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To cite this article: C. E. Kenig, D. Pilod, G. Ponce & L. Vega (2020) On the unique continuation of solutions to non-local non-linear dispersive equations, Communications in Partial Differential Equations, 45:8, 872-886, DOI: [10.1080/03605302.2020.1739707](https://doi.org/10.1080/03605302.2020.1739707)

To link to this article: <https://doi.org/10.1080/03605302.2020.1739707>



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Published online: 13 Mar 2020.



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On the unique continuation of solutions to non-local non-linear dispersive equations

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ABSTRACT

We prove unique continuation properties of solutions to a large class of nonlinear, non-local dispersive equations. The goal is to show that if u_1, u_2 are two suitable solutions of the equation defined in $\mathbb{R}^n \times [0, T]$ such that for some non-empty open set $\Omega \subset \mathbb{R}^n \times [0, T]$, $u_1(x, t) = u_2(x, t)$ for $(x, t) \in \Omega$, then $u_1(x, t) = u_2(x, t)$ for any $(x, t) \in \mathbb{R}^n \times [0, T]$. The proof is based on static arguments. More precisely, the main ingredient in the proofs will be the unique continuation properties for fractional powers of the Laplacian established by Ghosh, Salo and Uhlmann, and some extensions obtained here.

ARTICLE HISTORY

Received 13 February 2020
Accepted 1 March 2020

KEYWORDS

Nonlinear dispersive equation; non-local operators

1991 MATHEMATICS

SUBJECT

CLASSIFICATION

Primary: 35Q55;
Secondary: 35B05

1. Introduction

This work continues the study initiated in [1] concerning unique continuation properties of solutions to nonlinear non-local dispersive equations. Roughly, the aim is to prove that if u_1, u_2 are two suitable solutions of a nonlinear dispersive model for $(x, t) \in \mathbb{R}^n \times [0, T]$ such that for some open non-empty set $\Omega \subset \mathbb{R}^n \times [0, T]$

$$u_1(x, t) = u_2(x, t), \quad (x, t) \in \Omega, \quad (1.1)$$



then $u_1(x, t) = u_2(x, t)$ for all $(x, t) \in \mathbb{R}^n \times [0, T]$.

As in [1] the equations to be examined here have their dispersive relation described by a non-local differential operator.

Our original motivation is to establish the desired result in real solutions for the so called generalized dispersion Benjamin-Ono (BO) equation whose initial value problem (IVP) can be written as

$$\begin{cases} \partial_t u - \partial_x D_x^\alpha u + u \partial_x u = 0, & x, t \in \mathbb{R}, \alpha \in (-1, 2) - \{0, 1\}, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.2)$$

where in any dimension n the operator D_x^α is defined for $\alpha \in (0, 2)$ as

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$$\begin{aligned}
 D_x^\alpha(x) &= (-\Delta)^{\alpha/2}f(x) = ((2\pi|\xi|)^\alpha \hat{f})^\vee(x) \\
 &= \lim_{\epsilon \downarrow 0} \frac{1}{c_{n,\alpha}} \int_{|y| \geq \epsilon} \frac{f(x+y) - f(x)}{|y|^{n+\alpha}} dy,
 \end{aligned}
 \tag{1.3}$$

and for $\alpha \in (-n, 0)$ as

$$\begin{aligned}
 D_x^\alpha(x) &= (-\Delta)^{\alpha/2}f(x) = ((2\pi|\xi|)^\alpha \hat{f})^\vee(x) \\
 &= \frac{1}{c_{n,\alpha}} \int_{\mathbb{R}^n} \frac{f(x+y)}{|y|^{n+\alpha}} dy,
 \end{aligned}
 \tag{1.4}$$

with

$$c_{n,\alpha} = \pi^{n/2} 2^{-\alpha} \Gamma(-\alpha/2) / \Gamma((n + \alpha)/2),$$

see for example [2].

In (1.2) the limiting cases: $\alpha = 2$ corresponds to the Korteweg-de Vries (KdV) equation [3], $\alpha = 1$ corresponds to the BO equation [4, 5], $\alpha = 0$, after a change of variable, coincides with the inviscid Burgers' equation which is hyperbolic, and $\alpha = -1$ is the so called Hilbert-Burgers (HB) equation, see [6, 7].

The KdV equation and the BO equation arise both as mathematical models for the unidirectional propagation of long waves and in inverse scattering theory. They are completely integrable (see [8, 9]) the only ones in the family $\alpha \in [-1, \infty)$. In particular, their solutions satisfy infinitely many conservations laws.

The general case of the equation in (1.2) appears as a model for vorticity waves in the coastal zone, see [10].

Formally, real valued solutions of (1.2) satisfy three conservation laws:

$$\begin{aligned}
 I_1(u) &= \int_{-\infty}^{\infty} u(x, t) dx, & I_2(u) &= \int_{-\infty}^{\infty} u^2(x, t) dx, \\
 I_3(u) &= \int_{-\infty}^{\infty} \left((D_x^{\alpha/2} u)^2 - \frac{1}{3} u^3 \right) (x, t) dx.
 \end{aligned}
 \tag{1.5}$$

The well-posedness of the IVP (1.2) has been extensively considered. For the case $\alpha \in (1, 2)$ we refer to [11–18] and references therein, and for $\alpha \in (0, 1)$ we refer to [19, 20] and references therein, (for the case $\alpha = 1$ see the reference [1]).

