-water system which appears in the general form (1) when defining the principal unknown vector $\mathbf{u}$ and the flux function $\mathbf{f}$, respectively by

$$\mathbf{u} = \begin{bmatrix} h \\ hu \end{bmatrix}, \qquad \mathbf{f}(\mathbf{u}) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}.$$

In physical terms, the unknown $h(x, t)$ represents the local flow depth at a point x in space and at a time t. The unknown $u(x, t)$ represents the horizontal fluid velocity at x and t, averaged over the fluid column.

The solution of the Riemann problem for the shallow-water equations is well known, and can be found in many texts on conservation laws (cf. [2, 12]). One way to normalize the problem is to consider the left state $\mathbf{u}_L$ given, and look at

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all possible right states. Since h represents the total flow depth of the fluid, an additional admissibility condition is usually imposed, requiring both $\mathbf{u}_L$ and $\mathbf{u}_R$ to feature non-negative flow depth. Indeed, if imposing the requirement that $h_L \geq 0$ and $h_R \geq 0$, then it can be shown that the Riemann problem can be solved for all right states $\mathbf{u}_R$ satisfying this admissibility condition.

As can be gleaned from Figure 1, the condition that h be non-negative restricts the analysis to the right half-plane in the (h, u) phase space. However, if one looks closely at the solution of the Riemann problem, it appears that the solution features dry states for many possible right states (see Figure 1). In particular, in order to resolve the Riemann problem with a right state in the shaded region in Figure 1, one needs to incorporate a dry region $(h = 0)$ into the solution. Even though the solution is well defined mathematically, from a physical point of view, it does not seem reasonable for a dry region to develop from initial conditions which otherwise seem perfectly normal (just as it does not seem reasonable to include states with $h < 0$). Note that one obvious admissibility condition would be to simply specify that any right state in the shaded region of Figure 1 should be disregarded. To set such a criterion we would set the threshold by $u_R \leq u^\star + 2\sqrt{gh}$, representing the boundary of the shaded region with $u^\star$ given by the $\mathcal{R}_1$ curve evaluated at $h = 0$ (see general solution in Section 4 for context). However, there does not appear to exist any physical motivation for this condition. Therefore, in the present work, we aim to identify conditions on the right state $\mathbf{u}_R$ which are rooted in the physical interpretation of the equations, and also guarantee that the solution of the Riemann problem does not include a dry state.

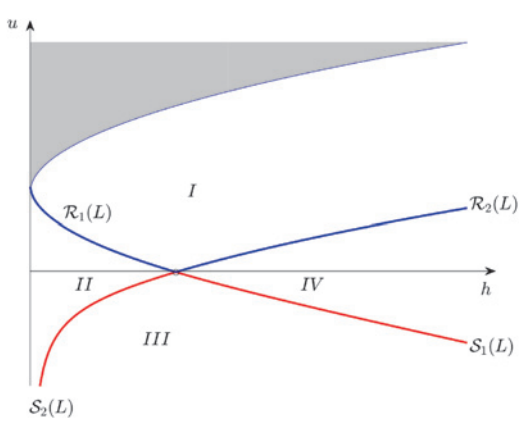

Figure 1: Phase space for a particular left state (h_L, u_L). The red curves denoted by $\mathcal{S}_1$ and $\mathcal{S}_2$ indicate possible right states which can be reached through a single discontinuity. The blue curves denoted by $\mathcal{R}_1$ and $\mathcal{R}_2$ show right states which can be reached through a continuous solution.

Recall that the shallow-water assumption applies to surface waves that are slowly varying, and a shock represents a bore (i. e., a traveling hydraulic jump) where the shock structure which may feature oscillations or turbulent structures) has been neglected [7, 22, 28]. If this physical interpretation is taken as a point of departure, then it appears that a Riemann problem may develop from the collision of two bores, and a natural admissibility condition would be to consider only such Riemann problems. Thus we will study the history of the Riemann problem, or more succinctly the backwards problem for $t < 0$. Examining possible solutions of the backwards problem will lead to clear conditions on whether or not a Riemann problem can develop. As it turns out, these conditions will exclude Riemann problems whose solution involves a dry state.

Note that a Riemann problem could also develop from certain initial data which are arranged in such a way that the solution lines up at some time so as to give a perfect Riemann problem. While this is possible, it would clearly be unstable to even the smallest perturbations. On the other hand, one might also consider the collision of three or more traveling hydraulic jumps, or the collision between rarefaction waves and shocks, but these situations are so unlikely to happen that they would constitute a set of measure zero in the configuration space. In the current work, we focus on the origin of the Riemann problem which can be represented by a set of non-zero measure in the configuration space given by the phase plane.

The outline of this paper is as follows. In sections 2 and 3, a short discussion of the properties of basic admissible waves for the shallow-water system is given, and the standard solution of the Riemann problem is explained in section 4. This is standard fare, but we need various formulas in order to set up the problem to be attacked. In sections 5, 6, and 7, Riemann problems originating from various configurations are investigated. Some ramifications of our results are discussed in the Conclusion.

2 Shock waves and bore properties

As already mentioned above, the shallow-water system can be written in terms of mass and momentum conservation in the form

$$h_t + (hu)_x = 0, \tag{4}$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0. \tag{5}$$

A derivation of this system from first principles can be found in [25], where it is also shown that the conservation of energy is formulated as

$$\left(\frac{1}{2}hu^2+\frac{1}{2}gh^2\right)_t+\left(\frac{1}{2}hu^3+gh^2u\right)_x=0. \tag{6}$$

Discontinuous solutions develop naturally in this system even in the case of flat bathymetry which is under study here. In the case when the solution features jumps, the imposition of mass and momentum conservation leads to an energy loss (see [23]) which has been the subject of a number of studies [3–6, 15, 26]. In the context of the conservation laws, the energy loss means that (6) becomes an inequality, which is then taken as the mathematical entropy in order to pick out physically reasonable discontinuous solutions.

In the context of the shallow-water equations (4) and (5) the Rankine-Hugoniot condition (2) yields the following relations:

$$(h_R-h_L)\sigma=h_Ru_R-h_Lu_L,$$

$$(h_Ru_R-h_Lu_L)\sigma=\left(h_Ru_R^2+\frac{1}{2}gh_R^2\right)-\left(h_Lu_L^2+\frac{1}{2}gh_L^2\right).$$

Combining these two equations enables us to find an expression for u_R in terms of h, h_L and u_L as shown in [2, 12]. Indeed, one may define the Hugoniot locus of all possible right states (h, u) for a given left state (h_L, u_L) in terms of the shock curves $\mathcal{S}_1$ and $\mathcal{S}_2$ as follows.

$$\mathcal{S}_1(L): u(h)=u_L-(h-h_L)\sqrt{\frac{g}{2}\left(\frac{1}{h}+\frac{1}{h_L}\right)}, \tag{7}$$

$$\mathcal{S}_2(L): u(h)=u_L+(h-h_L)\sqrt{\frac{g}{2}\left(\frac{1}{h}+\frac{1}{h_L}\right)}. \tag{8}$$

A useful observation to be used later is that the fluid velocity of u on $\mathcal{S}_1$ is strictly decreasing, while the velocity on $\mathcal{S}_2$ is strictly decreasing. In fact taking the derivative yields the expression

$$u'(h)=\mp\frac{\sqrt{\frac{g}{2}}\left(2h^2+hh_L+h_L^2\right)}{2h^2h_L\sqrt{\frac{1}{h}+\frac{1}{h_L}}},$$

where the minus sign refers to the $\mathcal{S}_1$ curve and the plus sign to the $\mathcal{S}_2$ curve. Inspecting the term on the right in the above relation confirms that the sign of the derivative $u'(h)$ depends only on whether the derivative is taken on $\mathcal{S}_1$ or on $\mathcal{S}_2$.

