Spin-off learning about epidemics from modelling with differential equations

Aprendizagem subsidiária sobre epidemias resultante da modelação com equações diferenciais

Mette Andresen **0000_0002_9822_1439**
University of Bergen
Norway
Mette.andresen@uib.no

Abstract. This article presents a case study of students modelling epidemics with differential equation systems. The study introduces *spin-off* learning as a process intertwined with learning mathematics and examines the conditions for 'spin-off' learning to occur when modelling epidemics. Learning mathematics is conceptualised in terms of emergent modelling and models and identified by the associated progressive, horizontal and vertical mathematising in textual analysis. Signs of 'spin-off' learning are linked with the mathematising. Furthermore, the 'spin-off' learning is related to the direction of modelling being either expressive or explorative. As a result, the stratification of students' mathematising activities and the introduction of the notion of direction served to pinpoint a variety of conditions for 'spin-off' learning internal to the modelling processes. In addition, the results showed the huge potential in mathematical modelling for 'spin-off' learning which might be realised and utilised at a societal level for preparation against new crises.
Keywords: learning mathematics by modelling; emergent models; expressive and explorative modelling; differential equations models; epidemical concepts and notions.

Resumo. Este artigo apresenta um estudo de caso sobre estudantes que realizam a modelação de epidemias, usando sistemas de equações diferenciais. O estudo propõe a ideia de aprendizagem subsidiária ('spin-off') sobre epidemias como um processo entrelaçado com a aprendizagem da matemática e examina as condições para que essa aprendizagem ocorra durante a modelação de epidemias. A aprendizagem da matemática é conceptualizada em termos de modelação e modelos emergentes e identificada, por meio de análise textual, a partir da sua associação com a matematização progressiva, horizontal e vertical. Os indicadores de aprendizagem subsidiária estão ligados à matematização. Além disso, a aprendizagem subsidiária está relacionada com a direção, expressiva ou exploratória, que a modelação assume. Como resultado, a estratificação das atividades de matematização dos alunos e a introdução da noção de direção permitiram apontar diversas condições para que ocorra uma aprendizagem subsidiária no âmbito dos processos de modelação. Mais ainda, os resultados mostraram o enorme potencial da modelação matemática para a

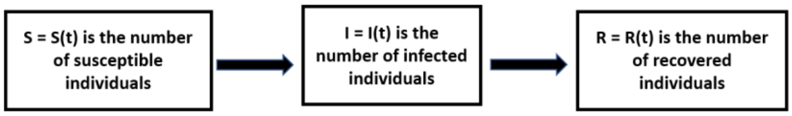
https://doi.org/10.48489/quadrante.23603
Received: February 2021 / Accepted: October 2021

aprendizagem subsidiária que pode ser alcançada e utilizada, no plano social, como preparação para enfrentar novas crises.
Palavras-chave: aprendizagem por meio da modelação; modelos emergentes; modelação expressiva e exploratória; modelos de equações diferenciais; noções e conceitos epidemiológicos.

Introduction

This article presents a case study of students modelling epidemics with differential equation (DE) systems. This study took place with a group of Norwegian teachers in a masters' program that requires 60 ECTS in mathematics and two years of professional practice as a mathematics teacher[1]. The teachers, in the following called "students", attended the course *Modelling in and for mathematics teaching and learning* which is a 15 ECTS course in a masters' programme in mathematics education. In cases of *learning by modelling*, learning would most often refer to either learning mathematics or learning about/developing competence in modelling (see Blomhøj and Ärlebäck, 2018). In the present study, the learning of mathematics was, in contrast, conceptualised in terms of emergent models, by vertical and horizontal mathematising (Gravemeijer & Stephan, 2002).

The article introduces the term *spin-off* learning, referring to the students building model-associated knowledge via the modelling process. The study aimed to identify conditions for such *spin-off* learning to take place, internal to the modelling process. The notion *direction of modelling* was introduced to account for students' work with already existing models like SIR (Susceptible, Infected and Recovered individuals, see Figure 1). Direction of modelling refers to the distinction between expressive and explorative modelling (see below).



With parameters α for the infection rate and β for the recovery rate is the SIR model expressed by the system:

$$\begin{cases} dS/dt=-\alpha SI \\ dI/dt=\alpha SI-\beta I \\ dR/dt=\beta I \end{cases}$$

The SIR model can be modified to take into account factors such as incubation time or vaccination

Figure 1. The SIR model

During the COVID-19 pandemic, the public knowledge about epidemics was perceived as an important prerequisite in Denmark for the authorities' gaining control over the corona-virus (HOPE, 2020). People's compliance with COVID-19 regulations, willingness to acknowledge restrictions on their freedom and to be vaccinated (95% acceptance in Denmark), all rested on the authorities' trustworthy health communication which seemed to be successful: the share of people who feel well informed about the background for restrictions and so on was more than 60% (Lindholt, M. F., Jorgensen, F., Bor, A., & Petersen, M. B. (2020)) during the pandemic in Demark, and more than 95% have felt knowledgeable about COVID-19 (HOPE, 2020). The COVID-19 pandemic exemplifies how public knowledge about a mathematical model's mechanisms may play a decisive role. Epidemic models can offer insights, crucial for public vaccination programs, not only in Denmark. In a similar manner, public knowledge about, for example, the mechanisms of Energy Balance Climate Models, with derived subsections concerning CO_2 emissions and food climate imprint, may play a decisive role for the green transition and the fulfilment of long-term climate policy objectives. Hence, important *big mathematical models* could and should be considered an element of general education and integrated into the school curriculum.