In [15] it was shown that the IVP (1.2) with $\alpha \in (1, 2)$ is globally well-posed in $L^2(\mathbb{R})$. In [20] it was established that the IVP (1.2) with $\alpha \in (0, 1)$ is locally well-posed in $H^s(\mathbb{R}), s > 3/2 - 5\alpha/4$, and globally well-posed in $H^{\alpha/2}(\mathbb{R})$ if $\alpha > 6/7$.

In the case $\alpha \in (-1, 0)$ the local well posedness in $H^s(\mathbb{R})$ with $s > 3/2$ follows by a standard argument based on energy estimates.

Our first result is the following:

Theorem 1.1. *Let $\alpha \in (-1, 2) - \{0, 1\}$. Let u_1, u_2 be two real solutions of the IVP (1.2) such that*

$$u_1, u_2 \in C([0, T] : H^s(\mathbb{R})) \cap C^1([0, T] : H^{s'}(\mathbb{R})), \tag{1.6}$$

with $s > \max\{\alpha + 1; 3/2\}$ and $s' = \min\{s - (\alpha + 1); s - 1\}$. Moreover, if $\alpha \in (-1, -1/2]$ assume that

$$x\partial_x u_1, \quad x\partial_x u_2 \in L^\infty([0, T] : L^2(\mathbb{R})). \tag{1.7}$$

If there exists a non-empty open set $\Omega \subset \mathbb{R} \times [0, T]$ such that

$$u_1(x, t) = u_2(x, t), (x, t) \in \Omega, \tag{1.8}$$

then $u_1(x, t) = u_2(x, t)$ for all $(x, t) \in \mathbb{R} \times [0, T]$

Remark 1.2. (i) For the value $\alpha = 2$, i.e. the KdV equation, the result in [Theorem 1.1](#) was proved by Saut-Scheurer in [21]. In this case, the operator modeling the dispersion relation is local, and the proof is based on appropriate Carleman estimates.

In the cases $\alpha = \pm 1$, i.e. for the BO and HB equations resp., the result in [Theorem 1.1](#) was recently obtained in [1].

In the case $\alpha = 0$ the result fails.

(ii) The hypothesis in (1.6) guarantees that $\partial_x u_j(\cdot, t)$, $j = 1, 2$, $t \in [0, T]$ is a continuous function and that $\partial_t u(\cdot, t) \in L^2(\mathbb{R})$, $j = 1, 2$ for any $t \in [0, T]$. This is satisfied if $u_1(x, 0), u_2(x, 0) \in H^s(\mathbb{R})$, i.e. the solution flow preserves (locally in time) the class $H^s(\mathbb{R})$ with $s > \max\{\alpha + 1; 3/2\}$.

The hypothesis (1.7) implies that for any $t \in [0, T]$ one has that $\partial_x u_1(t), \partial_x u_2(t) \in L^1(\mathbb{R})$ which is essential in the proof of [Theorem 1.1](#) for the case $\alpha \in (-1, -1/2]$. Property (1.7) is ensured if one assumes that

$$x\partial_x u_1(x, 0), x\partial_x u_2(x, 0) \in L^2(\mathbb{R}).$$

This will be proved in the [Appendix A](#).

(iii) It will be clear from our proof below that the result in [Theorem 1.1](#) extends to the equation in (1.2) with a more general non-linearity. In fact, [Theorem 1.1](#) applies to any pair of appropriate solutions u_1, u_2 of the IVP associated to the equation

$$\partial_t u - \partial_x D_x^\alpha u + F(u, \dots, \partial_x^j u) = 0, x, t \in \mathbb{R}, \tag{1.9}$$

with $\alpha \in [-1, \infty) - 2\mathbb{Z}$, $j \in \mathbb{N}$ and $F(\cdot)$ inline a regular enough function representing the non-linearity.

(iv) For some non-local models, our version of unique continuation is too strong, and it is only realized by assuming that $u_2 \equiv 0$. This is the case of the IVP associated to the Camassa-Holm equation [22]

$$\partial_t u + u\partial_x u + \partial_x(1 - \partial_x^2)^{-1} \left(u^2 + \frac{1}{2}(\partial_x u)^2 \right) = 0, t, x \in \mathbb{R}, \tag{1.10}$$

see [23]. This will also occur in part (ii) of [Theorems 1.8](#) and [1.9](#) below for the [Equation \(1.20\)](#) when a Hartree term is present.

An argument similar to that used in the proof of [Theorem 1.1](#) will also yield the following result:

Corollary 1.3. *Let $\alpha \in (-1, 2) - \{0\}$. Let u be a real solution of the IVP (1.2) in the class defined in (1.6). Moreover, if $\alpha \in (-1, -1/2]$ assume that*

$$x\partial_x u \in L^\infty([0, T] : L^2(\mathbb{R})). \tag{1.11}$$

If there exist a constant $c_0 \in \mathbb{R}$ and a non-empty open set $\Omega \subset \mathbb{R} \times [0, T]$ such that

$$u(x, t) = c_0, (x, t) \in \Omega, \tag{1.12}$$

then $c_0 = 0$ and $u(x, t) = 0$ for all $(x, t) \in \mathbb{R} \times [0, T]$.

Remark 1.4. (i) By working with $\partial_x u$ instead of u the result in [Corollary 1.3](#) extends to the end-point cases $\alpha = 2$ (KdV) and $\alpha = -1$ (HB), for appropriate solutions, using the arguments in [\[21\]](#) and [\[1\]](#), respectively.