The Hugoniot loci may also be described in terms of the momentum $q=hu$. Indeed, for a given left (h_L, q_L), the possible right states must satisfy one of the following relations

$$\mathcal{S}_1(L): q(h)=\frac{q_L}{h_L}h-h(h-h_L)\sqrt{\frac{g}{2}\left(\frac{1}{h}+\frac{1}{h_L}\right)}, \tag{9}$$

$$\mathcal{S}_2(L): q(h)=\frac{q_L}{h_L}h+h(h-h_L)\sqrt{\frac{g}{2}\left(\frac{1}{h}+\frac{1}{h_L}\right)}. \tag{10}$$

Taking the second derivative of these expressions shows that these curves are strictly concave and convex, respectively:

$$q''(h)=\mp\frac{\sqrt{\frac{g}{2}}\left(8h^3+12h^3h_L+3hh_L^2+h_L^3\right)}{4h^3h_L^2\left(\frac{1}{h}+\frac{1}{h_L}\right)^{\frac{3}{2}}}.$$

Finally, the speed of the discontinuity may be found from the Rankine-Hugoniot condition as

$$\sigma=u_L\mp h_R\sqrt{\frac{g}{2}\left(\frac{1}{h_R}+\frac{1}{h_L}\right)}=u_R\pm h_L\sqrt{\frac{g}{2}\left(\frac{1}{h_R}+\frac{1}{h_L}\right)}. \tag{11}$$

Next, let us discuss the entropy condition for shock waves. It is well known [12, 24] that it is necessary to impose both the Rankine-Hugoniot and the entropy condition to ensure uniqueness of a solution. In the context of the shallow-water theory, the mechanical energy serves as a mathematical entropy. In fact, it is well known that energy is lost in a shock either due to turbulence or the continuous creation of surface oscillations [3, 6, 11, 13, 14, 25]. Similar considerations can be used in various other applications, such as for example in the context of porous media [1].

In the present case, the expected loss of mechanical energy is enforced by imposing the inequality

$$\Delta E=\underbrace{\left(\frac{1}{2}hu^2+\frac{1}{2}gh^2\right)_t}_{\eta(\mathbf{u})_t}+\underbrace{\left(\frac{1}{2}hu^3+gh^2u\right)_x}_{\psi(\mathbf{u})_x}<0, \tag{12}$$

for discontinuous solutions. It is also convenient to introduce the relative mass flux m by

$$m=h_R(u_R-\sigma)=h_L(u_L-\sigma)=\pm h_Rh_L\sqrt{\frac{g}{2}\left(\frac{1}{h_R}+\frac{1}{h_L}\right)}. \tag{13}$$

Using m, we can express the rate at which energy is lost at the shock by

$$\Delta E=\psi(\mathbf{u}_R)-\psi(\mathbf{u}_L)-\sigma(\eta(\mathbf{u}_R)-\eta(\mathbf{u}_L)),$$

$$=-\frac{mg}{4}\frac{(h_R-h_L)^3}{h_Rh_L}.$$

Note that since we always require $\Delta E<0$ for discontinuous solutions, if $h_L<h_R$, then we must have $m>0$ from the previous relation. Invoking (13) then shows that $u_R>\sigma$ and $u_L>\sigma$. On the other hand, similar considerations show that if $h_L>h_R$, then (12) requires that $u_R<\sigma$ and $u_L<\sigma$.

These relations show that fluid particles always move across the shock from the region of lower depth to the region of higher depth, a fact already noted in [25]. Moreover, combining equations (7), (8) and (13) and using the condition (12) shows that we must have $u_R < u_L$ for all discontinuous solutions. The most important properties of the shock curves are summarized in Tables 1 and 2 and Figure 2.

One should also remark that both $\mathcal{S}_1$ and $\mathcal{S}_2$ shocks satisfy the Lax entropy condition (cf. [12]). This condition states that the speed σ_i of an $\mathcal{S}_i$ shock must satisfy the relation

$$\lambda_i(R) \le \sigma_i \le \lambda_i(L), \quad i = 1, 2, \tag{14}$$

where λ_i are the eigenvalues of the flux Jacobian matrix $\nabla\mathbf{f}$. For the shallow water equations this eigenvalues are given by

$$\lambda_1 = u - \sqrt{gh}, \quad \lambda_2 = u + \sqrt{gh}. \tag{15}$$

A geometrical representation of the Lax entropy condition in the (x, t)–plane is shown in Figures 3 and 4.

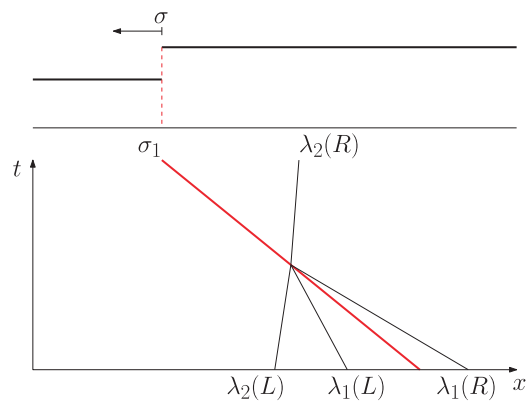

Figure 3: Left moving bore with speed σ_1, $\lambda_i(L)$ and $\lambda_i(R)$ for a $\mathcal{S}_1$ shock.

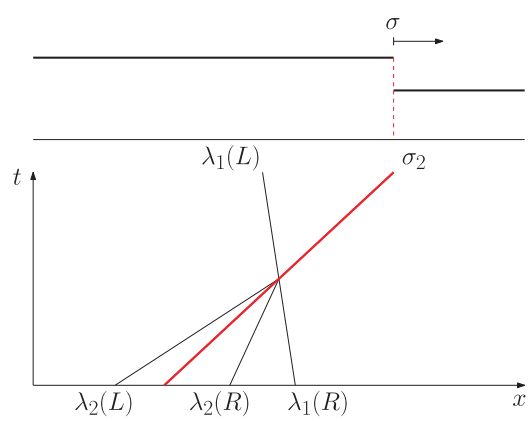

Figure 4: Right moving bore with speed σ_2, $\lambda_i(L)$ and $\lambda_i(R)$ for a $\mathcal{S}_2$ shock.

3 Rarefaction waves

Following the classical theory (presented for example in [10, 28, 12]) we seek traveling wave solutions of the form $\mathbf{u}(x, t) = \mathbf{v}(\xi)$ with $\xi = \frac{x}{t}$. Substituting this term the into the conservation law one can easily verify that the system reduces to a system of ODEs of the form

$$\dot{\mathbf{v}} = \mathbf{r}(\mathbf{v}). \tag{16}$$

The solution is then given by the integral of (16). We may now exploit this insight using the theory of Riemann invariants $w : \mathbb{R}^2 \to \mathbb{R}$, which is a smooth function that is constant along the integral curves [10]. For the shallow-water equations the Riemann invariants are given by

Table 1: Properties of shock curves $\mathcal{S}_1$ and $\mathcal{S}_2$.

Hugoniot locus	Fluid velocity $u(h)$	Momentum $q(h)$
$\mathcal{S}_1$	$u'(h) < 0$ $u''(h) > 0$	$q''(h) < 0$
$\mathcal{S}_2$	$u'(h) > 0$ $u''(h) < 0$	$q''(h) > 0$

Table 2: Jump properties on $\mathcal{S}_1$ and $\mathcal{S}_2$.