Theoretical framework

In this section the basic idea of emergent models is presented and operationalised into a tool for textual analysis. *Spin-off* learning is introduced, as well as the notion of conditions for it to take place. The notion of *direction of modelling* is introduced and included in the tool, to take students' work with ready-made models into account in the analysis.

Emergent mental models of mathematical concepts and relations

In the study, modelling was expressed in terms of *emergent models* (Gravemeijer & Stephan, 2002). The notions of emergent models and modelling originated in Realistic Mathematics Education (RME) which sees mathematics as a human activity (see Freudenthal, 1991). One of the basic principles of RME was students' horizontal and vertical mathematising, described by Gravemeijer and Stephan (2002) as the passing of four levels of activity where a new mathematical reality is created at each level. The levels of activity were:

1. *situational level* with descriptions in natural language and own wording;
2. *referential level* where a *model of* is created and inquired. A *model of* is identified by the students' use of situation related terms and half-way formalised explanations, for example, that "the number of sick persons will grow exponentially over time";
3. *general level* with creation and handling of a *model for*. A *model for* is identified by the students' use of general expressions and terms with no visible relation to the

situation, for example, that “We find that the graph of I(t) hits the maximum value if the parameter has a value of 0.259”;

4. *formal level* with general reasoning and considerations. For example, considerations and reflections about modelling with DE.

Horizontal mathematising is described as the passing from first to second level, and vertical mathematising as passing further up. This four-layer model was the basis for the design heuristics of emergent models that aim to support the students’ processes of emergent modelling (Gravemeijer & Stephan, 2002). Gravemeijer (2020) described emergent modelling as an incremental process in which models and mathematical conceptions co-evolve. Central to the emergent modelling design heuristic is the use of a series of sub-models. Together they substantiate an overarching model which develops from a model of informal mathematical activity to a model for more formal mathematical reasoning. The overarching model is mental, with Gravemeijer (2020) mentioning, for example, the concepts of *distribution* and *function*. In this view, modelling is not separated from mathematics nor from *reality*. The two goals, modelling for the sake of mathematics and mathematics for the sake of modelling, mentioned by Niss and Blum (2020), are here intertwined.

According to Cobb (2002, p. 193) the four-layer model might “facilitate . . . the analysis of mathematical learning in instructional situations . . . The explication of a mapping between a situation and a model might then be viewed as a description of the way that the situation became structured during modelling activity”. Based hereon, the four-layer model was in this study operationalised into a tool for textual analysis of reports to interpret the students’ mathematical concept formation. Signs, displayed by the wording in reports, of students’ activity were stratified regarding the levels. The progressive mathematisation, then, was detected as progressive variation between the levels and interpreted as steps of the students’ concept formation, in the form of sub-models evolving into an emergent model. Accordingly, the mathematical learning outcome was conceptualised as emergent models of essential mathematical concepts, following Cobb (2002). In this study of modelling with DE, the essential mathematical concepts encompassed solution, slope field, equilibrium solution and nullclines. The goal, in the form of an overarching (mental) model, was the mathematical idea to model an epidemical reality using DE, for example including versions of the SIR model.

Spin-off **learning**

The outcome of the students’ *spin-off* learning in the form of knowledge about epidemics was closely related with the specific mathematical model. The students did not gain substantial medical knowledge. The introduction of medical facts and concepts and the links between them happened via the model and interpretations of it, rather than via observations or medically substantiated considerations. In contrast with the conceptualisation of mathematics learning described above, the learning was, therefore, conceptualised in the present study as building model-associated knowledge in the form of facts and concepts detached from a coherent, theoretical foundation and independent of authentic experience.

The *spin-off* learning about epidemics, hence, means to become familiar with central epidemic terms via the mathematical model. The distinction between the conceptualisation of mathematics learning and that of learning about epidemics, and the asymmetric intertwinement between the two, justifies the use of the term *spin-off*.

Inspired by insight into the HOPE project (HOPE, 2020), this article advocates for a stronger focus on the importance of the theme for teaching mathematical modelling. In line with this and seen from a mathematics education perspective, the notion of *spin-off* learning is closely related to the discussion about goals for modelling (Niss & Blum, 2020).

Traditionally, five types of arguments have been invoked for the inclusion of modelling in school mathematics: the formative argument, the critical competence argument, the utility argument, the *picture of mathematics* argument and the *promoting mathematics learning* argument (Blum & Niss, 1991). The notion of *spin-off* learning has potential to nuance and enrich the discussion of the critical competence and the utility argument by focusing on the relations between societal issues and modelling as a learning activity. For example, any argument for students' becoming critical to the use of growth-models should take the *spin-off* learning into account regarding the theme for modelling. Likewise, the utility argument may be supported and sharpened by examining the actual *spin-off* learning.

