Diagram spaces and multiplicative structures

Mirjam Solberg

Thesis for the degree of Philosophiae Doctor (PhD)
University of Bergen, Norway
2022

UNIVERSITY OF BERGEN

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Date of defense: 14.01.2022

Year: 2022

Title: Diagram spaces and multiplicative structures

Name: Mirjam Solberg

Print: Skipnes Kommunikasjon / University of Bergen

Acknowledgements

Throughout my work on this thesis I have received a great deal of support and assistance.

First and foremost my sincere thanks goes to my supervisor Christian Schlichtkrull. Thank you for introducing me to an interesting topic, for suggestions, feedback, valuable insights, encouragement and support while I was working on it. I have learnt a great deal from you, and from our collaboration and discussions. Thank you for the time and effort you have put in to help me complete this project. For this, I am extremely grateful.

I am also grateful for the friendly and stimulating environment of the topology group at the department. Special thanks to my co-supervisors Bjørn Ian Dundas and Morten Brun.

I thank Steffen Sagave for facilitating my exchange stay at the University of Bonn.

I would like to thank my colleagues at HVL for showing interest and encouraging me.

During my time at the mathematics department I have gotten to know many wonderful people. I thank you all for the shared experience of life as a phd student, with all its ups and downs. In particular I would like to acknowledge the friendships of Anastasia, Victor, Valentin, Nazanin, Andrea, Francesca and Eugenia.

Finally I want to thank my family for encouragement and support during all these years.

Abstract

This thesis explores diagrammatic E_n structures as models for E_n spaces.

Paper A: Braided injections and double loop spaces. (Christian Schlichtkrull and Mirjam Solberg.)
We consider a framework for representing double loop spaces (and more generally E_2 spaces) as commutative monoids. There are analogous commutative rectifications of braided monoidal structures and we use this framework to define iterated double deloopings. We also consider commutative rectifications of E_∞ spaces and symmetric monoidal categories and we relate this to the category of symmetric spectra.

Paper B: Weak braided monoidal categories and their homotopy colimits.(Mirjam Solberg.)
We show that the homotopy colimit construction for diagrams of categories with an operad action, recently introduced by Fiedorowicz, Stelzer and Vogt, has the desired homotopy type for diagrams of weak braided monoidal categories. This provides a more flexible way to realize E_2 spaces categorically.

Paper C: Operads and algebras in n-fold monoidal categories. (Mirjam Solberg.)
We develop the concept of n-fold monoidal operads and algebras over n-fold monoidal operads in n-fold monoidal categories. We give examples of n-fold monoidal operads whose algebras generalize the concepts of monoids, commutative monoids and n-fold monoidal structures, to the setting of an n-fold monoidal category.

Paper D: Higher monoidal injections and diagrammatic E_n structures. (Christian Schlichtkrull and Mirjam Solberg.)
We use the framework of n-fold monoidal categories to examine E_n structures in a diagrammatic setting. A major objective is to introduce the category $\mathcal{I}_n$ of n-fold monoidal injections as a counterpart to the symmetric monoidal category of finite sets and injective functions. This then leads to an n-fold monoidal version of the classical James construction. We also discuss applications to n-fold commutative strictification of E_n structures.

List of included papers

Paper A: Braided injections and double loop spaces.
Christian Schlichtkrull and Mirjam Solberg.
Published in: Trans. Amer. Math. Soc. 368 (2016), no. 10, 7305–7338.
Published by the American Mathematical Society.

Paper B: Weak braided monoidal categories and their homotopy colimits.
Mirjam Solberg.
Published in: Theory and Applications of Categories, Vol. 30, No. 3, 2015, pp. 40-48.

Paper C: Operads and algebras in n-fold monoidal categories.
Mirjam Solberg.

Paper D: Higher monoidal injections and diagrammatic E_n structures.
Christian Schlichtkrull and Mirjam Solberg.

Contents

Part I

Introduction and main results

Chapter 1

Introduction

1.1 Preliminaries

The theme of this thesis is multiplicative structures and commutativity. The simplest setting for this is sets, where we have the well known concept of a monoid, a set with an associative multiplication and an identity element. If the multiplication is commutative, we have a commutative monoid. Commutativity is here a strictly binary question, the monoid is commutative or it is not. Changing the setting to topological spaces we get a more interesting picture. We can consider topological monoids and commutative topological monoids, but also monoids where the multiplication is not strictly commutative, but commutative up to homotopy. There can also be levels of this commutativity, commutative up to a certain level of homotopies or commutative up to all higher homotopies.

Another setting for looking at commutative structure, which will feature heavily in this thesis, is categories. A monoidal category is a category equipped with a multiplication, it is called commutative if the monoidal product is commutative. Here there are also levels of commutativity. The most commonly considered structure is the symmetric monoidal category, where the monoidal product is commutative up to coherent isomorphisms. Mac Lane's famous description of the coherence theorem for symmetric monoidal categories, see [ML98, Chapter XI.1], is often shortened to "all coherence diagrams commute". Where symmetric monoidal categories represent the highest degree of commutativity, short of strict commutativity, braided monoidal categories represent the lowest possible degree. Filling out the spectrum, there are the n-fold monoidal categories introduced in [BFSV03] for each integer $n \geq 1$. The two lowest levels in the hierarchy, 1-fold monoidal categories and 2-fold monoidal categories, are equivalent to monoidal categories, and braided monoidal categories respectively. A symmetric monoidal category can be considered as an n-fold monoidal category for any $n \geq 1$.

In this preliminary section we will recall the definitions of the various monoidal category structures and also the operads associated with them. Operads are a very useful tool when it comes to the study of multiplicative structures and an integral part of this thesis. Finally we recall some definitions and results related to iterated loop spaces, the topological counterpart and inspiration for the definition of n-fold monoidal categories.

1.1.1 Monoidal category structures

Definition of a monoidal category from Chapter VII.1 in [ML98]:

Definition 1. A monoidal category, $(\mathcal{A}, \otimes, I, \alpha, \lambda, \varrho)$, consists of a category $\mathcal{A}$ together with a functor $\otimes\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, called the monoidal product, an object I, called the unit object, and isomorphisms

$$\alpha_{A,B,C}\colon (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C),$$

$$\lambda_A\colon I \otimes A \xrightarrow{\cong} A \qquad \text{and} \qquad \varrho_A\colon A \otimes I \xrightarrow{\cong} A$$

natural in $A, B, C \in \mathcal{A}$, called the associativity, left unit and right unit isomorphisms respectively. These isomorphisms must be such that the associativity pentagon

$$\begin{array}{ccccc} & & ((A \otimes B) \otimes C) \otimes D & & \\ & \swarrow^{\alpha_{A,B,C} \otimes \mathrm{id}} & & \searrow^{\alpha_{A \otimes B,C,D}} & \\ (A \otimes (B \otimes C)) \otimes D & & & & (A \otimes B) \otimes (C \otimes D) \\ \downarrow{\scriptstyle \alpha_{A,B \otimes C,D}} & & & & \downarrow{\scriptstyle \alpha_{A,B,C \otimes D}} \\ A \otimes ((B \otimes C) \otimes D) & & \xrightarrow{\mathrm{id} \otimes \alpha_{B,C,D}} & & A \otimes (B \otimes (C \otimes D)) \end{array}$$

commutes for all objects A, B, C, D in $\mathcal{A}$, and the triangle

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\ & {\scriptstyle \varrho_A \otimes \mathrm{id}} \searrow \quad \swarrow {\scriptstyle \mathrm{id} \otimes \lambda_B} & \\ & A \otimes B & \end{array} .$$

commutes for all objects A in $\mathcal{A}$. If the natural isomorphisms α, λ and ϱ are all identities, the monoidal category is called strict.

Definition of a monoidal functor from Chapter XI.2 in [ML98]:

Definition 2. A strong monoidal functor

$$F\colon (\mathcal{A}, \otimes, I, \alpha, \lambda, \varrho) \to (\mathcal{A}', \otimes', I', \alpha', \lambda', \varrho')$$

consists of a functor $F\colon \mathcal{A} \to \mathcal{A}'$ of the underlying categories, together with isomorphisms

$$\psi\colon I' \xrightarrow{\cong} F(I) \quad \text{and} \quad \varphi_{A,B}\colon F(A) \otimes' F(B) \xrightarrow{\cong} F(A \otimes B)$$

natural in $A, B \in \mathcal{A}$, such that the diagrams

$$\begin{array}{ccc}
(F(A) \otimes' F(B)) \otimes' F(C) & \xrightarrow{\alpha'_{F(A),F(B),F(C)}} & F(A) \otimes' (F(B) \otimes' F(C)) \\
\downarrow{\scriptstyle \varphi_{A,B} \otimes' \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id} \otimes' \varphi_{B,C}} \\
F(A \otimes B) \otimes' F(C) & & F(A) \otimes' F(B \otimes C) \\
\downarrow{\scriptstyle \varphi_{A \otimes B, C}} & & \downarrow{\scriptstyle \varphi_{A, B \otimes C}} \\
F((A \otimes B) \otimes C) & \xrightarrow{F(\alpha_{A,B,C})} & F(A \otimes (B \otimes C)),
\end{array}$$

$$\begin{array}{ccc}
I' \otimes' F(A) & \xrightarrow{\psi \otimes' \mathrm{id}} & F(I) \otimes' F(A) \\
& \searrow{\scriptstyle \lambda'_{F(A)}} & \downarrow{\scriptstyle \varphi_{I,A}} \\
& & F(I \otimes A) \\
& & \downarrow{\scriptstyle F(\lambda_A)} \\
& & F(A)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
F(A) \otimes' F(I) & \xleftarrow{\mathrm{id} \otimes' \psi} & F(A) \otimes' I' \\
\downarrow{\scriptstyle \varphi_{A,I}} & \swarrow{\scriptstyle \varrho'_{F(A)}} & \\
F(A \otimes I) & & \\
\downarrow{\scriptstyle F(\varrho_A)} & & \\
F(A) & &
\end{array}$$

commute for all $A, B, C \in \mathcal{A}$. If for each $A, B \in \mathcal{A}$, $\varphi_{A,B}$ and ψ are identities, the monoidal functor is called strict.

Definition of a symmetric monoidal category from Chapter XI.1 in [ML98]:

Definition 3. A monoidal category $\mathcal{A}$ is symmetric monoidal if it is equipped with a symmetry isomorphism

$$\vartheta_{A,B}\colon A \otimes B \xrightarrow{\cong} B \otimes A$$

natural in $A, B \in \mathcal{A}$, such that $\vartheta_{B,A} = \vartheta_{A,B}^{-1}$ and the hexagonal diagram

$$\begin{array}{ccccc}
& (B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) & \\
{\scriptstyle \vartheta_{A,B} \otimes \mathrm{id}} \nearrow & & & & \searrow {\scriptstyle \mathrm{id} \otimes \vartheta_{A,C}} \\
(A \otimes B) \otimes C & & & & B \otimes (C \otimes A) \\
{\scriptstyle \alpha_{A,B,C}} \searrow & & & & \nearrow {\scriptstyle \alpha_{B,C,A}} \\
& A \otimes (B \otimes C) & \xrightarrow[\vartheta_{A, B \otimes C}]{} & (B \otimes C) \otimes A &
\end{array}$$

commutes for all objects A, B, C in $\mathcal{A}$.

A strong monoidal functor $F\colon \mathcal{A} \to \mathcal{A}'$ between symmetric monoidal categories is a symmetric monoidal functor if the diagram

$$\begin{array}{ccc} F(A) \otimes' F(B) & \xrightarrow{\vartheta'_{F(A),F(B)}} & F(B) \otimes' F(A) \\ \Big\downarrow{\scriptstyle \varphi_{A,B}} & & \Big\downarrow{\scriptstyle \varphi_{B,A}} \\ F(A \otimes B) & \xrightarrow{F(\vartheta_{A,B})} & F(B \otimes A) \end{array}$$

commutes for all $A, B \in \mathcal{A}$.

A symmetric monoidal category is called a permutative category if the associativity and unit isomorphisms are identities.

Definition of a braided monoidal category from Chapter XI.1 in [ML98]:

Definition 4. A monoidal category $\mathcal{A}$ is braided monoidal if it is equipped with a braiding

$$\chi_{A,B}\colon\ A \otimes B \xrightarrow{\cong} B \otimes A$$

natural in $A, B \in \mathcal{A}$, such that both hexagonal diagrams

$$\begin{array}{ccccc} & (B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) & \\ {\scriptstyle \chi_{A,B}\otimes \mathrm{id}} \nearrow & & & & \searrow {\scriptstyle \mathrm{id}\otimes\chi_{A,C}} \\ (A \otimes B) \otimes C & & & & B \otimes (C \otimes A) \\ {\scriptstyle \alpha_{A,B,C}} \searrow & & & & \nearrow {\scriptstyle \alpha_{B,C,A}} \\ & A \otimes (B \otimes C) & \xrightarrow[\chi_{A,B\otimes C}]{} & (B \otimes C) \otimes A & \end{array}$$

and

$$\begin{array}{ccccc} & A \otimes (C \otimes B) & \xrightarrow{\alpha^{-1}_{A,C,B}} & (A \otimes C) \otimes B & \\ {\scriptstyle \mathrm{id}\otimes\chi_{B,C}} \nearrow & & & & \searrow {\scriptstyle \chi_{A,C}\otimes \mathrm{id}} \\ A \otimes (B \otimes C) & & & & (C \otimes A) \otimes B \\ {\scriptstyle \alpha^{-1}_{A,B,C}} \searrow & & & & \nearrow {\scriptstyle \alpha^{-1}_{C,A,B}} \\ & (A \otimes B) \otimes C & \xrightarrow[\chi_{A\otimes B,C}]{} & C \otimes (A \otimes B) & \end{array}$$

commute for all objects A, B, C in $\mathcal{A}$.

A monoidal functor $F\colon \mathcal{A} \to \mathcal{A}'$ between braided monoidal categories is a braided monoidal functor if the diagram

$$\begin{array}{ccc} F(A)\otimes' F(B) & \xrightarrow{\chi'_{F(A),F(B)}} & F(B)\otimes' F(A) \\ \Big\downarrow{\scriptstyle \varphi_{A,B}} & & \Big\downarrow{\scriptstyle \varphi_{B,A}} \\ F(A\otimes B) & \xrightarrow{F(\chi_{A,B})} & F(B\otimes A) \end{array}$$

commutes for all $A, B \in \mathcal{A}$.

The concept of n-fold monoidal categories was defined and developed by Balteanu, Fiedorowicz, Schwänzel and Vogt in [BFSV03]. The following is Definition 1.7 from that paper.

Definition 5. An n-fold monoidal category is a category $\mathcal{E}$ with the following structure: There are n monoidal products

$$\Box_1, \ldots, \Box_n \colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}$$

which are strictly associative and there is an object $0 \in \mathcal{E}$ which is a common strict unit for all the monoidal products. For each pair (i, j) of natural numbers such that $1 \leq i < j \leq n$ there is a natural transformation

$$\eta^{i,j}_{A,B,C,D} \colon (A\Box_j B)\Box_i(C\Box_j D) \to (A\Box_i C)\Box_j(B\Box_i D).$$

These natural transformations $\eta^{i,j}$ are subject to the following conditions:

Internal unit condition $\eta^{i,j}_{A,B,0,0} = \eta^{i,j}_{0,0,A,B} = \mathrm{id}_{A\Box_j B}$.

External unit condition $\eta^{i,j}_{A,0,B,0} = \eta^{i,j}_{0,A,0,B} = \mathrm{id}_{A\Box_i B}$.

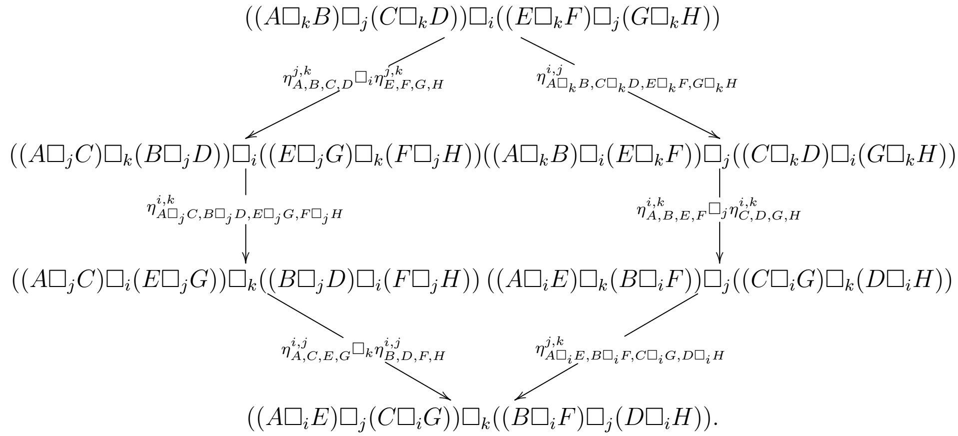
Internal associativity condition Commutativity of the diagram

$$\begin{array}{ccc} (U\Box_j V)\Box_i(W\Box_j X)\Box_i(Y\Box_j Z) & \xrightarrow{\eta^{i,j}_{U,V,W,X}\Box_i \mathrm{id}_{Y\Box_j Z}} & ((U\Box_i W)\Box_j(V\Box_i X))\Box_i(Y\Box_j Z) \\ \Big\downarrow{\scriptstyle \mathrm{id}_{U\Box_j V}\Box_i \eta^{i,j}_{W,X,Y,Z}} & & \Big\downarrow{\scriptstyle \eta^{i,j}_{U\Box_i W, V\Box_i X, Y, Z}} \\ (U\Box_j V)\Box_i((W\Box_i Y)\Box_j(X\Box_i Z)) & \xrightarrow{\eta^{i,j}_{U,V,W\Box_i Y, X\Box_i Z}} & (U\Box_i W\Box_i Y)\Box_j(V\Box_i X\Box_i Z). \end{array}$$

External associativity condition Commutativity of the diagram

$$\begin{array}{ccc} (U\Box_j V\Box_j W)\Box_i(X\Box_j Y\Box_j Z) & \xrightarrow{\eta^{i,j}_{U\Box_j V, W, X\Box_j Y, Z}} & ((U\Box_j V)\Box_i(X\Box_j Y))\Box_j(W\Box_i Z) \\ \Big\downarrow{\scriptstyle \eta^{i,j}_{U, V\Box_j W, X, Y\Box_j Z}} & & \Big\downarrow{\scriptstyle \eta^{i,j}_{U,V,X,Y}\Box_j \mathrm{id}_{W\Box_i Z}} \\ (U\Box_i X)\Box_j((V\Box_j W)\Box_i(Y\Box_j Z)) & \xrightarrow{\mathrm{id}_{U\Box_i X}\Box_j \eta^{i,j}_{V,W,Y,Z}} & (U\Box_i X)\Box_j(V\Box_i Y)\Box_j(W\Box_i Z). \end{array}$$

Coherence For each triple (i,j,k) satisfying $1 \le i < j < k \le n$, the following coherence diagram must commute



Remark 6. The assumption in [BFSV03] of strict associativity and a strict unit in the definition of an n-fold monoidal category is made for convenience. In paper [SS] we spell out the definition with associativity and identity isomorphisms that are not necessarily identities.

Remark 7. A 1-fold monoidal category is by definition the same thing as a strict monoidal category. A braided strict monoidal category $(\mathcal{B},\otimes,I,\chi)$ has an induced structure of a 2-fold monoidal category, see [BFSV03, Remark 1.5]. This is achieved by setting $\square_1 = \square_2 = \otimes$, and

$$\eta^{1,2}_{A,B,C,D} = \mathrm{id}_A \otimes \chi_{B,C} \otimes \mathrm{id}_D.$$

A symmetric strict monoidal category, i.e. a permutative category, $(\mathcal{C},\otimes,I,\vartheta)$ has an induced structure of an n-fold monoidal category for each $n \ge 1$, see [BFSV03, Remark 1.9]. Similarly to above, we have $\square_i = \otimes$ for each $1 \le i \le n$ and

$$\eta^{i,j}_{A,B,C,D} = \mathrm{id}_A \otimes \vartheta_{B,C} \otimes \mathrm{id}_D$$

for each pair $1 \le i < j \le n$.

Although this gives a strong connection between braided/symmetric monoidal categories and n-fold monoidal categories, there is a crucial difference. The braiding χ and symmetry ϑ are required to be isomorphisms, whereas there is no such requirement for $\eta^{i,j}$. This is important, because if such a requirement was made, the resulting structure would be equivalent to that of a symmetric monoidal category for $n \ge 3$, as shown by Joyal and Street in [JS93, Proposition 5.4]. For $n = 2$ such a structure would be equivalent to a braided monoidal category. This difference, however, turns out to be not so important as the homotopy category of braided monoidal categories and the homotopy category of 2-fold monoidal categories are already equivalent, see [FSV13, Theorem 8.22].

Definition of an n-fold monoidal functor from [BFSV03, Definition 1.8]:

Definition 8. An n-fold monoidal functor $(F, \lambda^1, \ldots, \lambda^n)\colon \mathcal{E} \to \mathcal{F}$ between n-fold monoidal categories consists of a functor F such that $F(0) = 0$ together with natural transformations

$$\lambda^i_{A,B}\colon\ F(A)\Box_i F(B) \to F(A\Box_i B) \text{ for } i = 1, 2, \ldots, n$$

satisfying the same associativity and unit conditions as monoidal functors. In addition, the following hexagonal interchange diagram commutes:

$$\begin{array}{ccc}
(F(A)\Box_j F(B))\Box_i(F(C)\Box_j F(D)) & \xrightarrow{\eta^{i,j}} & (F(A)\Box_i F(C))\Box_j(F(B)\Box_i F(D)) \\
\downarrow{\scriptstyle \lambda^j_{A,B}\Box_i\lambda^j_{C,D}} & & \downarrow{\scriptstyle \lambda^i_{A,C}\Box_j\lambda^i_{B,D}} \\
F(A\Box_j B)\Box_i F(C\Box_j D) & & F(A\Box_i C)\Box_j F(B\Box_i D) \\
\downarrow{\scriptstyle \lambda^i_{A\Box_j B, C\Box_j D}} & & \downarrow{\scriptstyle \lambda^j_{A\Box_i C, B\Box_i D}} \\
F((A\Box_j B)\Box_i(C\Box_j D)) & \xrightarrow{F(\eta^{i,j})} & F((A\Box_i C)\Box_j(B\Box_i D))
\end{array}$$

Note that the λ^i's are not required to be isomorphisms.

1.1.2 Operads

Many of the definitions in this subsection are taken from [May72], but the setting has been generalized from topological spaces to a symmetric monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \varrho, \vartheta)$. When there is a monoidal product of more than two objects in $\mathcal{C}$, we suppress the parenthesis in the resulting product. Any two parenthesised versions are canonically isomorphic, so it is a matter of convenience. We say that an operad with a sequence of objects in a category $\mathcal{C}$ is internal to $\mathcal{C}$. In this section we will focus on operads internal to the category of sets *Set* and the category of small categories *Cat*, as these are the ones most relevant to the work in this thesis.

The following definition of a non-Σ operad is based on [May72, Definition 1.1]. The setting is generalized as noted above, and the symmetric group operation is removed together with the equivariance diagrams to get a non-Σ operad instead of a symmetric operad.

Definition 9. A non-Σ operad C internal to $\mathcal{C}$ consists of a sequence of objects $\mathsf{C}(j)$ in $\mathcal{C}$ for $j \geq 0$, together with the following data:

1. For each integer $k \geq 0$ and each k-tuple of integers $j_1, \dots, j_k \geq 0$ a morphism

$$\gamma\colon\ \mathsf{C}(k) \otimes \mathsf{C}(j_1) \otimes \cdots \otimes \mathsf{C}(j_k) \to \mathsf{C}(j_1 + \cdots + j_k),$$

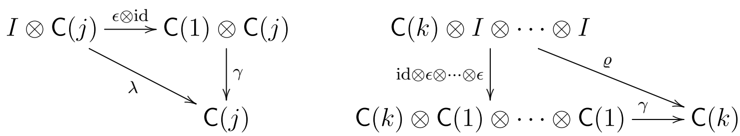
   where, for $k = 0$, $\gamma\colon\ \mathsf{C}(0) \to \mathsf{C}(0)$ is the identity. These operad structure maps must satisfy an associativity condition: The following composite

$$\begin{aligned}
&\mathsf{C}(k) \otimes \mathsf{C}(j_1) \otimes \cdots \otimes \mathsf{C}(j_k) \otimes \mathsf{C}(i_{1,1}) \otimes \cdots \otimes \mathsf{C}(i_{k,j_k}) \xrightarrow{\vartheta} \\
&\quad \mathsf{C}(k) \otimes \mathsf{C}(j_1) \otimes \mathsf{C}(i_{1,1}) \otimes \cdots \otimes \mathsf{C}(i_{1,j_1}) \otimes \cdots \otimes \mathsf{C}(j_k) \otimes \mathsf{C}(i_{k,1}) \otimes \cdots \otimes \mathsf{C}(i_{k,j_k}) \\
&\quad \xrightarrow{\mathrm{id}\otimes\gamma\otimes\cdots\otimes\gamma} \mathsf{C}(k) \otimes \mathsf{C}(i_{1,1} + \cdots + i_{1,j_1}) \otimes \cdots \otimes \mathsf{C}(i_{k,1} + \cdots + i_{k,j_k}) \\
&\quad \xrightarrow{\gamma} \mathsf{C}\big((i_{1,1} + \cdots + i_{1,j_1}) + \cdots + (i_{k,1} + \cdots + i_{k,j_k})\big)
\end{aligned}$$

   is equal to

$$\begin{aligned}
&\mathsf{C}(k) \otimes \mathsf{C}(j_1) \otimes \cdots \otimes \mathsf{C}(j_k) \otimes \mathsf{C}(i_{1,1}) \otimes \cdots \otimes \mathsf{C}(i_{k,j_k}) \xrightarrow{\gamma\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}} \\
&\quad \mathsf{C}(j_1 + \cdots + j_k) \otimes \mathsf{C}(i_{1,1}) \otimes \cdots \otimes \mathsf{C}(i_{k,j_k}) \xrightarrow{\gamma} \\
&\quad \mathsf{C}(i_{1,1} + \cdots + i_{1,j_1} + \cdots + i_{k,1} + \cdots + i_{k,j_k}).
\end{aligned}$$

2. An identity morphism $\epsilon\colon\ I \to \mathsf{C}(1)$ such that the diagrams



   commute for all $j, k \geq 0$.

The definition of a non-Σ operad morphism is also from [May72, Definition 1.1], dropping the equivariance condition on the maps.

Definition 10. A operad morphism $\Psi\colon\ \mathsf{C} \to \mathsf{C}'$ between non-Σ operads is a sequence of morphisms

$$\Psi_j\colon\ \mathsf{C}(j) \to \mathsf{C}'(j)$$

in $\mathcal{C}$, such that $\Psi_1 \circ \epsilon = \epsilon' \colon I \to \mathsf{C}'(1)$ and the diagram

$$\begin{array}{ccc} \mathsf{C}(k) \otimes \mathsf{C}(j_1) \otimes \cdots \otimes \mathsf{C}(j_k) & \xrightarrow{\gamma} & \mathsf{C}(j_1 + \cdots + j_k) \\ \downarrow{\scriptstyle \Psi_k \otimes \Psi_{j_1} \otimes \cdots \otimes \Psi_{j_k}} & & \downarrow{\scriptstyle \Psi_{j_1+\cdots+j_k}} \\ \mathsf{C}'(k) \otimes \mathsf{C}'(j_1) \otimes \cdots \otimes \mathsf{C}'(j_k) & \xrightarrow{\gamma'} & \mathsf{C}'(j_1 + \cdots + j_k) \end{array}$$

commutes for integers $k, j_1, \ldots, j_k \geq 0$.

An important example of a non-Σ operad is the endomorphism operad, as this is what lets us define an operad action on an object in $\mathcal{C}$. The definition is taken from [MSS02, Definition 1.7], disregarding the symmetric group action.

Definition 11. Let $(\mathcal{C}, \otimes, I)$ be a symmetric closed monoidal category, i.e. it has an internal hom functor compatible with the monoidal structure. The endomorphism operad End_X for an object X in $\mathcal{C}$ is defined by

$$\mathsf{End}_X(j) = \hom(X^{\otimes j}, X),$$

with

$$\gamma \colon \mathsf{End}_X(k) \otimes \mathsf{End}_X(j_1) \otimes \cdots \otimes \mathsf{End}_X(j_k) \to \mathsf{End}_X(j)$$

defined as the composite

$$\hom(X^{\otimes k}, X) \otimes \hom(X^{\otimes j_1}, X) \otimes \cdots \otimes \hom(X^{\otimes j_k}, X) \to \\ \hom(X^{\otimes k}, X) \otimes \hom(X^{\otimes j}, X^{\otimes k}) \to \hom(X^{\otimes j}, X)$$

for $k, j_1, \ldots, j_k \geq 0$, where $j = j_1 + \cdots + j_k$. The identity morphism $\epsilon \colon I \to \hom(X, X)$ is the adjoint of the identity on X.

Definition 12 (Definition 1.20 [MSS02])**.** Let C be an operad internal to a symmetric closed monoidal category $\mathcal{C}$. An action of C on an object X in $\mathcal{C}$ is an operad morphism

$$\mathsf{C} \to \mathsf{End}_X.$$

The object X together with the action is called a C-algebra.

Using the adjoint relationship between hom and $\otimes$, an action is often rewritten as a sequence of morphisms

$$\mathsf{C}(k) \otimes X^{\otimes k} \to X$$

satisfying conditions corresponding to the conditions for an operad morphism.

The multiplicative structures we have mentioned earlier are all encoded by operads. Associative multiplications arise from actions of non-Σ operads described in the example below.

Example 13. We now consider the symmetric monoidal category *Set* with cartesian product $\times$ as monoidal product, and unit object $I = \{*\}$, a one element set. Let $\mathsf{A}(k) = \{*\}$, for all $k \geq 0$. A one element set is a terminal object in the category of small sets *Set*. Therefore there is a unique non-Σ operad structure on A with

$$\gamma\colon\ \mathsf{A}(k) \times \mathsf{A}(j_1) \times \cdots \times \mathsf{A}(j_k) \to \mathsf{A}(j_1 + \cdots + j_k)$$

being the unique morphism

$$\gamma\colon\ \{*\} \times \{*\} \times \cdots \times \{*\} \to \{*\},$$

and the identity ϵ the unique morphism $I \to \mathsf{A}(1) = \{*\}$. Since $\{*\}$ is terminal, the associativity and identity diagrams will be commutative by default. The A-algebras are *Set*-monoids.

In *Cat* we can similarly get a non-Σ operad by setting $\mathsf{A}(k) = \{*\}$ for $k \geq 0$. Now $\{*\}$ is a terminal category with one object and only the identity morphism. The rest of the operad structure is analogously defined to the *Set* version of the operad. The A-algebras for this categorical version are the small strict monoidal categories. Notice that this operad action induce strict associativity. This however is not a significant restriction when we work with monoidal categories, since any monoidal category is equivalent, via strong monoidal functors, to a strict monoidal category, see [ML98, Theorem 1 Chapter XI.3]

More common than non-Σ operads are symmetric operads, often just referred to as operads. The definition below is a generalized version of [May72, Definition 1.1].

Definition 14. A symmetric operad internal to $\mathcal{C}$ is a non-Σ operad C together with a right action of the symmetric group Σ_j on $\mathsf{C}(j)$ for each $j \geq 0$, satisfying the following two equivariance conditions. For $\sigma \in \Sigma_k$, the diagram

$$\begin{array}{ccc}
\mathsf{C}(k) \otimes \mathsf{C}(j_1) \otimes \cdots \otimes \mathsf{C}(j_k) & \xrightarrow{\sigma \otimes \mathrm{id}} & \mathsf{C}(k) \otimes \mathsf{C}(j_1) \otimes \cdots \otimes \mathsf{C}(j_k) \\
\sigma \big\downarrow \cong & & \\
\mathsf{C}(k) \otimes \mathsf{C}(j_{\sigma^{-1}(1)}) \otimes \cdots \otimes \mathsf{C}(j_{\sigma^{-1}(k)}) & & \big\downarrow \gamma \\
\gamma \big\downarrow & & \\
\mathsf{C}(j_{\sigma^{-1}(1)} + \cdots + j_{\sigma^{-1}(k)}) & \xrightarrow{\sigma(j_1,\ldots,j_k)} & \mathsf{C}(j_1 + \cdots + j_k)
\end{array}$$

must commute for all $k, j_1, \ldots, j_k \geq 0$, where $\sigma(j_1, \ldots, j_k)$ denotes the block permutation in $\Sigma_{j_1+\cdots+j_k}$ induced by σ. If we have $\tau_i \in \Sigma_{j_i}$ for $i = 1, \ldots, k$, let $\tau_1 \oplus \cdots \oplus \tau_k$ denote the image of $(\tau_1, \cdots, \tau_k)$ under the canonical inclusion $\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \subseteq \Sigma_{j_1+\cdots+j_k}$. The diagram

$$\begin{array}{ccc} \mathsf{C}(k) \otimes \mathsf{C}(j_1) \otimes \cdots \otimes \mathsf{C}(j_k) & \xrightarrow{\gamma} & \mathsf{C}(j_1 + \cdots + j_k) \\ \downarrow{\scriptstyle \mathrm{id} \otimes \tau_1 \otimes \cdots \otimes \tau_k} & & \downarrow{\scriptstyle \tau_1 \oplus \cdots \oplus \tau_k} \\ \mathsf{C}(k) \otimes \mathsf{C}(j_1) \otimes \cdots \otimes \mathsf{C}(j_k) & \xrightarrow{\gamma} & \mathsf{C}(j_1 + \cdots + j_k) \end{array}$$

must commute for all $k, j_1, \ldots, j_k \geq 0$. A symmetric operad is often just called an operad.

A non-Σ operad morphism $\Psi\colon \mathsf{C} \to \mathsf{C}'$ between symmetric operads is a symmetric operad morphism if each morphism

$$\Psi_j\colon \mathsf{C}(j) \to \mathsf{C}'(j)$$

is Σ_j equivariant.

Remark 15. The endomorphism operad from Definition 11 is a symmetric operad. The action of Σ_j is precomposition with the action of Σ_j on $X^{\otimes j}$ that comes from the symmetric monoidal structure. An action of a symmetric operad on an object X is defined as a symmetric operad morphism from the symmetric operad to End_X.

Associative multiplicative structures can also be given by actions of symmetric operads.

Example 16. Let A be the set operad with $\mathsf{A}(k) = \Sigma_k$, where the Σ_k action is given by right multiplication and

$$\gamma(\tau; \tau_1, \ldots, \tau_k) = \tau(j_1, \ldots, j_k)(\tau_1 \oplus \cdots \oplus \tau_k),$$

for $\tau \in \Sigma_k$, $\tau_i \in \Sigma_{j_i}$ for $i = 1, \ldots, k$. As in the definition of a symmetric operad, $\tau_1 \oplus \cdots \oplus \tau_k$ denotes the image of $(\tau_1, \cdots, \tau_k)$ under the canonical inclusion $\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \subseteq \Sigma_{j_1+\cdots+j_k}$. And $\tau(j_1, \ldots, j_k)$ denotes the block permutation in $\Sigma_{j_1+\cdots+j_k}$ induced by τ. The algebras over A are *Set*-monoids, i.e. a sets with an associative multiplication and a unit element.

Viewing $\mathsf{A}(k) = \Sigma_k$ as a discrete category with only identity morphisms, we obtain a *Cat*-operad A. Categories with an action of A are the strict monoidal small categories.

