Computational Science in the 17th century. Numerical solution of algebraic equations: digit–by–digit computation

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Abstract. In this paper we give a complete overview of test–problems by Viète from 1600, Harriot from 1631 and Oughtred from 1647. The original material is not easily accessible due to archaic language and lack of conciseness. Viéte's method was gradually elucidated by the subsequent writers Harriot and Oughtred using symbols and being more concise. However, the method is presented in tables and from the layout of the tables it is difficult to find the general principle. Many authors have therefore described Viète's process inaccurately and in this paper we give a precise description of the divisor used in the process which has been verified on all the test–problems. The process of Viète is an iterative method computing one digit of the root in each iteration and has a linear rate of convergence and we argue that the digit–by–digit process lost its attractiveness with the publications in 1685 and 1690 of the Newton-Raphson method which doubles the number of digits for each iteration.

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1 Introduction

Viète wrote two treatises on solving equations: one theoretical and the other numerical. The second treatise *De numerosa potestatum ad exegesim resolutione* or *On the numerical resolution of powers* was published in 1600 and offered something quite new. Here Viète took equations that could be solved only with difficulty, or not at all, by standard methods and showed how numerical solutions could be found to whatever degree of accuracy was required [26]. Viète exemplifies his technique on solving equations on numerous examples more like what we find in more modern papers on numerical solution of nonlinear equations. All examples by Viète have integer solutions. Viète's work was the first comprehensive method of solving such equations that had been attempted, and it involved no restrictions as to terms, signs, or degree [17]. Viète's method closely resembles the method of Šaraf–al–Din al-Tūsī (died in the last quarter of the 12th century)

[21]. This method is described in a manuscript on algebra entitled *On equations*. However, Viète's treatise from 1600 contains the first printed version of such a method.

The *Appendice Algébraique* of 1594, an appendix to *L'arithmetique* from 1585, Simon Stevin writes that after the publication of L' Arithmétique he has found a general rule to solve all equations either perfectly or with any degree of approximation. The appendix itself was reproduced in French and Latin in 1608 and in the reprint of L'Arithmétique by A. Girard of 1625. The processes presented by Stevin and Viète compute the solution or root one digit at time. There are basically two stages in such a process, first ascertain the number of digits in the solution and determine the first digit of the solution. The next stage is to determine one digit at a time. If the sought root is an integer, the process terminates after a few steps.

The first printed method for numerical solution of equations is that of Gerolamo Cardano (1501–1576) in *Ars Magna* from 1545 under the title *De regula aurea*. This is the first successful general methods of approximating roots of algebraic equations. The method was known in manuscripts and commonly referred to as the Rule of Double False Position since the 11th century. In *Ars Magna* there are four examples using the double false position. The double false position is a bracketing methods where the solution of the equation will be in the interval. The secant method uses the same linear interpolation as the double false position, but is not a bracketing technique. In "Newton's Waste Book" ([39, p. 489-49] and there tentatively dated to early 1665) Ypma [40] identifies the method used by Newton to be the secant method. It is well known that these techniques are not digit–by–digit computation.

The invention of decimal fractions is usually ascribed Simon Stevin [2, p. 314], but most importantly he introduced their use in mathematics in Europe. Simon Stevin wrote a booklet called *De Thiende* or "the art of tenths", first published in Dutch in 1585, translated into French the same year and to English in 1608. With the work of Simon Stevin, the classical restriction of "numbers" to integers or to rational fractions was eliminated. For Stevin, the real numbers formed a continuum. His general notion of a real number was accepted, tacitly or explicitly, by all later scientists [31, p.69].

Viète does not use decimals. In [32, 33] Viète writes (translation by Witmer [34])

> Thus if you are seeking the root of 2, a square, extract, if you wish, the root of[1] 2 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00, [which is] $141,421,356,237,309,505$. So the root of 2 is said to be approximately $\frac{141,421,356,237,309,505}{100,000,000,000,000,000}$.

Both Harriot [6] and Oughtred [18] use the same process as Viète but make different arrangement of the computation. Harriot shows the use of fractions and Oughtred also shows computing a solution with decimals. Wallis [35] gives

[1] The integer $2 \cdot 10^{28}$ in Viète is replaced by $2 \cdot 10^{34}$

a better estimate of the new digit than Viète, Harriot, and Oughtred by not excluding terms in the divisor.

The mathematical notation of equations changes in the period we consider. In addition the methods or processes are illustrated using examples leaving some details to interpretations and an example of the arrangement is given in Figure 2. Figure 1 contains examples of the notation used by the cited authors in the different sections of the paper.

Fig. 1. Examples of mathematical notation in the 17th century

Viète	$x^5 + 500x = 254832$	1QC+500N, aequetur 254832
Harriot	$x^5 + 500x = 254832$	$aaaaa + 500a \ === 254832$
Oughtred	$x^5 - 15x^4 + 160x^3 - 1250x^2$ $+6480x = 170304782$	1qc-15qq+160c-1250q+6480ℓ = 170304782
Newton MS	$x^3 + 30x = 14356197$	$\mathcal{L}$c+30$\mathcal{L}$=14356197
Wallis	$x^3 - 2x^2 = 186494880$	Rc-2Rq=186494880
Newton [16]	$x^4 - x^3 - 19x^2 + 49x - 30 = 0$	$x^4 - x^3 - 19xx + 49x - 30 = 0$

Cajori [1] in 1916 was one of the first to point out that the Viète process has been described inaccurately by leading historians at that time, including Cantor in 1900[2]. Due to the inexplicitness of Viète process, writers like Augustus De Morgan in 1847[3], Cajori [1, p.40], and Nordgaard [17, p.28] have misinterpreted the process. Rashed in 1974 [21], Goldstin in 1977 [4, p.66] and Ypma in 1995 [40] make the error of stating an explicit formula to determine a digit. As will be shown the general equations stated in Section 3 serve as estimates to determine the digits used by Viète, Harriot and Oughtred. Section 3 contains a description of Viète's method and all test examples. However, the first section on solving equations is on Stevin's method in Section 2. The next two sections contains the examples used by Harriot in Section 4 and Oughtred in Section 5. Compared to extensive reuse of the test problems [27] of Joseph Raphson [20] from 1690, there are few authors that reuse the test problems of Viète.

In Section 6 is Newton's annotations from Viète and Oughtred. The annotation represents 'state of the art' in mid 17th century but the manuscript was never published. Newton's method was first published in print in Wallis' algebra in 1685, but the algebra book also introduces a modification of the divisor used by Viète, Harriot and Oughtred. This is treated in Section 7.

Already in 1670 we find evidence in a letter from Collins to Leibniz, that the computational work using the Viète process *unfit for a Christian, and more proper to one that can undertake to remove the Italian Alps into England* [22]. In

[2] M. Cantor, Vorlesungen über Geschichte der Mathematik, II, 1900, p. 640–641.