Hugoniot locus	$\mathcal{S}_1$	$\mathcal{S}_2$
Increase/decrease in flow depth	$h_L < h_R$	$h_L > h_R$
Front speed	$\sigma < 0$ or $\sigma > 0$	$\sigma < 0$ or $\sigma > 0$
Relative mass flux	$m > 0$	$m < 0$
Velocity relation	$u_R < u_L$	$u_R < u_L$
Velocity relation	$u_R > \sigma$	$u_R < \sigma$
Velocity relation	$u_L > \sigma$	$u_L < \sigma$

Figure 2: Left panel: bore with flow depth $h_L < h_R$ which corresponds to the right state being on the shock curve $\mathcal{S}_1$. Right panel: bore with flow depth $h_R < h_L$ corresponding to the right state being on the shock curve $\mathcal{S}_2$. In both cases the bore front may feature positive (right-moving), zero, or negative (left-moving) propagation velocity.

$$w_1 = u - 2\sqrt{gh}, \qquad w_2 = u + 2\sqrt{gh}. \tag{17}$$

Along a Riemann invariant, the solution must therefore satisfy

$$u_L \pm 2\sqrt{gh_L} = u_R \pm 2\sqrt{gh_R}. \tag{18}$$

Hence, for a given left state we may write the rarefaction wave solution as follows

$$\mathcal{R}_1(L): \; u(h) = u_L - 2\sqrt{gh} + 2\sqrt{gh_L}, \tag{19}$$

$$\mathcal{R}_2(L): \; u(h) = u_L + 2\sqrt{gh} - 2\sqrt{gh_L}. \tag{20}$$

By comparison, one can also show that $\dot{\mathbf{v}}$ is the right eigenvector $\mathbf{r}(\mathbf{v})$, and ξ is the corresponding eigenvalue $\lambda(\mathbf{v})$ belonging to the Jacobi matrix of the flux function. Having $\lambda = \xi$ would mean that the eigenvalues must be increasing from left to right. This implies $\lambda_i(L) < \lambda_i(R)$ and by equation (15) that $u_L < u_R$ whenever there is a rarefaction wave. Figures 5 and 6 depicts two rarefaction waves propagating left and right in the (x, t)–plane. Following the characteristics one can see how the waves moves forward in time.

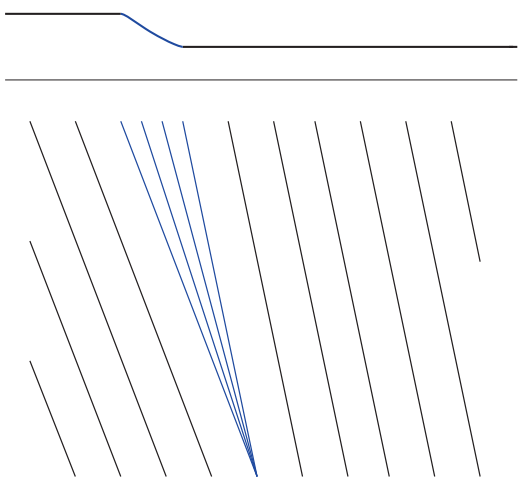

Figure 5: Left moving rarefaction wave smoothly varying between $\lambda_1(L)$ and $\lambda_1(R)$.

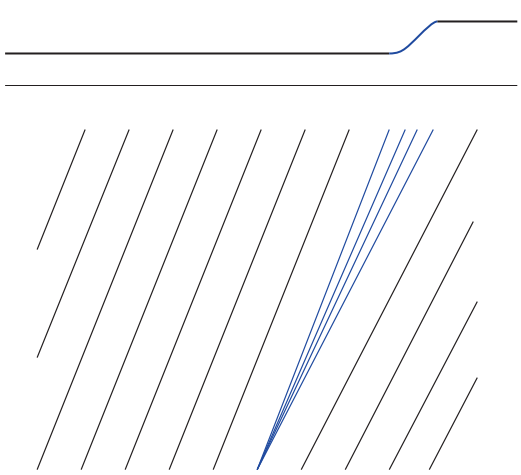

Figure 6: Right moving rarefaction wave providing a smooth transition between $\lambda_2(L)$ and $\lambda_2(R)$.

In fluid mechanics, some authors refer to these waves as negative surges resulting from a decrease in flow depth [9]. Interestingly, Peregrine was able to show that a negative surge together with a bore advancing in positive direction originates from the collision of two fast shocks [21]. Therefore, we will discuss the development of the Riemann problem from a collision of two $\mathcal{S}_2$ shocks in Section 6.

4 General solution of the Riemann problem

Using the results from sections 3 and 4, the general solution of the Riemann problem can be fond using the rarefaction curves defined by (19) and (20)

$$\mathcal{R}_1(L): u(h) = u_L - 2\sqrt{gh} + 2\sqrt{gh_L}, \; u > u_L$$
$$\mathcal{R}_2(L): u(h) = u_L + 2\sqrt{gh} - 2\sqrt{gh_L}, \; u > u_L$$

and the shock curves (7) and (8)

$$\mathcal{S}_1(L): u(h) = u_L - (h - h_L)\sqrt{\frac{g}{2}\left(\frac{1}{h} + \frac{1}{h_L}\right)}, \; u < u_L$$
$$\mathcal{S}_2(L): u(h) = u_L + (h - h_L)\sqrt{\frac{g}{2}\left(\frac{1}{h} + \frac{1}{h_L}\right)}, \; u < u_L$$

In the following, it will be convenient to plot the integral curves and shock curves for a particular left state (h_L, u_L) plotted in two different coordinate systems. Figure 7 shows the integral curves in (h, q)–coordinates, where $q = hu$ is the momentum, while Figure 8 shows the integral curves in (h, u)–coordinates. The benefit of the

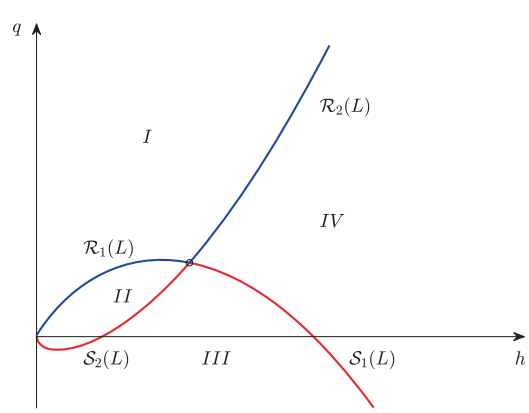

Figure 7: Phase-space in (h, q)–coordinates.

former representation lies in the fact that the shock speed between q_1 and q_2 is given by

$$\sigma = \frac{q_1 - q_2}{h_1 - h_2},$$

which is simply the secant line joining the two states. For example, any right state given on the $\mathcal{S}_2(L)$ curve in Figure 8 would give rise to a right- moving bore since the slope of the secant line joining left and right states would be positive, i. e., $\sigma_R > 0$. Of course, if the right state is not given on any of the integral or shock curves, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{R}_1$ and $\mathcal{R}_2$ need to be combined to give a solution of the Riemann problem. Indeed as explained in [12, 18], given a left state we may consider all possible right states and then find the solution depending on whether the right state is in region I, II, III or IV. For instance, let us say we have a right state somewhere in region four. Then in order to find an entropy solution we must define a middle state at some point (h_M, u_M) on the shock curve $\mathcal{S}_1(L)$ such that it is connected to a rarefaction curve $\mathcal{R}_2(M)$. Similarly, the entropy solution for each region is found by two elementary waves going through some middle state. For region I, we follow

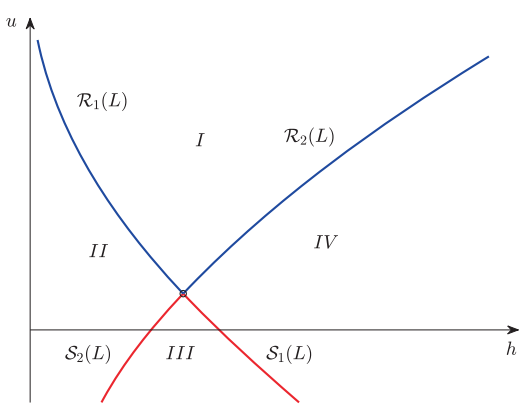

Figure 8: Phase-space in (h, u)–coordinates.