(ii) In the case $\alpha = 2$ (KdV) under more restricted hypothesis stronger results are known. More precisely, in [\[24\]](#) it was shown that the solution of the IVP for the KdV equation corresponding to data $u_0 \in L^2(\mathbb{R})$ with the additional decay assumption

$$\int_{-\infty}^{\infty} (1 + |x|)|u_0(x)|dx + \int_0^{\infty} e^{\delta|x|^{1/2}} |u_0(x)|dx < \infty \text{ for some } \delta > 0,$$

becomes analytic with respect to the variable x for all $t > 0$. This result is based on the inverse scattering method and it is unknown for other nonlinearities.

In the case of $\alpha = \pm 1$, i.e. for BO and HB equations, respectively, the proof of [Theorem 1.1](#) in [\[1\]](#) is based on the identities

$$(-\partial_x^2)^{1/2} = D_x = \partial_x \mathcal{H} \text{ and } \partial_x D_x^{-1} = \mathcal{H},$$

where \mathcal{H} is the Hilbert transform

$$\mathcal{H}f(x) = \frac{1}{\pi} \text{p.v.} \left(\frac{1}{x} * f \right) (x) = (-i \operatorname{sgn}(\xi) \hat{f}(\xi))^\vee(x),$$

and the following well-known unique continuation principle for the Hilbert transform:

Lemma A. *Let $f \in L^2(\mathbb{R})$ be a real valued function. If there exists an open (non-empty) interval $I \subset \mathbb{R}$ such that*

$$f(x) = \mathcal{H}f(x) = 0, \text{ for a.e. } x \in I,$$

then $f(x) = 0$ for any $x \in \mathbb{R}$.

A key argument in the proof of [Theorem 1.1](#) will be following unique continuation result regarding the fractional powers of the Laplacian in \mathbb{R}^n established by Ghosh-Salo-Uhlmann in [\[25\]](#):

Theorem B ([\[25\]](#)). *Let $\alpha \in (0, 2)$ and $f \in H^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$. If there exists a non-empty open set $\Theta \subset \mathbb{R}^n$ such that*

$$(-\Delta)^{\alpha/2} f(x) = f(x) = 0, \text{ in } \mathcal{D}'(\Theta), \tag{1.13}$$

then $f \equiv 0$ in $H^s(\mathbb{R}^n)$.

Remark 1.5. It is clear that the statement of [Theorem B](#) extends to $\alpha \in (0, \infty) - 2\mathbb{Z}$, and by “duality” to $\alpha \in (-n/2, \infty) - 2\mathbb{Z}$, (see the argument at the end of the proof of [Theorem 1.6](#) in [Section 2](#)).

Moreover, if $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then it also holds for $\alpha \in (-n, -n/2]$, see [\[26\]](#) for less restrictive regularity assumptions.

Unique continuation for the fractional Laplacian has been studied in other works, see [\[26–31\]](#) and references therein.

To illustrate the argument to prove [Theorem B](#) and its extensions it is convenient to examine the case $\alpha = 1$ and $n \geq 1$ (when $n = 1$ the result follows by [Lemma A](#)) assuming that $f \in H^{1/2}(\mathbb{R}^n)$. By inspecting the elliptic extension problem

$$\begin{cases} (\Delta_x + \partial_y^2)U(x, y) = 0, & (x, y) \in \mathbb{R}^n \times (0, \infty), \\ U(x, 0) = f(x), \end{cases} \tag{1.14}$$

one has that

$$(-\Delta_x)^{1/2}f(x) = - \partial_y U(x, 0). \tag{1.15}$$

Then, from the hypothesis (1.13) using the reflexion principle for harmonic functions one has that $U(x, y)$ has a harmonic extension $\tilde{U}(x, y)$ defined in the open set $\mathbb{R}^n \times (-\infty, 0) \cup \Theta \times \{0\} \cup \mathbb{R}^n \times (0, \infty)$. From the equation, the identity (1.15) and the hypothesis (1.13) it follows that $\tilde{U}(x, y)$ has zero of infinite order at any point of $\Theta \times \{0\}$. Hence, since \tilde{U} is real analytic, it follows that $\tilde{U}(x, y) \equiv 0$ and consequently $f(x) = 0$ for $x \in \mathbb{R}^n$, which proves **Theorem B** for $\alpha = 1$ under the assumption $f \in H^{1/2}(\mathbb{R}^n)$.

In the general case $\alpha \in (0, 2)$ the proof of **Theorem B**, with $f \in H^\alpha(\mathbb{R}^n)$ follows by combining the ideas of Caffarelli-Silvestre in [32], with a unique continuation principle obtained in [30], see also [27, 31], and some regularity results found in [32], see [25].

Further, we notice that the argument given above to prove **Theorem B** in the particular case $\alpha = 1/2$ still applies if one considers $(1 - \Delta)^{1/2}$ instead of $(-\Delta)^{1/2}$. More precisely, one regards the elliptic problem

$$\begin{cases} (\Delta_x - 1 + \partial_y^2)U(x, y) = 0, & (x, y) \in \mathbb{R}^n \times (0, \infty), \\ U(x, 0) = f(x), \end{cases} \tag{1.16}$$

to get that

$$(1 - \Delta_x)^{1/2}f(x) = - \partial_y U(x, 0). \tag{1.17}$$

Assuming that $f \in H^{1/2}(\mathbb{R}^n)$ and $(1 - \Delta_x)^{1/2}f(x) = f(x) = 0$ for $x \in \Theta$ open (non-empty) sub-set of \mathbb{R}^n the above approach yields the desired result $f(x) = 0$ for any $x \in \mathbb{R}^n$.