[3] Augustus De Morgan, Involution and Evolution, in The Penny Cyclopaedia of the Society for the Diffusion of Useful Knowledge, London 1846, Volume 2 p.103.

Section 8 we argue that with the presence of higher order methods at the end of the 17th century the use of digit–by–digit calculation for algebraic equation diminished. However, new variations of digit–by–digit methods for algebraic equations appears in the two books by John Ward [37, 38], a new method by Horner [8] and by Holdred [7]. The digit–by–digit process survived in textbooks for hand calculation until the age of calculators and was practised a lot to compute square roots. This process is discussed in the final Section 9.

2 Stevin's Method 1594

The *Appendice Algébraique* of 1594, Stevin uses two examples. The first equation has an integer solution. The second equation has a root that is not integral, but Stevin does not use the decimal notation of his *De Thiende*. The two examples are $x^3 = 300x + 33915024$ and $x^3 = 300x + 33900000$. To find a first approximation for x, try $x = 10^k$, for $k = 0, 1, 2 \ldots$. The result is that for $k = 2$ the value of x^3 is less than that of $300x + 33915024$, but for $k = 3$, the value of x^3 is larger. Hence there will be 3 digits in the root (if integer) To find the first digit, or approximation for x, he now substitutes $x = 100, 200, 300, 400$ and finds $300 < x < 400$. The first digit is then 3. Now he tries $x = 3 \cdot 10^2 + 10,\ 3 \cdot 10^2 + 20,\ 3 \cdot 10^2 + 30$ and finds $320 < x < 330$ and the second digit is 2. Then $x = 3 \cdot 10^2 + 2 \cdot 10 + 1,\ 3 \cdot 10^2 + 2 \cdot 10 + 2,\ 3 \cdot 10^2 + 2 \cdot 10 + 3,\ 3\ 10^2 + 2\ 10 + 4$. It appears that for $x = 3 \cdot 10^2 + 2 \cdot 10 + 4 = 324$ both sides of the equation finally are equal so that $x = 324$ is the root.

Stevin points out that the method can also be applied if the root is not an integral number. Consider $x^3 = 300x + 33900000$ and we find $323 < x < 324$. Then write $x = 323 + \frac{d_1}{10}$ and test for $d_1 = 0, 1, \ldots, 9$ and we find the first decimal digit $d_1 = 9$. Proceed with $x = 323.9 + \frac{d_2}{100}$ as above, with $d_2 = 0, 1, 2, 3, 4, 5, 6$ to find $d_2 = 5$, then $x = 323.95 + d_3 \cdot 10^{-3}$ etc. This can go on indefinitely.

Stevin's method was made popular in the algebra in four volumes by John Kersey in 1673 and 1674 where four examples are given with quadratic, cubic and fourth order polynomials and using decimals [13, Book II, Ch.X].

3 Viète's Method 1600

Viète's method is an extension of Stevin's method. Where Stevin systematic examines all digits $0, 1, \ldots, 9$, Viète makes an estimate of the digit and either increases or decreases it. Hutton in 1795 [9, Algebra, p. 87] and [10, Tract 33, p. 270] writes:

> The method is very laborious, and is but little more than what was before done by Stevinus on this subject, depending not a little upon trials.

However, it is today agreed that François Viète was not familiar with the works of Stevin [34, Translator's introduction].

Viète divides the examples in two; pure and affected equations. For the pure equations Viète uses the technique presented in Section 9.

3.1 Pure equations

In the section *Purarum resolutione* in *De numerosa potestatum purarum, atque adfectarum ad exegesin resolutione tractatus* Viète [33, p.163–172] demonstrates digit-by-digit computation on five problems using the technique in Section 9. Table 1 contains problem number used by Viète and the solution. In Nordgaard [17, p.25] is the arrangement of the computation in Viète's Problem II $x^3 = 157464$ with a close paraphrase in modern notation.

Table 1. Pure equations in Viète 1600 [32, p.3r-6v] and 1646 [33, p.166–172]

Name	$p(x) = N$	Solution
Problem I	$x^2 = 2916$	54
Problem II	$x^3 = 157464$	54
Problem III	$x^4 = 331776$	24
Problem IV	$x^5 = 7962624$	24
Problem V	$x^6 = 191102976$	24

3.2 Affected Equations

In the section *Adfectarum resolutione* in *De numerosa potestatum purarum, atque adfectarum ad exegesin resolutione tractatus* Viète [33, p.173–223] gives numerous examples of digit–by–digit computation for positive roots of polynomials. It is generally agreed that the language used by Viète is archaic and there is an absence of clear symbolism and conciseness [3]. Taking Problem IX which in the notation of Viète is

> Quidam numerus ductus in sui Quadrato-cubum, & in 6000 facit 191,246,976. Queritur quis fit numerus ille. In notis 1CC+6000N æquatur 191,246,976 & fit 1N unitatatum quout?

will in modern language be *A certain number multiplied by its sixth power and by 6000 makes 191,246,976. The question is what that number is. In symbols,* $x^6 + 6000x = 191,246,976$. *What is x?*[4] An equation is "duly prepared" according to Viète if the coefficients of the polynomial are integers and N is positive and Viète partition the equations in affected positively (I to IX), affected negatively (X to XII), mixed (XIII to XV) and avulsed (XVI to XX). The positively and negatively affected equations have only one positive root.

[4] Translated by T.Richard Witmer [34].

Consider the quadratic equation $x^2 + cx + d = 0$. Let $f(x) = x^2 + cx + d$ and consider $f(x + h) = 0$ for given $x > 0$. A bound on h can be found from $h(2x + c) \le -f(x)$, and provided $2x + c > 0$ and the upper bound on h is

$$h \le \frac{-f(x)}{2x + c}. \tag{1}$$

All the coefficients in the examples used by Viète are positive. For the quadratic equation $-x^2 + cx + d = 0$ the corresponding bound will be an upper bound on h, $h \le \frac{-f(x)}{-2x+c}$ when $-2x+c \ge 0$. We can observe that the bound in the quadratic case is the correction to the current iterate x in the Newton-Raphson method $-\frac{f(x)}{f'(x)}$. Rashed [21, p.269] points out that the method of Šaraf–al–Din al-Tūsī and Viète are identical for quadratic equations. Problem 1a) is reproduced in [21, p.266–267] and compared to the method of Šaraf–al–Din al-Tūsī. In Table 2 the first column gives the name of the problems which are found in the second column. The third column is the solutions and the two last columns give the page numbers in the 1600 and 1646 editions.