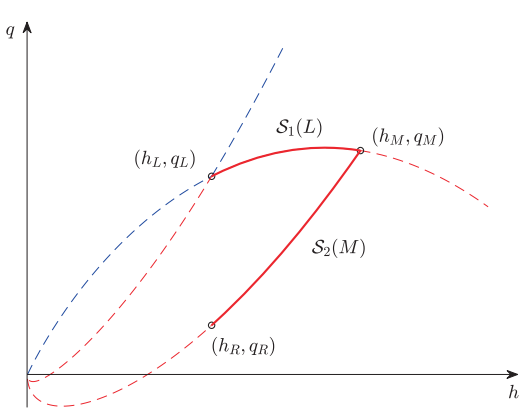

Figure 9: Solution in region *III* using (h, q)–coordinates.

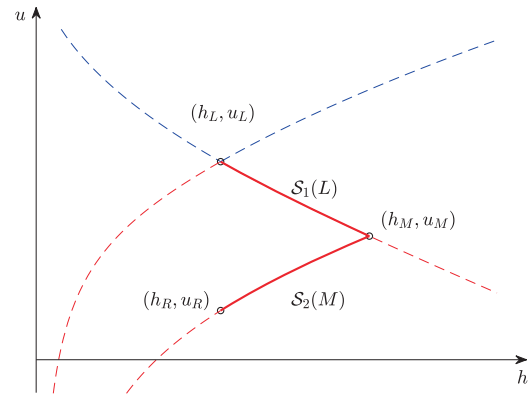

Figure 10: Solution in region *III* using (h, u)–coordinates.

$\mathcal{R}_1(L)$ connecting the right state with $\mathcal{R}_2(M)$ for a middle state. In region II, we first go along $\mathcal{R}_1(L)$ then $\mathcal{S}_2(M)$. Finally, in region III we connect $\mathcal{S}_1(L)$ with $\mathcal{S}_2(M)$ for some middle state (see example in Figures 9 and 10). Concluding this section we remark that the solution is in fact unique since all of these solutions satisfy the admissibility conditions, and it has been shown that there is only one middle state M for each region.

5 Development of the Riemann problem from a collision of $\mathcal{S}_2$ and $\mathcal{S}_1$ shocks

In this section, we will discuss the origin of the Riemann problem from a collision of two bores. It is most convenient to focus the discussion by assuming that a left state is given. With this proviso, we will prove that the Riemann problem associated to certain right states in region III arises from the head-on collision of two counter-propagating bores, while other right states are connected to an

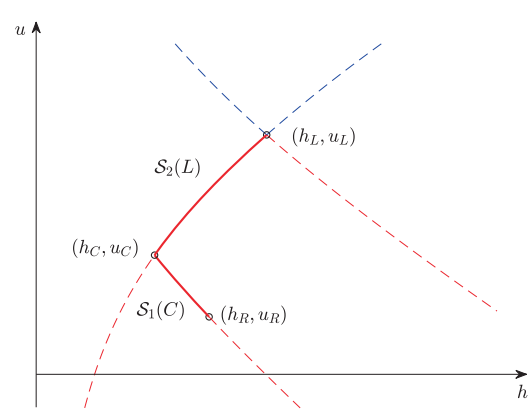

Figure 11: Backwards problem in (h, u)–coordinates.

overtaking collision of co-propagating bores (see Figure 11). Indeed, we will show that these two scenarios cover all possible right states in region III. Finally, we show that Riemann problems with right states in regions I, II and IV cannot develop from either head-on or overtaking collisions of an $\mathcal{S}_1$ and an $\mathcal{S}_2$ shock.

In order to understand how a given Riemann problem develops, we consider the backwards problem for $t < 0$. In order to solve the backwards problem, the usual disposition of a slow shock on the left left and a fast shock on the right has to be reversed. Indeed, to solve the backwards problem, the left state is connected to a center state lying on the locus $\mathcal{S}_2(L)$. The center state is then connected to the right state by ensuring that the right state lies on the locus $\mathcal{S}_1(C)$. This configuration then leads to the collision of the two shocks at time $t = 0$. Note that we have chosen to use the term *center state* for the backwards problem versus *middle state* for the forward problem.

As indicated in Figure 12, the solution of the Riemann problem for a right state in region III consists of a 1–shock and a 2–shock connected by a middle state on $\mathcal{S}_1(L)$. Note that the flow depth of the middle state will always be higher than for both the left and the right state. Specifically we always have $h_M > h_L$ and $h_M > h_L$ in region III. In fact, it can be observed that fluid particles from both sides will move back towards the middle, thus contributing to the raised flow depth of the middle state. In that respect, it seems natural that the Riemann problem should result

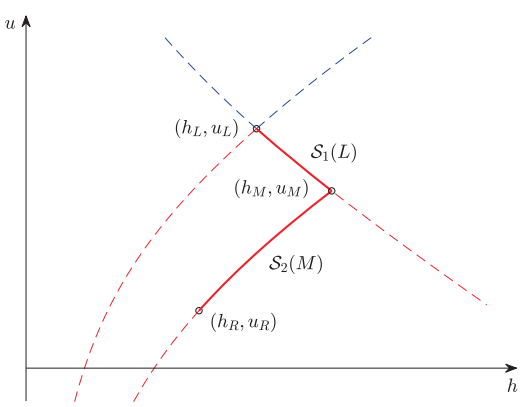

Figure 12: Forward problem in (h, u)–coordinates.

from two colliding bores. Figure 13 depicts the case of a head-on collision of a left-moving shock and a right-moving shock. Note that the backwards solution has the two shocks connected by a center state (h_C, u_C), then moving towards each other resulting in a Riemann problem at time $t = 0$.

Theorem 1. *Suppose that a left state $L = (h_L, u_L)$ for the Riemann problem is given. For any right state $R = (h_R, u_R)$ in region III, there exists a center state $C = (h_C, u_C)$ such that for $t < 0$, there is an $\mathcal{S}_2 - \mathcal{S}_1$ connection between L and R via C. The two shocks collide at $t = 0$, giving rise to a Riemann problem. On the other hand, it is not possible for a Riemann problem to develop from a $\mathcal{S}_2 - \mathcal{S}_1$ connection if the right state is in region I, II or IV.*

Proof. Step 1. *Existence of a center state*: We need to prove there is a center state connecting two colliding shock waves satisfying the bore conditions. Guided by the discussion above, and using an argument similar to one used in [16], we seek a point (h_C, u_C) on $\mathcal{S}_2(L)$ giving rise to a 1–shock, $\mathcal{S}_1(C)$ through (h_R, u_R). The equation defining the locus $\mathcal{S}_2(L)$ is given by

$$\mathcal{S}_2(L) : u = u_L + (h - h_L)\sqrt{\frac{g}{2}\left(\frac{1}{h} + \frac{1}{h_L}\right)}. \tag{21}$$

As already indicated in Table 1, taking the derivative $u'(h)$ shows that u is strictly increasing on $\mathcal{S}_2(L)$ for $h \in (0, h_L]$ and with range $(-\infty, u_L)$. On the other hand, any right state $(h_R, u_R) \in III$ in the locus $\mathcal{S}_1(C)$ will satisfy the relation

$$\mathcal{S}_1(C) : u_R = u_C - (h_R - h_C)\sqrt{\frac{g}{2}\left(\frac{1}{h_R} + \frac{1}{h_C}\right)}. \tag{22}$$

Keeping the right state fixed, and varying h_C shows that u_C is is strictly decreasing as a function of h_C with $h_C \in (0, h_L]$, and $u_C \in [u_R, \infty)$. Since $u_R < u_L$, the two loci defined above must necessarily intersect, thus defining the center state (h_C, u_C).