Conditions for *spin-off* learning

The aim was to identify conditions for collective *spin-off* learning in a group of students about epidemics. Conditions are understood to be circumstances internal to the modelling processes. This is in contrast with possible influences from external societal factors (e.g., classroom culture and students' socio-economical or ethnic background) and in contrast also with effects of the individuals' psychological or mental dispositions.

To identify conditions for *spin-off* learning, the study aimed to link signs of the students' knowledge about epidemics with their emergent modelling. The qualitative textual analysis of the reports would register signs of *spin-off* knowledge about epidemics when the text contained: i) convincing (i.e., first person) wording of concepts and notions from the area; and ii) convincing descriptions, arguments and explanations involving such concepts and notions. For each sign of *spin-off* learning, plausible links with mathematical terms and concepts were considered. Besides the level of activity (a), the conditions for *spin-off* learning were also examined regarding the modelling direction (b).

Two directions: expressive and explorative modelling

This study draws on a distinction between two *directions* of students' modelling: *expressive* and *explorative*. The students were expected stepwise to establish an overarching model of DE epidemic models in a process, contrasting with the process of picking out a ready-made model, and determine values of its constants to give the best fit with data. However, the students would not be able to do this purely based on progressive mathematising and formation of sub-models. Therefore, they were free to involve ready-made models like SIR and its modifications. In contrast with the distinction by Niss and Blum (2020) between descriptive and prescriptive modelling, which concerns more general aims for the relations between the model and the extra-mathematical domain, this distinction between expressive and explorative modelling relies on aims internal to the process:

i) Expressive modelling is understood as a student's activity aiming to capture a problem through a mathematical model, be it ready-made or under creation. Expressive modelling encompasses the student's own expression, in mathematical terms, of quantities and relations in connection with some sort of problem solving. Depending on the starting point this may involve what Niss and Blum (2020) refer to as implemented anticipation where the student needs to imagine in advance what elements and relations they can mathematise. The important point is that the model is perceived by the students as their own, novel creation. Expectedly, expressive modelling would support learning mathematics: expressive modelling is in harmony with Niss' description of a Danish mathematics education narrative of students who are expected to work intellectually independently (Niss, 2020), but also with the design heuristics of emergent models which build on the principle of guided reinvention (Freudenthal 1991; Gravemeijer & Stephan, 2002; Selter & Walter, 2020).

ii) Explorative modelling is understood as an activity aiming to explore a mathematical model, be it ready-made or under creation. It includes testing the model against data with the aim of evaluating the model. Testing may imply the interpretation in an actual context of the model, parts of its mechanisms or of its output. The important point here is that for students, the mathematical model is pre-existing, although it may be new to them. In the case of exploring a ready-made model, the interpretations would, expectedly, support *spin-off* learning. A simplified version of explorative modelling is widespread in school mathematics modelling tasks, where the task presents a ready-made model and asks the students to find the results of a given input, according to Berget and Bolstad (2019).

During the study's analyses, the internal aims were identified, either by having it explicitly stated in the text or based on an interpretation of the textual context. The notion of directions of modelling was introduced and included in the study's analysis to capture the role of ready-made models. The introduction intended to nuance the analysis and make it more fine-grained regarding the students' learning mathematics and their *spin-off* learning under the two directions of modelling.

Methodology

The study aimed to throw light on the research question: *What are the conditions for spin-off learning from students' mathematical modelling in the case of modelling epidemics?*

This section gives a basic introduction of the data for the study and an outline of the guidelines for the textual analysis of four cases of students' modelling.

Data

The DE part of the course was based on Blanchard et al. (2002) which progressively build up the DE models and examples, and balance between qualitative, quantitative, and numerical methods. Hereby inspired, groups of students (2-3 persons) under sparse supervision were asked to formulate, complete, and present a project that encompassed a simple DE model, and to report the project. The educational purpose was double: 1) to learn about DE by doing a modelling project; and 2) to gain personal experiences with learning mathematics from doing a modelling project. Each group chose the DE model they wanted to study. The course's individual examination included an interview about this project (10 minutes).

This study analyses four cases of such students' group reports dealing with modelling epidemics, including the compartment model SIR (Figure 1). The four cases were picked out of 17 reports prepared in 2014–2019 and evaluated by the author. The other 13 reports concern other DE models. Before the course started, the students were not familiar with modelling; neither did they know the DE models in advance. None of the students had prior knowledge about epidemics.

In the study's four cases, the students' collective learning processes were documented by their own descriptions of the mathematical modelling activities, and by the reflections reproduced in the reports. The descriptions and reflections were reported in a convincing way: for example, they used a first-person perspective in their writings in the report which is mentioned as a sign of being an active learner by Ju and Kwon (2007), and their reflections were reproduced using plain language, and their own words. Descriptions of their learning pathways also seemed convincing in the sense that the students, according to their wording, gradually became familiar with the mathematics involved. In all four cases, this impression of validity in the documentation was supported by the interviews conducted during the individual examinations. The reliability of the analysis rested on the condensation of meaning from units of the convincing texts in accordance with the qualitative methodology described in Kvale (2001). The meaning of each unit was interpreted in accordance with the emergent modelling framework.

Analysis

Each of the four projects in the reports was analysed as a case of modelling. In the initial summary of the case, the students' overall strategy and direction of modelling were identified.