For the general case, we shall prove the following related version of **Theorem B**:

Theorem 1.6. *Let $\alpha \in (0, 2)$ and $f \in H^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$. If there exists a non-empty open set $\Theta \subset \mathbb{R}^n$ such that*

$$(1 - \Delta)^{\alpha/2}f(x) = f(x) = 0, \text{ in } \mathcal{D}'(\Theta), \tag{1.18}$$

then $f \equiv 0$ in $H^s(\mathbb{R}^n)$.

Remark 1.7. (i) Related unique continuation principles in the context of **Theorem 1.6** have been studied in previous works, see [28] and references therein.

(ii) The range of α in **Theorem 1.6** can be extended to $\mathbb{R} - 2\mathbb{Z}$. In fact, **Theorem 1.6** can be re-stated in the following general form: let $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 - \alpha_2 \notin 2\mathbb{Z}$ and $f \in H^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$. If there exists a non-empty open set $\Theta \subset \mathbb{R}^n$ such that

$$(1 - \Delta)^{\alpha_j/2}f(x) = 0, \text{ in } \mathcal{D}'(\Theta) \text{ for } j = 1, 2, \tag{1.19}$$

then $f \equiv 0$ in $H^s(\mathbb{R}^n)$.

(iii) Our proof of **Theorem 1.6** starts by using the abstract approach in [33] to formulate a general setting and then uses some ideas which reduce the proof of this theorem to that of **Theorem B** in a higher dimension. This argument is quite natural and enables

us to consider the extension of this problem to fractional powers of elliptic operators with variable coefficients in \mathbb{R}^n and \mathbb{S}^n as well as in manifold. This will be addressed in a forthcoming work.

Theorem B and **Theorem 1.6** allow us to extend the unique continuation result in **Theorem 1.1** to a large class of non-linear non-local higher dimensional dispersive models. More precisely, we consider the general fractional Schrödinger equation

$$i\partial_t u + (\mathcal{L}_m)^{\alpha/2} u + Vu + (W * F(|u|))u + P(u, \bar{u}) = 0, \tag{1.20}$$

where $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, $\mathcal{L}_m = (-\Delta + m^2)$, $m \geq 0$, with $\alpha \in \mathbb{R} - 2\mathbb{Z}$ if $m > 0$ and $\alpha \in (-n, \infty) - \mathbb{Z}$ if $m = 0$, $V = V(x, t)$ is the potential energy, $W = W(|x|)$ defines the Hartree integrand and $P(z, \bar{z})$ is a polynomial or a regular enough function with $P(0, 0) = 0$.

The model in (1.20) arises in several different physical contexts, for example:

- when $m = 0$, $W = P = 0$, it was used in [34] to describe particles in a class of Levi stochastic processes,
- when $m > 0$, $W = P = 0$, it was derived as a generalized semi-relativistic (Schrödinger) equation, see [35] and references therein,
- when $m = 0$, $\alpha = 1$, $V = W = 0$, and $P(u, \bar{u}) = \pm |u|^a u$, $a > 0$, it is known as the half-wave equation, see [36, 37] and reference therein,
- when $m = 0$, $V = W = 0$, and

$$P(z, \bar{z}) = c_0 |z|^2 z + c_1 z^3 + c_3 z \bar{z}^2 + c_3 \bar{z}^3, \quad c_0 \in \mathbb{R}, \quad c_1, c_2, c_3 \in \mathbb{C},$$

it appears in [38] on the study of the long-time behavior of solutions to the water waves equations in \mathbb{R}^2 , where $(-\partial_x^2)^{1/2}$ models the dispersion relation of the linearized gravity water waves equations for one-dimensional interfaces,

- when $m = 1$, $V = P = 0$, $F(|z|) = |z|^2$ the model arises in gravitational collapse, see [39, 40] and references therein.

The well-posedness of the IVP (1.20) has been considered in several publications, see for example, [35, 41–44] and references therein.

Thus, we have the following results concerning unique continuation properties of solutions to the IVP associated to the Equation (1.20). First, we shall consider the case $m > 0$:

Theorem 1.8. (i) Let $\alpha \in (-2, 2) - \{0\}$ and $m > 0$. Let u_1, u_2 be two solutions of the IVP associated to the equation (1.20) with $W \equiv 0$ such that

$$u_1, u_2 \in C([0, T] : H^s(\mathbb{R}^n)) \cap C^1([0, T] : H^{s-\alpha}(\mathbb{R}^n)), \tag{1.21}$$

with $s > \max\{\alpha; n/2\}$. If there exists a non-empty open set $\Omega \subset \mathbb{R}^n \times [0, T]$ such that

$$u_1(x, t) = u_2(x, t), \quad (x, t) \in \Omega, \tag{1.22}$$

then $u_1(x, t) = u_2(x, t)$ for all $(x, t) \in \mathbb{R}^n \times [0, T]$.

(ii) For the general form of the Equation (1.20) the result in (i) still holds if one assumes that $u_2(x, t) = 0$ for $(x, t) \in \mathbb{R}^n \times [0, T]$.

Next, we consider the case $m = 0$:

Theorem 1.9. (i) Let $\alpha \in (0, 2)$ and $m = 0$. Let u_1, u_2 be two solutions of the IVP associated to the Equation (1.20) with $W \equiv 0$ such that

$$u_1, u_2 \in C([0, T] : H^s(\mathbb{R}^n)) \cap C^1([0, T] : H^{s-\alpha}(\mathbb{R}^n)), \tag{1.23}$$

with $s > \max\{\alpha; n/2\}$. If there exists a non-empty open set $\Omega \subset \mathbb{R}^n \times [0, T]$ such that

$$u_1(x, t) = u_2(x, t), \quad (x, t) \in \Omega, \tag{1.24}$$

then $u_1(x, t) = u_2(x, t)$ for all $(x, t) \in \mathbb{R}^n \times [0, T]$.