Table 2. Quadratic equations in Viète 1600 [32] and 1646 [33]

Name	$p(x) = N$	Solution	1600	1646
Problem Ia)	$x^2 + 7x = 60750$	243	p. 7v	p. 174
Example b)	$x^2 + 954x = 18487$	19	p. 8r	p. 175
Problem Xa)	$x^2 - 7x = 60750$	250	p. 18v	p. 195
Example b)	$x^2 - 240x = 484$	242	p. 19v	p. 196
Example c)	$x^2 - 60x = 1600$	80	p. 20r	p. 197
Example d)	$x^2 + 8x = 128$	8	p. 20r	p. 197
Problem XVIa)	$-x^2 + 370x = 9261$	27	p. 27v	p. 211
Example b)	$-x^2 + 370x = 9261$	343	p. 28r	p. 212

For Problem XVIa the bound on $h = \alpha_0$ (the second and last digit) will be a lower bound. Problem XVIa) and b) are Problem 4 in Harriot [6, p.128].

We now consider the cubic equations given in Table 3. Let $f(x) = x^3 + bx^2 + cx + d$ and consider the cubic equation $f(x + h) = 0$ for given x. Then

$$f(x + h) = f(x) + h(3x^2 + 2bx + c) + h^2(3x + b) + h^3 = 0. \tag{2}$$

Viète eliminates h^3 so $-f(x) \ge h(3x^2 + 2bx + c) + h^2(3x + b)$. If $3x + b \ge 0$, initial $\underline{h} \le h$ and $3x^2 + 2bx + c \ge 0$ then an upper bound on h is,

$$h \le \frac{-f(x)}{3x^2 + 2bx + c + (3x + b)\underline{h}}.$$

In Section 6 on Newton's annotation the above bound is the same as the bound (4) when $b = 0$. $\underline{h}$ will depend on the number of digits in the solution, k, and the order $j > 1$, 10^{k-j}. For the cubic equation, the bound is the Newton-Raphson correction only when $\underline{h} = 0$. The method of Šaraf–al–Din al-Tūsī and Viète deviates for cubic equations where the estimate of the next digit is the Newton-Raphson correction $-f(x)/f'(x)$ scaled [21]. To show the differences, Rashed [21, p.268–270] used Problem IIa. The arrangement of Viète's Problem IIa) and IIb)

Table 3. Cubic equations in Viète 1600 [32] and 1646 [33]

Name	$p(x) = N$	Solution	1600	1646
Problem IIa)	$x^3 + 30x = 14356197$	243	p. 9r	p. 176
Example b)	$x^3 + 95400x = 1819459$	19	p. 10r	p. 178
Problem IIIa)	$x^3 + 30x^2 = 86220288$	432	p. 10v	p. 180
Example b)	$x^3 + 10000x^2 = 5773824$	24	p. 11v	p. 182
Problem XIa)	$x^3 - 10x = 13584$	24	p. 20r	p. 198
Example b)	$x^3 - 116620x = 352947$	343	p. 21r	p. 199
Example c)	$x^3 - 6400x = 153000$	90	p. 22r	p. 200
Example d)	$x^3 + 64x = 1024$	8	p. 22r	p. 201
Problem XIIa)	$x^3 - 7x^2 = 14580$	27	p. 22r	p. 201
Example b)	$x^3 - 10x^2 = 288$	12	p. 22v	p. 202
Example c)	$x^3 - 7x^2 = 720$	12	p. 23v	p. 203
Example d)	$x^3 + 8x^2 = 1024$	8	p. 24r	p. 204
Problem XVIIa)	$-x^3 + 13104x = 155520$	12	p. 29r	p. 214
Example b)	$-x^3 + 13104x = 155520$	108	p. 29v	p. 215
Problem XVIIIa)	$-x^3 + 57x^2 = 24300$	30	p. 30r	p. 216
Example b)	$-x^3 + 57x^2 = 24300$	45	p. 30v	p. 217

and a close paraphrase in modern notation is found in Nordgaard [17, p.26–27].

Let f be a polynomial of degree n on the form $f(x) = x^n + q(x)$ where q is a polynomial of degree $n - 1$. We can write

$$q(x + h) = \sum_{i=0}^{n-1} \frac{h^i}{i!} q^{(i)}(x)$$

where $q^{(i)}$ is the ith derivative of q. Further

$$(x + h)^n = x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^{n-i} h^i + h^n \geq x^n + h \sum_{i=1}^{n-1} \binom{n}{i} x^{n-i} h_0^{i-1}$$

for $0 \leq \underline{h} \leq h$. So if all $q^{(i)}(x) \geq 0$ then

$$-f(x) \geq h \left[\sum_{i=1}^{n-1} \binom{n}{i} x^{n-i} h_0^{i-1} + \sum_{i=1}^{n-1} \frac{h_0^{i-1}}{i!} q^{(i)}(x) \right]. \tag{3}$$

This general case (refeq:general) will give an estimate which is either an upper bound or lower bound on h. The first sum is by Viète called the lower part and second sum is upper part based on the table Viète uses. To recognize the computation in the tables of Viète, Harriot and Oughtred note that $f(x_{j+1}) = f(x_j) + (f(x_{j+1}) - f(x_j))$ and some of the terms in $f(x_{j+1}) - f(x_j)$ are also needed in computing the divisor. Further details are given in Section 7.

Consider $f(x) = x^4 + ax^3 + bx^2 + cx + d$. Then the two parts will be

$$\text{(lower) } 4x^3 + 6x^2\underline{h} + 4x\underline{h}^2 \text{ and (upper) } 3ax^2 + 2bx + c + \underline{h}(3ax + b).$$

In all examples Viète uses the sum of lower and upper part to find an estimate on the (next) digit. An annotated version of Problem XV is found in [3, p.214–216]

Table 4. Higher order algebraic equations in Viète 1600 [32] and 1646 [33]

Name	$p(x) = N$	Solution	1600	1646
Problem IVa)	$x^4 + 1000x = 355776$	24	p. 12v	p. 183
Example b)	$x^4 + 100000x = 2731776$	24	p. 13v	p. 185
Problem V	$x^4 + 10x^3 = 470016$	24	p. 14r	p. 186
Problem VIa)	$x^4 + 200x^2 = 446976$	24	p. 14v	p. 187
Example b)	$x^4 + 200x^2 + 100x = 449376$	24	p. 15r	p. 188
Problem VII	$x^5 + 500x = 254832$	12	p. 16r	p. 190
Problem VIII	$x^5 + 5x^3 = 257472$	12	p. 16v	p. 191
Problem IX	$x^6 + 6000x = 191246976$	24	p. 17v	p. 193
Problem XIII	$x^4 - 68x^3 + 202752x = 5308416$	32	p.24r	p. 205
Problem XIV	$x^4 + 10x^3 - 200x = 1369856$	32	p.25r	p. 207
Problem XV	$x^5 - 5x^3 + 500x = 7905504$	24	p. 26r	p. 208
Problem XIXa)	$-x^4 + 27755x = 217944$	8	p. 31v	p. 219
Example b)	$-x^4 + 27755x = 217944$	27	p. 32r	p. 220
Problem XX	$-x^4 + 65x^3 = 1481544$	38	p. 32v	p. 221
Example b)	$-x^4 + 65x^3 = 1481544$	57	p. 33v	p. 222

and with explanations omitted in [15, p.37]. The last section in Viète's book [33, p.228] there is an example on how to transform the equation to get the root correct to the tenths and to the hundredths by scaling the variables. Given $x^3 + 6x = 8$. Substitute x by $\frac{x}{10}$ and the equation will be $x^3 + 6 \cdot 10^2 x = 8 \cdot 10^3$ and solve the new equation using the Viète process and the approximate root of the original equation will be $\frac{11}{10} = 1\frac{1}{10}$ which will be correct to the tenths.

