Brauer Groups of Bielliptic Surfaces and Twisted Derived Equivalences

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Abstract

This thesis focuses on two interrelated projects. The first project concerns the study of bielliptic surfaces, their Brauer groups and the pullback maps from their Brauer groups to those of their canonical covers. We prove results classifying injectivity and triviality of these maps. In order to do this, we provide some results of a very classical flavor: first we give generators for the torsion of the second integral cohomology of bielliptic surfaces, and secondly we give structure theorems for the Picard group of the product of two elliptic curves.

The second project revolves around the study of the twisted derived category of bielliptic surfaces. We expose some of the structure of these derived categories, and prove that an untwisted bielliptic surface does not admit any twisted Fourier-Mukai partner. This is done utilizing the results of the first part and the geometry of moduli spaces of sheaves.

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Chapter 1

Introduction

By the celebrated Gabriel-Rosenberg reconstruction theorem ([Ros98]), a smooth projective variety X is completely determined up to isomorphism by its abelian category of coherent sheaves. By looking at chain complexes and inverting quasi-isomorphisms, we can construct the (bounded) derived category $D^b \; X$ of X. Generally accepted as the right framework for any type of derived functors and the like, this invariant was considered as a rather formal object initially. It wasn't until S. Mukai's original paper [Muk81] and subsequent ones, where geometrically motivated equivalences were constructed between non-isomorphic varieties, that one began to see the interesting and geometric internal structure of this object. More specifically, Mukai showed in [Muk81] that the Poincaré bundle over the product of an abelian variety and its dual, $A \quad \hat{A}$, gave an equivalence of A with $\hat{A}$ on the level of derived categories. The functor here is given by

$$: D^b \; \hat{A} \qquad D^b \; A$$

by the formula

$$\mathbf{R}p \qquad {}^{\mathbf{L}} q \qquad ,$$

where p and q are the natural projections from $A \quad \hat{A}$ and denotes the Poincaré bundle.

This naturally prompted the question and investigation of under which conditions two varieties would produce an equivalence of their derived categories, and, more generally, what geometric information the derived category could encode. As is clear from Mukai the derived category as an invariant is coarser and less rigid than the underlying category of coherent sheaves, but at the same time it turns out to be a rather reasonable invariant.

As an important and famous example, Bondal and Orlov showed in [BO01] that varieties with ample or anti-ample canonical bundle are completely determined by the derived category. This relies on the fact by Orlov that all equivalences come from geometry in the sense that they are of the same form as Mukai's equivalence above. That is, given an equivalence $: D^b \; Y \qquad D^b \; X$ between varieties X and Y, there is an object $D^b \; Y \quad X$ such that is isomorphic to $\mathbf{R}q \qquad {}^{\mathbf{L}} q$. These are then called Fourier-Mukai transforms, and X and Y are called Fourier-Mukai partners.

More generally, the dimension of a variety, order of the canonical bundle and nefness of the canonical bundle are all examples of derived invariants, that is of properties that do not change under derived equivalence. An idea and a heuristic in the study of derived equivalences is that it allows one to replace a problem about sheaves on one variety with another problem involving sheaves on a different variety. In some cases an equivalence of derived categories is produced, as in the case of surfaces, where we have the following result from Bridgeland and Maciocia:

Proposition 1.0.1 (Bridgeland-Maciocia). Let X be a smooth projective surface with a xed polarization, and let Y be a smooth, ne, complete, two-dimensional moduli space of special, stable sheaves on X. Then there is a universal sheaf on Y X and the associated functor $: D^b\ Y \quad D^b\ X$ is an equivalence.

Indeed, this result is used to great effect in the complete classification of surfaces in terms of derived equivalence.

Theorem 1.0.2 (Bridgeland-Maciocia). Let X be a smooth projective surface and Y a smooth projective variety. If $D^b\ X \quad D^b\ Y$, then $X \quad Y$ unless

X is a K3 or an abelian surface. Then Y is also a K3 or abelian surface, and it is isomorphic to a moduli space of sheaves on X.

X is an elliptic surface of Kodaira dimension 1. Then Y is isomorphic to a relative Jacobian $J_X\ b$ of X.

Moreover, the set of such Y s are nite.

The caveat in examples such as these is that the moduli spaces of sheaves considered are *fine*, i.e., there is a universal sheaf on the product that induces the equivalence. Relaxing this condition brings us, potentially, into the territory of *twisted sheaves*.

To motivate this, consider a moduli space M of semistable sheaves on a variety X. For M to fail being fine, it could be the case that some points of M represent more than one semistable sheaf on X. However, even if this is not the case, a universal sheaf may not exist. A reason for this is that, while universal sheaves exist locally, they fail to glue well along all of M. Informally, twisted sheaves can be considered as a bunch of local sheaves together with glueing isomorphism with the defect that they ”don’t quite match up’. And indeed, the failure of the local universal sheaves glueing properly along all of M makes them into such sheaves.

Being a little bit more precise (we will study this in more detail in 2.4), suppose we have a variety X and an element in the Brauer group of X (roughyl the same as the cohomology group $H^2\ X, \quad _X$, represented by a Čech 2-cocycle on some open (analytic if over , étale otherwise) covering U_i . An -twisted sheaf is then a collection

$$F_i\ , \quad _{ij}$$

consisting of sheaves F_i on U_i, together with isomorphisms

$$_{ij}: F_j\ _{U_i \quad U_j} \qquad F_i\ _{U_i \quad U_j}$$

satisfying the usual glueing conditions except over the triple intersections, where

$$_{ij} \quad _{jk} \quad _{ki} \qquad _{ijk}\ \mathrm{id}\,.$$

This condition is what we see as ”failing to glue properly”. Bringing this back to our moduli space, as observed by Căldăraru in his thesis, there is an analytic or étale open covering

U_i of M with local universal sheaves $_i$ over each X U_i together with isomorphism $_{ij}$: $_j\, U_i\ U_j$ $_i\, U_i\ U_j$ making the collection $_i$, $_{ij}$ into a $_M$ -twisted sheaf for an $H^2\ M,\ _M$.

Proceeding in general, one naturally turns these twisted sheaves into an abelian category Coh $X,$ of -twisted sheaves. As in the case of usual sheaves, this category completely determines X up to isomorphism, and one can take construct the (bounded) derived category $D^b\ X,$ in the same way as before. The question then becomes, what does this category entail? What does it encode of X?

Less is done and less is known in the case of twisted derived equivalences. Noteworthy to mention for our purposes is that all twisted equivalences are on the same geometric form as the untwisted ones due to Canonaco-Stellari, and the analogous classification of surfaces done by Navas in his thesis [Nav10], where he shows that some of Bridgeland and Maciocia's result, along with other general results, carry over to the twisted setting.

Proposition 1.0.3 (Navas). Let X be surface of general type and Y a smooth projective variety. If $D^b\ X,$ $D^b\ Y,$, then X Y.

For surfaces of Kodaira dimension 1, denote by $M\ v$ the moduli space of stable sheaves with Mukai vector v. Then

Proposition 1.0.4 (Navas). Let $: Y$ C be a relatively minimal elliptic surface with Kodaira dimension 1, and let $: D^b\ X,$ $D^b\ Y$ be an equivalence. Then there exists a Mukai vector v $0, rf, d$ such that gcd r, d 1 and X $M\ v$

For surfaces of Kodaira dimension there are no twistings possible (as the cohomological Brauer group is trivial), so the result remains. Other results carry over more or less verbatim such as invariance of dimension, order of canonical bundle, a lot of the internal structure, invariance of canonical rings among other things as demonstrated by Navas. The case of Enriques surfaces is covered by Addington and Wray in [AW18]. For K3 and abelian surfaces, much work has been done by Căldăraru, Huybrechts, Stellari et.al, but as of this writing and this authors knowledge, a complete result such as the one provided by Bridgeland-Maciocia together with a Twisted Derived Torelli Theorem is not fully settled. When twisted sheaves get involved, the framework surrounding these investigations typically change enough to make murkier waters.

Another missing type of surface in this classification are the bielliptic ones. A bielliptic surface S occurs as a quotient of the product A B of elliptic curves by a finite group G acting on A by translations and on B by automorphisms such that B/G ^{1}. In the Enriques-Kodaira classification they fill up the spot of minimal smooth surfaces with Kodaira dimension S 0, irregularity $q\ S$ 1 and geometric genus $p_g\ S$ 0. By Bridgeland-Maciocia, the derived category completely determines S up to isomorphism, and one could ask if this remains true in the twisted case. This is where we come in in the last chapter of this thesis, where we conjecture the following.

Conjecture 1.0.5. Complex bielliptic surfaces do not admit non-isomorphic twisted Fourier-Mukai partners.

The original approach to this problem relied on the lifting property of derived equivalences between varieties that are étale cyclically covered by another variety. Namely, given varieties X and Y with étale cyclic coverings $_X: \tilde{X}$ X and $_Y: \tilde{Y}$ Y, if

$: D^b(Y, \ ) \quad D^b(X, \ )$ is an equivalence, under mild conditions often realized in practice, there is an equivariant lifting $: D^b(\tilde{Y}, \ _Y) \quad D^b(\tilde{X}, \ _X)$ fitting in to a commuting diagram

$$\begin{array}{ccc} D^b(\tilde{Y}, \ _Y) & \longrightarrow & D^b(\tilde{X}, \ _X) \\ {}_Y \downarrow\uparrow {}_Y & & {}_X \downarrow\uparrow {}_X \\ D^b(Y, \ ) & \longrightarrow & D^b(X, \ ) \end{array} .$$

Since the order of the canonical bundle of a bielliptic surface S is finite, such an étale cyclic covering of degree equal to this order exists, and is called the canonical cover of S. To use this lifting property it becomes of interest to understand the induced pullback maps on Brauer groups: For a bielliptic surface S with canonical cover $: X \quad S$, the Brauer map is the induced map $_{\mathrm{Br}}: \mathrm{Br}(S) \quad \mathrm{Br}(X)$. Asking the properties of this map leads us into a jungle of its own, and is the content of the third chapter this thesis.

1.1 New results

Let us highlight some of the new results obtained in this thesis. The thesis is organized into three parts. The first part is a short preliminary part, exposing some of the theory of bielliptic surfaces, canonical covers, Brauer groups, the norm map and twisted sheaves. The second part involves the study of the aforementioned Brauer map $_{\mathrm{Br}}: \mathrm{Br}(S) \quad \mathrm{Br}(X)$ induced by the canonical cover $: X \quad S$. We characterize when this map is injective and when it is trivial. The first step in studying this map is to understand the Brauer group of S, which is isomorphic to the torsion of $H^2(S, \ )$. Bielliptic surfaces comes with two elliptic fibrations $a_S: S \quad A/G$ and $g: S \quad B/G \quad ^1$, and using the multiple fibers of g, we find the generators of the torsion subgroup.

Proposition 1.1.1. Let $S \quad A \quad B/G$ be a bielliptic surface. Denote by D_i the reduced multiple bers of $g: S \quad ^1$ with the same multiplicity. Then the torsion of $H^2(S, \ )$ is generated by the classes of di erences $D_i \quad D_j$ for $i \quad j$.

From here we base our investigation on a result of Beuville (Proposition 2.3.3) which describes the kernel of the Brauer map as the quotient $\mathrm{Ker}\,\mathrm{Nm} / (1 \quad) \mathrm{Pic}(X)$, where $\mathrm{Nm}: \mathrm{Pic}(X) \quad \mathrm{Pic}(S)$ is the norm map and is the induced action on X. It then becomes important to understand the action of on $\mathrm{Pic}(X)$. When $X \quad A \quad B$, $\mathrm{Pic}(X) \quad \mathrm{Pic}(A) \quad \mathrm{Pic}(B) \quad \mathrm{Hom}(B, A)$, and we show the following structure theorem for the Hom-part of $\mathrm{Pic}(X)$.

Theorem 1.1.2. Let A and B be two isogenous elliptic curves with $j(B) \quad 0$ or $j(B) \quad 1728$. Then there exists an isogeny $: B \quad A$ such that

$$\mathrm{Hom}(B, A) \qquad , \qquad _B .$$

This is used to give a good description of Neron-Severi group in the end of Section 3.2. Our investigation then proceeds to study the Brauer map proper, and we split the investigation into different parts according to the type of S (Table 2.1) and the properties of the elliptic curves A and B. Since the notation gets rather involved in some of our results, we will include some easier to state results here and refer the reader to Chapter 3 for the full account. First, considering bielliptic surfaces of type 2 or 3, these admit a degree 2 étale cover $\tilde{} : \tilde{S} \quad S$ where $\tilde{S}$ is a bielliptic surface of type 1. Then we have

Theorem 1.1.3. (a) If S is of type 2, then $\tilde{}_{\mathrm{Br}} : \mathrm{Br}\ S \quad \mathrm{Br}\ \tilde{S}$ is trivial.

(b) If S is of type 3, then $\tilde{}_{\mathrm{Br}} : \mathrm{Br}\ S \quad \mathrm{Br}\ \tilde{S}$ is injective.

As another example, the Brauer map of the general bielliptic surface is injective, as seen in the following result.

Theorem 1.1.4. Given a bielliptic surface S, let $: X \quad S$ be its canonical cover. If the two elliptic curves A and B are not isogenous, then the pullback map

$$_{\mathrm{Br}} : \mathrm{Br}\ S \quad \mathrm{Br}\ X$$

is injective.

In the third and final part of this thesis we study twisted derived equivalences of bielliptic surfaces, ending with a short informal discussion of where to proceed from there. Our main result here is the fact that an untwisted bielliptic surface does not admit any twisted Fourier-Mukai partner.

Theorem 1.1.5. Let X be a complex bielliptic surface, and let Y be a complex smooth projective variety, and take two Brauer classes and on X and Y respectively, such that there is an exact equivalence $: D^b\ X, \quad D^b\ Y,$. If either or is trivial then they are both trivial and furthermore X and Y are isomorphic.

Notation. We are working over the eld of the complex numbers . If X is a complex abelian variety over , and n , then $X\ n$ will denote the subscheme of n-torsion points of X, while $n_X : X \quad X$ will stand for the "multiplication by n isogeny". Given $x \quad X$ a point, then the translation by x will be denoted as t_x. In addition, if $\dim X \quad 1$ that is, X is an elliptic curve then P_x will be the line bundle $_X\ x \quad p_0 \quad t\ _x\ \ _X\ p_0 \quad _X \quad p_0$ in $\mathrm{Pic}^0\ X$, where $p_0 \quad X$ is the identity element.

For any smooth complex projective variety Y we will denote the identity homomorphism as 1_Y (or simply 1 if there is no chance of confusion), while K_Y and $_Y$ will stand for the canonical divisor class and the dualizing sheaf on Y, respectively. If D and E are two linearly equivalent divisors on Y we will write $D \quad E$; in addition $_Y\ D$ will denote the line bundle associated to the divisor D.

Chapter 2

Preliminaries

2.1 Bielliptic Surfaces

A complex *bielliptic* (or *hyperelliptic*) surface S is a minimal smooth projective surface over the field of complex numbers with Kodaira dimension $\kappa(S) = 0$, irregularity $q(S) = 1$, and geometric genus $p_g(S) = 0$. By the work of Bagenera– De Franchis (see for example [Bad13, 10.24-10.27]), the canonical bundle ω_S has order either $2, 3, 4$ or 6 in Pic(S), and S occurs as a finite étale quotient of a product $A \times B$ of elliptic curves by a finite group G acting on A by translations and on B such that $B/G \cong \mathbb{P}^1$. More precisely we have the following classification result.

Theorem 2.1.1 (Bagnera De Franchis [BDF10], [Suw69, Theorem at p. 473], [BM77, p. 37]). A bielliptic surface is of the form $S = (A \times B)/G$, where A and B are elliptic curves and G a finite group of translations of A acting on B by automorphisms. They are divided into seven types according to G as shown in Table 2.1.

Type	G	Order of ω_S in $Pic(S)$	$H^2(S, \mathbb{Z})_{\mathrm{tor}}$
1	$\mathbb{Z}/2$	2	$\mathbb{Z}/2 \times \mathbb{Z}/2$
2	$\mathbb{Z}/2 \times \mathbb{Z}/2$	2	$\mathbb{Z}/2$
3	$\mathbb{Z}/4$	4	$\mathbb{Z}/2$
4	$\mathbb{Z}/4 \times \mathbb{Z}/2$	4	0
5	$\mathbb{Z}/3$	3	$\mathbb{Z}/3$
6	$\mathbb{Z}/3 \times \mathbb{Z}/3$	3	0
7	$\mathbb{Z}/6$	6	0

Table 2.1: Types of bielliptic surfaces and torsion of their second cohomology.

There are natural maps $a_S \colon S \quad A/G$ and $g \colon S \quad B/G \quad ^1$ which are both elliptic fibrations. The morphism a_s is smooth, and coincides with the Albanese morphism of S. On the other hand, g admits multiple fibers, corresponding to the branch points of the quotient $B \quad B/G$, with multiplicity equal to that of the associated branch point. The smooth fibers of a_S and g are isomorphic to B and A, respectively. We will denote by a and b the classes of these fibers in Num S , $H^2 \; S,$ and $H^2 \; S,$.

It is well known (see for example [Ser90a, p. 529]) that a and b span $H^2 \; S,$ and satisfy $a^2 \quad b^2 \quad 0$, $ab \quad G$. Furthermore, we have the following description of the second cohomology of S:

Proposition 2.1.2. The decomposition of $H^2 \; S,$ is described according to the type of S and the multiplicities $m_1, \ldots, m_s$ of the singular bers of $g \colon S \quad ^1$ as follows:

Type	$m_1, \ldots, m_s$	$H^2 \; S,$	$H^2 \; S, \quad {}_{\text{tor}}$
1	$2,2,2,2$	$\frac{1}{2}a \quad b \quad /2 \quad /2$	$/2 \quad /2$
2	$2,2,2,2$	$\frac{1}{2}a \quad \frac{1}{2}b \quad /2$	$/2$
3	$2,4,4$	$\frac{1}{4}a \quad b \quad /2$	$/2$
4	$2,4,4$	$\frac{1}{4}a \quad \frac{1}{2}b$	0
5	$3,3,3$	$\frac{1}{3}a \quad b \quad /3$	$/3$
6	$3,3,3$	$\frac{1}{3}a \quad \frac{1}{3}b$	0
7	$2,3,6$	$\frac{1}{6}a \quad b$	0

Proof. See [Ser90a, Tables 2 and 3]. The computation of the torsion of $H^2 \; S,$ can be also found in [Iit70, Ser91, Suw69, Ume75]. □

Since $H^2 \; S, \quad {}_S \quad 0$, the first Chern class map $c_1 \colon$ Pic $S \quad H^2 \; S,$ is surjective, so the Néron-Severi group $NS \; S \quad H^2 \; S,$. Modulo torsion we then get

$$\text{Num } S \qquad a_0 \qquad b_0$$

where $a_0 \quad \frac{1}{\text{ord} \quad _S} a$ and $b_0 \quad \frac{\text{ord} \quad _S}{G} b$.

2.1.1 Canonical covers

In general, let X be a smooth projective variety and let $L \quad$ Pic X such that $L^n \quad {}_X$. Then there is a uniquely determined étale cyclic covering (see e.g. [Huy06, §7.3]) $\; : X \quad X$ of degree d such that

$$_X \quad \overset{d \;\; 1}{\underset{i \;\; 0}{}} \; L^{\; i},$$

$$L \qquad _X .$$

Moreover, there is a free action of the cyclic group $G \quad /n$ on Y such that $X \quad X/G$ and is the quotient map. More precisely, if we let $\quad : \quad {}_{i}^{i} \, {}_{0}^{n} \, {}^{1} L^{i}$, then X is the relative spectrum Spec and the action of $G \quad /n$ is given on by L.

For a bielliptic surface S, denote by n the order of its canonical bundle. Then by the above ${}_S$ induces an étale cyclic cover ${}_S : X \quad S$, called the *canonical cover of* S. From now on, when there is no confusion, we will omit the subscript S and write simply $: X \quad S$.

If we let ${}_S : \quad G \, / \, \mathrm{ord} \quad {}_S$, we have that $G \quad /n \quad / \, {}_S$, and X is the abelian surface sitting as an intermediate quotient

$$A \quad B \longrightarrow S \quad A \quad B/G$$
$$X$$

where $H \quad / \, {}_S$. The abelian surface X thus comes with homomorphisms of abelian varieties $p_A : X \quad A/H$ and $p_B : X \quad B/H$ with kernels isomorphic to B and A, respectively. Denoting by a_X and b_X the classes of the fibers A and B in Num X, we have $a_X \; b_X \quad {}_S$ and the embedding $: \mathrm{Num}\; S \quad \mathrm{Num}\; X$ satisfies

$$a_0 \quad a_X, \quad b_0 \quad \frac{n}{S} b_X. \tag{2.1}$$

There is a fixed-point-free action of the group $/n$ on the abelian variety X such that the quotient is exactly S. We will denote by $\quad \mathrm{Aut}\; X$ a generator of $/n$. In what follows it will be useful to have an explicit description of when S is of type 1, 2, 3, or 5.

Suppose first that S is of type 1, 3, or 5, so G is cyclic, H is trivial, and $X \quad A \quad B$. If S is of type 3, then the j-invariant of B is 1728, and B admits an automorphism $: B \quad B$ of order 4. If S is of type 5, B has j-invariant 0 and admits an automorphism of order 3 (see for example [BM77, p. 37], [Bad13, List 10.27] or [BHPvdV15, p. 199]). With this notation we have that the automorphism of $A \quad B$ inducing the covering is given by

$$x, y \quad \begin{cases} x \quad , \; y \, , & \text{if } S \text{ is of type 1,} \\ x \quad , \quad y \quad , & \text{if } S \text{ is of type 3,} \\ x \quad , \quad y \quad , & \text{if } S \text{ is of type 5,} \end{cases} \tag{2.2}$$

where , , and are points of A of order 2, 4, and 3 respectively. We remark that different choices for the automorphism and - there are two possible choices in each case- will lead to isomorphic bielliptic surfaces.

If S is otherwise of type 2, then there are points ${}_1 \quad A$ and ${}_2 \quad B$, both of order two, such that X is the quotient of $A \quad B$ by the involution $x, y \quad x \quad {}_1, y \quad {}_2$. If we denote by x, y the image of x, y through the quotient map, we have that

$$x, y \quad x \quad , \; y \, , \tag{2.3}$$

where $\quad A$ is a point of order 2, $\quad {}_1$.

2.1.2 Covers of bielliptic surfaces by other bielliptic surfaces

When G is not a cyclic group, or when G is cyclic, but the order of G is not a prime number, then the bielliptic surface S admits a cyclic cover $\tilde{} : \tilde{S} \quad S$, where $\tilde{S}$ is another bielliptic surface. This construction, together with the statement of Lemma 2.1.3, appears explicitly in the unpublished work of Nuer [Nue],and is implicit in the work of Suwa [Suw69, p. 475]. The main point that we will need in Section 3.4 is the description of the pull-back map Num S Num $\tilde{S}$.

Lemma 2.1.3. (i) Let S be a bielliptic surface such that ord $_S$ is not a prime number and take d a proper divisor of n. Then there is a bielliptic surface $\tilde{S}$ sitting as an intermediate étale cover between S and X,

$$X \xrightarrow{\ \tilde{S}\ } \tilde{S} \xrightarrow{\ \tilde{}\ } S \quad (\text{composite: } S)$$

such that ord $_{\tilde{S}}$ $\frac{\text{ord} \ _S}{d}$ and

$$\tilde{}\ a_0 \quad \tilde{a}_0, \ \tilde{}\ b_0 \quad d\tilde{b}_0,$$

where $\tilde{a}_0, \tilde{b}_0$ are the natural generators of Num $\tilde{S}$.

(ii) Let S be a bielliptic surface with $_S$ 1, i.e., with G not cyclic. Then there is a bielliptic surface $\tilde{S}$ sitting as an intermediate étale cover between S and A B

$$A \quad B \xrightarrow{\ \tilde{S}\ } \tilde{S} \xrightarrow{\ \tilde{}\ } S \quad (\text{composite: } S)$$

such that $_{\tilde{S}}$ 1, ord $_{\tilde{S}}$ ord $_S$ and

$$\tilde{}\ a_0 \quad _S\tilde{a}_0, \ \tilde{}\ b_0 \quad \tilde{b}_0,$$

where $\tilde{a}_0, \tilde{b}_0$ are the natural generators of Num $\tilde{S}$.

Proof. For (i) let ord $_X$ kd and $\tilde{}: \tilde{S}$ S be the cyclic covering of order d associated to ^k_S. Here $^k_{\tilde{S}}$ $\tilde{}$ ^k_S $_{\tilde{S}}$, and by looking at the table for bielliptic surfaces we see that k 2 or k 3, hence $6K_{\tilde{S}}$ 0 and $\tilde{S}$ 0. Since $_{\tilde{S}}$ is not trivial, $\tilde{S}$ is an Enriques or bielliptic surface. It cannot be Enriques, because taking the canonical cover of $\tilde{S}$ we get the canonical cover X of S by composition, and X is not a K3 surface. In short, if we let g be a generator of G/H $/n$, $\tilde{S}$ is the quotient $X/$ g^d .

For (ii), by the assumption $_S$ 1, S is of type 2, 4 or 6. For these types, the action of G on B may be described as

$$x \quad x, \ x \quad x \quad 2 \ \text{ with } 2 \quad 0,$$

$$x \quad ix, \ x \quad x \quad \frac{1 \quad i}{2},$$

$$x \quad e^{\frac{2 \ i}{3}}, \ x \quad x \quad \frac{1 \quad e^{\frac{2 \ i}{3}}}{3}.$$

Viewing G via its action on B as above, we can take $\tilde{G}$ to be the subgroup of G generated by 1, i or $e^{\frac{2 \ i}{3}}$, respectively. Then by [GH11, p. 589], $\tilde{S}$: A $B/\tilde{G}$ is a bielliptic surface of type 1, 3 or 5, respectively, and the map $_S$: A B S factors as required. □

In what follows we will need a more explicit construction of $\tilde{S}$, when S is either of type 2 or 3.