Remark 17. Note that the category of algebras for the symmetric operad A with $\mathsf{A}(k) = \Sigma_k$ is the same as the category of algebras for the non-Σ operad A with $\mathsf{A}(k) = \{*\}$. Also note that $\{*\} \times \Sigma_k \cong \Sigma_k$. This fact generalizes to an adjunction between non-Σ operads and symmetric operads as noted in the abstract of [Bat07]. There is a forgetful functor from symmetric operads to non-Σ operads. We can call the left adjoint functor a symmetrization functor. If we have a non-Σ operad C, the symmetrization functor takes C to $\mathsf{C} \times \Sigma$ where $(\mathsf{C} \times \Sigma)(k) = \mathsf{C}(k) \times \Sigma_k$. The operad structure map γ is the product of the operad structure maps from C and the symmetric operad A. The action of the symmetric group on each level is trivial on the $\mathsf{C}(k)$ factor and given by right multiplication on Σ_k. The categories of algebras of the non-Σ operad C and the symmetric operad $\mathsf{C} \times \Sigma$ are isomorphic.

Now we look at operads that give various degrees of commutativity. First out is strict commutativity. Here we have both a *Set*-operad and a *Cat*-operad. In the rest of the examples in this section we will consider operads that induce non strict commutativity. Then *Set* is no longer a relevant setting and we will exclusively look at categorical operads.

Example 18. Similarly to Example 13 there is a canonical symmetric operad structure associated with the sequence $\mathsf{C}(k) = \{*\}$ for $k \geq 0$, both in *Set* and *Cat*. The C-algebras in *Set* are commutative *Set*-monoids. The C-algebras in *Cat* are commutative monoidal small categories, that is, permutative categories where the symmetry isomorphisms are actually identities.

Next is the categorical version of the well-know Barratt-Eccles operad, see [BE74], which give us permutative categories.

Example 19. Let $\tilde{\Sigma}(k)$ denote the translation category of Σ_k. That is, the objects in $\tilde{\Sigma}(k)$ are the elements of the symmetric group Σ_k. Furthermore, given objects ς and τ in $\tilde{\Sigma}(k)$, a morphism $\upsilon\colon \varsigma \to \tau$ is an element $\varsigma \in \Sigma_k$ such that $\upsilon\varsigma = \tau$. There is a symmetric operad structure associated with the sequence $\tilde{\Sigma}(k)$ for $k \geq 0$. The operad structure map γ is determined by what it does on objects, and here the definition is the same as for the categorical operad A from Example 16. The right action of an element σ is defined on objects and morphisms by taking $\upsilon\colon \varsigma \to \tau$ to $\upsilon\colon \varsigma\sigma \to \tau\sigma$. The $\tilde{\Sigma}$-algebras are the permutative categories.

Recall that permutative categories are symmetric strict monoidal small categories. The associativity is strict, but the symmetry is not necessarily strict. It is worth repeating that strict associativity is not a significant restriction, while strict symmetry (commutativity) is.

The following example is a symmetric *Cat*-operad whose algebras are braided strict monoidal small categories, see the paragraph preceding Lemma 8.12 in [FSV13]. The operad is similar to the previous example of the Barratt-Eccles operad, with the crucial difference that the morphisms in the categories are braids and not permutations.

Example 20. Let $\mathsf{Br}(k)$ be a category with objects the permutations in Σ_k. Let B_k denote the braid group on k strings and given a braid α, let $\bar{\alpha}$ be the underlying permutation. A morphism in $\mathsf{Br}(k)$ from an object ς to τ is a braid $\alpha \in B_k$ such that $\bar{\alpha}\varsigma = \tau$. The symmetric operad structure is defined similarly to the operad structure of the categorical Barratt-Eccles operad $\tilde{\Sigma}$. The Br-algebras are the braided strict monoidal small categories.

The next example is the symmetric operad M_n associated with n-fold monoidal categories. We refer to Section 3 of [BFSV03] for details, and will here try to give an impression of the operad.

Example 21. The objects of $\mathsf{M}_n(k)$ serve as templates for the various monoidal multiplications in an n-fold monoidal category with k factors. Examples of objects in $\mathsf{M}_4(3)$ include $1\Box_1 3\Box_1 2$ and $(2\Box_4 1)\Box_2 3$. The object will consist of exactly the digits 1, 2 and 3, in some order, in a product using some of the monoidal products $\Box_1$, $\Box_2$, $\Box_3$ and $\Box_4$. Examples of objects in $\mathsf{M}_2(5)$ include $(1\Box_1 3\Box_1 2)\Box_2(4\Box_1 5)$ and $(4\Box_2 1)\Box_1(2\Box_2 5)\Box_1 3$. The operad structure map

$$\gamma\colon\ \mathsf{M}_n(k) \times \mathsf{M}_n(j_1) \times \cdots \times \mathsf{M}_n(j_k) \to \mathsf{M}_n(j)$$

combines the objects from $\mathsf{M}_n(j_1), \dots, \mathsf{M}_n(j_k)$ using the template of the object from $\mathsf{M}_n(k)$. The digits are then shifted appropriately, so that the new object consists of the digits from 1 to j. Here is an example of γ

$$\gamma\colon\ \mathsf{M}_4(3) \times \mathsf{M}_4(1) \times \mathsf{M}_4(3) \times \mathsf{M}_4(3) \to \mathsf{M}_4(7)$$

on a tuple of objects:

$$\gamma\big((2\Box_4 1)\Box_2 3; 1, 2\Box_3 1\Box_3 3, (1\Box_2 3)\Box_1 2\big) = \big((3\Box_3 2\Box_3 4)\Box_4 1\big)\Box_2\big((5\Box_2 7)\Box_1 6\big).$$

The morphisms in $\mathsf{M}_n(k)$ codify the interchange maps in n-fold monoidal categories.

Remark 22. There is a strong analogy when it comes to the relationship between the symmetric operad $\tilde{\Sigma}$ and free permutative categories, the relationship between the symmetric operad Br and free braided strict monoidal categories and the relationship between the symmetric operad M_n and free n-fold monoidal categories. The free permutative category on one element is isomorphic to the disjoint union of all the symmetric groups, which again is isomorphic to $\amalg_{k\geq 0}\tilde{\Sigma}_k/\Sigma_k$. Similarly, the free braided strict monoidal category on one element is isomorphic to the disjoint union of all the braid groups, which again is isomorphic to $\amalg_{k\geq 0}\mathsf{Br}(k)/\Sigma_k$. For M_n we have that the free n-fold monoidal category on one element is isomorphic to $\amalg_{k\geq 0}\mathsf{M}_n(k)/\Sigma_k$, see [BFSV03, Section 3].

So far we have looked at non-Σ operads and symmetric operads, where the latter incorporates an action of the corresponding symmetric group at each level. When it comes to symmetric monoidal categories and braided monoidal categories respectively, the symmetric groups and braid groups play similar roles. Taking advantage of this, Fiedorowicz defines the concept of braided operads in [Fie, Definition 3.2]. The definition is similar to that of symmetric operads, with actions of braid groups instead of symmetric groups.

Definition 23. Let $\mathcal{C}$ be a symmetric monoidal category. A braided operad is a non-Σ operad C, internal to $\mathcal{C}$, together with a right action of the braid group B_k on $\mathsf{C}(k)$ for each $k \geq 0$, satisfying the following two equivariance conditions. For a braid $\alpha \in B_k$, let $\bar{\alpha}$ denote the underlying permutation. The diagram

$$\begin{array}{ccc}
\mathsf{C}(k) \otimes \mathsf{C}(j_1) \otimes \cdots \otimes \mathsf{C}(j_k) & \xrightarrow{\alpha\otimes\mathrm{id}} & \mathsf{C}(k) \otimes \mathsf{C}(j_1) \otimes \cdots \otimes \mathsf{C}(j_k) \\
\downarrow{\bar{\alpha}}\ \cong & & \\
\mathsf{C}(k) \otimes \mathsf{C}(j_{\bar{\alpha}^{-1}(1)}) \otimes \cdots \otimes \mathsf{C}(j_{\bar{\alpha}^{-1}(k)}) & & \downarrow{\gamma} \\
\downarrow{\gamma} & & \\
\mathsf{C}(j_{\bar{\alpha}^{-1}(1)} + \cdots + j_{\bar{\alpha}^{-1}(k)}) & \xrightarrow{\alpha(j_1,\ldots,j_k)} & \mathsf{C}(j_1 + \cdots + j_k)
\end{array}$$

must commute for all $k, j_1, \ldots, j_k \geq 0$. The braid $\alpha(j_1, \ldots, j_k)$ in $B_{j_1+\cdots+j_k}$ is obtained from α by replacing the mth strand in α by j_m strands for $m = 1, \ldots, k$. If we have $\beta_i \in B_{j_i}$ for $i = 1, \ldots, k$, let $\beta_1 \oplus \cdots \oplus \beta_k$ denote the image of $(\beta_1, \cdots, \beta_k)$ under the canonical inclusion $B_{j_1} \times \cdots \times B_{j_k} \subseteq B_{j_1+\cdots+j_k}$. The diagram

$$\begin{array}{ccc}
\mathsf{C}(k) \otimes \mathsf{C}(j_1) \otimes \cdots \otimes \mathsf{C}(j_k) & \xrightarrow{\gamma} & \mathsf{C}(j_1 + \cdots + j_k) \\
\downarrow{\mathrm{id}\otimes\beta_1\otimes\cdots\otimes\beta_k} & & \downarrow{\beta_1\oplus\cdots\oplus\beta_k} \\
\mathsf{C}(k) \otimes \mathsf{C}(j_1) \otimes \cdots \otimes \mathsf{C}(j_k) & \xrightarrow{\gamma} & \mathsf{C}(j_1 + \cdots + j_k)
\end{array}$$

must commute for all $k, j_1, \ldots, j_k \geq 0$.

A non-Σ operad morphism $\Psi\colon \mathsf{C} \to \mathsf{C}'$ between braided operads is a braided operad morphism if each morphism

$$\Psi_k\colon \mathsf{C}(k) \to \mathsf{C}'(k)$$

is B_k equivariant.

Note that any symmetric operad can be given the structure of a braided operad by setting the action of an braid equal to the action of the underlying permutation. In this way we can consider the endomorphism operad as a braided operad. An action of a braided operad, internal to $\mathcal{C}$, on an object $X \in \mathcal{C}$ is thus defined as a braided operad morphism from the braided operad to the endomorphism operad on X.

The next example, from [Fie, Example 3.2], is a braided version of the symmetric operad Br from Example 20. We also denote the braided version with Br.

Example 24. Let $\mathsf{Br}(k)$ be the translation category of the braid group B_k. That is, the objects of $\mathsf{Br}(k)$ are the braids on k strings. A morphism from a braid ς to τ is a braid α such that $\alpha\varsigma = \tau$. The operad structure map γ is defined similarly to that of the symmetric operad A in Example 16: For $\tau \in B_k$, $\tau_i \in B_{j_i}$ we have

$$\gamma(\tau; \tau_1, \ldots, \tau_k) = \tau(j_1, \ldots, j_k)(\tau_1 \oplus \cdots \oplus \tau_k),$$

where $\tau_1 \oplus \cdots \oplus \tau_k$ denotes the image of $(\tau_1, \cdots, \tau_k)$ under the canonical inclusion $B_{j_1} \times \cdots \times B_{j_k} \subseteq B_{j_1+\cdots+j_k}$. And $\tau(j_1, \ldots, j_k)$ denotes the block braid in $B_{j_1+\cdots+j_k}$ induced by τ. The right action of a braid $\beta \in B_k$ is defined on objects and morphisms by taking $\alpha\colon \varsigma \to \tau$ to $\alpha\colon \varsigma\beta \to \tau\beta$. The algebras of the braided operad Br are the braided strict monoidal small categories. So the algebras for the braided operad Br are the same as for the symmetric operad Br.

Thinking of the n-fold monoidal setting as the bridge between the braided monoidal and symmetric monoidal setting, it is natural to ask the following. Is it possible to find a family of groups to play the role that the symmetric groups do for symmetric operads and the braid groups do for the braided operads, but in the n-fold setting? The answer (to a more precise version of the question) is no, see the introduction of [Bat10].

1.1.3 Iterated loop spaces and iterated monoidal categories

In topology, the concept of a loop space has been and continue to be an important area of research. The following definition of a loop space is from [MSS02, Definition 2.1].

Definition 25. A loop space ΩX is the space of based maps from the circle S^1 to a space X. More generally an n-fold loop space $\Omega^n X$ is the space of based maps from the sphere S^n to a space X, $1 \leq n < \infty$.

It is helpful to interpret 'n-fold loop space' as the sequence $\{Y_i = \Omega Y_{i+1} | 0 \leq i < n\}$ with

$$Y_n = X,\ Y_{n-1} = \Omega X, \ldots,\ Y_0 = \Omega^n X.$$

An infinite loop space ($n = \infty$) is then a sequence $\{Y_i = \Omega Y_{i+1} | 0 \leq i\}$.

An n-fold loop space is also called an iterated loop space for $n > 1$.

A loop space has a natural product, induced by a projection of the circle onto a wedge of circles, which is unital and associative up to homotopy. Not only does associativity homotopies exist, but a diagram of associativity homotopies commutes up to homotopy, and so on. The product is said to be associative up to all higher homotopies. Taking the loop space of a a loop space, we obtain what is called a 2-fold loop space. An Eckmann-Hilton type argument shows that the multiplication of a 2-fold loop space is commutative up to homotopy. Iterating further, an n-fold loop space is the loop space of an $(n-1)$-fold loop space. For increasing n the multiplication is commutative up to higher and higher homotopies. In the limit case, an infinite loop space is commutative up to all higher homotopies.

There is a strong relationship between iterated loop spaces and various monoidal category structures. A model for the group completion the classifying space of a monoidal category is the loop space of the classifying space, so it is a loop space. Furthermore, it is well known that the group completion of the classifying space of a symmetric monoidal category is an infinite loop space, see the introduction of [Tho95]. Analogously, as pointed out by Stasheff [Sta92] and proved by Fiedorowicz [Fie, Example 3.2 and Proposition 3.4] and Berger [Ber99, Therorem 1.2], the group completion of the classifying space of a braided monoidal category is a double loop space. In [Tho95] Thomason proved that each infinite loop space is weakly equivalent to the group completion of the classifying space of a symmetric monoidal category. Inspired by this relationship, Balteanu, Fiedorowicz, Schwänzel and Vogt set out to find a categorical structure corresponding to n-fold loop spaces for a general n, see [BFSV03]. Analogous to the iterative definition of an n-fold loop space as a loop space on an $(n-1)$-fold loop space, an iterative definition of an n-fold monoidal category is given in Section 1 of [BFSV03]. Loosely speaking an n-fold monoidal category is a monoid in the category of $(n-1)$-fold monoidal categories. The induction start is given by letting a 1-fold monoidal category be a strict monoidal category. There is a subtle point about the functors in the category of $(n-1)$-fold monoidal categories. They are lax (or weak) monoidal functors. The result is that the interchange maps in an n-fold monoidal category are not required to be isomorphisms, unlike the symmetry isomorphisms of symmetric monoidal categories and braids of braided monoidal categories, which are isomorphisms. See Remark 7 in the Preliminaries for a further comment on this. The iterative definition of an n-fold monoidal category is translated into a more explicit description in [BFSV03, Definition 1.7], this is the definition we recalled earlier in this introduction (Definition 5).

After the definition of n-fold monoidal categories in [BFSV03], it is shown that the group completion of the classifying space of an n-fold monoidal category is an n-fold loop space, see [BFSV03, Theorem 2.2]. In a later article, [FSV13], it is shown that each n-fold loop space is weakly equivalent to the group completion of an n-fold monoidal category, see [FSV13, Theorem 8.22]. Similarly each 2-fold loop space is weakly equivalent to the group completion of a braided monoidal category. The method used in the article also provides a new proof ([FSV13, Theorem 8.23]) for Thomasons analogous result about infinite loop spaces and symmetric monoidal categories.

The multiplicative structures of loop spaces can, like the monoidal structures recalled earlier in this introduction, also be encoded by operads. In fact, the first use case for operads was the study of iterated loop spaces. For the rest of this section we focus on topological operads. In [BV68] Boardman and Vogt defined a family of operads, the little n-cubes operads C_n, which, by construction, act on n-fold loop spaces. Furthermore May proved that any connected C_n-space has the weak homotopy type of an n-fold loop space. This result is called the recognition theorem and is found in [May72, Theorem 1.3]. As May states in [May72, Remarks 13.3], the geometry of the little n-cubes operads is so closely tied to the geometry of iterated loop spaces that a recognition principle based solely on these operads would be of little practical value.

For a more general recognition principle, we consider E_n-operads and E_∞-operads. An E_n-operad is a symmetric operad weakly equivalent to the little n-cubes operad. An important example for our work is the nerve of the operad M_n which is an E_n operad by Theorem 3.14 in [BFSV03]. The nerve of M_n naturally acts on the classifying space of an n-fold monoidal category, see [BFSV03, Definition 3.1]. An E_∞-operad is a Σ-free symmetric operad which is contractible at each level. The Barratt-Eccles operad is an E_∞-operad (see the end of Chapter 15 in [May72]) and it naturally acts on the classifying space of a permutative category, see Theorem 4.9 in [May74]. May's recognition principle implies that a connected E_n space has the weak homotopy type of an n-fold loop space ([May72, Theorem 13.1]) and a connected E_∞ space has the weak homotopy type of an infinite loop space ([May72, Theorem 14.4]). For n equals 1 and 2, E_1- and E_2-operads can be modeled by A_∞- and B_∞-operads respectively. An A_∞-operad is a non-Σ operad that is contractible at each level, see [May72, Definition 3.5]. The nerve of the categorical non-Σ operad A from Example 13 is clearly an A_∞-operad. A B_∞-operad is a braided operad such that each level is contractible, and the actions of the braid group at each level is free, see the paragraph after Definition 3.2 in [Fie]. An example of a B_∞-operad is the nerve of the braided operad Br from Example 24. This braided operad naturally acts on the classifying space of a braided monoidal category, see the paragraph before Example 3.2 in [Fie]. May's recognition principle in particular implies that a connected A_∞ space has the weak homotopy type of a loop space ([May72, Theorem 13.4]) and a connected B_∞ space has the weak homotopy type of a double loop space ([Fie, Proposition 3.4]).

Chapter 2

Presentation of main results

2.1 Main results

In this section we present the main results of each paper of the thesis.

2.1.1 Braided injections and double loop spaces

In the preliminary section we saw how permutations and the symmetric groups are associated with symmetric monoidal categories, E_∞ structures and infinite loop spaces. Similarly braids and the braid groups are associated with braided monoidal categories, E_2 structures and double loop spaces. The main goal of this paper is to provide a commutative rectification of E_2 structures somewhat similar to what Sagave and Schlichtkrull does for E_∞ structures in [SS12]. In that article Sagave and Schlichtkrull works with $\mathcal{I}$-spaces which are functors from $\mathcal{I}$, the permutative category of finite sets and injections, to a suitable category of spaces or simplicial sets $\mathcal{S}$.

For our purposes we define a braided monoidal category of finite sets and braided injections $\mathfrak{B}$, and work with $\mathfrak{B}$-spaces. The definition of a braided injection is given in terms of homotopy classes of tuples of paths, quite similar to the definition of a braid given in [Bir74]. Loosely speaking, a braid in the nth braid group $\mathcal{B}_n$ can be represented by n paths starting at n distinct points and ending at n distinct points. In a similar way, a braided injection from $\mathbf{m} = \{1, \ldots, m\}$ to $\mathbf{n} = \{1, \ldots, n\}$ can be represented by m paths starting at m distinct points and the ending points are m distinct points out of n possible endpoints. An illustration of such representatives can be seen in Figure 2.1.1. The two leftmost drawings represent the same braided injection. Thinking of the paths as physical strings of thread fixed at the endpoints, one can see that strings of the leftmost picture can be moved into the position of the strings in the middle picture. The rightmost drawing represents a different braided injection, the strings of this one can not be moved to resemble either of the two others.

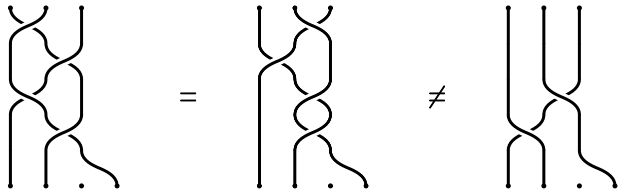


Figure 2.1: Illustration of braided injections from **3** to **4**.

The category $\mathfrak{B}$ mimics a key property of $\mathcal{I}$, namely that an injection can be uniquely decomposed into a permutation followed by an order-preserving injection. Similarly, as stated in the lemma below, a braided injection can be uniquely decomposed into a braid followed by an order-preserving injection. Let $\mathcal{M}(\mathbf{m},\mathbf{n})$ be the set of order-preserving injections from $\mathbf{m}$ to $\mathbf{n}$. The functor Υ embeds an order-preserving injection into $\mathfrak{B}$ in the obvious way.

Lemma (Lemma 2.3)**.** Every braided injection $\alpha\colon \mathbf{m} \to \mathbf{n}$ can be written uniquely as a composition $\alpha = \Upsilon(\mu) \circ \zeta$ with μ in $\mathcal{M}(\mathbf{m},\mathbf{n})$ and ζ in the braid group $\mathcal{B}_m$.

It is worth remarking that since every braid has an underlying permutation, every braided injection has an underlying injection. This induces a functor $\mathfrak{B} \to \mathcal{I}$. Thus any $\mathcal{I}$-space $X\colon \mathcal{I} \to \mathcal{S}$ gives rise to a $\mathfrak{B}$-space $\mathfrak{B} \to \mathcal{I} \xrightarrow{X} \mathcal{S}$.

Since we want to work with E_2 structures in $\mathfrak{B}$-spaces, we need a braided monoidal structure on the category of $\mathfrak{B}$-spaces $\mathcal{S}^{\mathfrak{B}}$. This is achieved in the usual way following the general set up in [Day70]. The monoidal product $\boxtimes$ is defined as a left Kan extension utilizing the monoidal products in $\mathfrak{B}$ and $\mathcal{S}$. The braiding $\mathfrak{b}$ is similarly derived from the braiding in $\mathfrak{B}$ and the symmetric twist in $\mathcal{S}$. The unit $U^{\mathfrak{B}}$ is a constant $\mathfrak{B}$-space with a single point at each level.

Proposition (Proposition 3.12)**.** The category $\mathcal{S}^{\mathfrak{B}}$ equipped with the $\boxtimes$-product, the unit $U^{\mathfrak{B}}$, and the braiding $\mathfrak{b}$ is a braided monoidal category.

In Section 4 of the paper we shift the focus from $\mathfrak{B}$-spaces to $\mathfrak{B}$-categories, i.e. functors from $\mathfrak{B}$ to the category of small categories *Cat*. Here we also have a braided monoidal structure, with Proposition 4.7 being the $\mathfrak{B}$-category version of Proposition 3.12. We define braided $\mathfrak{B}$-category monoids as a generalization of braided strict monoidal categories. That is, a constant $\mathfrak{B}$-category with a braided $\mathfrak{B}$-category monoid structure corresponds to having a braided strict monoidal category.

Recall the operad Br from Example 20 in the Preliminaries. It is similar to the Barratt-Eccles operad, but with the symmetric groups replaced by braid groups. There is both a symmetric operad version and a braided operad version (Example 24 in the Preliminaries), both denoted by Br. The structure of a braided strict monoidal small category can be encoded by an action of the symmetric operad Br. The braided version of the operad can act on $\mathfrak{B}$-categories. The next lemma shows that these algebras are isomorphic to a type of structure we call braided $\mathfrak{B}$-category monoids. This justifies considering these braided $\mathfrak{B}$-category monoids to be E_2 structures.

Lemma (Lemma 5.3)**.** The category $\mathsf{Br}\text{-}Cat^{\mathfrak{B}}$ is isomorphic to the category of braided $\mathfrak{B}$-category monoids.

As a way to relate the braided $\mathfrak{B}$-category monoids to braided strict monoidal categories, we use the Grothendieck construction. This is a general categorical construction that defines a functor $\mathcal{A}\!\int\colon Cat^{\mathcal{A}} \to Cat$ for a small category $\mathcal{A}$. One can think of it as a categorical version of the homotopy colimit. In fact Thomason's homotopy colimit theorem [Tho79, Theorem 1.2] relates the two concepts. The next result shows that when we apply the Grothendieck construction to a braided $\mathfrak{B}$-category monoid, the category we get inherits a braided strict monoidal category structure. One can say that the Grothendieck construction preserves the braided monoidal structures, or the E_2 structures if you want.

Proposition (Proposition 4.10)**.** The Grothendieck construction gives rise to a functor

$$\mathfrak{B}\!\int\colon \mathsf{Br}\text{-}Cat^{\mathfrak{B}} \to \mathsf{Br}\text{-}Cat.$$

We introduce weak equivalences in the following way. A morphism in $\mathsf{Br}\text{-}Cat$ is a weak equivalence if the nerve of the underlying functor is a weak equivalence of simplicial sets. A morphism in $\mathsf{Br}\text{-}Cat^{\mathfrak{B}}$ is a weak equivalence, called a $\mathfrak{B}$-equivalence, if the induced map on the Grothendieck construction is a weak equivalence. We call the classes of weak equivalences in $\mathsf{Br}\text{-}Cat$ and $\mathsf{Br}\text{-}Cat^{\mathfrak{B}}$ for w and $w_{\mathfrak{B}}$ respectively. The homotopy categories with respect to these weak equivalences are then equivalent as shown in the next proposition. The functor Δ is the constant embedding.

Proposition (Proposition 4.12)**.** The functors $\mathfrak{B}\!\int$ and Δ induce an equivalence of the localized categories

$$\mathfrak{B}\!\int : \mathsf{Br}\text{-}Cat^{\mathfrak{B}}[w_{\mathfrak{B}}^{-1}] \simeq \mathsf{Br}\text{-}Cat[w^{-1}] : \Delta.$$

We have now related E_2 structures in Cat to E_2 structures in $\mathsf{Br}\text{-}Cat^{\mathfrak{B}}$. The next step is to show that any braided $\mathfrak{B}$-category monoid is weakly equivalent to a strictly commutative $\mathfrak{B}$-category monoid, thus rectifying the E_2 structure. In order to achieve this we construct a functor Φ from $\mathsf{Br}\text{-}Cat$ to commutative $\mathfrak{B}$-category monoids, see Section 4.14. Letting $\mathcal{C}(Cat^{\mathfrak{B}})$ denote the category of commutative $\mathfrak{B}$-category monoids, the proposition below shows that this construction is functorial in $\mathcal{A}$ and that $\Phi(\mathcal{A})$ is a commutative $\mathfrak{B}$-category monoid.

Proposition (Proposition 4.16)**.** The $\mathfrak{B}$-category $\Phi(\mathcal{A})$ is a commutative monoid in $Cat^{\mathfrak{B}}$ and Φ defines a functor $\Phi\colon \mathsf{Br}\text{-}Cat \to \mathcal{C}(Cat^{\mathfrak{B}})$.

The theorem below, one of the main results of this paper, relates a braided $\mathfrak{B}$-category monoid A to the commutative $\mathfrak{B}$-category monoid $\Phi(\mathfrak{B}\!\int\! A)$ via the following chain of $\mathfrak{B}$-equivalences

$$A \simeq \Delta(\mathfrak{B}\!\int\! A) \simeq \Delta\big(\mathfrak{B}\!\int\!\Phi(\mathfrak{B}\!\int\! A)\big) \simeq \Phi(\mathfrak{B}\!\int\! A).$$

Theorem (Theorem 4.19)**.** Every braided $\mathfrak{B}$-category monoid is related to a strictly commutative $\mathfrak{B}$-category monoid by a chain of natural $\mathfrak{B}$-equivalences in $\mathsf{Br}\text{-}Cat^{\mathfrak{B}}$.

Section 5 is devoted to getting similar results for rectification of E_2 structures in the $\mathfrak{B}$-space setting. We define categories of E_2 structures $\mathrm{N}\mathsf{Br}\text{-}\mathcal{S}^{\mathfrak{B}}$ and $\mathrm{N}\mathsf{Br}\text{-}\mathcal{S}$ using the nerve of the braided and symmetric version of the Br operad respectively. The homotopy colimit preserves the algebra structure.

Lemma (Lemma 5.6)**.** The homotopy colimit functor can be promoted to a functor

$$(-)_{h\mathfrak{B}}\colon \mathrm{N}\mathsf{Br}\text{-}\mathcal{S}^{\mathfrak{B}} \to \mathrm{N}\mathsf{Br}\text{-}\mathcal{S}.$$

The relationship between the Grothendieck construction and the homotopy colimit, shown in the diagram below, follows from Thomason's work in [Tho79] checking that it is compatible with the braided structures.

Proposition (Proposition 5.7)**.** The diagram

$$\begin{array}{ccc} \mathsf{Br}\text{-}Cat^{\mathfrak{B}} & \xrightarrow{\mathrm{N}} & \mathrm{N}\mathsf{Br}\text{-}\mathcal{S}^{\mathfrak{B}} \\ {\scriptstyle \mathfrak{B}\int}\downarrow & & \downarrow{\scriptstyle (-)_{h\mathfrak{B}}} \\ \mathsf{Br}\text{-}Cat & \xrightarrow{\mathrm{N}} & \mathrm{N}\mathsf{Br}\text{-}\mathcal{S} \end{array}$$

commutes up to natural weak equivalence.

A morphism of $\mathfrak{B}$-spaces is a $\mathfrak{B}$-equivalence if the induced map on homotopy colimits is a weak equivalence. We write w for the class of morphisms in $\mathsf{NBr}\text{-}\mathcal{S}$ whose underlying maps of spaces are weak equivalences and $w_{\mathfrak{B}}$ for the class of morphisms in $\mathsf{NBr}\text{-}\mathcal{S}^{\mathfrak{B}}$ whose underlying maps of $\mathfrak{B}$-spaces are $\mathfrak{B}$-equivalences. The main results of Section 5 are summed up in the following theorem.

Theorem (Theorem 1.2)**.** The homotopy colimit $(-)_{h\mathfrak{B}}$ and the constant embedding Δ define an equivalence of the localized categories

$$(-)_{h\mathfrak{B}} \colon \mathsf{NBr}\text{-}\mathcal{S}^{\mathfrak{B}}[w_{\mathfrak{B}}^{-1}] \simeq \mathsf{NBr}\text{-}\mathcal{S}[w^{-1}] : \Delta$$

and every object in $\mathsf{NBr}\text{-}\mathcal{S}^{\mathfrak{B}}$ is naturally $\mathfrak{B}$-equivalent to a strictly commutative $\mathfrak{B}$-space monoid.

In Section 6 we define the bar construction on a $\mathfrak{B}$-space monoid. If the $\mathfrak{B}$ space monoid is commutative, the bar construction can be iterated twice. This provides a concrete example of a double delooping for the group completion of the nerve of a braided monoidal small category.

Corollary (Corollary 6.6)**.** If $\mathcal{A}$ is a braided monoidal small category, then

$$\mathrm{B}^{\boxtimes}(\mathrm{B}^{\boxtimes}(\mathrm{N}\Phi(\mathcal{A})))_{h\mathfrak{B}}$$

is a double delooping of the group completion of $\mathrm{N}\mathcal{A}$.

2.1.2 Weak braided monoidal categories and their homotopy colimits

In this paper we provide an answer to a question about homotopy properties of the homotopy colimit for weak braided monoidal categories, left open in [FSV13]. A weak braided monoidal category is a monoidal category with a family of natural morphisms $X \otimes Y \to Y \otimes X$, not necessarily isomorphisms, but satisfying the other axioms for a braiding. As is the case for a braided monoidal category, the nerve of a weak braided monoidal category is also an E_2 space, so we can consider a weak monoidal category to be a categorical E_2 structure.

A weak braided monoidal structure is more flexible than a braided monoidal structure and can be seen as a step towards the 2-fold monoidal structures which we will study later in the thesis. Where a braided monoidal category has one monoidal product and braidings that are isomorphisms, a weak braided monoidal category has one monoidal product and weak braidings that are not necessarily isomorphisms, while 2-fold monoidal categories have 2 monoidal products and interchange maps that are not necessarily isomorphisms.

Recall that braided strict monoidal small categories are algebras over the operad Br, the braid group version of the categorical Barratt-Eccles operad, see Example 20 in the Preliminaries. There is also a weak braided version of this operad, namely Br^+, with braid groups replaced by braid monoids. The category of algebras Br^+-Cat is isomorphic to the category of weak braided strict monoidal small categories, see [FSV13, Section 8]. The nerve of an object in Br^+-Cat inherits an action of the simplicial operad $N\mathsf{Br}^+$, the nerve functor applied to Br^+.

In [FSV13, Definition 4.10] a general homotopy colimit $\mathrm{hocolim}^{\mathsf{M}}X$ was constructed for a Cat-operad M and a diagram X of M-algebras. This was shown to have the desired homotopy properties if the operad in question satisfies the factorization condition in [FSV13, Definition 6.8]. The factorization condition was shown to be satisfied by the operads encoding symmetric strict monoidal categories, braided strict monoidal categories and n-fold monoidal categories, but the question if this is also true for Br^+ was left open. The key result of this paper is showing that Br^+ does satisfy the factorization condition, and thus we get the following result concerning the homotopy properties of $\mathrm{hocolim}^{\mathsf{Br}^+}X$ for a functor X from a small category into Br^+-Cat.

Theorem (Theorem 1.1)**.** There is a natural weak equivalence

$$\mathrm{hocolim}^{N\mathsf{Br}^+} NX \to N(\mathrm{hocolim}^{\mathsf{Br}^+} X)$$

of $N\mathsf{Br}^+$-algebras.

The proof involves an analysis of braid monoids. We consider a poset category $\mathcal{C}$ with objects a certain subset of a braid monoid, and show the existence of a unique minimal object in $\mathcal{C}$.

As a corollary we get an equivalence between Br^+-algebras and $N\mathsf{Br}^+$-algebras localized with respect to suitable classes of weak equivalences, see Section 3.1. Note that the same equivalence of localized categories was obtained by Fiedorowicz, Stelzer and Vogt in [FSV16, Section 11] without using the homotopy colimit construction of Br^+-algebras.

Corollary (Corollary 3.3)**.** We have an equivalence of localized categories

$$(\mathsf{Br}^+\text{-}Cat)[we^{-1}] \simeq (N\mathsf{Br}^+\text{-}\mathcal{S})[we^{-1}].$$

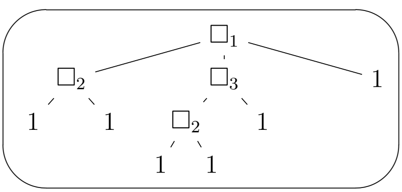
2.1.3 Operads and algebras in n-fold monoidal categories

The goal of this paper is to define a concept of n-fold monoidal operads and actions of n-fold monoidal operads on objects in n-fold monoidal categories. We want to preserve as much of the structure we find in symmetric operads, braided operads (see Definition 23 in the Preliminaries) and non-Σ operads as possible. In particular we want a 1-fold monoidal operad to be the same thing as a non-Σ operad. In order to formulate a familiar associativity axiom for the n-fold monoidal operad structure we need a canonical way to reorder a tensor product of objects. This is hard to achieve in an n-fold monoidal category. So we settled on defining an n-fold monoidal operad E as a structure with objects in a symmetric monoidal category $\mathcal{C}$. We say that E is internal to $\mathcal{C}$. As we will see later, in order for E to act on an object in an n-fold monoidal category $\mathcal{E}$, the category $\mathcal{E}$ has to be enriched over $\mathcal{C}$.