4 Test Examples from Harriot 1631

The test examples by Harriot are from the chapter *Exegetice numerosa* [6, p.117–167] or Numerical Exegesis [23, p.129–182] in *Praxis* [6]. In the manuscripts Harriot refers all his examples to Viète and each manuscript page with an example is marked *De numerosa potestatum resolutione* [23]. However, only three of the examples in *Praxis* are from Viète. Where Viète splits the computation of the divisor into a lower part (corresponding to x^n) and the upper part (the remaining divisor), Harriot also splits the order of the terms in two parts, without the correcting term $\underline{h}$ and the part with the correction $\underline{h}$. Further the tables in *Praxis* contains symbols commenting the computation where Viète has a verbal description.

In one example, Problem 6 in Table 7, Harriot suggests another table corresponding to $\underline{h} = 0$. This will also yield an upper bound. Three problems in Praxis in Table 7 are identical to problems in Viète and two of these problems are used by Newton in his manuscript [39, p. 63–71]:

- Problem 4 (p.128) in Table 7 is Viète's Problem XVI a) and b) in Table 3
- Example (p.138) in Table 7 is Viète's Problem IIb in Table 3 and also used by Newton in Table 5.
- Example (p.143) in Table 7 is Viète's Problem XIa in Table 3 and also used by Newton in Table 5.

Problem 13 (p.155) in Table 7 is reproduced in [15, p.38–39] and in [17, p.30] using Harriot's original formulation and notation

$$aaaa - 1024aa + 6254a = 19633735875.$$

Hankel's book on history of mathematics from 1874 illustrates Viète's method using one of Harriots's examples [6, p.164] $x^2 + 14x = 7929$ in Table 7 computing the approximate root 82.319 [5, p.370] using decimals.

The four 'pure' powers, Problem 1, 5, 11, and 15 in Table 7, Harriot uses the same technique as for 'affected' equations. The computation will be the same as described in Section 9 where we always get an upper bound on the digit.

5 Test Examples from Oughtred 1647/48

No one did more to popularize the new method of Viète than did the clergyman mathematician William Oughtred. This he accomplished by giving private tuition to ambitious young men and these spread his teachings throughout Great Britain; among them were Seth Ward, Christopher Wren, and John Wallis [17, p.31]. John Wallis [35] devotes a chapter on Mr. Oughtred and his Clavis [35, Ch.xv] and points out that Oughtred's contributions to Viète's method were in the simplification of the notation.

The second edition of William Oughtred (1574-1660) *Clavis Mathematicæ* (The Key to Mathematics) was published in 1648 and an English translation in 1647. In the chapter *Some examples of equations resolved in numbers* Oughtred [19, p.139–172] considers 16 examples using a digit-by-digit computation. In Table 9 the page numbers refer to the translated version from 1647. The first edition from 1631 contains only two examples using digit–by–digit computation, $\sqrt{3272869681} = 57209$ and $\sqrt[3]{187237601580329} = 57209$. These two examples are also in the second edition. Before the year 1700 five editions of this little volume had been published. The Clavis opens with an explanation of the Hindu-Arabic notation of decimal fractions. Oughtred would write $15|\underline{7}$ for 15.7.

The 16 examples are also used by Jeake [12, 11] which also includes some additional examples and comments on the computation. The same arrangement of the computation of Example 1 in Table 9 is found in De Morgan [15, p.39–40] using the Oughtred's notation in Figure 1. Oughtred's computation in Example 2 in Table 9 is discussed by Caljori [1, p.458–459]. This example has the same form as discussed in Section 6 and in this case (5) is an equality.

6 On Newton's Annotations 1664

In an unpublished note from 1664(?) reproduced in [39, p. 63–71] Newton annotates Viète's *Opera Mathematica* from 1646 using the simplified notation in Oughtred's *Clavis Mathematicæ* from 1648. Newton gives 7 examples computing the root digit–by–digit. This unpublished note represents the 'state of the art' in mid 17th century. In the table below the references are to Viète 1600 and 1646 [32, 33], to Harriot [6] and MS is Harriot's manuscripts collected by Stedall [24]. The first column in Table 5 gives the function and the second contains the references. For the first three problems the algorithm is the one used in Section

Table 5. Examples in Newton's note [39, p. 63–71]

$f(x)$	Reference(s)
$x^2 - 2916$	Problem I in Viète
$x^3 - 157464$	Problem II in Viète
$x^5 - 7962624$	Problem IIII in Viète
$x^3 + 30x - 14356197$	Problem IIa) in Viète and in MS
$x^3 + 95400x - 1819459$	Example IIb) in Viète, and in Harriot and MS
$x^3 - 10x - 13584$	Problem XIa) in Viète and MS
$x^3 - 116620x - 352947$	Example XIb) in Viète, and in Harriot and MS

9. The other problems are all on the form $x^3 + cx = d$ and a meta description of the algorithm is:

– Step 1: Determine the number of digits in the root, say $x = \alpha_2 10^2 + \alpha_1 10 + \alpha_0$.
– Step 2: Determine the first digit α_2: Choose the largest $0 < \alpha_2 \leq 9$ so that

$$(\alpha_2 10^2)^3 + c(\alpha_2 10^2) \leq d$$

– Step 3: Determine the second digit α_1: Choose the largest $0 \leq \alpha_1 \leq 9$ so that

$$(\alpha_2 10^2 + \alpha_1 10)^3 + c(\alpha_2 10^2 + \alpha_1 10) \leq d$$

– Step 4: Determine the last digit α_0: Choose the largest $0 \leq \alpha_0 \leq 9$ so that

$$(\alpha_2 10^2 + \alpha_1 10 + \alpha_0)^3 + c(\alpha_2 10^2 + \alpha_1 10 + \alpha_0) \leq d$$

The convergence of this technique follows from the observation that this is a bracketing process where the root will be in an interval on the form $[\cdot, \cdot)$ (the right end is open) and the assumed existence of a root and monotonicity of $x^3 + cx$ in the interval. The first interval will be $[\alpha_2 10^2, (\alpha_2+1)10^2)$, then $[\alpha_2 10^2 + \alpha_1 10,\ \alpha_2 10^2 + (\alpha_1+1)10)$ and the final $[\alpha_2 10^2 + \alpha_1 10 + \alpha_0,\ \alpha_2 10^2 + \alpha_1 10 + \alpha_0 + 1)$.