Example 2.1.4. (a) Suppose that S is a bielliptic surface of type 3. Then the canonical bundle has order 4. In addition the canonical cover X of S is a product of elliptic curves, that is $X \quad A \quad B$. Using the notation of (2.2), we obtain $\tilde{S}$ from $A \quad B$ by taking the quotient with respect to the involution $x, y \quad x \quad 2 \ , \quad y$. Thus we have that $\tilde{S}$ is a bielliptic surface of type 1. The map $\tilde{} : \tilde{S} \quad S$ is an étale double cover with associated involution $\tilde{}$. Hence, given $s \quad \tilde{S}$, we can see it as an equivalence class x, y of a point $x, y \quad A \quad B$. Then we have an explicit expression for $\tilde{}$:

$$\tilde{} \ s \qquad x \quad , \quad y \ . \tag{2.4}$$

(b) Suppose that S is a bielliptic surface of type 2, so the group G is isomorphic to the product $/2 \qquad /2$. Then we obtain $\tilde{S}$ from $A \quad B$ by taking the quotient with respect to $x, y \qquad x \qquad , \quad y$, where we are using the notation of (2.3). Thus, as in 2.1.4, $\tilde{S}$ is a bielliptic surface of type 1 and each $s \quad \tilde{S}$ can be written as an equivalence class x, y of a point $x, y \quad A \quad B$. If we denote again by $\tilde{}$ the involution induced by the cover $\tilde{} : \tilde{S} \quad S$, we have the following:

$$\tilde{} \ s \qquad x \quad {}_1, y \quad {}_2 \ . \tag{2.5}$$

2.2 Brauer Groups

In this section we define and introduce the Brauer group, a central player to come. To begin, let X be a scheme. Then the *cohomological Brauer group* $\mathrm{Br} \quad X$ is defined as the torsion part of the étale cohomology group $H^2_{et} \ X, \quad {}_X$. The exact sequence of sheaves

$$0 \qquad /n \qquad {}_X \ {}^n \qquad {}_X \quad 0$$

which yields the long exact cohomology sequence

$$\mathrm{Pic} \ X \quad {}^n \ \mathrm{Pic} \ X \quad {}^{c_1} \ H^2 \ X, \quad /n \qquad \mathrm{Br} \quad X \quad {}^n \ \mathrm{Br} \quad X \qquad 0$$

gives that the n-torsion part $\mathrm{Br} \quad X \ {}_n$ of $\mathrm{Br} \quad X$ fits into

$$0 \qquad \mathrm{Pic} \ X \qquad /n \qquad H^2 \ X, \quad /n \qquad \mathrm{Br} \quad X \ {}_n \qquad 0.$$

Taking the direct limit over all n gives the sequence

$$0 \qquad \mathrm{Pic} \ X \qquad / \qquad H^2 \ X, \quad / \qquad \mathrm{Br} \quad X \ {}_{tor} \qquad 0.$$

By [Gro66, 1.4] $\mathrm{Br} \quad X$ is torsion when X is a smooth scheme, and using that $H^2 \ X, \quad /$ and $\mathrm{Pic} \ X$ are the same in the analytic and the étale topology, we see from the above sequence that for complex varieties $\mathrm{Br} \quad X$ is isomorphic to the torsion of $H^2 \ X, \quad {}_X$ in the analytic topology. In addition, when X is quasi-compact and separated, by a theorem of Gabber (see, for example, [dJ]) for more details) the cohomological Brauer group of X is canonically isomorphic to the *Brauer group* $\mathrm{Br} \ X$ of Morita-equivalence classes of Azumaya algebras on X. For what it concerns the present thesis, we will only be concerned with smooth complex projective varieties, therefore all these three groups will be isomorphic and will be denoted simply by Br X . Furthermore we will only speak of the *Brauer group of* X, without any additional connotation.

Here are some examples of interpreting and calculating the cohomological Brauer group:

Example 2.2.1. From the exponential sequence we get the long exact sequence in cohomology

$$H^2 \; X, \qquad H^2 \; X, \quad _X \qquad H^2 \; X, \quad _X \qquad H^3 \; X, \qquad .$$

If $H^3 \; X,$ is torsion free, for example if X is a $K3$ surface or an abelian variety, then an element Br X goes to 0 in $H^3 \; X,$, hence comes from a class in $H^2 \; X, \; _X$. This allows us to consider a class in Br X as a class in $H^2 \; X, \; _X$ for which a certain positive integer multiple lands in $H^2 \; X,$.

Example 2.2.2. For a smooth complex projective curve X, one has Br X 0 from the exponential sequence above since $H^2 \; X, \; _X$ $H^3 \; X,$ 0.

Example 2.2.3. Let X be an Enriques surface, so that X is minimal with Kodaira dimension 0 and p_a p_g 0 with $2K$ 0. Any such X can be realized as the quotient of a K3-surface $\tilde{X}$ by a xed point free involution, so we have a degree 2 unbranched covering $\tilde{X}$ X. Since $\tilde{X}$ is simply connected, the fundamental group $_1 \; X$ of X is $/2$. Hence we have that $H_1 \; X,$ $/2$, which coincides by Poincaré duality with $H^3 \; X,$. Using the long exact sequence in cohomology induced by the exponential sequence, we get from p_g 0 that $H^2 \; X, \; _X$ $H^3 \; X,$ $/2$, so that Br X $/2$.

Example 2.2.4. If X is a surface with $H^2 \; X, \; _X$ 0, e.g., a bielliptic surface, then clearly Br X $H^3 \; X, \; _{tor}$ from the exponential cohomology sequence, which in turn coincides with the torsion of $H^2 \; X,$. However, it may be nontrivial to calculate this torsion. It is known for bielliptic surfaces by Proposition 2.1.2 , and depends on the type of bielliptic surface as can be seen in Table 2.1.

2.3 The norm map and a result of Beuville

In this section we expose the result of Beuville describing the kernel of the Brauer map in terms of the norm homomorphism in the case of cyclic coverings. The definition and properties of the norm homomorphism that we will need may or may not be well known to the reader, so we will state its construction and properties first. To this end, let $: X \quad Y$ be a finite locally free morphism of projective varieties of degree n. To it we can associate a group homomorphism Nm : Pic X Pic Y called the *norm homomorphism associated to* . This is constructed as follows. First, one lets $: \quad _X$, and defines a morphism of sheaves of multiplicative monoids $N : \quad _Y$: given s a section of on an open set U, let m_s be the endomorphism of U induced by the multiplication by s; we set $N \; s$: det m_s $_Y \; U$ (see [Gro61, § 6.4, and §6.5] or [Sta19, Lemma 0BD2]). The restriction of N to invertible sections induces a morphism of sheaves of groups $N : \quad _Y$. Now, given L an invertible sheaf on X, L is an invertible -module and, as such is represented by a cocycle u_{ij}, U_i for an open cover U_i of Y. Observe that u_{ij} U_{ij} . The fact that N is multiplicative ensures that also the v_{ij} : $N \; u_{ij}$ satisfies the cocycle condition and so uniquely identifies a line bundle Nm L on Y. The map L Nm L is a group homomorphism by [Gro61, (6.5.2.1)]. In addition [Gro61, (6.5.2.4)] ensures that

$$\mathrm{Nm} \quad M \quad M^{n}, \tag{2.6}$$

and we also have the following important property:

Proposition 2.3.1. Given two nite locally free morphisms $_1 : X \quad Y$ and $_2 : Y \quad Z$, then

$$\mathrm{Nm}_{2 \quad 1} \quad \mathrm{Nm}_{2} \; \mathrm{Nm}_{1}$$

Proof. See [Gro67, Lemma 21.5.7.2]. □

Suppose now that $: X \quad Y$ is an étale cyclic cover of degree n. Then there is a fixed-point-free automorphism $: X \quad X$ of order n such that $Y \quad X/$. In addition we can write $\quad {}^{n}_{h} \, {}^{1}_{0} M^{h}$ with M a line bundle of order n in Pic Y . In this particular setting the norm homomorphism satisfies some additional useful properties. First, as Nm behaves well with base change ([Gro61, Proposition 6.5.8]), it is not difficult to see that

$$\mathrm{Nm} \quad 1_X \qquad 0. \tag{2.7}$$

In additon, as discussed by Beauville in [Bea09], we have that

$$\mathrm{Nm} \quad L \quad \sum_{h \;\; 0}^{n} \quad n \quad 1^{\;h} \quad L \tag{2.8}$$

In fact, by the definiton of pushforward of divisors ([Gro67, Definition 21.5.5]), if L $_X \quad a_i \quad D_i$ with prime divisors on X, then Nm L $_Y \quad a_i \quad D_i$. Therefore (2.8) follows form the fact that for a prime divisor D we have that $D \quad {}^{n}_{h} \, {}^{1}_{0} \; {}^{h} \; D$.

Remark 2.3.2 (Pic^0 trick). In what follows it will be important to provide elements in the kernel of the Norm homomorphism. We will often use the following trick. Let $: X \quad Y$ be an étale morphism of degree n and suppose that there is a line bundle L on X such that Nm L Pic^0 Y . Then there is an element Pic^0 X such that Nm L is trivial. In fact, as abelian varieties are divisible groups, it is possible to nd Pic^0 Y such that ^{n} Nm L ^{1}. Then, by (2.6) we get

$$\mathrm{Nm} \quad L \qquad \mathrm{Nm} \quad L \qquad ^{n} \qquad _{Y}.$$

From now on, if there is no possibility of confusion, we will omit the subscript when denoting the norm. That is we will write Nm instead of Nm .

With the norm homomorphism explained, we can now state Beuville's result:

Proposition 2.3.3 ([Bea09, Prop. 4.1]). Let $: X \quad S$ be an étale cyclic covering of smooth projective varieties. Let be a generator of the Galois group of , Nm: Pic X Pic S be the norm map and $_{\mathrm{Br}}$: Br S Br X be the pullback. Then we have a canonical isomorphism

$$\mathrm{Ker} \quad _{\mathrm{Br}} \qquad \mathrm{Ker}\,\mathrm{Nm} \, / \; 1 \qquad \mathrm{Pic} \;\; X \; .$$

2.4 Twisted sheaves

Let X be a smooth projective variety. Using elements of the Brauer group properly interpreted as Čech 2-cocycles on an open covering of X, we can define the notion of twisted sheaves on X and consequently the twisted derived category of X.

De nition 2.4.1. A twisted variety $X,$ consists of a variety X together with a Brauer class Br X .

Let $X,$ be a twisted variety and represent Br X as a Čech 2-cocycle for an open analytic cover U_i $_{i}$ $_{I}$ by means of sections $_{ijk}$ U_i U_j $U_k,$ $_X$. A $_{ijk}$ -twisted sheaf (or simply a $_{ijk}$ -sheaf) F for the covering is a tuple F_i , $_{ij}$ where the F_i are $_{U_i}$-modules and $_{ij}$: F_j $_{U_i}$ $_{U_j}$ F_i $_{U_i}$ $_{U_j}$ are isomorphisms subject to the following conditions:

(i) $_{ii}$ id.

(ii) $_{ji}$ $_{ij}$ ^{1}.

(iii) $_{ij}$ $_{jk}$ $_{ki}$ $_{ijk}$ id on each triple intersection U_i U_j U_k.

We say that the $_{ijk}$ -sheaf F is coherent if all the local sheaves F_i are coherent. The class of $_{ijk}$ -twisted sheaves on X together with the obvious notions of homomorphism, kernel and cokernel yields an abelian category, denoted by Mod $X,$ $_{ijk}$, , the category of $_{ijk}$ -twisted sheaves on X. Restricting to coherent $_{ijk}$ -twisted sheaves, we get the category of coherent $_{ijk}$ -twisted sheaves on X, denoted Coh $X,$ $_{ijk}$, . As seen in the following two results, different choices of representative or coverings yields equivalent categories.

Lemma 2.4.2 ([C00, Lemma 1.2.3]). Let U_j $_j$ $_J$ be a re nement of the covering U_i $_i$ $_I$ on which can be represented. Then there is an equivalence of categories

$$\text{Mod } X, \quad _{ijk} , \qquad \text{Mod } X, \quad _{ijk} , \quad .$$

The way to go about this is to construct a natural refinment functor: Since is a refinement of , we have a map : J I such that for each j J, U_j U $_j$. If F is a $_{ijk}$ -sheaf along the covering , then the refinment of F to is given by

$$F \quad j \quad U_j \ , \qquad i \quad j \quad U_i \ U_j \quad ,$$

and we get our refinment functor

$$\text{Mod } X, \quad _{ijk} , \qquad \text{Mod } X, \quad _{ijk} , \quad .$$

On the other hand, if we fix the covering and choose another representative $_{ijk}$ for , there exists $_{ij}$ $_X$ U_i U_j such that $_{ijk}$ $_{ijk}$ ^{1} $_{ij}$ $_{jk}$ $_{ki}$. Sending a $_{ijk}$ -sheaf F_i , $_{ij}$ to the $_{ijk}$ -sheaf F_i , $_{ij}$ $_{ij}$ yields an equivalence Mod $X,$ $_{ijk}$, Mod $X,$ $_{ijk}$, . Thus we have:

Lemma 2.4.3 ([C00, Lemma 1.2.8]). If $_{ijk}$ and $_{ijk}$ represent the same element $\mathrm{Br}\ X$ along the open covering , the categories $\mathrm{Mod}\ X,\ _{ijk}$, and $\mathrm{Mod}\ X,\ _{ijk}$, are equivalent.

Remark 2.4.4. Note that this equivalence is non-canonical, as a di erent choice of 1-cochains $_{ij}$ yields a di erent equivalence, any two of which di ers by tensoring with a line bundle on X.

From these lemmas we can talk about twisted sheaves without specific mention of a Čech representative, so we shall simply speak of -twisted sheaves and denote their category by $\mathrm{Mod}\ X,$ and its coherent subcategory by $\mathrm{Coh}\ X,$, with the understanding that we really are considering an equivalence class of categories.

Before proceeding, we need to have a quick look at another way to look at these twisted sheaves in terms of their Azumaya algebra counterpart, as alluded to when we introduced the cohomological Brauer group of X. An *Azumaya algebra* on X is an associate, $_X$-algebra such that locally (in the analytic topology for our purposes, but also in the étale topology) it is isomorphic to the matrix algebra $M_n\ _X$ for some n. In particular, Azumaya algebras on X are locally free of some constant rank n^2. Two such Azumaya algebras are isomorphic if they are isomorphic as $_X$-algebras, hence by the Skolem-Noether theorem which gives the identification $\mathrm{Aut}\ M_n \quad PGL_n$, isomorphism classes of Azumaya algebras are in bijection with the set $H^1\ X, PGL_n$. We set the trivial Azumaya algebra of rank n^2 to be the algebra $\mathrm{End}\ E$ associated with any locally free sheaf E on X of rank n^2. This gives us an equivalence relation on Azumaya algebras as follows: Two Azumaya algebras $_1$ and $_2$ are *Morita-equivalent* if there exists locally free sheaves E_1 and E_2 on X such that $_1\ \mathrm{End}\ E_1 \quad _2\ \mathrm{End}\ E_2$ as Azumaya algebras.

Initially we defined the cohomological Brauer group of X to be $\mathrm{Br}\ X\ :\ H^2\ X,\ _X\ _{tors}$. The Brauer group $\mathrm{Br}\ X$ of X is defined to be the set of isomorphism classes of Azumaya algebras on X modulo Morita-equivalence. The relation between the two can be seen by considering the exact sequence

$$0 \quad _X \quad GL_n \quad PGL_n \quad 0$$

from which we get the long exact cohomology sequence

$$H^1\ X,\ _X \quad H^1\ X, GL_n \quad H^1\ X, PGL_n \quad ^n \quad H^2\ X,\ _X\ .$$

First, the maps $_n$ are always mapping into the torsion of $H^2\ X,\ _X$, as can be seen by an application of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & /n & \longrightarrow & SL_n & \longrightarrow & PGL_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & _X & \longrightarrow & GL_n & \longrightarrow & PGL_n & \longrightarrow & 0 \end{array} .$$

Hence $\mathrm{im}\ _n \quad \mathrm{Br}\ X$. Noting that the Morita-equivalence amounts to quotienting out by the image of the map $H^1\ X, GL_n \quad H^1\ X, \mathrm{PGL}_n$, the various maps $_n$ induces an injection $\mathrm{Br}\ X \quad \mathrm{Br}\ X$. This was conjectured by Grothendieck ([Gro68]) to be an isomorphism, and it is proven to be so for X quasi-compact and separated scheme. As in this paper we will only be working with smooth projective varieties, the aforementioned inclusion will indeed be an isomorphism.

To proceed to the twisted sheaves, we have the following which is essentially to rewrite the above correspondance Br X Br X :

Theorem 2.4.5 ([C00, Theorem 1.3.5]). Let be an Azumaya algebra over X, and let Br X be the corresponding class. Then there is a locally free -sheaf of nite rank, not necessarily unique, such that nd . Conversely, given any Br X such that there exists a locally free -sheaf of nite rank, nd is an Azumaya algebra whose class in Br X is .

Since Br X Br X via the maps $_n$, we see that in our case there will always exist such a locally free twisted sheaf of finite rank. Using this, one has a nice equivalence of categories

Theorem 2.4.6 ([C00, Theorem 1.3.7]). Let Br X and pick a locally free -sheaf of nite rank such that : nd is an Azumaya algebra whose class is . Let Coh X, be the category of coherent -modules, Then the functor F: Coh X, Coh X, de ned by

$$F \qquad {}_X$$

de nes an equivalence of categories with inverse G given by

$$G \qquad .$$

As an immediate little corollary we shall need later, we have:

Corollary 2.4.7. The order of divides the rank of any -sheaf.

As in the case of regular sheaves of modules, we can define tensor products, sheaf-homs, pushforward and pull-backs, as well as restriction and stalks in these new categories (see [C00, Proposition 1.2.10]):

- Let F F_i , $_i$ be an -sheaf and G G_i , $_i$ be an -sheaf, where we have chosen a cover of X that is refined to work for both F and G. We define F G to be the -sheaf given by the 'glueing' of F_i G_i along $_i$ $_i$.
- We define om F, G as an ^{1} -sheaf by 'glueing' the om F_i, G_i along the natural isomorphisms

  $$om \ F_i, G_i \quad {}^i \quad om \ F_i, G_j \quad {}^{i\ 1} \quad om \ F_j, G_j \ .$$

  For an -sheaf E we then define the dual E to be E : om E, $_X$ as usual, which is an -sheaf.

- For a morphism f: X Y and an -sheaf F on Y, we get the f -sheaf on X by taking f F_i , f $_{ij}$ on each f ^{1} U_i . The pushforward is a bit more subtle, and it is defined for an f -sheaf on X and yields an -sheaf on Y. The construction is roughly as follows: Choose an open cover U_i of Y for which is trivial along U_i for all i. Then f is trivial on f ^{1} U_i for all i as well. Let F be an f -sheaf on X. Then we can represent F as F_i , $_{ij}$ on f ^{1} U_i , and we take f F to be f F_i , f $_{ij}$ on U_i.

- For a point x X and an -sheaf F, we define the stalk F_x at X to be the stalk of F_i on U_i where U_i is a member of the open cover defining F and x U_i. Choosing another open U_j yields an isomorphic F_x because of the transition morphisms. We can then define the support supp F of a coherent -sheaf F to be the (closed) subset of X such that F_x 0, as usual.

- Given an -sheaf F on X and U X an open subset, we define the restriction F $_U$: i F where $i\colon U$ X is the inclusion. In concrete terms, given an open cover U_i of X for which is represented and F is given by F_i , $_{ij}$, we get that F $_U$ is the $_{ijk}$ U_{ijk} U -sheaf given by F_i U_i U , $_{ij}$ U_{ij} U .

Example 2.4.8. Taking a closed point x X and any class Br X , the skyscraper sheaf $_x$ on x is naturally an -sheaf on X by means of the injection $i\colon$ x X. This is because, since we are working over an algebraically closed eld, Br x 0, so $_x$ considered on x is an i 0 sheaf, so by the pushforward construction above we have that i k x is an -sheaf.

Lemma 2.4.9 ([C00, Lemma 2.1.1]). The category Mod $X,$ of -twisted sheaves has enough injectives for all Br X .

Chapter 3

Brauer Maps

3.1 Introduction

In this chapter we present the result of [BFTV19] which were obtained in collaboration with E. Ferrari and S. Tirabassi.

A morphism of projective varieties $f : Z \quad Y$ induces, via pullbacks, a homomorphism of the corresponding Brauer groups (see 2.2) $f_{\mathrm{Br}} : \mathrm{Br}\ Y \quad \mathrm{Br}\ Z$, which we call the *Brauer map induced by* f. In [Bea09] Beauville studies this map in the case of a complex Enriques surface S and that of its K3 canonical cover $\ : X \quad S$. More precisely the author of [Bea09] identifies the locus in the moduli space of Enriques surfaces in which $_{\mathrm{Br}}$ is not injective (and so trivial). Here we carry out a similar investigation in the case of bielliptic surfaces.

As seen in Section 2.1, the canonical bundle of a bielliptic surface S is a torsion element in Pic S , and therefore can be used to define an étale cyclic cover $\ : X \quad S$, where X is an abelian variety. We then obtain a homomorphism between the respective Brauer groups: $_{\mathrm{Br}} : \mathrm{Br}\ S \quad \mathrm{Br}\ X$. A very natural question is the following.

Question. When is $_{\mathrm{Br}}$ injective? When is it trivial?

As for Enriques surfaces, using the long exact exponential sequence, Poincaré duality and the universal coefficient theorem, we have a non-canonical isomorphism

$$\mathrm{Br}\ S \quad H^2\ S, \quad _{\mathrm{tor}},$$

so from the fourth column of Table 2.1, we easily see that this map is trivial when S is of type 4, 6 or 7. Thus we will limit ourselves to surfaces of type 1, 2, 3, and 5. We will find that the behavior of the Brauer map depends heavily on the geometry of the bielliptic surface S.

Our first step in this investigation is to focus on bielliptic surfaces of type 2 and 3. As we saw in 2.1.2, they admit a degree 2 étale cover $\tilde{\ } : \tilde{S} \quad S$, with $\tilde{S}$ a bielliptic surface of type 1 (see Examples 2.1.4 a) and 2.1.4 b) for more details). We investigate the properties of the induced Brauer map $\tilde{\ }_{\mathrm{Br}} : \mathrm{Br}\ S \quad \mathrm{Br}\ \tilde{S}$ finding how this behaves differently in the two cases:

Theorem A. (a) If S is of type 2, then $\tilde{\ }_{\mathrm{Br}} : \mathrm{Br}\ S \quad \mathrm{Br}\ \tilde{S}$ is trivial.

(b) If S is of type 3, then $\tilde{\ }_{\mathrm{Br}} : \mathrm{Br}\ S \quad \mathrm{Br}\ \tilde{S}$ is injective.

The main tool behind our argument is the result of Beauville in Section 2.3, stating that the kernel of the Brauer map of a cyclic étale cover $X \quad X/$ is naturally isomorphic to the kernel of the norm map Nm : Pic X Pic $X/$ quotiented by Im 1 , where is the induced action. We prove that a line bundle on $\tilde{S}$ is in the kernel of the norm map only if it is numerically trivial. Then we reach our conclusion by carefully computing the norm map of numerically trivial line bundles. The different behavior of the two type of surfaces is motivated by the different "values" taken by the norm map on torsion elements of H^2 $\tilde{S},$: in the type 2 case they are sent to topologically trivial line bundles, while this is not true in the type 3 case.

Aside from having its own interest, Theorem A, or more precisely some parts of its proof, will be useful in order to study the Brauer map to the canonical cover for bielliptic surfaces of type 2.

We then turn our attention to the main focus of this paper, and study the norm map to the canonical cover of a bielliptic surface. We give necessary and sufficient conditions for it to be injective, trivial, and, in the case of type 1 surfaces, neither trivial nor injective. This is done in Theorems 3.5.3, 3.5.9, 3.5.15, 3.5.20, and 3.5.22. Unfortunately the statements are particularly involved and it is not possible reproduce them here without a lengthy explanation of the notation used. An illustrative example of our results is the following.

Theorem B. Given a bielliptic surface S, let : X S be its canonical cover. If the two elliptic curves A and B are not isogenous, then the pullback map

$$_{\mathrm{Br}} : \mathrm{Br}\ S \qquad \mathrm{Br}\ X$$

is injective.

As an easy corollary one finds that the Brauer map of the general bielliptic surface is injective. The proof of Theorem B uses the same ideas of the proof of Theorem A. In fact we can leverage on the fact that X and S have the same Picard number (as it happened for the case of a bielliptic cover) to show that line bundles in the kernel of the norm map are topologically trivial. The result is then obtained by showing that line bundles in Pic^0 X which are also in the kernel of the norm map are always in Im 1 . As a corollary of both Theorem A and B we find an example of isogeny between two abelian varieties : X Y such that the corresponding group homomorphism $_{\mathrm{Br}}$ is not injective.

When the two curves A and B are isogenous, we see the first examples of bielliptic surfaces with a non injective Brauer map to the canonical cover.

This chapter is organized as follows. In Section 3.2 we give a description of the Neron–Severi group of a product of elliptic curves, using Section 3.2.1, which is a joint work of the second author of the main paper with J. Bergström. There a structure theorem for the homomorphism ring of two elliptic curves is given in the case of j-invariant 0 or 1728, which in turn gives a really useful description of the Picard group of the product of such curves, fundamental to study the Brauer map of bielliptic surfaces of type 3 and 5. In Section 3.3 we provide explicit generators for H^2 $S,$ $_{\mathrm{tor}}$, when S is a bielliptic surface of type 1, 2, 3 or 5. We prove Theorem A in Section 3.4, while we completely describe the norm map to the canonical cover in Section 3.5. Here we also construct examples of bielliptic surfaces of every type in which the Brauer map behaves differently.

3.2 The Neron–Severi of a product of elliptic curves

In this section we aim to describe $\mathrm{Num}(A \times B)$ when A and B are two elliptic curves. We will do so by using the identification of $\mathrm{Num}(X) \cong NS(X)$ which holds for abelian surfaces.

Our first step is to report the results of the Appendix of [BFTV19], written by J. Bergström and S. Tirabassi, whose main goal is to give a structure theorem for the homomorphism lattice of two elliptic curves. We will then, in Subsection 3.2.2 use the isomorphism

$$\mathrm{Pic}(A \times B) \cong \mathrm{Pic}(A) \oplus \mathrm{Pic}(B) \oplus \mathrm{Hom}(B, A)$$

in order to pick clever generators for the Neron–Severi group of the product of two elliptic curves. This in turn will allow an accurate description of the action of the automorphism on the Neron–Severi group of the product $A \times B$ when S is a bielliptic surface of type 3 or 5.