Based on condensing the meaning of naturally delineated subsections of the text, the direction of modelling was assigned to the associated modelling subsequences. The delineation of subsections did not divide the text into disjointed classes. For example, a subsequence of explorative modelling included in a larger expressive modelling activity would count as a unit. The subsection's direction was labelled either a) expressive modelling, i.e., the process was driven by the problem, or b) explorative modelling, i.e., the model was the starting point.

The textual analysis, presupposing that the students' mental activities were reflected by the wording in their reports, also served to stratify the mathematical learning process in terms of passing through the four levels, situational, referential, general, and formal. Each mathematical concept or relation in the text was assigned to a level of activity in the four-layer model (Gravemeijer & Stephan, 2002).

The marking served to stratify the signs of mental activity into the four levels of progressive mathematising and, in parallel, to identify signs of *spin-off* learning. Such signs were identified, and marked, in the form of statements with non-mathematical epidemic terms as described above.

Analysis of data

In each of the four cases, the analysis was initiated by sketching the students' modelling pathway in the project. Next, a table was produced containing the smallest units of text, naturally delineated after its meaning and the direction of modelling. Each unit was marked by a) level of activity in the four-level-model, b) direction of modelling, and c) content, i.e., mathematical concept and notion, non-mathematical epidemic terms, and comments. The following sections present each of the four cases through an initial summary of the report followed by shortened (chronological) excerpts from those tables, chosen to illustrate and justify a final summary of the cases' findings. The excerpts were chosen to demonstrate a variety of combinations of direction and level of activity which form the conditions for *spin-off* learning. This was on the expense of the possibility to trace a learning trajectory for individual mathematical concepts and terms.

Case 1. 15 pages report: *Analysis of models of influenza outbreak at a boarding school in England*

The modelling project was based on data from the textbook. The students took the model (SIR) as their starting point rather than the problem (epidemic at the boarding school). Their strategy was to choose the model and fill in data to explore it, rather than to assert or hypothesise relations between factors and figures in expressive modelling, which could be tested against data. In the second part of the report, the students explored the problem (not the model) via a vaccination-modified version of the SIR model by use of analytical methods. They had solved initial value problems and found threshold values, analysed equilibrium points, and found nullclines in the SIR model with vaccine. Finally, they interpreted the results and gave meaning to the models' parameters. They expanded the model to SEIR (SIR with an incubation period) and adjusted the parameters to fit the model with data. Then, they set up the Jacobi matrix and found eigenvalues and eigenvectors. They reduced the dimension of the system from three to two, sketched nullclines and summed up their results, while giving meaning to and interpreting their results (Table 1).

Table 1. Marked units from case 1

Case 1				
	Excerpt (translated from Norwegian by the author)	**Level of activity:** **1) situational** **2) referential** **3) general** **4) formal**	**Explorative or expressive modelling**	**Sub-models, concepts and notions, comments**
1.1	We assume that pupils at boarding school belong to one of the three groups S, I or R	The SIR model is at referential level, 2) → 1)	The model is the starting point, and it is applied to the situation	The SIR model is linked with the situation
1.2	This leads to the following model: Equation 1: $dS/dt = -\alpha SI$ Equation 2: $dI/dt = \alpha SI - \beta I$ Equation 3: $dR/dt = \beta I$	This is at general level, 3)	SIR model explored	SIR model
1.3	The parameter α describes how quickly the population gets infected, while β describes how quickly the infected get healthy	Interpretation into the situation of the *model for*, 3) → 1)	The SIR model explored	The infection rate α, the recovery rate β
1.4	After 6 days, the peak for I(t) appears to be 282763≈0.37 when β = 0.44 (see Figure 1)	General terms, except one use of the term 'days' for time 2), 3)	The SIR model explored	Recovery rate β

1.5	The equilibrium point (0,0) is after an epidemic, when R = 1. The equilibrium point (1.0) implies a closed system where no one is an infectious agent and therefore becomes a disease-free system	Interpretations of the general results into referential level (the terms R = 1) and into situation, 2)	Application of results on situation can be a test of the model: Explorative	Equilibrium points, development of the epidemic, disease-free system
1.6	Vaccination will then be a factor that only affects S(t) (...). The starting model from section 2.1 can then be rewritten as: $dS/dt = -1.66SI - \mu S$ $dI/dt = 1.66SI - 0.44I$ $dR/dt = 0.44I + \mu S$	The first part is referential, the second part mathematises into general level, 2), 2) → 3)	The SIR model is expressively changed to fit with the new situation with vaccinations	Vaccination, modified SIR model
1.7	Analysis of the point (0,0) gives the matrix M1. $M1 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.44 \end{bmatrix}$. The matrix has zero along the secondary, the eigenvalues will be $\lambda 1$= - 0.1 and $\lambda 2$= - 0.44	General level, 3)	Exploration of details of the SIR model with vaccine	Jacobi matrix, eigenvalues
1.8	(...) vaccination pays off to keep students healthy. Both Figures 6 and 11 show that the maximum value for I is just under 100 infected after six days compared to 282 without vaccination	Interpretation of results from explorations at the general level into the situation, 3) → 1)	Focus on the model rather than on the situation, the model is explored	Effects of vaccination