Part of the structure of a symmetric operad C is a sequence of objects $\mathsf{C}(n)$ in $\mathcal{C}$, each with an action of the permutation group Σ_n. This means that a symmetric operad has an underlying functor from $\Sigma^{\mathrm{op}} = (\amalg_{n\geq 0}\Sigma_n)^{\mathrm{op}}$ to $\mathcal{C}$. The category Σ is the free permutative category on one element. Similarly a braided operad has an underlying functor from the free braided strict monoidal category on one element $\mathcal{B}^{\mathrm{op}} = (\amalg_{n\geq 0}\mathcal{B}_n)^{\mathrm{op}}$, where $\mathcal{B}_n$ is the nth braid group. For a non-Σ operad, the corresponding underlying functor is from the set of natural numbers, including 0, which can be thought of as the free strict monoidal category on one element. In line with this pattern, an n-fold monoidal operad internal to $\mathcal{C}$ is, in Definition 4.3 of this paper, defined with an underlying functor

$$\mathsf{E}\colon \mathcal{F}_n^{\mathrm{op}} \to \mathcal{C} \qquad \text{where} \qquad A \mapsto \mathsf{E}_A.$$

The category $\mathcal{F}_n$ is the free n-fold monoidal category on one element 1, see the paragraph before Section 3.4. The objects in this category are products of tuples of 1's using the n-monoidal products. An example of an object is $(1\square_2 1)\square_1\big((1\square_2 1)\square_3 1\big)\square_1 1$ which can be visualized using a tree, as in the illustration below.



To define the additional structure we make use of the symmetric monoidal structure of $\mathcal{C}$. First there should be a unit element morphism from the monoidal unit I in $\mathcal{C}$ to E_1.

Furthermore there should be operad structure maps

$$\gamma\colon\ \mathsf{E}_A \otimes \mathsf{E}_{B_1} \otimes \cdots \otimes \mathsf{E}_{B_k} \to \mathsf{E}_{\Box_A(B_1,\ldots,B_k)},$$

where $\otimes$ is the monoidal product in $\mathcal{C}$. The object $\Box_A(B_1,\ldots,B_k)$ is, loosely speaking, the monoidal product in $\mathcal{F}_n$ of objects $B_1,\ldots,B_k$, according to the structure of the object A. If for example $A=(1\Box_2 1)\Box_1 1$ and $B_1=1\Box_1 1$, $B_2=1$ and $B_3=1\Box_3 1\Box_3 1$, then

$$\Box_A(B_1,B_2,B_3)=((1\Box_1 1)\Box_2 1)\Box_1(1\Box_3 1\Box_3 1).$$

The first 1 in A is replaced with B_1, the second 1 in A is replaced with B_2 and the third 1 in A is replaced with B_3. The structure maps must satisfy unit conditions, associativity conditions and equivariance conditions as detailed in Definition 4.3 of the paper. These conditions are modeled on the corresponding conditions for symmetric and braided operads. Per this definition, a 1-fold monoidal operad is the same as a non-Σ operad.

Section 5 in the paper deals with how an n-fold monoidal operad internal to $\mathcal{C}$ can act on an object in an n-fold monoidal category $\mathcal{E}$. For this to work the category has to be enriched over $\mathcal{C}$, for details see Definition 2.3 in this paper. As we shall see, an n-fold monoidal operad, the endomorphism operad, can in this case be associated with an object X in $\mathcal{E}$. For an object A in $\mathcal{F}_n$ we write $X^{\Box_A}$ for the monoidal product of a tuple of X's according to the n-fold monoidal structure of A. Using once more the example of $A=(1\Box_2 1)\Box_1 1$, we have $X^{\Box_A}=(X\Box_2 X)\Box_1 X$. The endomorphism operad End_X is defined with

$$(\mathsf{End}_X)_A=\underline{\mathcal{E}}(X^{\Box_A},X)$$

in Definition 5.1. For a morphism $\Phi\colon\ A\to A'$,

$$(\mathsf{End}_X)_\Phi\colon\ (\mathsf{End}_X)_{A'}\to(\mathsf{End}_X)_A$$

is precomposition by a morphism induced by Φ. The rest of the n-fold monoidal operad structure maps are quite analogous to the structure maps of the standard symmetric endomorphism operad associated with an object in a symmetric monoidal category. The endomorphism operad is then used to define an action of an n-fold monoidal operad.

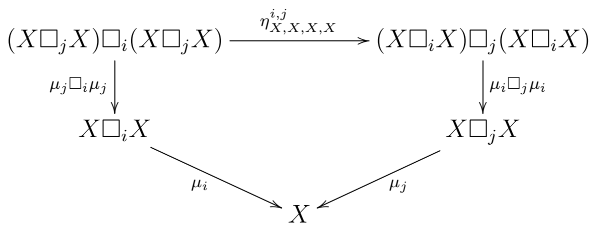
Definition (Definition 5.4)**.** An action of an n-fold monoidal operad C, internal to $\mathcal{C}$, on an object X in $\underline{\mathcal{E}}$, consists of a map $\theta\colon\ \mathsf{C}\to\mathsf{End}_X$ of n-fold monoidal operads.

This is the analogous definition to that of an action of a symmetric monoidal operad given by May in [May72]. However, a symmetric operad action is often described adjointly as a collection of maps $\mathsf{C}(j)\times X^j\to X$. If the n-fold monoidal category $\underline{\mathcal{E}}$ is not only enriched over $\mathcal{C}$, but also tensored over $\mathcal{C}$ in a manner compatible with the enriched structure (see Definition 6.1), there is an analogous adjoint description of an action of an n-fold monoidal operad. The proof that these two approaches are equivalent is more involved than in the symmetric monoidal case, and is dealt with in Section 6 of the paper.

In Section 7 we define the concepts of n-fold monoids and n-fold commutative monoids, and also n-fold monoidal operads that induce these structures. An n-fold monoid (Definition 7.1) is an object X in an n-fold monoidal category with n associative monoid multiplications, one for each monoidal product,

$$\mu_i\colon\ X\Box_i X \to X.$$

The monoid products should have a common strict unit. An n-fold monoid is called commutative if the diagram



commutes for all $1 \leq i < j \leq n$ (Definition 7.4). The operads for these structures are defined internal to the category *Set*. Every category can be considered to be enriched over *Set*, so we can have n-fold monoidal *Set*-operads act on objects in any n-fold monoidal category. The n-fold monoidal operad for an n-fold commutative monoid Comm is defined in Definition 7.6 with $\mathsf{Comm}_A = \{*\}$ for each $A \in \mathcal{F}_n$. This is similar to the symmetric operad C for commutative *Set*-monoids, see Example 18 in the Preliminaries. The n-fold monoidal operad Assoc for n-fold monoids is likewise similar to the symmetric operad A from Example 16 in the Preliminaries. For the symmetric operad we have $\mathsf{A}(k) = \Sigma_k$ which is the morphism set $\Sigma(\mathbf{k}, \mathbf{k})$ in the permutative category Σ. The n-fold monoidal operad is in Definition 7.2 defined with $\mathsf{Assoc}_A = \amalg_{A' \in \mathcal{F}_n} \mathcal{F}_n(A, A')$. The algebras of the operads Assoc and Comm are shown to be the n-fold monoids and the n-fold commutative monoids respectively, in Proposition 7.3 and Proposition 7.7.

The last section of the paper is devoted to E_n structures. These are structures where there is a degree of commutativity to the multiplications, similar to the structure of an E_n-space or the categorical counterpart, an n-fold monoidal category. We can now no longer consider an n-fold monoidal category which is just enriched over *Set*, there needs to be more flexibility. Instead we consider an n-fold monoidal category $\underline{\mathcal{E}}$ enriched over *Cat* and model the definition of an E_n object in $\mathcal{E}$ on the definition of an n-fold monoidal category in *Cat*. Definition 8.1 defines an E_n object in $\underline{\mathcal{E}}$ as an n-fold monoid with multiplications

$$\mu_i\colon\ X\Box_i X \to X \qquad \text{for } i = 1, \ldots, n,$$

a common unit, and for each pair of integers i, j with $1 \leq i < j \leq n$, a 2-cell $\Delta^{i,j}$ as illustrated by the diagram

$$\begin{array}{ccccc}
(X\Box_j X)\Box_i(X\Box_j X) & & \xrightarrow{\eta^{i,j}_{X,X,X,X}} & & (X\Box_i X)\Box_j(X\Box_i X) \\
\Big\downarrow{\scriptstyle \mu_j\Box_i\mu_j} & & & & \Big\downarrow{\scriptstyle \mu_i\Box_j\mu_i} \\
X\Box_i X & & \overset{\Delta^{i,j}}{\Longrightarrow} & & X\Box_j X \\
& \searrow^{\mu_i} & & \swarrow_{\mu_j} & \\
& & X. & &
\end{array}$$

Note that the outer diagram is the same as that for a commutative n-fold monoid. For a commutative n-fold monoid the diagram must be commutative, for an E_n object the requirement is a 2-cell from one composite to the other.

Furthermore there are internal and external unit conditions, internal and external associativity conditions, and a coherence condition, all similar to the axioms for an n-fold monoidal category. In fact, an E_n object thus defined is a generalization of an n-fold monoidal category.

Proposition (Proposition 8.2)**.** Let $\underline{Cat}$ be the 2-category of small categories, functors and natural transformations. We consider $\underline{Cat}$ as an n-fold monoidal category with $\Box_1 = \cdots = \Box_n = \times$, and the terminal category $\{*\}$ as the unit. An E_n object $\mathcal{X}$ in $\underline{Cat}$ is exactly the same as an n-fold monoidal category structure on $\mathcal{X}$.

The structure of an E_n object can also be encoded by an n-fold monoidal operad. In Definition 8.3 we define the n-fold monoidal operad E_n internal to *Cat*. For an object $A \in \mathcal{F}_n$ we set

$$(\mathsf{E}_n)_A = (A \downarrow \mathcal{F}_n).$$

This was inspired by the categorical version of the Barratt-Eccles operad, which has the comma category $(\mathbf{k} \downarrow \Sigma)$ at level k.

The final result of this paper is that the E_n-algebras are the E_n-objects.

Proposition (Proposition 8.5)**.** Let X be an object in an n-fold monoidal *Cat*-category $\underline{\mathcal{E}}$. An E_n object structure on X in the sense of Definition 8.1 is equivalent to an action of E_n on X.

2.1.4 Higher monoidal injections and diagrammatic E_n structures

Commutative rectification of E_∞ structures and E_2 structures was achieved in [SS12] and [SS16] respectively, by using diagram spaces and diagram categories with these structures. The main objective of this paper is to continue this line of work and consider E_n structures on diagram categories and diagram spaces. Specifically we will use diagrams with an n-fold monoidal category, see Definition 5 in Preliminaries, as indexing category. Proposition 4.2 in the paper shows that if we have a closed symmetric monoidal category $\mathcal{C}$, where the underlying category is cocomplete, and we have a small strict n-fold monoidal category $\mathcal{A}$, then the diagram category $\mathcal{C}^{\mathcal{A}}$ inherits an n-fold monoidal structure. Throughout this section $\otimes$ stands for the monoidal product in such a category $\mathcal{C}$.

A crucial step in setting up the theory in this paper is to determine what we mean by an E_n structure on a diagram $X \in \mathcal{C}^{\mathcal{A}}$, where $\mathcal{A}$ is n-fold monoidal. For this we use the notion of n-fold monoidal operads developed in [Sol]. In Section 4.16 we consider the n-fold monoidal operad E with underlying functor

$$E\colon \mathcal{F}_n^{\mathrm{op}} \to Cat, \quad a \mapsto (a \downarrow \mathcal{F}_n).$$

This is the operad that was denoted E_n in the previous paper. Recall that it is an n-fold version of the categorical Barratt-Eccles operad and that its algebras generalize the concept of strict n-fold monoidal categories, see Propositions 8.5 and 8.2 in [Sol]. The latter fact is also reflected in the symmetrization of E, which we will discuss later in this presentation.

The action of an n-fold monoidal operad C, internal to $\mathcal{C}$, on a diagram $X \in \mathcal{C}^{\mathcal{A}}$, can be described in more explicit terms than the action in a general n-fold monoidal category. According to Definition 4.7 in the paper it consists of a family of natural transformations

$$\theta_c\colon C(c) \otimes X(a_1) \otimes \cdots \otimes X(a_k) \to X(\Box^c(a_1, \ldots, a_k))$$

indexed by objects $c \in \mathcal{F}_n(k)$ and $a_i \in \mathcal{A}$ for $i = 1, \ldots k$. The object $\Box^c(a_1, \ldots, a_k)$ is constructed as an n-fold monoidal product of the objects $a_1, \ldots, a_k$ according to the structure of c. The natural transformations must be unital, associative and equivivariant as specified in the definition. A diagram X equipped with a C-action is called a C-algebra.

In order to relate this concept of E_n structures as algebras over the n-fold monoidal operad E to traditional E_n structures we define symmetrization of n-fold monoidal operads in Section 4.8. An n-fold monoidal operad C internal to $\mathcal{C}$ has an underlying functor $C\colon \mathcal{F}_n^{\mathrm{op}} \to \mathcal{C}$. Left Kan extension along the canonical functor $\varsigma_n\colon \mathcal{F}_n^{\mathrm{op}} \to \Sigma^{\mathrm{op}}$ gives a functor $\varsigma_{n!}(C)\colon \Sigma^{\mathrm{op}} \to \mathcal{C}$ natural in C. The n-fold monoidal operad structure on C induce a symmetric operad structure on $\varsigma_{n!}(C)$, and this is what we call the symmetrization of C. Proposition 4.9 shows that $\varsigma_{n!}$ is the left adjoint in an adjunction between the category of n-fold operads internal to $\mathcal{C}$ and symmetric operads internal to $\mathcal{C}$:

$$\varsigma_{n!}\colon\ Op_n(\mathcal{C}) \rightleftarrows Op_\Sigma(\mathcal{C}) :\varsigma_n^*$$

As a corollary we get that, in a symmetric setting, C-algebras and $\varsigma_{n!}C$-algebras are naturally isomorphic. In the result below the permutative category $\mathcal{A}$ is considered to have the canonical n-fold monoidal structure associated to a symmetric monoidal category, see Remark 7 in the Preliminaries.

Corollary (Corollary 4.11)**.** Let $\mathcal{A}$ be a small permutative category and let C be an n-fold monoidal operad internal to C. Then the categories of algebras C-$\mathcal{C}^{\mathcal{A}}$ and $\varsigma_{n!}C$-$\mathcal{C}^{\mathcal{A}}$ are naturally isomorphic.

These results provide the justification for considering a C-algebra structure as an E_n structure, if the nerve of the symmetrization of C is an E_n operad. As mentioned in Section 1.1.3 of the Preliminaries, the nerve of the symmetric operad $\mathcal{M}_n$ is an E_n operad. Therefore the following result lets us consider the algebras over the n-fold monoidal operad E as E_n structures.

Proposition (Proposition 4.17)**.** The symmetrization of the n-fold monoidal operad E is isomorphic to the operad $\mathcal{M}_n$ governing n-fold monoidal categories.

For technical reasons we need to shift to using the n-fold monoidal operad E^{op}. This is defined similarly to E, but with the opposite category at each level. The symmetrization of E^{op} is $\mathcal{M}_n^{\mathrm{op}}$ which is isomorphic to $\mathcal{M}_n$, so E^{op}-algebras are also E_n structures. Proposition 5.3 shows that for an E^{op}-algebra X in $Cat^{\mathcal{A}}$, the Grothendieck construction $\mathcal{A} \int X$ inherits the structure of an E^{op}-algebra in Cat, which is equivalent to the structure of an n-fold monoidal category by the above corollary.

We write E^{op}-$Cat^{\mathcal{A}}$ for the category of E^{op}-algebras in $Cat^{\mathcal{A}}$, and similarly E^{op}-Cat for the category of E^{op}-algebras in Cat. A morphism in E^{op}-Cat is a weak equivalence if the nerve of the underlying functor is a weak equivalence of simplicial sets. A morphism in E^{op}-$Cat^{\mathcal{A}}$ is a weak equivalence if the induced functor on the Grothendieck construction is a weak equivalence. Localizing with respect to these classes of weak equivalences respectively, yields homotopy categories of E^{op}-algebras. The main result of the paper is that there is an equivalence between the homotopy categories of E_n structures in *Cat* and $Cat^{\mathcal{A}}$.

Theorem (Theorem 5.11)**.** Let $\mathcal{A}$ be a small and strict n-fold monoidal category with contractible classifying space. Then the functors $\mathcal{A}\int$ and Δ induce an equivalence between the localized categories

$$\mathcal{A}\textstyle\int : E^{\mathrm{op}}\text{-}Cat^{\mathcal{A}}[w_{\mathcal{A}}^{-1}] \simeq E^{\mathrm{op}}\text{-}Cat[w^{-1}] : \Delta.$$

So far the results mentioned have been about diagrams indexed by any small strict n-fold monoidal category $\mathcal{A}$. Now we consider a specific indexing category we call $\mathcal{I}_n$, the category of n-fold monoidal injections, see Definition 3.4 in the paper. The objects of $\mathcal{I}_n$ are the objects of the free n-fold monoidal category on one element, $\mathcal{F}_n$. A morphism in $\mathcal{I}_n$, called an n-fold monoidal injection, consists of a pair of one morphism in $\mathcal{F}_n$ and an injective order preserving function of ordered finite sets. This is similar to how a braided injection in the category $\mathfrak{B}$ or an injection in the category $\mathcal{I}$ can be decomposed: A braided injection can be written as a pair of an element in the braid group and an order preserving function. An injection can be written as a pair of an element in the symmetric group and an order preserving function. In all three cases the number of elements in the domain of the order preserving function should match the permutation/braid/morphism in $\mathcal{F}_n$. A further common trait of these categories is that they have similar universal properties. As per Remark 3.11 in this paper, $\mathcal{I}$ is a free permutative category generated by the morphism $0 \to 1$ and $\mathfrak{B}$ is a free braided strict monoidal category the morphism $0 \to 1$. For $\mathcal{I}_n$ the universal property is stated in the following result.

Proposition (Proposition 3.9)**.** The category $\mathcal{I}_n$ is the free n-fold monoidal category generated by the morphism $0 \to 1$.

A goal for future research is to study n-fold commutative monoids in $Cat^{\mathcal{I}_n}$ further. Particularly to check if the homotopy category of n-fold commutative monoids in $Cat^{\mathcal{I}_n}$ is equivalent to the homotopy category of n-fold monoidal categories, localizing each of the categories with respect to the relevant weak equivalences. In this paper we have taken a step in that direction by showing that it is possible to realize a free n-fold monoidal category as the Grothendieck construction of an n-fold commutative monoid in $Cat^{\mathcal{I}_n}$: Given a category X with a distinguished object $* \in X$, there is a functor $X^{\bullet} \colon \mathcal{I} \to Cat$ that maps $\mathbf{p}$ to the product category X^p and takes a morphism $f \colon \mathbf{p} \to \mathbf{q}$ in $\mathcal{I}$ to a functor $f_* \colon X^p \to X^q$. For a p-tuple of objects $\mathbf{x} = (x_1, \ldots, x_p)$, the components of $f_*\mathbf{x} = \mathbf{y}$ are given by $y_j = x_i$ if $f(i) = j$ and $y_j = *$ if j is not in the image of f. Precomposing with the canonical functor $\varsigma_n \colon \mathcal{I}_n \to \mathcal{I}$ (see Corollary 3.10 in this paper), we get an element in $Cat^{\mathcal{I}_n}$. We can give $X^{\bullet}$ an n-fold commutative monoid structure where the product is induced by concatenation of tuples of objects, see the discussion at the start of Section 6 of this paper. Let X be a small category and X_+ the disjoint union of X with the terminal category. Treating the disjoint object as the distinguished object we form the $\mathcal{M}_n$-algebra $\mathcal{I}_n\int(X_+)^\bullet$. The free n-fold monoidal category on X is $\mathbb{M}_n(X)$, where $\mathbb{M}_n$ is the monad associated to the symmetric operad $\mathcal{M}_n$. The inclusion $X \to \mathcal{I}_n\int(X_+)^\bullet$ induces a map of $\mathcal{M}_n$-algebras $\mathbb{M}_n(X) \to \mathcal{I}_n\int(X_+)^\bullet$.

Theorem (Theorem 6.6)**.** The canonical map of $\mathcal{M}_n$-algebras

$$\mathbb{M}_n(X) \to \mathcal{I}_n\textstyle\int(X_+)^\bullet$$

is a weak equivalence.

The free n-fold monoidal category $\mathbb{M}_n(X)$ is therefore weakly equivalent to the Grothendieck construction of the n-fold commutative monoid $(X_+)^\bullet$ and we have a concrete model for $\mathbb{M}_n(X)$ in $Cat^{\mathcal{I}_n}$.

Analogously, in the simplicial set setting, we have the following result.

Theorem (Theorem 6.3)**.** For a based simplicial set X there is a natural weak equivalence $\rho\colon X^\bullet_{h\mathcal{I}_n} \xrightarrow{\sim} \mathrm{N}\,\mathbb{M}_{n*}(X)$ of $N\mathcal{M}_n$-algebras.

2.2 Future research

Here we list some topics of interest for future research.

- It would be good to have more explicit examples of n-fold monoidal operads. One idea is to try and make an n-fold monoidal operad version of the little n-cubes operad.

- Recall that an E_∞ operad is a Σ-free operad which is contractible at each level. For n equals 1 and 2, E_1- and E_2-operads can be modeled by A_∞- and B_∞-operads respectively. An A_∞-operad is a non-Σ operad that is contractible at each level. A B_∞-operad is a braided operad such that each level is contractible, and the actions of the braid group at each level is free.

  An n-fold monoidal E_n-operad can be defined as an n-fold monoidal operad with a contractible space at each level, such that the underlying functor is cofibrant in a suitable model structure. The cofibrant condition is analogous to the condition that the group action should be free for E_∞ and B_∞ operads. It would be interesting to further examine the relationship between n-fold monoidal E_n-operads and symmetric monoidal E_n-operads.

- In [BM03] Berger and Moerdijk define model structures on operads internal to symmetric monoidal model categories, given that certain conditions are satisfied. It would be interesting to see if a similar approach can be taken to define model structures on n-fold monoidal operads.

- In [Bat10] Batanin defines locally constant n-operads as higher braided operads. Both n-fold monoidal operads and locally constant n-operads generalize non-Σ operads ($n = 1$), braided operads ($n = 2$) and symmetric operads ($n = \infty$), but different aspects are generalized. It would be interesting to explore the relationship between these different generalizations.

- Finally, there is the continuation of the red thread that runs through this thesis, rectifying E_n structures. In [SS] we introduced the category $\mathcal{I}_n$ of n-fold monoidal injections as an n-fold analog of the indexing categories $\mathcal{I}$ for $\mathcal{I}$-spaces and $\mathfrak{B}$ for $\mathfrak{B}$-spaces, used in rectifying E_∞ ([SS12]) and E_2 ([SS16]) structures respectively. With the use of the n-fold monoidal operads, we have explicitly defined E_n objects in a diagram category indexed over a small n-fold monoidal category. In particular we can apply this to the diagram category $Cat^{\mathcal{I}_n}$. In [SS, Theorem 4.11] we showed that the homotopy category of these E_n diagram categories and the homotopy category of the corresponding E_n structures in Cat are equivalent. A natural next step is to try and generalize the rectification of E_∞ structures in [SS12] or the rectification of E_2 structures in [SS16] to E_n structures, using the setting of $\mathcal{I}_n$-categories and $\mathcal{I}_n$-spaces. As a first step one could start with comparing $\mathfrak{B}$-categories and $\mathcal{I}_2$-categories to see what can be generalized there.

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Part II

Included papers

Paper A: Braided injections and double loop spaces

Christian Schlichtkrull and Mirjam Solberg

Published in: Trans. Amer. Math. Soc. 368 (2016), no. 10, 7305–7338.
Published by the American Mathematical Society.

BRAIDED INJECTIONS AND DOUBLE LOOP SPACES

CHRISTIAN SCHLICHTKRULL AND MIRJAM SOLBERG

Abstract. We consider a framework for representing double loop spaces (and more generally E_2 spaces) as commutative monoids. There are analogous commutative rectifications of braided monoidal structures and we use this framework to define iterated double deloopings. We also consider commutative rectifications of E_∞ spaces and symmetric monoidal categories and we relate this to the category of symmetric spectra.

1. Introduction

The study of multiplicative structures on spaces has a long history in algebraic topology. For many spaces of interest the notion of a strictly associative and commutative multiplication is too rigid and must be replaced by the more flexible notion of an E_∞ multiplication encoding higher homotopies between iterated products. This is analogous to the situation for categories where strictly commutative multiplications rarely occur in practice and the more useful E_∞ notion is that of a symmetric monoidal structure. Similar remarks apply to multiplicative structures on other types of objects. However, for certain kinds of applications it is desirable to be able to replace E_∞ structures by strictly commutative ones, and this can sometimes be achieved by modifying the underlying category of objects under consideration. An example of this is the introduction of modern categories of spectra (in the sense of stable homotopy theory) [EKMM97, HSS00, MMSS01] equipped with symmetric monoidal smash products. These categories of spectra have homotopy categories equivalent to the usual stable homotopy category but come with refined multiplicative structures allowing the rectification of E_∞ ring spectra to strictly commutative ring spectra. This has proven useful for the import of ideas and constructions from commutative algebra into stable homotopy theory. Likewise there are symmetric monoidal refinements of spaces [BCS10, SS12] allowing for analogous rectifications of E_∞ structures.

Our main objective in this paper is to construct similar commutative rectifications in braided monoidal contexts. In order to provide a setting for this we introduce the category $\mathfrak{B}$ of *braided injections*, see Section 2. This is a braided monoidal small category that relates to the category $\mathcal{I}$ of finite sets and injections in the same way the braid groups relate to the symmetric groups. We first explain how our rectification works in the setting of small categories *Cat* and let Br-*Cat* denote the category of braided (strict)

2010 *Mathematics Subject Classification.* Primary 18D10, 18D50, 55P48; Secondary 55P43.
Key words and phrases. Braided monoidal categories, double loop spaces, diagram spaces.
First published in: Trans. Amer. Math. Soc. 368 (2016), no. 10, 7305–7338.

monoidal small categories. Let $Cat^{\mathfrak{B}}$ be the diagram category of functors from $\mathfrak{B}$ to Cat and let us refer to such functors as $\mathfrak{B}$-*categories*. The category $Cat^{\mathfrak{B}}$ inherits a braided monoidal convolution product from $\mathfrak{B}$ and there is a corresponding category $\mathsf{Br}\text{-}Cat^{\mathfrak{B}}$ of braided monoidal $\mathfrak{B}$-categories. A morphism $A \to A'$ in $\mathsf{Br}\text{-}Cat^{\mathfrak{B}}$ is said to be a $\mathfrak{B}$-*equivalence* if the induced functor of Grothendieck constructions $\mathfrak{B}\!\int\! A \to \mathfrak{B}\!\int\! A'$ is a weak equivalence of categories in the usual sense. We write $w_{\mathfrak{B}}$ for the class of $\mathfrak{B}$-equivalences and w for the class of morphisms in $\mathsf{Br}\text{-}Cat$ whose underlying functors are weak equivalences. The following rectification theorem is obtained by combining Proposition 4.12 and Theorem 4.19.

Theorem 1.1. *The Grothendieck construction* $\mathfrak{B}\!\int$ *and the constant embedding* Δ *define an equivalence of the localized categories*

$$\mathfrak{B}\!\int\colon \mathsf{Br}\text{-}Cat^{\mathfrak{B}}[w_{\mathfrak{B}}^{-1}] \simeq \mathsf{Br}\text{-}Cat[w^{-1}] : \Delta$$

and every object in $\mathsf{Br}\text{-}Cat^{\mathfrak{B}}$ *is naturally* $\mathfrak{B}$*-equivalent to a strictly commutative* $\mathfrak{B}$*-category monoid.*

Thus, working with braided monoidal categories is weakly equivalent to working with braided monoidal $\mathfrak{B}$-categories and the latter category has the advantage that we may assume multiplications to be strictly commutative. This implies in particular that every braided monoidal small category is weakly equivalent to one of the form $\mathfrak{B}\!\int\! A$ for a commutative $\mathfrak{B}$-category monoid A.

Let Br be the categorical operad such that the category of Br-algebras can be identified with $\mathsf{Br}\text{-}Cat$ (see Section 5.1 for details). For the analogous rectification in the category of spaces $\mathcal{S}$ (which we interpret as the category of simplicial sets) we consider the operad $\mathrm{N}\mathsf{Br}$ in $\mathcal{S}$ obtained by evaluating the nerve of Br. This is an E_2 operad in the sense of being equivalent to the little 2-cubes operad and we may think of the category of algebras $\mathrm{N}\mathsf{Br}\text{-}\mathcal{S}$ as the category of E_2 spaces. In order to rectify E_2 spaces to strictly commutative monoids we work in the diagram category of $\mathfrak{B}$-spaces $\mathcal{S}^{\mathfrak{B}}$ equipped with the braided monoidal convolution product inherited from $\mathfrak{B}$. There is an analogous category of E_2 $\mathfrak{B}$-spaces $\mathrm{N}\mathsf{Br}\text{-}\mathcal{S}^{\mathfrak{B}}$. After localization with respect to the appropriate classes of $\mathfrak{B}$-equivalences $w_{\mathfrak{B}}$ in $\mathrm{N}\mathsf{Br}\text{-}\mathcal{S}^{\mathfrak{B}}$ and weak equivalences w in $\mathrm{N}\mathsf{Br}\text{-}\mathcal{S}$, Proposition 5.8 and Theorem 5.9 combine to give the following result.

Theorem 1.2. *The homotopy colimit* $(-)_{h\mathfrak{B}}$ *and the constant embedding* Δ *define an equivalence of the localized categories*

$$(-)_{h\mathfrak{B}}\colon \mathrm{N}\mathsf{Br}\text{-}\mathcal{S}^{\mathfrak{B}}[w_{\mathfrak{B}}^{-1}] \simeq \mathrm{N}\mathsf{Br}\text{-}\mathcal{S}[w^{-1}] : \Delta$$

and every object in $\mathrm{N}\mathsf{Br}\text{-}\mathcal{S}^{\mathfrak{B}}$ *is naturally* $\mathfrak{B}$*-equivalent to a strictly commutative* $\mathfrak{B}$*-space monoid.*

This implies in particular that every double loop space is equivalent to an E_2 space of the form $A_{h\mathfrak{B}}$ for a commutative $\mathfrak{B}$-space monoid A. To give an example why this may be useful, notice that if A is a commutative $\mathfrak{B}$-space monoid, then the category $\mathcal{S}^{\mathfrak{B}}/A$ of $\mathfrak{B}$-spaces over A inherits the structure of a braided monoidal category. It is less obvious how to define such a structure for the corresponding category of spaces over an E_2 space.

The above rectification theorems have corresponding versions for symmetric monoidal categories and E_∞ spaces that we spell out in Section 7. As an application of this we show how to rectify certain E_∞ ring spectra to strictly commutative symmetric ring spectra. However, the braided monoidal setting is somewhat more subtle and is the main focus of this paper.

Our main tool for replacing braided monoidal structures by strictly commutative ones is a refinement of the usual strictification construction used to replace monoidal categories by strictly monoidal ones, see e.g. [JS93, Section 1]. While it is well-known that this construction cannot be used to turn braided monoidal categories into categories with a strictly commutative multiplication, we shall see that it can be reinterpreted so as to take values in commutative $\mathfrak{B}$-category monoids instead. This gives rise to the *$\mathfrak{B}$-category rectification functor* Φ introduced in Section 4.14. In order to obtain an analogous rectification on the space level we apply the results of Fiedorowicz-Stelzer-Vogt [FV03, FSV13] that show how to associate braided monoidal categories to E_2 spaces. Our rectification functor Φ then applies to these braided monoidal categories and we can apply the nerve functor level-wise to get back into the category of commutative $\mathfrak{B}$-space monoids.

It was pointed out by Stasheff and proved by Fiedorowicz [Fie] and Berger [Ber99] that the classifying space of a braided monoidal small category becomes a double loop space after group completion. As an application of our techniques we show in Section 6 how one can very simply define the double delooping: Given a braided monoidal category $\mathcal{A}$, we apply the rectification functor Φ and the level-wise nerve to get a commutative $\mathfrak{B}$-space monoid $\mathrm{N}\Phi(\mathcal{A})$. The basic fact (valid for any commutative monoid in a braided monoidal category whose unit is terminal) is now that the bar construction applied to $\mathrm{N}\Phi(\mathcal{A})$ is a simplicial monoid and hence can be iterated once to give a bisimplicial $\mathfrak{B}$-space. Evaluating the homotopy colimit of this $\mathfrak{B}$-space we get the double delooping. This construction in fact gives an alternative proof of Stasheff's result independent of the operadic recognition theorem for double loop spaces.

Another ingredient of our work is a general procedure for constructing equivalences between localized categories that we detail in Appendix A. This improves on previous work by Fiedorowicz-Stelzer-Vogt [FSV13, Appendix C] and has subsequently been used by these authors in [FSV] to sharpen some of their earlier results.

1.3. **Organization.** We begin by introducing the category of braided injections in Section 2 and establish the basic homotopy theory of $\mathfrak{B}$-spaces in Section 3. Then we switch to the categorical setting in Section 4 where we prove Theorem 1.1. In Section 5 we return to the analysis of $\mathfrak{B}$-spaces and prove Theorem 1.2, whereas Section 6 is dedicated to double deloopings of commutative $\mathfrak{B}$-space monoids. Finally, we consider the symmetric monoidal version of the theory and relate this to the category of symmetric spectra in Section 7. The material on localizations of categories needed for the paper is collected in Appendix A.

2. The category of braided injections

We generalize the geometric definition of the braid groups by introducing the notion of a *braided injection*. In this way we obtain a category $\mathfrak{B}$ of braided injections such that the classical braid groups appear as the endomorphism monoids.

In the following we write I for the unit interval. Let $\mathbf{n}$ denote the ordered set $\{1,\dots,n\}$ for $n \geq 1$. A braided injection α from $\mathbf{m}$ to $\mathbf{n}$, written $\alpha\colon \mathbf{m} \to \mathbf{n}$, is a homotopy class of m-tuples $(\alpha_1,\dots,\alpha_m)$, where each α_i is a path $\alpha_i\colon I \to \mathbb{R}^2$ starting in $(i,0)$ and ending in one of the points $(1,0)$, ..., $(n,0)$ with the requirement that $\alpha_i(t) \neq \alpha_j(t)$ for all t in I, whenever $i \neq j$. Two m-tuples $(\alpha_1,\dots,\alpha_m)$ and $(\beta_1,\dots,\beta_m)$ are homotopic if there exists an m-tuple of homotopies $H_i\colon I \times I \to \mathbb{R}^2$ from α_i to β_i, fixing endpoints, such that $H_i(s,t) \neq H_j(s,t)$ for all (s,t) in $I \times I$ whenever $i \neq j$. The requirement that H_i fixes endpoints ensures that a braided injection α from $\mathbf{m}$ to $\mathbf{n}$ defines an underlying injective function $\bar{\alpha}\colon \mathbf{m} \to \mathbf{n}$ by writing $\alpha_i(1) = (\bar{\alpha}(i),0)$. When visualising an injective braid, we think of the points $\alpha_i(t)$ for $i = 1,\dots,m$ as a family of distinct points in $\mathbb{R}^2$ moving downwards from the initial position $(1,0)$, ..., $(m,0)$, for $t = 0$, to the final position $(\bar{\alpha}(1),0)$,..., $(\bar{\alpha}(m),0)$, for $t = 1$.