3.2.1 The homomorphism lattice of two elliptic curves

In this subsection we report the results of the Appendix of [BFTV19]. As said before, the main goal is to give a structure theorem for the $\mathbb{Z}$-module $\mathrm{Hom}(B, A)$ where A and B are two complex elliptic curves with $j(B) \neq 0, 1728$. If B is an elliptic curve with j-invariant 0 or 1728, then B admits an automorphism ρ_B of order 3 or 4 respectively. The main result is that the group $\mathrm{Hom}(B, A)$ can be completely described in terms of ρ_B and an isogeny $\varphi : B \to A$. More precisely we have the following statement:

Theorem 3.2.1. Let A and B two isogenous complex elliptic curves with $j(B)$ either 0 or 1728. Then there exists an isogeny $\varphi : B \to A$ such that

$$\mathrm{Hom}(B, A) = \langle \varphi, \varphi \circ \rho_B \rangle.$$

This subsection is organized in three main parts. In the first we outline some classical results about imaginary quadratic fields and their orders. The second is concerned with complex elliptic curves with complex multiplication. Theorem 3.2.1 is proven in 3.2.1.3. The key idea of our argument is to describe $\mathrm{Hom}(B, A)$ as a fractional ideal of $\mathrm{End}(B)$ homothetic to $\mathrm{End}(B)$. This is done by observing that the class number of $\mathrm{End}(B)$ is 1.

3.2.1.1 Preliminaries on orders in imaginary quadratic fields

An *imaginary quadratic field* is a subfield $K \subset \mathbb{C}$ of the form $\mathbb{Q}(\sqrt{-d})$, with d a positive, square-free integer. The *discriminant of* K is the integer d_k defined as

$$d_K = \begin{cases} -d, & \text{if } d \equiv 3 \mod 4, \\ -4d, & \text{otherwise.} \end{cases}$$

The *ring of integers of* K, $\mathcal{O}_K$ is the largest subring of K which is a finitely generated abelian group. Then we have that $\mathcal{O}_K = \mathbb{Z}[\omega]$, where

$$\omega = \begin{cases} \frac{1+\sqrt{-d}}{2}, & \text{if } d \equiv 3 \mod 4, \\ \sqrt{-d}, & \text{otherwise.} \end{cases} \tag{3.1}$$

An *order* in an imaginary quadratic field K is a subring of $_K$ which properly contains . It turns out that n for some positive integer n.

Given an order in an imaginary quadratic field K, a *fractional ideal* of is a non-zero finitely generated sub -module of K. For every fractional ideal M of there is an K and an ideal of such that M . We will need the following notions.

De nition 3.2.2. (i) Two fractional -ideals M and M are homothetic if there is K such that M M .

(ii) A fractional -ideal is invertible if there is a fractional ideal M such that M M . The set of invertible -ideals is denoted by I .

(iii) A fractional -ideal M is principal if it is of the form for some K . So principal ideals are precisely the fractional ideals homothetic to . The set of principal -ideals is denoted by P .

Principal ideals are clearly invertible. In general not all fractional ideals are invertible, but they are so if $_K$ (see also [Cox11, Proposition 5.7]). The quotient

$$: I /P$$

describes the homothety classes of invertible -ideals. It is a group with the product and it is called the *ideal class group of* . Its order is finite and is called the *class number of* . When $_K$, then the class number of is exactly the class number of the field K, which is a function of the discriminant of K (see [Cox11, Theorem 5.30(ii)]). More generally the class number of is a general function of d_K and $:$ $_K$.

Example 3.2.3. If K is either i or $\overline{3}$, then all the fractional ideals of $_K$ are homothetic to $_K$. In fact the class number of the eld K in this case is 1, as it was computed by Gauss in his book Disquisitiones arithmeticae .

3.2.1.2 Elliptic curves with complex multiplication

The importance of orders in the study of the geometry of elliptic curves is that they describe the endomorphism rings of a complex elliptic curves:

Theorem 3.2.4. Let A be an elliptic curve over , then End A is either isomorphic to or to an order in an imaginary quadratic eld.

Proof. See [Sil09, Theorem VI.5.5]. □

We say that a (complex) elliptic curve has *complex multiplication* if its endomorphisms ring is larger than . Observe that in this case End A is a quadratic field K and End A is an order in K.

Given a complex elliptic curve A there is a canonical way to identify its endomorphisms ring with a subring of . More generally let A and B two elliptic curves, then there are two lattices $_A$ and $_B$ in such that A $/$ $_A$ and B $/$ $_B$. Given a complex number such that $_B$ $_A$, the map $:$ defined by z z descends to an (algebraic) homomorphism $: B$ A. It is possible to show (see [Sil09, VI.5.3(d)]) that any morphism of elliptic curves preserving the origin is obtained in this way, and in particular we get an isomorphism of abelian groups

$$\mathrm{Hom}\ B, A \qquad {}_B \qquad {}_A \qquad . \tag{3.2}$$

By setting B A we get a ring isomorphism

$$\mathrm{End}\ A \qquad : \qquad {}_A \qquad {}_A \qquad .$$

The isomorphism is characterized as the unique isomorphism f : End A such that, for any and for every invariant form on A we have that f ([Sil13, II.1.1]).

Notation 3.2.5. For an elliptic curve with complex multiplication A such that End A n , we will denote by ${}_A$ the isogeny ${}_n$: A A and we will say that A has complex multiplication by ${}_A$.

It is clear that, with this identification, End A 1_A, ${}_A$ as a -module.

Example 3.2.6. (a) Suppose that B is an elliptic curve with j-invariant 0. Then we can write B / ${}_B$, with ${}_B$ $1, e^{\frac{2 i}{3}}$. Then End B $\overline{\ 3}$ and End B ${}_K$ $\frac{1 \overline{\ 3}}{2}$. We have that ${}_B$ is induced by the multiplication by $\frac{1 \overline{\ 3}}{2}$ and is an automorphism of B satisfying ${}^2_B$ ${}_B$ 1_B 0. This is exactly the automorphism which in 2.1.1 was denoted by and which was used to construct bielliptic surfaces of type 5.

(b) Suppose now that the j-invariant of B is 1728. Then we can take ${}_B$ $1, i$ and we have that End B i . The endomorphisms ring of B is isomorphic to i and the multiplication by i induces an automorphism ${}_B$ such that ${}^2_B$ 1_B. This is the automorphism of B used to construct bielliptic surfaces of type 3 in 2.1.1.

3.2.1.3 Proof of Theorem 3.2.1

We are now ready to provide a proof for Theorem 3.2.1. Our key point will be the following:

Claim: the -module Hom B, A is isomorphic to a fractional ideal of ${}_K$.

Before proceeding with showing that this Claim is true, let us see how it implies the statement. We do this applying Example 3.2.3 and deducing that all fractional ${}_K$-ideals are homothetic to ${}_K$. Therefore there exist K such that

$$M \qquad {}_K \qquad 1, \qquad , \qquad ,$$

where is like in (3.1). But then we have that Hom B, A , ${}_B$, and the statement is true.

Proof of the Claim. Let ${}_A$ $1,$ a lattice in such that A / ${}_A$, and denote by K the quadratic eld End B . Then the ring End B is exactly the ring of integers ${}_K$. Observe that this is isomorphic to a lattice in , and that B / ${}_K$ (See Example 3.2.6).

By (3.2) we can identify M : Hom B, A as a nitely generated subgroup of . Composition on the right with endomorphism of B gives to M a structure of ${}_K$-module.

Let 0 denote an element of : Hom A, B , identifyied with a subgroup of . Then clearly M $_K$. We deduce that M K is a fractional ideal of $_K$, and the Claim is proven. □

Remark 3.2.7. (a) For any order in a quadratic extension of a representative of each homothety class of fractional ideals can be given as I , where is an over order and I is an invertible fractional ideal (see [Mar18]). The over order S can be given a -basis of the form 1, f where f is a positive integer.

For any pair of isogenous complex elliptic curves with complex multiplication we have that Hom B, A is a fractional End B ideal. In addition, if we assume that End B has class number 1, we have that B / End B . In fact, under this assumption [Cox11, Corollary 10.20] yields that, there is just one elliptic curve up to isomorphism with endomorphism ring End B .

In conclusion, demanding that End B has class number 1 (instead of j B being either 0 or 1728) is su cient for Theorem 3.2.1.3 to hold.

So Theorem 3.2.1.3 will hold for the 13 isomorphism classes of complex elliptic curves B for which End B has discriminant -3, -4,-7, -8, -11, -12, -16, -19, -27, -28, -43, -67, -164 (see [Sil13, Example 11.3.2]).

(b) It is clear from the proof that the role of A and B can be exchanged, so we have proven a structure theorem for Hom A, B when the endomorphism ring of one of the two curves has class number 1.

Theorem 3.2.1.3 is not constructive, in the sense that it does not provide a way to determine the isogeny such that and $_B$ generate Hom B, A . In the next example we see how generators can be indeed constructed.

Example 3.2.8. Let be the lattice 1, 2i , and consider A : / . Consider the 2-torsion point : 0, i of A and let B be the quotient A/ . It is clear that B has j-invariant 1728. We claim that Hom B, A 2, 2 $_B$.

We use rst (3.2) and identify Hom B, A with a lattice in . Given a bi Hom B, A , we have that both and i must be elements of . We deduce that both a and b must be even integers and so Hom B, A 2, 2 i . We conclude by observing that $_B$ is the automorphism of B induced by multiplication by i.

3.2.2 The structure of the Neron Severi group of the product of two elliptic curves

We can now proceed with the discussion of the structure of the Neron–Severi group of the product of two elliptic curves. Our main goal is to provide generators that behaves well with the group action defining a bielliptic surface S. In this way we will be able to compute the norm map Nm : Pic A B Pic S . Let A be an elliptic curve over with identity element p_0, and be the lattice such that A / . Identify A with its dual and consider $_A$ the *normalized Poincaré bundle* on A A:

$$_A \quad _{A \ A \ A} \quad \mathrm{pr}_1 \ _A \ p_0 \quad \mathrm{pr}_2 \ _A \ p_0$$

where $_A$ A A is the diagonal divisor and $\mathrm{pr}_1, \mathrm{pr}_2$ are the projections of A A onto the first and second factor respectively. Observe that if x is a point in A, then the topologically trivial line bundle P_x is simply $_{A\ A}$ $_x$ $_A$ $_x$ $_A$.

Given another elliptic curve B, line bundles L_A and L_B on A and B respectively, and a morphism $: B$ A, we define a line bundle on the product A B

$$L\ L_A, L_B, \quad : \quad 1_A \qquad _A \quad \mathrm{pr}_A L_A \quad \mathrm{pr}_B L_B \tag{3.3}$$

where pr_A and pr_B are the projections onto A and B respectively. Recall the see-saw principle, stating that to check if two line bundles L and M on A B are isomorphic, it is enough to check that the restrictions L_a M_a for all a A, and L_b M_b for one b B. Using this on the case where $: B$ A is another homomorphism, one finds that

$$1_A \qquad _A \quad 1_A \qquad _A \quad 1_A \qquad _A,$$

and consequently that if M_A and M_B are two other line bundles on A and B, then

$$L\ L_A \quad M_A, L_B \quad M_B, \qquad L\ L_A, L_B, \qquad L\ M_A, M_B, \quad .$$

In addition, the universal property of the dual abelian variety ensures that every line bundle L Pic A B is of the form $L\ L_A, L_B,$ for some invertible sheaves L_A and L_B and a morphism . To see this, let L_A denote the restriction of L to A p_0 and L_B denote the restriction of L to p_0 B. Setting L : L $\mathrm{pr}_A L_A{}^1$ $\mathrm{pr}_B L_B{}^1$, note that the restriction of L to p_0 B is trivial, whilst the restrictions of L to A b is in Pic^0 B for all b B. Thus, invoking the universal property of the dual here there is a unique morphism $\tilde{}: B$ A such that

$$L \qquad 1_A \qquad ^1 \ \tilde{} \qquad _A$$

where is the isomorphism x $_A$ x p_0 used to identify A with its dual. Our desired morphism is then ^{1} $\tilde{}$. In all we therefore have an isomorphism

$$\mathrm{Pic}\ A \quad B \qquad \mathrm{Pic}\ A \qquad \mathrm{Pic}\ B \qquad \mathrm{Hom}\ B, A\ .$$

If we quotient by numerically trivial line bundles, we find that

$$H^2\ A \quad B, \qquad \mathrm{Num}\ A \quad B \qquad B \qquad A \quad \mathrm{Hom}\ B, A\ , \tag{3.4}$$

where A and B are the classes of the fibers of the two projections. Let us denote by l deg L_A ,deg L_B , the first Chern class of L $L_A, L_B,$. Then every class in Num A B can be written as l $m, n,$ for some integers n and m and an isogeny . In what follows we will often refer to line bundles (or numerical classes) in Hom B, A as elements of the *Hom-part* of Pic A B (or of Num A B . For our purposes it will be really important to pick explicit generators for Num A B to see how the automorphism acts on H^2 A $B,$. In order to do that, we need to investigate the -module structure on Hom B, A .

So suppose that there is a nontrivial isogeny $: B$ A. Then we know that Hom B, A has rank 1 if A does not have complex multiplication, and 2 otherwise (more details about elliptic curves with complex multiplication can be found in Section 3.2.1).

Suppose the first, so that there exists an isogeny $: B$ A such that l 0, 0, generates the Hom-part of H^2 A $B,$. We will call such isogeny a *generating isogeny for*

Num $A \times B$. Observe that, since $l(0,0,\phi)$ is necessarily a primitive class, ϕ cannot factor through any "multiplication by n" map. That is, we cannot write $\phi = n\psi$ for any n. In particular, for any integer n we have that Ker ϕ does not contain $B[n]$ as a subscheme.

Suppose now that A has complex multiplication, and again fix a non trivial isogeny ϕ : $B \to A$. Then also B has complex multiplication, and Hom(B, A) is a rank 2 free $\mathbb{Z}$-module. We pick generators ϕ_1 and ϕ_2, and we have that for any line bundle L on $A \times B$ there are two integers h and k such that

$$L \cong L(M_A, M_B, h\phi_1 + k\phi_2), \tag{3.5}$$

where M_A and M_B are element of Pic(A) and Pic(B) respectively. In addition we can write

$$H^2(A \times B, \mathbb{Z}) = \langle l(1,0,0), l(0,1,0), l(0,0,\phi_1), l(0,0,\phi_2) \rangle. \tag{3.6}$$

In the particular cases in which the j-invariant of B is either 0 or 1728, then Theorem 3.2.1 yields a more accurate description. In fact, if we denote by $\rho_B : B \to B$ the automorphism ρ or ω (see again the Subsection 3.2.1 or Paragraph 2.1.1), we have that there exist an isogeny $\phi : B \to A$ such that, in (3.5) and (3.6) we can take $\phi_1 = \phi$ and $\phi_2 = \phi \circ \rho_B$. So we have that

$$H^2(A \times B, \mathbb{Z}) = \langle l(1,0,0), l(0,1,0), l(0,0,\phi), l(0,0,\phi \circ \rho_B) \rangle. \tag{3.7}$$

In this case we say that ϕ is again a *generating isogeny* for $H^2(A \times B, \mathbb{Z})$. Observe again the isogenies ϕ_i, as well as ϕ, cannot factor through the multiplication by an integer or they could not generate the whole Hom(B, A).

3.3 Generators for the torsion of the second cohomology for bielliptic surfaces

In this section we give explicit generators for the torsion of $H^2(S, \mathbb{Z})$ in terms of the reduced multiple fibers of the elliptic fibration $g : S \to \mathbb{P}^1$. More precisely we will prove the following statement:

Proposition 3.3.1. Let $S = (A \times B)/G$ be a bielliptic surface. Denote by D_i the reduced multiple fibers of $g : S \to \mathbb{P}^1$ with the same multiplicity. Then the torsion of $H^2(S, \mathbb{Z})$ is generated by the classes of differences $D_i - D_j$ for $i \neq j$.

The reader who is familiar with the work of Serrano might find similarities between the above statement and Serrano's description of the torsion of $H^2(X, \mathbb{Z})$ when there is an elliptic fibration $\pi : X \to C$ with multiple fibers (cfr. [Ser90b, Corollary 1.5 and Proposition 1.6]). However in [Ser90b] it is used the additional assumption that $h^1(X, \mathcal{O}_X) = h^1(C, \mathcal{O}_C)$. This clearly does not hold in our context.

Before proving Proposition 3.3.1 we need two preliminary Lemmas.

Lemma 3.3.2. Let $g : S \to \mathbb{P}^1$ be an elliptic pencil with connected fibers. Let D_1 and D_2 be two reduced multiple fibers. Let m_1 and m_2 be the corresponding multiplicities. Then, for all non negative integers n,

$$D_1 \neq nD_2. \tag{3.8}$$

Proof. The statement is obvious for n 0, so one has to check for n 0. By contradiction, assume D_1 nD_2, and let F be the generic ber of g. Then

$$h^0\ S,\ \ _S\ F \qquad h^0\ \ ^1, g\ \ _S\ F$$
$$h^0\ \ ^1,\ \ _1\ 1\ \ g\ \ _S$$
$$h^0\ \ ^1,\ \ _1\ 1\ \ 2.$$

Since $h^0\ S,\ _S\ D_1$ $h^0\ S,\ _S\ m_1 D_1$ $h^0\ S,\ _S\ F$, it follows that $h^0\ S,\ _S\ D_1$ 2.

The absurd hypothesis is used here: if D_1 nD_2, then, since the supports of D_1 and D_2 are disjoint, $H^0\ S,\ _S\ D_1$ has at least two independent sections, and therefore the dimension of $H^0\ S,\ _S\ D_1$ is 2. Thus, since D_1^2 0 implies that there are no basepoints (see for example [Bea96, II.5]), the map is actually a morphism $_{D_1}: S$ ^{1}. Note that both D_1 and nD_2 are bers of this morphism.

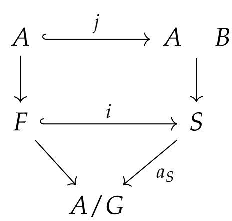
Let now C be the generic ber of (which is irreducible by semicontinuity). Since $C\ D_1$ 0, one gets $C\ F$ 0 for any ber F of g. This implies that g and $_{D_1}$ have the same generic ber. So one can write C F for a ber F of g. But then

$$D_1 \quad F \quad m_1 D_1,$$

which in turn implies that $_S\ D_1$ $^{m_1\ \ 1}$ $_S$, which is a contradiction. □

Lemma 3.3.3. Let S A B/G be a bielliptic surface with its brations $f\colon S$ A/G and $g\colon S$ ^{1}. Let D_1 and D_2 be two reduced multiple bers of g. Then the restriction of $_S\ D_1$ D_2 to the generic ber of a_S is trivial.

Proof. Let F g ^{1} p be a smooth ber of g. Here p is the orbit G y of a point y B not xed under any element of G. We will choose an embedding of A into S via an isomorphism $: A$ F such that we get a commutative diagram



where i is just the natural inclusion of the ber F into S and is the quotient map. To this end we let $: A$ F be the isomorphism a G x, y and j be the embedding a x, y . Since the multiple bers D_i are images of A y_i , i $1, 2$, where the y_i B are points xed under a subgroup of G of order equal to the multiplicity of D_i, we have that $_S\ D_1$ D_2 p $_B\ y_1$ y_2 where p_B is the projection A B B and y_1, y_2 B are the points corresponding to D_1, D_2, respectively. Then

$$i \quad _S\ D_1 \quad D_2 \qquad j \quad _S\ D_1 \quad D_2$$
$$j\ p_B \quad _B\ y_1 \quad y_2$$

As p j is the constant map we have that this is clearly trivial. Hence i $_S\ D_1$ D_2 is trivial, and since is an isomorphism we deduce the statement. □

For the remainder, we identify F and A via the isomorphism defined in the proof above. So we get the following commutative triangle.

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & S \\ & \searrow \quad \swarrow_{a_S} & \\ & A/G & \end{array} \tag{3.9}$$

Note that is an isogeny of degree G . In particular we have also that the dual isogeny $: \mathrm{Pic}^0\ S \quad \mathrm{Pic}^0\ A$ has degree G (see, for example [BL13, Proposition 2.4.3]).

With these observations we are now ready to start proving Proposition 3.3.1. We first remark that by the canonical bundle formula for elliptic fibrations (see e.g. [Bad13, Thm. 7.15]) applied to $g : S \quad {}^1$ we can write

$${}_S \quad g \quad {}_1 \quad 2 \qquad {}_S \quad \underset{k}{} \quad m_k \quad 1 \ D_k$$

where the D_k are the multiple fibers of g of multiplicity m_k. Choosing points p, q on ^{1} giving rise to the fibers $m_i D_i$ and $m_j D_j$ we get that

$$K_S \qquad D_i \quad D_j \qquad \underset{k \quad i,j}{} \quad m_k \quad 1 \ D_k. \tag{3.10}$$

Since ${}_S$ is a nontrivial element in $\mathrm{Pic}^0\ S$ we conclude that the classes of $D_i \quad D_j$ and ${}_{k \quad i,j}\ m_k \quad 1\ D_k$ coincide in $H^2\ S,$. Moreover, we observe that K_S restricts trivially to A, so ${}_S$ yields a nontrivial element in Ker . Note that if D_i and D_j have the same multiplicity m, the difference $D_i \quad D_j$ induces a (possibly trivial) torsion element in $H^2\ S,$ of order m. We prove Proposition 3.3.1 by showing that a sufficient number of these is nontrivial so to generate the torsion of $H^2\ S,$. We proceed by a case by case analysis, studying separately bielliptic surfaces of type 1, 2, 3, and 5. The key point in the argument is the observation that, if $D_i \quad D_j$ is trivial, then the line bundle ${}_S\ D_i \quad D_j$ belongs to $\mathrm{Pic}^0\ S$. In addition, using Lemma 3.2 and the diagram (3.9), we would have that ${}_S\ D_i \quad D_j \qquad {}_S$, in particular ${}_S\ D_i \quad D_j \qquad$ Ker , while Lemma 3.3.3 ensures that ${}_S\ D_i \quad D_j$ cannot be ${}_S$. A closer study of the structure of Ker $\hat{G}$ will bring us to the desired conclusion.

3.3.1 Type 1 bielliptic surfaces

In this case we have that Ker is the reduced group scheme $/2$ and the fibration $g : S \quad {}^1$ has four multiple fibers all of multiplicity 2. Hence, up to reordering the indices (3.10) yields

$$K_S \quad D_i \quad D_j \quad D_k \quad D_l. \tag{3.11}$$

In particular, as the canonical divisor is algebraically equivalent to 0, for distinct indices i, j, k, and l we have that $D_j \quad D_i$ is algebraically equivalent to $D_k \quad D_l$. Thus we get three classes in $H^2\ S,$

$$\begin{aligned} D_1 \quad D_2 &\qquad D_1 \quad D_2,\ D_3 \quad D_4\ , \\ D_1 \quad D_3 &\qquad D_1 \quad D_3,\ D_2 \quad D_4\ , \\ D_1 \quad D_4 &\qquad D_1 \quad D_4,\ D_2 \quad D_3\ , \end{aligned} \tag{3.12}$$

which a priori are neither distinct nor nontrivial. Since $H^2\ S,\quad_{tors}$ is isomorphic to the Klein 4-group, we need to show that they are indeed different classes and are not zero. Note that, if two classes are equal, since they both are 2-torsion and the third classes is clearly equal to the sum of the first two, then the remaining class would be trivial. Thus it will be enough to show that for any two distinct indices the divisor $D_i \quad D_j$ is not algebraically equivalent to 0. Suppose otherwise that for some indices we have that $_S\ D_i \quad D_j \qquad \mathrm{Pic}^0\ S$, then (3.11) would imply that also $_S\ D_k \quad D_l$ would be in $\mathrm{Pic}^0\ S$. The above discussion yields that both $_S\ D_i \quad D_j$ and $_S\ D_i \quad D_j$ are nontrivial elements of Ker , which has only one nontrivial element, $_S$. Then we can write

$$_S \qquad _S\ D_i \quad D_j \qquad _S\ D_k \quad D_l \qquad _S{}^2 \qquad _S,$$

which brings a contradiction, and thus we may conclude.

3.3.2 Type 2 bielliptic surfaces

Here $H^2\ S,\quad_{tors} \qquad /2$, Ker $\qquad /2 \qquad /2$ and like in the previous case there are four multiple fibers, each of multiplicity 2. As above we get the three classes induced by $D_1 \quad D_2$, $D_1 \quad D_3$ and $D_1 \quad D_4$, and we want to show that they cannot be all trivial. Suppose that two of these classes, say $D_1 \quad D_2$ and $D_1 \quad D_3$, are trivial in $H^2\ S,\quad$. For $i \quad 2, 3$ set $L_i :\quad _S\ D_1 \quad D_i$ and $M_i :\quad _S\ D_i \quad D_4$, then the L_i's and the M_i's determine nontrivial elements of Ker , which has only three nonzero elements. We deduce that some of these must be the same line bundle. The only option which would not contradict Lemma 3.3.3 would be that $L_i \quad M_j$ for some $i \quad j$. But then we would have

$$_S \quad L_i \quad M_j \quad L_i{}^2 \qquad _S,$$

which would be a contradiction. Hence at most one of the three classes can be trivial, and indeed one is actually trivial because the two nontrivial classes must coincide, implying the third is trivial.