Case 1 summary

The group's starting point from the model dominated their working style, which was mainly explorative, and included activities at levels 1), 2), and 3). They managed to link their work at the general level and referential level closely with the situation (illustrated by exc. (excerpts) 1.1; 1.3 and 1.8, Table 1). They unfolded the meaning of the model's terms and mechanisms in plain language during the report (illustrated by exc 1.2, 1.5, 1.6, and 1.7). This demonstrates that the explorative modelling produced *spin-off* learning about non-mathematical concepts and notions (for example, infection rate and recovery rate), intertwined with the mathematisation up to general level (found in exc 1.2, 1.3, 1.4, 1.6, 1.7). Incidents of expressive modelling (like in 1.6) also link the situation with the referential level in the form of the *model of* with vaccination in a way that demonstrates knowledge about vaccination.

Case 2. 32 pages report: *Project assignment in differential equations*

The students in Case 2 took the problem of epidemic diseases as their starting point; the overall goal was to model the Ebola epidemics in Congo 2019. Firstly, they examined mechanics of the SIR model without data (qualitative inquiry of the model, explorative), and with random variations of its parameters. During this investigation, they became familiar with the meaning and impact of the main parameters of the model. To further investigate the SIR model (explorative), the students thereafter filled in data from the textbook from the boarding school. They concluded that the SIR model was suitable for modelling this epidemic. Next, they wanted to see whether it might be useful to also model the ongoing epidemic in Congo (SIR was here a tool for expressive modelling of the epidemic). They discarded the SIR model because it was too simple and turned to the SEIR model (SIR expanded with incubation period). They concluded that, when comparing the application of SEIR with the application of SIR on data, the only difference was the time span of the epidemic (Table 2).

Table 2. Marked units from case 2

	Case 2			
	Excerpt (translated from Norwegian by the author)	**Level of activity: 1) situational 2) referential 3) general 4) formal**	**Explorative or expressive modelling**	**Sub-models, concept and notions, comments**
2.1	We can thus set up the following expression for the change in the number of susceptible with respect to time: $\frac{dS(t)}{dt}=-\alpha S(t)I(t)$	*Change in number of* is to some degree already mathematised, hence part of the *model of*: referential level 2) → 3)	Expressive; the model of is mathematised further to express the mechanism	Change of S formalised in terms of s, I, t and α Susceptible
2.2	A large α corresponds to a population where there is a lot of interaction and a disease that has a high risk of infection	Interpret elements of the *model for* into the situation: 3) → 1)	Explanation of the meaning of α; exploration of the model	α: infection rate parameter
2.3	We can also set up the change in R(t), the recovered (or dead): $\frac{dR(t)}{dt}=\beta I(t)$ which gives how fast people go from I (t) to R (t)	*model of* transformed into *model for*; then *model for* interpreted at referential level: 2) → 3) → 2)	Expressive: the change expressed in formal terms	Removed, Infected, β: recovery rate
2.4	We find that the graph hits the maximum value if the parameter has a value of 0.259	General level, 3)	Exploration inside the SIR model	Maximum number of infected, value of α (infection rate parameter)
2.5	If this is true, then the reproduction number is $R_0=\frac{\alpha}{\beta}=\frac{0.259}{0.0714}\approx 3.6$ for this flu	*Reproduction number* is referential,	Exploration inside the SIR model	Reproduction number, infection

		2)		rate and recovery rate
2.6	In comparison, smallpox has an R_0 between 3 - 5, (...)	Comparison of element of *model of* with situation: 2) → 1)	(Indirectly) exploration or test of the SIR model	Characteristics of other epidemic diseases
2.7	We assume 14 days of illness, so about 76/14 = 5.4 new sick people per day Of course, this is an estimate, but it gives us a value for *dI/dt* we can use to estimate α	Referential level, 2)	Expressively modelling the epidemic (in Congo)	Time span of being sick, number of new sick per day
2.8	We assume that the rest of the population is in the S group. (...) not all 11 million people (...) are exposed to infection	Changes between situational and referential level, 1) ⇔ 2)	Adjusting the terms and conditions	Population's behaviour
2.9	The parameter α has much impact on the development of the epidemic. If this parameter is set a little lower, the course will be quite different	*Impact* (..) is understood to be in the SIR model; *model of*, 2)	Exploration of the SIR model	The parameter α, dynamics of the SIR model

Case 2 Summary

The group's expressive modelling of the epidemic with SIR was supported by use of graphs (exc. 2.4, 2.9). Concepts and notions at referential level were present both in the expressive (exc. 2.1, 2.3, 2.7, 2.8) and in the explorative parts (exc. 2.2, 2.5, 2.6, 2.9) of the group's work. Explanations in plain language were intertwined with mathematisations up to the general level (exc. 2.1, 2.3, 2.4). The explanations in plain language of non-mathematical concepts and notions about epidemics should be interpreted as signs of *spin-off* learning.