Step 2. *Head-on collision and overtaking bores:* We now analyze whether the center state found in Step 1 actually leads to a collision of shocks. As will be shown presently,

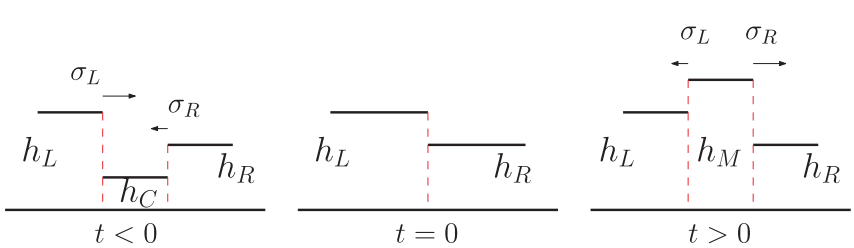

Figure 13: The Riemann problem at $t = 0$ originates from two counter-propagating shocks ($t < 0$). The solution is given for $t > 0$.

the center state will always give rise to a Riemann problem originating from either a head-on collision or an overtaking collision of two shocks. In order to prove this statement, we first note that having $(h_C, u_C) \in \mathcal{S}_2(L)$ implies that $h_C < h_L$ and $u_C < u_L$. From the bore properties in Section 2 we see that the left shock is described by

$$\sigma_L = \frac{h_L u_L - h_C u_C}{h_L - h_C} = u_C + h_L \sqrt{\frac{g}{2}\left(\frac{1}{h_C} + \frac{1}{h_L}\right)}, \tag{23}$$

when substituting u_L from equation (21). Keeping this in mind, we now consider σ_R. Since the right state (h_R, u_R) is in the locus $\mathcal{S}_1(C)$, we must have $h_C < h_R$ and we may now use an argument reminiscent of the derivation of the Lax entropy condition (see [12], for example). The idea is to show that $\sigma_L > \sigma_R$ by considering the difference of these two quantities, and then using the mean-value theorem. Since q is continuous on $[h_C, h_R]$ and differentiable on the open interval (h_C, h_R), it follows from the mean-value theorem that there exists $h_* \in (h_C, h_R)$ such that

$$\sigma_R = \frac{q_R - q_C}{h_R - h_C} = \left.\frac{dq}{dh}\right|_{h_*}.$$

In addition, differentiating $q(h)$ twice shows that the momentum q is a strictly concave function of h on the locus $\mathcal{S}_1(C)$ (see Table 1). Therefore, an upper bound on the derivative may be obtained by evaluating it at the leftmost point, h_C. Combining this observation with equation (23) yields the estimate

$$\begin{aligned}\sigma_L - \sigma_R &> \sigma_L - \left.\frac{dq}{dh}\right|_{hC} \\ &= h_L \sqrt{\frac{g}{2}\left(\frac{1}{h_C} + \frac{1}{h_L}\right)} + \sqrt{g h_C} \\ &> 0.\end{aligned}$$

Hence we conclude that $\sigma_L > \sigma_R$ whenever the right state is in region III. This relation ensures that the center state chosen above gives rise to a Riemann problem. If σ_L and σ_R have the same sign, then the Riemann problem develops from an overtaking shock collision. If σ_L and σ_R have opposite sign, the Riemann problem develops from a head-on collision.

Step 3. *Inadmissible connections:* Regarding the last statement of the theorem, we will now argue that for a right state in region I, II or IV there is no admissible connection. If we first consider region I, we must choose u_C such that $u_C < u_L$ in order to satisfy the entropy condition in Section 2. Furthermore, in region I we have $u_R > u_L$, which means that $u_C < u_R$. This violates the entropy condition as a result of the center state being to the left, relative to the right state. In fact, the entropy condition ensures that the only admissible connection using one 1–shock and one 2–shock is a center state satisfying $u_L > u_C > u_R$, and this can obviously only be true for a right state in region III.

Before proceeding, we will offer some clarifying remarks. For the shallow-water equation, given a left state we can always connect any right state with a middle state as mentioned earlier. Once you know the right state it is then possible go back through a center state. We find it instructive to describe the solution for two particular states in both phase-space and in (x, t)–coordinates. Figure 15

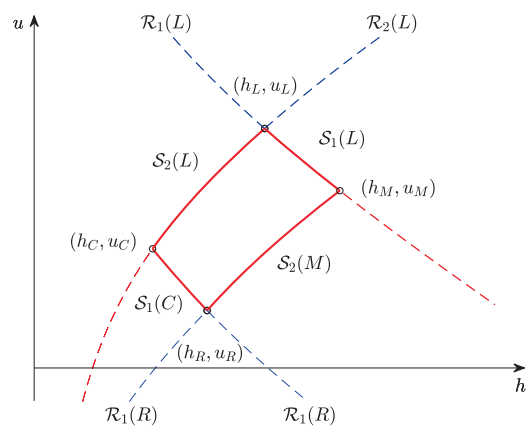

Figure 14: Development of the Riemann problem in phase space.

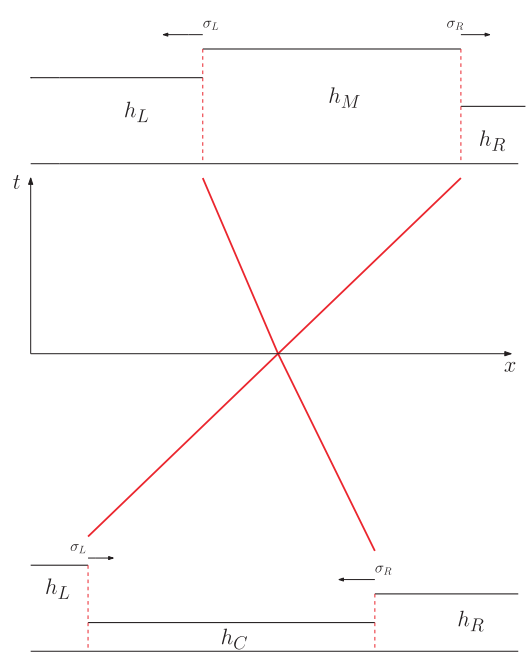

Figure 15: Development of the Riemann problem in (x, t)–coordinates.

depicts the special case from earlier in (x,t)–plane with two counter-propagating bores colliding at $t = 0$. Also observe that for $t < 0$, we need to consider the admissible connection from the perspective of the right state. Then of course, a $\mathcal{S}_1(C) - \mathcal{S}_2(L)$ connection is entropy-satisfying, and is shown in Figure 14. On the other hand, for the forward problem, we connect $\mathcal{S}_1(L) - \mathcal{S}_2(M)$ as discussed in Section 4.