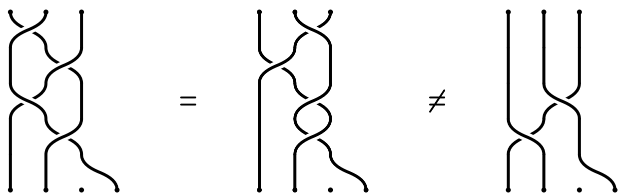


Figure 1. Braided injections with the same underlying injective map: $1 \mapsto 2$, $2 \mapsto 4$, $3 \mapsto 1$.

We can compose two braided injections $\alpha\colon \mathbf{m} \to \mathbf{n}$ and $\beta\colon \mathbf{n} \to \mathbf{p}$ by choosing representatives $(\alpha_1,\dots,\alpha_m)$ and $(\beta_1,\dots,\beta_n)$, and set $\beta \circ \alpha$ to be the homotopy class of the paths

$$(\beta_{\bar{\alpha}(1)} \cdot \alpha_1, ..., \beta_{\bar{\alpha}(m)} \cdot \alpha_m).$$

Here $\beta_{\bar{\alpha}(i)} \cdot \alpha_i$ denotes the usual composition of paths,

$$\beta_{\bar{\alpha}(i)} \cdot \alpha_i(t) = \begin{cases} \alpha_i(2t), & \text{for } 0 \leq t \leq 1/2, \\ \beta_{\bar{\alpha}(i)}(2t-1), & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

We let $\mathbf{0}$ denote the empty set and say that there is exactly one braided injection from $\mathbf{0}$ to $\mathbf{n}$ for $n \geq 0$.

Definition 2.1. The category $\mathfrak{B}$ of braided injections has objects the finite sets $\mathbf{n}$ for $n \geq 0$ and morphisms the braided injections between these sets.

Next we recall the definitions of some categories closely related to $\mathfrak{B}$.

Definition 2.2. The categories $\mathcal{B}$, Σ, $\mathcal{I}$ and $\mathcal{M}$ all have as objects the finite sets $\mathbf{n}$ for $n \geq 0$. Here the *braid category* $\mathcal{B}$ and the *permutation category* Σ have respectively the braid group $\mathcal{B}_n$ and the permutation group Σ_n as the endomorphism set of $\mathbf{n}$, and no other morphisms. The morphisms in $\mathcal{I}$ and $\mathcal{M}$ are the injective functions and the order preserving injective functions, respectively.

There is a canonical functor Π from $\mathfrak{B}$ to $\mathcal{I}$ that takes a braided injection $\alpha\colon \mathbf{m} \to \mathbf{n}$ to the underlying injective function $\bar{\alpha}\colon \mathbf{m} \to \mathbf{n}$. By definition, $\mathcal{B}$ is a subcategory of $\mathfrak{B}$ and Σ is a subcategory of $\mathcal{I}$. Clearly Π restricts to a functor from $\mathcal{B}$ to Σ, which we also denote by Π. The category $\mathcal{M}$ is a subcategory of $\mathcal{I}$ and there is a canonical embedding $\Upsilon\colon \mathcal{M} \to \mathfrak{B}$ with $\Upsilon(\mathbf{n}) = \mathbf{n}$. For an injective order preserving function $\mu\colon \mathbf{m} \to \mathbf{n}$, let μ_i be the straight path from $(i,0)$ to $(\mu(i),0)$ for $1 \leq i \leq m$. Since μ is order preserving, $\mu_i(t)$ is different from $\mu_j(t)$ whenever $i \neq j$, and we can define $\Upsilon(\mu)$ as the braided injection represented by the tuple $(\mu_1, \dots, \mu_m)$. These functors fit into the following commutative diagram

$$\begin{array}{ccccc} \mathcal{B} & \subseteq & \mathfrak{B} & & \\ {\scriptstyle \Pi}\downarrow & & {\scriptstyle \Pi}\downarrow & \nwarrow{\scriptstyle \Upsilon} & \\ \Sigma & \subseteq & \mathcal{I} & \supseteq & \mathcal{M}. \end{array} \tag{2.1}$$

The categories $\mathcal{B}$, Σ, $\mathcal{I}$ and $\mathcal{M}$ are all monoidal categories with monoidal product $\sqcup$ given on objects by $\mathbf{m} \sqcup \mathbf{n} = \mathbf{m} + \mathbf{n}$. In addition, $\mathcal{B}$ is braided monoidal and Σ and $\mathcal{I}$ are symmetric monoidal. We will extend these monoidal structures to a braided monoidal structure on $\mathfrak{B}$ such that all functors in the diagram are strict monoidal functors and functors between braided monoidal categories are braided strict monoidal functors. In order to do this, we will show that every morphism in $\mathfrak{B}$ can be uniquely written in terms of a braid and a morphism in $\mathcal{M}$.

Lemma 2.3. *Every braided injection $\alpha\colon \mathbf{m} \to \mathbf{n}$ can be written uniquely as a composition $\alpha = \Upsilon(\mu) \circ \zeta$ with μ in $\mathcal{M}(\mathbf{m}, \mathbf{n})$ and ζ in the braid group $\mathcal{B}_m$.*

Proof. Let $\mu\colon \mathbf{m} \to \mathbf{n}$ be the unique order preserving injective function whose image equals that of $\bar{\alpha}$, and let $\{j_1, \dots, j_m\}$ be the permutation of the set $\mathbf{m} = \{1, \dots, m\}$ determined by $\bar{\alpha}(i) = \mu(j_i)$ for $i = 1, \dots, m$. Choose representatives $(\mu_1, \dots, \mu_m)$ and $(\alpha_1, \dots, \alpha_m)$ for $\Upsilon(\mu)$ and α respectively. Let μ'_i be the reverse path of μ_i for $1 \leq i \leq m$. Since the path μ'_{j_i} starts in $(\mu(j_i), 0) = \alpha_i(1)$ and ends in $(j_i, 0)$, the homotopy class of the concatenated paths $(\mu'_{j_1} \cdot \alpha_1, \dots, \mu'_{j_m} \cdot \alpha_m)$ is a braid on m strings and we define this to be ζ. The composite $\Upsilon(\mu) \circ \zeta$ is represented by $(\mu_{j_1} \cdot \mu'_{j_1} \cdot \alpha_1, \dots, \mu_{j_m} \cdot \mu'_{j_m} \cdot \alpha_m)$, which is clearly homotopic to $(\alpha_1, \dots, \alpha_m)$. The morphism μ is uniquely determined by $\bar{\alpha}$ and we see from the construction that ζ is then also uniquely determined. ∎

The above lemma implies that there is a canonical identification

$$\mathfrak{B}(\mathbf{m},\mathbf{n}) \cong \mathcal{M}(\mathbf{m},\mathbf{n}) \times \mathcal{B}_m. \tag{2.2}$$

Now consider a pair (μ,ζ) in $\mathcal{M}(\mathbf{m},\mathbf{n}) \times \mathcal{B}_m$ and a pair (ν,ξ) in $\mathcal{M}(\mathbf{n},\mathbf{p}) \times \mathcal{B}_n$. By Lemma 2.3 there exists a unique morphism $\xi_*(\mu)$ in $\mathcal{M}(\mathbf{m},\mathbf{n})$ and a unique braid $\mu^*(\xi)$ in $\mathcal{B}_m$ such that the diagram

$$\begin{array}{ccc} \mathbf{m} & \xrightarrow{\Upsilon(\mu)} & \mathbf{n} \\ {\scriptstyle \mu^*(\xi)}\downarrow & & \downarrow{\scriptstyle \xi} \\ \mathbf{m} & \xrightarrow{\Upsilon(\xi_*(\mu))} & \mathbf{n} \end{array}$$

commutes in $\mathfrak{B}$. Hence we see that composition in $\mathfrak{B}$ translates into the formula

$$(\nu,\xi)\circ(\mu,\zeta) = (\nu\circ\xi_*(\mu), \mu^*(\xi)\circ\zeta)$$

under the identification in (2.2).

In order to define functors out of the categories considered in Definition 2.2, it is sometimes convenient to have these categories expressed in terms of generators and relations. Consider first the case of $\mathcal{M}$ and write $\partial_n^i\colon \mathbf{n}\to\mathbf{n}\sqcup\mathbf{1}$ for the morphism that misses the element i in $\{1,\dots,n+1\}$. It is well known that $\mathcal{M}$ is generated by the morphisms ∂_n^i subject to the relations

$$\partial_{n+1}^i\partial_n^j = \partial_{n+1}^{j+1}\partial_n^i \quad \text{for } i\le j.$$

Now consider the category $\mathfrak{B}$ and let $\zeta_n^1,\dots,\zeta_n^{n-1}$ be the standard generators for the braid group $\mathcal{B}_n$, see e.g. [Bir74, Theorem 1.8].

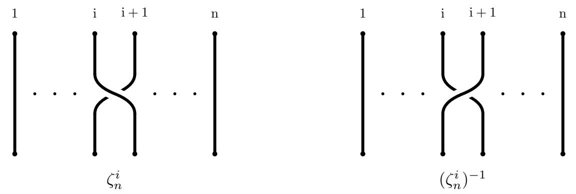


FIGURE 2. The generator ζ_n^i and its inverse

We also write $\partial_n^i\colon \mathbf{n}\to\mathbf{n}\sqcup\mathbf{1}$ for the braided injections obtained by applying the functor Υ to the corresponding morphisms in $\mathcal{M}$.

Lemma 2.4. *The category $\mathfrak{B}$ is generated by the morphisms $\zeta_n^i\colon \mathbf{n}\to\mathbf{n}$ for $n\ge 2$ and $1\le i\le n-1$, and the morphisms $\partial_n^i\colon \mathbf{n}\to\mathbf{n}\sqcup\mathbf{1}$ for $n\ge 0$ and $1\le i\le n+1$, subject to the relations*

$$\begin{aligned} &\zeta_n^i\zeta_n^j = \zeta_n^j\zeta_n^i && \text{for } |i-j|\ge 2\\ &\zeta_n^i\zeta_n^{i+1}\zeta_n^i = \zeta_n^{i+1}\zeta_n^i\zeta_n^{i+1} && \text{for } 1\le i\le n-2\\ &\partial_{n+1}^i\partial_n^j = \partial_{n+1}^{j+1}\partial_n^i && \text{for } i\le j \end{aligned}$$

and

$$\zeta^i_{n+1}\partial^j_n = \begin{cases} \partial^j_n \zeta^{i-1}_n & \text{for } j < i \\ \partial^{j+1}_n & \text{for } j = i \\ \partial^{j-1}_n & \text{for } j = i+1 \\ \partial^j_n \zeta^i_n & \text{for } j > i+1. \end{cases}$$

Proof. The identification $\mathfrak{B}(\mathbf{m},\mathbf{n}) \cong \mathcal{M}(\mathbf{m},\mathbf{n}) \times \mathcal{B}_m$ makes it clear that any morphism can be written in terms of the generators. The two first relations are the relations for the braid groups (see e.g., [Bir74, Theorem 1.8]), the next are the relations in $\mathcal{M}$, so that leaves the relations between the ∂^i_n's and ζ^i_n's. It is easy to see that these relations hold in $\mathfrak{B}$ and that they can be used to decompose any product of the ∂^i_n's and the ζ^i_n's into the form $\Upsilon(\mu) \circ \zeta$ for a braid ζ and a morphism μ in $\mathcal{M}$. Since such a decomposition is unique, the relations are also sufficient. □

Finally, we consider the category $\mathcal{I}$ and write $\sigma^i_n \colon \mathbf{n} \to \mathbf{n}$ for the image of ζ^i_n under the projection $\Pi \colon \mathfrak{B} \to \mathcal{I}$. We obtain a presentation of $\mathcal{I}$ from the presentation of $\mathfrak{B}$ by imposing the relation $\sigma^i_n\sigma^i_n = \mathrm{id}_n$, just as the symmetric group Σ_n is obtained from $\mathcal{B}_n$.

We use the above to define a strict monoidal structure on $\mathfrak{B}$ with unit $\mathbf{0}$. Just as for the monoidal categories considered in Diagram (2.1), the monoidal product $\mathbf{m}\sqcup\mathbf{n}$ of two objects $\mathbf{m}$ and $\mathbf{n}$ in $\mathfrak{B}$ is $\mathbf{m}+\mathbf{n}$. The decomposition of a braided injection given in (2.2) lets us define the monoidal product $(\mu,\zeta) \sqcup (\nu,\xi)$ of two morphisms (μ,ζ) and (ν,ξ) in $\mathfrak{B}$ as $(\mu \sqcup \nu, \zeta \sqcup \xi)$ using the monoidal structures on $\mathcal{M}$ and $\mathcal{B}$, for an illustration of this see Figure 3.

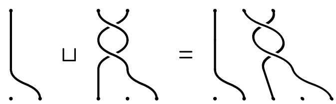


FIGURE 3. The monoidal product of two braided injections.

It is well known that the subcategory $\mathcal{B}$ is braided with braiding $\chi_{\mathbf{m},\mathbf{n}} \colon \mathbf{m}\sqcup\mathbf{n} \to \mathbf{n}\sqcup\mathbf{m}$ moving the first m strings over the last n strings while keeping the order among the m strings and the n strings respectively. This family of isomorphisms is in fact also a braiding on $\mathfrak{B}$. The hexagonal axioms for a braiding only involve morphisms in $\mathcal{B}$ so it remains to check that $\chi_{\mathbf{m},\mathbf{n}}$ is natural with respect to the generators ∂^i_k. This is quite clear geometrically (see Figure 4 for an illustration) and can be checked algebraically by writing $\chi_{\mathbf{m},\mathbf{n}}$ in terms of the generators.

3. THE HOMOTOPY THEORY OF $\mathfrak{B}$-SPACES

In this section we introduce $\mathfrak{B}$-spaces as functors from $\mathfrak{B}$ to the category of spaces and equip the category of $\mathfrak{B}$-spaces with a braided monoidal model structure. We assume some familiarity with the basic theory of cofibrantly generated model categories as presented in [Hov99, Section 2.1] and [Hir03, Section 11].

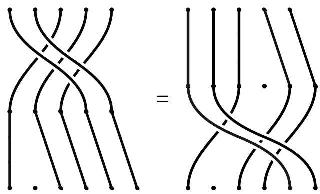


FIGURE 4. The equality $(\partial_3^2 \sqcup \mathrm{id}_{\mathbf{2}}) \circ \chi_{\mathbf{2},\mathbf{3}} = \chi_{\mathbf{2},\mathbf{4}} \circ (\mathrm{id}_{\mathbf{2}} \sqcup \partial_3^2)$.

3.1. **The category of $\mathfrak{B}$-spaces.** A $\mathfrak{B}$-space is a functor $X\colon \mathfrak{B} \to \mathcal{S}$, where $\mathcal{S}$ is the category of simplicial sets. We call a natural transformation between two such functors a morphism between the two $\mathfrak{B}$-spaces and write $\mathcal{S}^{\mathfrak{B}}$ for the category of $\mathfrak{B}$-spaces so defined.

The category $\mathcal{S}^{\mathfrak{B}}$ inherits much structure from $\mathcal{S}$. All small limits and colimits exists and are constructed level-wise. Furthermore, $\mathcal{S}^{\mathfrak{B}}$ is enriched, tensored and cotensored over $\mathcal{S}$. For a $\mathfrak{B}$-space X and a simplicial set K, the tensor $X \times K$ and cotensor X^K are the $\mathfrak{B}$-spaces given in level $\mathbf{n}$ by

$$(X \times K)(\mathbf{n}) = X(\mathbf{n}) \times K \quad \text{and} \quad X^K(\mathbf{n}) = \mathrm{Map}_{\mathcal{S}}(K, X(\mathbf{n})),$$

where $\mathrm{Map}_{\mathcal{S}}$ is the standard simplicial function complex. The simplicial set of maps from X to Y is the end

$$\mathrm{Map}_{\mathcal{S}^{\mathfrak{B}}}(X, Y) = \int_{\mathbf{n} \in \mathfrak{B}} \mathrm{Map}_{\mathcal{S}}\big(X(\mathbf{n}), Y(\mathbf{n})\big).$$

Lemma 3.2. *The category of $\mathfrak{B}$-spaces is a bicomplete simplicial category with the above defined structure.* □

3.3. **The $\mathfrak{B}$-model structure on $\mathcal{S}^{\mathfrak{B}}$.** We will use the free $\mathfrak{B}$-space functors $F_{\mathbf{n}}\colon \mathcal{S} \to \mathcal{S}^{\mathfrak{B}}$ given by $F_{\mathbf{n}}(K) = \mathfrak{B}(\mathbf{n}, -) \times K$ to transport the usual model structure on simplicial sets to $\mathcal{S}^{\mathfrak{B}}$. The functor $F_{\mathbf{n}}$ is left adjoint to the evaluation functor $\mathrm{Ev}_{\mathbf{n}}$ taking a $\mathfrak{B}$-space X to the simplicial set $X(\mathbf{n})$. Note that since $\mathbf{0}$ is initial in $\mathfrak{B}$, the functor $F_{\mathbf{0}}$ takes a simplicial set to a constant $\mathfrak{B}$-space. We often use the notation Δ for $F_{\mathbf{0}}$.

It is a standard fact, see for instance [Hir03, Theorem 11.6.1], that $\mathcal{S}^{\mathfrak{B}}$ has a level model structure where a morphism is a weak equivalence (or respectively a fibration) if it is a weak equivalence (or respectively a fibration) of simplicial sets when evaluated at each level $\mathbf{n}$. This model structure is cofibrantly generated with generating cofibrations

$$I = \{F_{\mathbf{n}}(i) \mid \mathbf{n} \in \mathfrak{B},\, i\colon\ \partial\Delta^k \to \Delta^k \text{ for } 0 \leq k\}$$

and generating acyclic cofibrations

$$J = \{F_{\mathbf{n}}(j) \mid \mathbf{n} \in \mathfrak{B},\, j\colon\ \Lambda^k_l \to \Delta^k \text{ for } k > 0 \text{ and } 0 \leq l \leq k\}$$

where i and j denote the inclusion of the boundary of Δ^k and the lth horn of Δ^k in Δ^k respectively. The cofibrations in the level model structure have a concrete description using latching maps. The nth latching space of a $\mathfrak{B}$-space X is defined as

$$L_{\mathbf{n}}(X) = \operatorname*{colim}_{(\mathbf{m}\to\mathbf{n})\in\partial(\mathfrak{B}\downarrow\mathbf{n})} X(\mathbf{m}),$$

where $\partial(\mathfrak{B}\downarrow\mathbf{n})$ is the full subcategory of the comma category $(\mathfrak{B}\downarrow\mathbf{n})$ with objects the non-isomorphisms. For a map of $\mathfrak{B}$-spaces $f\colon X\to Y$, the nth latching map is the $\mathcal{B}_n$-equivariant map $L_{\mathbf{n}}f\colon L_{\mathbf{n}}(Y)\amalg_{L_{\mathbf{n}}(X)} X(\mathbf{n})\to Y(\mathbf{n})$. A map $f\colon X\to Y$ is a cofibration if for every $n\geq 0$, the nth latching map $L_{\mathbf{n}}f$ is a cofibration of simplicial sets such that the $\mathcal{B}_n$-action on the complement of the image is free. We refer to such cofibrations as $\mathfrak{B}$-cofibrations.

The level model structure is primarily used as a convenient first step in equipping $\mathcal{S}^{\mathfrak{B}}$ with a model structure making it Quillen equivalent to $\mathcal{S}$. In such a model structure we need a wider class of weak equivalences. Recall that the Bousfield-Kan construction of the homotopy colimit of a functor X from a small category $\mathcal{C}$ to $\mathcal{S}$ is the simplicial set $\operatorname{hocolim}_{\mathcal{C}} X$ with k-simplices

$$\coprod_{\mathbf{m}_0\leftarrow\cdots\leftarrow\mathbf{m}_k} X(\mathbf{m}_k)_k \tag{3.1}$$

for morphisms $\mathbf{m}_0\leftarrow\mathbf{m}_1,\dots,\mathbf{m}_{k-1}\leftarrow\mathbf{m}_k$ in $\mathcal{C}$, cf. [BK72, Section XII.5.1]. When the functor X is a $\mathfrak{B}$-space we will often denote its homotopy colimit by $X_{h\mathfrak{B}}$.

Definition 3.4. A morphisms $X\to Y$ of $\mathfrak{B}$-spaces is a $\mathfrak{B}$-equivalence if the induced map $X_{h\mathfrak{B}}\to Y_{h\mathfrak{B}}$ is a weak equivalence of simplicial sets.

We say that a morphism $X\to Y$ of $\mathfrak{B}$-spaces is a $\mathfrak{B}$-fibration if $X(\mathbf{n})\to Y(\mathbf{n})$ is a fibration of simplicial sets for every $\mathbf{n}\in\mathfrak{B}$ and if the square

$$\begin{array}{ccc} X(\mathbf{m}) & \xrightarrow{X(\alpha)} & X(\mathbf{n}) \\ \downarrow & & \downarrow \\ Y(\mathbf{m}) & \xrightarrow{Y(\alpha)} & Y(\mathbf{n}) \end{array} \tag{3.2}$$

is homotopy cartesian for every braided injection $\alpha\colon \mathbf{m}\to\mathbf{n}$.

In order to make the $\mathfrak{B}$-equivalences and $\mathfrak{B}$-fibrations part of a cofibrantly generated model structure we have to add more generating acyclic cofibrations compared to the level model structure. We follow the approach taken for diagram spectra in [HSS00, Section 3.4] and [MMSS01, Section 9] and for diagram spaces in [SS12, Section 6.11]: Each braided injection $\alpha\colon \mathbf{m}\to\mathbf{n}$ gives rise to a map of $\mathfrak{B}$-spaces $\alpha^*\colon F_{\mathbf{n}}(*)\to F_{\mathbf{m}}(*)$. The latter map factors through the mapping cylinder $M(\alpha^*)$ as $\alpha^* = r_\alpha j_\alpha$, where j_α is a cofibration in the level model structure and r_α is a simplicial homotopy equivalence. We now set

$$\bar{J} = \{j_\alpha \square i \mid \alpha\colon \mathbf{m}\to\mathbf{n}\in\mathfrak{B},\ i\colon \partial\Delta^k\to\Delta^k \text{ for } 0\leq k\},$$

where $\square$ denotes the pushout-product, see e.g. [Hov99, Definition 4.2.1].

Proposition 3.5. *There is a model structure on $\mathcal{S}^{\mathfrak{B}}$, the $\mathfrak{B}$-model structure, with weak equivalences the $\mathfrak{B}$-equivalences, fibrations the $\mathfrak{B}$-fibrations and cofibrations the $\mathfrak{B}$-cofibrations. This model structure is simplicial and cofibrantly generated where $I_{\mathfrak{B}} = I$ is the set of generating cofibrations and $J_{\mathfrak{B}} = J \cup \bar{J}$ is the set of generating acyclic cofibrations.*

Proof. The proof is similar to the proofs of Propositions 6.16 and 6.19 in [SS12]. (We refer the reader to Remark 3.14 for a summary of the extent to which the results for symmetric monoidal diagram categories established in [SS12] carries over to the present setting.) □

As promised this model structure makes $\mathfrak{B}$-spaces Quillen equivalent to simplicial sets.

Proposition 3.6. *The adjunction* $\operatorname{colim}_{\mathfrak{B}} \colon \mathcal{S}^{\mathfrak{B}} \rightleftarrows \mathcal{S} : \Delta$ *is a Quillen equivalence.*

Proof. The category $\mathfrak{B}$ has an initial object so $\mathrm{N}\mathfrak{B}$ is a contractible simplicial set. Arguing as in the proof of Proposition 6.23 in [SS12] yields the result. □

Example 3.7. In general an $\mathcal{I}$-space $Z \colon \mathcal{I} \to \mathcal{S}$ pulls back to a $\mathfrak{B}$-space $\Pi^* Z$ via the functor $\Pi \colon \mathfrak{B} \to \mathcal{I}$ from Section 2. Consider in particular a based space X with base point $*$ and the $\mathcal{I}$-space $X^{\bullet} \colon \mathcal{I} \to \mathcal{S}$ such that $X^{\bullet}(\mathbf{n}) = X^n$. A morphism $\alpha \colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ acts on an element $\mathbf{x} = (x_1, \dots, x_m)$ by

$$\alpha_*(\mathbf{x}) = (x_{\alpha^{-1}(1)}, \dots, x_{\alpha^{-1}(n)}),$$

where $x_{\alpha^{-1}(j)} = x_i$ if $\alpha(i) = j$ and $x_{\alpha^{-1}(j)} = *$ if j is not in the image of α. It is proved in [Sch07] that if X is connected, then the geometric realization $|X^{\bullet}_{h\mathcal{I}}|$ is equivalent to the infinite loop space $\Omega^{\infty}\Sigma^{\infty}(|X|)$. In contrast to this we shall prove in Example 5.10 that $|(\Pi^* X^{\bullet})_{h\mathfrak{B}}|$ is equivalent to $\Omega^2\Sigma^2(|X|)$ for connected X.

3.8. **The flat $\mathfrak{B}$-model structure on $\mathcal{S}^{\mathfrak{B}}$.** We will now consider another structure on $\mathfrak{B}$-spaces, the flat $\mathfrak{B}$-model structure, which takes into account that each level of a $\mathfrak{B}$-space has a left action of a braid group. The weak equivalences are again the $\mathfrak{B}$-equivalences, but the flat $\mathfrak{B}$-model structure has more cofibrant objects than the $\mathfrak{B}$-model structure. In some places, in particular in Section 6, we get more general results by considering these "flat" objects instead of only the $\mathfrak{B}$-cofibrant objects. The flat $\mathfrak{B}$-model structure is constructed similarly to the $\mathfrak{B}$-model structure, but the starting point is Shipley's mixed model structure on the category $\mathcal{B}_n$-$\mathcal{S}$ of simplicial sets with left $\mathcal{B}_n$-action, see [Shi04, Proposition 1.3]. Shipley only considers finite groups, but the construction applies equally well to discrete groups in general if one allows all subgroups to be considered. An equivariant map is a weak equivalence (or respectively a cofibration) in the mixed model structure if the underlying map of simplicial sets is. Recall that given a group H and an H-space K, the space of homotopy fixed points K^{hH} is the homotopy limit of K viewed as a diagram over the one-object category H. An equivariant map $K \to L$ is a fibration in the mixed model structure if the induced maps $K^H \to L^H$ of fixed points are fibrations and the diagrams

$$\begin{array}{ccc} K^H & \longrightarrow & K^{hH} \\ \downarrow & & \downarrow \\ L^H & \longrightarrow & L^{hH} \end{array}$$

are homotopy cartesian for all subgroups H of $\mathcal{B}_n$. This model structure is cofibrantly generated, see the proof of [Shi04, Proposition 1.3] for a description of the generating (acyclic) cofibrations.

The forgetful functor $\mathrm{Ev}_{\mathbf{n}}\colon \mathcal{S}^{\mathfrak{B}} \to \mathcal{B}_n\text{-}\mathcal{S}$ evaluating a $\mathfrak{B}$-space X at the nth level has a right adjoint $G_{\mathbf{n}}$ given by $G_{\mathbf{n}}(K) = \mathfrak{B}(\mathbf{n}, -) \times_{\mathcal{B}_n} K$. We proceed as in the previous subsection and get a new level model structure on $\mathcal{S}^{\mathfrak{B}}$ where a morphism is a weak equivalence (or respectively a fibration) if it is a weak equivalence (or respectively a fibration) in the mixed model structure on $\mathcal{B}_n$-$\mathcal{S}$ when evaluated at each level $\mathbf{n}$. This model structure is cofibrantly generated with generating (acyclic) cofibrations I_{m} (J_{m}) obtained by applying $G_{\mathbf{n}}$ to the generating (acyclic) cofibrations for the mixed model structure on $\mathcal{B}_n$-$\mathcal{S}$ for all $\mathbf{n}$ in $\mathfrak{B}$. A morphisms $f\colon X \to Y$ is a cofibration in this level model structure if for every $n \geq 0$, the nth latching map $L_{\mathbf{n}}f$ is a cofibration of simplicial sets. We refer to such cofibrations as flat $\mathfrak{B}$-cofibrations. A morphism $X \to Y$ of $\mathfrak{B}$-spaces is said to be a flat $\mathfrak{B}$-fibration if $X(\mathbf{n}) \to Y(\mathbf{n})$ is a fibration in the mixed model structure on $\mathcal{B}_n$-$\mathcal{S}$ for every $\mathbf{n}$ in $\mathfrak{B}$ and if the square (3.2) is homotopy cartesian for every braided injection $\alpha\colon \mathbf{m} \to \mathbf{n}$.

Proposition 3.9. *There is a model structure on $\mathcal{S}^{\mathfrak{B}}$, the flat $\mathfrak{B}$-model structure, with weak equivalences the $\mathfrak{B}$-equivalences, fibrations the flat $\mathfrak{B}$-fibrations and cofibrations the flat $\mathfrak{B}$-cofibrations. This model structure is simplicial and cofibrantly generated where $I_{\mathrm{flat}} = I_{\mathrm{m}}$ is the set of generating cofibrations and $J_{\mathrm{flat}} = J_{\mathrm{m}} \cup \bar{J}$ is the set of generating acyclic cofibrations.*

Proof. The proof is similar to the proofs of Propositions 6.16 and 6.19 in [SS12]. □

We will refer to the flat $\mathfrak{B}$-cofibrant objects simply as flat objects. These will play an important role also when we are considering the $\mathfrak{B}$-model structure. The next result gives a criterion for an object to be flat which is easier to check than the one given above.

Proposition 3.10. *A $\mathfrak{B}$-space X is flat if and only if each morphism $\mathbf{m} \to \mathbf{n}$ induces a cofibration $X(\mathbf{m}) \to X(\mathbf{n})$ and for each diagram of the following form (with maps induced by the evident order preserving morphisms)*

$$\begin{array}{ccc} X(\mathbf{m}) & \longrightarrow & X(\mathbf{m} \sqcup \mathbf{n}) \\ \downarrow & & \downarrow \\ X(\mathbf{l} \sqcup \mathbf{m}) & \longrightarrow & X(\mathbf{l} \sqcup \mathbf{m} \sqcup \mathbf{n}) \end{array} \tag{3.3}$$

the intersection of the images of $X(\mathbf{l} \sqcup \mathbf{m})$ and $X(\mathbf{m} \sqcup \mathbf{n})$ in $X(\mathbf{l} \sqcup \mathbf{m} \sqcup \mathbf{n})$ equals the image of $X(\mathbf{m})$.

Proof. Recall from Definition 2.2 the canonical embedding $\Upsilon\colon \mathcal{M} \to \mathfrak{B}$, where $\mathcal{M}$ is the category with the same objects as $\mathfrak{B}$ and injective order preserving functions as morphisms. This induces an embedding $(\mathcal{M} \downarrow \mathbf{n}) \to (\mathfrak{B} \downarrow \mathbf{n})$ whose image is a skeletal subcategory by Lemma 2.3. Identifying $(\mathcal{M} \downarrow \mathbf{n})$ with the poset category of subsets of $\mathbf{n}$, we see that a $\mathfrak{B}$-space gives rise to an $\mathbf{n}$-cubical diagram for all $\mathbf{n}$. Furthermore, it follows from the definitions that a map of $\mathfrak{B}$-spaces is a flat $\mathfrak{B}$-cofibration if and only if the induced maps of cubical diagrams are cofibrations in the usual sense. Given this, the proof proceeds along the same lines as the proof of the analogous result for $\mathcal{I}$-spaces, see [SS12, Proposition 3.11]. □

3.11. **The braided monoidal structure on $\mathcal{S}^{\mathfrak{B}}$.** Any category of diagrams in $\mathcal{S}$ indexed by a braided monoidal small category inherits a braided monoidal convolution product from the indexing category. We proceed to explain how this works in the case of $\mathcal{S}^{\mathfrak{B}}$. Given $\mathfrak{B}$-spaces X and Y, we define the $\mathfrak{B}$-space $X \boxtimes Y$ to be the left Kan extension of the $(\mathfrak{B} \times \mathfrak{B})$-space

$$\mathfrak{B} \times \mathfrak{B} \xrightarrow{X \times Y} \mathcal{S} \times \mathcal{S} \xrightarrow{\times} \mathcal{S}$$

along the monoidal structure map $\sqcup\colon \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$. Thus, the data specifying a map of $\mathfrak{B}$-spaces $X \boxtimes Y \to Z$ is equivalent to the data giving a map of $(\mathfrak{B} \times \mathfrak{B})$-spaces $X(\mathbf{m}) \times Y(\mathbf{n}) \to Z(\mathbf{m} \sqcup \mathbf{n})$. We also have the level-wise description

$$X \boxtimes Y(\mathbf{n}) = \operatorname*{colim}_{\mathbf{n}_1 \sqcup \mathbf{n}_2 \to \mathbf{n}} X(\mathbf{n}_1) \times Y(\mathbf{n}_2)$$

where the colimit is taken over the comma category $(\sqcup \downarrow \mathbf{n})$ associated to the monoidal product $\sqcup\colon \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$. The monoidal unit for the $\boxtimes$-product is the terminal $\mathfrak{B}$-space $U^{\mathfrak{B}} = \mathfrak{B}(\mathbf{0}, -)$. Using that $\mathcal{S}$ is cartesian closed one easily defines the coherence isomorphisms for associativity and unity required to make $\mathcal{S}^{\mathfrak{B}}$ a monoidal category. We specify a braiding $\mathfrak{b}\colon X \boxtimes Y \to Y \boxtimes X$ on $\mathcal{S}^{\mathfrak{B}}$ by requiring that the diagram of $(\mathfrak{B} \times \mathfrak{B})$-spaces

$$\begin{array}{ccccc} X(\mathbf{m}) \times Y(\mathbf{n}) & & \xrightarrow{\text{twist}} & & Y(\mathbf{n}) \times X(\mathbf{m}) \\ \downarrow & & & & \downarrow \\ X \boxtimes Y(\mathbf{m} \sqcup \mathbf{n}) & \xrightarrow{\mathfrak{b}(\mathbf{m} \sqcup \mathbf{n})} & Y \boxtimes X(\mathbf{m} \sqcup \mathbf{n}) & \xrightarrow{Y \boxtimes X(\chi_{\mathbf{m},\mathbf{n}})} & Y \boxtimes X(\mathbf{n} \sqcup \mathbf{m}) \end{array} \tag{3.4}$$

be commutative. The following proposition can either be checked by hand or deduced from the general theory in [Day70].

Proposition 3.12. *The category $\mathcal{S}^{\mathfrak{B}}$ equipped with the $\boxtimes$-product, the unit $U^{\mathfrak{B}}$, and the braiding $\mathfrak{b}$ is a braided monoidal category.* □

We use the term *$\mathfrak{B}$-space monoid* for a monoid in $\mathcal{S}^{\mathfrak{B}}$. By the universal property of the $\boxtimes$-product, the data needed to specify the unit $u\colon U^{\mathfrak{B}} \to A$ and the multiplication $\mu\colon A \boxtimes A \to A$ on a $\mathfrak{B}$-space monoid A amounts to a zero simplex u in $A(\mathbf{0})$ and a map of $(\mathfrak{B} \times \mathfrak{B})$-spaces $\mu\colon A(\mathbf{m}) \times A(\mathbf{n}) \to A(\mathbf{m} \sqcup \mathbf{n})$ satisfying the usual associativity and unitality conditions. By the definition of the braiding, A is commutative (that is, $\mu \circ \mathfrak{b} = \mu$) if and only if the diagram of $(\mathfrak{B} \times \mathfrak{B})$-spaces

$$\begin{array}{ccc} A(\mathbf{m}) \times A(\mathbf{n}) & \xrightarrow{\mu} & A(\mathbf{m} \sqcup \mathbf{n}) \\ \downarrow{\scriptstyle \text{twist}} & & \downarrow{\scriptstyle A(\chi_{\mathbf{m},\mathbf{n}})} \\ A(\mathbf{n}) \times A(\mathbf{m}) & \xrightarrow{\mu} & A(\mathbf{n} \sqcup \mathbf{m}) \end{array} \tag{3.5}$$

is commutative.