Case 3. 19 pages report: *Development of HIV-infected persons in the world*

In the first part of Case 3, the students wanted to describe the development over time of the global population of HIV-infected people. Their strategy was to test different models on data and evaluate them. This was an overall expressive modelling strategy that included sequences with explorations of the involved models. They started with logistic growth and used regression to find solution curves for various initial values and values of the model's parameters. Next, they tried out polynomial regression, but discarded the model for being too simple and then turned to the SIR model. The SIR model was evaluated qualitatively and quantitatively. The qualitative analysis included determination of equilibrium points and nullclines, and classification of the equilibrium points by linearisation. They found the Jacobi matrix, eigenvalues, and eigenvectors. In this part of the report, they completed an analytical analysis. This part was bare technical with no references to HIV. They determined the roots of the characteristic polynomial by graphs. For the quantitative analysis, they calculated each of the two parameters (infection rate and recovery rate) separately and gave a brief interpretation of the results in the summary (Table 3).

Table 3. Marked units from case 3

Case 3				
	Excerpt (translated from Norwegian by the author)	**Level of activity: 1) situational 2) referential 3) general 4) formal**	**Explorative or expressive modelling**	**Sub-models, concept and notions, comments**
3.1	We want to make different models and evaluate which one of them will give the best picture of the situation.	Situation, 1)	Expressive modelling	Modelling epidemics
3.2	We assume that the growth in HIV infected is proportional to the number of HIV-infected in the beginning, (…). The differential equation is: $I' = -k(M-I)\cdot I$ (1) The general solution to this differential equation is $I(t) = \frac{M}{\mathrm{C \cdot e - Mkt+1}}$ (2)	Proportionality is in the *model of*, Referential level to general, 2) → 3)	Mathematisation of the features of the population of HIV-infected; expressive	Prerequisites for logistic growth (LG) models, Differential equation, general solution
3.3	Following this model [Figure 2], the maximum number will be around 34.6 million of HIV-infected in the world since when $t \to \infty$ will $2{,}7e^{-0{,}25t} \to 0$.	Interpretation of graph of the *model for* into situation, 3) → 1)	Application of the LG model and kind of hypothetical verification of the model	Development of LG
3.4	(…). We should take this into account in further processing of the project	Interpretation of input data for the DE model, 1) → 2)	Expressive modelling	Development over time of group of HIV-infected
3.5	We would like to investigate whether another model for $I\,(t)$ can be a better model for the development of people living with HIV. (…). Then there will be differential equations *dI/dt* = $N(t) - R(t)$ (5)	Mathematising the situation, 1) → 2)	Expressive modelling	Living with HIV (I), new cases (N), death by AIDS (R), Differential equation *dI/dt*
3.6	HIV infection and AIDS will not be a completely traditional SIR system since one cannot recover from the disease, just keep it in check. We then get this system: (…)	Mathematisation from situational to referential level and further, from referential to general,	The situation is the starting point for the use of the SIR model to express the development of	Dynamics of the development of the group of HIV-infected over time, The SIR model

Since the entire population is either uninfected, infected or dead, we get that $R(t) = 1 - S(t) - I(t)$, and the system can be rewritten to: $dS/dt = -\alpha SI$ $dI/dt = \alpha SI - \beta I$ (6)	1) → 2) → 3)	the group of HIV-infected.

Case 3 summary

The group's modelling direction was overall expressive and the mathematisation of the situation into a *model of* was described in plain language (exc. 3.1, 3.4, 3.5, 3.6). Their choice of problem (HIV-infected) led to discussions in plain language about the characteristics of an epidemic intertwined with the mathematisation up to the general level (exc. 3.2, 3.6). Interpretations in plain language of results from the *model for* (exc. 3.3) about HIV infection and AIDS should be interpreted as signs of *spin-off* learning about non-mathematical concepts and notions.

Case 4. 15 pages report: *The Ebola virus*

In Case 4, the students studied Ebola modelled by the SIR model. They took as the starting point a diagram illustrating mathematical modelling, with five stages (phenomenon from reality, formulation of mathematical model, analysis of the model, interpretation of the results, and test of the model). These five stages structured the main part of the report, which was followed by two final paragraphs about i) possible expansions of the use of the model, and ii) conclusion. Although the diagram starts with the real-world problem, the group choose the SIR model from the beginning, and explored it qualitatively in the paragraphs on formulation and analysis (including the determination of equilibrium points and Jacobi matrix, etc.) of the model. During the formulation and analysis, they explained, interpreted, and discussed relevant non-mathematical terms. They continued their exploration of the SIR model with a step-by-step determination of its parameters based on data from WHO and others. To test the model, they included data from the Ebola epidemic in Liberia but recurrently, they found discrepancies in some parts of the development of the epidemic. They reproduced explanations and reflections that pointed to key features of their model, distinct from the situation. Finally, they discussed the applicability of the SIR model to other problems and elaborated on the idea to model a case of residents who are intoxicated, with high risk of becoming addicted, where some groups are exposed to drugs, and where the situation is contagious to others in the form of recruitment, whereas some rehabilitation measures have been established. The conclusion reproduced reflection about modelling as such, in addition to discussion of the features of Ebola and the epidemic in terms of mathematised concepts and non-mathematical terms (Table 4).