(ii) For the general form of the Equation (1.20) the result in (i) still holds if one assumes that $u_2(x, t) = 0$ for $(x, t) \in \mathbb{R}^n \times [0, T]$.

Remark 1.10. (i) In Theorems 1.8 and 1.9 we are assuming the existence of solutions of the IVP (1.20) in the class described in (1.21) and (1.23). This depends on the dimension n , on the values of $m \geq 0$ and $\alpha \in (0, 2)$, on the regularity and decay properties of V and W , and on the structure of the terms $F(\cdot)$ and $P(\cdot, \cdot)$. In particular, the hypothesis $s > n/2$ guarantees that $H^s(\mathbb{R}^n)$ is an algebra respect to the usual product of functions so the polynomial $P(u, \bar{u})$ belongs to $L^2(\mathbb{R}^n)$ at each $t \in [0, T]$

(ii) The results in Theorem 1.9 ($m=0$) extend to the case $\alpha \in (-n/2, 0) - \mathbb{Z}$ by assuming (instead of (1.23)) that

$$u_1, u_2 \in C([0, T] : H(\mathbb{R}^n)) \cap C^1([0, T] : L^p(\mathbb{R}^n)),$$

with $s > n/2$ and $1/p = 1/2 + \alpha/2$.

(iii) As was mentioned before, in part (ii) of Theorems 1.8 and 1.9 we achieve a weaker version of the desired unique continuation result. Roughly speaking, in the general case one works with the equation solved by $w(x, t) = (u_1 - u_2)(x, t)$. However, when the non-local Hartree term in (1.20) is present, the equation for $w(x, t)$ does not satisfy the necessary property enjoyed by a solution v of (1.20), that is, for any $\Omega \subset \mathbb{R}^n \times [0, T]$ open set

$$\text{if } v(x, t) = 0, \quad (x, t) \in \Omega, \quad \text{then } (\mathcal{L}_m)^{\alpha/2} v(x, t) = 0, \quad (x, t) \in \Omega.$$

(iv) The comments in Remark 1.2 part (iii) concerning more general nonlinearities in (1.20) also apply to Theorems 1.8 and 1.9.

Next, we consider the extension of Corollary 1.3 to solutions of the IVP associated to the equation in (1.20).

Corollary 1.11. Let $\alpha \in (-2, 2) - \{0\}$ if $m > 0$ and $\alpha \in (0, 2) - \{0\}$ if $m = 0$. Let u be a solution of the IVP associated to the Equation (1.20) with $V \equiv W \equiv 0$ such that

$$u \in C([0, T] : H^s(\mathbb{R}^n)) \cap C^1([0, T] : H^{s-\alpha}(\mathbb{R}^n)). \tag{1.25}$$

If there exist a constant $c_0 \in \mathbb{R}$ and a non-empty open set $\Omega \subset \mathbb{R} \times [0, T]$ such that

$$u(x, t) = c_0, \quad (x, t) \in \Omega, \tag{1.26}$$

then $c_0 = 0$ and $u_1(x, t) = 0$ for all $(x, t) \in \mathbb{R} \times [0, T]$.

Remark 1.12. The observations in Remark 1.10 (ii) also hold for Corollary 1.11 in the case $m = 0$.

Next, we shall give an application of Theorem B to a non-local non-dispersive model. We consider the 2D quasi-geostrophic (QG) equation

$$\partial_t \theta + u \cdot \nabla \theta = 0, \quad (x, t) \in \mathbb{R}^2 \times \mathbb{R}, \quad (1.27)$$

and

$$u = (u_1, u_2) = \nabla^\perp (-\Delta)^{-1/2} \theta = (-\partial_y, \partial_x) (-\Delta)^{-1/2} \theta, \quad (1.28)$$

where θ denotes a scalar (potential temperature) convected by the velocity field u .

The QG was derived in [45] as a model for atmospheric turbulence. Its relation with the Euler system for incompressible (homogeneous) flow

$$\begin{cases} \partial_t v + v \cdot \nabla v = \nabla p, & k = 1, \dots, n, \quad n = 2, 3, \\ \nabla \cdot v = 0, \end{cases}$$

where $v = (v_1, \dots, v_n)$ is the velocity and p denotes the pressure, and its vorticity $\omega = \nabla \times v$ formulation in 3D

$$\partial_t \omega + v \cdot \nabla \omega = (Dv)\omega, \quad (1.29)$$

has been extensively studied, see [46, 47] and references therein. Notice that by differentiating the equation in (1.27) one gets

$$\partial_t (\nabla^\perp \theta) + u \cdot \nabla (\nabla^\perp \theta) = (Du)(\nabla^\perp \theta), \quad (1.30)$$

which resembles to the equation in (1.29).

We also observe that in the 2D case of the Euler system the vorticity ω is a scalar function satisfying the equation

$$\partial_t \omega + v \cdot \nabla \omega = 0, \quad (1.31)$$

which resembles that in (1.27). In both dimensions 2 and 3 the Biot-Savart law

$$v = \nabla \times (-\Delta)^{-1} \omega, \quad (1.32)$$

allows to recover the velocity v in terms of the vorticity ω .

The following theorem presents a difference between the behavior of the solutions of the QG equation and those for the vorticity system in 2 and 3 dimensions.