3.3.3 Type 3 bielliptic surfaces

Here $H^2\ S,\quad_{tors} \qquad /2$ and Ker $\qquad /4$, but now we have two fibers of multiplicity 4 and one of multiplicity 2. Denote by E the reduced multiple fiber of multiplicity 2 and by D_1, D_2 the reduced multiple fibers of multiplicity 4. By the canonical bundle formula, we get

$$K_S \quad E \quad D_1 \quad D_2.$$

Then in $H^2\ S,\quad$ we have the following equalities

$$E \quad 2D_1 \qquad D_2 \quad D_1\ , \quad \text{and} \quad E \quad 2D_2 \qquad D_1 \quad D_2\ .$$

We need to show that they are not both trivial. Suppose by contradiction they are both zero in $H^2\ S,\quad$, then, as before we have that $_S\ E \quad 2D_1$ and $_S\ E \quad 2D_2$ are non trivial elements of Ker . Since both these line bundles have order two in Pic S , and Ker has only one element of order 2, we deduce that

$$_S\ E \quad 2D_1 \qquad _S\ E \quad 2D_2\ .$$

But then

$$_S{}^2 \quad _S E \quad D_1 \quad D_2 \quad ^2 \quad _S E \quad 2D_1 \quad _S E \quad 2D_2 \quad _S E \quad 2D_1 \quad ^2 \quad _S$$

which is impossible because $_S$ is of order 4. Therefore $E \quad 2D_1$ and $E \quad 2D_2$ induce the same nontrivial torsion element of $H^2 \; S,$.

3.3.4 Type 5 bielliptic surfaces

Here $H^2 \; S, \quad _{tors} \quad /3$, Ker $\quad /3$ and there are three multiple fibers, each of multiplicity 3. By the canonical bundle formula, we get

$$K_S \quad D_i \quad D_j \quad 2D_k \quad D_k \quad D_i \quad D_k \quad D_j \; .$$

Again, K_S is algebraically equivalent to zero, so we get that $D_k \quad D_i \quad D_j \quad D_k$ in $H^2 \; S,$. Running through the indices we get the two classes

$$D_1 \quad D_2 \quad D_1 \quad D_2, \; D_3 \quad D_1, \; D_2 \quad D_3 \; ,$$
$$D_1 \quad D_3 \quad D_1 \quad D_3, \; D_3 \quad D_2, \; D_2 \quad D_1 \; .$$

We need to show that they are distinct and both nontrivial. Observe that if they were the same class then both classes would be trivial, so it is enough to show that they are not the zero class. Again suppose by contradiction that $D_k \quad D_i \quad 0$ in $H^2 \; S,$, then we can write

$$_S \quad _S \; D_1 \quad D_2 \quad _S \; D_1 \quad D_3 \; ,$$

with $_S \; D_1 \quad D_2$ and $_S \; D_1 \quad D_3$ for nontrivial elements in Ker . Neither $_S \; D_1 \quad D_2$ nor $_S \; D_1 \quad D_3$ can be isomorphic to the canonical bundle $_S$, or we would have $_S \; D_k \quad D_i \quad _S$, contradicting Lemma 3.3.3. As Ker has only two nontrivial elements, we necessarily have

$$_S \; D_1 \quad D_2 \quad _S \; D_1 \quad D_3$$

and so $_S \; D_2 \quad D_3 \quad _S$, which contradicts Lemma 3.3.3 again, thus we can conclude.

3.4 The Brauer map to another bielliptic surface

Let S be a bielliptic surface of type 2 or 3. Then by Example 2.1.4 there is a 2:1 cyclic cover $\tilde{} : \tilde{S} \quad S$, where $\tilde{S}$ is a bielliptic surface of type 1. As in paragraph 2.1.2 , we will denote by $\tilde{}$ the involution induced by $\tilde{}$. In this section we are concerned with studying the Brauer map $\tilde{}_{\mathrm{Br}} : \mathrm{Br} \; S \quad \mathrm{Br} \; \tilde{S}$. Surpisingly we reach two antipodal conclusions, depending on the type of the bielliptic surface in object.

Recall that, as $\tilde{S}$ is a bielliptic surface of type 1, the elliptic fibration $q_B : \tilde{S} \quad ^1$ has four multiple fibers $D_1, \ldots, D_4$ of multiplicity 2, corresponding to the four 2-torsion points of B. We will denote by $_{ij}$ the line bundle $_{\tilde{S}} \; D_i \quad D_j$.

3.4.1 Bielliptic surfaces of type 2

Suppose that S is of type 2, and note that the involution $\tilde{}$ acts on the set of the D_i's by exchanging them pairwise. Up to relabeling we can assume that $\tilde{}$ D_1 D_2 and $\tilde{}$ D_3 D_4. By (2.8), we therefore have that

$$\tilde{} \quad \mathrm{Nm} \quad {}_{13} \quad {}_{13} \quad \tilde{} \quad {}_{13} \quad {}_{13} \quad {}_{24} \quad {}_{\tilde{S},} \tag{3.13}$$

where the last equality is a consequence of (3.11).Thus, if we denote by the generator of Ker $\tilde{}$, we get that

$$\mathrm{Nm} \quad {}_{13} \quad {}_{S,} \quad {}_{S} \quad \mathrm{Pic}^0 \; S \; .$$

Then we can use the Pic^0 trick (Remark 2.3.2) and find a Pic^0 S such that Nm $\tilde{}$ ${}_{13}$ is trivial.

Lemma 3.4.1. In the above notation, the line bundle $\tilde{}$ ${}_{13}$ does not belong to the image of 1 $\tilde{}$

Before proceeding with the proof, let us note that since S is of type 2, the Brauer group is isomorphic to $/2$, and using Beuville's result 2.3.3 with the nontrivial element $\tilde{}$ ${}_{13}$ in the quotient we get immediately the following corollary:

Corollary 3.4.2. If S is of type 2, then the induced map ${}_{\mathrm{Br}}$: Br S Br $\tilde{S}$ is trivial.

Proof of Lemma 3.4.1. We will show that the class of ${}_{13}$ in H^2 $\tilde{S},$ is not in the image of 1 $\tilde{}$. Denote by ${}_{ij}$ the algebraic equivalence class of the line bundle ${}_{ij}$. Then, by Proposition 2.1.2 and (3.12), for every L in Pic $\tilde{S}$ there are integers n, m, and h, and k such that

$$c_1 \; L \quad \frac{n}{2} \; a \quad m \; b \quad h \quad {}_{13} \quad k \quad {}_{14}$$

Since $\tilde{}$ exchanges the D_i pairwise, we have that $\tilde{}$ ${}_{13}$ ${}_{24}$ and $\tilde{}$ ${}_{14}$ ${}_{23}$. But from (3.12) ${}_{24}$ ${}_{13}$ and ${}_{23}$ ${}_{14}$, and clearly $\tilde{}$ keeps the classes a and b xed, hence

$$1 \quad \tilde{} \quad c_1 \; L \quad 0.$$

But on the other side we have that c_1 $\tilde{}$ ${}_{13}$ ${}_{13}$, which is not trivial, thus $\tilde{}$ ${}_{13}$ cannot possibly lie in the image of 1 $\tilde{}$, and the lemma is proved. □

3.4.2 Bielliptic surface of type 3

In this paragraph we aim to show the following statement

Theorem 3.4.3. If S is a bielliptic surface of type 3, then the Brauer map $\tilde{}_{\mathrm{Br}}$: Br S Br $\tilde{S}$ induced by the cover $\tilde{}$: $\tilde{S}$ S, where $\tilde{S}$ is bielliptic of type 1, is injective.

We will use 2.3.3 and show that Ker Nm / Im 1 is trivial. There are two main key steps:

1. We first study the norm map when applied to numerically trivial line bundles;
2. then we prove that all the line bundles L in Ker Nm are numerically trivial.

3.4.2.1 Norm of numerically trivial line bundles

We will use the notation of Example 2.1.4. Observe that we have the following diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\;\tilde{}\;} & S \\ \downarrow{\scriptstyle a_{\tilde{S}}} & & \downarrow{\scriptstyle a_S} \\ A/G & \longrightarrow & A/H, \end{array} \tag{3.14}$$

Where G $/2$ and H is $/4$.

Remark 3.4.4. Note that the bottom arrow, , is an isogeny of degree 2. As the vertical arrows are the Albanese maps of $\tilde{S}$ and S respectively, we have that $\tilde{}$: Pic^0 S Pic^0 $\tilde{S}$ coincides with the isogeny dual to . In particular it is surjective.

Our first step in the study of the norm homomorphism for numerically trivial line bundles is to see how it behaves when applied to the generator of the torsion of H^2 $\tilde{S},$. In order to do that, we remark that the automorphism acts on B 2 with at least one fixed point, the one corresponding to the identity element of B. Since has order 4, it cannot act transitively on the remaining three points on B 2 . Thus the action has at least two fixed points. We deduce that $\tilde{}$ acts on the set of the reduced multiple fibers by leaving fixed at least two of them, let us say D_1 and D_2. If the action were trivial, then we would have that all the line bundles $_{ij}$ are invariant under the action of $\tilde{}$ and as a consequence they would be pullbacks of line bundles coming from S. We would deduce that all the torsion classes of H^2 $\tilde{S},$ are pullbacks of classes from H^2 $S,$, which is impossible. Thus we know that $\tilde{}$ exchanges D_3 and D_4. Then we can prove the following Lemma.

Lemma 3.4.5. Let n and m be two integers. Then the norm of the line bundle $_{13}{}^n$ $_{14}{}^m$ is zero if and only if n and m have the same parity. In addition we have that Nm $_{13}{}^n$ $_{14}{}^m$ is not in Pic^0 S if n and m are not congruent modulo 2.

Proof. Observe rst of all that, thanks to the above discussion, the line bundle $_{34}$ $_{13}$ $_{14}$ is invariant with respect to the action of $\tilde{}$. In particular we can write $_{34}$ $\tilde{}$ where is a line bundle on S whose algebraic equivalence class is the only nontrivial class in H^2 $S,$.

Now, if n and m are both even, then $_{13}{}^n$ $_{14}{}^m$ is the trivial line bundle, and there is nothing to prove. Otherwise, if n and m are odd, then

$$\mathrm{Nm} \quad {}_{13}{}^{n} \quad {}_{14}{}^{m} \quad \mathrm{Nm} \quad {}_{34} \qquad {}^{2} \qquad {}_{S}.$$

Conversely suppose that n and m are not congruent modulo 2. Up to exchanging n and m we can assume that m is even, while n is odd. Then $_{13}{}^n$ $_{14}{}^m$ $_{13}$. Again by (2.8) we get

$$\tilde{} \;\; \mathrm{Nm} \quad {}_{13} \qquad {}_{13} \quad \tilde{} \quad {}_{13} \qquad {}_{34} \qquad \tilde{} \quad .$$

We deduce that Nm $_{13}$ is either equal to or to $_S{}^2$. In any case it is not algebraically equivalent to zero and so the statement is proven. □

Remark 3.4.6. (a) Observe that $_{34}$ is in the image of 1 $\tilde{}$, as we have that $_{34}$ $_{\tilde{S}}$ D_3 $\tilde{}$ $_{\tilde{S}}$ D_3 .

(b) We will see in what follows that the di erent behavior of the norm map applied to torsion classes is what determines the contrast between the type 2 and type 3 bielliptic surfaces. In particular, the fact that the norm map of a torsion class is not necessarily algebraically trivial is what does not allow us to use Remark 2.3.2 in order to provide a non trivial class in Ker Nm / Im 1 $\tilde{}$

Now we turn our attention to the elements of Pic^0 $\tilde{S}$ whose norm is trivial. We will show that they never determine nonzero classes in Ker Nm / Im 1 $\tilde{}$.

Lemma 3.4.7. Denote by Nm : Pic $\tilde{S}$ Pic S the norm homomorphism. Let L Pic^0 $\tilde{S}$, such that Nm L $_S$. Then the class of L in H^1 /2 , Pic $\tilde{S}$ is trivial.

Proof. We have to show that such L is in the image of the morphism 1 $\tilde{}$. By Remark 3.4.4, we can write L $\tilde{}$ M with M Pic^0 S . Then our assumption warrants that

$$_S \quad \mathrm{Nm}\ L \quad M^2.$$

We deduce that M is a 2-torsion point in Pic^0 S . Now we know that Pic^0 S 2 is a group scheme isomorphic to /2 /2 . Let be the element $_S{}^2$ Pic^0 S 2 then we can nd Pic^0 S 2 , nontrivial, such that

$$\mathrm{Pic}^0\ S\ 2 \qquad _S, \ , \ , \qquad .$$

In particular, as $\tilde{}$ $_{\tilde{S}}$,

$$\mathrm{Ker\ Nm} \quad \mathrm{Pic}^0\ \tilde{S} \qquad _{\tilde{S}}, \tilde{} \qquad . \tag{3.15}$$

Now we aim at producing a line bundle Pic^0 $\tilde{S}$ Im 1 $\tilde{}$, $_{\tilde{S}}$. Thus we will have that Pic^0 $\tilde{S}$ Im 1 $\tilde{}$ is a nontrivial subgroup of Ker Nm Pic^0 $\tilde{S}$. From (3.15) we deduce that

$$\mathrm{Ker\ Nm} \quad \mathrm{Pic}^0\ \tilde{S} \quad \mathrm{Pic}^0\ \tilde{S} \quad \mathrm{Im}\ 1 \quad \tilde{}$$

and so the statement.

To this aim let $\bar{}$ A : A/G the image of the point A de ning the involution $\tilde{}$ (see (2.4)). Denote also by p_0 the identity element of A ; observe that by the construction of bielliptic surfaces $\bar{}$ p_0. Consider the following line bundle on $\tilde{S}$:

$$: \quad a_{\tilde{S}} \quad _A \ p_0 \quad t_{-} \quad _A \quad p_0 \quad .$$

Clearly is a nontrivial element in Pic^0 $\tilde{S}$. In addition by (2.4) we see that

$$1 \quad \tilde{} \quad a_{\tilde{S}} \quad _A \quad p_0$$

therefore it is in the image of 1 $\tilde{}$. Thus we can conclude. □

3.4.2.2 Injectivity of the Brauer map

We are now ready to prove Theorem 3.4.3. We will do so by showing the following statement.

Proposition 3.4.8. If L Ker Nm , then L is numerically trivial.

Before proceeding with the proof, let us show how this implies Theorem 3.4.3. So let L be a line bundle in the kernel of the norm map. Then, assuming the proposition, L is numerically trivial, implying that $c_1\ L$ is torsion. In particular we have that

$$L \qquad {}_{13}{}^{n} \qquad {}_{14}{}^{m}$$

for some positive integers n and m, and for some Pic^0 $\tilde{S}$. Again writing $\tilde{}$ for some Pic^0 S , we have that

$$\mathrm{Nm}\ L \qquad {}^{2} \quad \mathrm{Nm} \quad {}_{13}{}^{n} \qquad {}_{14}{}^{m} \ .$$

Since Nm L is trivial, it is in particular algebraically trivial, and so the second part of Lemma 3.4.5 implies that n and m must have the same parity. The first part of Lemma 3.4.5 then gives that Nm L Nm $_S$, so Ker Nm , which by Lemma 3.4.7 then gives that Im 1 $\tilde{}$. Now, as n and m have the same parity, either $_{13}{}^{n}$ $_{14}{}^{m}$ is trivial or isomorphic to $_{13}$ $_{14}$ $_{34}$, and consequently L is isomorphic to either or $_{34}$. By Remark 3.4.6(a) the latter must also be in the image of 1 $\tilde{}$, and so Theorem 3.4.3 is proven.

Proof of Proposition 3.4.8. Let L in the kernel of the norm map. Lemmas 2.1.3 and 2.1.3 imply that $\tilde{}$ Num S is a sublattice of index 2 of Num $\tilde{S}$. In particular $L\ ^2$ is numerically equivalent to the pullback of a line bundle from S. Thus we can write

$$L\ ^{2} \quad \tilde{}\ M \qquad {}_{13}{}^{n} \qquad {}_{14}{}^{m}$$

for some positive integers n and m, and for some Pic^0 $\tilde{S}$. Again, by Remark 3.4.4 we can write $\tilde{}$ for some Pic^0 S , and so, up to substituting M with M we have that

$$L\ ^{2} \quad \tilde{}\ M \quad {}_{13}{}^{n} \qquad {}_{14}{}^{m}.$$

If we show that M is numerically trivial we can conclude. Now, since the line bundle $_{13}{}^{n}$ $_{14}{}^{m}$ is torsion, so is its norm, and because L Ker Nm , we have that

$$\mathrm{Nm}\ L\ ^{2} \qquad M\ ^{2} \quad \mathrm{Nm} \quad {}_{13}{}^{n} \qquad {}_{14}{}^{m} \qquad {}_{S}.$$

So we see that M is also torsion, and hence it is numerically trivial.

□

3.5 The Brauer map to the canonical cover

In this section we study the Brauer map $_{\mathrm{Br}}$: Br S Br X when S is a bielliptic surface and X is its canonical cover. Then there is an n to 1 étale cyclic cover : X S, where n denotes the order of the canonical bundle $_S$. Thus, as in the previous section, we can use Beauville's work [Bea09] to study the kernel of the $_{\mathrm{Br}}$ via the norm homomorphism Nm : Pic X Pic S . As in the other cases the Brauer group is trivial, we can assume that S is of type 1, 2, 3, or 5. Recall that, independently from the case at hand, there are two elliptic curves A and B such that X is isogenous to A B. In what follows we will see that the geometry of the Brauer maps depends much on the geometry of A B, and in particular on whether there are isogenies between A and B or not. Throughout this section we will use the notation established in paragraph 2.1.1.

3.5.1 The norm of numerically trivial line bundles

Our first step will be proving the following proposition, which will allows us to study the norm map from - since numerical and algebraic equivalence coincide for abelian varieties, in particular for X - a strictly numerical point of view.

Proposition 3.5.1. Let L Pic^0 X Ker Nm . Then L is in Im 1 .

Before going any further we need to describe more precisely our setting and introduce some notation.

Observe first that, if we let as in 2.1.1 $p_A : X$ A/H and $p_B : X$ B/H be the two elliptic fibrations of the abelian variety X, then Pic^0 X is generated by p_A Pic^0 A/H and p_B Pic^0 B/H , thus we can write any L Pic^0 X as p_A p_B , where Pic^0 A/H , Pic^0 B/H . In this notation we have the following.

Lemma 3.5.2. For every Pic^0 B/H we have that p_B is in the image of 1 . In particular these line bundles are in the kernel of the norm homomorphism.

Proof. We suppose rst that G is cyclic and so the group H is trivial, and X A B. We proceed with a case by case analysis.

Type 1 case. Since abelian varieties are divisible groups, there exist Pic^0 B such that 2_B . Then by (2.2) we have that

$$1 \quad p_B \quad p_B\, 1 \quad 1_B \quad p_B 2_B \quad p_B ,$$

and the statement is proven in this case.

Type 3 case. In this case the j-invariant of B is 1728 and there is an automorphism of B of order 4. Consider the map 1 $: B$ B. Since this is not trivial it is an isogeny, and in particular 1 $: \mathrm{Pic}^0$ B Pic^0 B is surjective. Let Pic^0 B such that 1 , then by (2.2) we have

$$1 \quad p_B \quad p_B\, 1 \quad p_B ,$$

and the statement is proven in this case.

Type 5 case. This case is similar to the previous one in which instead of we use the automorphism . We note that 1 $: B$ B is non trivial, and so an isogeny. In particular the dual map 1 $: \mathrm{Pic}^0$ B Pic^0 B is surjective and we can nd such that 1 . Again (2.2) yields:

$$1 \quad p_B \quad p_B\, 1 \quad p_B ,$$

and the statement is proven.

In order to conclude we need to analyze the case of bielliptic surfaces of type 2. Under this assumption the group H is not trivial but it is cyclic of order 2. Let B : B/H and observe that we have the following diagram

$$\begin{array}{ccc} X & \longrightarrow & X \\ \big\downarrow{\scriptstyle p_B} & & \big\downarrow{\scriptstyle p_B} \\ B & \xrightarrow{1_B} & B , \end{array}$$

So let, as in the type 1 case, Pic^0 B such that 2_B , then we will have again that 1 p_B p_B and the proof is concluded. □

Now let $L \quad p_A \quad p_B \quad \mathrm{Pic}^0 \; X$ such that $\mathrm{Nm} \; L \qquad _S$. Lemma 3.5.2 implies that also p_A is in the kernel of the norm homomorphism. In addition we have that the class of L in $H^1 \; G, \mathrm{Pic} \; A \quad B$ is just the class of p_A. We have a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X \\ \downarrow{\scriptstyle p_A} & & \downarrow{\scriptstyle a_S} \\ A/H & \longrightarrow & A/G, \end{array}$$

where the bottom arrow is an isogeny of degree n. In particular we can write $p_A \qquad M$ with $M \quad \mathrm{Pic}^0 \; S$. In addition we have that

$$_S \quad \mathrm{Nm} \; p_A \qquad M \; ^n,$$

thus we have that

$$p_A \; \mathrm{Pic}^0 \; A/H \qquad \mathrm{Ker} \; \mathrm{Nm} \qquad \mathrm{Pic}^0 \; S \; \; n \quad .$$

It easy to see that the right-hand-side above is a group isomorphic to the cyclic group of order n. Since $\mathrm{Im} \; 1$ is a subgroup of the kernel of the norm, if we provide an element of order n in $p_A \; \mathrm{Pic}^0 \; A/H \qquad \mathrm{Im} \; 1$ we would conclude that

$$p_A \; \mathrm{Pic}^0 \; A/H \qquad \mathrm{Im} \; 1 \qquad p_A \; \mathrm{Pic}^0 \; A/H \qquad \mathrm{Ker} \; \mathrm{Nm}$$

and consequently the statement of Proposition 3.5.1. Let p_0 be the identity element of A/H, using the notation of 2.2 and (2.3) we set

$$: \quad \begin{array}{ll} _A \; p_0 \quad t \quad _A \quad p_0 \; , & \text{if } S \text{ is of type } 1, \\ _{A/H} \; p_0 \quad t \quad _{A/H} \quad p_0 \; , & \text{if } S \text{ is of type } 2, \\ _A \; p_0 \quad t \quad _A \quad p_0 \; , & \text{if } S \text{ is of type } 3, \\ _A \; p_0 \quad t \quad _A \quad p_0 \; , & \text{if } S \text{ is of type } 5; \end{array}$$

where is the image of under the isogeny $A \qquad A/H$. Then is a nontrivial element of $\mathrm{Pic}^0 \; A/H$ with the desired property. In addition, by (2.2) (2.3), we have that $p_A \qquad 1 \qquad p_A \;\; _A \; p_0$, and so we can conclude. □

Now we are ready to start our investigation of the Brauer map $_{\mathrm{Br}} : \mathrm{Br} \; S \qquad \mathrm{Br} \; X$. We first put ourselves in the special situation in which there are no nontrivial morphisms between A and B.

3.5.2 The Brauer map when the two elliptic curves are not isogenous

If there are no isogenies between A and B, the the lattice $\mathrm{Num} \; X$ has rank 2 and it is generated by the classes of the two fibers, a_X and b_X. In addition, $\mathrm{Num} \; S$ is a sublattice of $\mathrm{Num} \; X$ of index n. So, let L be in the kernel of the norm map. We have that $L \; ^n$ is numerically equivalent to the pullback of a line bundle from S. More precisely we can write

$$L \; ^n \qquad L \quad p_A \quad p_B \qquad M \quad p_B \; ,$$

with Pic^0 B/H . Lemma 3.5.2 ensures that M is in the kernel of the norm map. In particular, M is an n-torsion element in Pic S . We deduce that it is numerically trivial, and so L was numerically trivial to start with. Now we apply Proposition 3.5.1 and deduce the following statement.

Theorem 3.5.3. If S : A B/G is a bielliptic surface such that the elliptic curves A and B are not isogenous, then the Brauer map to the canonical cover $_{\mathrm{Br}}$: Br S Br X is injective.

Before going to the next case, observe that if S is a bielliptic surface of type 2, then we have the following diagram

$$\begin{array}{ccc} B & \xrightarrow{\quad \tilde{S} \quad} & \tilde{S} \\ \downarrow & & \downarrow \tilde{} \\ X & \xrightarrow{\quad S \quad} & S, \end{array}$$

If A and B are not isogenous Theorem 3.5.3) above implies that the Brauer map induced by $_S$ is injective. On the other side, the results of this paragraph imply that the Brauer map induced by $_S$ is trivial. Then the Brauer map induced by cannot be injective and we have

Corollary 3.5.4. If : X Y is an isogeny of abelian varieties, the map $_{\mathrm{Br}}$: Br Y Br X is not necessarily injective.

3.5.3 The Brauer map when the two elliptic curves are isogenous

Suppose now that A and B are isogenous. Our first step will be to use the description the Picard group and of the Neron–Severi of A B that we outlined in 3.2 in order to find the image of 1 and the numerical type of line bundles in the kernel of the Norm homomorphism when S is a cyclic bielliptic surface.

So let L Pic A B . Then there exists line bundles L_A and L_B and a morphism : B A such that L L $L_A, L_B,$, where

$$L \;\; L_A, L_B, \qquad 1_A \qquad\quad {}_A \quad p_A L_A \quad p_B L_B.$$

Letting be the automorphism of A B inducing S, we may write t where t is the corresponding translation on A, and is the automorphism of B of order n ord $_S$. First we show the following lemma.

Lemma 3.5.5. In the notation above, we have that

$$L \;\; L_A, L_B, \qquad L \;\; t \;\; L_A, \qquad L_B \qquad P \;\; , \qquad .$$

Proof. Letting pr_2 be the second projection from A A, by an application of the See-Saw Principle we nd that

$$t \quad 1_A \quad\; {}_A \quad\; {}_A \quad \mathrm{pr}_2 \, P \; .$$

Using this we get

$$
\begin{aligned}
1 \quad {}_A \quad t \quad 1 \quad {}_A \\
1_A \quad 1_A \quad t \quad 1_A \quad {}_A \\
1_A \quad 1_A \quad {}_A \quad \mathrm{pr}_2 P \\
1_A \quad {}_A \quad 1_A \quad \mathrm{pr}_2 P \\
1_A \quad {}_A \quad p_B \quad P \,,
\end{aligned}
$$

where the last passage follows from $\mathrm{pr}_2 \quad 1_A \quad p_B$. Moreover, we have that

$$p_A L_A \quad p_A t \; L_A \text{ and } \quad p_B L_B \quad p_B \quad L_B,$$

so by applying to each of the three pieces of $L \; L_A, L_B,$ and collecting terms we are done. □

Next we find the numerical type of L when L Ker Nm .

Lemma 3.5.6. Suppose that G is a cyclic group, so that $X \quad A \quad B$. If L Pic $A \quad B$ is in the kernel of the norm map, then $c_1 \; L \quad l \; 0, 0,$ for some isogeny $: B \quad A$.