6 Development of the Riemann problem from a collision of two $\mathcal{S}_2$ shocks

Consideration will now be given to the Riemann problem arising from a $\mathcal{S}_2 - \mathcal{S}_2$ connection. As it will turn out, the resulting Riemann problem will have a right state in region *II*. As before, we consider the left state given. It is then straightforward to see that the center state in the backwards problem must lie in the Rankine-Hugoniot locus $\mathcal{S}_2(L)$. Thus the center state is given by the formula

$$u_C = u_L + (h_C - h_L)\sqrt{\frac{g}{2}\left(\frac{1}{h_C} + \frac{1}{h_L}\right)}. \tag{24}$$

On the other hand, if the center state is to be connected to the right state by an $\mathcal{S}_2$ -shock, then the right state must lie on the $\mathcal{S}_2(C)$ shock curve and therefore satisfy the relation

$$u_R = u_C + (h_R - h_C)\sqrt{\frac{g}{2}\left(\frac{1}{h_R} + \frac{1}{h_C}\right)}. \tag{25}$$

Putting these two formulas together defines the region of all possible right states as

$$\begin{aligned}\Omega_2 = \bigcup_{h_C \in (0,h_L)} \Bigg\{ &(h_R, u_R) | u_R \\ &= u_L + (h_C - h_L)\sqrt{\frac{g}{2}\left(\frac{1}{h_C} + \frac{1}{h_L}\right)} \\ &\quad + (h_R - h_C)\sqrt{\frac{g}{2}\left(\frac{1}{h_R} + \frac{1}{h_C}\right)},\ 0 < h_R < h_C \Bigg\}.\end{aligned}$$

We have the following theorem.

Theorem 2. *Suppose that a left state $L = (h_L, u_L)$ for the Riemann problem is given. The set of all possible right states $R = (h_R, u_R)$ such that the Riemann problem originates from the collision of two $\mathcal{S}_2$ shocks is given by Ω_2. This set lies in region II, and the shock speeds of the backwards problem line up such that the two shocks meet at $t = 0$.*

On the other hand, it is not possible for a Riemann problem to develop from a $\mathcal{S}_2 - \mathcal{S}_2$ connection if the right state is in the complement of the set Ω_2.

Proof. First of all, the definition of the set Ω_2 is straightforward from the relations for the Hugoniot loci $\mathcal{S}_2(L)$ and $\mathcal{S}_2(C)$. Any state which does not lie in Ω_2 can therefore not be reached via a $\mathcal{S}_2 - \mathcal{S}_2$ connection.

To see that the state $R = (h_R, u_R)$ lies in region II, consider the difference between u_R given by (25) and (24) and u in the locus $\mathcal{S}_2(L)$ as defined by (8). Denoting this difference by $F(h) = u - u_R$, we obtain the formula

$$\begin{aligned}F(h) = u_L &+ (h - h_L)\sqrt{\frac{g}{2}\left(\frac{1}{h} + \frac{1}{h_L}\right)} - u_L \\ &- (h_C - h_L)\sqrt{\frac{g}{2}\left(\frac{1}{h_C} + \frac{1}{h_L}\right)} - (h - h_C)\sqrt{\frac{g}{2}\left(\frac{1}{h} + \frac{1}{h_C}\right)}.\end{aligned}$$

It needs to be shown that $F(h) < 0$ for $h < h_C$. Note first the $F(h_C) = 0$. If it can be shown that $F'(h) > 0$ for $h < h_C$ then we can conclude that $F(h)$ is strictly monotone increasing, and can therefore only cross the abscissa one time, so that $F(h)$ will have to be negative in the interval in question.

Evaluating the first and second derivative of $F(h)$ yields

$$\begin{aligned}F'(h) = &\sqrt{\frac{g}{2}\left(\frac{1}{h} + \frac{1}{h_L}\right)} - \frac{\sqrt{\frac{g}{2}}(h - h_L)}{2h^2\sqrt{\frac{1}{h} + \frac{1}{h_L}}} - \sqrt{\frac{g}{2}\left(\frac{1}{h} + \frac{1}{h_C}\right)} \\ &+ \frac{\sqrt{\frac{g}{2}}(h - h_C)}{2h^2\sqrt{\frac{1}{h} + \frac{1}{h_C}}}\end{aligned}$$

and

$$F''(h) = -\frac{\sqrt{\frac{g}{2}}(5h_L + 3h)}{4h^4\left(\frac{1}{h} + \frac{1}{h_L}\right)^{3/2}} + \frac{\sqrt{\frac{g}{2}}(5h_C + 3h)}{4h^4\left(\frac{1}{h} + \frac{1}{h_C}\right)^{3/2}}.$$

By inspection, we see that $F''(h) < 0$ so that the derivative is strictly monotone decreasing. Therefore, we have $F'(h) > F'(h_C)$ for all $h < h_C$, and if it can be shown that $F'(h_C) > 0$, then we are done.

Lemma 1. *Given $F(h) = u - u_R$, we have $F'(h_C) > 0$.*

Proof. Evaluating the derivative $F'(h)$ given above at $h = h_C$ and multiplying with $\sqrt{\frac{2}{g}}$ for the sake of clarity yields

$$\sqrt{\frac{2}{g}}F'(h_C) = \sqrt{\frac{1}{h_C}+\frac{1}{h_L}} - \frac{(h_C-h_L)}{2h_C^2\sqrt{\frac{1}{h_C}+\frac{1}{h_L}}} - \sqrt{\frac{1}{h_C}+\frac{1}{h_C}}$$

$$= \sqrt{\frac{1}{h_C}+\frac{1}{h_L}} - \sqrt{\frac{2}{h_C}} - \frac{h_C}{2h_C^2\sqrt{\frac{1}{h_C}+\frac{1}{h_L}}} + \frac{h_L}{2h_C^2\sqrt{\frac{1}{h_C}+\frac{1}{h_L}}}.$$

Next multiply with the positive number $\sqrt{\frac{1}{h_C}+\frac{1}{h_L}}$ to obtain

$$\sqrt{\frac{2}{g}}\sqrt{\frac{1}{h_C}+\frac{1}{h_L}}F'(h_C) = \frac{1}{h_C}+\frac{1}{h_L} - \sqrt{\frac{2}{h_C}}\sqrt{\frac{1}{h_C}+\frac{1}{h_L}} - \frac{h_C}{2h_C^2} + \frac{h_L}{2h_C^2}$$

$$= \frac{1}{2h_C}+\frac{1}{h_L} - \sqrt{\frac{2}{h_C}}\sqrt{\frac{1}{h_C}+\frac{1}{h_L}} + \frac{h_L}{2h_C^2}.$$

Letting $h_C = \epsilon h_L$ for $\epsilon \in (0, 1)$, we find

$$\sqrt{\frac{2}{g}}\sqrt{\frac{1}{h_C}+\frac{1}{h_L}}F'(h_C) = \frac{1}{2\epsilon h_L}+\frac{1}{h_L} - \sqrt{\frac{2}{\epsilon h_L}}\sqrt{\frac{1}{\epsilon h_L}+\frac{1}{h_L}} + \frac{h_L}{2\epsilon^2 h_L^2}.$$

Evidently, the proof will be achieved if it can be shown that the function

$$f(\epsilon) = \epsilon + 1 + 2\epsilon^2 - 2\sqrt{2}\epsilon\sqrt{\epsilon+1}$$

is positive for all $\epsilon \in (0, 1)$. To this end, we take the first and second derivatives:

$$f'(\epsilon) = 4\epsilon - 2\sqrt{2}\sqrt{\epsilon+1} - \frac{\sqrt{2}\epsilon}{\sqrt{\epsilon+1}} + 1,$$

$$f''(\epsilon) = \frac{\sqrt{2}\epsilon}{2(\epsilon+1)^{3/2}} - \frac{2\sqrt{2}}{\sqrt{\epsilon+1}} + 4.$$

Note that $f''(\epsilon) > 0$ by inspection, and f is therefore strictly convex on $(0, 1)$. Thus by convexity we know that $f'(\epsilon)$ is strictly increasing, so that $f'(\epsilon) < f'(1)$ for all $\epsilon \in (0, 1)$. But evaluating $f'(\epsilon)$ at 1 yields $f'(1) = 0$. Hence, $f'(\epsilon) < f'(1) = 0$. So the function f is strictly decreasing on $(0, 1)$ meaning $f(\epsilon) > f(1) = 0$.