Theorem 1.13. *Let θ be a solution of the IVP associated to the Equation (1.27) such that*

$$\theta \in C([0, T] : H^s(\mathbb{R}^2)) \cap C^1([0, T] : H^{s-1}(\mathbb{R}^2)), \quad (1.33)$$

with $s > 1$. If there exist $t_0 \in [0, T]$, $c_0 \in \mathbb{R}$ and a non-empty open set $\Theta \subset \mathbb{R}^2$ such that

$$\theta(\cdot, t_0)|_\Theta = c_0 \text{ and } u(\cdot, t_0)|_\Theta = (0, 0), \quad (1.34)$$

then $c_0 = 0$, $\theta(x, t) = 0$ and $u(x, t) = (0, 0)$ for all $(x, t) \in \mathbb{R}^2 \times [0, T]$.

Remark 1.14. (i) We observe that **Theorem 1.13** fails for the Euler system in 2 and 3 dimensions by just picking up a data $v(\vec{x}, 0) = \vec{c}_0$ for $\vec{x} \in \Theta$, (since $\omega(\vec{x}, 0) = \nabla \times v(\vec{x}, 0)$), and using that the equations are time reversible. As it was shown above in the case $n = 3$ one can think of $\nabla^\perp \theta$ as the vorticity and in the case $n = 2$ one sees θ as the vorticity itself.

(ii) The proof of the existence of solution (in fact, the local well-posedness in $H^s(\mathbb{R}^n)$, with $s > n/2 + 1$) to the IVP associated to the Equation (1.27) in the class (1.33) follows a classical argument.

(iii) **Theorem 1.13** applies to the generalized form of the QG equation considered in [47]

$$\partial_t \theta + u \cdot \nabla \theta = 0, \quad u = (u_1, u_2) = \nabla^\perp (-\Delta)^{-1+\alpha/2} \theta, \tag{1.35}$$

with $\alpha \in (0, 2)$, where $\alpha = 1$ corresponds to the QG equation and $\alpha = 0$ to the vorticity equation in 2D (1.31).

The rest of this paper is organized as follows: **Section 2** contains the proof of **Theorem 1.6**. The proof of **Theorem 1.1**, **Corollary 1.3**, **Theorems 1.8** and **1.9**, **Corollary 1.11** and **Theorem 1.13** will be in **Section 3**. **Appendix A** contains the proof of the last statement in **Remark 1.2** part (ii) concerning condition (1.7).

2. Preliminary estimates

Proof of Theorem 1.6. First note that, without loss of generality, we can assume that $f \in H^\infty(\mathbb{R}^n)$, by convolving with a compactly supported approximate identity.

Let $\alpha \in (0, 2)$ and Θ be a non-empty open sub-set of \mathbb{R}^n . Let L be a non-negative second order elliptic operator with a dense domain $dom(L)$ in $L^2(\mathbb{R}^n)$. Following [33] for $f \in dom(L)$ we define for $y \in [0, \infty)$

$$\begin{aligned} V(x, y) &= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} (L^{\alpha/2} f)(x) e^{-y^2/4t} \frac{dt}{t^{1-\alpha/2}} \\ &= \frac{y^\alpha}{2^\alpha \Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) e^{-y^2/4t} \frac{dt}{t^{1+\alpha/2}}. \end{aligned} \tag{2.1}$$

Thus, by **Theorem 1.1** in [33] one has

$$V \in C^\infty((0, \infty) : dom(L)) \cap C^\infty([0, \infty) : L^2(\mathbb{R}^n)), \tag{2.2}$$

with

$$\begin{cases} -L_x V + \frac{1-\alpha}{y} \partial_y V + \partial_y^2 V = 0, & (x, y) \in \mathbb{R}^n \times (0, \infty), \\ V(x, 0) = f(x). \end{cases} \tag{2.3}$$

and

$$L^{\alpha/2} f(x) = -c_\alpha^* \lim_{y \downarrow 0} y^{1-\alpha} \partial_y V(x, y), \tag{2.4}$$

where $c_\alpha^* = 2^\alpha \Gamma(\alpha/2) / \alpha \Gamma(-\alpha/2)$.

As it was remarked in [33] the identities above should be understood in $L^2(\mathbb{R}^n)$. Moreover, $V(x, y)$ solves the singular boundary value problem (2.3) and the first equality in (2.1) involves the heat semi-group e^{-tL} acting on $L^{\alpha/2} f$ and the second does not involve fractional powers of L . Thus, (2.4) describes the fractional powers of L in terms of the solution of the boundary value (2.3) which is the original idea in [32].

Notice that in the case $L = -\Delta_x$ the equation in (2.3) can be written in the divergence form

$$div_{x,y}(y^{1-\alpha} \nabla_{x,y} V) = 0. \tag{2.5}$$

and that when $\alpha = 1$ (2.3) and (2.4) agree with (1.14) and (1.15), respectively.

