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Short note

Convergence of energy stable finite-difference schemes with interfaces

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ABSTRACT

We extend the convergence results in Svärd and Nordström (2019) [7] for single-domain energy-stable high-order finite difference schemes, to include domains split into several grid blocks. The analysis also demonstrates that reflective boundary conditions enjoy the same convergence properties. Finally, we briefly indicate that these results (and the previous ones in [7]) also hold in multiple dimensions.

1. Background

The convergence rate of finite-difference schemes that are closed at the boundaries with stencils of lower order accuracy[1] than the interior stencils is a long-standing problem that has been treated extensively in the literature in e.g. [3,4,1,5]. Recently the problem was addressed in [7], where it was shown that energy-stable schemes, satisfying some natural constraints, automatically "gain" as many orders on the boundary as the highest spatial derivative.

The case when the domain is split into (at least) two computational domains that are conjoined at an interface, was not treated in [7]. The interface case is fundamentally different from the boundary case. In the theory for boundary conditions, the semi-discretisation of the initial-boundary value problem is Laplace transformed. This allows a decomposition of the modes into those that decay into the domain from the boundary, and should be supplied with boundary conditions, and those that do not. The boundedness of the boundary data and the energy stability of the scheme, enabled a proof of the convergence result.

This proof, however, does not immediately encompass interfaces, since they take data from the conjoining domain, which may or may not be sufficiently bounded. Moreover, the treatment at an interface need not abide to the minimal number of boundary conditions requirement, since data at an interface can be considered exact. Likewise, the previous theory did not consider reflective boundaries, i.e., those that feed out-going waves back into the domain via the in-going characteristics, since sufficient boundedness of the solution that is fed back has to be established.

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[1] Convergence rate is the rate by which the solution error decreases with grid refinements. The order of accuracy is given by the truncation error and can be different at different points. It represents how accurately the equation (not the solution) is approximated.

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Here, we address both interfaces and reflective boundary conditions and we discuss possible generalisations to multiple dimensions. Specifically, we require

(1) that the semi-discrete scheme is energy stable, has a polynomial nullspace (possibly empty), is nullspace consistent, and nullspace invariant. (All the assumptions used in [7].)
(2) that the number of interface conditions, as seen from one side, is consistent with the minimal number of boundary conditions. That is, data from across the interface can be replaced by smooth and bounded data, and result in a well-posed problem.

Remark. *The second requirement immediately rids one of the problems discussed above. It prohibits over-specification at the interface. Furthermore, it is the most robust choice that is most often used in practice.*

The programme for analysing interface accuracy is to view the interface as an external boundary and, with the aid of the requirements above, the a priori estimates and previous theory, infer the optimal convergence rates.

We will demonstrate this for a few examples. The generalisations are obvious and will only be discussed briefly.

2. The hyperbolic case

We consider the advection equation on the real line,

$$u_t + au_x = 0, \quad a > 0. \tag{1}$$

We split the domain into two: $x < 0$ and $x > 0$ with an interface at $x = 0$ and discretise each with a Summation-by-Parts (SBP) scheme (see [6]). (We assume appropriate trail-off conditions to the left and right and denote the left and right domain as $\Omega_{L,R}$.) The grid is equidistant with indexing $x_i = ih$, $i = 0, \pm 1, \pm 2, ...$, such that x_0 represents the interface in both domains.

$$\begin{aligned} \mathbf{v}_t^L + aP^{-1}Q\mathbf{v}^L &= 0 \\ \mathbf{v}_t^R + aP^{-1}Q\mathbf{v}^R &= -P^{-1}a(v_0^R - v_0^L)\mathbf{e}_0 \end{aligned} \tag{2}$$

where $\mathbf{v}^L = (...v_{-1}^L, v_0^L)^T$ and $\mathbf{v}^R = (v_0^R, v_1^R, ...)^T$ are the unknowns to the left and right and $\mathbf{e}_0 = (1, 0, ...0)^T$. The SBP operator has the following properties: $P = P^T > 0$, $Q + Q^T = B = diag(-1, 0, ..., 0, 1)$ and $P^{-1}Q\mathbf{v}$ is a first-derivative approximation. (See [7] for more information.) To adhere with the second requirement above, there is only a penalty term to the right in agreement with the characteristic direction.

Remark. *Generally, one can penalise both equations at x_0 and, with a proper scaling, stability can be proven. Although stable and convergent, such a "two-way" coupling renders the left problem ill-posed viewed as a stand-alone problem. Hence, Requirement (2) is not satisfied.*

We apply the energy method in a somewhat unconventional way. Multiply the first equation in (2) by $\mathbf{v}^L P$ to obtain

$$\left(\|\mathbf{v}_L\|^2\right)_t + a(v_0^L)^2 = 0.$$

The left problem is well-posed as a stand alone problem and we obtain the bound $v_0^L \in L^2(0, T)$. Next, we multiply the second equation in (2) by $(\mathbf{v}^R)^T P$,

$$\left(\|\mathbf{v}_R\|^2\right)_t - a(v_0^R)^2 = -2av_0^R(v_0^R - v_0^L).$$

By recasting the boundary terms, we obtain

$$\left(\|\mathbf{v}_R\|^2\right)_t + a((v_0^R - v_0^L)^2 = a(v_0^L)^2.$$

This problem is bounded, and well-posed, since $v_0^L \in L^2(0, T)$. Hence, the theory of [7] applies to both the left and right problems individually and we conclude that (2) will converge with optimal rates.