Consider $f(x) = x^3 + cx - d$ and $f(x+h) = 0$ for given $x > 0$ and unknown h. Then

$$f(x+h) = f(x) + h(3x^2 + 3xh + h^2 + c) = 0.$$

Let $h \geq \underline{h} \geq 0$ be an initial estimate, then an upper bound on h will be

$$h \leq \frac{-f(x)}{3x^2 + 3x\underline{h} + c} = \hat{h}, \tag{4}$$

provided c is not too negative. To determine digit number $j > 1$, α_{k-j}, consider

$$x_j = \sum_{i=k-j}^{k-1} 10^i \alpha_i = x_{j-1} + 10^{k-j}\alpha_{k-j}.$$

Define $h_j = 10^{k-j}$ then from (4)

$$\alpha_{k-j} \leq \left\lfloor \frac{1}{h_j} \quad \frac{-f(x_j)}{3x_j^2 + 3x_j h_j + c} \right\rfloor. \tag{5}$$

In the following table we show the actual computation of the digits in the solution. We assume that the magnitude of the root and the first digit are known.

	$x^3 + 30x - 14356197$ $x^* = 243$		$x^3 + 95400x - 1819459$ $x^* = 19$
x	200	240	10
$-f(x)$	6350197	524997	864459
$\underline{h}$	10	1	1
$3x^2 + 3xh_0\underline{h} + c$	126030	173550	95730
$\hat{h}$	50.4	3.03	9.03
α_i	4	3	9

Consider finding root x_* of $x^3 + 30x - 14356197$. The number of digits in x_*, when $(x_*)^3 >> 30x_* > 0$, will be the number of digits in $\sqrt[3]{14\,356\,197}$ which is 3 and the leading digit will be $\alpha_2 = 2$. To find the next digit of x_* use (4) with $x = 200$ and $\underline{h} = 10$. An upper bound of $\alpha_1 \leq \lfloor 5.04 \rfloor = 5$. However, 5 is too large, and the second digit is found to be 4.

	$x^3 - 10x - 13584$ $x^* = 24$	$x^3 - 116620x - 352947$ $x^* = 343$	
x	20	300	340
$-f(x)$	5784	8338947	699747
$\underline{h}$	1	10	1
$3x^2 + 3xh_0\underline{h} + c$	1250	162380	231200
$\hat{h}$	4.63	51.4	3.03
α_i	4	4	3

Newton's transcripts of Viète's solution of $x^3 + 30x = 14356197$ is found in [39, p.66] and reproduced in [40, p.534]. The notebook (MS Add. 4000) with transcripts is available online[5].

7 Contributions of John Wallis 1685

In his algebra and history of algebra book [35] from 1685, John Wallis discusses the work by Viète, Harriot and Oughtred and summarizes the method in one example. In [35, p.103–105] he gives the example $x^3 - 2x^2 = 186494880$ and computes the root 572 following the same basic principle as in [32, 6, 18] to compute the solution digit by digit. Consider $f(x) = x^3 + bx^2 + d$. Contrary to Viète, Harriot and Oughtred, Wallis does not exclude the h^3 term in (2) and uses

$$\alpha_{k-j} \leq \left\lfloor \frac{1}{h_j} \quad \frac{-f(x_j)}{3x_j^2 + 2bx_j + (3x_j + b)h_j + h_j^2} \right\rfloor, \tag{6}$$

where $h_j = 10^{k-j}$ as in Section 6. Further,

$$f(x + h) = f(x) + (f(x + h) - f(x)) = 3x^2h + 3xh^2 + h^3 + 2bxh + bh^2. \tag{7}$$

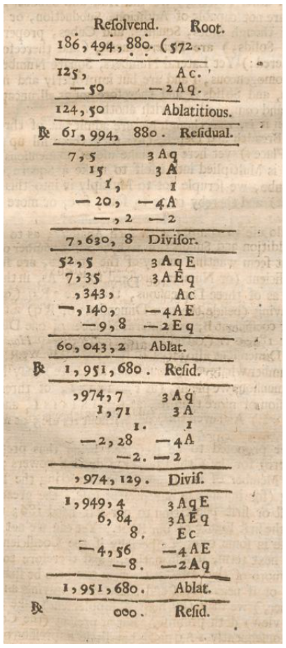
Since $500 < \sqrt[3]{186000000} \leq \sqrt[3]{186494880 + 2x^2}$ it follows that the sought root has three digits (α_2, α_1, and α_0) and the first digit will be 5. So $\alpha_2 = 5$, $x_1 = 500$, and $h_1 = 10$ and $h_0 = 1$. In Wallis the known is denoted A and E is to be determined which corresponds to x and h properly scaled. In the first column in Figure 2 is the computation in [35, p.104] and in the next column the same computation using the notation in this paper.

A problem proposed by Pell and later proposed to Wallis by Colonel Silas Titus is to find a, b, and c so that [25]

$$a^2 + bc = 16, \quad b^2 + ac = 17, \text{ and } c^2 + ab = 18.$$

[5] https://cudl.lib.cam.ac.uk/view/MS-ADD-04000/1

Fig. 2. Wallis 1685 $x^3 - 2x^2 = 186494880$



Resolvend.	Root.
186,494,880.	(572.
125,	Ac.
—50	—2Aq.
124,50	Ablatitious.
℞ 61,994, 880.	Residual.
7,5	3Aq
15	3A
1,	1
—20,	—4A
—,2	—2
7,630, 8	Divisor.
52,5	3AqE
7,35	3AEq
,343,	Ac
—,140,	—4AE
—9,8	—2Eq
60,043,2	Ablat.
℞ 1,951,680.	Resid.
,974,7	3Aq
1,71	3A
1.	1
—2,28	—4A
—2.	—2
,974,129.	Divis.
1,949,4	3AqE
6,84	3AEq
8.	Ec
—4,56	—4AE
—8.	—2Aq
1,951,680.	Ablat.
℞ 000.	Resid.