Next, we fix the operator $L = 1 - \Delta_x$, assume without loss of generality that $\Theta = B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$ and $f \in H^\infty(\mathbb{R}^n)$ with

$$f|_{B_1(0)} = (1 - \Delta)^{\alpha} f|_{B_1(0)} \equiv 0. \tag{2.6}$$

In this case

$$\begin{cases} \Delta_x V - V + \frac{1 - \alpha}{y} \partial_y V + \partial_y^2 V = 0, & (x, y) \in \mathbb{R}^n \times (0, \infty), \\ V(x, 0) = f(x), \end{cases} \tag{2.7}$$

with

$$(1 - \Delta_x)^{\alpha/2} f(x) = -c_{\alpha}^* \lim_{y \downarrow 0} y^{1-\alpha} \partial_y V(x, y), \quad x \in \mathbb{R}^n. \tag{2.8}$$

We note that under our regularity assumptions and the hypothesis (2.6) the local infinite order of vanishing argument in [30] (proof of Proposition 2.2) applies to the equation (2.7). In particular, it tells us that the solution $V(x, y)$ vanishes of infinite order at $(x, y) \in B_1(0) \times \{0\}$, i.e.

$$\lim_{y \downarrow 0} \frac{V(x, y)}{y^k} = 0, \quad \forall x \in B_1(0), \quad \forall k \in \mathbb{N}. \tag{2.9}$$

Now we define the function $\tilde{V}(x, t, y)$ as

$$\tilde{V}(x, t, y) = \cos(t) V(x, y), \quad (x, t, y) \in \mathbb{R}^{n+1} \times [0, \infty). \tag{2.10}$$

Therefore,

$$\begin{cases} \Delta_{x,t,y} \tilde{V} + \frac{1 - \alpha}{y} \partial_y \tilde{V} = 0, & (x, t, y) \in \mathbb{R}^{n+1} \times (0, \infty), \\ \tilde{V}(x, t, 0) = \cos(t) f(x), \end{cases} \tag{2.11}$$

with

$$\lim_{y \downarrow 0} \frac{\tilde{V}(x, t, y)}{y^k} = 0, \quad \forall (x, t) \in B_1(0) \times (-1, 1) \quad \forall k \in \mathbb{N}. \tag{2.12}$$

Thus, we are in the setting of the proof of Theorem B. Applying Proposition 2.2. in [30] one gets that

$$\tilde{V}(x, t, y) = 0, \quad \text{for } (x, t, y) \in \{(x, t, y) : |x|^2 + t^2 + y^2 < 1, y > 0\}.$$

But \tilde{V} is real analytic in $\mathbb{R}^{n+1} \times (\epsilon, \infty)$ for any $\epsilon > 0$. Therefore $\tilde{V} \equiv 0$ and so $V \equiv 0$ and consequently $f(x) = 0$ for any $x \in \mathbb{R}$.

This completes the proof for the case $\alpha \in (0, 2)$.

Next, for $\alpha \in (0, \infty) - 2\mathbb{Z}$ and $f \in H^s(\mathbb{R}^n), s \in \mathbb{R}$, we first reduce it to the case $f \in H^\infty(\mathbb{R}^n)$. Thus, applying the result for $\alpha \in (0, 2)$ to

$$\tilde{f}(x) = (1 - \Delta)^k f(x), \quad \text{with } k \in \mathbb{N} \text{ and } \alpha \in (2k, 2(k + 1)),$$

the desired result follows.

Finally, if $\alpha \in (-\infty, 0) - 2\mathbb{Z}$ we define

$$g(x) = (1 - \Delta)^{\alpha/2} f(x).$$

Thus, from hypothesis (1.18) one has that

$$g(x) = 0 \text{ and } (1 - \Delta)^{-\alpha/2} g(x) = f(x) = 0, \quad x \in \Theta,$$

with $-\alpha \in (0, \infty) - 2\mathbb{Z}$ and $g \in H^{-\alpha}(\mathbb{R}^n)$. Therefore, from the previous case one has that $g(x) = 0$ for $x \in \mathbb{R}^n$ and consequently, $f(x) = 0$ for $x \in \mathbb{R}^n$. □

3. Proof of Theorems 1.1, 1.8, 1.9, 1.13 and Corollaries 1.3, 1.11

Proof of Theorem 1.1. We define the function

$$w(x, t) = (u_1 - u_2)(x, t)$$

which satisfies the equation

$$\partial_t w - \partial_x D_x^\alpha w + \partial_x u_1 w + u_2 \partial_x w = 0, \quad x, t \in \mathbb{R}, \quad \alpha \in (-1, 2) - \{0\}. \tag{3.1}$$

From the hypothesis (1.8) and the Equation (3.1) it follows that

$$w(x, t) = D_x^\alpha \partial_x w(x, t) = 0, \quad (x, t) \in \Omega \subseteq \mathbb{R} \times [0, T]. \tag{3.2}$$

Hence, there exist $t_0 \in (0, T)$ and an open non-empty interval (a, b) such that $(a, b) \times \{t_0\} \subset \Omega$.

First, we observe by the Hardy-Littlewood-Sobolev inequality that if $\alpha \in (-1/2, 2)$ and $f(x) = \partial_x w(x, t_0)$, then $D_x^\alpha f$ is a well defined function in $L^p(\mathbb{R})$ with $p=2$ if $\alpha \in [0, 2)$ and with p such that $1/p = 1/2 + \alpha$ if $\alpha \in (-1/2, 0)$.

Therefore, applying Theorem B and Remark 1.5 to the function $\partial_x w(x, t_0)$ one obtains that $\partial_x w(x, t_0) = 0$ for any $x \in \mathbb{R}$ and consequently $w(x, t_0) = 0$ for any $x \in \mathbb{R}$ which yields the desired result in the case $\alpha \in (-1/2, 2)$.

Next, we consider the case $\alpha \in (-1, -1/2]$. By the assumptions (1.6) and (1.7) it follows that if $f(x) = \partial_x w(x, t_0)$, then $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Thus, $D_x^\alpha f \in L^q(\mathbb{R})$ for any $q \in (1/(1 + \alpha), 2/(1 + 2\alpha))$.