Proof. We can write $c_1 \; L \quad l \; d_1, d_2,$ for two integers d_1 and d_2 and an isogeny . Since L Ker Nm ,

$$0 \quad c_1 \quad \mathrm{Nm}\, L \quad c_1 \; L \quad c_1 \; L \quad \ldots \quad {}^{n \; 1} \quad c_1 \; L \; .$$

Clearly does not change the degrees d_1 and d_2, and since the line bundle P is algebraically trivial, the lemma above implies that $c_1 \; L \quad l \; d_1, d_2,$. Since is the automorphism 1_B, or , we have 1 $\ldots \quad {}^{n \; 1}$ 0 and so

$$
\begin{aligned}
0 \quad & c_1 \; L \quad c_1 \; L \quad \ldots \quad {}^{n \; 1} \quad c_1 \; L \\
& l \; nd_1, nd_2, \quad 1 \quad \ldots \quad {}^{n \; 1} \\
& l \; nd_1, nd_2, 0
\end{aligned}
$$

which gives $d_1 \quad d_2 \quad 0$.

□

We now turn our attention to the Brauer map in general and we study it by performing a case by case analysis on the different type of bielliptic surfaces.

3.5.3.1 Bielliptic surfaces of type 1

In this paragraph we study the Brauer map to the canonical cover of bielliptic surfaces of type 1. If B does not have complex multiplication, we fix, once and for all, a generating isogeny $: B \quad A$. Otherwise we fix two generators ${}_i : B \quad A$, $i \quad 1, 2$. Our first step is to describe 1 Pic $A \quad B$. For the calculations ahead, keep in mind that the automorphism on B is 1_B and that the torsion point on A is , which is of order 2.

Lemma 3.5.7. Let S be a bielliptic surface of type 1, and consider L 1 Pic $A \quad B$, then there exist three integers m, h and k, and a line bundle Pic^0 B such that

$$
L \quad
\begin{cases}
L \; P \; {}^{n}, \; , 2h & \text{if } B \text{ does not have complex multiplication;} \\
L \; P \; {}^{n}, \; , 2h \quad {}_1 \quad 2k \quad {}_2 & \text{if } B \text{ has complex multiplication.}
\end{cases}
$$

Proof. We do the complex multiplication case, the other is similar. Let M Pic A B. As before, M L M_A, M_B, h $_1$ k $_2$. We can write M_A $_A$ n p_0 and M_B $_B$ m q_0 for p_0, q_0 the identity elements of A, B, respectively, integers n and m and some topologically trivial line bundles and . With this notation Lemma 3.5.5 gives

$$M \quad L \; t \quad {}_A \, n \;\; p_0 \quad , \; {}_B \, m \;\; q_0 \quad {}^1 \quad h \quad {}_1 \quad k \quad {}_2 \; P \, , \; h \quad {}_1 \quad k \quad {}_2 \, .$$

Observe that as ranges in all Pic^0 B also : ^{2} h $_1$ k $_2$ P ranges in the whole Pic^0 B. In addition we have that

$$1 \qquad M \quad L \;\; P \;\; {}^n, \;\; , 2h \quad {}_1 \quad 2k \quad {}_2$$

□

Remark 3.5.8. It is not di cult to check that, for any two integers h and k

$$L \;\; 0, 0, 2h \quad {}_1 \quad 2k \quad {}_2 \qquad L \;\; 0, 0, h \quad {}_1 \quad k \quad {}_2 \qquad L \;\; 0, 0, h \quad {}_1 \quad k \quad {}_2 \quad {}^1,$$

and so it is in Im 1 .

We are now ready to prove one of the main statements of this section:

Theorem 3.5.9. Suppose that S is a bielliptic surface of type 1 whose canonical cover is A B with A and B isogenous elliptic curves. Then the Brauer map to the canonical cover of S is not injective if, and only if, one of the following mutually exclusive conditions is satis ed:

1. the elliptic curve B (and so A) does not have complex multiplication and P is trivial;
2. the elliptic curve B (and so A) has complex multiplication and we have that at least one of the following line bundles is trivial

$$L_1 : \quad {}_1 P \, , \quad L_2 : \quad {}_2 P \, , \quad L_3 : \quad {}_1 \quad {}_2 \quad P \qquad (3.16)$$

Proof. We deal with the complex multiplication case that is slightly more involved. The argument for the other case is very similar.

Before explaining the details of our reasoning we would like to give, for the reader convenience, a quick outline of the proof. The key observation is that the assumption on the line bundles (3.16) are equivalent to the norm of one of the following invertible sheaves

$$M_1 : \quad 1 \quad {}_1 \qquad {}_A, \; M_2 : \quad 1 \quad {}_2 \qquad {}_A, \; M_3 : \quad 1 \quad {}_1 \quad {}_2 \qquad {}_A \qquad (3.17)$$

being topologically trivial. Therefore, if the assumptions are veri ed, we can use the Pic^0 trick (Remark 2.3.2) to provide an element in the kernel of the norm map. Such an element will give by construction a nontrivial class in Ker Nm / Im 1 . Conversely, if neither of the line bundles is trivial, then an element in the kernel of the norm map will be forced to be numerically equivalent to 1 2 $_A$ for some isogeny Hom B, A. Then we will apply Lemma 3.5.7 and see that such a line bundle lies in Im 1 , so no element of Pic A B yields a nontrivial class in Ker Nm / Im 1 .

Now, for the complete argument, observe rst that by (2.8) and Lemma 3.5.5 , for every in Pic^0 A and every isogeny $: B \quad A$,

$$\mathrm{Nm} \quad 1 \qquad {}_A \quad p_A \qquad \mathrm{Nm} \quad 1 \qquad {}_A \qquad \mathrm{Nm}\ p_A$$
$$\mathrm{Nm} \quad 1 \qquad {}_A \quad p_A \quad {}^2 \tag{3.18}$$
$$p_B \quad P \quad p_A \quad {}^2$$

Suppose rst that one of the three line bundles in (3.16) is trivial. To x the ideas we can assume that ${}_1P$ is trivial, the argument is identical in the other cases. Then by (3.18) we have that $\mathrm{Nm} \quad 1 \quad {}_1 \quad {}_A$ is in the kernel of , and so it is in Pic^0 S . We can therefore apply the Pic^0 trick and nd Pic^0 S such that the norm of $1 \quad {}_1 \quad {}_A$ is trivial. But by Lemma 3.5.7 we have that $1 \quad {}_1 \quad {}_A$ is not in the image of 1 and so it de nes a non trivial class in $\mathrm{Ker}\,\mathrm{Nm}$ / Im 1 , and one direction of the statement is proven.

Conversely suppose that there is a line bundle L on X which identi es a non trivial class in $\mathrm{Ker}\,\mathrm{Nm}$ / Im 1 . By Lemma 3.5.6

$$L \quad 1 \quad h \quad {}_1 \quad k \quad {}_2 \quad {}_A \quad p_A \quad p_B \ ,$$

for two integers h and k, and two topologically trivial line bundles and . Not that h and k cannot be both even, for otherwise Lemma Remark 3.5.8 yield that $L \quad p_A \quad p_B$ $\mathrm{Ker}\,\mathrm{Nm}$ / Im 1 which, by Proposition 3.5.1, implies that $L \quad 0$. Thus we can assume that one between h and k is odd. Then by Lemma 3.5.2 and Lemma 3.5.7 we have that

$$L \quad 1 \quad {}_i \quad {}_A \quad p_A \quad M, \quad \text{or} \quad L \quad 1 \quad {}_1 \quad {}_2 \quad {}_A \quad p_A \quad M,$$

with M in Im 1 . From (3.18) we get that $p_B \quad P \quad p_A \quad {}^2 \quad {}_X$, where is equal to ${}_1$, ${}_2$ or ${}_1 \quad {}_2$. But then $p_B \quad P$ (and p is trivial, and consequently one of the line bundles in (3.16) is trivial and the statement is proved. ☐

Example 3.5.10. (a) Suppose that $A \quad B$. If A does not have complex multiplication, then we can take 1_A. In particular we have that P is never trivial and the Brauer map is injective.

(b) Suppose again that $A \quad B$ and that the j-invariant of A is 1728. Then End A i and the multiplication by i induces an automorphism of A of order 4, and we can take 1_A and as generators of End A . Suppose that P is a xed point of the dual automorphism . Then $1_A \quad P$ is zero and the Brauer map is not injective.

In order to complete our description of the Brauer map for type 1 bielliptic surfaces we need to give necessary and sufficient conditions for it to be trivial. To this aim we want to provide two distinct non-zero classes in $\mathrm{Ker}\,\mathrm{Nm}$ / Im 1 . We can assume that the Brauer map is already non-injective, and so the condition of Theorem 3.5.9 are satisfied. Suppose first that B does not have complex multiplication. And consider L in the kernel of

For example we can identify A with its dual and with and take $\frac{1}{2}, \frac{1}{2}$, where $1, i \quad A \quad / \ .$

the norm map, yielding a non trivial class in Ker Nm / Im 1 . Then, as before, we have that

$$L \quad 1 \quad h \qquad {}_A \quad p_A \quad p_B \;.$$

Again by Lemma 3.5.7 we can assume that h is odd, and in Ker Nm / Im 1 the class of L and that of 1 $_A$ p_A are the same. Since P is trivial, (3.18) implies that 1 $_A$ p_A is in the kernel of the norm map for some Pic^0 A . So Nm p_A 1 $_S$ and by Proposition 3.5.1 p_A ^{1} lies in the image of 1 . We deduce that, in Ker Nm / Im 1 ,

$$L \quad 1 \qquad {}_A \quad p_A \qquad 1 \qquad {}_A \quad p_A$$

for every Pic^0 A such that 1 $_A$ p_A is in the kernel of the norm homomorphism. In particular there is only one non-trivial element in Ker $_{\mathrm{Br}}$.

Thus we can assume that B has complex multiplication and that, as before, we have fixed $_1$ and $_2$ a system of generators for Hom A, B . Take L in the kernel of the norm map defining a non-trivial element in Ker Nm / Im 1 . By shaving off potential even parts we can write L M_i p_A M with M in the image of 1 and M_i one of the line bundles appearing in (3.17). Suppose that only one of the linde bundles in 3.16, say L_1, is trivial. Then reasoning as in the previous case the class of L in Ker Nm / Im 1 is equal to the class of M_1 p_A for every Pic^0 A such that Nm M_1 p_A is trivial. Thus, there is just one non-zero class and the Brauer map is again non trivial. Finally suppose that two (and so all) line bundles in (3.16) are trivial. We have that both M_1 and M_2 are in the kernel of the norm map. In addition

$$M_1 \quad M_2{}^1 \qquad 1 \qquad {}_1 \qquad {}_2 \qquad {}_A,$$

which by Lemma 3.5.7 is not in the image of 1 . Therefore we deduce that they determine two different classes in Ker Nm / Im 1 , and hence the Brauer map is trivial. We have thus proven the following statement.

Theorem 3.5.11. The Brauer map to the canonical cover of a type 1 bielliptic surface is trivial if, and only if, the elliptic curves A and B are isogenous, B has complex multiplication, and all the line bundles in (3.16) are trivial.

Example 3.5.12. (a) If A B then the Brauer map is never trivial. Suppose otherwise that there are $_1$ and $_2$ generators of End A such that both $_1P$ and $_2P$ are zero. Then we can write 1_A h $_1$ k $_2$ and we would get that P 1_AP is trivial, reaching an obvious contradiction.

(b) Let now A / $2i$ and let the point $0, i$ $2i$. The elliptic curve B : $A/$ has j-invariant 1728 and Hom B, A is generated by the isogenies $_1$: $_2$ and $_2$: $_2$ $_B$, where $_2 : B$ A denotes the isogeny induced by multiplication by 2 (see Example 3.2.8 in the Subsection 3.2.1). Observe that

$$_2\,P \qquad _2 \quad _A \qquad p_0 \qquad _A \quad _2 \qquad p_0 \qquad _B$$

Thus we have that $_1P$ $_2P$ $_B$ and the Brauer map is trivial.

3.5.3.2 Bielliptic surfaces of type 3

Let now S be a bielliptic surface of type 3. Then the canonical cover of S is isomorphic to $A \quad B$ with $j\ B \quad 1728$ and multiplication by i induces and automorphism of B of order 4, . By the discussion in 3.2, it is possible to find a generating isogeny such that

$$\mathrm{Num}\ X \quad l\ 1,0,0\ ,\ l\ 0,1,0\ \ l\ 0,0, \quad ,\ l\ 0,0, \qquad ,$$

We fix, once and for all, such a and prove the following Lemma, which yields a precise description of $1 \qquad \mathrm{Pic}\ X$.

Lemma 3.5.13. Let $: B \quad A$ and isogeny. Then there are two integers h and k such that $h \quad k$. Then the line bundle $1 \qquad {}_A \quad \mathrm{Im}\ 1$ if and only if h+k is even.

Proof. Let $T : \mathrm{Hom}\ B, A \quad \mathrm{Hom}\ B, A$ the linear operator obtained by composing on the right with 1_B . Then an isogeny as in the statement is in the image of T if, and only if, $h \quad k$ is even. To see this, suppose there is a such that $h \quad k \qquad 1_B$ and compose with 1_B on both sides. For the other direction, since $h \quad k$ is even, so is $h \quad k$, and we can divide by 2 on both sides of the equation $1_B \quad 1_B \quad 2$ to nd $\frac{1_B}{2}$. Now, using that for any $\mathrm{Hom}\ B, A$

$$1 \qquad 1 \qquad {}_A \quad 1 \qquad 1_B \qquad {}_A \quad p_B \qquad 1_B \qquad P$$

and that elements of the form p_B with $\mathrm{Pic}^0\ B$ are in the image of 1 by Lemma 3.5.2, we get the statement. □

Remark 3.5.14. Observe that this Lemma implies easily that the quotient $\mathrm{Hom}\ B, A\ / \mathrm{Im}\ 1$, where we are identifying $\mathrm{Hom}\ B, A$ with the corresponding subgroup of $\mathrm{Num}\ A$ B , is cyclic generated by the coset $1_A \qquad {}_A \quad \mathrm{Im}\ 1$.

Now we are ready to start studying the kernel for the Brauer map ${}_{\mathrm{Br}} : \mathrm{Br}\ S \quad \mathrm{Br}\ X$. Our main result is the following

Theorem 3.5.15. Let S is a bielliptic surface of type 3 with canonical cover $A \quad B$ such that A and B are isogenous. Then the Brauer map to the canonical cover is identically zero if, and only if, $1_B \qquad P_2$ is trivial

Proof. For any isogeny $: B \quad A$, $\mathrm{Pic}^0\ A$ and $\mathrm{Pic}^0\ B$, using that the norm of p_B is trivial by Lemma 3.5.2, we have that

$$
\begin{aligned}
\mathrm{Nm}\ 1 \qquad {}_A \quad p_A \quad p_B \quad & 1 \qquad {}_A \quad p_A \\
& 1 \qquad {}_A \quad p_B \qquad P \quad p_A \\
& 1 \qquad {}_A \quad p_B \quad 1_B \qquad P_2 \quad p_A \\
& 1 \qquad {}_A \quad p_B \qquad P_3 \quad p_A \\
& p_A \quad {}^4 \quad p_B\ 1_B \qquad P_2\ .
\end{aligned}
\tag{3.19}
$$

Suppose that $1_B \qquad P_2 \qquad {}_B$. Since P_2 is a two torsion point, this is equivalent to asking that $1_B \qquad P_2$ is also trivial. Then (3.19) implies that the norms of

1 $_A$ and of 1 $_A$ lie in Pic^0 S . Then using the Pic^0-trick (Remark 2.3.2) and Lemma 3.5.13 we can nd a non zero class in Ker Nm / Im 1 , and the Brauer map is trivial.

Conversely, let L be a line bundle de ning a nontrivial class in Ker Nm / Im 1 . Then as we did in the case of type 1 surfaces, we can write

$$L \quad 1 \quad h \quad k \quad {}_A \quad p_A \quad p_B$$

with and in Pic^0 A and Pic^0 B . Lemma 2.1.3 implies that the integer h k is odd or we would have that p_A is in the kernel of the norm map, and consequently, by Proposition 3.5.1, L Im 1) . Thus we can write

$$L \quad M \quad M$$

where M is in the image of 1 , and M is numerically equivalent to 1 $_A$ (this is a consequence of Lemma 3.5.13 and Remark 3.5.14). We deduce that M is in the kernel of the norm map. But then (3.19) implies that 1 P_2 is trivial, proving the statement. □

Example 3.5.16. Suppose that A B, so we can take 1_A. If P_2 is a xed point of , then we have that $_A$ yields a nontrivial element in Ker Nm / Im 1 . Conversely, if P_2 is not a xed point of we will have that the Brauer map is injective.

3.5.3.3 Bielliptic surfaces of type 5

Let S be a bielliptic surface of type 5. We will solve this case in a similar fashion as for bielliptic surfaces of type 3. In the type-5 case, the canonical cover is isomorphic to an abelian surface A B with j B 0. As already seen, B admits an automorphism of order 3 such that ^{2} 1 0. Again, thanks to Theorem 3.5.3 we need to study only the case in which A and B are isogenous. Also in this case, by the results of 3.2, there is generating isogeny : B A such that

$$\mathrm{Num}\ X \quad l\ 1,0,0\ ,\ l\ 0,1,0\ \ l\ 0,0,\ \ ,\ l\ 0,0, \qquad .$$

With this notation, we prove a statement analogous to Lemma 3.5.13:

Lemma 3.5.17. Let : B A and isogeny. Then there are two integers h and k such that h k . If h k is not divisible by 3, then 1 $_A$ / Im 1 . Conversely if 3 divides h k, then 1 $_A$ p_B Im 1 , for every Pic^0 B .

Proof. The argument is completely analogous to the proof of Lemma 3.5.13, after observing that, if T : Hom B, A Hom B, A is the operator de ned by pre composing with 1_B , then the image of T are exactly the homomorphism h k such that 3 divides k h. □

Remark 3.5.18. This Lemma implies easily that the quotient of the Hom-part of Num A B by the action of 1 is isomorphic to /3 with elements 1_A $_A$ Im 1 and 1_A $_A$ Im 1 1_A 2 $_A$ Im 1 .

We will also need the following statement:

Lemma 3.5.19. Let B an elliptic curve with j-invariant 0 and an element $\mathrm{Pic}^0\ B$. Consider the following line bundles

$$P_1 : \quad 2 \qquad 1_B \quad , \quad , P \ : \quad 2 \qquad 1_B \qquad , \quad \text{and} \quad P_1 \quad : \quad 2 \qquad 1_B \quad 1_B \qquad .$$

If one of them is trivial then they are all trivial.

Proof. Observe rst that 2 1_B 2 1_B . Since is an automorphism the triviality of P is equivalent to the triviality of P_1. In addition as P_1 P_1 P we have that if P_1 and P are both trivial, then also P_1 is trivial. It remains to show that if P_1 $_B$, then also P_1 and P are trivial. We not that P_1 $_B$ if, and only if, P_1 $_B$. On the other side we have

$$P_1 \qquad 2 \qquad 1_B \quad 1_B \qquad\qquad 1_B \qquad 2 \qquad 1_B \qquad P_1{}^{1}.$$

We conclude that the triviality of P_1 is equivalent to the triviality of P_1 as required by the statement. □

Now we are ready to prove the main result of this paragraph:

Theorem 3.5.20. Let S be an bielliptic surface of type 5 such that the two elliptic curves A and B are isogenous. Let be a generating isogeny, then we have that the Brauer map $_{\mathrm{Br}} : \mathrm{Br}\ S \quad \mathrm{Br}\ A \quad B$ is trivial if, and only if, the line bundle 2 1_B P $_B$.

Proof. The argument is really similar to what happens for type 3 bielliptic surfaces. We rst note that, for any isogeny $: B \quad A$, and every and in $\mathrm{Pic}^0\ A$ and $\mathrm{Pic}^0\ B$ respectively, we have that

$$\mathrm{Nm} \quad 1 \qquad\quad {}_A \quad p_A \qquad p_B \qquad p_A \quad {}^3 \quad p_B \ 2 \qquad 1_B \qquad P \ . \tag{3.20}$$

Suppose rst that 2 1_B P is trivial. Then (3.20) ensures that the norm of $M_1 : \quad 1 \qquad {}_A$ is topologically trivial. By Lemma 3.5.17 we know that no line bundle numerically equivalent to M_1 is in the image of 1 . Thus we use the Remark 2.3.2 to provide an element in $\mathrm{Ker\,Nm}$ inducing a non trivial class in $\mathrm{Ker\,Nm} \,/\, \mathrm{Im}\ 1$.

Conversely, assume that L is a line bundle in $\mathrm{Ker\,Nm}$ whose class in $\mathrm{Ker\,Nm} \,/\, \mathrm{Im}\ 1$ is not trivial. As before we can write

$$L \qquad 1 \quad h \qquad k \qquad\qquad {}_A \quad p_A \qquad p_B \ .$$

We apply Lemma 3.5.17 and write $L \quad M \quad M$ with $M \quad \mathrm{Im}\ 1$ and M a line bundle numerically equivalent to one of the following

$$M_1 : \quad 2 \qquad 1_B \qquad P \ , \quad \text{and} \quad M_1 \quad : \quad 2 \qquad 1_B \quad 1 \qquad\quad P \ . \tag{3.21}$$

Clearly M is in the kernel of the norm map, which, by (3.20) implies that one among the following is trivial:

$$P_1 : \quad 2 \qquad 1_B \qquad P \ , \quad \text{and} \quad P_1 \quad : \quad 2 \qquad 1_B \quad 1_B \qquad\quad P \ .$$

We conclude by applying Lemma 3.5.19 and deducing that P_1 $_B$. □

Example 3.5.21. Suppose that $A \quad B$. Note that the isogeny $: \quad 2 \qquad 1_B \ : B$ has degree 3, and its kernel is contained in $B\ 3$ which has order 9. If is in the kernel of then the bielliptic surface obtained by the action of $x, y \qquad x \quad , \quad y$ has trivial Brauer map. Otherwise the Brauer map is injective.

3.5.3.4 Bielliptic surfaces of type 2

We kept last the bielliptic surfaces of type two since for them we need an *ad hoc* argument. Let therefore S be a bielliptic surface of type 2 and denote by X its canonical cover. Then $X \quad A \quad B/ \quad t_{\ 1,\ 2}$ for two elliptic curves A and B and $_1$ and $_2$ points of order 2 in A and B respectively. Let us fix generators for Hom B, A : if B does not have complex multiplication then Hom B, A with $: B \quad A$ an isogeny; otherwise there are two isogenies $_1, \ _2 : B \quad A$ such that Hom $B, A \quad _1, \ _2$. Our goal is to prove the following statement.

Theorem 3.5.22. In the above notation the Brauer map $_{\mathrm{Br}}$: Br S Br X is not injective if, and only if, one of the following conditions is satis ed:

1. the elliptic curve B does not have complex multiplication and either $_2$ is not the identity element of A or $P_{\ 1}$ is not trivial.
2. the elliptic curve B has complex multiplication and not all of the following elements are the identity element in the elliptic curve they belong to

$$1 \quad 2, \quad 2 \quad 2, \quad _1P_{\ 1}, \quad _2P_{\ 1}, \quad 1 \quad 2 \quad 2, \quad 1 \quad 2 \quad P_{\ 1}$$

Before proceeding with the proof we need to set up some notation. Recall that we have the following diagram

$$\begin{array}{ccc} A \quad B & \xrightarrow{\ \tilde{S}\ } & \tilde{S} \\ \downarrow & & \downarrow \tilde{} \\ X & \xrightarrow{\ S\ } & S, \end{array}$$

where $\tilde{S}$ is a bielliptic surface of type 1. We have that $S \quad X/ \ , \tilde{S} \quad A \quad B/\tilde{}$ and X $A \quad B/ \ $, where denotes the translation $t_{\ 1,\ 2}$. We are going to deal just with the case in which B hax complex multiplication. The proof in the other case will be identical, provided that one drops one of the two generators. We first observe the following fact:

Lemma 3.5.23. In the notation above suppose that B has complex multiplication and let L_i be the line bundle $1 \quad _i \quad _A$, for $i \quad 1, 2$. Then the conditions of Theorem 3.5.22 are satis ed if, and only if, for every Pic^0 $A \quad B$ one of the following line bundles is not -invariant:

$$L_1 \quad , L_2 \quad , L_1 \quad L_2 \quad . \tag{3.22}$$

Proof. By see-saw, it is easy to see that

$$1 \quad _i \quad _A \qquad 1 \quad _i \quad _A \qquad p_A P_{\ _i \ 2} \quad p_B \ _i P_{\ 1},$$
$$1 \quad _1 \quad _2 \quad _A \qquad 1 \quad _1 \quad _2 \quad _A$$
$$p_A P_{\ _1 \ _2 \ 2} \quad p_B \ 1 \quad 2 \ P_{\ 1};$$

the statement follows directly. □

Proof of the su ciency of the conditions of the Theorem 3.5.22. Suppose that the conditions of the statement are satis ed. Then, by Lemma 3.5.23, one of the line bundles (3.22) is not -invariant. Suppose rst that L_1 is not -invariant for every topologically trivial . Thus we have that l $0, 0,$ $_1$ is not in Num X . We deduce that

$$2 \quad _1 / \ 1 \quad \tilde{} \qquad \mathrm{Num}\ X\ . \tag{3.23}$$

Otherwise we would have

$$\begin{aligned} 2 \quad _1 \quad & 1 \ \tilde{} \\ & 1 \ \tilde{} \quad h \quad _1 \ k \quad _2 \\ & 2h \quad _1 \ 2k \quad _2 . \end{aligned}$$

Therefore h 1, k 0 and $_1$, contradicting our previous conclusion. Now consider the line bundle $L :$ Nm 1 $_1$ $_A$. We want to show that there is Pic^0 X such that Nm $_S$ L is trivial. We use the functoriality of the norm map (Proposition 2.3.1) and we obtain that

$$\mathrm{Nm}\ _S\ L \qquad \mathrm{Nm}\tilde{}\ \ \mathrm{Nm}\ _{\tilde{S}}\ \ 1 \quad _1 \quad _A\ .$$

Observe that by (3.18) we have that $_{\tilde{S}}$Nm $_{\tilde{S}}$ 1 $_1$ $_A$ is numerically trivial. Therefore we have that Nm $_{\tilde{S}}$ 1 $_1$ $_A$ is itself numerically trivial. This implies that

$$\mathrm{Nm}\tilde{}\ \ \mathrm{Nm}\ _{\tilde{S}}\ \ 1 \qquad _A \quad \mathrm{Pic}^0\ S\ .$$

In fact if we have that Nm $_{\tilde{S}}$ 1 $_1$ $_A$: Pic^0 $\tilde{S}$ then we write $\tilde{}$ and we have that

$$\mathrm{Nm}\ _S\ L \qquad \mathrm{Nm}\tilde{}\ \ \mathrm{Nm}\ _{\tilde{S}}\ \ 1 \quad _1 \quad _A \qquad ^2.$$

On the other side if Nm $_{\tilde{S}}$ 1 $_1$ $_A$: T a numerically trivial but not algebraically trivial line bundle, then as in (3.13) we have that Nm$\tilde{}$ T is topologically trivial. Thus, as before, we obtain such that Nm $_S$ L $_S$ via the Pic^0 trick (Remark 2.3.2).