Table 4. Marked units from case 4

	Case 4			
	Excerpt (translated from Norwegian by the author)	**Level of activity:** **1) situational** **2) referential** **3) general** **4) formal**	**Explorative or expressive modelling**	**Sub-models, concept and notions, comments**
4.1	In our case, we focus on the SIR epidemic model that concerns the Ebola virus. From this we form a simplified model where we show the relationship between M = susceptible, I = infected, F = removed	Ebola is identified with the SIR model. Referential level, 2)	Study of Ebola becomes reproducing and exploring the SIR model	SIR: susceptible, infected and removed
4.2	We therefore want to take a closer look at the virus, find out how it can spread and try to create a possible model to *predict* possible outcomes. (…). We want to start from the simplest model possible and try to develop our model from there	Situational level, 1)	The intention is to model expressively	Non-mathematical facts and features about Ebola
4.3	(…) assumptions: 1) We assume that the population is constant (…). 2) The population is spread equally over a geographical area. (…). 3) The virus kills a fixed percentage of those infected. 4) People do not panic and do not take infection-reducing measures	Link from referential level to situation, 2) → 1)	Explorative: the prerequisites for the SIR model are interpreted in the situation – this interpretation gives information and understanding about the SIR model, rather than about the situation	Prerequisites for the SIR model: Size of the population is constant A constant for the contact between sick and healthy Virus kills a fixed percentage of the infected
4.4	Therefore, we assume that the probability of one susceptible person being infected is proportional to the number of infected, and must therefore multiply by M to find the total rate	Situation to model of, 1) → 2)	Expressive modelling	Dynamics of the model
4.5	$M < \frac{a}{b}$, hence $M'<0$, meaning that the epidemic shrinks	3) → 1)	Explorative: interpretation of *model for*	Mechanics of the SIR model, *a* and *b*
4.6	Here we quickly see that we have 4 options:	Situational level: 1) Number of susceptible (M) is part of the	Explorative: Interpretation of *model of*	Infection rate, number of susceptible, vaccination,

1) By reducing the infection rate (*b*) 2) By reducing the number of susceptible (M) by vaccinating people 3) Increase *a*, i.e., increase the healing rate. 4) Wait! The epidemic will eventually run out of susceptibility, thus die out	*model of*, that is referential level linked with situational level: 2) → 1)	healing rate, antibiotics Long term development of the epidemic (it dies out)

Case 4 summary

The focus on the SIR model and on modelling processes implied explorative group work; they explored the SIR model through modelling (exc. 4.1, 4.3, 4.5, 4.6). The explorations alternated with expressive modelling (exc. 4.2, 4.4) when mathematising the situation and establishing the *model of*. The exploration of the SIR model involved interpretations in normal language of the model's mechanisms and its mathematised elements. Thereby, the students unfolded the mathematised elements from both referential and general levels into situational level. The explanations should be interpreted as signs of *spin-off* learning about non-mathematical concepts and notions.

Spin-off learning in the four cases

Table 5 contains the concepts and notions present in one or more of the four reports in a form that fulfils the criteria concerning plain language, own words etc. mentioned above.

Table 5. Learning outcome about epidemics

Concepts and notions included in the *spin-off* learning
About epidemic (and other) diseases
• *Obtained immunity*, meaning that a recovered person does not get infected again
• *Incubation time*, meaning the time it takes from being infected until the disease breaks out
• How *contagious* a disease is, the characteristic of the disease that tells how many other people each person is estimated to infect
• *How others are infected*, by drop infection, airborne or other
• *When to infect*, i.e., it is a characteristic of the disease when in the course of being ill you can infect others
• *Symptoms*, and *absence of symptoms*, meaning signs of being ill and instances of being ill without any external signs
• *Antibodies* that confer immunity to a disease
About progress of epidemics over time
• What does it take to call it an *epidemic*, and what to talk about a *pandemic?*

• Which of those factors that *determine* the development of an epidemic are in the form of *internal characteristics* of the disease?

• What are the *determining* factors in terms of *external characteristics* such as the population's age distribution, geographical distribution, ethnicity, and the country's climate?

• What are the *determining* factors in the form of *controllable behaviour and measures*, such as hygiene standards, information campaigns, access to vaccines and funeral rituals?

Summary of the results

Each of the four case comprises *spin-off* learning, detected as wording in plain language and in the students' own words of concepts and notions, listed in Table 5. In the cases, the students take distinct approaches to the modelling project regarding starting points, and they span from being mainly explorative to mainly expressive. The *spin-off* learning is intertwined with the mathematisation, and the result of this study points to different conditions for the *spin-off* learning, internal to the modelling process.

Horizontal mathematising in expressive modelling

Mathematising from situational to referential level is horizontal mathematising. When students build up the *model of*, as part of a sequence of expressive modelling, they presumably need to seek information about the problem they want to study. Horizontal mathematising in expressive modelling implies that the students, in principle, have no idea about what entities, relations and factors to include in the model. In the case of epidemics, though, the compartments in the form of populations of susceptible, infected, and recovered (or removed) persons, and concepts of infection rate and recovery rate, will appear in the students' first search. In continuation of the modelling sequence other aspects are discovered to be included, e.g., how Ebola is spread from infected people in exc. 4.2 and how the population mingle in exc. 2.8.