Recall that given maps $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ of $\mathfrak{B}$-spaces, the pushout-product is the induced map

$$f_1 \Box f_2\colon (X_1 \boxtimes Y_2) \amalg_{(X_1 \boxtimes X_2)} (Y_1 \boxtimes X_2) \to Y_1 \boxtimes Y_2.$$

Following [Hov99, Definition 4.2.6] we say that a model structure on $\mathcal{S}^{\mathfrak{B}}$ is a monoidal model structure if given any two cofibrations f_1 and f_2, the pushout-product $f_1 \Box f_2$ is a cofibration that is in addition acyclic if f_1 or f_2 is.

Lemma 3.13. *Both the $\mathfrak{B}$-model structure and the flat $\mathfrak{B}$-model structure are monoidal model structures.*

Proof. We give the proof for the $\mathfrak{B}$-model structure, the proof for the flat case is similar. By Lemma 3.5 in [SS00] it suffices to verify the condition for the generating (acyclic) cofibrations. For two generating cofibrations i, i' in $\mathcal{S}$ it is easy to check that $F_{\mathbf{m}}(i) \Box F_{\mathbf{n}}(i')$ is isomorphic to $F_{\mathbf{m} \sqcup \mathbf{n}}(i \Box i')$. This uses that $F_{\mathbf{m}}(K) \boxtimes F_{\mathbf{n}}(L)$ is naturally isomorphic to $F_{\mathbf{m} \sqcup \mathbf{n}}(K \times L)$ for two simplicial sets K and L and also that $F_{\mathbf{m} \sqcup \mathbf{n}}$ is a left adjoint and hence commutes with colimits. Simplicial sets is a monoidal model category, therefore $i \Box i'$ is a cofibration and then so is $F_{\mathbf{m} \sqcup \mathbf{n}}(i \Box i')$, since $F_{\mathbf{m} \sqcup \mathbf{n}}$ preserves cofibrations. Similarly $F_{\mathbf{m}}(i) \Box F_{\mathbf{n}}(j)$ is an acyclic cofibration if j is a generating acyclic cofibration in $\mathcal{S}$.

Now let $\alpha\colon \mathbf{m} \to \mathbf{m}'$ be a morphism in $\mathfrak{B}$. We check that $(j_\alpha \Box i) \Box F_{\mathbf{n}}(i')$ is an acyclic cofibration for $i\colon \partial\Delta^k \to \Delta^k$ and $i'\colon \partial\Delta^l \to \Delta^l$ generating cofibrations in $\mathcal{S}$. Using that $j_\alpha \Box i \cong j_\alpha \Box F_{\mathbf{0}}(i)$, we get the identifications

$$(j_\alpha \Box i) \Box F_{\mathbf{n}}(i') \cong j_\alpha \Box (F_{\mathbf{0}}(i) \Box F_{\mathbf{n}}(i')) \cong j_\alpha \Box F_{\mathbf{n}}(i \Box i') \cong j_\alpha \boxtimes F_{\mathbf{n}}(*) \times (i \Box i').$$

Since j_α is a cofibration by construction, it follows from the first part of the lemma and the fact that the $\mathfrak{B}$-model structure is simplicial, that this is a cofibration. For the same reason it therefore suffices to show that $j_\alpha \boxtimes F_{\mathbf{n}}(*)$ is a $\mathfrak{B}$-equivalence. For this we apply the two out of three property for $\mathfrak{B}$-equivalences to the diagram

$$\begin{array}{ccccc} F_{\mathbf{m}}(*) \boxtimes F_{\mathbf{n}}(*) & \xrightarrow{j_\alpha \boxtimes \mathrm{id}_{F_{\mathbf{n}}(*)}} & M(\alpha^*) \boxtimes F_{\mathbf{n}}(*) & \xrightarrow{r_\alpha \boxtimes \mathrm{id}_{F_{\mathbf{n}}(*)}} & F_{\mathbf{m}'}(*) \boxtimes F_{\mathbf{n}}(*) \\ \downarrow{\scriptstyle \cong} & & & & \downarrow{\scriptstyle \cong} \\ F_{\mathbf{m} \sqcup \mathbf{n}}(*) & & \xrightarrow[\sim]{(\alpha \sqcup \mathrm{id}_{\mathbf{n}})^*} & & F_{\mathbf{m}' \sqcup \mathbf{n}}(*). \end{array}$$

The vertical maps are isomorphisms and the lower horizontal map is a $\mathfrak{B}$-equivalence since both $F_{\mathbf{m}\sqcup\mathbf{n}}(*)_{h\mathfrak{B}}$ and $F_{\mathbf{m}'\sqcup\mathbf{n}}(*)_{h\mathfrak{B}}$ are contractible. Furthermore, $r_\alpha \boxtimes \mathrm{id}_{F_{\mathbf{n}}(*)}$ is a simplicial homotopy equivalence since r_α is a simplicial homotopy equivalence and $- \boxtimes \mathrm{id}_{F_{\mathbf{n}}(*)}$ preserves simplicial homotopy equivalences. This completes the proof. □

Remark 3.14. In [SS12] a projective model structure is defined for a general diagram category $\mathcal{S}^{\mathcal{K}}$ indexed by a small symmetric monoidal category $\mathcal{K}$ that is well-structured as per Definition 5.5 in [SS12]. Similarly a flat model structure is defined for $\mathcal{S}^{\mathcal{K}}$ if in addition $\mathcal{K}$ together with its subcategory of automorphisms form a well-structured relative index category as per Definition 5.2 in [SS12]. These definitions can be canonically extended to allow braided monoidal categories as index categories such that similar model structures exist. This will not make $\mathfrak{B}$ a well-structured index category because the comma category $(\mathbf{k} \sqcup - \downarrow \mathbf{l})$ will in general not have a terminal object for $\mathbf{k}$ and $\mathbf{l}$ in $\mathfrak{B}$. This property is however not used to establish the model structures, so Proposition 3.5 and Proposition 3.10 are proved as the similar results in [SS12]. But the proofs of results concerning how the monoidal structure interacts with the model structures do use the mentioned property. Above we have shown that the model structures we consider are monoidal model structures by an alternative argument. It is not clear if the arguments in [SS12] can be generalized to define model structures on monoids and commutative monoids in $\mathfrak{B}$-spaces.

Let X and Y be $\mathfrak{B}$-spaces and consider the natural transformation

$$\nu_{X,Y}\colon\ X_{h\mathfrak{B}} \times Y_{h\mathfrak{B}} \xrightarrow{\cong} (X\times Y)_{h(\mathfrak{B}\times\mathfrak{B})} \to ((-\sqcup -)^*(X\boxtimes Y))_{h(\mathfrak{B}\times\mathfrak{B})} \to (X\boxtimes Y)_{h\mathfrak{B}}$$

where the second map is induced by the universal natural transformation of $\mathfrak{B}\times\mathfrak{B}$ diagrams $X(\mathbf{m}) \times Y(\mathbf{n}) \to (X\boxtimes Y)(\mathbf{m}\sqcup\mathbf{n})$. These maps gives rise to a monoidal structure on the functor $(-)_{h\mathfrak{B}}$, c.f. [Sch09, Proposition 4.17].

Lemma 3.15. *If both X and Y are flat, then $\nu_{X,Y}\colon\ X_{h\mathfrak{B}} \times Y_{h\mathfrak{B}} \to (X\boxtimes Y)_{h\mathfrak{B}}$ is a weak equivalence.*

Proof. The fact that the flat $\mathfrak{B}$-model structure is monoidal combined with Ken Brown's Lemma implies that the functor $X \boxtimes (-)$ takes $\mathfrak{B}$-equivalences between flat $\mathfrak{B}$-spaces to $\mathfrak{B}$-equivalences since X is itself flat. Therefore we can take a cofibrant replacement of Y in the $\mathfrak{B}$-model structure and it will suffice to prove the result when Y is $\mathfrak{B}$-cofibrant. Applying a symmetric argument we reduce to the case where both X and Y are $\mathfrak{B}$-cofibrant, which in turn implies that also $X\boxtimes Y$ is $\mathfrak{B}$-cofibrant. By Proposition 18.9.4 in [Hir03] the canonical map $\mathrm{hocolim}_{\mathfrak{B}}\, Z \to \mathrm{colim}_B\, Z$ is a weak equivalence for any $\mathfrak{B}$-cofibrant $\mathfrak{B}$-space Z. The claim now follows since the colimit functor is strong symmetric monoidal. □

4. $\mathfrak{B}$-categories and braided monoidal structures

In this section we introduce the notion of a $\mathfrak{B}$-category and equip the category of such with a braided monoidal structure. We then relate the braided (strict) monoidal objects in this setting to braided (strict) monoidal categories in the usual sense. Finally we introduce the $\mathfrak{B}$-category rectification functor and use this to show that any braided monoidal structure can be rectified to a strictly commutative structure up to $\mathfrak{B}$-equivalence.

4.1. **$\mathfrak{B}$-categories and the Grothendieck construction.** Let *Cat* denote the category of small categories and let $Cat^{\mathfrak{B}}$ be the functor category of $\mathfrak{B}$-diagrams in *Cat*. We shall refer to an object in $Cat^{\mathfrak{B}}$ as a $\mathfrak{B}$-*category*. Recall that the Grothendieck construction $\mathfrak{B}\int X$ on a $\mathfrak{B}$-category X is a category with objects $(\mathbf{n},\mathbf{x})$ given by an object $\mathbf{n}$ in $\mathfrak{B}$ and an object $\mathbf{x}$ in the category $X(\mathbf{n})$. A morphism $(\alpha,s)\colon(\mathbf{m},\mathbf{x})\to(\mathbf{n},\mathbf{y})$ is a morphism $\alpha\colon\mathbf{m}\to\mathbf{n}$ in $\mathfrak{B}$ together with a morphism $s\colon X(\alpha)(\mathbf{x})\to\mathbf{y}$ in $X(\mathbf{n})$. The composition of morphisms is defined by

$$(\beta,t)\circ(\alpha,s)=(\beta\circ\alpha,t\circ X(\beta)(s)).$$

This construction defines a functor $\mathfrak{B}\int\colon Cat^{\mathfrak{B}}\to Cat$ in the obvious way. We think of $\mathfrak{B}\int X$ as the homotopy colimit of X in *Cat*. This is justified by Thomason's homotopy colimit theorem [Tho79, Theorem 1.2] which states that there is a natural weak equivalence

$$\eta\colon \operatorname*{hocolim}_{\mathbf{n}\in\mathfrak{B}} \mathrm{N}(X(\mathbf{n}))\xrightarrow{\simeq}\mathrm{N}(\mathfrak{B}\textstyle\int X). \tag{4.1}$$

Let us say that a functor $Y\to Y'$ between small categories is a weak equivalence if the induced map of nerves $\mathrm{N}(Y)\to\mathrm{N}(Y')$ is a weak equivalence of simplicial sets. We say that a map of $\mathfrak{B}$-categories $X\to X'$ is a $\mathfrak{B}$-*equivalence* if the map of Grothendieck constructions $\mathfrak{B}\int X\to\mathfrak{B}\int X'$ is a weak equivalence in this sense. By the natural weak equivalence in (4.1) this is equivalent to the level-wise nerve $\mathrm{N}(X)\to\mathrm{N}(X')$ being a $\mathfrak{B}$-equivalence in the sense of the previous section. Let w denote the class of weak equivalences in *Cat*, and let $w_{\mathfrak{B}}$ be the class of $\mathfrak{B}$-equivalences in $Cat^{\mathfrak{B}}$. With the given definition of $\mathfrak{B}$-equivalences it is not surprising that the categories $Cat^{\mathfrak{B}}$ and *Cat* become equivalent after localization with respect to these classes of equivalences. For the convenience of the reader we have collected the relevant background material on localization in Appendix A. Let us write $\Delta\colon Cat\to Cat^{\mathfrak{B}}$ for the functor that takes a small category to the corresponding constant $\mathfrak{B}$-category.

Proposition 4.2. *The functors $\mathfrak{B}\int$ and Δ induce an equivalence of the localized categories*

$$\mathfrak{B}\textstyle\int\colon Cat^{\mathfrak{B}}[w_{\mathfrak{B}}^{-1}]\simeq Cat[w^{-1}]:\Delta.$$

For the proof of the proposition we need to introduce an auxiliary endofunctor on $Cat^{\mathfrak{B}}$. Let $(\mathfrak{B}\downarrow\bullet)$ be the $\mathfrak{B}$-category defined by the comma categories $(\mathfrak{B}\downarrow\mathbf{n})$. By definition, an object of $(\mathfrak{B}\downarrow\mathbf{n})$ is a pair $(\mathbf{m},\gamma)$ given by an object $\mathbf{m}$ and a morphism $\gamma\colon \mathbf{m}\to\mathbf{n}$ in $\mathfrak{B}$. A morphism $\alpha\colon(\mathbf{m}_1,\gamma_1)\to(\mathbf{m}_2,\gamma_2)$ is a morphism $\alpha\colon\mathbf{m}_1\to\mathbf{m}_2$ in $\mathfrak{B}$ such that $\gamma_1=\gamma_2\circ\alpha$. Let $\pi_n\colon(\mathfrak{B}\downarrow\mathbf{n})\to\mathfrak{B}$ be the forgetful functor mapping $(\mathbf{m},\gamma)$ to $\mathbf{m}$. Clearly these functors assemble to a map of $\mathfrak{B}$-categories $\pi\colon(\mathfrak{B}\downarrow\bullet)\to\Delta(\mathfrak{B})$. Given a $\mathfrak{B}$-category X, the *bar resolution* $\overline{X}$ is the $\mathfrak{B}$-category defined by the level-wise Grothendieck constructions

$$\overline{X}(\mathbf{n})=(\mathfrak{B}\downarrow\mathbf{n})\textstyle\int X\circ\pi_n.$$

The structure maps making $\overline{X}$ a $\mathfrak{B}$-category are inherited from the $\mathfrak{B}$-category $(\mathfrak{B}\downarrow\bullet)$ in the obvious way. Our use of the term "bar resolution" is motivated by the analogous bar resolution for $\mathfrak{B}$-spaces that we shall consider in Section 5.4.

Lemma 4.3. *There is a natural level-wise weak equivalence* $\mathrm{ev}\colon\overline{X}\to X$.

Proof. For each $\mathbf{n}$ we define a functor $\mathrm{ev}(\mathbf{n})\colon(\mathfrak{B}\downarrow\mathbf{n})\int X\circ\pi_n\to X(\mathbf{n})$. An object in the domain has the form $((\mathbf{m},\gamma),\mathbf{x})$ with $(\mathbf{m},\gamma)$ in $(\mathfrak{B}\downarrow\mathbf{n})$ and $\mathbf{x}$ an object in $X(\mathbf{m})$. We map this to the object $X(\gamma)(\mathbf{x})$ in $X(\mathbf{n})$. A morphism from $((\mathbf{m}_1,\gamma_1),\mathbf{x}_1)$ to $((\mathbf{m}_2,\gamma_2),\mathbf{x}_2)$ amounts to a morphism $\alpha\colon(\mathbf{m}_1,\gamma_1)\to(\mathbf{m}_2,\gamma_2)$ in $(\mathfrak{B}\downarrow\mathbf{n})$ together with a morphism $s\colon X(\alpha)(\mathbf{x}_1)\to\mathbf{x}_2$ in $X(\mathbf{m}_2)$. We map such a morphism to the morphism

$$X(\gamma_2)(s)\colon X(\gamma_1)(\mathbf{x}_1)=X(\gamma_2)(X(\alpha)(\mathbf{x}_1))\to X(\gamma_2)(\mathbf{x}_2)$$

in $X(\mathbf{n})$. These functors are compatible when $\mathbf{n}$ varies and give rise to the map of $\mathfrak{B}$-categories in the lemma. To show that $\mathrm{ev}(\mathbf{n})$ is a weak equivalence, we consider the canonical functor

$$j(\mathbf{n})\colon X(\mathbf{n})\to(\mathfrak{B}\downarrow\mathbf{n})\textstyle\int X\circ\pi_n,\qquad \mathbf{x}\mapsto(1_n,\mathbf{x})$$

where 1_n denotes the terminal object in $(\mathfrak{B}\downarrow\mathbf{n})$. Then $\mathrm{ev}(\mathbf{n})\circ j(\mathbf{n})$ is the identity functor on $X(\mathbf{n})$ and it is easy to see that there is a natural transformation from the identity functor on $(\mathfrak{B}\downarrow\mathbf{n})\int X\circ\pi_n$ to $j(\mathbf{n})\circ\mathrm{ev}(\mathbf{n})$. Hence $j(\mathbf{n})$ defines a homotopy inverse of $\mathrm{ev}(\mathbf{n})$. □

Lemma 4.4. *There is a natural* $\mathfrak{B}$*-equivalence* $\pi\colon\overline{X}\to\Delta(\mathfrak{B}\int X)$.

Proof. For each $\mathbf{n}$ the forgetful functor $\pi_n\colon(\mathfrak{B}\downarrow\mathbf{n})\to\mathfrak{B}$ gives rise to a functor

$$(\mathfrak{B}\downarrow\mathbf{n})\textstyle\int X\circ\pi_n\to\mathfrak{B}\int X$$

by mapping an object $((\mathbf{m},\gamma),\mathbf{x})$ to $(\mathbf{m},\mathbf{x})$. Letting $\mathbf{n}$ vary this defines the map of $\mathfrak{B}$-categories in the lemma. We must show that the functor $\mathfrak{B}\int\pi$ is a weak equivalence and for this we consider the diagram of categories

$$\begin{array}{ccc}
\mathfrak{B}\int\big((\mathfrak{B}\downarrow\bullet)\int X\circ\pi_\bullet\big) & \xrightarrow{\mathfrak{B}\int\pi} & \mathfrak{B}\int\Delta(\mathfrak{B}\int X)\\
{\scriptstyle\mathfrak{B}\int\mathrm{ev}}\big\downarrow & & \big\downarrow{\scriptstyle\cong}\\
\mathfrak{B}\int X & \xleftarrow{\ \mathrm{proj}\ } & \mathfrak{B}\times(\mathfrak{B}\int X)
\end{array}$$

where proj denotes the projection away from the first variable. This diagram is not commutative but we claim that it commutes up to a natural transformation. Indeed, consider an object $(\mathbf{n}, (\mathbf{m}, \gamma), \mathbf{x})$ with $\mathbf{n}$ in $\mathfrak{B}$, $(\mathbf{m}, \gamma)$ an object in $(\mathfrak{B} \downarrow \mathbf{n})$, and $\mathbf{x}$ an object in $X(\mathbf{m})$. The functor $\mathfrak{B} \int \mathrm{ev}$ maps this to $(\mathbf{n}, X(\gamma)(\mathbf{x}))$ whereas the other composition maps it to $(\mathbf{m}, \mathbf{x})$. It is easy to see that the morphisms

$$(\gamma, \mathrm{id}_{X(\gamma)(\mathbf{x})}) \colon (\mathbf{m}, \mathbf{x}) \to (\mathbf{n}, X(\gamma)(\mathbf{x}))$$

define a natural transformation between these functors. Since $\mathfrak{B} \int \mathrm{ev}$ is a weak equivalence by Lemma 4.3 and proj is a weak equivalence because $\mathfrak{B}$ has an initial object, it follows that also $\mathfrak{B} \int \pi$ is a weak equivalence. □

Proof of Proposition 4.2. We first observe that the localization of *Cat* with respect to w actually exists since Thomason has realized it as the homotopy category of a suitable model structure, see [Tho80]. With terminology from Appendix A, Lemmas 4.3 and 4.4 give a chain of natural $\mathfrak{B}$-equivalences relating $\Delta(\mathfrak{B} \int X)$ to X. The other composition $\mathfrak{B} \int \Delta Y$ can be identified with the product category $\mathfrak{B} \times Y$ which is weakly equivalent to Y since $\mathfrak{B}$ has an initial object. Hence the result follows from Proposition A.1. □

Remark 4.5. Let $(\bullet \downarrow \mathfrak{B})$ denote the $\mathfrak{B}^{\mathrm{op}}$-category defined by the comma categories $(\mathbf{n} \downarrow \mathfrak{B})$. The universal property of the Grothendieck construction established in [Tho79, Proposition 1.3.1] implies that $\mathfrak{B} \int X$ can be identified with the coend $(\bullet \downarrow \mathfrak{B}) \times_{\mathfrak{B}} X$ in *Cat*. This in turn implies that the functor $\mathfrak{B} \int$ participates as the left adjoint in an adjunction

$$\mathfrak{B} \textstyle\int \colon \mathit{Cat}^{\mathfrak{B}} \rightleftarrows \mathit{Cat} : \mathit{Cat}((\bullet \downarrow \mathfrak{B}), -).$$

The right adjoint takes a small category Y to the $\mathfrak{B}$-category for which the objects of $\mathit{Cat}((\mathbf{n} \downarrow \mathfrak{B}), Y)$ are the functors from $(\mathbf{n} \downarrow \mathfrak{B})$ to Y and the morphisms are the natural transformations. However, this adjunction is not so useful for our purposes since it cannot be promoted to an adjunction between the braided monoidal structures we shall consider later.

4.6. **Braided monoidal structures.** As in the case of $\mathfrak{B}$-spaces considered in Section 3.11, the braided monoidal structure of $\mathfrak{B}$ induces a braided monoidal structure on $\mathit{Cat}^{\mathfrak{B}}$: Given $\mathfrak{B}$-categories X and Y, we define $X \boxtimes Y$ to be the left Kan extension of the $(\mathfrak{B} \times \mathfrak{B})$-category

$$\mathfrak{B} \times \mathfrak{B} \xrightarrow{X \times Y} \mathit{Cat} \times \mathit{Cat} \xrightarrow{\times} \mathit{Cat}$$

along the monoidal structure map $\sqcup \colon \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$. Thus, the data specifying a map of $\mathfrak{B}$-categories $X \boxtimes Y \to Z$ is equivalent to the data giving a map of $(\mathfrak{B} \times \mathfrak{B})$-categories $X(\mathbf{m}) \times Y(\mathbf{n}) \to Z(\mathbf{m} \sqcup \mathbf{n})$. We also have the level-wise description

$$X \boxtimes Y(\mathbf{n}) = \operatorname*{colim}_{\mathbf{n}_1 \sqcup \mathbf{n}_2 \to \mathbf{n}} X(\mathbf{n}_1) \times Y(\mathbf{n}_2).$$

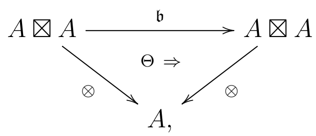
The monoidal unit for the $\boxtimes$-product is the terminal $\mathfrak{B}$-category $U^{\mathfrak{B}} = \mathfrak{B}(\mathbf{0}, -)$. Using that *Cat* is cartesian closed one easily defines the coherence isomorphisms for associativity and unity required to make $Cat^{\mathfrak{B}}$ a monoidal category. We specify a braiding $\mathfrak{b}\colon X \boxtimes Y \to Y \boxtimes X$ on $Cat^{\mathfrak{B}}$ by requiring that the categorical analogue of the diagram (3.4) be commutative. The following is the categorical analogue of Proposition 3.12.

Proposition 4.7. *The category $Cat^{\mathfrak{B}}$ equipped with the $\boxtimes$-product, the unit $U^{\mathfrak{B}}$, and the braiding $\mathfrak{b}$ is a braided monoidal category.* □

We use the term $\mathfrak{B}$-*category monoid* for a monoid in $Cat^{\mathfrak{B}}$. By the universal property of the $\boxtimes$-product, the data needed to specify the unit $U^{\mathfrak{B}} \to A$ and the multiplication $\otimes\colon A \boxtimes A \to A$ on a $\mathfrak{B}$-category monoid A amounts to a unit object $\mathbf{u}$ in $A(\mathbf{0})$ and a map of $(\mathfrak{B} \times \mathfrak{B})$-categories $\otimes\colon A(\mathbf{m}) \times A(\mathbf{n}) \to A(\mathbf{m} \sqcup \mathbf{n})$ satisfying the usual associativity and unitality conditions. By the definition of the braiding, A is commutative (that is, $\otimes \circ \mathfrak{b} = \otimes$) if and only if the categorical version of the diagram (3.5) is commutative.

In order to talk about braided $\mathfrak{B}$-category monoids we need the notion of a natural transformation between maps of $\mathfrak{B}$-categories: Given maps of $\mathfrak{B}$-categories $f, g\colon X \to Y$, a natural transformation $\phi\colon f \Rightarrow g$ is a family of natural transformations $\phi(\mathbf{n})\colon f(\mathbf{n}) \Rightarrow g(\mathbf{n})$ such that for any morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathfrak{B}$ we have an equality of natural transformations $\phi(\mathbf{n}) \circ X(\alpha) = Y(\alpha) \circ \phi(\mathbf{m})$ between the functors $f(\mathbf{n}) \circ X(\alpha) = Y(\alpha) \circ f(\mathbf{m})$ and $g(\mathbf{n}) \circ X(\alpha) = Y(\alpha) \circ g(\mathbf{m})$. Here the symbol $\circ$ denotes the usual "horizontal" composition, and we use the notation $X(\alpha)$ and $Y(\alpha)$ both for the functors defined by X and Y and for the corresponding identity natural transformations. A braiding of a $\mathfrak{B}$-category monoid A is then a natural transformation $\Theta\colon \otimes \Rightarrow \otimes \circ \mathfrak{b}$ as depicted in the diagram

(4.2)



such that Θ has an inverse and the familiar axioms for a braided monoidal structure holds. In order to formulate this in a convenient manner we observe that the data defining a natural isomorphism Θ as above amounts to a natural isomorphism

$$\Theta_{\mathbf{m},\mathbf{n}}\colon \mathbf{a} \otimes \mathbf{b} \to A(\chi_{\mathbf{m},\mathbf{n}}^{-1})(\mathbf{b} \otimes \mathbf{a})$$

of functors $A(\mathbf{m}) \times A(\mathbf{n}) \to A(\mathbf{m} \sqcup \mathbf{n})$ for all $(\mathbf{m}, \mathbf{n})$, with the requirement that for each pair of morphisms $\alpha\colon \mathbf{m}_1 \to \mathbf{m}_2$ and $\beta\colon \mathbf{n}_1 \to \mathbf{n}_2$ we have

$$A(\alpha \sqcup \beta) \circ \Theta_{\mathbf{m}_1,\mathbf{n}_1} = \Theta_{\mathbf{m}_2,\mathbf{n}_2} \circ (A(\alpha) \times A(\beta))$$

as an equality of natural transformations.

Definition 4.8. A braiding of a $\mathfrak{B}$-category monoid A is a natural isomorphism Θ as in (4.2) such that the diagrams

$$\begin{array}{ccc} \mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c} & \xrightarrow{\Theta_{\mathbf{l},\mathbf{m}}\otimes\mathrm{id}_{\mathbf{c}}} & A(\chi_{\mathbf{l},\mathbf{m}}^{-1}\sqcup 1_{\mathbf{n}})(\mathbf{b}\otimes\mathbf{a}\otimes\mathbf{c}) \\ \downarrow{\scriptstyle \Theta_{\mathbf{l},\mathbf{m}\sqcup\mathbf{n}}} & & \downarrow{\scriptstyle A(\chi_{\mathbf{l},\mathbf{m}}^{-1}\sqcup 1_{\mathbf{n}})(\mathrm{id}_{\mathbf{b}}\otimes\Theta_{\mathbf{l},\mathbf{n}})} \\ A(\chi_{\mathbf{l},\mathbf{m}\sqcup\mathbf{n}}^{-1})(\mathbf{b}\otimes\mathbf{c}\otimes\mathbf{a}) & = & A(\chi_{\mathbf{l},\mathbf{m}}^{-1}\sqcup 1_{\mathbf{n}})A(1_{\mathbf{m}}\sqcup\chi_{\mathbf{l},\mathbf{n}}^{-1})(\mathbf{b}\otimes\mathbf{c}\otimes\mathbf{a}) \end{array}$$

and

$$\begin{array}{ccc} \mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c} & \xrightarrow{\mathrm{id}_{\mathbf{a}}\otimes\Theta_{\mathbf{m},\mathbf{n}}} & A(1_{\mathbf{l}}\sqcup\chi_{\mathbf{m},\mathbf{n}}^{-1})(\mathbf{a}\otimes\mathbf{c}\otimes\mathbf{b}) \\ \downarrow{\scriptstyle \Theta_{\mathbf{l}\sqcup\mathbf{m},\mathbf{n}}} & & \downarrow{\scriptstyle A(1_{\mathbf{l}}\sqcup\chi_{\mathbf{m},\mathbf{n}}^{-1})(\Theta_{\mathbf{l},\mathbf{n}}\otimes\mathrm{id}_{\mathbf{b}})} \\ A(\chi_{\mathbf{l}\sqcup\mathbf{m},\mathbf{n}}^{-1})(\mathbf{c}\otimes\mathbf{a}\otimes\mathbf{b}) & = & A(1_{\mathbf{l}}\sqcup\chi_{\mathbf{m},\mathbf{n}}^{-1})A(\chi_{\mathbf{l},\mathbf{n}}^{-1}\sqcup 1_{\mathbf{m}})(\mathbf{c}\otimes\mathbf{a}\otimes\mathbf{b}) \end{array}$$

commute for all objects $\mathbf{a}\in A(\mathbf{l})$, $\mathbf{b}\in A(\mathbf{m})$, and $\mathbf{c}\in A(\mathbf{n})$.

Notice that for A a constant $\mathfrak{B}$-category monoid this definition recovers the usual notion of a braided strict monoidal category. We write $\mathsf{Br}\text{-}\mathit{Cat}^{\mathfrak{B}}$ for the category of braided $\mathfrak{B}$-category monoids and braiding preserving (strict) maps of $\mathfrak{B}$-category monoids. Thus, a morphism $f\colon A\to B$ in $\mathsf{Br}\text{-}\mathit{Cat}^{\mathfrak{B}}$ is a map of $\mathfrak{B}$-category monoids such that for all $\mathbf{m},\mathbf{n}$ we have

$$f(\mathbf{m}\sqcup\mathbf{n})\circ\Theta^{A}_{\mathbf{m},\mathbf{n}} = \Theta^{B}_{\mathbf{m},\mathbf{n}}\circ(f(\mathbf{m})\times f(\mathbf{n}))$$

as an equality of natural transformations between functors from $A(\mathbf{m})\times A(\mathbf{n})$ to $B(\mathbf{m}\sqcup\mathbf{n})$. Similarly, we write $\mathsf{Br}\text{-}\mathit{Cat}$ for the category of braided strict monoidal small categories and braiding preserving strict monoidal functors.

Remark 4.9. The natural transformations between maps of $\mathfrak{B}$-categories make $\mathit{Cat}^{\mathfrak{B}}$ a 2-category in the obvious way. Furthermore, this enrichment is compatible with the $\boxtimes$-product such that $\mathit{Cat}^{\mathfrak{B}}$ is a braided monoidal 2-category in the sense of [JS93, Section 5]. In such a setting there is a notion of braided monoidal objects with coherence isomorphisms generalizing those for a braided monoidal category. With the terminology from [JS93], our notion of a braided $\mathfrak{B}$-category monoid is thus the same thing as a braided strict monoidal object in $\mathit{Cat}^{\mathfrak{B}}$. We shall not be concerned with the coherence theory for $\mathfrak{B}$-categories and leave the details for the interested reader.

Our main goal in this subsection is to show that the functor $\mathfrak{B}\!\int$ induces an equivalence between the categories $\mathsf{Br}\text{-}\mathit{Cat}^{\mathfrak{B}}$ and $\mathsf{Br}\text{-}\mathit{Cat}$ after localization as in Proposition 4.2. Consider in general a $\mathfrak{B}$-category monoid A. Then $\mathfrak{B}\!\int\! A$ inherits the structure of a strict monoidal category with product $\otimes\colon \mathfrak{B}\!\int\! A\times\mathfrak{B}\!\int\! A\to\mathfrak{B}\!\int\! A$ defined on objects and morphisms by

$$\begin{aligned} &\big[(\mathbf{m}_1,\mathbf{a}_1)\xrightarrow{(\alpha,s)}(\mathbf{m}_2,\mathbf{a}_2)\big]\otimes\big[(\mathbf{n}_1,\mathbf{b}_1)\xrightarrow{(\beta,t)}(\mathbf{n}_2,\mathbf{b}_2)\big] \\ &= \big[(\mathbf{m}_1\sqcup\mathbf{n}_1,\mathbf{a}_1\otimes\mathbf{b}_1)\xrightarrow{(\alpha\sqcup\beta,s\otimes t)}(\mathbf{m}_2\sqcup\mathbf{n}_2,\mathbf{a}_2\otimes\mathbf{b}_2)\big]. \end{aligned}$$

The monoidal unit for $\otimes$ is the object $(\mathbf{0},\mathbf{u})$ defined by the unit object $\mathbf{u}\in A(\mathbf{0})$. Now suppose that A has a braiding given by a compatible family of natural isomorphisms $\Theta_{\mathbf{m},\mathbf{n}}\colon \mathbf{a}\otimes\mathbf{b}\to A(\chi^{-1}_{\mathbf{m},\mathbf{n}})(\mathbf{b}\otimes\mathbf{a})$. Then we define a braiding of $\mathfrak{B}\!\int\! A$ by the natural transformation

$$(\mathbf{m},\mathbf{a})\otimes(\mathbf{n},\mathbf{b})=(\mathbf{m}\sqcup\mathbf{n},\mathbf{a}\otimes\mathbf{b})\xrightarrow{(\chi_{\mathbf{m},\mathbf{n}},A(\chi_{\mathbf{m},\mathbf{n}})(\Theta_{\mathbf{m},\mathbf{n}}))}(\mathbf{n}\sqcup\mathbf{m},\mathbf{b}\otimes\mathbf{a})=(\mathbf{n},\mathbf{b})\otimes(\mathbf{m},\mathbf{a}).$$

We summarize the construction in the next proposition.

Proposition 4.10. *The Grothendieck construction gives rise to a functor*

$$\mathfrak{B}\!\int\colon \mathsf{Br}\text{-}\mathit{Cat}^{\mathfrak{B}}\to\mathsf{Br}\text{-}\mathit{Cat}.\qquad\square$$

Remark 4.11. It is clear from the definition that the functor $\mathfrak{B}\!\int$ is monoidal and hence takes monoids in $\mathit{Cat}^{\mathfrak{B}}$ to monoids in Cat. However, $\mathfrak{B}\!\int$ is not braided monoidal and consequently does not take commutative monoids to commutative monoids. The main point of the above proposition is that it nonetheless preserves braided monoidal structures.

For the next proposition we write w for the class of morphisms in Br-Cat whose underlying functors are weak equivalences in Cat. Similarly we write $w_{\mathfrak{B}}$ for the class of morphisms in Br-$\mathit{Cat}^{\mathfrak{B}}$ whose underlying maps of $\mathfrak{B}$-categories are $\mathfrak{B}$-equivalences.

Proposition 4.12. *The functors $\mathfrak{B}\!\int$ and Δ induce an equivalence of the localized categories*

$$\mathfrak{B}\!\int\colon \mathsf{Br}\text{-}\mathit{Cat}^{\mathfrak{B}}[w_{\mathfrak{B}}^{-1}]\simeq\mathsf{Br}\text{-}\mathit{Cat}[w^{-1}]:\Delta.$$

The proof of the proposition is based on the following lemma.