Finally, denote the shock speeds of the backwards problem by

$$\sigma_R = \frac{q_R - q_C}{h_R - h_C},$$

and

$$\sigma_L = \frac{q_L - q_C}{h_L - h_C}.$$

Now recall that it was proved in Section 3 that the function $q(h)$ is convex, and note that the convexity on the Hugoniot locus $\mathcal{S}_2(C)$, including the admissible and the entropy-violating part guarantees that $\sigma_R > \sigma_L$, as is required for the two shocks to meet at $t = 0$.

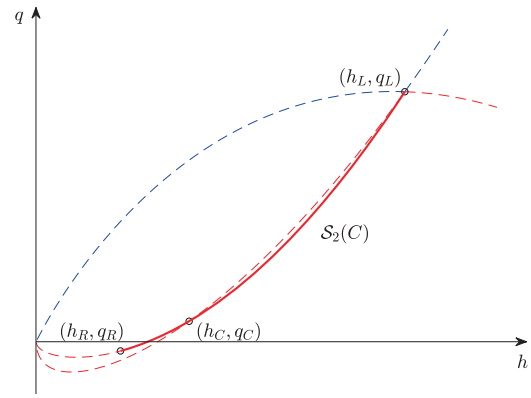

Figure 16: Backwards problem in (h, q)–coordinates.

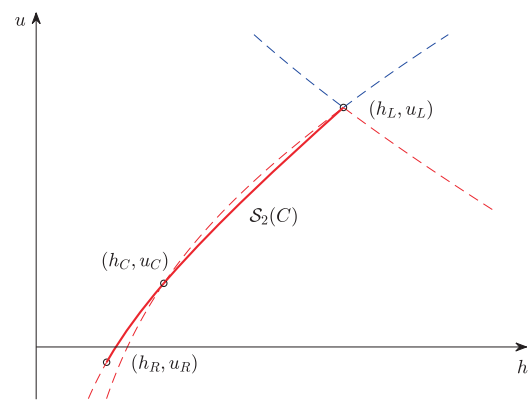

Figure 17: Backwards problem in (h, u)–coordinates.

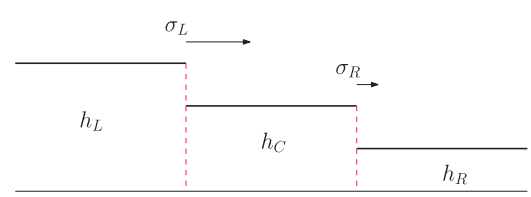

Figure 18: Two colliding bores forming the Riemann Problem.

A particular example of the backwards problem is represented in phase space for (h, u) and (h, q) coordinates.

In Figure 17, we observe that $h_R < h_C < h_L$ and $u_R < u_C < u_L$. Also note that in Figure 16 the line joining each state has a positive slope, which implies that both states moves in the positive direction. From this we may hope to create a Riemann problem if the left state is moving faster than the right state, causing an overtaking (see Figure 18). Though, this is clear due to the fact that s_2 is convex, i. e., the rate of change given by the shock speed σ is increasing from left to right. We state the general formulation in the next theorem.

7 Development of the Riemann problem from a collision of two $\mathcal{S}_1$ shocks

The final case to be considered is when a Riemann problem develops from the collision between two $\mathcal{S}_1$ shocks. The situation is similar, and the arguments in the proofs are virtually the same as in the previous section. Finding a center center state turns out to only be possible for some right states in region IV.

As before, we consider the left state given. It is then straightforward to see that the center state in the backwards problem must lie in the Rankine-Hugoniot locus $\mathcal{S}_2(L)$. Thus the center state is given by the formula

$$u_C = u_L - (h_C - h_L)\sqrt{\frac{g}{2}\left(\frac{1}{h_C} + \frac{1}{h_L}\right)}.$$

On the other hand, if the center state is to be connected to the right state by an $\mathcal{S}_2$ -shock, then the right state must lie on the $\mathcal{S}_2(C)$ shock curve and therefore satisfy the relation

$$u_R = u_C - (h_R - h_C)\sqrt{\frac{g}{2}\left(\frac{1}{h_R} + \frac{1}{h_C}\right)}.$$

Putting these two formulas together defines the region of all possible right states as

$$\begin{aligned}\Omega_4 = \bigcup_{h_C \in (0, h_L)} \Bigg\{ &(h_R, u_R) | u_R \\ &= u_L - (h_C - h_L)\sqrt{\frac{g}{2}\left(\frac{1}{h_C} + \frac{1}{h_L}\right)} \\ &\quad - (h_R - h_C)\sqrt{\frac{g}{2}\left(\frac{1}{h_R} + \frac{1}{h_C}\right)}, \ 0 < h_R < h_C \Bigg\}.\end{aligned}$$

We have the following theorem.

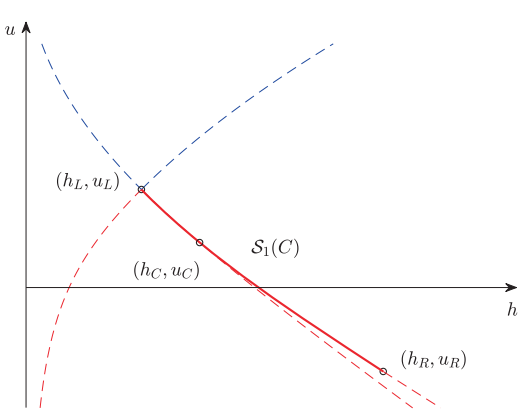

Figure 19: Backwards problem in (h, q)–coordinates.

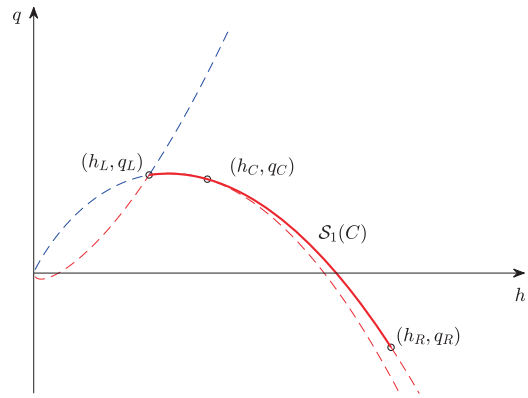

Figure 20: Backwards problem in (h, u)–coordinates.