	186 494 880	$-d$
Ac	125 000 000	$x_1^3,\ x_1 = 500$
-2Aq	-50 000	bx_1^2
Residual	61 994 880	$-f(x_1),\ h_1 = 10$
3Aq	7 500 000	$3x_1^2 \cdot h_1$
3A	150 000	$3x_1h_1 \cdot h_1$
I	1 000	$h_1^2 \cdot h_1$
-4A	-20 000	$2bx_1 \cdot h_1$
-2	-200	$bh_1 \cdot h_1$
Divisor	7 630 800	$\alpha_1 \le 8,\ x_2 = 570,\ h = \alpha_1 h_1 = 70$
3AqE	52 500 000	$3x_1^2h$
3AEq	7 350 000	$3x_1h^2$
Ec	343 000	h^3
-4AE	-140 000	$2bx_1h$
-2Eq	-9 800	$3x_1h^2$
Residual	1 951 680	$-f(x_2),\ h_0 = 1$
3Aq	974 700	$3x_2^2 \cdot h_0$
3A	1 710	$3x_2h_0 \cdot h_0$
I	1	$h_0^2 \cdot h_0$
-4A	-2 280	$2bx_2 \cdot h_0$
-2	-2	$bh_0 \cdot h_0$
Divisor	974 129	$\alpha_0 \le 2,\ x_3 = 572$

Wallis [35, p.225–252] treats this problem and in [35, Ch.62] reduces the three equations to a fourth order algebraic equation

$$x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0$$

to determine $a = \sqrt{x^*/2}$ using Viète's method. Wallis computes

$$x^* = 12.756441794480744$$

with 17 correct digits. The second equations follows from multiplying the first quadratic equation by a and the second by b and eliminate abc to get the equation

$$17b - b^3 = 16a - a^3,\ \text{where } a = \sqrt{\frac{1}{2}x^*}.$$

This equation is solved for b to 16 digits again using Viète's method. Having found a and b, c is found from the first quadratic $a^2 + bc = 16$. The third example of Viète's method is found using synthetic division

$$f(x) = \frac{x^4 - 80x^3 + 1998x^2 - 14937x + 5000}{x - x^*}$$

and finding a second root of the quartic polynomial 0.350987046. In all three examples Wallis is using all terms in the divisor.

If Newton's method is applied to the system $F(a,b,c) = (a^2 + bc - 16, b^2 + ac - 17, c^2 + ab - 18)$ with the starting point $(a,b,c) = (2,3,4)$ the error in F is of order 10^{-14} after 5 iterations.

8 End of an Era

1n the 1670s John Collins (1625-1683) wrote an account of Pell's achievements for Leibniz, and after describing one of Pell's table (a yard long, according to Collins) and its use, he made the remark that in an attempt to solve the equations with Viète's method,*Mr Warner used to call work unfit for a Christian, and more proper to one that can undertake to remove the Italian Alps into England* [22, Ch.LXXXV,p.248]. Similar statement from 1758 on Viète's method, Montucla [14, p.492] regards the calculation of the root of a biquadratic polynomial to eleven decimal places as a work of the most extravagant labour or as Hutton says in 1795 the method is very laborious.

On Wednesday, 17 December 1690, in a meeting of the Royal Society we find the following announcement of Raphson's book [20] (quote from [30])

> Mr Ralpson's Book was this day produced by E Halley, wherein he gives a Notable Improvemt of ye method of Resolution of all sorts of Equations Shewing, how to Extract their Roots by a General Rule, which doubles the known figures of the Root known by each Operation, So yt by repeating 3 or 4 times he finds them true to Numbers of 8 or 10 places. The Society being highly pleased with this his performance Ordered him their thanks with their Desires, that he would please to Continue to prosecute those Studys, wherein he hath been so Successful.

This marks the end of an active area on numerical solution of algebraic equations using digit–by–digit computations. As mentioned in the introduction improved methods appeared, but these methods were soon replaced by the Newton-Raphson method, the Rule of Double False Position or the secant method. However, the digit–by–digit computation of square square roots continued to be popular and was used in schools right up to the 1960s [29, 3].

9 Computing the Square Root

Why no one before Viète should have thought of applying to the solution of algebraic equations the classical method of finding roots of large numbers may seem strange [17, p.24]. In this section we discuss this classical approach to compute square root of any positive integer digit by digit. The history of the method goes back in Europe to the 13th century with the method of Ibn al–Bannã [3]. Already in 1695 Wallis pointed out that the digit–by–digit computation advocated by Viète, Harriot and Oughtred was not an efficient method [36]. Other iterative methods that are not digit-by-digit based method are based on repeated approximation of the root [28].

Let N be a positive integer and assume that $\sqrt{N}$ is an integer. Assume for $k \geq 1$ that

$$N = \sum_{i=0}^{2k-1} \beta_i \, 10^i = \sum_{i=0}^{k-1} (\beta_{2i} + 10\beta_{2i+1}) \, 10^{2i}, \ \ \beta_i \in \{0, 1, 2, \ldots, 9\}$$

where not both $\beta_{2(k-1)}$ and $\beta_{2(k-1)+1}$ are equal 0. The number of digits in $\sqrt{N}$ will then be k, say

$$\sqrt{N} = \sum_{i=0}^{k-1} \alpha_i \, 10^i, \ \ \alpha_i \in \{0, 1, \ldots, 9\}$$

In the following an approximation x_j of $\sqrt{N}$ will be the number with k digits where the j leftmost digits $\alpha_{k-1}, \ldots, \alpha_{k-j}$ are determined and $\alpha_{k-j-1} = \ldots = \alpha_0 = 0$,

$$x_j = \sum_{i=k-j}^{k-1} \alpha_i 10^i = 10^{k-j} \sum_{i=0}^{j-1} \alpha_{i+k-j} \, 10^i, \quad j = 1, 2, \ldots, k$$

and the remaining $k - j$ digits are 0. Let

$$a_j = \sum_{i=0}^{j-1} \alpha_{i+k-j} \, 10^i, \text{ then } x_j = 10^{k-j} a_j, \quad j = 1, \ldots, k-1.$$

Let $d_j = \alpha_{k-j-1} 10^{k-j-1}$ where α_{k-j-1} is the digit to be determined. Since $x_{j+1} = x_j + d_j$ we have $a_{j+1} = 10a_j + \alpha_{k-j-1}$.

To determine α_{k-j-1} choose largest α_{k-j-1} so that

$$(x_j + d_j)^2 \leq N \text{ or } d_j(2x_j + d_j) \leq N - x_j^2$$

Then we have

$$10^{2(k-j-1)}(20a_j + \alpha_{k-j-1})\alpha_{k-j-1} \leq N - x_j^2$$

Now use the assumption that the last $k - j$ digits in x_j are 0. Hence

$$\begin{aligned} N - x_j^2 &= 10^{2(k-j)} r_j + \sum_{i=0}^{2(k-j)-1} \beta_i 10^i \\ &= 10^{2(k-j-1)} \left(10^2 r_j + 10\beta_{2(k-j-1)+1} + \beta_{2(k-j-1)}\right) + \sum_{i=0}^{2(k-j-1)-1} \beta_i 10^i \end{aligned}$$

Then α_{k-j-1} is the largest integer so that

$$(20a_j + \alpha_{k-j-1})\alpha_{k-j-1} \leq \hat{r}_j,$$

where

$$\hat{r}_j = 10^2\ r_j + 10\beta_{2(k-j-1)+1} + \beta_{2(k-j-1)}.$$

Further, we have

$$r_{j+1} = \hat{r}_j - (20a_j + \alpha_{k-j-1})\alpha_{k-j-1}, \quad j = 1, \dots k-1.$$

To find the largest α_{k-j-1}, Wallis [35, p.98] chooses $\alpha_{k-j-1} \approx \lfloor \frac{(\hat{r}_j - \beta_{2(k-j-1)})/10}{2a_j} \rfloor$ and increase or decrease if needed. To determine an approximation to the second digit in Fig. 9 this will be $\lfloor \frac{77}{10} \rfloor$ and for the third digit $\lfloor \frac{(2386-6)/10}{57 \cdot 2} \rfloor$.