Testing the model against data

Testing a model against data can imply a shift from general or referential activity level to situational level in the four-layer model. In connection with testing, the students may be encouraged to think about reasons for a bad fit and, therefore, seek more information about the problem. Testing or validating the model may be part of an explorative modelling sequence with the aim to investigate the model. It may also be a subsequence of an expressive modelling sequence with the overall goal to find a model that captures the problem (e.g., the number of HIV-infected in exc. 3.4).

Interpretation of a model for application

Application of a mathematical model on data can also imply a shift from general or referential level to situational level. The application may be part of an exploration of the model, i.e., part of explorative modelling, or it may be part of an inquiry of a situation represented by data, i.e., part of expressive modelling.

For application of a ready-made mathematical model like SIR on a real-world problem, the students must in principle be able to interpret its terms. In the case of reporting from the modelling process, the students must be able to calculate a result (e.g., finding the maximum number of infected and the value of the infection rate parameter in exc. 2.4), to explain the meanings in plain language and in their own words (e.g., explaining the meaning of the infection rate parameter in exc. 2.2), and be able to explain the model's mechanisms (e.g., explain in plain language what the prerequisites for the SIR model mean in exc. 4.3). The request for reporting hence forces the students to look further into the mathematical model and its implications.

Interpretation of a result by unfolding mathematised entities

When results are interpreted in a situation, either from processing a *model of* or a *model for*, the students are mapping the model with the problem in the real world and unfolding the mathematised entity. The student can bring such results with him or her in daily life (e.g., how to prevent the spread of an epidemic in exc. 4.6 and vaccination pays off in exc. 1.8, respectively). The students can get further insights by variation of the model's parameters or mechanisms producing other results.

Conclusions and perspectives

This final section briefly examines the article's four main aspects separately with the aim to evaluate the potentials of each one.

Operationalisation of the study's conceptualisation of learning

In this study, the learning of mathematics was conceptualised in terms of emergent models and the passing through four levels of mental activity associated with horizontal and vertical mathematising and the design heuristics of RME. The operationalisation of this into a tool for analysis of learning mathematics was inspired by Cobb's (2002) linking between the design heuristics on the one hand, and, on the other hand, analysis of learning and the description of structuring a situation by modelling. The analysis was based on this tool which allowed a fine-grained examination of modelling processes from a learning perspective.

Introduction of *direction* of modelling

The study introduced the notion of *direction* of modelling, understood as detecting and attributing an aim to the modelling. The notion of *direction* served to distinguish between explorative modelling and expressive modelling which, in turn, allowed for nuancing the examination of learning potentials in modelling processes. According to this study's analysis, the students based their expressive modelling decisions, about creation of structures and relations, on subsequences of explorative modelling that encompassed comparison and revision of models. The explorative modelling, thereby, was seen as part of the students' creativity in contrast with bare routine work as stated in Berget and Bolstad (2019).

Introduction of *spin-off*

The study introduced the notion of *spin-off* learning and justified the use of the term by its intertwinement with mathematical learning. *spin-off* learning is conceptualised distinct from the learning of mathematics in terms of emergent modelling. It comes via mathematical modelling, lending authority from the subject although it is neither coherent nor theoretically founded. Thereby, the introduction of *spin-off* learning can serve to pinpoint and challenge myths about purity of mathematics, and their implications. By pointing out the importance of the theme for modelling the notion of *spin-off* learning is, further, closely related to the critical competence and the utility argument in Blum and Niss (1991).

To illustrate outcomes from the *spin-off* learning, a list of concepts and notions about epidemics was extracted from students' reports, based on textual analysis principles. This outcome is distinct from learning outcomes from transfer of simplified daily life incidents into a classroom context like, for example, the making of Margaritas in Carreira et al. (2011). In contrast with the description of the Margarita recipes, the content of Table 5 is valuable for the students. It originates from the horizontal mathematising and its simplifications, and it is not used only as a vehicle for mathematics.

Conditions for *spin-off* learning

The conditions for the *spin-off* learning were studied in four cases. Students' reports were examined for appearances of epidemical, non-mathematical concepts and notions. Conditions for *spin-off* learning were identified by interpretation of the meaning of the textual context regarding i) the level of mathematisation, and ii) the direction of modelling.

The results point out conditions for *spin-off* learning in the form of four distinct circumstances internal to the modelling process: a) the initial (horizontal) mathematising; b) validation of the model (final or preliminary); c) application of a ready-made model; and d) interpretation of the results of a modelling sequence.

The supportive conditions were not restricted to modelling being either explorative or expressive; mathematical learning, as well as *spin-off* learning, took place under both directions of modelling. Surprisingly, the study does not pinpoint the one *direction* of modelling as superior to the other regarding learning opportunities. Neither does it pinpoint an explorative or an expressive teaching design.

Mathematical modelling projects have huge potentials for *spin-off* learning for students who are experienced with mathematics and its teaching and learning but, importantly, unfamiliar with mathematical modelling, DE, and DE models in advance. Expectedly, similar potentials reside in school mathematics when the scene is set for pupils' modelling activities. Such potentials should be realised by carefully choosing the theme for the project and direct the *spin-off* learning towards areas of societal importance.

Notes

[1] European credit transfer and accumulation system for higher education. Sixty ECTS credits are the equivalent of a full standard academic year of study or work.

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