In order to determine the non injectivity of the Brauer map we have to ensure that L is not in Im 1 . Suppose that this were not the case, and consider the following commutative diagram

$$\begin{array}{ccc} A \quad B & \xrightarrow{\ \tilde{}\ } & A \quad B \\ \downarrow & & \downarrow \\ X & \longrightarrow & X, \end{array}$$

Then c_1 L 1 $\tilde{}$ Num X . However the properties of the norm (see (2.8)) ensures that c_1 L l $0, 0, 2$ $_1$, thus we would have that l $0, 0, 2$ $_1$ Num X , contradicting (3.23).

If L_2 is not -invariant for every Pic^0 A B , then we proceed as before by exchanging the role of $_1$ and $_2$. Thus, it remain only to see what happen if L_1 L_2 is not -invariant for every . In this case we will have that l $0, 0,$ $_1$ $_2$ / Num A B , and so either l $0, 0,$ $_1$ or l $0, 0,$ $_2$ are not in the image of . Without loss of generality we can assume the rst. Then we will still have (3.23) and we can repeat the above argument.

□

In order to complete the proof of Theorem 3.5.22 we need to show that if all $1 \quad {}_1 \quad {}_A$, $1 \quad {}_2 \quad {}_A$, and $1 \quad {}_1 \quad {}_2 \quad {}_A$ are -invariant then the Brauer map to X is injective. Observe that, under this assumptions, we can write

$$1 \quad {}_1 \quad {}_A \quad L_1, \quad 1 \quad {}_2 \quad {}_A \quad L_2, \quad \text{and} \quad 1 \quad {}_1 \quad {}_2 \quad {}_A \quad L_3.$$

for some line bundles L_1, L_2, and L_3 in Pic X . Then for Pic^0 X write $p_A \ {}_1$ $p_B \ {}_2$ we have

$$\begin{aligned} {}_S\mathrm{Nm} \ {}_S \ L_i \quad & L_i \quad L_i \quad p_A \ {}_1{}^2 \quad p_B \ {}_1 P \ , \\ {}_S\mathrm{Nm} \ {}_S \ L_1 \quad L_2 \quad & L_1 \quad L_2 \quad L_1 \quad L_2 \\ & p_A \ {}_1{}^2 \quad p_B \ {}_1 P \quad {}_2 P \ ; \end{aligned}$$

where, in both cases, the last equality is again given by (3.18). Observe that neither the ${}_i$'s nor ${}_1 \quad {}_2$ can factor through the multiplication by 2 isogeny, or we would have that ${}_1$ and ${}_2$ cannot generate Hom B, A . In particular, we cannot have that neither ${}_i P$ nor ${}_1 \quad {}_2 \ P$ can be trivial. We deduce that

$$\begin{aligned} {}_S\mathrm{Nm} \ {}_S \ L_i \quad & A \ \ B, \\ {}_S\mathrm{Nm} \ {}_S \ L_1 \quad L_2 \quad & A \ \ B. \end{aligned}$$

In particular we obtained the following lemma:

Lemma 3.5.24. In the above notation, if the conditions of Theorem 3.5.22 are not satis ed, then line bundles numerically equivalent to L_i or $L_1 \quad L_2$ are not in the kernel of the norm map Nm ${}_S$.

Before going further we need an intermediate step:

Lemma 3.5.25. For any integer n, $L_i{}^{2n}$ and $L_1 \quad L_2{}^{2n}$ are in Im 1

Proof. Obviously it is enough to show that $L_i{}^2$ is in the image of 1 . To this aim, we pull $L_i \quad L_i$ back to $A \quad B$ and apply (3.18). We see that

$$L_i \quad L_i \quad p_B \,\mathrm{Pic}^0 \ B \quad A \quad B,$$

and we deduce that : $L_i \quad L_i$ is a line bundle in $p_B \,\mathrm{Pic}^0 \ B/H$. By 3.5.2 we know that Im 1 . Thus we can write

$$L_i{}^2 \quad L_i \quad L_i{}^1.$$

□

Conclusion of the Proof of Theorem 3.5.22. Let M is a line bundle such that Nm ${}_S$ M ${}_S$, we will show that M is in the image of 1 . Using (2.8), we know that $M \quad M$ ${}_X$. By pulling back via we get that $M \ \tilde{} \ M$ is again trivial and by the proof of 3.5.6 we see that $c_1 \quad M \quad l \ 0, 0, h \ {}_1 \ k \ {}_2$ for two integers h and k. Then we can write

$$M \quad 1 \ h \ {}_1 \ {}_A \quad 1 \ k \ {}_2 \ {}_A \quad L_1{}^h \quad L_2{}^k \quad ,$$

for some in $\mathrm{Pic}^0\ A\quad B$. Therefore $M\quad L_1\ {}^h\quad L_2\ {}^k$, and we deduce that $M\quad L_1{}^h\quad L_2{}^k$ for some $\mathrm{Pic}^0\ X$. If h and k are both even, then by Lemma 3.5.25 we know that $\mathrm{Ker}\,\mathrm{Nm}\ {}_S$, and the class of M in $\mathrm{Ker}\,\mathrm{Nm}\ {}_S\ /\ \mathrm{Im}\ 1$ is exactly . We apply Proposition 3.5.1 and deduce that $M\quad 0$.

We will now show that neither one between h and k can be odd. Suppose otherwise that h and k are not both even. For example, assume that h is odd and k is even, the proof in the other cases is very similar. Under this hypothesis, Lemma 3.5.25 ensures that L_1 is in the kernel of the norm map. But this contradicts Lemma 3.5.24, and our proof is complete □

Example 3.5.26. (a) Suppose that $A\quad B$, then the isogenies $_1$ and $_2$ are indeed isomorphisms and thus the Brauer map can never be injective.

(b) Let B be an elliptic curve without complex multiplication and consider $_2$ a point of order 2 in B. Let A be the elliptic curve $B/\quad _2$ and $:B\quad A$ the quotient map. The dual map has degree 2. Let $_1\quad A$ the point such that $P\ _1$ is trivial and let be another order 2 element of A. All this data identify a bielliptic surface of type 2 whose Brauer map to the canonical cover is injective.

Chapter 4

Twisted Derived Equivalences

4.1 Introduction

Given a smooth projective variety X over a field k, and an element of the Brauer group of X, , one can construct the twisted derived category $D^b\ X,$ (more details on the construction will be given in 4.2). One of the key questions one may pose is the following:

Question. How much geometry does $D^b\ X,$ encode?

More precisely, given two varieties X and Y and two elements of their respective Brauer group and such that there is an exact equivalence $D^b\ X,$ $D^b\ Y,$, what can be said about the mutual relationship of X and Y?

For example, it is well known that *twisted Fourier–Mukai partners* (i. e. varieties with equivalent twisted derived categories) shares the same dimenison, the same Kodaira dimension, and the same order of the canonical bundle (see [Huy06] for the proofs when the Brauer classes are both trivial, the properly twisted case is due to Navas [Nav10]). In certain cases, for example when X is Fano, or oppositely, when X has ample canonical bundle, it is possible to reconstruct it from its derived category, meaning that an equivalence of the twisted derived categories as above will imply an isomorphism between X and Y (see [Orl97] when both Brauer classes are trivial and again [Nav10] for the properly twisted case).

In this chapter we focus on bielliptic surfaces. We conjecture the following:

Conjecture 4.1.1. Complex bielliptic surfaces do not admit non-isomorphic twisted Fourier Mukai partners.

This result has been proved by Bridgeland–Maciocia in [BM98a] in the case in which both the Brauer classes involved are trivial. Our main results is the following

Main Theorem. Let X be a complex bielliptic surface, and let Y be a complex smooth projective variety, and take two Brauer classes and on X and Y respectively, such that there is an exact equivalence : $D^b\ X,$ $D^b\ Y,$. If either or is trivial then they are both trivial and furthermore X and Y are isomorphic.

The first step of our argument will be showing that twisted Fourier–Mukai partners of bielliptic surfaces are again bielliptic surfaces. This is a consequence of the fact that twisted derived euivalences preserve the order of the canonical bundle and the Hochschild cohomology ([Huy06] for the regular case, and [Nav10] for the properly twisted one) - hence in low dimension the Betti numbers are also preserved. Then we use a computation by Addington–Wray ([AW18]) about the topological Grothendieck group to show that two bielliptic surfaces as in the statement must be of different types (cfr. 2.1). Finally, we will use derived equivalence induced by moduli space of sheaves to show that bielliptic surfaces of different types cannot be twisted derived equivalent.

It is important to remark that, thanks to the work of Honigs–Lieblich–Tirabassi ([HLT17] our main theorem can be directly extended into positive characteristic with mild restrictions:

Corollary. Given a bielliptic surface X over an algebraically closed field k of characteristic greater than 3. If there exists a smooth projective variety Y and a twisted derived equivalence $: D^b\ X, \quad D^b\ Y,$ such that at least one of the Brauer classes involved is trivial, then both classes are trivial. In addition X and Y are isomorphic.

This chapter consists of three parts. In the first we will outline background material about twisted derived categories, derived equivalences and moduli spaces of sheaves. In the second we will prove our main result. In the last and final part we will think a bit on the difficulties in the case where both surfaces are twisted.

4.2 Preliminary results and background material

In this section we presents some preliminary material which will be used to formulate and prove our main result. In particular in 2.4 we will define rigorously twisted sheaves. In 4.2.1 and 4.2.2 we will follow the work of Căldăraru ([C00]) to construct the derived categories of twisted sheaves and to define some useful functor between them. This will allow us to introduce the notion of Fourier–Mukai transform between derived categories of twisted sheaves, which we do in 4.2.3. In 4.2.4 we see, following the work of Navas ([Nav10]), Krug–Sosna ([KS15]), and Addington-Wray ([AW18]) how one can lift a (twisted) Fourier–Mukai functor between the (twisted) derived categories of some varieties to an equivariant Fourier–Mukai functors of their respective canonical covers. In 4.2.5 we see how a Fourier–Mukai functor acts at cohomological level. Finally in 4.2.6 we discuss the smoothness of some moduli spaces of sheaves which will be used to construct specific Fourier–Mukai functor between some twisted derived categories of bielliptic surfaces.

4.2.1 The Twisted Derived Category

We recall the construction of the derived category in the context of twisted sheaves, in complete analogy to the construction found in e.g. [Huy06] in the untwisted case. Let $X,$ be a twisted variety and let $C\ X,$ denote the abelian category of complexes of twisted sheaves in Coh $X,$, i.e., the objects are complexes

$$: \quad \dots \quad d^{i\ 2} \quad i\ 1\ d^{i\ 1} \quad i\ d^i \quad i\ 1\ d^{i\ 1} \quad \dots$$

of -twisted coherent sheaves and the morphisms f : are given by commuting diagrams:

$$\begin{array}{ccccccccc} \dots & \xrightarrow{d^{i\ 2}} & i\ 1 & \xrightarrow{d^{i\ 1}} & i & \xrightarrow{d^{i}} & i\ 1 & \xrightarrow{d^{i\ 1}} & \dots \\ & & \downarrow f^{i\ 1} & & \downarrow f^{i} & & \downarrow f^{i\ 1} & & \\ \dots & \xrightarrow{d^{i\ 2}} & i\ 1 & \xrightarrow{d^{i\ 1}} & i & \xrightarrow{d^{i}} & i\ 1 & \xrightarrow{d^{i\ 1}} & \dots \end{array}$$

A morphism f : induces a morphism $H^i\ f\ : H^i\ \ H^i$ between their i-th cohomology sheaves, which for a general complex is defined as the sheaf

$$H^i \quad : \frac{\ker\ d^i}{\mathrm{im}\ d^{i\ 1}}.$$

To pass from the category $C\ X,$ of complexes to the derived category $D\ X,$ the construction passes through homotopy category $K\ X,$. The objects in $K\ X,$ are the complexes of $C\ X,$ and its morphisms are equivalence classes of morphisms

$$\mathrm{Mor}_{K\ X,} \quad , \quad : \mathrm{Mor}_{C\ X,} \quad , \quad /$$

where $f\ \ g$ if they are homotopically equivalent, i.e., if there exists morphisms ^{i}: ^{i} $^{i\ 1}$ such that $f^i\ \ g^i\ \ ^{i\ 1}\ d^i\ \ d^{i\ 1}\ \ ^i$. Localizing the homotopy category with respect to the class of quasi-isomorphisms, i.e., morphisms f : such that the induced map $H^i\ f$ is an isomorphism for each i, one gets the twisted derived category $D\ X,$. There exists a functor

$$Q\ _{X,}\ : C\ X, \qquad D\ X,$$

preserving quasi-isomorphisms which is universal with respect to this property, i.e., for any category D and functor $F\colon C\ X, \qquad D$ preserving quasi-isomorphisms, there is a functor $R\colon D\ X, \qquad D$ such that $F\ \ R\ \ Q\ _{X,}$.

For the rest of this thesis we shall be concerned with the subcategory $D^b\ X,$ of $D\ X,$ consisting of complexes for which all but finitely many sheaves are different from 0, called the *bounded derived category* of -twisted coherent sheaves on X. Similarily we have the subcategories $D\ \ X,$ and $D\ \ X,$ of complexes bounded below and bounded above, respectively.

4.2.2 Derived Functors

Derived functors works just as in the untwisted case. If $X,$ and $Y,$ are two twisted varieties and $F\colon$ Coh $X, \qquad$ Coh $Y,$ is a left exact functor, the *right derived functor* of F, if it exists, is the functor $RF\colon D^b\ X, \qquad D^b\ Y,$ uniquely determined up to isomorphism by the following properties:

(i) RF is right exact as a functor between triangulated categories,

(ii) If Kom F is the natural functor extending F on complexes, there exists a morphism $Q\ _{Y,}\ \ $ Kom $F\ \ \ RF\ \ Q\ _{X,}$,

(iii) if $G\colon D^b\ X, \qquad D\ Y,$ is an exact functor, any morphism $Q\ _{Y,}\ \ $ Kom F $G\ \ Q\ _{X,}$ factorizes through a morphism $RF\ \ \ G$.

Completely analogues we have the *left derived functor* $LG\colon D^b\ X, \quad D^b\ Y, \quad$ of a right exact functor $G\colon$ Coh $X, \quad$ Coh $Y, \quad$.

In the following we will summarize the derived functors and their properties we shall need, following [C00]. In short, we get all the usual derived functors that we care about, defined exactly as their counterparts in the untwisted case, together with their usual properties.

Proposition 4.2.1 ([C00, Theorem 2.2.6]). Let X and Y be smooth schemes or analytic spaces of nite dimension and $f\colon X \quad Y$ be a proper morphism. If $\ ,\quad$ Br X and $\quad$ Br Y , the following derived functors are de ned:

$$\mathbf{R} \quad om \ : D^b\ X, \quad D^b\ X, \quad D^b\ X, \quad ^1 \quad ,$$

$$^L \quad : D^b\ X, \quad D^b\ X, \quad D^b\ X, \quad ,$$

$$\mathbf{L}f \ : D^b\ Y, \quad D^b\ X, f \quad ,$$

$$\mathbf{R}f \ : D^b\ X, f \quad D^b\ Y, \quad .$$

Furthermore, if X is a scheme or a compact analytic space, then

$$\mathbf{R}Hom \ : D^b\ X, \quad ^{op} \quad D^b\ X, \quad D^b\ Ab$$

where $D^b\ X, \quad ^{op}$ denotes the opposite category, and Ab is the category of abelian groups.

Proposition 4.2.2 ([C00, Section 2.3]). Let $\ X, \quad , Y, \quad , Z, \quad$ be twisted varieties and $f\colon X \quad Y, g\colon Y \quad Z$ be proper morphisms. Then we have the following:

- $\mathbf{R}\ g \quad f \quad \mathbf{R}g \quad \mathbf{R}f$ as functors from $D\ X, f \quad g \quad$ to $D\ Z, \quad$,
- $\mathbf{L}\ f \quad g \quad \mathbf{L}f \quad \mathbf{L}g$ as functors from $D \quad Z,$ to $D \quad X, f \quad g \quad$,
- $\mathbf{R}Hom \quad F, G \quad \mathbf{R} \quad X, \mathbf{R} \quad om \quad F, G$ for $F, G \quad D^b\ X, \quad$,
- $\mathbf{R}f\ \mathbf{R} \quad om \quad F, G \quad \mathbf{R} \quad om \quad \mathbf{R}f \quad F\ , \mathbf{R}f \quad G$, for $F \quad D \quad X, f$), $G \quad D \quad X, f \quad$,
- (Projection Formula) $\mathbf{R}f \quad F \quad ^L G \quad Rf \quad F \quad ^L \mathbf{L}f \quad G$ for $F \quad D \quad X, f$ and $G \quad D \quad Y, \quad$,
- $\mathbf{R} \quad om \quad \mathbf{L}f \quad F\ , G \quad \mathbf{R} \quad om \quad F, \mathbf{R}f \quad G$, for $F \quad D \quad Y, \quad , G \quad D \quad Y, \quad$,
- $\mathbf{L}f \quad F \quad ^L G \quad \mathbf{L}f \quad F \quad ^L \mathbf{L}f\ G$ for $F \quad D \quad X, \quad , G \quad D \quad X, \quad$,
- $F \quad ^L G \quad G \quad ^L F, \ F \quad ^L \ G \quad ^L H \quad F \quad ^L G \quad ^L H$ for $F \quad D \quad X, \quad , G \quad D \quad X, \quad$ and $H \quad D \quad X, \quad$,
- $\mathbf{R} \quad om \quad F, G \quad ^L H \quad \mathbf{R} \quad om \quad F, G \quad ^L H$ for $F \quad D \quad X, \quad , G \quad D \quad X, \quad$ and $H \quad D\ X, \quad$,

$\mathbf{R}\quad om\quad F, \mathbf{R}\quad om\quad G, H\qquad \mathbf{R}\quad om\quad F\quad {}^{L} G, H$ for $F\quad D\quad X, \quad , G\quad D\quad X,$ and $H\quad D\quad X, \quad ,$

$\mathbf{R}\quad om\quad F, G\quad {}^{L} H\qquad \mathbf{R}\quad om\quad F\quad {}^{L} H\ , G$ for $F\quad D\quad X, \quad , G\quad D\quad X,$ and where for a bounded -complex H, $H\ :\ \mathbf{R}\quad om\ H, \quad {}_{X}$.

Moreover, we have the Flat Base Change Theorem for the derived category of twisted sheaves. Namely, if $u\colon Y\quad Y$ is a at morphism in the commutative diagram

$$\begin{array}{ccc} X\quad {}_{Y} Y & \xrightarrow{v} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{u} & Y \end{array}$$

there is a functorial isomorphism $u\ \mathbf{R}f\quad F\qquad \mathbf{R}g\ v\quad F$ for any $F\quad D^{b}\ X, f\quad$.

Duality for proper smooth morphisms also holds in this setting.

Theorem 4.2.3 ([C00, Theorem 2.4.1]). Let $f\colon X\quad Y$ be a smooth morphism of relative dimension n between smooth schemes (or if you will, smooth analytic spaces), and let $\quad$ Br Y . De ne a functor $f^{!}\colon D^{b}\ Y, \qquad D^{b}\ X, f\qquad$ by

$$f^{!}\qquad :\ \mathbf{L}f\qquad\qquad {}_{X/Y}\ n$$

where ${}_{X/Y}$ is the n-th exterior power ${}^{n}\quad {}_{X/Y}$ of the sheaf of relative di erentials. Then for any $G\quad D^{b}\ Y, \quad$ there is a natural homomorphism

$$\mathbf{R}f\ f^{!}G\qquad G,$$

which by Proposition 4.2.8 induces a homomorphism

$$\mathbf{R}f\ \mathbf{R}\quad om\quad F, f^{!}G\qquad \mathbf{R}\quad om\quad \mathbf{R}f\ F, G$$

which is an isomorphism for any $F\quad D^{b}\ X, f\qquad$.

As an immediate corollary of this, under the conditions of the above theorem, we have that $f^{!}$ is a right adjoint to $\mathbf{R}f$ as functors between $D^{b}\ X, f\qquad$ and $D^{b}\ Y, \quad$.

Thus far we see that the twisted derived category behaves completely analogous to the regular derived category. Next up we will look at Fourier-Mukai transforms in the twisted setting and their geometric significance.

4.2.3 Twisted Fourier-Mukai Transforms

Fix two smooth projective varieties X and Y with Brauer classes $\quad$ Br X , $\quad$ Br Y . Let $p\colon X\quad Y\quad X$ and $q\colon X\quad Y\quad Y$ be the natural projections.

Definition 4.2.4. A functor $F\colon D^{b}\ X, \qquad D^{b}\ Y, \quad$ is a Fourier-Mukai functor or of Fourier-Mukai type if there exists an object $\quad D^{b}\ X\quad Y, \quad {}^{1}\qquad$ and an isomorphism of functors $F\qquad$, where $\quad$ is the exact functor de ned by

$$: \mathbf{R}q \quad {}^{L} \; p \quad .$$

If F is an equivalence, we say that it is a Fourier-Mukai transform. The object is called the kernel of . If there exists such an equivalence relating $D^b \; X,$ and $D^b \; Y,$, we say that they are twisted Fourier-Mukai partners.

Remark 4.2.5. Note that since p is at we need not take derived pullback. Also, the same kernel can be used to de ne a functor in the opposite direction, so the notation above might be ambiguous. To be precise, many authors write $\;{}^{X} \; {}^{Y}$ to indicate the direction.

In the case of an untwisted derived equivalence $: D^b \; X \quad D^b \; Y$, Orlov showed in [Orl03, Orl97] that there is an object $D^b \; X \quad Y$, unique up to isomorphism, such that . In fact, he showed a more general result, namely that any exact and fully faithful functor which admits right and left adjoints is of Fourier-Mukai type. Luckily, the same holds true in the twisted case, as shown by Cananaco and Stellari in [CS07], where they also generalize Orlov's original result:

Theorem 4.2.6. Let $X,$ and $Y,$ be twisted varieties and let $: D^b \; X, \quad D^b \; Y,$ be an exact functor such that, for any $F, G \quad \mathrm{Coh} \; X,$,

$$\mathrm{Hom}_{D^b \; X,} \quad F \; , \quad G \; j \quad 0 \; if \; j \quad 0.$$

Then there exists an object $D^b \; X \quad Y, \quad {}^{1}$ such that . Moreover, is uniquely determined up to isomorphism.

In particular, for any equivalence $: D^b \; X, \quad D^b \; Y,$, we know that it is a Fourier-Mukai transform with kernel $D^b \; X \quad Y, \quad {}^{1}$. The geometric significance of this cannot be overstated, as the existence of such a kernel, just as in the untwisted case, allows us to extract geometric information from the twisted derived equivalence.

Recall that for a general k-linear category , where k is a field (which in our case is the field of complex numbers), a *Serre functor* is a k-linear equivalence S: such that for any two objects A, B , there exists an isomorphism

$$_{A,B}: \mathrm{Hom} \; A, B \quad \mathrm{Hom} \; B, S \; A$$

of k-vector spaces, which is functorial in both A and B.

The functor $S_{\; X,}$ on the category $D^b \; X,$ defined as

$$S_{\; X,} \quad : \quad {}_{X} \dim \; X$$

is then a Serre functor, just like in the untwisted case. Indeed, since the functor S on $D^b \; X$ given by ${}_{X} \dim \; X$ is a Serre functor, given $F, G \quad D^b \; X,$, using the properties of our derived functors and the fact that $G \quad F$ is naturally an untwisted sheaf, we get

$$\begin{aligned}\mathrm{Hom}_{D^b}\; X, \quad F, S\; X, \quad G \quad & \mathrm{Hom}_{D^b}\; X, \quad F, G \quad X \dim X \\ & \mathrm{Hom}_{D^b}\; X \quad G \quad F, \quad X \dim X \\ & \mathrm{Hom}_{D^b}\; X \quad X \dim X \;, S\; G \quad F \\ & \mathrm{Hom}_{D^b}\; X \quad X \dim X \;, G \quad F \quad X \dim X \\ & \mathrm{Hom}_{D^b}\; X, \quad G, F \quad .\end{aligned}$$

Taking duals on both sides then gives the required isomorphism.

Having a Serre functor gives a very rich geometric structure on the category also in the twisted case, and we will see that many of the properties we would expect still holds here and many of the proofs go through essentially without change. As an example of the geometric significance of having such a Serre functor, we recall that if and are k-linear categories with finite dimensional Hom's, endowed with Serre functors S and S , respectively, we have $F \quad S \qquad S \quad F$ for any k-linear equivalence F: by an easy application of Yoneda's Lemma. Using this one can show an important geometric fact of twisted derived equivalences following the proof in [Huy06, Proposition 4.1] mutatis mutandis:

Proposition 4.2.7. Let $X,$, $Y,$ be two twisted smooth projective varieties, and suppose $: D^b\; X, \qquad D^b\; Y,$ is an equivalence. Then $\dim X \qquad \dim Y$, and ord X ord Y .