To determine the first digit α_{k-1} we note that

$$d_0^2 \le N, \text{ or } \alpha_{k-1}^2 \le 10\beta_{2k-1} + \beta_{2(k-1)},$$

so the first digit can be easily be determined directly.

Then we have the following digit by digit square root algorithm

$r := 0$
$a := 0$
for $j = 1, 2, \dots, k$
  $r := 100r + 10\beta_{2(k-j)+1} + \beta_{2(k-j)}$
  Find the largest α_{k-j} so that
    $\alpha_{k-j}(20a + \alpha_{k-j}) \le r$
  $a := 10a + \alpha_{k-j}$
  $r := r - \alpha_{k-j}(20a + \alpha_{k-j})$

We give two examples computing the square root by Wallis in 1685 [35, p.99] in Fig. 9 and Newton in 1707 [16, p.32] in Fig. 9.

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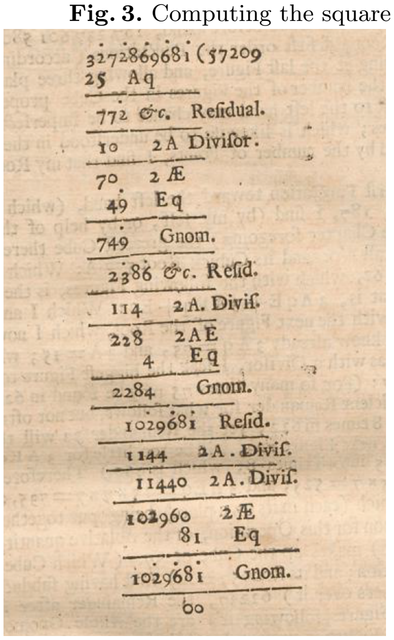
Table 6. Test Examples from Oughtred 1647/48

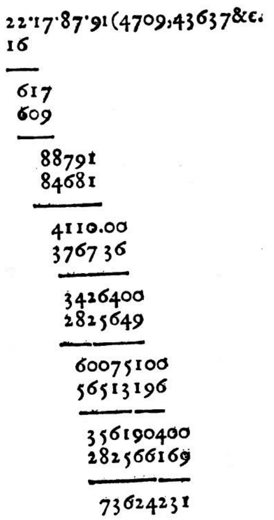
Name	$p(x) = N$	Solution
Example 1 (p.140)	$x^5 - 15x^4 + 160x^3 - 1250x^2 + 6480x = 170304782$	47
Example 2 (p.142)	$x^3 + 420000x = 247651713$	417
Example 3 (p.143)	$x^3 + 1007x^2 = 247617936$	417
Example 4 (p.145)	$x^4 - 44299005x = 22252086$	354
Example 5 (p.146)	$x^4 - 124600x^2 = 89726256$	354
Example 6 (p.147)	$x^4 - 340x^3 = 621066096$	354
Example 7 (p.149)	$x^4 - 77108000x = 85530576$	426
Example 8 (p.150)	$-x^3 + 3200x = 46577$	47
Example 9 (p.151)	$-x^3 + 3200x = 46577$	15.7
Example 10 (p.152)	$-x^3 + 53x^2 = 13254$	47
Example 11 (p.153)	$-x^3 + 53x^2 = 13254$	20.05
Example 12 (p.154)	$-x^3 + 60034x = 1023768$	236
Example 13 (p.155)	$-x^3 + 60034x = 1023768$	17.135
Example 14 (p.156)	$x^4 - 72x^3 + 238600x = 8725815.7056$	47.6
Example 15 (p.158)	$-x^3 + 3x = 1.258640782100$	0.4499
Example 16 (p.154)	$x^5 - 5x^3 + 5x = 1.147152872702092$	0.2437

Fig. 3. Computing the square root of 3272869681 in Wallis 1685[35, p.99]



j	α_{k-j}	$\hat{r}_{j-1}$	a_{j-1}	$(20a_{j-1}+\alpha_{k-j})\alpha_{k-j}$	r_j
1	5	32	0	25	7
2	7	772	5	749	23
3	2	2386	57	2284	102
4	0	10296	572	0	10296
5	9	1029681	5720	1029681	0

Fig. 4. Square root of 22178791 with five decimals in Newton 1707[16, p.32]



j	α_{k-j}	$\hat{r}_{j-1}$	a_{j-1}	$(20a_{j-1}+\alpha_{k-j})\alpha_{k-j}$	r_j
1	4	22	0	16	6
2	7	617	4	609	8
3	0	887	47	0	887
4	9	88791	470	84681	4110
5	4	411000	4709	376736	34264
6	3	3426400	47094	2825649	600751
7	6	60075100	470943	56513196	3561904
8	3	356190400	4709436	282566169	73624231
9	7	7362423100	47094363	6593210869	76921223

Table 7. Test Examples from Harriot 1631 *Praxis*

Name	$p(x) = N$	Solution
Problem 1 (p.117)	$x^2 - 48233025$	6945
Problem 2 (p.119)	$x^2 + 432x = 13584208$	3476
Example 1 (p.121)	$x^2 + 75325x = 41501984$	547
Example 2 (p.122)	$x^2 + 675325x = 369701984$	547
Problem 3 (p.124)	$x^2 - 624x = 16305126$	4362
Example A (p.125)	$x^2 - 6253x = 6254$	6254
Example R (p.127)	$x^2 - 732x = 86005$	835
Problem 4 (p.128)	$-x^2 + 370x = 9261$	27 and 343
Problem 5 (p.131)	$x^3 = 105689636352$	4728
Problem 6 (p.132)	$x^3 + 68x^2 + 4352x = 186394079$	547
Problem 7 (p.134)	$x^3 + 45796x = 449324752$	746
Example (p.138)	$x^3 + 95400x = 1819459$	19
Example (p.139)	$x^3 + 274576x = 301163392$	536
Problem 8 (p.141)	$x^3 - 2648x = 91148512$	452