Theorem 3. *Suppose that a left state $L = (h_L, u_L)$ for the Riemann problem is given. The set of all possible right states $R = (h_R, u_R)$ such that the Riemann problem originates from the collision of two $\mathcal{S}_2$ shocks is given by Ω_4. This set lies in region IV, and the shock speeds of the backwards problem line up such that the two shocks meet at $t = 0$.*

On the other hand, it is not possible for a Riemann problem to develop from a $\mathcal{S}_2 - \mathcal{S}_2$ connection if the right state is in the complement of the set Ω_4.

The proof of theorem 3 is virtually the same as that of Theorem 2, except for changing signs in the right places. From Figure 19, we observe that both bores are moving to the left due to a negative slope. However, the right state moves faster than the left. This is also true in general since $\mathcal{S}_1$ is strictly concave in momentum coordinates. Again, we must also choose an admissible connection. Similar to the case in Section 6, we follow $\mathcal{S}_1(L)$ from left state to center state, continuing along the $\mathcal{S}_1(C)$ curve from the center state to the right state (see Figure 20).

8 Conclusion

In this article, we have considered the Riemann problem associated to the shallow-water equations. The study of the Riemann problem is important when trying to understand the behavior of solutions of a system of conservation laws. For example the Riemann problem can used as a tool in the front-tracking method where general initial data are decomposed into piecewise constant functions which gives rise to a series of Riemann problems [12]. This approach is used in existence proofs and numerical schemes, but one may face difficulties interpreting solutions of the Riemann problem for the shallow-water equations in the case when the solution includes a dry region ($h = 0$). In gas dynamics,

this situation is known as cavitation and is a well-defined concept, but the creation of a dry zone between two propagating waves does not seem reasonable from a physical point of view in the case of shallow water theory.

In the present work, we have imposed the condition that the Riemann problem should arise from the collision of two bores. With this condition in place, we were able to show that solutions of the Riemann problem do not feature cavitation. In summary, for a given left state, the collision of an $\mathcal{S}_1(L)$ and an $\mathcal{S}_2(L)$ shock gives rise to a Riemann problem in Region III (Theorem 1). The collision of two $\mathcal{S}_2(L)$ shocks gives rise to a Riemann problem in Region II (Theorem 2), and the collision of two $\mathcal{S}_1(L)$ shocks gives rise to a Riemann problem in Region IV (Theorem 3). It is clear that a right state in region I is not permitted if these admissibility conditions are used. In particular, we avoid a right state in the shaded region of Figure 1 which is the region where the resolution of the Riemann problem features a dry state.

References

[1] I. Aavatsmark, "Kapillarenergie als Entropiefunktion," *Z. Angew. Math. Mech.*, vol. 69, pp. 319–327, 1989, https://doi.org/10.1002/zamm.19890691002.

[2] I. Aavatsmark, *Bevarelsesmetoder for hyperbolske differensialligninger*, Lecture Notes, University of Bergen, 2003, p. 140.

[3] A. Ali and H. Kalisch, "Energy balance for undular bores," *Compt. Rend. Mecanique*, vol. 338, pp. 67–70, 2010, https://doi.org/10.1016/j.crme.2010.02.003.

[4] A. Ali and H. Kalisch, "Mechanical balance laws for Boussinesq models of surface water waves," *J. Nonlinear Sci.*, vol. 22, pp. 371–398, 2012, https://doi.org/10.1007/s00332-011-9121-2.

[5] A. Ali and H. Kalisch, "A dispersive model for undular bores," *Anal. Math. Phys.*, vol. 2, pp. 347–366, 2012, https://doi.org/10.1007/s13324-012-0040-7.

[6] T. Benjamin and J. Lighthill, "On cnoidal waves and bores," *Proc. R. Soc. A*, vol. 224, pp. 448–460, 1954, https://doi.org/10.1098/rspa.1954.0172.

[7] M. Bjørkavåg and H. Kalisch, "Wave breaking in Boussinesq models for undular bores," *Phys. Lett. A*, vol. 375, pp. 1570–1578, 2011, https://doi.org/10.1016/j.physleta.2011.02.060.

[8] S. Bianchini and A. Bressan, "Vanishing viscosity solutions of nonlinear hyperbolic systems," *Ann. of Math.*, vol. 161, pp. 223–352, 2005, https://doi.org/10.4007/annals.2005.161.223.

[9] H. Chanson, Hydraulics of open channel flow, Arnold, 1999.

[10] L. C. Evans, *Partial Differential Equations*. 1st ed. Graduate Studies in Mathematics, vol. 19, Providence, RI, American Mathematical Society, 1998.

[11] H. Favre, *Ondes de Translation*, Paris, Dunod, 1935.

[12] H. Holden and N. H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*, New York, Springer, 2015.

[13] F. M. Henderson, *Open Channel Flow*, Prentice Hall, 1996.

[14] H. G. Hornung, C. Willert and S. Turner, "The flow field downstream of a hydraulic jump," *J. Fluid Mech.*, vol. 287, pp. 299–316, 1995, https://doi.org/10.1017/s0022112095000966.

[15] H. Kalisch, Z. Khorsand and D. Mitsotakis, "Mechanical balance laws for fully nonlinear and weakly dispersive water waves," *Physica D*, vol. 333, pp. 243–253, 2016, https://doi.org/10.1016/j.physd.2016.03.001.

[16] H. Kalisch and D. Mitrović, "Singular solutions of a fully nonlinear 2x2 system of conservation laws," *Proc. Edinb. Math. Soc.*, vol. 55, pp. 711–729, 2012, https://doi.org/10.1017/s0013091512000065.

[17] P. D. Lax, "Hyperbolic systems of conservation laws II," *Comm. Pure Appl. Math.*, vol. 10, pp. 537–566, 1957, https://doi.org/10.1002/cpa.3160100406.

[18] R. J. LeVeque, *Finite Volume Methods for Hyperbolic Problems*, Cambridge, Cambridge University Press, 2002.

[19] T. P. Liu, "Existence and uniqueness theorems for Riemann problems," *Trans. Amer. Math. Soc.*, vol. 212, pp. 375–382, 1975, https://doi.org/10.1090/s0002-9947-1975-0380135-2.

[20] T. P. Liu, "The Riemann problem for general systems of conservation laws," *J. Differ. Equ.*, vol. 18, pp. 218–234, 1975, https://doi.org/10.1016/0022-0396(75)90091-1.

[21] D. H. Peregrine, "Water-wave interaction in the surf zone," in *Coastal Engineering Proceedings 1974*, 1975, pp. 500–517.

[22] D. H. Peregrine, "Calculations of the development of an undular bore," *J. Fluid Mech.*, vol. 25, p. 321, 1966, https://doi.org/10.1017/s0022112066001678.

[23] L. Rayleigh, "Note on Tidal Bores," *Proc. Roy. Soc. London Ser. A*, vol. 81, pp. 448–449, 1908, https://doi.org/10.1098/rspa.1908.0102.

[24] M. Renardy and R. C. Rogers, *An Introduction to Partial Differential Equations*, Texts in Applied Mathematics, vol. 13, New York, Springer-Verlag, 1993.

[25] J. J. Stoker, *Water Waves: The Mathematical Theory with Applications*, New York, Interscience Publishers, 1957.

[26] B. Sturtevant, "Implications of experiments on the weak undular bore," *Phys. Fluids*, vol. 8, pp. 1052–1055, 1965, https://doi.org/10.1063/1.1761354.

[27] C. Tsikkou, "Hyperbolic conservation laws with large initial data. Is the Cauchy problem well-posed?," *Quart. Appl. Math.*, vol. 68, pp. 765–781, 2010, https://doi.org/10.1090/s0033-569x-2010-01208-9.

[28] G. B. Whitham, *Linear and Nonlinear Waves*, New York, Wiley, 1974.