Lemma 4.13. *The bar resolution functor taking a $\mathfrak{B}$-category X to $\overline{X}$ can be promoted to an endofunctor on Br-$\mathit{Cat}^{\mathfrak{B}}$.*

Proof. Consider in general a $\mathfrak{B}$-category monoid A with unit object $\mathbf{u}\in A(\mathbf{0})$ and multiplication specified by functors $\otimes\colon A(\mathbf{m})\times A(\mathbf{n})\to A(\mathbf{m}\sqcup\mathbf{n})$. Then $\overline{A}$ inherits a monoid structure with unit object $(1_{\mathbf{0}},\mathbf{u})$ in $\overline{A}(\mathbf{0})$, and multiplication

$$\overline{\otimes}\colon(\mathfrak{B}\!\downarrow\!\mathbf{m})\!\int\! A\circ\pi_{\mathbf{m}}\times(\mathfrak{B}\!\downarrow\!\mathbf{n})\!\int\! A\circ\pi_{\mathbf{n}}\to(\mathfrak{B}\!\downarrow\!\mathbf{m}\sqcup\mathbf{n})\!\int\! A\circ\pi_{\mathbf{m}\sqcup\mathbf{n}}$$

defined on objects and morphisms by

$$\begin{aligned}&\big[((\mathbf{m}_1,\gamma_1),\mathbf{a}_1)\xrightarrow{(\alpha,s)}((\mathbf{m}_2,\gamma_2),\mathbf{a}_2)\big]\overline{\otimes}\big[((\mathbf{n}_1,\delta_1),\mathbf{b}_1)\xrightarrow{(\beta,t)}((\mathbf{n}_2,\delta_2),\mathbf{b}_2)\big]\\&=\big[((\mathbf{m}_1\sqcup\mathbf{n}_1,\gamma_1\sqcup\delta_1),\mathbf{a}_1\otimes\mathbf{b}_1)\xrightarrow{(\alpha\sqcup\beta,s\otimes t)}((\mathbf{m}_2\sqcup\mathbf{n}_2,\gamma_2\sqcup\delta_2),\mathbf{a}_2\otimes\mathbf{b}_2)\big].\end{aligned}$$

Now suppose that in addition A has a braiding specified by a family of natural isomorphisms $\Theta_{\mathbf{m},\mathbf{n}}\colon\mathbf{a}\otimes\mathbf{b}\to A(\chi^{-1}_{\mathbf{m},\mathbf{n}})(\mathbf{b}\otimes\mathbf{a})$. Then we define a braiding $\overline{\Theta}$ of $\overline{A}$ by the natural isomorphisms

$$\begin{array}{ccc} ((\mathbf{m},\gamma),\mathbf{a})\overline{\otimes}((\mathbf{n},\delta),\mathbf{b}) & \xrightarrow{\overline{\Theta}} & \overline{A}(\chi_{\mathbf{m},\mathbf{n}}^{-1})\big[((\mathbf{n},\delta),\mathbf{b})\overline{\otimes}((\mathbf{m},\gamma),\mathbf{a})\big] \\ \| & & \| \\ ((\mathbf{m}\sqcup\mathbf{n},\gamma\sqcup\delta),\mathbf{a}\otimes\mathbf{b}) & \xrightarrow{(\chi_{\mathbf{m},\mathbf{n}},A(\chi_{\mathbf{m},\mathbf{n}})(\Theta_{\mathbf{m},\mathbf{n}}))} & ((\mathbf{n}\sqcup\mathbf{m},\chi_{\mathbf{m},\mathbf{n}}^{-1}\circ(\delta\sqcup\gamma)),\mathbf{b}\otimes\mathbf{a}). \end{array}$$

It is straight forward to check the axioms for a braiding as formulated in Definition 4.8. □

Proof of Proposition 4.12. We first observe that the work of Fiedorowicz-Stelzer-Vogt [FSV13] shows that the localization of $\mathsf{Br}\text{-}\mathit{Cat}$ exists, cf. Example A.2 in the appendix. Given this, the proof of the proposition follows the same pattern as the proof of Proposition 4.2: For a braided $\mathfrak{B}$-category monoid A we know from Lemma 4.13 that $\overline{A}$ has the structure of a braided $\mathfrak{B}$-category monoid and it is clear from the definitions that the $\mathfrak{B}$-equivalences ev and π in Lemmas 4.3 and 4.4 are morphisms in $\mathsf{Br}\text{-}\mathit{Cat}^{\mathfrak{B}}$. With the terminology from Appendix A we therefore have a chain of natural $\mathfrak{B}$-equivalences in $\mathsf{Br}\text{-}\mathit{Cat}^{\mathfrak{B}}$ relating A and $\Delta(\mathfrak{B}\textstyle\int A)$. Given a braided strict monoidal category $\mathcal{A}$, the other composition $\mathfrak{B}\textstyle\int\Delta(\mathcal{A})$ can be identified with the product category $\mathfrak{B}\times\mathcal{A}$ as an object in $\mathsf{Br}\text{-}\mathit{Cat}$. Clearly the projection $\mathfrak{B}\times\mathcal{A}\to\mathcal{A}$ is a weak equivalence in $\mathsf{Br}\text{-}\mathit{Cat}$ and the proposition therefore follows from Proposition A.1. □

4.14. **Rectification and strict commutativity.** Now we proceed to introduce the $\mathfrak{B}$-category rectification functor and show how it allows us to replace braided monoidal structures by strictly commutative structures up to $\mathfrak{B}$-equivalence. Let $(\mathcal{A},\otimes,\mathbf{u})$ be a braided strict monoidal small category. We shall define the $\mathfrak{B}$-*category rectification* of $\mathcal{A}$ to be a certain $\mathfrak{B}$-category $\Phi(\mathcal{A})$ such that the objects of $\Phi(\mathcal{A})(\mathbf{n})$ are n-tuples $(\mathbf{a}_1,\dots,\mathbf{a}_n)$ of objects in $\mathcal{A}$. By definition $\Phi(\mathcal{A})(\mathbf{0})$ has the "empty string" $\emptyset$ as its only object. The morphisms in $\Phi(\mathcal{A})(\mathbf{n})$ are given by

$$\Phi(\mathcal{A})(\mathbf{n})\big((\mathbf{a}_1,\dots,\mathbf{a}_n),(\mathbf{b}_1,\dots,\mathbf{b}_n)\big)=\mathcal{A}(\mathbf{a}_1\otimes\dots\otimes\mathbf{a}_n,\mathbf{b}_1\otimes\dots\otimes\mathbf{b}_n)$$

with composition inherited from $\mathcal{A}$. Here we agree that the $\otimes$-product of the empty string is the unit object $\mathbf{u}$ so that $\Phi(\mathcal{A})(\mathbf{0})$ can be identified with the monoid of endomorphisms $\mathcal{A}(\mathbf{u},\mathbf{u})$. For a morphism $\alpha\colon\mathbf{m}\to\mathbf{n}$ in $\mathfrak{B}$, the induced functor

$$\Phi(\mathcal{A})(\alpha)\colon\Phi(\mathcal{A})(\mathbf{m})\to\Phi(\mathcal{A})(\mathbf{n})$$

is given on objects by

$$\Phi(\mathcal{A})(\alpha)(\mathbf{a}_1,\dots,\mathbf{a}_m)=(\mathbf{a}_{\bar{\alpha}^{-1}(1)},\dots,\mathbf{a}_{\bar{\alpha}^{-1}(n)})$$

where $\bar{\alpha}\colon\mathbf{m}\to\mathbf{n}$ denotes the underlying injection, $\mathbf{a}_{\bar{\alpha}^{-1}(j)}=\mathbf{a}_i$ if $\bar{\alpha}(i)=j$, and $\mathbf{a}_{\bar{\alpha}^{-1}(j)}=\mathbf{u}$ if j is not in the image of $\bar{\alpha}$. In order to describe the action on morphisms we use Lemma 2.3 to get a factorization $\alpha=\Upsilon(\nu)\circ\xi$ with $\nu\in\mathcal{M}(\mathbf{m},\mathbf{n})$ and $\xi\in\mathcal{B}_m$. The action of $\Phi(\mathcal{A})(\alpha)$ on a morphism f from $(\mathbf{a}_1,\dots,\mathbf{a}_m)$ to $(\mathbf{b}_1,\dots,\mathbf{b}_m)$ is then determined by the commutativity of the diagram

$$\begin{array}{ccccc}
\mathbf{a}_{\bar{\alpha}^{-1}(1)} \otimes \cdots \otimes \mathbf{a}_{\bar{\alpha}^{-1}(n)} & = & \mathbf{a}_{\bar{\xi}^{-1}(1)} \otimes \cdots \otimes \mathbf{a}_{\bar{\xi}^{-1}(m)} & \xleftarrow{\xi_*} & \mathbf{a}_1 \otimes \cdots \otimes \mathbf{a}_m \\
\Big\downarrow{\scriptstyle \Phi(\mathcal{A})(\alpha)(f)} & & & & \Big\downarrow{\scriptstyle f} \\
\mathbf{b}_{\bar{\alpha}^{-1}(1)} \otimes \cdots \otimes \mathbf{b}_{\bar{\alpha}^{-1}(n)} & = & \mathbf{b}_{\bar{\xi}^{-1}(1)} \otimes \cdots \otimes \mathbf{b}_{\bar{\xi}^{-1}(m)} & \xleftarrow{\xi_*} & \mathbf{b}_1 \otimes \cdots \otimes \mathbf{b}_m
\end{array}$$

where ξ_* denotes the canonical action of ξ on the m-fold $\otimes$-product. In particular, this describes the action of $\Phi(\mathcal{A})(\alpha)$ on the generating morphisms in Lemma 2.4 and one easily checks that the relations in this lemma are preserved. Hence the above construction does indeed define a $\mathfrak{B}$-category. The construction is clearly functorial in $\mathcal{A}$ so that we have defined a functor $\Phi\colon \mathsf{Br}\text{-}\mathit{Cat} \to \mathit{Cat}^{\mathfrak{B}}$. This functor was first considered in the unpublished Master's Thesis by the second author [Sol11].

The $\mathfrak{B}$-category $\Phi(\mathcal{A})$ is homotopy constant in positive degrees in the sense of the next lemma. Here we let $\mathfrak{B}_+$ denote the full subcategory of $\mathfrak{B}$ obtained by excluding the initial object $\mathbf{0}$.

Lemma 4.15. *The functor $\Phi(\mathcal{A})(\alpha)\colon \Phi(\mathcal{A})(\mathbf{m}) \to \Phi(\mathcal{A})(\mathbf{n})$ is a weak equivalence for any morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathfrak{B}_+$.*

Proof. We first consider a morphism of the form $j\colon \mathbf{1} \to \mathbf{m}$ and claim that the functor $\Phi(\mathcal{A})(j)$ is in fact an equivalence of categories. Indeed, let $p\colon \Phi(\mathcal{A})(\mathbf{m}) \to \Phi(\mathcal{A})(\mathbf{1})$ be the obvious functor taking $(\mathbf{a}_1, \dots, \mathbf{a}_m)$ to $(\mathbf{a}_1 \otimes \cdots \otimes \mathbf{a}_m)$. Then $p \circ j$ is the identity on $\Phi(\mathcal{A})(\mathbf{1})$ and it is clear that the other composition $j \circ p$ is naturally isomorphic to the identity on $\Phi(\mathcal{A})(\mathbf{m})$. For a general morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathfrak{B}_+$ we have a commutative diagram

$$\begin{array}{ccc}
 & \Phi(\mathcal{A})(\mathbf{1}) & \\
{\scriptstyle \Phi(\mathcal{A})(j)}\swarrow & & \searrow{\scriptstyle \Phi(\mathcal{A})(\alpha j)} \\
\Phi(\mathcal{A})(\mathbf{m}) & \xrightarrow{\Phi(\mathcal{A})(\alpha)} & \Phi(\mathcal{A})(\mathbf{n})
\end{array}$$

and the result follows. ☐

The next proposition shows that Φ takes braided monoidal structures to strictly commutative structures and is the reason why we refer to Φ as a "rectification functor". Let us write $\mathcal{C}(\mathit{Cat}^{\mathfrak{B}})$ for the category of commutative $\mathfrak{B}$-category monoids.

Proposition 4.16. *The $\mathfrak{B}$-category $\Phi(\mathcal{A})$ is a commutative monoid in $\mathit{Cat}^{\mathfrak{B}}$ and Φ defines a functor $\Phi\colon \mathsf{Br}\text{-}\mathit{Cat} \to \mathcal{C}(\mathit{Cat}^{\mathfrak{B}})$.*

Proof. We define functors $\otimes\colon \Phi(\mathcal{A})(\mathbf{m}) \times \Phi(\mathcal{A})(\mathbf{n}) \to \Phi(\mathcal{A})(\mathbf{m} \sqcup \mathbf{n})$ by

$$(\mathbf{a}_1, \dots, \mathbf{a}_m) \otimes (\mathbf{b}_1, \dots, \mathbf{b}_n) = (\mathbf{a}_1, \dots, \mathbf{a}_m, \mathbf{b}_1, \dots, \mathbf{b}_n)$$

on objects and by applying the monoidal structure $f \otimes g$ of $\mathcal{A}$ on morphisms. These functors are natural in $(\mathbf{m}, \mathbf{n})$ as one verifies by checking for the generating morphisms in Lemma 2.4. By the universal property of the $\boxtimes$-product we therefore get an associative product on $\Phi(\mathcal{A})$. It is clear that the object $\emptyset$ in $\Phi(\mathcal{A})(\mathbf{0})$ specifies a unit for this multiplication. The categorical analogue of the criteria for commutativity expressed by the commutativity of (3.5) clearly holds on objects and on morphisms it follows from the naturality of the braiding on $\mathcal{A}$. □

Remark 4.17. The definition of $\Phi(\mathcal{A})$ can be extended to braided monoidal small categories $\mathcal{A}$ that are not necessarily strict monoidal. Indeed, the objects of $\Phi(\mathcal{A})(\mathbf{n})$ are again n-tuples $(\mathbf{a}_1,\dots,\mathbf{a}_n)$ of objects in $\mathcal{A}$ and a morphism from $(\mathbf{a}_1,\dots,\mathbf{a}_n)$ to $(\mathbf{b}_1,\dots,\mathbf{b}_n)$ is defined to be a morphism

$$(\cdots((\mathbf{a}_1\otimes\mathbf{a}_2)\otimes\mathbf{a}_3)\otimes\cdots\otimes\mathbf{a}_{n-1})\otimes\mathbf{a}_n\to(\cdots((\mathbf{b}_1\otimes\mathbf{b}_2)\otimes\mathbf{b}_3)\otimes\cdots\otimes\mathbf{b}_{n-1})\otimes\mathbf{b}_n$$

in $\mathcal{A}$. Proceeding as in the strict monoidal case, the coherence theory for braided monoidal categories ensures that $\Phi(\mathcal{A})$ canonically has the structure of a commutative $\mathfrak{B}$-category monoid. This is functorial with respect to braided strong monoidal functors that strictly preserve the unit objects.

We shall view $\mathcal{C}(\mathit{Cat}^{\mathcal{B}})$ as the full subcategory of $\mathsf{Br}\text{-}\mathit{Cat}^{\mathfrak{B}}$ given by the braided $\mathfrak{B}$-category monoids with identity braiding $\otimes=\otimes\circ\mathfrak{b}$.

Proposition 4.18. *The composite functor*

$$\mathsf{Br}\text{-}\mathit{Cat}\xrightarrow{\Phi}\mathcal{C}(\mathit{Cat}^{\mathfrak{B}})\to\mathsf{Br}\text{-}\mathit{Cat}^{\mathfrak{B}}\xrightarrow{\mathfrak{B}\int}\mathsf{Br}\text{-}\mathit{Cat}$$

is related to the identity functor on $\mathsf{Br}\text{-}\mathit{Cat}$ *by a natural weak equivalence.*

Proof. For a braided strict monoidal category $\mathcal{A}$ we define a functor $P\colon\mathfrak{B}\int\Phi(\mathcal{A})\to\mathcal{A}$ such that P takes an object $(\mathbf{m},(\mathbf{a}_1,\dots,\mathbf{a}_m))$ to $\mathbf{a}_1\otimes\cdots\otimes\mathbf{a}_m$. A morphism (α,f) from $(\mathbf{m},(\mathbf{a}_1,\dots,\mathbf{a}_m))$ to $(\mathbf{n},(\mathbf{b}_1,\dots,\mathbf{b}_n))$ is given by a morphism $\alpha\colon\mathbf{m}\to\mathbf{n}$ in $\mathfrak{B}$ together with a morphism f from $\mathbf{a}_{\bar\alpha^{-1}(1)}\otimes\cdots\otimes\mathbf{a}_{\bar\alpha^{-1}(n)}$ to $\mathbf{b}_1\otimes\cdots\otimes\mathbf{b}_n$. Using Lemma 2.3 we get a factorization $\alpha=\Upsilon(\nu)\circ\xi$ with $\nu\in\mathcal{M}(\mathbf{m},\mathbf{n})$ and $\xi\in\mathcal{B}_m$, and let $P(\alpha,f)$ be the composition

$$\begin{array}{ccc}\mathbf{a}_1\otimes\cdots\otimes\mathbf{a}_m & \xrightarrow{P(\alpha,f)} & \mathbf{b}_1\otimes\cdots\otimes\mathbf{b}_n\\ \downarrow{\scriptstyle \xi_*} & & \uparrow{\scriptstyle f}\\ \mathbf{a}_{\bar\xi^{-1}(1)}\otimes\cdots\otimes\mathbf{a}_{\bar\xi^{-1}(m)} & = & \mathbf{a}_{\bar\alpha^{-1}(1)}\otimes\cdots\otimes\mathbf{a}_{\bar\alpha^{-1}(n)}.\end{array}$$

It is straight forward to check that P is a braided strict monoidal functor. Furthermore, it follows from the definition of Thomason's equivalence (4.1) that the composition

$$\mathrm{N}(\Phi(\mathcal{A})(\mathbf{1}))\to\operatorname*{hocolim}_{\mathbf{n}\in\mathfrak{B}}\mathrm{N}(\Phi(\mathcal{A})(\mathbf{n}))\xrightarrow{\eta}\mathrm{N}(\mathfrak{B}\textstyle\int\Phi(\mathcal{A}))\xrightarrow{P}\mathrm{N}(\mathcal{A})$$

is the canonical identification. Hence it suffices to prove that the first map, induced by the inclusion $\{\mathbf{1}\}\to\mathfrak{B}$, is a weak equivalence. To this end we first restrict $\mathrm{N}(\Phi(\mathcal{A}))$ to $\mathfrak{B}_+$ such that all the structure maps are weak equivalences by Lemma 4.15. Then it follows from [GJ99, Lemma IV.5.7] that the diagram

$$\begin{array}{ccc} \mathrm{N}(\Phi(\mathcal{A})(\mathbf{1})) & \longrightarrow & \mathrm{N}(\Phi(\mathcal{A}))_{h\mathfrak{B}_+} \\ \downarrow & & \downarrow \\ \{\mathbf{1}\} & \longrightarrow & \mathrm{N}(\mathfrak{B}_+) \end{array}$$

is homotopy cartesian, and since $\mathrm{N}(\mathfrak{B}_+)$ is contractible this in turn implies that $\mathrm{N}(\Phi(\mathcal{A})(\mathbf{1})) \to \mathrm{N}(\Phi(\mathcal{A}))_{h\mathfrak{B}_+}$ is a weak equivalence. Secondly, it is easy to see that the inclusion of $\mathfrak{B}_+$ in $\mathfrak{B}$ is homotopy cofinal such that the induced map $\mathrm{N}(\Phi(\mathcal{A}))_{h\mathfrak{B}_+} \to \mathrm{N}(\Phi(\mathcal{A}))_{h\mathfrak{B}}$ is a weak equivalence by [Hir03, Theorem 19.6.13]. □

Combining the result obtained in this section we get the following theorem.

Theorem 4.19. *Every braided $\mathfrak{B}$-category monoid is related to a strictly commutative $\mathfrak{B}$-category monoid by a chain of natural $\mathfrak{B}$-equivalences in* Br-$Cat^{\mathfrak{B}}$.

Proof. Given a braided $\mathfrak{B}$-category monoid A, we have the following chain of $\mathfrak{B}$-equivalences

$$A \simeq \Delta(\mathfrak{B}\textstyle\int A) \simeq \Delta\big(\mathfrak{B}\textstyle\int \Phi(\mathfrak{B}\textstyle\int A)\big) \simeq \Phi(\mathfrak{B}\textstyle\int A).$$

The first and last equivalences are the chains of $\mathfrak{B}$-equivalences $\Delta(\mathfrak{B}\textstyle\int(-)) \simeq (-)$ from the proof of Proposition 4.12 and the $\mathfrak{B}$-equivalence in the middle is obtained by applying Δ to the weak equivalence $\mathfrak{B}\textstyle\int\Phi(-) \simeq (-)$ in Proposition 4.18. □

5. E_2 SPACES AND BRAIDED COMMUTATIVITY

Building on the categorical foundations in the last section, we proceed to show that every E_2 space can be represented by a strictly commutative $\mathfrak{B}$-space monoid up to $\mathfrak{B}$-equivalence.

5.1. **Operadic interpretation of braided monoidal structures.** In order to relate our results from the previous section to multiplicative structures on spaces, it is convenient to work with an operadic interpretation of braided monoidal structures. By a *Cat*-operad we understand an operad internal to the category *Cat*. Thus, a *Cat*-operad M is given by a sequence of small categories $\mathsf{M}(k)$ for $k \geq 0$ together with functors

$$\gamma\colon \mathsf{M}(k) \times \mathsf{M}(j_1) \times \cdots \times \mathsf{M}(j_k) \to \mathsf{M}(j_1 + \dots j_k),$$

a unit object $\mathbf{1} \in \mathsf{M}(1)$, and a right Σ_k-action on $\mathsf{M}(k)$. These data are required to satisfy the usual axioms for associativity, unity, and equivariance as listed in [May72, Definition 1.1]. We shall always assume that a *Cat*-operad M is reduced in the sense that $\mathsf{M}(0)$ is the terminal category with one object and one morphism. A *Cat*-operad as above gives rise to a monad $\mathbb{M}$ on *Cat* by letting

$$\mathbb{M}(X) = \coprod_{n\geq 0} \mathsf{M}(k) \times_{\Sigma_k} X^k$$

for a small category X. Here X^0 denotes the terminal category. By definition, an M-algebra in Cat is an algebra for this monad and we write M-Cat for the category of M-algebras. An algebra structure $\theta\colon \mathbb{M}(X) \to X$ is determined by a family of functors $\theta_k\colon \mathsf{M}(k) \times X^k \to X$ satisfying the axioms listed in [May72, Lemma 1.4].

Following [FSV13, Section 8] we introduce a Cat-operad Br such that Br-algebras are braided strict monoidal small categories. The objects of $\mathsf{Br}(k)$ are the elements $a \in \Sigma_k$ and given objects a and b, a morphism $\alpha\colon a \to b$ is an element $\alpha \in \mathcal{B}_k$ such that $\bar{\alpha}a = b$. Composition in $\mathsf{Br}(k)$ is inherited from $\mathcal{B}_k$ and the right action of an element $g \in \Sigma_k$ is defined on objects and morphisms by taking $(\alpha\colon a \to b)$ to $(\alpha\colon ag \to bg)$. The structure map

$$\gamma\colon \mathsf{Br}(k) \times \mathsf{Br}(j_1) \times \cdots \times \mathsf{Br}(j_k) \to \mathsf{Br}(j_1 + \cdots + j_k)$$

is defined on objects by

$$\gamma(a, b_1, \ldots, b_k) = a(j_1, \ldots, j_k) \circ b_1 \sqcup \cdots \sqcup b_k$$

where $a(j_1, \ldots, j_k)$ denotes the block permutation of $\mathbf{j}_1 \sqcup \cdots \sqcup \mathbf{j}_k$ specified by a. The action on morphisms is analogous except for the obvious permutation of the indices. Let A be the discrete Cat-operad given by the objects of Br. It is well-known and easy to check that A-algebras are the same thing as monoids in Cat, that is, strict monoidal small categories. Hence a Br-algebra X has an underlying strict monoidal category with unit object determined by the structure map $\theta_0\colon \mathsf{Br}(0) \times X^0 \to X$ and monoidal structure $\otimes = \theta_2(1_2, -, -)$ determined by restricting the structure map $\theta_2\colon \mathsf{Br}(2) \times X^2 \to X$ to the unit object $1_2 \in \mathsf{Br}(2)$. With t the non-unit object of $\mathsf{Br}(2)$ and ζ the generator of $\mathcal{B}_2$, the morphism $\zeta\colon 1_2 \to t$ determines a natural transformation

$$\theta_2(\zeta, \mathrm{id}_{\mathbf{x}_1}, \mathrm{id}_{\mathbf{x}_2})\colon \mathbf{x}_1 \otimes \mathbf{x}_2 \to \mathbf{x}_2 \otimes \mathbf{x}_1$$

which gives a braiding of X. Conversely, for a braided strict monoidal category X we define a Br-algebra structure by the functors $\theta_k\colon \mathsf{Br}(k) \times X^k \to X$ taking a tuple of morphisms $\alpha\colon a \to b$ in $\mathsf{Br}(k)$ and $f_i\colon \mathbf{x}_i \to \mathbf{y}_i$ in X for $i = 1, \ldots, k$, to the composition in the commutative diagram

$$\begin{array}{ccc}
\mathbf{x}_{a^{-1}(1)} \otimes \cdots \otimes \mathbf{x}_{a^{-1}(k)} & \xrightarrow{\alpha_*} & \mathbf{x}_{b^{-1}(1)} \otimes \cdots \otimes \mathbf{x}_{b^{-1}(k)} \\
{\scriptstyle f_{a^{-1}(1)} \otimes \cdots \otimes f_{a^{-1}(k)}} \downarrow & & \downarrow {\scriptstyle f_{b^{-1}(1)} \otimes \cdots \otimes f_{b^{-1}(k)}} \\
\mathbf{y}_{a^{-1}(1)} \otimes \cdots \otimes \mathbf{y}_{a^{-1}(k)} & \xrightarrow{\alpha_*} & \mathbf{y}_{b^{-1}(1)} \otimes \cdots \otimes \mathbf{y}_{b^{-1}(k)}.
\end{array}$$

Here α_* denotes the canonical action of α defined by the braided monoidal structure. Summarizing, we have the following consistency result that justifies our use of the notation Br-Cat in the previous section.

Lemma 5.2. *The category* Br-Cat *of* Br-*algebras is isomorphic to the category of braided strict monoidal categories.* ☐

It is natural to ask for an analogous operadic characterization of braided $\mathfrak{B}$-category monoids. However, since the symmetric groups do not act on the iterated $\boxtimes$-products in $Cat^{\mathfrak{B}}$, we instead have to work with braided operads as introduced by Fiedorowicz [Fie]. By definition, a braided Cat-operad M is a sequence of small categories $\mathsf{M}(k)$ for $k \geq 0$ together with structure maps and a unit just as for a Cat-operad. The difference from an (unbraided) Cat-operad is that in the braided case we require a right $\mathcal{B}_k$-action on $\mathsf{M}(k)$ for all k such that the braided analogue of the equivariance axiom for a Cat-operad holds. A braided Cat-operad M defines a monad on $Cat^{\mathfrak{B}}$ by letting

$$\mathbb{M}(X) = \coprod_{k \geq 0} \mathsf{M}(k) \times_{\mathcal{B}_k} X^{\boxtimes k}$$

for a $\mathfrak{B}$-category X. By definition, an M-algebra in $Cat^{\mathfrak{B}}$ is an algebra for this monad and we write M-$Cat^{\mathfrak{B}}$ for the category of M-algebras. It follows from the universal property of the $\boxtimes$-product that an M-algebra structure on a $\mathfrak{B}$-category X can be described in terms of functors

$$\theta_k \colon \mathsf{M}(k) \times X(\mathbf{n}_1) \times \cdots \times X(\mathbf{n}_k) \to X(\mathbf{n}_1 \sqcup \cdots \sqcup \mathbf{n}_k) \tag{5.1}$$

such that the usual associativity and unity axioms hold as well as the equivariance axiom stating that the diagram

$$\begin{array}{ccc} \mathsf{M}(k) \times X(\mathbf{n}_1) \times \cdots \times X(\mathbf{n}_k) & \xrightarrow{\theta_k \circ (\sigma \times \mathrm{id})} & X(\mathbf{n}_1 \sqcup \cdots \sqcup \mathbf{n}_k) \\ \downarrow{\scriptstyle \mathrm{id} \times \bar{\sigma}} & & \downarrow{\scriptstyle X(\sigma(n_1,\dots,n_k))} \\ \mathsf{M}(k) \times X(\mathbf{n}_{\bar{\sigma}^{-1}(1)}) \times \cdots \times X(\mathbf{n}_{\bar{\sigma}^{-1}(k)}) & \xrightarrow{\theta_k} & X(\mathbf{n}_{\bar{\sigma}^{-1}(1)} \sqcup \cdots \sqcup \mathbf{n}_{\bar{\sigma}^{-1}(k)}) \end{array}$$

is commutative for all $\sigma \in \mathcal{B}_k$. We also use the notation Br for the braided Cat-operad for which the category $\mathsf{Br}(k)$ has objects the elements $a \in \mathcal{B}_k$ and a morphism $\alpha \colon a \to b$ is an element $\alpha \in \mathcal{B}_k$ such that $\alpha a = b$. The structure maps making this a braided Cat-operad are defined as for the analogous unbraided operad. Let A be the discrete braided Cat-operad given by the objects in Br. It is easy to see that an A-algebra in $Cat^{\mathfrak{B}}$ is the same thing as a $\mathfrak{B}$-category monoid and hence that a Br-algebra is a $\mathfrak{B}$-category monoid with extra structure. Indeed, suppose that X is a Br-algebra in $Cat^{\mathfrak{B}}$ and write $\otimes \colon X \boxtimes X \to X$ for the monoid structure defined by restricting $\theta_2 \colon \mathsf{Br}(2) \times X^{\boxtimes 2} \to X$ to the unit object $1_2 \in \mathsf{Br}(2)$. With ζ the standard generator of $\mathcal{B}_2$, the morphism $\zeta \colon 1_2 \to \zeta$ determines a natural isomorphism $\Theta = \theta(\zeta, -, -)$ as in the diagram (4.2) and Θ satisfies the axioms for a braiding of X. Arguing as in the unbraided setting we get the following analogue of Lemma 5.2.

Lemma 5.3. *The category Br-$Cat^{\mathfrak{B}}$ is isomorphic to the category of braided $\mathfrak{B}$-category monoids.* □

5.4. **Rectification of E_2 algebras.** Applying the nerve functor N to the unbraided Cat-operad Br we get an operad $\mathrm{N}\mathsf{Br}$ in simplicial sets with kth space $\mathrm{N}\mathsf{Br}(k)$. This is an E_2 operad in the sense that its geometric realization is equivalent to the little 2-cubes operad, cf. [FSV13, Proposition 8.13]. Since the nerve functor preserves products it is clear that it induces a functor $\mathrm{N}\colon \mathsf{Br}\text{-}\mathit{Cat} \to \mathrm{N}\mathsf{Br}\text{-}\mathcal{S}$. This was first observed by Fiedorowicz [Fie], and is the braided version of the analogous construction for permutative categories considered by May [May74]. Similarly, the braided version of the Cat-operad Br gives rise to the braided operad $\mathrm{N}\mathsf{Br}$ in simplicial sets. By the level-wise characterization of Br-algebras in (5.1) it is equally clear that the level-wise nerve induces a functor $\mathrm{N}\colon \mathsf{Br}\text{-}\mathit{Cat}^{\mathfrak{B}} \to \mathrm{N}\mathsf{Br}\text{-}\mathcal{S}^{\mathfrak{B}}$.

Now we want to say that the homotopy colimit functor induces a functor from $\mathrm{N}\mathsf{Br}\text{-}\mathcal{S}^{\mathfrak{B}}$ to $\mathrm{N}\mathsf{Br}\text{-}\mathcal{S}$, but to explain this properly requires some preparation. Recall that the pure braid group $\mathcal{P}_k$ is the kernel of the projection $\Pi\colon \mathcal{B}_k \to \Sigma_k$. Following [FSV13], a braided operad M can be "debraided" to an (unbraided) operad $\mathsf{M}/\mathcal{P}_k$ with kth term the orbit space $\mathsf{M}(k)/\mathcal{P}_k$. The structure maps are inherited from the structure maps of M and Σ_k acts from the right via the isomorphism $\Sigma_k \cong \mathcal{B}_k/\mathcal{P}_k$. For instance, the debraiding of the braided Cat-operad Br is the corresponding unbraided Cat-operad Br and similarly for the braided operad $\mathrm{N}\mathsf{Br}$. In the following lemma we consider the product of the latter with an arbitrary braided operad M and form the debraided operad $(\mathrm{N}\mathsf{Br} \times \mathsf{M})/\mathcal{P}$.

Lemma 5.5. *Let* M *be a braided operad in simplicial sets. Then the homotopy colimit functor can be promoted to a functor*

$$(-)_{h\mathfrak{B}}\colon \mathsf{M}\text{-}\mathcal{S}^{\mathfrak{B}} \to (\mathrm{N}\mathsf{Br} \times \mathsf{M})/\mathcal{P}\text{-}\mathcal{S}.$$

Proof. Let X be a $\mathfrak{B}$-space with M-action defined by natural maps

$$\theta_k\colon \mathsf{M}(k) \times X(\mathbf{n}_1) \times \cdots \times X(\mathbf{n}_k) \to X(\mathbf{n}_1 \sqcup \cdots \sqcup \mathbf{n}_k).$$

To X we associate the simplicial category (that is, simplicial object in Cat) $\mathfrak{B}\textstyle\int X$ obtained by applying the Grothendieck construction in each simplicial degree of X thought of as a $\mathfrak{B}$-diagram of simplicial discrete categories. It is clear from the definition that the nerve of $\mathfrak{B}\textstyle\int X$ can be identified with $X_{h\mathfrak{B}}$. Let us further view $\mathsf{Br}(k)$ as a constant simplicial category and $\mathsf{M}(k)$ as a simplicial discrete category. Then we define maps of simplicial categories

$$\theta_k\colon \mathsf{Br}(k) \times \mathsf{M}(k) \times (\mathfrak{B}\textstyle\int X)^k \to \mathfrak{B}\textstyle\int X$$

such that a tuple of objects $a \in \mathsf{Br}(k)$, $\mathsf{m} \in \mathsf{M}(k)$, and $(\mathbf{m}_i, x_i) \in \mathfrak{B}\textstyle\int X$ for $i = 1, \dots, k$, is mapped to the object

$$\big(\mathbf{m}_{\bar{a}^{-1}(1)} \sqcup \cdots \sqcup \mathbf{m}_{\bar{a}^{-1}(k)}, X(a(m_1, \dots, m_k))\theta_k(\mathsf{m}, x_1, \dots, x_k)\big).$$

A tuple of morphisms $\alpha\colon a \to b$ in $\mathsf{Br}(k)$ and $\beta_i\colon (\mathbf{m}_i, x_i) \to (\mathbf{n}_i, y_i)$ in $\mathfrak{B}\textstyle\int X$ for $i = 1, \dots, k$, is mapped to the morphism specified by

$$\alpha(m_{\bar{\alpha}^{-1}(1)}, \dots, m_{\bar{\alpha}^{-1}(k)}) \circ \beta_{\bar{\alpha}^{-1}(1)} \sqcup \cdots \sqcup \beta_{\bar{\alpha}^{-1}(k)}.$$

Evaluating the nerves of these simplicial categories we get a map of bisimplicial sets and by restricting to the simplicial diagonal a map of simplicial sets

$$(\mathrm{N}\mathsf{Br}(k) \times \mathsf{M}(k)) \times \mathrm{N}(\mathfrak{B}\textstyle\int X)^k \to \mathrm{N}(\mathfrak{B}\textstyle\int X).$$

It is not difficult to check that these maps satisfy the conditions for a braided operad action and hence descends to an action of the debraided operad $(\mathsf{NBr}\times\mathsf{M})/\mathcal{P}$. Clearly this is functorial in X. $\square$

When M is the braided operad $\mathrm{N}\mathsf{Br}$ we can compose with the diagonal map of (unbraided) operads $\mathrm{N}\mathsf{Br}/\mathcal{P}\to(\mathrm{N}\mathsf{Br}\times\mathrm{N}\mathsf{Br})/\mathcal{P}$ to get the next lemma.