Name	$p(x) = N$	Solution
Example (p.143)	$x^3 - 116620x = 352947$	343
Example (p.145)	$x^3 - 127296x = 85700000$	536
Problem 9 (p.146)	$-x^3 + 52416x = 1244160$	216 and 24
Problem 10 (p.149)	$x^3 - 68x = 134454528$	536
Problem 11 (p.151)	$x^4 = 19565295376$	374
Problem 12 (p.153)	$x^4 - 426x = 2068948$	38
Example (p.154)	$x^4 - 43602354x = 4172008$	352
Problem 13 (p.155)	$x^4 - 1024x^2 + 6254x = 19633735875$	375
Problem 14 (p.158)	$x^4 - 1024x^2 - 6254x = 19629045375$	1375
Problem 15 (p.160)	$x^5 = 15755509298176$	436
Problem 16 (p.162)	$x^5 - 57x^3 + 5263x = 900050558322$	246
Example (p.164)	$x^2 + 14x = 7929$	$82\frac{319}{1000}$
Example (p.166)	$x^3 + 135x = 98754$	$45\frac{24}{100}$

References

1. Florian Cajori. *William Oughtred, a great seventeenth-century teacher of mathematics*. Chicago Open Court Pub. Co, 1916.
2. Florian Cajori. *A history of mathematical notations, Volume 1. Notations in elementary mathematics*. The Open court publishing company, Chicago, 1928.
3. Jean-Luc Chabert (Ed.). *A History of Algorithms. From the Pebble to the Microchip*. Springer-Verlag, Berlin, 1999.
4. Herman Heine Goldstine. *A History of Numerical Analysis from the 16th through the 19th Century*. Springer-Verlag, New York, 1977.
5. Hermann Hankel. *Geschichte der Mathematik in Alterthum und Mittelalter*. Leipzig, 1874.
6. Thomas Harriot. *Artis Analyticae Praxis*. London, 1631.
7. Theophilus Holdred. *A New Method of Solving Equations with Ease and Expedition; by which the True Value of the Unknown Quantity is Found without Previous Reduction. With a Supplement, Containing Two Other Methods of Solving Equations, Derived from the Same Principle*. London, 1820.
8. William George Horner. A new method of solving numerical equations of all orders, by continuous approximation. *Philosophical Transactions*, 109:308–335, 1819.
9. Charles Hutton. *A mathematical and philosophical dictionary. Volume 1*. London, 1795.
10. Charles Hutton. *Tracts on mathematical and philosophical subjects. Volume 2*. London, 1812.
11. Samuel Jeake. *A Compleat Body of Arithmetic*. London, 1701.
12. Samuel Jeake, Senior. *ΛΟΓΙΣΤΙΚΗΛΟΓΙ'Α, or Arithmetick surveighed and reviewed: in four books, etc. [Edited by Samuel Jeake, the Younger.]*. London, 1696.
13. John Kersey. *The Elements of That Mathematical Art Commonly Called Algebra*. London, 1673.
14. Jean Étienne Montucla. *Histoire des mathématiques, Volume I*. Paris, 1758.
15. Augustus De Morgan. Notices of the progress of the problem of evolution. *The companion to the almanac*, pages 34–52, 1839.
16. Isaac Newton. *Arithmetica universalis; Sive de compositione et resolutione arithmetica liber*. London, 1707. Edited by William Whiston.
17. Martin Andrew Nordgaard. *A historical survey of algebraic methods of approximating the roots of numerical higher equations up to the year 1819*. PhD thesis, Columbia University, New York City, 1922.
18. William Oughtred. *Clavis mathematicae*. London, 1631.
19. William Oughtred. *The key of the mathematics new forged and filed*. London, 1647.
20. Joseph Raphson. *Analysis Æquationum Universalis, seu ad Æquationes Algebraicas Resolvendas Methodus Generalis, et Expedita, Ex nova Infinitarum serierum Doctrina, Deducta ac Demonstrata*. London, 1690.
21. Roshdi Rashed. Résolution des équations numérique et algèbre: Šaraf–al–Din al-Tūsī, Viéte. *Archive for History of Exact Sciences*, 12(3):244–290, 1974.
22. Stephen Jordan Rigaud. *Correspondence of Scientific Men of the Seventeenth Century, Vol.1*. Oxford, 1841.
23. Muriel Seltman and Robert Goulding. *Thomas Harriot's Artis Analyticae Praxis. An English Translation with Commentary*. Springer, 2007.
24. Jacqueline Anne Stedall. *The Greate Invention of Algebra: Thomas Harriot's Treatise on equations*. Oxford University Press, Oxford, 2003.

25. Jacqueline Anne Stedall. Tracing mathematical networks in seventeenth-century England. In Eleanor Robson and Jacqueline Stedall, editors, *The Oxford handbook of the history of mathematics*, pages 133–151. Oxford University Press, 2009.
26. Jacqueline Anne Stedall. Reconstructing Thomas Harriot's treatise on equations. In Robert Fox, editor, *Thomas Harriot and His World: Mathematics, Exploration, and Natural Philosophy in Early Modern England*, pages 53–63. Ashgate Publishing (Routledge), 2012.
27. Trond Steihaug. Computational science in the eighteenth century. Test cases for the methods of Newton, Raphson, and Halley: 1685 to 1745. *Numerical Algorithms*, 83:1259–1275, 2020.
28. Trond Steihaug and D. G. Rogers. Approximating cube roots of integers, after Heron's Metrica III.20. *Normat*, 61(2):87–110, 2013.
29. Philip D. Straffin. Liu Hui and the first golden age of Chinese mathematics. *Mathematics Magazine*, 71(3):163–181, 1998.
30. David J. Thomas and Judith M. Smith. Joseph Raphson, F.R.S. *Notes and Records of the Royal Society of London*, 44(2):151–167, 1990.
31. Bartel Leendert van der Waerden. *A History of Algebra. From al-Khwarizmi to Emmy Noether.* Springer, Berlin, 1985.
32. François Viète. *De numerosa potestatum ad exegesim resolutione.* Paris, 1600.
33. François Viète. *Opera mathematica, edited by Frans van Schooten.* Leiden, 1646.
34. François Viète. *The analytic art. Translated by T.Richard Witmer.* The Kent State University Press, 1983.
35. John Wallis. *A treatise of algebra, both historical and practical.* London, 1685.
36. John Wallis. A discourse concerning the methods of approximation in the extraction of surd roots. *Philosophical Transactions (1683-1775)*, 19:2–11, 1695.
37. John Ward. *A Compendium of Algebra.* London, 1695.
38. John Ward. *The Young Mathematician's Guide. Being a Plain and Easie Introduction to the Mathematicks.* London, 1707. Reprinted 1709.
39. Derek Thomas Whiteside. *The Mathematical Papers of Isaac Newton 1664-1666, Volume I.* Cambridge University Press, Cambridge, 1967.
40. Tjalling J. Ypma. Historical development of the Newton-Raphson method. *SIAM Review*, 37(4):531–551, 1995