Hence, applying Theorem B and Remark 1.5 to the function $\partial_x w(x, t_0)$ one gets that $w(x, t_0) = 0$ for any $x \in \mathbb{R}$, which completes the proof. □

Proof of Corollary 1.3. By applying the argument in the proof of Theorem 1.1 to the Equation (1.2) it follows that $f(x) = \partial_x u(x, t_0) = 0$ for all $x \in \mathbb{R}^n$. Since $\partial_x u(\cdot, t_0) \in H^1(\mathbb{R})$ one gets the desired result. □

The proofs of Theorems 1.8 and 1.9 part (i) are similar, so they will be omitted.

Proof of Theorem 1.8 (ii). In this case we work directly with the general form of the equation in (1.20). By hypotheses $u_2 \equiv 0$ and from the Equation (1.20) it follows that in $\Omega \subset \mathbb{R}^n \times [0, T]$

$$\begin{aligned} u_1(x, t) &= (\mathcal{L}_m)^{\alpha/2} u_1(x, t) \\ &= (-\Delta + m^2)^{\alpha/2} u_1(x, t) = 0, \quad (x, t) \in \Omega. \end{aligned} \tag{3.3}$$

Thus, there exists $t_0 \in (0, T)$ and an open (non-empty) set $\Theta \subset \mathbb{R}^n$ such that $\Theta \times \{t_0\} \subset \Omega$.

Hence, applying [Theorem B](#) if $m=0$ and [Theorem 1.6](#) if $m>0$ to the function $u_1(x, t_0)$ one obtains that $u_1(x, t_0) = 0$ for any $x \in \mathbb{R}^n$, which yields the desired result. \square

The proof of [Theorem 1.9](#) (ii) is similar to those described above, so it will be omitted. This is also the case of [Corollary 1.11](#).

Proof of Theorem 1.13. By hypothesis for any $(x, y) \in \Theta$

$$u_1(x, y, t_0) = -(-\Delta)^{-1/2} \partial_y \theta(x, y, t_0) = 0 \text{ and } \partial_y \theta(x, y, t_0) = 0.$$

Therefore, using the extension of [Theorem B](#) with $\alpha = -1/2$, see [\[26\]](#) and [Remark 1.5](#), one has that

$$\partial_y \theta(x, y, t_0) = 0 \quad (x, y) \in \mathbb{R}^2.$$

Similarly, one gets that

$$\partial_x \theta(x, y, t_0) = 0 \quad (x, y) \in \mathbb{R}^2.$$

Since $\theta \in H^s(\mathbb{R}^2)$, $s > 2$, it follows that

$$\theta(x, y, t_0) = 0 \quad (x, y) \in \mathbb{R}^2.$$

Finally, using [\(1.28\)](#) we complete the proof. \square

Funding

C.E.K. was supported by the NSF grant DMS-1800082. D.P. was supported by a Trond Mohn Foundation grant. L.V. was supported by an ERCEA Advanced Grant 2014 669689 - HADE, by the MINECO projects PGC 2018-095422-B-100 and SEV-2017-0718, and by the Eusko Jauriaritza Grant IT1247-19.

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Appendix A

For $\alpha \in (-1, -1/2]$ we shall prove that if

$$u \in C([0, T] : H^s(\mathbb{R})) \cap C^1([0, T] : H^{s-1}(\mathbb{R})) \text{ with } s > 3/2 \quad (\text{A.1})$$

is a strong solution of the IVP associated to the Equation (1.2) and $x\partial_x u(x, 0) \in L^2(\mathbb{R})$ then

$$x\partial_x u \in L^\infty([0, T] : L^2(\mathbb{R})). \quad (\text{A.2})$$

First, we observe that if $f \in H^1(\mathbb{R})$, then

$$x\partial_x D^\alpha(\partial_x f) = -(1 + \alpha)D^\alpha(\partial_x f) + \partial_x D^\alpha(x\partial_x f). \quad (\text{A.3})$$

Therefore taking derivative of the Equation (1.2), multiplying the result by $x^2\partial_x u$ and integration in the x -variable one gets

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (x\partial_x u)^2(x, t) dx \\ &= \int x\partial_x D^\alpha \partial_x u \ x\partial_x u dx - \int x\partial_x(u\partial_x u)x\partial_x u dx \\ &= E_1(t) + E_2(t). \end{aligned} \quad (\text{A.4})$$

Integration by parts leads to the estimate

$$\begin{aligned} |E_2(t)| &\leq c \|\partial_x u(t)\|_\infty \int (x\partial_x u)^2(x, t) dx \\ &\quad + c \|u(t)\|_\infty \|\partial_x u(t)\|_2 \left(\int (x\partial_x u)^2(x, t) dx \right)^{1/2}. \end{aligned} \quad (\text{A.5})$$

Thus, using (A.3) we write

$$E_1(t) = -(1 + \alpha) \int D^\alpha \partial_x u \ x\partial_x u dx + \int \partial_x D^\alpha(x\partial_x u) \ x\partial_x u dx. \quad (\text{A.6})$$

Since $\partial_x D^\alpha$ is skew-symmetric the last term in (A.6) vanishes. Hence,

$$|E_1(t)| \leq c \|\partial_x D^\alpha u(t)\|_2 \left(\int (x\partial_x u)^2(x, t) dx \right)^{1/2}. \quad (\text{A.7})$$

Finally, inserting (A.5) and (A.7) in (A.4) and using the hypothesis (A.1) one gets the desired result (A.2).