Lemma 5.6. *The homotopy colimit functor can be promoted to a functor*

$$(-)_{h\mathfrak{B}}\colon \mathrm{N}\mathsf{Br}\text{-}\mathcal{S}^{\mathfrak{B}}\to \mathrm{N}\mathsf{Br}\text{-}\mathcal{S}. \qquad \square$$

The natural maps introduced so far are compatible in the expected way.

Proposition 5.7. *The diagram*

$$\begin{array}{ccc} \mathsf{Br}\text{-}\mathit{Cat}^{\mathfrak{B}} & \xrightarrow{\mathrm{N}} & \mathrm{N}\mathsf{Br}\text{-}\mathcal{S}^{\mathfrak{B}} \\ {\scriptstyle \mathfrak{B}\int}\downarrow & & \downarrow{\scriptstyle (-)_{h\mathfrak{B}}} \\ \mathsf{Br}\text{-}\mathit{Cat} & \xrightarrow{\mathrm{N}} & \mathrm{N}\mathsf{Br}\text{-}\mathcal{S} \end{array}$$

commutes up to natural weak equivalence.

Proof. Given a braided $\mathfrak{B}$-category X, we claim that Thomason's equivalence η in (4.1) is in fact a morphism in $\mathrm{N}\mathsf{Br}\text{-}\mathcal{S}$. In order to verify the claim we first use Proposition 4.10 and Lemma 5.2 to get an explicit description of the $\mathrm{N}\mathsf{Br}$-algebra structure on $\mathrm{N}(\mathfrak{B}\int X)$. Secondly, we use Lemmas 5.3 and 5.6 to get an explicit description of the $\mathrm{N}\mathsf{Br}$-algebra structure on $(\mathrm{N}X)_{h\mathfrak{B}}$. It is then straight forward (although somewhat tedious) to check that Thomason's explicit description of η in [Tho79, Lemma 1.2.1] is compatible with the algebra structures. $\square$

We proceed to show that the functor $(-)_{h\mathfrak{B}}$ in Lemma 5.6 induces an equivalence after suitable localizations of the domain and target. Let us write w for the class of morphisms in $\mathrm{N}\mathsf{Br}\text{-}\mathcal{S}$ whose underlying maps of spaces are weak equivalences and $w_{\mathfrak{B}}$ for the class of morphisms in $\mathrm{N}\mathsf{Br}\text{-}\mathcal{S}^{\mathfrak{B}}$ whose underlying maps of $\mathfrak{B}$-spaces are $\mathfrak{B}$-equivalences. The following is the $\mathfrak{B}$-space version of Proposition 4.12. As usual Δ denotes the constant functor embedding.

Proposition 5.8. *The functors $(-)_{h\mathfrak{B}}$ and Δ induce an equivalence of the localized categories*

$$(-)_{h\mathfrak{B}}\colon \mathrm{N}\mathsf{Br}\text{-}\mathcal{S}^{\mathfrak{B}}[w_{\mathfrak{B}}^{-1}]\simeq \mathrm{N}\mathsf{Br}\text{-}\mathcal{S}[w^{-1}]:\Delta.$$

For the proof of the proposition we need to invoke the *bar resolution* for $\mathfrak{B}$-spaces. Given a $\mathfrak{B}$-space X, this is the $\mathfrak{B}$-space $\overline{X}$ defined by

$$\overline{X}(\mathbf{n})=\operatorname*{hocolim}_{(\mathfrak{B}\downarrow\mathbf{n})} X\circ\pi_{\mathbf{n}}$$

with notation as for the categorical bar resolution considered in Section 4.1. (See e.g. [HV92] for the interpretation of this as an actual bar construction.) Arguing as in the proof of Lemma 5.5 one sees that this construction can be promoted to an endofunctor on $\mathsf{NBr}\text{-}\mathcal{S}^{\mathfrak{B}}$.

Proof of Proposition 5.8. First recall that the localization $\mathsf{NBr}\text{-}\mathcal{S}[w^{-1}]$ exists since it can be realized as the homotopy category of a suitable model structure. As for the categorical analogue in Proposition 4.12 there are natural $\mathfrak{B}$-equivalences $\mathrm{ev}\colon \overline{A} \to A$ and $\pi\colon \overline{A} \to \Delta(A_{h\mathfrak{B}})$ in $\mathsf{NBr}\text{-}\mathcal{S}^{\mathfrak{B}}$. For a Br-algebra Y in $\mathcal{S}$, the other composition $\Delta(Y)_{h\mathfrak{B}}$ can be identified with the product algebra $\mathrm{N}\mathfrak{B} \times Y$ such that the projection defines a weak equivalence of Br-algebras $\Delta(Y)_{h\mathfrak{B}} \xrightarrow{\sim} Y$. The statement therefore follows from Proposition A.1. □

With these preparations we can finally prove that NBr-algebras in $\mathcal{S}^{\mathfrak{B}}$ can be rectified to strictly commutative $\mathfrak{B}$-space monoids. Our proof of this result differs from the proof of the analogous categorical statement in Theorem 4.19 since we do not have a space-level version of the rectification functor Φ. Instead we shall make use of the functor $F\colon \mathsf{NBr}\text{-}\mathcal{S} \to \mathsf{Br}\text{-}\mathit{Cat}$ introduced by Fiedorowicz-Stelzer-Vogt [FSV13] and then compose the latter with Φ. The relevant facts about the functor F are discussed in the context of localization in Example A.2.

Theorem 5.9. *Every* NBr*-algebra in* $\mathcal{S}^{\mathfrak{B}}$ *is related to a strictly commutative* $\mathfrak{B}$*-space monoid by a chain of natural* $\mathfrak{B}$*-equivalences in* $\mathsf{NBr}\text{-}\mathcal{S}^{\mathfrak{B}}$*.*

Proof. Let A be an NBr-algebra in $\mathcal{S}^{\mathfrak{B}}$. Then $A_{h\mathfrak{B}}$ is an NBr-algebra in $\mathcal{S}$ and applying the functor F we get a Br-algebra $F(A_{h\mathfrak{B}})$ in Cat. We claim that A is related to the commutative $\mathfrak{B}$-space monoid $\mathrm{N}\Phi(F(A_{h\mathfrak{B}}))$ by a chain of $\mathfrak{B}$-equivalences in $\mathsf{NBr}\text{-}\mathcal{S}^{\mathfrak{B}}$. To this end we first proceed as in the proof of Proposition 5.8 to get a chain of $\mathfrak{B}$-equivalences $A \simeq \Delta(A_{h\mathfrak{B}})$. Then we compose the chains of weak equivalences

$$A_{h\mathfrak{B}} \simeq \mathrm{N}F(A_{h\mathfrak{B}}) \simeq \mathrm{N}(\mathfrak{B}\textstyle\int\Phi(F(A_{h\mathfrak{B}}))) \simeq \mathrm{N}\Phi(F(A_{h\mathfrak{B}}))_{h\mathfrak{B}}$$

defined respectively in [FSV13, C.2], Proposition 4.18, and Proposition 5.7. This in turn gives a chain of $\mathfrak{B}$-equivalences

$$A \simeq \Delta(A_{h\mathfrak{B}}) \simeq \Delta(\mathrm{N}\Phi(F(A_{h\mathfrak{B}}))_{h\mathfrak{B}}) \simeq \mathrm{N}\Phi(F(A_{h\mathfrak{B}})),$$

again by Proposition 5.8. □

Example 5.10. In general an (unbraided) operad M in $\mathcal{S}$ gives rise to a functor $\mathsf{M}\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{S}$ as explained in [CMT78]. Given a based space X we have the $\mathcal{I}$-space $X^{\bullet}$ from Example 3.7 and may form the coend $\mathsf{M}\otimes_{\mathcal{I}} X^{\bullet}$ (whose geometric realization is denoted $\mathsf{M}|X|$ by May [May72]). In the same way a braided operad M gives rise to a functor $\mathsf{M}\colon \mathfrak{B}^{\mathrm{op}} \to \mathcal{S}$ and using the same notation for the pullback of $X^{\bullet}$ to a $\mathfrak{B}$-space we may form the coend $\mathsf{M} \otimes_{\mathfrak{B}} X^{\bullet}$ considered by Fiedorowicz [Fie]. Writing $\mathsf{M}/\mathcal{P}$ for the debraided operad, the fact that the pure braid groups $\mathcal{P}_n$ act trivially on X^n implies that there is a natural isomorphism $\mathsf{M} \otimes_{\mathfrak{B}} X^{\bullet} \cong \mathsf{M}/\mathcal{P}_n \otimes_{\mathcal{I}} X^{\bullet}$. Now specialize to the braided operad NBr and recall that the homotopy colimit $X^{\bullet}_{h\mathfrak{B}}$ can be identified with the coend $\mathrm{N}(\bullet\!\downarrow\!\mathfrak{B}) \otimes_{\mathfrak{B}} X^{\bullet}$.

Proceeding as in [Sch07, Section 4.2] we define a map of $\mathfrak{B}^{\mathrm{op}}$-spaces $\mathrm{N}(\bullet\downarrow\mathfrak{B}) \to \mathsf{NBr}$ such that the induced map of coends

$$\mathrm{N}(\bullet\downarrow\mathfrak{B}) \otimes_{\mathfrak{B}} X^{\bullet} \to \mathsf{NBr} \otimes_{\mathfrak{B}} X^{\bullet}$$

is an equivalence. The above remarks together with the fact that the geometric realization of the debraiding $\mathsf{NBr}/\mathcal{P}$ is equivalent to the little 2-cubes operad $\mathcal{C}_2$ imply that there are equivalences

$$|\mathsf{NBr} \otimes_{\mathfrak{B}} X^{\bullet}| \cong |\mathsf{NBr}/\mathcal{P} \otimes_{\mathcal{I}} X^{\bullet}| \simeq \mathcal{C}_2 \otimes_{\mathcal{I}} |X|^{\bullet}.$$

For connected X it therefore follows from [May72, Theorem 2.7] that the geometric realization of $X^{\bullet}_{h\mathfrak{B}}$ is homotopy equivalent to $\Omega^2\Sigma^2(|X|)$. We may interpret this as saying that the commutative $\mathfrak{B}$-space monoid $X^{\bullet}$ represents the 2-fold loop space $\Omega^2\Sigma^2(|X|)$.

6. Classifying spaces for braided monoidal categories

We consider a monoidal category $(\mathcal{A}, \otimes, I)$ and therein a monoid A, a right A-module M, and a left A-module N. Suppressing a choice of parentheses from the notation, the two-sided bar construction $B^{\otimes}_{\bullet}(M, A, N)$ is the simplicial object defined by

$$[k] \mapsto M \otimes A^{\otimes k} \otimes N$$

with structure maps as for the usual bar construction for spaces, see for instance [May72, Chapter 9]. If the unit I for the monoidal structure is both a right and left A-module we can define the bar construction on A as $B^{\otimes}_{\bullet}(A) = B^{\otimes}_{\bullet}(I, A, I)$. This works in particular when I is a terminal object in $\mathcal{A}$.

In order to say something about the multiplicative properties of $B^{\otimes}_{\bullet}(A)$ we investigate how monoids behave with respect to the monoidal product. If $\mathcal{A}$ is a braided monoidal category with braiding b the monoidal product $A \otimes B$ of two monoids A and B is again a monoid. Suppressing parentheses, the multiplication $\mu_{A\otimes B}$ is the morphism

$$A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id}_A \otimes b_{B,A} \otimes \mathrm{id}_B} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B$$

where μ_A and μ_B are the multiplications of the monoids A and B respectively. Unlike in a symmetric monoidal category, the monoidal product of two commutative monoids in $\mathcal{A}$ is not necessarily a commutative monoid. But it is straightforward to check that if A is a commutative monoid, then the multiplication $\mu_A \colon A \otimes A \to A$ is a monoid morphism. Suppose given a monoid A in $\mathcal{A}$ such that the unit I is a right and left A-module. Then the above implies that for each k, $B^{\otimes}_{k}(A)$ is a monoid. If in addition A is commutative, the family of multiplication maps assemble into a morphism $B^{\otimes}_{\bullet}(A) \otimes B^{\otimes}_{\bullet}(A) \to B^{\otimes}_{\bullet}(A)$ of simplicial objects, where the monoidal product is taken degreewise. The bar construction on a commutative monoid A is a simplicial monoid in $\mathcal{A}$ with this multiplication.

Now we specialize to the braided monoidal category $\mathcal{S}^{\mathfrak{B}}$ of $\mathfrak{B}$-spaces. Here we can realize a simplicial object $Z_{\bullet}$ by taking the diagonal $|Z_{\bullet}|$ of the two simplicial directions to obtain a $\mathfrak{B}$-space. We define the bar construction on a $\mathfrak{B}$-space monoid A as

$\mathrm{B}^{\boxtimes}(A) = |\mathrm{B}^{\boxtimes}_{\bullet}(A)|$. From now on we will refer to the simplicial version as the simplicial bar construction. The above discussion about the multiplicative properties of the simplicial bar construction implies the following result.

Lemma 6.1. *The bar construction* $\mathrm{B}^{\boxtimes}(A)$ *on a commutative* $\mathfrak{B}$*-space monoid* A *is a (not necessarily commutative) monoid in* $\mathcal{S}^{\mathfrak{B}}$*.* □

Recall that the natural transformation $\nu_{A,B} \colon A_{h\mathfrak{B}} \times B_{h\mathfrak{B}} \to (A \boxtimes B)_{h\mathfrak{B}}$ from Lemma 3.15 gives the homotopy colimit functor $(-)_{h\mathfrak{B}} \colon \mathcal{S}^{\mathfrak{B}} \to \mathcal{S}$ the structure of a lax monoidal functor. As usual when we have a lax monoidal functor, it follows that if A is a $\mathfrak{B}$-space monoid, then $A_{h\mathfrak{B}}$ inherits the structure of a monoid in $\mathcal{S}$. If M is a right A-module, then $M_{h\mathfrak{B}}$ inherits the structure of a right $A_{h\mathfrak{B}}$-module in $\mathcal{S}$ and similarly for a left A-module N. We can then apply the two-sided simplicial bar construction in $\mathcal{S}$ to $A_{h\mathfrak{B}}$, $M_{h\mathfrak{B}}$ and $N_{h\mathfrak{B}}$ and obtain $B_{\bullet}(M_{h\mathfrak{B}}, A_{h\mathfrak{B}}, N_{h\mathfrak{B}})$. The natural transformation ν gives rise to maps $B_k(M_{h\mathfrak{B}}, A_{h\mathfrak{B}}, N_{h\mathfrak{B}}) \to \mathrm{B}^{\boxtimes}_k(M, A, N)_{h\mathfrak{B}}$ that commute with the simplicial structure maps. Hence we obtain a morphism

$$B(M_{h\mathfrak{B}}, A_{h\mathfrak{B}}, N_{h\mathfrak{B}}) \to \mathrm{B}^{\boxtimes}(M, A, N)_{h\mathfrak{B}}$$

in $\mathcal{S}$. By specializing to the case where M and N is the unit $U^{\mathfrak{B}}$ we can relate $\mathrm{B}^{\boxtimes}(A)_{h\mathfrak{B}}$ to $B(A_{h\mathfrak{B}})$ via $B(U^{\mathfrak{B}}_{h\mathfrak{B}}, A_{h\mathfrak{B}}, U^{\mathfrak{B}}_{h\mathfrak{B}})$. The homotopy colimit of $U^{\mathfrak{B}}$ is homeomorphic to $\mathrm{N}\mathfrak{B}$ which is a contractible simplicial set. Hence the map $B(\mathrm{N}\mathfrak{B}, A_{h\mathfrak{B}}, \mathrm{N}\mathfrak{B}) \to B(A_{h\mathfrak{B}})$ induced by the projection $\mathrm{N}\mathfrak{B} \to *$ is a weak equivalence.

Proposition 6.2. *If* A *is a* $\mathfrak{B}$*-space monoid with underlying flat* $\mathfrak{B}$*-space the above defined maps*

$$\mathrm{B}^{\boxtimes}(A)_{h\mathfrak{B}} \xleftarrow{\simeq} B(\mathrm{N}\mathfrak{B}, A_{h\mathfrak{B}}, \mathrm{N}\mathfrak{B}) \xrightarrow{\simeq} B(A_{h\mathfrak{B}})$$

are weak equivalences.

Proof. The argument for the right hand map being a weak equivalence is given before the proposition. The map $(A_{h\mathfrak{B}})^{\times k} \to (A^{\boxtimes k})_{h\mathfrak{B}}$ is a weak equivalence for each $k \geq 0$ since A is flat, see Lemma 3.15. It follows that the left hand map is the diagonal of a map of bisimplicial sets which is a weak equivalence at each simplicial degree of the bar construction. Therefore it is itself a weak equivalence. □

Our goal is to use the bar construction in $\mathfrak{B}$-spaces to give a double delooping of the group completion of $A_{h\mathfrak{B}}$ for a commutative $\mathfrak{B}$-space monoid A with underlying flat $\mathfrak{B}$-space. In order to apply the previous proposition twice we will show that the bar construction on something flat is also flat.

Lemma 6.3. *If* A *is a* $\mathfrak{B}$*-space monoid with underlying flat* $\mathfrak{B}$*-space, then the underlying* $\mathfrak{B}$*-space of the bar construction* $\mathrm{B}^{\boxtimes}(A)$ *on* A *is also flat.*

Proof. When A is flat, it follows from Lemma 3.13 that $\mathrm{B}^{\boxtimes}_k(A)$ is flat for each $k \geq 0$. The criterion for flatness given in Proposition 3.10 can be checked in each simplicial degree.

Thus, $\mathrm{B}^{\boxtimes}(A)$ is the diagonal of a bisimplicial object which is flat at each simplicial degree of the bar construction and is therefore flat. □

We use the well known fact that the group completion of a homotopy commutative simplicial monoid M may be modelled by the canonical map $M \to \Omega(B(M)^{\mathrm{fib}})$, where the fibrant replacement $B(M)^{\mathrm{fib}}$ is the singular simplicial set of the geometric realization of $B(M)$. By a double delooping of a simplicial set K we mean a based simplicial set L such that $\Omega^2(L^{\mathrm{fib}}) \simeq K$.

Proposition 6.4. *If A is a commutative $\mathfrak{B}$-space monoid with underlying flat $\mathfrak{B}$-space, then $\mathrm{B}^{\boxtimes}(\mathrm{B}^{\boxtimes}(A))_{h\mathfrak{B}}$ is a double delooping of the group completion of $A_{h\mathfrak{B}}$.*

Proof. Letting A equal $\mathrm{B}^{\boxtimes}(A)$ in Proposition 6.2 and using Lemma 6.3 we get

$$\mathrm{B}^{\boxtimes}(\mathrm{B}^{\boxtimes}(A))_{h\mathfrak{B}} \simeq B(\mathrm{B}^{\boxtimes}(A)_{h\mathfrak{B}}).$$

Evaluating $\Omega((-)^{\mathrm{fib}})$ on this we get equivalences

$$\Omega(\mathrm{B}^{\boxtimes}(\mathrm{B}^{\boxtimes}(A))_{h\mathfrak{B}}^{\mathrm{fib}}) \simeq \Omega(B(\mathrm{B}^{\boxtimes}(A)_{h\mathfrak{B}})^{\mathrm{fib}}) \simeq \mathrm{B}^{\boxtimes}(A)_{h\mathfrak{B}}^{\mathrm{fib}} \simeq B(A_{h\mathfrak{B}})^{\mathrm{fib}}$$

where the map in the middle is an equivalence since $\mathrm{B}^{\boxtimes}(A)_{h\mathfrak{B}}$ is connected and hence group-like. Looping once more we see that $\mathrm{B}^{\boxtimes}(\mathrm{B}^{\boxtimes}(A))_{h\mathfrak{B}}$ is indeed a double delooping of the group completion of $B(A_{h\mathfrak{B}})$. □

Recall from Remark 4.17 that we can construct a commutative $\mathfrak{B}$-space monoid $\mathrm{N}\Phi(\mathcal{A})$ for any braided (not necessarily strict) monoidal small category. Next, we show that $\mathrm{N}\Phi(\mathcal{A})$ has underlying flat $\mathfrak{B}$-space so we can apply the above result to the double bar construction on $\mathrm{N}\Phi(\mathcal{A})$.

Lemma 6.5. *Let $\mathcal{A}$ is a braided monoidal small category. The commutative $\mathfrak{B}$-space monoid $\mathrm{N}\Phi(\mathcal{A})$ has underlying flat $\mathfrak{B}$-space.*

Proof. Here we prove the result for a braided strict monoidal small category, the non-strict case is left to the reader. We use the criterion given in Proposition 3.10. For each braided injection $\mathbf{m} \to \mathbf{n}$ the induced functor $\Phi(\mathcal{A})(\mathbf{m}) \to \Phi(\mathcal{A})(\mathbf{n})$ is injective on both objects and morphisms. Thus the nerve of that map is a cofibration of simplicial sets. The functor $\Phi(\mathcal{A})(\mathbf{m}) \to \Phi(\mathcal{A})(\mathbf{m} \sqcup \mathbf{n})$ induced by the inclusion of $\mathbf{m}$ in $\mathbf{m} \sqcup \mathbf{n}$ takes an object $(\mathbf{a}_1, \dots, \mathbf{a}_m)$ to $(\mathbf{a}_1, \dots, \mathbf{a}_m, U^{\mathfrak{B}}, \dots U^{\mathfrak{B}})$. Since we have a strict monoidal structure it takes a morphism f to the morphism $f \boxtimes \mathrm{id}_{U^{\mathfrak{B}} \boxtimes \cdots \boxtimes U^{\mathfrak{B}}} = f$. If we consider a diagram similar to (3.3) for the $\mathfrak{B}$-category $\Phi(\mathcal{A})$ it is clear that the intersection of the images of $\Phi(\mathcal{A})(\mathbf{l} \sqcup \mathbf{m})$ and $\Phi(\mathcal{A})(\mathbf{m} \sqcup \mathbf{n})$ in $\Phi(\mathcal{A})(\mathbf{l} \sqcup \mathbf{m} \sqcup \mathbf{n})$ equals the image of $\Phi(\mathcal{A})(\mathbf{m})$. The same then holds for the $\mathfrak{B}$-space $\mathrm{N}\Phi(\mathcal{A})$. □

Corollary 6.6. *If $\mathcal{A}$ is a braided monoidal small category, then $\mathrm{B}^{\boxtimes}(\mathrm{B}^{\boxtimes}(\mathrm{N}\Phi(\mathcal{A})))_{h\mathfrak{B}}$ is a double delooping of the group completion of $\mathrm{N}\mathcal{A}$.*

Proof. The underlying $\mathfrak{B}$-space of $\mathrm{N}\Phi(\mathcal{A})$ is flat, so we can apply the proposition and get that $\mathrm{B}^{\boxtimes}(\mathrm{B}^{\boxtimes}(\mathrm{N}\Phi(\mathcal{A})))_{h\mathfrak{B}}$ is a double delooping of the group completion of $\mathrm{N}\Phi(\mathcal{A})_{h\mathfrak{B}}$. But by combining Propositions 5.7 and 4.18, the latter is weakly equivalent to $\mathrm{N}\mathcal{A}$. □

7. $\mathcal{I}$-categories and E_∞ spaces

In this section we focus on diagrams indexed by the category $\mathcal{I}$ and we record the constructions and results analogous to those worked out for diagrams indexed by the category $\mathfrak{B}$ in the previous sections. The proofs are completely analogous to those in the braided case (if not simpler) and will be omitted throughout. We then relate this material to the category of symmetric spectra.

Let $Cat^{\mathcal{I}}$ denote the category of $\mathcal{I}$-categories with the symmetric monoidal convolution product inherited from $\mathcal{I}$. The Grothendieck construction defines a functor $\mathcal{I}\!\int\colon Cat^{\mathcal{I}} \to Cat$ and a map of $\mathcal{I}$-categories $X \to Y$ is said to be an *$\mathcal{I}$-equivalence* if the induced functor $\mathcal{I}\!\int X \to \mathcal{I}\!\int Y$ is a weak equivalence. We write Sym for the symmetric monoidal analogue of the Cat-operad Br. Thus, the category $\mathsf{Sym}(k)$ has as its objects the elements a in Σ_k and a morphism $\alpha\colon a \to b$ is an element $\alpha \in \Sigma_k$ such that $\alpha a = b$. It is proved in [May74] that a Sym-algebra in Cat is the same thing as a permutative (i.e., symmetric strict monoidal) category and that the nerve $\mathrm{N}\mathsf{Sym}$ can be identified with the Barratt-Eccles operad. The latter is an E_∞ operad in the sense that $\mathrm{N}\mathsf{Sym}(k)$ is Σ_k-free and contractible for all k. As in Proposition 4.12 one checks that there is an equivalence of localized categories

$$\mathcal{I}\!\int\colon \mathsf{Sym}\text{-}Cat^{\mathcal{I}}[w_{\mathcal{I}}^{-1}] \simeq \mathsf{Sym}\text{-}Cat[w^{-1}] :\Delta.$$

The rectification functor Φ from Section 4.14 also has a symmetric monoidal version, now in the form of a functor $\Phi\colon \mathsf{Sym}\text{-}Cat^{\mathcal{I}} \to \mathcal{C}(Cat^{\mathcal{I}})$ where the codomain is the category of commutative $\mathcal{I}$-category monoids. The composite functor

$$\mathsf{Sym}\text{-}Cat \xrightarrow{\Phi} \mathcal{C}(Cat^{\mathcal{I}}) \to \mathsf{Sym}\text{-}Cat^{\mathcal{I}} \xrightarrow{\mathcal{I}\int} \mathsf{Sym}\text{-}Cat \tag{7.1}$$

is weakly equivalent to the identity functor and arguing as in the proof of Theorem 4.19 we get the following result.

Theorem 7.1. *Every Sym-algebra in $Cat^{\mathcal{I}}$ is related to a strictly commutative $\mathcal{I}$-category monoid by a chain of $\mathcal{I}$-equivalences in $\mathsf{Sym}\text{-}Cat^{\mathcal{I}}$.* □

In particular, every symmetric monoidal category is weakly equivalent to one of the form $\mathcal{I}\!\int A$ for A a strictly commutative $\mathcal{I}$-category monoid. Now let $\mathcal{S}^{\mathcal{I}}$ be the category of $\mathcal{I}$-spaces equipped with the symmetric monoidal convolution product inherited from $\mathcal{I}$. A map of $\mathcal{I}$-spaces $X \to Y$ is an *$\mathcal{I}$-equivalence* if the induced map of homotopy colimits $X_{h\mathcal{I}} \to Y_{h\mathcal{I}}$ is a weak equivalence and the $\mathcal{I}$-space version of Proposition 5.8 gives an equivalence of the localized categories

$$(-)_{h\mathcal{I}}\colon \mathrm{N}\mathsf{Sym}\text{-}\mathcal{S}^{\mathcal{I}}[w_{\mathcal{I}}^{-1}] \simeq \mathrm{N}\mathsf{Sym}\text{-}\mathcal{S}[w^{-1}] :\Delta. \tag{7.2}$$

Furthermore, one checks that the $\mathcal{I}$-category version of Thomason's equivalence (4.1) gives a natural weak equivalence relating the two compositions in the diagram

$$\begin{array}{ccc} \mathsf{Sym}\text{-}\mathit{Cat}^{\mathcal{I}} & \xrightarrow{\mathrm{N}} & \mathrm{N}\mathsf{Sym}\text{-}\mathcal{S}^{\mathcal{I}} \\ {\scriptstyle \mathcal{I}\int}\downarrow & & \downarrow{\scriptstyle (-)_{h\mathcal{I}}} \\ \mathsf{Sym}\text{-}\mathit{Cat} & \xrightarrow{\mathrm{N}} & \mathrm{N}\mathsf{Sym}\text{-}\mathcal{S}. \end{array} \tag{7.3}$$

Arguing as in the proof of Theorem 5.9 one can use this to show that every $\mathrm{N}\mathsf{Sym}$-algebra in $\mathcal{S}^{\mathcal{I}}$ is $\mathcal{I}$-equivalent to one that is strictly commutative. However, a stronger form of this statement has been proved in [SS12]: There is a model structure on $\mathrm{N}\mathsf{Sym}\text{-}\mathcal{S}^{\mathcal{I}}$ such that the equivalence (7.2) can be derived from a Quillen equivalence, and a further model structure on $\mathcal{C}(\mathcal{S}^{\mathcal{I}})$ (the category of commutative $\mathcal{I}$-space monoids) making the latter Quillen equivalent to $\mathrm{N}\mathsf{Sym}\text{-}\mathcal{S}^{\mathcal{I}}$.

7.2. **Symmetric spectra and E_∞ spaces.** Let Sp^{Σ} be the category of symmetric spectra as defined in [HSS00]. The smash product of symmetric spectra makes this a symmetric monoidal category with monoidal unit the sphere spectrum. Given an (unbased) space X we write $\Sigma^{\infty}(X_+)$ for the suspension spectrum with nth space $X_+ \wedge S^n$ where X_+ denotes the union of X with a disjoint base point. If X is an E_∞ space (i.e., an algebra for an E_∞ operad in $\mathcal{S}$), then $\Sigma^{\infty}(X_+)$ is an E_∞ symmetric ring spectrum for the same operad. It is proved in [EM06] that in general an E_∞ symmetric ring spectrum is stably equivalent to a strictly commutative symmetric ring spectrum. However, the proof of this fact is not very constructive and it is of interest to find more memorable commutative models of the E_∞ ring spectra in common use. Here we shall do this for E_∞ symmetric ring spectra of the form $\Sigma^{\infty}(\mathrm{N}\mathcal{A}_+)$ for a permutative category $\mathcal{A}$. The relevant operad is the Barratt-Eccles operad $\mathrm{N}\mathsf{Sym}$ as explained above. In order to make use of the rectification functor Φ we recall from [SS12, Section 3] that the suspension spectrum functor extends to a strong symmetric monoidal functor $\mathbb{S}^{\mathcal{I}} \colon \mathcal{S}^{\mathcal{I}} \to Sp^{\Sigma}$ taking an $\mathcal{I}$-space X to the symmetric spectrum $\mathbb{S}^{\mathcal{I}}[X]$ with nth space $X(\mathbf{n})_+ \wedge S^n$. Given a permutative category $\mathcal{A}$ we may apply this functor to the commutative $\mathcal{I}$-space monoid $\mathrm{N}\Phi(\mathcal{A})$ to get the commutative symmetric ring spectrum $\mathbb{S}^{\mathcal{I}}[\mathrm{N}\Phi(\mathcal{A})]$.

Proposition 7.3. *Given a permutative category $\mathcal{A}$, the commutative symmetric ring spectrum $\mathbb{S}^{\mathcal{I}}[\mathrm{N}\Phi(\mathcal{A})]$ is related to $\Sigma^{\infty}(\mathrm{N}\mathcal{A}_+)$ by a chain of natural stable equivalences of E_∞ symmetric ring spectra.*

Proof. Composing the natural weak equivalence relating the composite functor (7.1) to the identity functor with Thomason's weak equivalence relating the two compositions in (7.3), we get a weak equivalence of $\mathrm{N}\mathsf{Sym}$-algebras

$$\mathrm{N}\mathcal{A} \xleftarrow{\simeq} \mathrm{N}(\mathcal{I}\textstyle\int\Phi(\mathcal{A})) \xleftarrow{\simeq} (\mathrm{N}\Phi(\mathcal{A}))_{h\mathcal{I}}.$$

Furthermore, using the bar resolution $\overline{(-)}$ as in Section 5.4 we get a chain of $\mathcal{I}$-equivalences

$$\Delta(\mathrm{N}\Phi(\mathcal{A})_{h\mathcal{I}}) \xleftarrow{\simeq} \overline{\mathrm{N}\Phi(\mathcal{A})} \xrightarrow{\simeq} \mathrm{N}\Phi(\mathcal{A})$$

in $\mathrm{NSym}\text{-}\mathcal{S}^{\mathcal{I}}$. This gives the result since the functor $\mathbb{S}^{\mathcal{I}}[-]$ takes $\mathcal{I}$-equivalences to stable equivalences. □

We also note that the symmetric spectrum $\mathbb{S}^{\mathcal{I}}[\mathrm{N}\Phi(\mathcal{A})]$ has several of the pleasant properties discussed in [HSS00, Section 5]: The fact that it is S-cofibrant (this is what some authors call flat) ensures that it is homotopically well-behaved with respect to the smash product, and the fact that it is semistable ensures that its spectrum homotopy groups can be identified with the stable homotopy groups of $\mathrm{N}\mathcal{A}$.

Example 7.4. The underlying infinite loop space $Q(S^0)$ of the sphere spectrum plays a fundamental role in stable homotopy theory. In order to realize the E_∞ ring spectrum $\Sigma^\infty(Q(S^0)_+)$ as a commutative symmetric ring spectrum, we use that $Q(S^0)$ is weakly equivalent to the classifying space of Quillen's localization construction $\Sigma^{-1}\Sigma$, where as usual Σ denotes the category of finite sets and bijections. (We refer to [Gra76] for a general discussion of Quillen's localization construction and to [SS12] for an explicit description of $\Sigma^{-1}\Sigma$.) The category $\Sigma^{-1}\Sigma$ inherits a permutative structure from Σ and it follows from Proposition 7.3 that the commutative symmetric ring spectrum $\mathbb{S}^{\mathcal{I}}[\mathrm{N}\Phi(\Sigma^{-1}\Sigma)]$ is a model of $\Sigma^\infty(Q(S^0)_+)$.

Appendix A. Localization of categories

We make some elementary remarks on localization of categories. Let $\mathcal{C}$ be a (not necessarily small) category and let $\mathcal{V}$ be a class of morphisms in $\mathcal{C}$. Recall that a *localization of $\mathcal{C}$ with respect to $\mathcal{V}$* is a category $\mathcal{C}'$ together with a functor $L\colon C \to \mathcal{C}'$ that maps the morphisms in $\mathcal{V}$ to isomorphisms in $\mathcal{C}'$ and is initial with this property: Given a category $\mathcal{D}$ and a functor $F\colon \mathcal{C} \to \mathcal{D}$ that maps the morphisms in $\mathcal{V}$ to isomorphisms in $\mathcal{D}$, there exists a unique functor $F'\colon \mathcal{C}' \to \mathcal{D}$ such that $F = F' \circ L$. Clearly a localization of $\mathcal{C}$ with respect to $\mathcal{V}$ is uniquely determined up to isomorphism if it exists. We sometimes use the notation $\mathcal{C} \to \mathcal{C}[\mathcal{V}^{-1}]$ for such a localization. It will be convenient to assume that $\mathcal{C}$ and $\mathcal{C}[\mathcal{V}^{-1}]$ always have the same objects.