This same fact can be shown using the existence and uniqueness of the Fourier-Mukai kernel such that . Indeed, using Theorem 4.2.9 has left and right adjoints with kernels given by

$$_L : \qquad q \quad Y \dim Y$$

and

$$_R : \qquad p \quad X \dim X \;,$$

respectively. Since is an equivalence, the adjoints are isomorphic, and by uniqueness of kernels we have $_L \qquad _R$. From this we get

$$p \quad X \quad q \quad Y \dim X \qquad \dim Y$$

and $\dim X \qquad \dim Y$ follows. At the same time we get that the kernel of a Fourier-Mukai transform satisfies $p \quad X \qquad q \quad Y$, and moreover from the projection formula that $x \qquad x \qquad Y$ for any $x \quad X$. In other words, the images of skyscraper sheaves under a Fourier-Mukai transform are *special* objects of the derived category, i.e., objects F such that $F \qquad Y \quad F$.

We will need the following lemma:

Lemma 4.2.8 ([Nav10, Lemma 1.4.6]). Let $: S \qquad T$ be a morphism, and for each $t \quad T$, let $i_t : S_t \qquad S$ be the inclusion of the ber $^{1}\; t$ in S. Let E be an object of $D\; S,$ such that for all $t \quad T$, $\mathbf{L} i_t\, E$ is a twisted sheaf on S_t. Then E is a twisted sheaf on S, at over T.

We will say that an FM-transform is a *sheaf transform* if there is an integer p such that for all closed points y Y, H^i $_y$ 0 unless i p. In other words, up to shift, $_y$ is an actual twisted sheaf and not a proper complex. Due the lemma above, an equivalent condition is that the kernel of is concentrated in some degree p, and is flat over Y. A useful consequence of this is a practical way to check if an equivalence $: D^b\ X,$ $D^b\ Y,$ induces an isomorphism X Y. The most rigid version is the following:

Corollary 4.2.9 ([Huy06, Corollary 5.23]). Suppose $: D^b\ X,$ $D^b\ Y,$ is an equivalence such that for any closed point x X, there is a closed point $f\ x$ Y with

$$_x \qquad _{f\ x}\ .$$

Then $f\colon X$ Y de nes an isomorphism and is the composition of f with the twist by some line bundle M Pic Y , i.e.,

$$M \qquad f\ .$$

The less rigid version of the above is the situation of single points x_0 X and y_0 Y such that $_{x_0}$ $_{y_0}$. This implies the existence an open neighbourhood U X and the existence of a morphism $f\colon U$ Y, with $f\ x_0$ y_0, such that $_x$ $_{f\ x}$ for all x U. To conclude this and in the process essentially proving the above corollary, we need some lemmas.

Lemma 4.2.10 ([Huy06, Lemma 3.29]). Let $i\colon T$ X be a closed subscheme. Then for any F $D^b\ X,$, one has

$$\mathrm{supp}\ F \quad T \quad \mathrm{supp}\ \mathbf{L}i\ F$$

Lemma 4.2.11. Let $: D\ X,$ $D\ Y,$ be an equivalence with kernel . Let x X be a closed point. Then we have that $_x$ $\mathbf{L}i$, where $i\colon$ x Y X Y is the natural inclusion, and $\mathbf{L}i$ is considered as a sheaf on Y via the second projection X Y Y.

Proof. Consider the base change diagram

$$\begin{array}{ccc} x \quad Y & \xrightarrow{\ i\ } & X \quad Y \\ \downarrow{\scriptstyle p_x} & & \downarrow{\scriptstyle p} \\ x & \xrightarrow{\ i_x\ } & X \end{array}$$

By de nition $_x$ $\mathbf{R}q$ $^{\mathbf{L}}$ p $_x$ where p and q are the natural projections from X Y. The support of p $_x$ is x Y, and i , i are exact, so $\mathbf{R}i$ $\mathbf{L}i$ p $_x$ i i $_x$, and since i p $_x$ $p_x i_x$ $_x$ $_x$ $_Y$, we get that p $_x$ i i $_{x\ Y}$. By the projection formula we have $^{\mathbf{L}}$ p $_x$ $\mathbf{R}i$ $\mathbf{L}i$. Finally,

$$_x \quad \mathbf{R}q \qquad q \quad _x \qquad \mathbf{R}q\ \mathbf{R}i\ \mathbf{L}i \qquad \mathbf{R}q\ \mathbf{L}i$$

where q q i is the isomorphism x Y Y. Hence $_x$ q $\mathbf{L}i$, i.e., $_x$ $\mathbf{L}i$ considered as a sheaf via the second projection q. □

We recall that an object $F \quad D^b \ X,$ is *simple* if $\mathrm{Hom}\ F, F$.

Lemma 4.2.12 ([Huy06, Lemma 4.5]). Suppose $F \quad D^b \ X,$ is a simple object with zero-dimensional support. If $\mathrm{Hom}\ F, F\ i \quad 0$ for $i \quad 0$, then

$$F \qquad {}_x\ m$$

for some closed point $x \quad X$ and some integer m.

Proposition 4.2.13. Suppose $: D\ X, \qquad D\ Y,$ is an equivalence such that there exists points $x_0 \quad X$ and $y_0 \quad Y$ with $\quad {}_{x_0} \qquad {}_{y_0}$. Then X and Y are birational. In particular, if X and Y are minimal, they are isomorphic.

Proof. Let be the kernel of and consider the support $\mathbf{supp}$ of , de ned as the union of the support of all the cohomology sheaves of . By Lemma 4.2.10 the ber of the natural projection $\mathbf{supp} \qquad X$ at $x_0 \quad X$ is given by $\mathbf{supp} \qquad x_0 \quad Y$ $\mathbf{supp}\ \mathbf{L}i$ where $i\colon \ x_0 \quad Y \qquad X \quad Y$ is the inclusion. Since $\quad {}_{x_0} \qquad \mathbf{L}i$ by Lemma 4.2.11, and $\quad {}_{x_0} \qquad {}_{y_0}$ by assumption, we get $\mathbf{supp} \qquad x_0 \quad Y$ $\mathbf{supp}\ \ {}_{y_0}$, so the ber above x_0 is zero-dimensional. This holds in an open neighbourhood $U \quad X$ of x_0, so the support of $\quad {}_x$ is zero-dimensional for all $x \quad U$. Since $\quad {}_x$ is a simple sheaf with zero-dimensional support, and $\mathrm{Hom} \quad {}_x\ , \quad {}_x\ i \quad 0$ for all $i \quad 0$, we must have by Lemma 4.2.12 that $\quad {}_x \qquad {}_x\ m_x$. Due to semi-continuity, the shift m_x is constant locally around $x_0 \quad U$. In conclusion, there is an open subset $U \quad X$ such that for all $x \quad U$, there is an $f\ x \quad Y$ such that (up to shift) $\quad {}_x \qquad {}_{f\ x}$. By restricting the kernel to $U \quad Y$, Lemma 4.2.8 then implies that is a twisted sheaf over $U \quad Y$, at over U, and hence $\quad {}_x \qquad {}_{x \quad Y} \qquad {}_{f\ x}$. By taking local sections of $\quad {}_{U \quad Y}$ we de ne a morphism $U \qquad Y$ whose graph coincides with the support of over U, and from $\quad {}_x \qquad {}_{f\ x}$ we get that this morphism induces f on closed points. Now following [Huy06, Corollary 5.23], we get a morphism $f\colon U \qquad Y$, hence a rational map $X \qquad Y$. Because is an equivalence, this rational map has an inverse, and X is birational to Y. □

Lemma 4.2.14. Let X,Y be smooth projective varieties and $: D^b\ Y, \qquad D^b\ X$ an equivalence. Then for any $y \quad Y$ there is an inequality

$$\sum_i \mathrm{Ext}^1 \quad {}^i \quad {}_y\ , \quad {}^i \quad {}_y \qquad 2,$$

where $\quad {}^i \quad {}_y \ : \ H^i \qquad {}_y$ denotes the i-th cohomology sheaf of $\quad {}_y$ and $\quad {}_y$ is the skyscraper sheaf at y.

Proof. The second statement is [C00, Thm. 3.2.1], whilst the rst statement can be deduced just as in [BM01b, Lemma 2.9] by using the spectral sequence

$$E_2^{p,q} \qquad \sum_i \mathrm{Ext}^p\ H^i \qquad {}_y\ , H^{i\ \ q} \qquad {}_y \qquad \mathrm{Hom} \qquad {}_y\ , \qquad {}_y\ p \quad q \ .$$

□

Corollary 4.2.15. If X and Y are Abelian surfaces, any equivalence $: D^b\ Y, \qquad D^b\ X$ is a sheaf transform.

Proof. On an abelian surface any non-zero sheaf E satis es $Ext^1\ E, E \quad 2$. Hence by the lemma above, for each $y \quad Y$, there can be only one integer p such that $^{p} \quad _{y} \quad 0$. □

To conclude this section, we briefly consider Hochschild (co)homology under twisted derived equivalence. For any smooth projective variety X, let $d\colon X \quad X \quad X$ be the diagonal embedding. Then one has the bigraded ring

$$HH\ X \ : \quad \underset{i,}{} H_{i,}\ A\ X \quad \text{with}\ HA_{i,} \quad X \ : \ \text{Ext}^i_{X \quad X}\ d \quad _X, d \quad _X\ .$$

The algebra structure is given by composition in $D^b\ X \quad X$. The *Hochschild cohomology* is the subring

$$HH \quad X \ : \quad \underset{i}{} HA_{i,0}\ X \qquad \text{Ext}^i_{X \quad X}\ d \quad _X, d \quad _X\ ,$$

and the *Hochschild homology* is

$$HH \quad X \ : \quad \underset{i}{} HA_{i,1}\ X \qquad \underset{i}{} \text{Ext}^i_{X \quad X}\ d \quad _X, d \quad _X\ .$$

In [Huy06, Prop. 6.1] it is shown that the canonical ring $R\ X \qquad HA_{0,} \quad X$, a subring of $HH\ X$, is invariant under derived equivalence. The proof relies on the existence of a Serre functor, a Fourier-Mukai kernel and its uniqueness and is a yoga in the formality of derived equivalences. In the case where $X,$ is a twisted variety, using that $d \quad _X$ and $d \quad _X$ naturally can be consideres a ^{1} -sheaves and the existence and uniqess of a kernel from Theorem 4.2.6, the proof goes through also in the twisted setting without change, as demonstrated in [Nav10, Thm. 1.6.15]. Both these isomorphism extends to isomorphisms on $HH\ X$ which respects the bigrading and multiplicative structure, so we conclude:

Theorem 4.2.16 ([Huy06, Prop. 6.1], [Nav10, Thm. 1.6.15]). Let $X,$ and $Y,$ be two twisted smooth projective varietes and suppose $D^b\ X, \qquad D^b\ Y,$. Then there exists an isomorphism $HH\ X \quad HH\ Y$ that respects the bigrading and multiplicative structure.

4.2.4 Cyclic Coverings and Liftings

Let X be a smooth projective variety and let $L \quad \text{Pic}\ X$ such that $L^n \quad _X$. Then there is a uniquely determined étale cyclic covering (see e.g. [Huy06, §7.3]) $\ : X \quad X$ of degree d such that

$$_X \quad \overset{d\ 1}{\underset{i\ 0}{}} L^{\ i},$$

$$L \qquad _X.$$

Moreover, there is a free action of the cyclic group $G \quad /n$ on Y such that $X \quad X/G$ and is the quotient map. More precisely, if we let $\ : \ \overset{n\ 1}{\underset{i\ 0}{}} L^i$, then X is the relative spectrum Spec and the action of $G \quad /n$ is given on by L.

We want to study the situation where X and Y are two smooth projective varieties, L Pic X and $M \quad \text{Pic}\ Y$ are n-torsion with associated étale cyclic coverings $_L\colon X \quad X$ and $_M\colon Y \quad Y$ together with an equivalence $D^b\ X, \qquad D^b\ Y,$, to say something about under which conditions we can lift the equivalence to the coverings, and conversely when derived equivalence of the coverings descends to the quotient. In order to do this we need the notion of when a functor is equivariant above and what it means to lift from below.

De nition 4.2.17. Let X and Y be two smooth projective varieties acted upon freely by the cyclic group G $/n$. Denote by $_X: X$ X and $_Y: Y$ Y the quotient maps. Suppose Br X and Br Y are G-invariant. A functor $: D^b\ X,$ $D^b\ Y,$ is equivariant if there exists Aut G and for all g G an isomorphism of functors

$$g \quad \tilde{} \quad \tilde{} \quad g \ .$$

De nition 4.2.18. In the situation above, let $: D^b\ X,$ $D^b\ Y,$ be a functor. A functor $: D^b\ X, p$ $D^b\ Y, q$ is a lift of if the following diagram commutes:

$$\begin{array}{ccc} D^b\ X,\ _X & \xrightarrow{\tilde{}} & D^b\ Y,\ _Y \\ _X\uparrow\downarrow\ _X & & _Y\uparrow\downarrow\ _Y \\ D^b\ X, & \longrightarrow & D^b\ Y, \end{array}$$

With these two notions in place, we can state the following:

Theorem 4.2.19. Let X and Y be smooth projective varieties with étale cyclic coverings $_L: X$ X and $_M: Y$ Y associated to n-torsion line bundles L Pic X and M Pic Y . Then

(i) If $: D^b\ Y,$ $D^b\ X,$ is an equivalence satisfying

$$p_1 L \qquad p_2 M$$

in $D^b\ X\ \ Y,\ ^1$, where p_1 and p_2 are the natural projections from X Y, there exists an equivariant lifting $: D^b\ X,\ _L$ $D^b\ Y,\ _M$ of .

(ii) If $: D^b\ X,\ _L$ $D^b\ Y,\ _M$ is an equivariant equivalence, is the lift of an equivalence $: D^b\ X,$ $D^b\ Y,$.

In particular in the situation with L $_X$ torsion, we recover Bridgeland and Maciocia's original result in [BM98b, Thm. 4.5] on canonical covers. The above theorem is formulated as in [LP15, Thm. 10], and for the proof in the twisted setting one can follow the proofs as in [LP15], [BM98b, Thm. 4.5] or Huybrecht's book [Huy06, Prop. 7.18]. Some more explanation is given in [AW18, Prop. 2.1] in the twisted setting where the covers are the canonical ones. Since, though we have not mentioned this, twisted sheaves may be formulated in the language of stacks using $_m$-gerbes, the stack inclined reader may find the approach in [KS15] the most rigourosly and formally satisfying.

4.2.5 Twisted Chern Characters and the Mukai Lattice

Let X be a smooth projective variety with Brauer class , and denote by $K\ X,$ the Grothendieck group of the category Coh $X,$ of coherent -sheaves on X. Following [Huy17, Section 2.1], we define a Chern character map

$$\mathrm{ch}: K\ X, \qquad H\ \ X, \ .$$

First, represent by a cocycle $U_i, \ _{ijk}$ where $^{n}_{ijk}$ 1 and let E be a locally free -sheaf. Then $E \ ^{n}$ is naturally untwisted and we define

$$\text{ch} \ E \ : \ ^{n}\overline{\text{ch} \ E \ ^{n}} \quad H \quad X, \quad .$$

Note that this makes sense since the rank of a locally free twisted sheaf is always nonzero. Using that any coherent -sheaf on a smooth projective variety admits a finite resolution by locally free -sheaves ([C00, Lemma 2.1.4]), this extends to give a Chern character map

$$\text{ch} \quad : K \ X, \qquad H \quad X,$$

that is additive, multiplicative with respect to tensor product and satisfying the Grothendieck-Riemann-Roch theorem. Consequently, we get a Chern character for every object E $D^b \ X,$ via the map $D^b \ X,$ $K \ X,$ $, E$ $1 \ ^{i} \ E^i$. Using this we define the Mukai vector of an object E $D \ X,$ in the usual way as $v \ E$ ch E $\overline{td \ X}$. As in the untwisted case, given an equivalence $: D \ Y,$ $D \ X,$ with kernel , the induced map $q \ v \ p$ in cohomology induces a commutative diagram

$$\begin{array}{ccc} : D^b \ Y, & \longrightarrow & D^b \ X, \\ \downarrow v & & \downarrow v \\ : H \quad Y, & \longrightarrow & H \quad X, \end{array}$$

Here is an (ungraded) isomorphism of -vector spaces.

For objects E, F $D^b \ X,$, we define the Euler characteristic E, F as in the untwisted case as

$$E, F \qquad \sum_{i} 1 \ ^{i} \dim \ \text{Ext}^i \ E, F \ .$$

Let v v_j $H \ X,$ and define v $\sum_j$ $\overline{\ \ 1} \ ^{j} v_j$. Then the *Mukai pairing* on $H \ X,$ is given by

$$v, w \qquad \int_X v \quad w,$$

where the dual v is defined by

$$v \ : \quad v \ \frac{\overline{\text{td} \ X}}{\overline{\text{td} \ X}}.$$

As in the untwisted case, we have the following result relating the Mukai pairing with the Euler characteristic.

Lemma 4.2.20. For E, F $D^b \ X,$, we have E, F $v \ E \ , v \ F$.

Proof. It is enough to show this when E is a locally free -sheaf and F any -sheaf. Then E, F X, E F , and E F is untwisted, so we can apply Hirzebruch-Riemann-Roch to get

$$X, E \quad F \qquad \int_X \text{ch} \ E \quad F \quad \text{td} \ X \ .$$

Using multiplicativity of the twisted Chern character, the following string of equalities holds:

$$E,F \quad X,E \quad F$$
$$_X \mathrm{ch}\ E \quad F \quad \mathrm{td}\ X$$
$$_X \mathrm{ch}\ E \quad \overline{\mathrm{td}\ X} \quad \mathrm{ch}\ F \quad \overline{\mathrm{td}\ X}$$
$$_X v\ E \quad v\ F\ .$$

Hence we need to show that $v\ E$ $v\ E$. Writing out $v\ E$, we get $v\ E$ $\mathrm{ch}\ E$ $\overline{\mathrm{td}\ X}$, so we are left to show $\mathrm{ch}\ E$ $\mathrm{ch}\ E$.

Let n ord and choose a cocycle $U_i,\ _{ijk}$ where $^n_{ijk}$ 1 as before . Then $\mathrm{ch}\ E$ $^n\ \overline{\mathrm{ch}\ E\ ^n}$. Here G : $E\ ^n$ is a usual locally free sheaf, and the Chern classes $c_k\ G$ satisfies $c_k\ G$ $1\ ^k c_k\ G$, so that in particular $\mathrm{ch}\ G$ $\mathrm{ch}\ G$. Using then that $E\ ^n$ $E\ ^n$, we see that

$$\mathrm{ch}\ E \quad ^n\ \overline{\mathrm{ch}\ E\ ^n}$$
$$^n\ \overline{\mathrm{ch}\ E\ ^n}$$
$$^n\ \overline{\mathrm{ch}\ E\ ^n}$$
$$^n\ \overline{\mathrm{ch}\ E\ ^n}$$
$$\mathrm{ch}\ E\ .$$

This concludes that $v\ E$ $v\ E$ and thus E,F $v\ E\ ,V\ F$.

□

As in the untwisted case, the induced cohomological transform ^{H} from an FM-transform : $D^b\ Y,$ $D^b\ X,$ is an isometry with respect to the Mukai pairing:

Lemma 4.2.21. Let : $H\ Y,$ $H\ X,$ be the induced cohomological transform from an equivalence : $D\ Y,$ $D\ X,$. Then is an isometry with respect to the Mukai pairing.

Proof. Let p,q be the projections Y X Y and Y X X, respectively, and let n $\dim X$ $\dim Y$. For v,w $H\ Y,$ and u $H\ Y\ X,$, using the equalities $v.w$ $v\ .w$, $q\ v$ $q\ v$ and $p\ u$ $1\ ^n p\ u$, this is proved as usual: If is the kernel of , then ^{1} has kernel $q\ _X n$, and the shift functor n acts on cohomology by multiplication by $1\ ^n$. Hence up to this sign, the cohomological transforms given by v and v are inverse to eachother, and one calculates

$$v, \quad w\ _X \quad ^1\ v\ ,w\ _Y.$$

□

Suppose X is an Abelian or bielliptic surface and consider the Chern character map

$$\mathrm{ch}\colon K\ X \qquad H^2\ X, \quad .$$

We define the *algebraic Mukai lattice* $H_{alg}\ X,$ as the image of this map, which coincides with $\quad$ Num X $\quad$. To see this, on $K\ X$ we have the Euler pairing

$$E\ ,\ F \quad : \qquad 1^{\ i} \dim \mathrm{Ext}^i\ E, F\ ,$$
$$i$$

and on $H_{alg}\ X,$ we have the non-degenerate Mukai pairing

$$v, w \ : \quad {}_X v\ .w,$$

where now for $v \quad v_0, v_2, v_4\ , v \quad v_0, \quad v_2, v_4$. As $\quad {}_X \quad 0$, the Todd class of X is $1, 0, 0$, so for $v \quad r, D, s\ , w \quad r\ , D\ , s$ we get

$$v, w \quad rs \quad r\ s \quad D\ \ D\ .$$

Moreover, we have Lemma 4.2.29 that $E\ ,\ F \qquad \mathrm{ch}\ E\ , \mathrm{ch}\ F$. Let $K_{num}\ X$ be the numerical Grothendieck group of X, i.e., the quotient $K\ X\ /$ of $K\ X$ by the left radical of the Euler form. By Serre duality, the Euler form descends to a non-degenerate bilinear form on $\mathrm{K}_{num}\ X$. Now, a class lies in the radical of the Euler form if and only if it lies in the radical of the Mukai pairing, and a class lies in the radical of the Mukai pairing if and only if it has zero Chern character. So ker ch $\quad$ and we have $K_{num}\ X \quad$ im ch . If we restrict the image of ch to $H^2\ X,$ we get Num X since the class $\ {}_X\ D$ of a numerically trivial divisor D is equivalent to $\ {}_X$. Thus $H_{alg}\ X, \qquad$ Num $X \qquad$.

4.2.6 Moduli Spaces of Sheaves

Moduli spaces of sheaves on a smooth projective variety X has some intimate connections with the problem of derived equivalences, both in the twisted and non-twisted cases. This was first explored by Mukai in his seminal work on derived equivalences of abelian varieties with their dual, where a universal family of sheaves was used as a kernel for a Fourier-Mukai transform. Later work by Mukai and Orlov extended this to K3 surfaces, where derived equivalence between K3 surfaces expresses one as the moduli space of (stable) sheaves on the other. Further work by Bridgeland & Macioca uses the interplay between moduli spaces of sheaves and derived equivalences to study bielliptic surfaces. We will do the same.

In general, the moduli space of sheaves on a variety will form a stack. To get an actual scheme, conditions of stability must be introduced on our sheaves. As a quick reminder, let X be a smooth projective variety with ample divisor H and E a coherent sheaf on X. The *Hilbert polynomial* of E is defined as

$$P_E \quad P_{H,E}\ m \ : \quad E \quad {}_X\ mH \quad .$$

The Hilbert polynomial, as is well known, may be written on the form

$$P_{H,E}\ m \qquad \overset{\dim\ E}{\underset{i\ \ 0}{}} \quad {}_i\ E\ \frac{m^i}{i!}$$

where $_i$ is an integer, for any i $0, \dots, \dim E$ and where the dimension $\dim E$ of a coherent sheaf is defined as the dimension of its support. It is said to be of *pure* of dimension d if for every subsheaf F of E, d $\dim E$ $\dim F$. The *normalized* Hilbert polynomial of a torsion-free coherent sheaf E with respect to H is defined to be

$$p_E \quad p_{H,E} \; m \quad \frac{P_{H,E} \; m}{\mathrm{rk} \; E}.$$

De nition 4.2.22. (Gieseker stability). Fix an ample divisor H. A torsion-free coherent sheaf E is stable (resp. semistable) if $p_{H,E} \; m$ $p_{H,F} \; m$ (resp. $p_{H,E} \; m$ $p_{H,F} \; m$) for m 0 and all proper subsheaves F E.

De nition 4.2.23. (Slope stability). Fix an ample divisor H. The slope of a torsion-free coherent sheaf E is de ned as

$$E \quad \frac{c_1 \; E \quad H}{\mathrm{rk} \; E}.$$

A torsion-free sheaf E is -stable (resp. -semistable) if F E (resp. F E) for all non-trivial subsheaves F E with 0 $\mathrm{rk} \; F$ $\mathrm{rk} \; E$.

The following lemma gives the relationship between these various stability notions:

Lemma 4.2.24 ([HL10, Lemma 1.2.13]). We have the following implications:

E is stable E is stable E is semistable E is semistable.

Proposition 4.2.25 ([Pot17, Proposition 3.4.6]). Fix an ample divisor H on X. If E is a coherent sheaf that is stable with respect to H, then $\mathrm{Hom} \; E, E$, i.e., E is simple.

In [Gie77], Gieseker considers families of semi-stable sheaves and constructs moduli spaces using Geometric Invariant Theory. A more modern treatment can be found in [HL10].

Recall that a family of sheaves on X parametrized by S is a coherent $_X$ $_S$-module , flat over S. Two such families and parametrized by S are equivalent if there is a line bundle L on S such that q L, where q is the projection X S S. For a closed point s S, we denote by $_s$ the restriction of to the fibre X_s over s. Now for the moduli functor: Keeping H fixed as before, let : Sch^{op} Sets be the functor that sends a scheme S to the set of equivalence classes of families on X parametrized by S, such that for all s S, $_s$ is semi-stable with respect to H. If $f\colon T$ S is a morphism of schemes, the map id_X f defines a map S T .

De nition 4.2.26. A scheme M_H is a coarse moduli space for semi-stable sheaves on X with respect to H if M_H corepresents the functor , i.e., there exists a morphism of functors : h_{M_H}, where h_{M_H} Hom $, M_H$, such that for any other morphism of functors : h_Y, there is a unique morphism $f\colon M_H$ Y such that f , where f : h_{M_H} h_Y is the functor given by composition with f. If M_H represents the functor , i.e., h_{M_H}, then M_H is a ne moduli space.

To translate this into the language of quasi-universal and universal sheaves, or families, we notice that if M_H is a fine moduli space, the identity morphism id Hom M_H, M_H corresponds to family on X M_H, unique up to a twist by a line bundle on M_H. This is the *universal family*. The quasi-universal family is different, being a family parametrized by M_H such that if is a family of sheaves parametrized by S and $: S$ M_H is the corresponding morphism which maps closed points s S to the sheaf $_s$, there is a locally free sheaf W on S of finite rank such that q W . This family is universal if W is a line bundle.