Let again $\mathcal{C}$ be a category equipped with a class of morphisms $\mathcal{V}$ and consider a category $\mathcal{A}$ together with a pair of functors $F, G\colon \mathcal{A} \to \mathcal{C}$. In this situation we say that F and G are *related by a chain of natural transformations in $\mathcal{V}$*, written $F \simeq_{\mathcal{V}} G$, if there exists a finite sequence of functors $H_1, \dots, H_n$ from $\mathcal{A}$ to $\mathcal{C}$ with $H_1 = F$ and $H_n = G$, and for each $1 \le i < n$ either a natural transformation $H_i \to H_{i+1}$ with values in $\mathcal{V}$ or a natural transformation $H_{i+1} \to H_i$ with values in $\mathcal{V}$. In the next proposition we write $I_{\mathcal{C}}$ for the identity functor on $\mathcal{C}$.

Proposition A.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories related by the functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$. Suppose that $\mathcal{V}$ is a class of morphisms in $\mathcal{C}$ and that $\mathcal{W}$ is a class of morphisms in $\mathcal{D}$ such that $F(\mathcal{V}) \subseteq \mathcal{W}$, $G(\mathcal{W}) \subseteq \mathcal{V}$, $G \circ F \simeq_{\mathcal{V}} I_{\mathcal{C}}$, and $F \circ G \simeq_{\mathcal{W}} I_{\mathcal{D}}$. Then the localization of $\mathcal{C}$ with respect to $\mathcal{V}$ exists if and only if the localization of $\mathcal{D}$*

with respect to $\mathcal{W}$ exists, and in this case F and G induce an equivalence of categories $F\colon \mathcal{C}[\mathcal{V}^{-1}] \rightleftarrows \mathcal{D}[\mathcal{W}^{-1}] :G$.

Proof. Suppose that a localization of $\mathcal{D}$ with respect to $\mathcal{W}$ exists and write $L\colon \mathcal{D} \to \mathcal{D}'$ for such a localization. Then we define a category $\mathcal{C}'$ with the same objects as $\mathcal{C}$ and morphism sets $\mathcal{C}'(C_1, C_2) = \mathcal{D}'(LF(C_1), LF(C_2))$. The composition in $\mathcal{C}'$ is inherited from the composition in $\mathcal{D}'$. We claim that the canonical functor $L^{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}'$, which is the identity on objects and given by LF on morphisms, is a localization of $\mathcal{C}$ with respect to $\mathcal{V}$. Thus, consider a category $\mathcal{E}$ and a functor $H\colon \mathcal{C} \to \mathcal{E}$ that maps the morphisms in $\mathcal{V}$ to isomorphisms in $\mathcal{E}$. We must define a functor $H'\colon \mathcal{C}' \to \mathcal{E}$ such that $H = H' \circ L^{\mathcal{C}}$, and it is clear that we must have $H'(C) = H(C)$ for all objects C in $\mathcal{C}'$. In order to define the action on morphisms, we factor the composite functor $K = H \circ G\colon \mathcal{D} \to \mathcal{E}$ over the localization of $\mathcal{D}'$ to get a functor $K'\colon \mathcal{D}' \to \mathcal{E}$. The relation $G \circ F \simeq_{\mathcal{V}} I_{\mathcal{C}}$ gives a natural isomorphism $\phi_C\colon HGF(C) \to H(C)$ and we define the action of H' on the morphism set $\mathcal{C}'(C_1, C_2)$ to be the composition

$$\mathcal{D}'(LF(C_1), LF(C_2)) \xrightarrow{K'} \mathcal{E}(HGF(C_1), HGF(C_2)) \xrightarrow{\phi_{C_2}\circ(-)\circ\phi_{C_1}^{-1}} \mathcal{E}(H(C_1), H(C_2)).$$

It is immediate from the definition that H' satisfies the required conditions and it remains to show that it is uniquely determined. The composition $L^{\mathcal{C}} \circ G$ factors over $\mathcal{D}'$ to give the left hand square in the commutative diagram

$$\begin{array}{ccccc} \mathcal{D} & \xrightarrow{G} & \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle L}\downarrow & & {\scriptstyle L^{\mathcal{C}}}\downarrow & & {\scriptstyle L}\downarrow \\ \mathcal{D}' & \xrightarrow{G'} & \mathcal{C}' & \longrightarrow & \mathcal{D}'. \end{array}$$

Notice that the relation $F \circ G \simeq_{\mathcal{W}} I_{\mathcal{D}}$ gives a natural isomorphism relating the composition $L \circ F \circ G$ to L. Let $\mathcal{J}$ be the category with objects 0 and 1, and two non-identity arrows $i\colon 0 \to 1$ and $j\colon 1 \to 0$. (Thus, $\mathcal{J}$ is a groupoid with inverse isomorphisms i and j.) Then we may interpret the natural isomorphism in question as a functor $\mathcal{D}\times\mathcal{J} \to \mathcal{D}'$, or, by adjointness, a functor $\mathcal{D} \to (\mathcal{D}')^{\mathcal{J}}$. The latter factors over $\mathcal{D}'$ to give a natural isomorphism relating the composition in the bottom row of the diagram to the identity functor on $\mathcal{D}'$. It follows that $G'\colon \mathcal{D}' \to \mathcal{C}'$ is fully faithful and consequently that H' is uniquely determined on the full subcategory of $\mathcal{C}'$ generated by objects of the form $G(D)$ for D in $\mathcal{D}$. Furthermore, the relation $G \circ F \simeq_{\mathcal{V}} I_{\mathcal{C}}$ implies that any morphism in $\mathcal{C}'$ can be written as a composition of morphisms in this subcategory with morphisms in the image of $L^{\mathcal{C}}$ and inverses of morphisms in the image of $L^{\mathcal{C}}$. This shows that H' is uniquely determined on the whole category $\mathcal{C}'$. The last statement in the proposition is an immediate consequence. □

Example A.2. The work of Fiedorowicz-Stelzer-Vogt [FSV13] fits into this framework. Let M be a *Cat*-operad that is Σ-free in the sense that Σ_k acts freely on $\mathsf{M}(k)$ for all k. In [FSV13, C.2] the authors define a functor $F\colon \mathsf{NM}\text{-}\mathcal{S} \to \mathsf{M}\text{-}Cat$ and show that if M satisfies a certain "factorization condition", then there are chains of weak equivalences $\mathrm{N} \circ F \simeq_w I$ and $F \circ \mathrm{N} \simeq_w I$ with I the respective identity functors. By [FSV13, Lemma 8.12] this applies in particular to the (unbraided) operad Br discussed in Section 5.1. It is well-known that the localization of $\mathrm{N}\mathsf{Br}\text{-}\mathcal{S}$ with respect to the weak equivalences exists, since it can be realized as the homotopy category of a suitable model category. Thus, it follows from Proposition A.1 that also the localization of $\mathsf{Br}\text{-}\mathcal{C}at$ with respect to the weak equivalences exists and that these localized categories are equivalent. This is shown in [FSV13, Proposition 7.4] except that the discussion of Grothendieck universes and "localization up to equivalence" is not really needed in order to state this result.

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Christian Schlichtkrull, Department of Mathematics, University of Bergen, P.O. Box 7800, N-5020 Bergen, Norway

Email address: `christian.schlichtkrull@math.uib.no`

Mirjam Solberg, Department of Mathematics, University of Bergen, P.O. Box 7800, N-5020 Bergen, Norway

Email address: `mirjam.solberg@math.uib.no`

Paper B: Weak braided monoidal categories and their homotopy colimits

Mirjam Solberg

Published in: Theory and Applications of Categories, Vol. 30, No. 3, 2015, pp. 40-48.

WEAK BRAIDED MONOIDAL CATEGORIES AND THEIR HOMOTOPY COLIMITS

MIRJAM SOLBERG

Abstract. We show that the homotopy colimit construction for diagrams of categories with an operad action, recently introduced by Fiedorowicz, Stelzer and Vogt, has the desired homotopy type for diagrams of weak braided monoidal categories. This provides a more flexible way to realize E_2 spaces categorically.

1. Introduction

Braided monoidal categories have been much studied and are used extensively in many areas of mathematics, for instance in knot theory, representation theory and topological quantum field theories. It has been known for a long time that the nerve of a braided monoidal category is an E_2 space, and it was shown recently [FSV13] that all homotopy types of E_2 spaces arise in this way. In this article we study a weaker categorical structure, namely weak braided monoidal categories. These are monoidal categories with a family of natural morphisms $X \otimes Y \to Y \otimes X$ satisfying the axioms for a braiding, except that they are not required to be isomorphisms. We will see that weak braided monoidal categories give a more flexible way to realize E_2 spaces categorically.

Homotopy colimit constructions have become increasingly important in homotopy theory. In order for the equivalence between weak braided monoidal categories and E_2 spaces to be really useful, one should be able to construct homotopy colimits on the categorical level. Such a homotopy colimit construction was defined in [FSV13] in general for diagrams of categories with an operad action. The question of the homotopy properties of the homotopy colimit was left open for weak braided monoidal categories. In this paper we provide an answer to that question. Let Br^+-Cat denote the category of weak braided monoidal categories and let X be a diagram of weak braided monoidal categories. Applying the nerve N to a weak braided monoidal category yields a space with an action of the E_2 operad $N\mathsf{Br}^+$, see Subsection 3.1. Let $\mathrm{hocolim}^{\mathsf{Br}^+} X$ denote the homotopy colimit of X defined in [FSV13], and let $\mathrm{hocolim}^{N\mathsf{Br}^+} NX$ denote the homotopy colimit of NX, for details see Subsection 3.1. Then our main result, Theorem 3.2, can be stated as follows.

Theorem 1.1. *There is a natural weak equivalence*

$$\mathrm{hocolim}^{N\mathsf{Br}^+} NX \to N(\mathrm{hocolim}^{\mathsf{Br}^+} X)$$

2010 *Mathematics Subject Classification.* Primary 18D10, 18D50; Secondary 55P48.
Key words and phrases. Weak braided monoidal categories, homotopy colimits, double loop spaces.
First published in: Theory and Applications of Categories, Vol. 30, No. 3, 2015, pp. 40-48.

of $N\mathsf{Br}^+$*-algebras.*

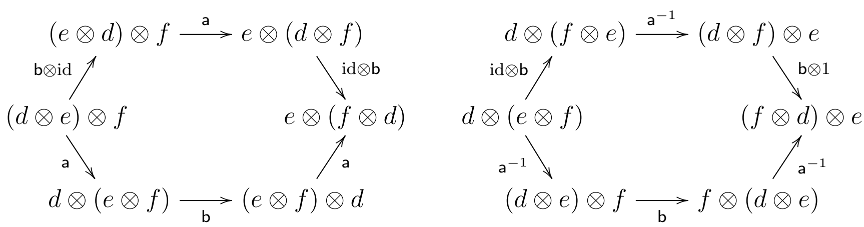
1.2. **Organization.** We begin by giving the definition of weak braided monoidal categories in Section 2 and provide some examples. In Section 3 we set up and prove our main result, Theorem 3.2. The proof involves an analysis of braid monoids which is interesting in its own right.

2. Weak braided monoidal categories

Let $\mathcal{D}$ be a monoidal category with monoidal product $\otimes$, monoidal unit i, associativity isomorphism a and left and right unit isomorphisms l and r respectively. A *weak braiding* for $\mathcal{D}$ consists of a family of morphisms

$$\mathsf{b}_{d,e}\colon\ d \otimes e \to e \otimes d$$

in $\mathcal{D}$, natural in d and e, such that $\mathsf{l}_d \mathsf{b}_{d,i} = \mathsf{r}_d$, and $\mathsf{r}_d \mathsf{b}_{i,d} = \mathsf{l}_d$ and the following two diagrams



commute for all d, e and f in $\mathcal{D}$. Here the sub indices of the weak braiding b and the associativity isomorphism a have been omitted. A *weak braided monoidal category* is a monoidal category equipped with a weak braiding. Note that if all the morphisms $\mathsf{b}_{d,e}$ are isomorphisms, then b is a braiding for the monoidal category.

Remark 2.1. The notion of a weak braided monoidal category found in [BFSV03] and [FSV13] differs from the definition given here, in that the underlying monoidal structure is required to be strictly associative and strictly unital. This is not a significant difference, since each weak braided monoidal category is equivalent to a weak braided strict monoidal category, along monoidal functors preserving the weak braiding. The proof of this is similar to the proof of the analogous result for braided monoidal structures.

Weak braided monoidal categories have not been much studied in the literature, so before we proceed we will look at some examples to show how such structures naturally arise. The first example, the disjoint union of the braid monoids, is somehow the canonical example.

Example 2.2. Let $\mathcal{B}_m^+$ denote the braid monoid on m strings with the following presentation:

$$\langle \sigma_1, \ldots, \sigma_{m-1} \,|\, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1 \text{ and } \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle.$$

The elements in $\mathcal{B}_m^+$ are called positive braids on m strings, or just positive braids.

Let $\mathcal{B}^+$ denote the category with an object $\mathbf{m}$ for each integer $m \geq 0$, with endomorphisms of $\mathbf{m}$ the elements of the braid monoid $\mathcal{B}_m^+$, and no other morphisms. This is a strict monoidal category with monoidal product given by $\mathbf{m} \sqcup \mathbf{n} = \mathbf{m}+\mathbf{n}$ and juxtaposition of positive braids, the unit is $\mathbf{0}$. The weak braiding from $\mathbf{m} \sqcup \mathbf{n}$ to $\mathbf{n} \sqcup \mathbf{m}$ is given by the positive braid

$$(\sigma_n \cdots \sigma_{m+n-1}) \cdots (\sigma_2 \cdots \sigma_{m+1})(\sigma_1 \cdots \sigma_m),$$

braiding the first m strings over the last n strings. This is the same as the usual braiding in the classical braid category, which is the disjoint union of the braid groups, see [JS93, Example 2.1].

Example 2.3. We consider the category of non-negatively graded abelian groups. An object G is a collection of abelian groups G_n for $n \geq 0$. A morphism $f\colon G \to H$ consists of group homomorphisms $f_n\colon G_n \to H_n$ for $n \geq 0$. This category has a monoidal product given by

$$(G \otimes H)_n = \bigoplus_{n_1+n_2=n} G_{n_1} \otimes H_{n_2}.$$

Now fix an integer k. For $g \in G_{n_1}$ and $h \in H_{n_2}$ the assignment $g \otimes h \mapsto k^{n_1 n_2} h \otimes g$ induces a map from $G_{n_1} \otimes H_{n_2}$ to $H_{n_2} \otimes G_{n_1}$, which in turn induces a homomorphism $(G \otimes H)_n \to (H \otimes G)_n$. The collection of such maps gives a weak braiding for the category of non-negatively graded abelian groups. Note that if k is a unit, i.e. ± 1, then the weak braiding is an actual braiding.

This example may be generalized to the category of non-negatively graded R-modules for any commutative ring R. Pick an element in R to play the role of k in the weak braiding.

A much studied construction is the center of a monoidal category, which can be endowed with a braided monoidal structure, see for instance Example 2.3 in [JS93]. Our next example is a weak version of this.

Example 2.4. Let $\mathcal{D}$ be a strict monoidal category with monoidal unit i. We consider pairs (d, δ) where d is an object in $\mathcal{D}$ and δ is a natural transformation $\delta\colon d \otimes (-) \to (-) \otimes d$ such that $\delta_i = \mathrm{id}_d$ and such that for any two objects $x, y \in \mathcal{D}$ the triangle

$$\begin{array}{ccc} d \otimes x \otimes y & \xrightarrow{\delta_x \otimes \mathrm{id}_y} & x \otimes d \otimes y \\ & {\scriptstyle \delta_{x \otimes y}} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_x \otimes \delta_y} & \\ & x \otimes y \otimes d & \end{array}$$

commutes. An arrow between two pairs $(d, \delta) \to (e, \epsilon)$ consists of a morphism $\phi\colon d \to e$ such that for all $x \in \mathcal{D}$ the identity $\epsilon_x \circ (\phi \otimes \mathrm{id}_x) = (\mathrm{id}_x \otimes \phi) \circ \delta_x$ holds. We can define a monoidal product of two such pairs by setting

$$(d, \delta) \otimes (e, \epsilon) = (d \otimes e, (\delta \otimes \mathrm{id}_e) \circ (\mathrm{id}_d \otimes \epsilon)).$$

The collection of morphisms

$$\delta_e\colon\ (d,\delta)\otimes(e,\epsilon)\to(e,\epsilon)\otimes(d,\delta)$$

satisfies the conditions for a weak braiding on this category of pairs and arrows. We call this the weak center of $\mathcal{D}$.

The requirement that $\mathcal{D}$ should be strictly associative and strictly unital was only a matter of convenience. A similar construction works for any monoidal category, details are left to the interested reader.

2.5. **Operadic interpretation of weak braided monoidal structures.** When the underlying monoidal multiplication is strict, weak braided monoidal categories are the algebras over a certain *Cat*-operad. By a *Cat*-operad we understand an operad internal to the category *Cat* of small categories. Following [FSV13, Section 8] we will introduce the *Cat*-operad Br^+ such that Br^+-algebras are weak braided strict monoidal categories. The objects of $\mathsf{Br}^+(k)$ are the elements A of the symmetric group Σ_k. Let $p\colon\ \mathcal{B}_k^+\to\Sigma_k$ denote the projection of the braid monoid onto the corresponding symmetric group. Then a morphism $\alpha\colon A\to B$ in $\mathsf{Br}^+(k)$ is a positive braid $\alpha\in\mathcal{B}_k^+$ such that $p(\alpha)A=B$. Composition in $\mathsf{Br}^+(k)$ is given by multiplication in $\mathcal{B}_k^+$. The category $\mathsf{Br}^+(k)$ has a right action of Σ_k defined on objects and morphisms by sending $\alpha\colon A\to B$ to $\alpha\colon Ag\to Bg$ for $g\in\Sigma_k$. The operad structure map

$$\gamma\colon \mathsf{Br}^+(k)\times\mathsf{Br}^+(j_1)\times\cdots\times\mathsf{Br}^+(j_k)\to\mathsf{Br}^+(j_1+\cdots+j_k)$$

takes the tuple $(A,B_1,\dots,B_k)$ to

$$A(j_1,\dots,j_k)\circ(B_1\sqcup\cdots\sqcup B_k).$$

Here $A(j_1,\dots,j_k)$ denotes the canonical block permutation obtained from A by replacing the ith letter with j_i letters. The action on morphisms is analogous except for the obvious permutation of the indices. It is easy to check that the category of weak braided monoidal categories with weak braiding preserving strict monoidal functors is isomorphic to Br^+-algebras. See for instance the argument given in Section 5.1 in [SS14] for the braided monoidal version. We denote the category of Br^+-algebras by Br^+-*Cat*.

3. Homotopy colimits of weak braided monoidal categories

In [FSV13, Definition 4.10] there is a general homotopy colimit construction for a diagram of algebras over a *Cat*-operad. Let $\mathcal{L}$ be a small category and consider the category $(\mathsf{Br}^+\text{-}Cat)^{\mathcal{L}}$ of functors $\mathcal{L}\to\mathsf{Br}^+\text{-}Cat$ and natural transformations. The above mentioned construction gives in particular a functor

$$\operatorname*{hocolim}_{\mathcal{L}}{}^{\mathsf{Br}^+}\colon\ (\mathsf{Br}^+\text{-}Cat)^{\mathcal{L}}\to\mathsf{Br}^+\text{-}Cat.$$

3.1. **The homotopy type of the homotopy colimit.** Let $\mathcal{S}$ be the category of simplicial sets and let N be the nerve functor from *Cat* to $\mathcal{S}$. If we apply N levelwise to the *Cat*-operad Br^+, we get an operad $N\mathsf{Br}^+$ internal to the category $\mathcal{S}$. We denote the category of algebras over $N\mathsf{Br}^+$ as $N\mathsf{Br}^+\text{-}\mathcal{S}$. A morphism of $N\mathsf{Br}^+$-algebras is called a weak equivalence if the underlying simplicial set map is a weak equivalence. These are the weak equivalences in the standard model structure on $N\mathsf{Br}^+$-algebras, for a reference to the topological case, see for instance [SV91, Theorem B]. Given a diagram $W\colon \mathcal{L} \to (N\mathsf{Br}^+\text{-}\mathcal{S})$, let $\operatorname{hocolim}_{\mathcal{L}}^{N\mathsf{Br}^+} W$ denote the coend construction $N(-/\mathcal{L}) \otimes_{\mathcal{L}} QW$, where Q is an object wise cofibrant replacement functor in the category of $N\mathsf{Br}^+$-algebras. This is the homotopy colimit of QW from Definition 18.1.2 in [Hir03]. If X is in $(\mathsf{Br}^+\text{-}Cat)^{\mathcal{L}}$, then there is a natural map

$$\operatorname*{hocolim}_{\mathcal{L}}{}^{N\mathsf{Br}^+} NX \to N(\operatorname*{hocolim}_{\mathcal{L}}{}^{\mathsf{Br}^+} X),$$

see the paragraph before Definition 6.7 [FSV13]. This is an operadic version of Thomason's map in Lemma 1.2.1 [Tho79]. The question if this map is a weak equivalence or not, was left open in [FSV13]. Our main result provides a positive answer to this problem.

Theorem 3.2. *The diagram*

$$\begin{array}{ccc} (\mathsf{Br}^+\text{-}Cat)^{\mathcal{L}} & \xrightarrow{N} & (N\mathsf{Br}^+\text{-}\mathcal{S})^{\mathcal{L}} \\ {\scriptstyle \operatorname{hocolim}_{\mathcal{L}}^{\mathsf{Br}^+}}\downarrow & & \downarrow{\scriptstyle \operatorname{hocolim}_{\mathcal{L}}^{N\mathsf{Br}^+}} \\ \mathsf{Br}^+\text{-}Cat & \xrightarrow{N} & N\mathsf{Br}^+\text{-}\mathcal{S} \end{array}$$

commutes up to weak equivalence of $N\mathsf{Br}^+$-algebras.

The operad $N\mathsf{Br}^+$ is an E_2 operad, see Proposition 8.13 in [FSV13]. The above theorem gives one way to relate weak braided monoidal categories and E_2 spaces as seen in the corollary below. Fiedorowicz, Stelzer and Vogt obtain the same equivalence without using the homotopy colimit construction of Br^+-algebras in [FSV].

Corollary 3.3. *We have an equivalence of localized categories*

$$(\mathsf{Br}^+\text{-}Cat)[we^{-1}] \simeq (N\mathsf{Br}^+\text{-}\mathcal{S})[we^{-1}].$$

Proof. Theorem 3.2 shows that Theorem 7.6 in [FSV13] applies to the operad Br^+. The corollary then follows from the latter theorem with the added observation that the localization $(\mathsf{Br}^+\text{-}Cat)[we^{-1}]$ exists, see Proposition A.1 in [SS14]. □

By general theory (details will be provided later), the proof of the theorem reduces to showing that certain categories have the property that each connected component has an initial object. Fix an $A \in \Sigma_m$, a $B \in \Sigma_n$, and non-negative integers $r_1, \ldots, r_n$ such that $r_1 + \cdots + r_n = m$. Let $\tilde{B}$ denote the canonical block permutation $B(r_1, \ldots, r_n) \in \Sigma_m$ obtained from B by replacing the ith letter with r_i letters. We define a poset category $\mathcal{C}$ depending on A, B and $r_1, \ldots, r_n$. The objects in $\mathcal{C}$ are the positive braids $\alpha \in \mathcal{B}_m^+$ such that

$$p(\alpha)A\tilde{B} \in (\Sigma_{r_1} \times \cdots \times \Sigma_{r_n}) \subseteq \Sigma_m.$$

There is a morphism $\alpha \leq \beta$ from α to β in $\mathcal{C}$ if there exist $\gamma_i \in \mathcal{B}_{r_1}^+$ for $i = 1, \ldots, n$ such that $(\gamma_1 \sqcup \cdots \sqcup \gamma_n)\alpha = \beta$ in $\mathcal{B}_m^+$.

3.4. **Analysis of minimal positive braids in $\mathcal{C}$.** First we note that all references to the presentation of a braid monoid, will be to the standard presentation, see Example 2.2. We call an object ν in $\mathcal{C}$ a *minimal object* if for all objects ν' in $\mathcal{C}$, $\nu' \leq \nu$ implies $\nu' = \nu$. Define the norm $|\beta| \in \mathbb{N}_0$ of an element β in $\mathcal{B}_m^+$, as the length of any word representing β, see Section 6.5.1 in in [KT08]. This is well defined because each of the relations in the presentation identifies words of equal length. It is immediate from the definition that ν is a minimal object if and only if $\nu \neq (\gamma_1 \sqcup \cdots \sqcup \gamma_n)\nu'$ for all $\gamma_i \in B_{r_i}^+$ and all $\nu' \in \mathcal{C}$ with $|\nu'| < |\nu|$.

Proposition 3.5. *Given an object α in $\mathcal{C}$ there is a unique minimal object ν_α such that $\nu_\alpha \leq \alpha$.*

This proposition is the key ingredient in the proof of our main result. But before we prove either, we will derive some auxiliary results from the nature of the standard presentation of a braid monoid.

Let w and w' be two words representing the same positive braid. According to Section 6.1.5 in [KT08], w' can be obtained from w by a finite number of consecutive substitutions of the form

$$w_1 r w_2 = w_1 r' w_2$$

where $r = r'$ is one of the relations in the presentation. Observe that for each of the relations $r = r'$ in the presentation, r and r' contain the same letters, only the order and number of occurrences of each letter differ. This implies that a letter σ_i is in w if and only if σ_i is in w'.

For integers $r_1, \ldots, r_n$ adding up to m, consider $\mathcal{B}_{r_1}^+ \times \cdots \times \mathcal{B}_{r_n}^+$ as a submonoid of $\mathcal{B}_m^+$ consisting of the positive braids in $\mathcal{B}_m^+$ which can be represented by a word not containing the letters σ_{r_i} for $i = 1, \ldots, n-1$. The above discussion shows that $\alpha \in \mathcal{B}_m^+$ lies in $\mathcal{B}_{r_1}^+ \times \cdots \times \mathcal{B}_{r_n}^+$ if and only if no word representing α contains any of the letters σ_{r_i} for $i = 1, \ldots, n-1$. Also, a positive braid $\alpha \in \mathcal{B}_m^+$ does not lie in $\mathcal{B}_{r_1}^+ \times \cdots \times \mathcal{B}_{r_n}^+$ if and only if any word representing α contains at least one of the letters σ_{r_i} for $i = 1, \ldots, n-1$. This immediately implies the next lemma.

Lemma 3.6. *Given two positive braids α and β in $\mathcal{B}_m^+$ such that their product $\beta\alpha$ lies in $\mathcal{B}_{r_1}^+ \times \cdots \times \mathcal{B}_{r_n}^+$, then both α and β lie in $\mathcal{B}_{r_1}^+ \times \cdots \times \mathcal{B}_{r_n}^+$ as well.*

A right common multiple of two elements x and x' in a monoid M, is an element in M that is of the form $xy = x'y'$ for some y and y' in M. A right least common multiple of x and x' is an element $\mathrm{lcm}(x, x') \in M$ such that $\mathrm{lcm}(x, x')$ is a right common multiple of x and x', and such that any right common multiple of x and x' is of the form $\mathrm{lcm}(x, x')z$ for some $z \in M$. A unique right least common multiple of γ and γ' exists for any two positive braids γ and γ' on k strings, see Theorem 6.5.4 in [KT08]. Since we will only be dealing with right least common multiples, and not left least common multiples, the notation lcm will not be ambiguous. We will however use a subindex k to indicate that the least common multiple lcm_k is taken in the braid monoid $\mathcal{B}_k^+$.

Corollary 3.7. *Given $\gamma_i, \gamma_i' \in \mathcal{B}_{r_i}^+$ for $i = 1, \dots, n$, let $\gamma = \gamma_1 \sqcup \dots \sqcup \gamma_n$ and similarly $\gamma' = \gamma_1' \sqcup \dots \sqcup \gamma_n'$. Then the least common multiple* $\mathrm{lcm}_m(\gamma, \gamma')$ *in $\mathcal{B}_m^+$ of γ and γ' is equal to*

$$\mathrm{lcm}_{r_1}(\gamma_1, \gamma_1') \sqcup \dots \sqcup \mathrm{lcm}_{r_n}(\gamma_n, \gamma_n').$$

Proof. It is clear that $\mathrm{lcm}_{r_1}(\gamma_1, \gamma_1') \sqcup \dots \sqcup \mathrm{lcm}_{r_n}(\gamma_n, \gamma_n')$ is the right least common multiple of γ and γ' in the monoid $\mathcal{B}_{r_1}^+ \times \dots \times \mathcal{B}_{r_n}^+$. Now $\mathrm{lcm}_{r_1}(\gamma_1, \gamma_1') \sqcup \dots \sqcup \mathrm{lcm}_{r_n}(\gamma_n, \gamma_n')$ is a right multiple of both γ and γ', so we get

$$\mathrm{lcm}_m(\gamma, \gamma')\phi = \mathrm{lcm}_{r_1}(\gamma_1, \gamma_1') \sqcup \dots \sqcup \mathrm{lcm}_{r_n}(\gamma_n, \gamma_n')$$

for some $\phi \in \mathcal{B}_m^+$. Since the product $\mathrm{lcm}_m(\gamma, \gamma')\phi$ lies in $\mathcal{B}_{r_1}^+ \times \dots \times \mathcal{B}_{r_n}^+$, then so does $\mathrm{lcm}_m(\gamma, \gamma')$, and the result follows. □

Proof of Proposition 3.5. We first prove the existence of ν_α. If α is not a minimal object, there exists an object $\alpha_1 \in \mathcal{C}$ such that $\alpha_1 \leq \alpha$ and $|\alpha_1| < |\alpha|$. We repeat this process as many times as necessary until we obtain a minimal α_k with $\alpha_k \leq \alpha_{k-1}$ and $|\alpha_k| < |\alpha_{k-1}|$. The process terminates after a finite number of steps since the norm of the α_i's decrease strictly each time. We set $\nu_\alpha = \alpha_k$, so by construction $\nu_\alpha \leq \alpha$.

We now turn to the uniqueness of ν_α. Suppose there are two minimal objects ν_α and ν_α' such that both $\nu_\alpha \leq \alpha$ and $\nu_\alpha' \leq \alpha$. Then α equals both $(\gamma_1 \sqcup \dots \sqcup \gamma_n)\nu_\alpha$ and $(\gamma_1' \sqcup \dots \sqcup \gamma_n')\nu_\alpha'$ for some $\gamma_i, \gamma_i' \in \mathcal{B}_{r_i}^+$, $i = 1, \dots, n$. Abbreviating $\gamma_1 \sqcup \dots \sqcup \gamma_n$ to γ and $\gamma_1' \sqcup \dots \sqcup \gamma_n'$ to γ', we recall that

$$\mathrm{lcm}_m(\gamma, \gamma') = \mathrm{lcm}_{r_1}(\gamma_1, \gamma_1') \sqcup \dots \sqcup \mathrm{lcm}_{r_n}(\gamma_n, \gamma_n').$$

Since α is a right common multiple of both γ and γ',

$$\alpha = (\mathrm{lcm}_{r_1}(\gamma_1, \gamma_1')) \sqcup \dots \sqcup (\mathrm{lcm}_{r_n}(\gamma_n, \gamma_n'))\omega$$

for some $\omega \in \mathcal{B}_m^+$. The right least common multiple of γ_i and γ_i' is in particular a right common multiple of γ_i and γ_i', so $\mathrm{lcm}(\gamma_i, \gamma_i') = \gamma_i\phi_i = \gamma_i'\phi_i'$ for some $\phi_i, \phi_i' \in \mathcal{B}_{r_i}^+$, $i = 1, \dots, n$. Combining this we get that

$$(\gamma_1 \sqcup \dots \sqcup \gamma_n)\nu_\alpha = \alpha = (\gamma_1 \sqcup \dots \sqcup \gamma_n)(\phi_1 \sqcup \dots \sqcup \phi_n)\omega \quad \text{and}$$
$$(\gamma_1' \sqcup \dots \sqcup \gamma_n')\nu_\alpha' = \alpha = (\gamma_1' \sqcup \dots \sqcup \gamma_n')(\phi_1' \sqcup \dots \sqcup \phi_n')\omega.$$

The braid monoid injects into the corresponding braid group [FSV13, Theorem 6.5.4], so we can apply left cancellation to the above equations to obtain

$$\nu_\alpha = (\phi_1 \sqcup \dots \sqcup \phi_n)\omega \quad \text{and} \quad \nu_\alpha' = (\phi_1' \sqcup \dots \sqcup \phi_n')\omega.$$

It is straightforward to check that $p(\omega)A\tilde{B}$ is in $\Sigma_{r_1} \times \dots \times \Sigma_{r_n}$, so that ω is an object in $\mathcal{C}$. Then the above equations say that $\omega \leq \nu_\alpha$ and $\omega \leq \nu_\alpha'$ in $\mathcal{C}$. But since ν_α and ν_α' are minimal objects these maps have to be identities. This proves the uniqueness of ν_α. □

Lemma 3.8. *Each connected component in $\mathcal{C}$ has an initial object.*

Proof. Given a morphism $\alpha \leq \beta$ in $\mathcal{C}$, Proposition 3.5 associates to α and β unique minimal objects ν_α and ν_β respectively. The two objects must be equal since $\nu_\alpha \leq \alpha \leq \beta$, but ν_β is the unique minimal object with $\nu_\beta \leq \beta$. Hence the minimal objects associated to any two objects in the same connected component has to be equal, and we have a unique minimal object in each connected component of $\mathcal{C}$. The minimal objects are initial in their respective connected components. □

Fix an $M \in \Sigma_m$, an $N \in \Sigma_n$, and non-negative integers $s_1, \dots, s_n$ such that $s_1 + \dots + s_n = m$. The *factorization category* $\mathcal{C}(M, N, s_1, \dots, s_n)$, as defined in [FSV13, Section 6], has as objects tuples $(C_1, \dots, C_n, \alpha)$ consisting of $C_i \in \Sigma_{s_i}$ for $i = 1, \dots n$, and $\alpha \in \mathcal{B}_m^+$ such that

$$p(\alpha)M = \tilde{N}(C_1 \sqcup \dots \sqcup C_n). \tag{1}$$

A morphism from $(C_1, \dots, C_n, \alpha)$ to $(D_1, \dots, D_n, \beta)$ consists of elements γ_i in $\mathcal{B}_{s_i}^+$ for $i = 1, \dots n$ such that $(\gamma_1 \sqcup \dots \sqcup \gamma_n)\alpha = \beta$.

Lemma 3.9. *The factorization category $\mathcal{C}(A, B^{-1}, r_{B^{-1}(1)}, \dots, r_{B^{-1}(n)})$ is isomorphic to the category $\mathcal{C}$ considered in this section.*

Proof. Here $\tilde{B}^{-1}(C_1 \sqcup \dots \sqcup C_n) = (C_{B(1)} \sqcup \dots \sqcup C_{B(n)})\tilde{B}^{-1}$, so Equation (1) can be rewritten as

$$p(\alpha)A\tilde{B} = C_{B(1)} \sqcup \dots \sqcup C_{B(n)}.$$

This equation determines the C_i's uniquely given α with $p(\alpha)A\tilde{B}$ in $\Sigma_{r_1} \times \dots \times \Sigma_{r_n}$. The two categories therefore have isomorphic objects, and the morphism sets are easily seen to be isomorphic as well. □

Proof of Theorem 3.2. Together Lemmas 3.8 and 3.9 show that each factorization category has an initial object in each of its connected components. Thus the operad Br^+ satisfies the factorization condition [FSV13, Definition 6.8] and the result follows from Theorem 6.10 [FSV13]. □

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Department of Mathematics, University of Bergen, P.O. Box 7800, N-5020 Bergen, Norway

Email address: mirjam.solberg@math.uib.no

Graphic design: Communication Division, UiB / Print: Skipnes Kommunikasjon AS

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ISBN: 9788230845820 (print)
9788230840306 (PDF)