By results due to Gieseker, Maruyama and Simpson, a coarse moduli space always exists:

Theorem 4.2.27 ([Huy06, Theorem 10.18]). A coarse moduli space M_H always exists. Moreover, M_H is a projective variety.

To understand the local structure of the moduli space, i.e., its dimension and smoothness at a point, we have the following characterization

Theorem 4.2.28 ([HL10, Theorem 4.5.4 and Corollary 4.5.2]). Let E be a stable sheaf represented by a point E M_H . Then the Zariski tangent space to M_H at E is given by

$$T_{E} M_H \quad \mathrm{Ext}^1 \; E, E \; .$$

If Ext^2 E, E 0, then M_H is smooth at E .

Specializing to the case of surfaces (smooth and projective), we will be interested in the moduli space M_H v of sheaves with a given Chern class v r, D, s NS X . This works, because for a family parametrized by a connected scheme S, the Chern classes of $_s$ is constant for all s S. Thus M_H v is just the union of those components of M_H containing sheaves whose Chern class is v.

Proposition 4.2.29 ([Huy06, Lemma 10.22 and Corollary 10.23]). Let X be a smooth surface and v r, D, s NS X . Suppose that there exists v such that v, v 1. Then every semi-stable sheaf is stable, there exists an ample class H such that gcd r, D H, s 1 and M_H v is a ne moduli space.

The relationship between derived equivalences and moduli spaces of sheaves can be illustrated by the following result by Bridgeland, giving sufficient criteria on a moduli space for the existence of a derived equivalence. First recall that a sheaf E on X is *special* if E $_X$ E.

Proposition 4.2.30 ([BM01b, Corollary 2.8]). Let X be a smooth projective surface with a xed polarization, and let Y be a smooth, ne, complete, two-dimensional moduli space of special, stable sheaves on X. Then there is a universal sheaf on Y X and the associated functor $: D^b$ Y D^b X is an equivalence.

Now, some properties of elements v H^2 $X,$ gives nice properties of the moduli space of (semi-)stable sheaves of class v on X. First the notion of a polarization H being general with respect to v.

De nition 4.2.31. Let v H^2 $X,$. We say that H is general with respect to v (or does not lie on a wall with respect to v) if for every -semistable sheaf E with ch E v and every 0 F E which satis es F E , then

$$\frac{c_1\ F}{\mathrm{rk}\ F} \quad \frac{c_1\ E}{\mathrm{rk}\ E}.$$

De nition 4.2.32. Let v r, D, S H^2 $X,$ with D NS X . Then

(i) A class v is primitive if v is undivisible, i.e., if v dv_0 with d , then d 1. Equivalently, there exists v such that v, v 1.

(ii) A class v is isotropic if v^2 v, v 0.

Proposition 4.2.33 ([Pot17, Proposition 3.7.2]). Let X be a bielliptic surface and : X X the canonical cover. Let v r, D, s with r 0 and suppose that v w for some w N X and that v is primitive and isotropic. Choose an ample divisor H which is general with respect to v. Then there exists a two-dimensional, projective, smooth, ne moduli space M_H v of stable, special sheaves on X of class v. Moreover, the universal sheaf on M_H v X induces an autoequivalence of D^b X such that the Chern class of $_x$ is v for any closed point x X.

Finally, a celebrated result by Atiyah will ensure non-emptiness of the moduli spaces we will consider later.

Theorem 4.2.34 ([Huy06, Thm. 12.23]). Let r, d be two coprime integers, and let D be a mooth elliptic curve. Then

(i) Any simple vector bundle of rank r and degree d on D is stable.

(ii) For any line bundle L Pic^d D there exists a unique stable vector bundle of rank r and with determinant isomorphic to D.

4.3 Twisted Derived Equivalences of Bielliptic Surfaces

Here we will study the derived equivalence D^b $Y,$ D^b X in the situation where X is a bielliptic surface. This situation was studied by Bridgeland and Maciocia in [BM01a] the untwisted setting, where they showed the following:

Theorem 4.3.1. Let X be a bielliptic surface. Then the only Fourier-Mukai partner of X is itself.

In the twisted case we first conclude that also Y must be a bielliptic surface, and in the case where is nontrivial that it is of different type than X.

In general, for a smooth projective variety X and a class Br X with image $^-$ H^3 $X,$, we let K^i_{top} X denote the topological K-theory of X, and we let K^i_{top} $X,^-$ denote the twisted topological K-theory of X. For the present purposes, we will not go into details here about (twisted) topological K-theory, but use the following facts:

Proposition 4.3.2 ([Mou19, Cor.1.2], [CS15, Section 6.3]). Let X and Y be smooth projective varieties with classes Br X , Br Y and images $\bar{}$ $H^3(X, \quad)$, $\bar{}$ $H^3(Y, \quad)$. Then if $D^b(X, \quad) \quad D^b(Y, \quad)$, $K^i_{top}(X, \bar{}) \quad K^i_{top}(Y, \bar{})$.

Proposition 4.3.3 ([AW18, Prop. 1.1]). If X is any compact complex surface, then

$$K^1_{top} \quad H^1(X, \quad) \quad H^3(X, \quad) .$$

If Br X has image $\bar{}$ $H^3(X, \quad)$, then

$$K^i_{top}(X, \bar{}) \quad H^1(X, \quad) \quad H^3(X, \quad)/\bar{}.$$

Using these two facts, we can show the following:

Lemma 4.3.4. Let X be a bielliptic surface and Y a smooth projective variety such that there is an equivalence $: D^b(Y, \quad) \quad D^b(X)$ with nontrivial. Then Y is a bielliptic surface of di erent type than X.

Proof. As noted earlier, dimension and order of canonical bundle is preserved by any derived equivalence, twisted or untwisted, so Y is a surface with torsion canonical bundle and is thus minimal. To conclude it is bielliptic, we only need the second Betti number $b_2(Y) \quad 2$. This is ensured by the invariance of Hochschild homology in Theorem 4.2.16. Thus Y is bielliptic. For the other statement, by the two results above we have an isomorphism

$$H^1(Y, \quad) \quad H^3(Y, \quad)/\bar{} \quad H^1(X, \quad) \quad H^3(X, \quad) .$$

Since the torsion of $H^3(Y, \quad)$ coincides with the torsion of $H^2(Y, \quad)$, inspecting the di erent types of bielliptic surfaces (see Table 2.1) yields that X and Y cannot be of the same type when $\quad 1$. E.g. if Y is of type 1 then $H^3(Y, \quad)_{tor} \quad /2 \quad /2$, and so X cannot be of type 1, but must be of type 2, as the torsions in the above isomorphism would not coincide. □

Remark 4.3.5. In the last lemma, by table 2.1 and the isomorphisms above with the fact that the order of the canonical bundle is preserved, we have the following possibilites for the types of X and Y.

(i) Y is of type 1 and X is of type 2.

(ii) Y is of type 3 and X is of type 4.

(iii) Y is of type 5 and X is of type 6.

Also worth to note is that the same argument gives that X and Y must be of the same type for an equivalence $D^b(Y, \quad) \quad D^b(X, \quad)$ where $, \quad 1$.

Next we will need to know something about the properties of the images of skyscraper points.

Lemma 4.3.6. Let $: D^b(Y, \quad) \quad D^b(X)$ be an equivalence with X Abelian or bielliptic, $y \quad Y$ a closed point, $E : \quad _y$ and $v :$ ch $E \quad H_{alg}(X, \quad)$. Then v cannot be primitive in $H_{alg}(X, \quad)$ unless $\quad 1$.

Proof. Suppose v is primitive in $H_{alg}\ X,$. Then there is v $H_{alg}\ X,$ such that v, v 1. Here v ch e where e $K\ X$, and because the natural map $D\ X$ $K\ X\ , F$ $_i$ $1^{\ i}\ H^i\ F$, is surjective, v ch E for some E $D\ X$. Then we have

$$1 \quad v, v \quad \text{ch}\ E\ , \text{ch}\ E \quad 0, 0, 1\ , \text{ch} \quad ^1\ E \quad r \quad ^1\ E \ ,$$

and since the rank of any locally free -twisted sheaf is divisible by the order of , we must have 1.

□

Lemma 4.3.7. Let $: D^b\ Y,$ $D^b\ X$ be an equivalence between bielliptic surfaces Y and X. Denote by $: D^b\ Y,\ _Y$ $D^b\ X$ the lifting of to the canonical covers $_Y : Y$ Y and $_X : X$ X. If 1, we have Y X and 1.

Proof. Let G $/n$, n ord $_Y$ ord $_X$. Recall that the lifting of is equivariant in the sense that there is an automorphism $: G$ G such that g g for all g G. This is equivalent to the equivariance of the kernel , i.e., g g . Note that this makes sense, as once we x a cocycle $U_i,\ _{ijk}$ representing , the cocycle $_Y^{\ 1}\ U_i\ ,\ _{ijk}$ $_Y$ representing $_Y$ is equal to the cocycle $g\ ^1\ _Y^{\ 1}\ U_i\ ,\ _{ijk}$ $_Y$ g representing g $_Y$ since $_Y$ g $_Y$, so there is no non-canonical choices being made.

Suppose now that $_Y$ 1. Then there is a cocycle $U_i,\ _{ijk}$ representing such that the cocycle $_Y^{\ 1}\ U_i\ ,\ _{ijk}$ $_Y$ representing $_Y$ satis es $_{ijk}$ $_Y$ $_{ij}\ _{jk}\ _{ki}$ for some $_{ij}$ $_Y$ $_Y^{\ 1}\ U_{ij}$ (to be more precise, we x a cocycle $U_i,\ _{ijk}$ representing , and since 1, there is a cover V_j of Y such that : $_Y$ satis es $_{ijk}$ $_{ij}\ _{jk}\ _{ki}$, and we take a re nment of $_Y\ V_j$ and U_i). But the cocycle $1\ _{ijk}$ also represents $_Y$ on this cover, and sending a regular sheaf F to $F_i,\ _{ij}$ id , where F_i is the restriction of F to $_Y^{\ 1}\ U_i$, yields an equivalence $D\ Y$ $D\ Y,$. Since the equality g g holds, it holds in particular for represented as a 1_{ijk} -sheaf, but a 1_{ijk} -sheaf for the cover V_i X, V_i : $_Y^{\ 1}\ U_i$, is nothing but the glueing data $V_i,\ _i$ for a regular sheaf on Y X, and the local equalitities g g $_i$ $_i$ glue along the gluing of the $_i$ s to g g . The induced exact functor $D\ Y$ $D\ X$ is clearly an equivalence, and it is equivariant since its kernel is. Hence we conclude that this equivalence is the lift of an equivalence $D\ Y$ $D\ X$, implying Y X and 1 by Lemma 4.3.4. □

Recall (see 2.1) that for a bielliptic surface X A B/G, the numerical group Num X is generated by the classes $A_0 : \frac{1}{\text{ord}\ _X} A$ and $B_0 : \frac{1}{_X} B$ where $_X$ $\frac{G}{\text{ord}\ _X}$ and A, B denotes the classes of the generic fibers of the projections X B and X A, respectively.

Lemma 4.3.8 ([Nue]). Let X A B/G be a bielliptic surface with its canonical cover $: X$ X. A Mukai vector v r, aA_0 bB_0, s $H_{alg}\ X,$ is primitive if and only if gcd r, a, b, s 1. For a primitive v, set

$$v \ : \ \text{gcd}\ r, a, \frac{\text{ord}\quad _X}{_X} b, \text{ord}\quad _X\ s\ .$$

Then v divides ord $_X$ and $\frac{v}{v}$ is primitive.

Proof. If v is primitive there exists a Mukai vector $v \quad r, a\, A_0 \quad b\, B_0, s$ such that $v, v \quad 1$. Then $rs \quad r\, s \quad ab \quad a\, b \quad 1$, so there cannot be a common factor of r, a, b and s. Conversely, if $d \quad 1$ divides r, a, b and s, we have that $v \quad dv$ and so $v, w \quad d\; v, w \quad 1$ for any Mukai vector w, and consequently v cannot be primitive. Suppose that v is primitive and that a prime p divides $\;v\;$. Then p must divide ord $\quad_X$ lest p divides r, a, b and s, which is absurd. By looking again at Table 2.1, it follows that $\;v\;$ can be written on the form $2^i 3^j$ with the understanding that i or j vanishes if 2 or 3 does not divide ord $\quad_X$, and that ord $\quad_X$ can be written as $2^k 3^l$ with the same convention of k or l vanishing. We cannot have both $i \quad k$ and $3 \quad l$, because that would imply that either 2 or 3 divides gcd r, a, b, s. Hence $\;v\;$ divides ord $\quad_X$.

Now we can write $r \quad v\; r\;, a \quad v\; a\;, \frac{\text{ord} \quad_X}{\quad_X} b \quad v\; b\;$, and ord $\quad_X\; s \quad v\; s$ for some integers $r\;, a\;, b\;, c\;$. Denoting by A_X and B_X the bers of the elliptic brations of $\tilde{X}$, the nal claim follows from the fact that $\;v$ cannot be divisible by more than $\;v$ and that

$$v \quad r, aA_X \quad b\frac{\text{ord} \quad_X}{\quad_X} B_X, \text{ord} \quad_X\; s \quad v \quad r\;, a\; A_X \quad b\; B_X, s$$

where $s \quad a\; b \quad c_2$ so an appropriate c_2 may be chosen.

□

We are now ready to prove the main result of this chapter:

Theorem 4.3.9. Let X be a bielliptic surface and Y a smooth projective variety. If there exists an equivalence $D^b\; Y, \quad D^b\; X\;, Y \quad X$ and consequently $\quad 1$.

Proof. Let $\;: D^b\; Y, \quad D^b\; X$ be the equivalence. Take a closed point $y \quad Y$, let $E : \quad_y$ and let v be the Mukai vector of E. Then v is isotropic with respect to the Mukai pairing because $v^2 \quad E, E \quad_y, \;_y \quad 0$. We may write $v \quad dv_0$ where v_0 is isotropic and primitive in $H_{alg}\; X, \quad$. Notice that d must divide $n :\;$ ord $\quad_X$, because ord $\quad$ divides n and so there is a locally free $\;$-twisted sheaf F of rank n on Y, and $n \quad 0, 0, 1\;$, ch $F \quad d\; v_0$, ch $\quad F\quad$.

Suppose rst that that the rank of E is 0 and write $v_0 \quad 0, aA \quad bB\;, s$. From $v_0^2 \quad 0$ we get that $a \quad 0$ or $b \quad 0$, and v_0 is of the form $0, rf, s$ where f denotes the class of a general ber of one of the elliptic brations, call it $\;: X \quad C$. Since v_0 is primitive, by Proposition 4.2.29 there is an ample H such that there exists a ne moduli space $M_H\; v_0$ parametrizing stable sheaves of class v_0 on X. Moreover, r and s are coprime and hence by Theorem 4.2.34 the moduli space is nonempty. The tangent space of $M_H\; v_0$ at a point $F \quad M_H\; v_0\;$, where F is a stable sheaf, is given by $T_F\, M_H\; v_0 \quad \text{Ext}^1\; F, F\;$.

Now, since the moduli space parametrizes stable sheaves with Chern character $0, rf, s\;$, the sheaves lives on bers of $\;$. By restricting to the component of $M_H\; v_0$ that contains the stable sheaves on smooth bers we may assume they are special, and because they are stable they are simple. Then Serre duality reads $\text{Ext}^2\; E, E \quad \text{Hom}\; E, E \quad_X$ Hom E, E so the dimension of the tangent space is

$$\dim \text{Ext}^1\; E, E \quad 2 \quad \text{ch}\; E\;, \text{ch}\; E \quad 2 \quad v, v \quad 2.$$

The moduli space has a natural elliptic bration $\;: M_H\; v_0 \quad C$ where $\quad$ maps $F\;$, a sheaf concentrated on a ber X_c, to the base point $x \quad C$. From Theorem 4.2.34 (ii), generically the determinant map $\mathrm{det}\colon \quad ^{1} \; c \qquad \mathrm{Pic}^{s} \; X_c$ identi es the bre $\quad ^{1} \; c$ with $\mathrm{Pic}^{s} \; X_c \qquad X_c$. Thus the dimension of $M_H \; v_0$ is at least 2 and we conclude it is a smooth projective surface.

Since we have smooth, ne, complete, two-dimensional moduli space of special, stable sheaves on X we can apply Proposition 4.2.30 and get that the universal sheaf induces an equivalence $\; : D^b \; M_H \; v_0 \qquad D^b \; X$ mapping $0, 0, 1$ to v_0 in cohomology. By Theorem 4.3.1, $M_X^H \; v_0 \qquad X$, and so gives an autoequivalence of $D^b \; X$, which after composition by gives an equivalence $D^b \; Y, \qquad D^b \; X$ mapping $\; _y$ to a sheaf with Chern character $0, 0, d$. Since the rank and rst Chern class of this sheaf is 0, it has 0-dimensional support, and because it is simple we must have $d \quad 1$. But then v is primitive, and Lemma 4.3.6 implies $\quad 1$ and consequently $Y \quad X$ by Theorem 4.3.1.

Suppose now that $r \quad 0$ and that $n : \; \mathrm{ord} \; _X$ is prime. Let $\; : \tilde{X} \qquad X$ be the canonical cover. We can write $\; v_0 \qquad v_0 \; w$ with w isotropic and primitive and $\; v_0 \quad 1$ or $\; v_0 \quad n$. If $\; v_0 \quad n$, we have $nv_0 \qquad v_0 \quad n \quad w$, so $v_0 \qquad w$. Then by Proposition 4.2.33, we get an autoequivalence $D^b \; X \qquad D^b \; X$ mapping $0, 0, 1$ to v_0 in cohomology, and we can conclude as above that $d \quad 1, \qquad 1$ and $Y \quad X$.

For $\; v_0 \quad 1$, denote by $\tilde{}$ the lifting of to the canonical covers. From $\; v_0 \quad w$ we get that $n \tilde{} \; 0, 0, 1 \qquad v \quad dw$, so $w \quad \frac{n}{d} \tilde{} \; 0, 0, 1$. Since w is primitive, $n \quad d$ and $w \quad \tilde{} \; 0, 0, 1$. But then Lemma 4.3.6 implies that $\quad 1$, and $\tilde{}$ becomes an equivariant equivalence $D^b \; \tilde{Y} \qquad D^b \; \tilde{X}$ that descends to an equivalence $D^b \; Y \qquad D^b \; X$, so $Y \quad X$ again by Theorem 4.3.1 which forces $\quad 1$ by Lemma 4.3.4.

Finally, in the case where n is composite, we can assume that $n \quad 4$ by Table 2.1 and Remark 4.3.5 because the other cases yield a trivial Brauer group for Y. More precisely, Y is of type 3 and X is of type 4. Let $p\colon Y \qquad Y$, $q\colon X \qquad X$ be the étale cyclic coverings of degree 2 associated to $\; ^2_Y$ and $\; ^2_X$, respectively. Then as we have seen, X and Y are bielliptic surfaces, and since is an equivalence, $p_1 \; ^2_Y \qquad p_2 \; ^2_X$ where is the kernel, so that lifts to an equivariant transform $\tilde{} : D \; Y \; , p \qquad D \; X$. But here $\mathrm{ord} \; _X \quad 2$, so we know that $p \quad 1$ from the previous part. Then by Theorem 3.4.3 $\quad 1$ because the Brauer map is injective in this case, so $Y \quad X$.

□

4.4 Going from here

The original goal of this thesis was to prove a theorem analogous to 4.3.1, i.e., that twisted bielliptic surfaces have no non-trivial twisted Fourier-Mukai partners. On that account we have shown that this is true in the situation where one of the bielliptic surfaces is twisted and the other is not. So in studying an equivalence $D^b \; Y, \qquad D^b \; X, \quad$ where Y is a bielliptic surface, it is safe to assume that $\; , \qquad 1$. In this section we will think a bit informally on the difficulties in this twisted-twisted case.

In the paper [BM01a] (corrected in [BM19]) of Bridgeland and Maciocia, the technique of the proof relies on their so-called *relative Fourier-Mukai transforms* for elliptic surfaces. Namely, for an elliptic surface $\; : X \qquad C$, define to be the smallest positive integer such that has a holomorphic -multisection. Equivalently, letting f be the class of a smooth fiber, we have that

$$\min \; f \quad D \quad 0 \; D \quad \mathrm{Num} \; X \quad .$$

Now, supposing we have integers $a \quad 0$ and b with gcd $a \quad , b \quad 1$ we can construct the relative Jacobian $J_X \; a, b$, the moduli space of pure dimension 1 stable sheaves of class a, b supported on smooth fibers of . So the general point of $J_X \; a, b$ correspond to a rank a, degree b stable vector bundle supported on a smooth fiber of . Here we actually have $J_X \; a, b \quad J_X \; b$, and the point is to construct equivalences between the derived category of X and that of the derived category of $J_X \; b$. This is done using the following theorem.

Theorem 4.4.1 ([BM01a, Thm. 4.1]). Let $: X \quad C$ be an elliptic surface and take an element

$$\begin{matrix} c & a \\ d & b \end{matrix} \quad \mathrm{SL}_2$$

such that divides d and $a \quad 0$. Then there exists an equivalence $: D^b \; J_X \; b \quad D^b \; X$ such that for any closed point $y \quad J_X \; b$, $\quad _y$ has Chern character $0, af, b$, where f is the class of a bre. Moreover, the functor satis es

$$\begin{matrix} r & E \\ d & E \end{matrix} \qquad \begin{matrix} c & a \\ d & b \end{matrix} \quad \begin{matrix} r \; E \\ d \; E \end{matrix}$$

for all objects $E \quad D^b \; J_X \; b$.

For bielliptic surfaces, relative Jacobians with respect to either fibration yields an isomorphic surface, so in this case the relative Fourier-Mukai transforms induces autoequivalences of the derived category.

The original proof of 4.3.1 only works in the case where the Néron-Severi space of the canonical cover has rank 2, i.e., in the cyclic case, and a corrected proof is provided in [BM19]. The main technique of the proof remains the same, however, and relies on some luxuries we do not have in the case of twisted sheaves. This provides some unique difficulties in proving the theorem.

So suppose we have an equivalence $: D^b \; Y \quad D^b \; X$. As we have seen is a sheaf-transform, so up to a shift we can assume that takes skyscraper sheaves of points on Y to sheaves on X of class

$$v \qquad _y \qquad r, aA_0 \quad bB_0, s \qquad \mathrm{Num} \; X \qquad .$$

The idea is to cleverly use the relative Fourier-Mukai transforms to induce autoequivalences of $D^b \; X$, that when composed with , will send $\quad _y$ to a sheaf of class $0, 0, 1$ (or at the very least to a sheaf with a class such that $r \quad 0$). This is the same we did when Y was twisted but X was not. The first difficulty we met in that case was the issue of v not being primitive straight away. When Y is not twisted we get primitivity for free, because we have the structure sheaf $\quad _Y$ on Y in that case so that

$$1 \qquad \mathrm{ch} \quad _Y \; , \mathrm{ch} \quad _y \qquad \mathrm{ch} \qquad _Y \; , \mathrm{ch} \qquad _y \quad .$$

The primitivity of v is used to great effect to get necessary divisibility conditions to apply Theorem 4.4.1 correctly. In the twisted case, not only do we have that v is not necessarily primitive a-priori, which we solved when X was untwisted, but v is in general a rational class. This makes the whole framework of moduli spaces of sheaves very different, and we do not have access to the many tools being used for regular sheaves.

However, it is possible to construct moduli spaces of twisted sheaves as Yoshioka does in [Yos04] by using the equivalence of categories between twisted sheaves and certain sheaves on an associated Brauer-Severi variety lying above X, or as done by Lieblich in [Lie07] using the language of $\ _m$-gerbes.

To avoid potential obstacles with moduli spaces of twisted sheaves, if that is indeed the correct way to go about this, we wanted initially to use the lifting of the equivalence to the equivariant equivalence on the canonical covers above. Hence we needed to understand the Brauer map. Indeed, if the pullbacks of the Brauer classes are trivial, one can descend the equivalence to an equivalence below between the untwisted varieties, and invoke Theorem 4.3.1. The initial hope was that one could classify how triviality of the Brauer map moves in the moduli space of bielliptics, and use some deformation argument to reduce to the untwisted case. With the twisted-untwisted result of this chapter, one would need just one pullback to be trivial, potentially simplifying such an approach. Alas, this dream did not come true.

Another idea was to use the Rouquier isomorphism in [Rou11], Théoréme 4.18, asserting that an equivalence $D^b\ Y \qquad D^b\ X$ gives an isomorphism of algebraic groups

$$\mathrm{Aut}^0\ Y \qquad \mathrm{Pic}^0\ Y \qquad \mathrm{Aut}^0\ X \qquad \mathrm{Pic}^0\ X\ .$$

As Aut^0 fixes a given Brauer class, this isomorphism holds in the twisted setting whenever the characteristic of the base fields does not divide the order of (Tirabassi, private correspondance). Since X and Y are of the same type, we have $Y \quad A_Y \quad B/G$ and $X \quad A_X \quad B/G$, as the group G is the same and the same lattice defining B can be taken. Using that $\mathrm{Aut}^0\ Y \quad A_Y, \mathrm{Aut}^0\ X \quad A_X$, if one could extract an isomorphism $\mathrm{Aut}^0\ Y \quad \mathrm{Aut}^0\ X$ one would get far in producing an isomorphism $Y \quad X$.

These may be naive attempts at something requiring new techniques, but we suspect the approach to involve moduli spaces because of the heuristic that there is an intimate connection between derived categories and moduli spaces of sheaves. Twisted sheaves can come into this picture when introducing non-fine moduli spaces as in [C00, Prop. 3.3.2]. Given any flat, projective morphism X/S, a relatively ample sheaf $\quad 1$ and Hilbert polynomial P, the relative moduli space M/S of stable sheaves with Hilbert polynomial P on the fibers of X/S admits a covering (analytic opens over , étale open otherwise) $X \quad U_i$ where there exists local universal sheaves $\ _i$. Furthermore, there exists an $\quad H^2\ M, \quad _M$ (only dependent on $X/S, \quad 1$ and P) together with transition isomorphisms $\ _{ij}$ making (U_i , $\ _{ij}$ into an -twisted sheaf. So the Brauer group functions as (in general part of) an obstruction group to the existence of universal sheaves.

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