This procedure can be generalised to symmetric systems, $u_t + Au_x = 0$ discretised by

$$\begin{aligned} \mathbf{v}_t^L + (A \otimes P^{-1}Q)\mathbf{v}^L &= +(A^- \otimes P^{-1})((v_0^L - v_0^R) \otimes \mathbf{e}_0^L) \\ \mathbf{v}_t^R + (A \otimes P^{-1}Q)\mathbf{v}^R &= -(A^+ \otimes P^{-1})((v_0^R - v_0^L) \otimes \mathbf{e}_0^R) \end{aligned} \tag{3}$$

where $\mathbf{e}^{L,R}$ are both one at $x = 0$ and zero elsewhere. (For more information on Kronecker products, see [6].) The division $A = A^+ + A^-$ refers to the signs of the eigenvalues such that only in-going characteristics are penalised on each side of the interface. As in the scalar case, the outgoing characteristics on each side serve as data for the other side and are independently bounded in $L^2(0, T)$ such that optimal rates are obtained.

2.1. Reflective boundary conditions

Another way of looking at (1) is obtained by transforming the left domain, $x<0$, to $\xi>0$. That is, by $x=-\xi$ and $\partial_x=-\partial_\xi$, such that,

$$u^L_t - a u^L_\xi = 0, \quad \xi > 0. \tag{4}$$

Relabelling ξ as x, $u^L=u^1$, $u^R=u^2$, we obtain the system,

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_t + \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_x = 0, \quad x>0$$

with the boundary condition, $u^1(0,t)=u^2(0,t)$. (In particular we note that the problem is well-posed if $u^2(0,t)=g(t)\in L^2(0,T)$.) The previously derived a priori estimate guarantees that $u^1(0,t)\in L^2(0,T)$ and we can view the problem as a well-posed problem with a *reflective* external boundary that consequently enjoy optimal convergence rates.

3. The parabolic case

We consider the same setup for the advection-diffusion equation, $u_t + a u_x = \epsilon u_{xx}$, with an interface at $x=0$ and trail-off conditions to the left and right. The SBP second-derivative operator is given as: $D_2 = P^{-1}(-A+BS)$ where A is symmetric positive semi-definite and S holds a first-derivative stencils at the boundary points. Following [2], the approximation becomes,

$$\begin{aligned} P\mathbf{v}^L + aQ\,\mathbf{v}^L &= \epsilon(-A+BS)\mathbf{v}^L + \sigma_1 \mathbf{e}_0(v^L_0 - v^R_0) + \sigma_2\epsilon\mathbf{e}_0((S\mathbf{v}^L)_0 - (S\mathbf{v}^R)_0) \\ P\mathbf{v}^R + aQ\,\mathbf{v}^R &= \epsilon(-A+BS)\mathbf{v}^R + \sigma_3 \mathbf{e}_0(v^L_0 - v^0_0) + \sigma_4\epsilon\mathbf{e}_0((S\mathbf{v}^R)_0 - (S\mathbf{v}^L)_0). \end{aligned} \tag{5}$$

In [2] the stable choices of the σ's are derived. Furthermore, these σ's have the following dependencies of h: $\sigma_{1,3}\sim\mathcal{O}(h^{-1})$ and $\sigma_{2,4}\sim\mathcal{O}(1)$.

Consider the left problem. It is an approximation of the advection-diffusion equation with boundary condition

$$\alpha^L u^L + h\beta^L u^L_x = \alpha^L u^R + h\beta^L u^R_x = f(u^R, u^R_x). \tag{6}$$

A sufficient condition for this stand-alone problem to be well-posed is $u^R\in L^2(0,T)$ and $u^R_x\in L^2(0,T)$. (Or more precisely, $hu^R_x\in L^2(0,T)$.) Furthermore, we simultaneously demand the mirrored conditions on the right side.

Remark. *Note that the interface condition is an approximation of $u^L(0,t)=u^R(0,t)$ and not a Robin-type condition since $\sigma_{1,3}$ are $\mathcal{O}(h^{-1})$ and $\sigma_{2,4}$ are $\mathcal{O}(1)$. Furthermore, parabolicity requires a boundary condition on all boundaries implying that an interface condition on both sides of the interface does not imply over-specification of the stand-alone problems, in accordance with Requirement (2).*

Next, we investigate the stability of the stand-alone problems. That is, we want the numerical approximations of (6) to be bounded in $L^2(0,T)$. By perusing the stability proof in [2], it is clear that $(S\mathbf{v}^L)_0$ and $(S\mathbf{v}^R)_0$ are indeed bounded in $L^2(0,T)$. (See p. 358-9. The eigenvalues associated with $(S\mathbf{v}^L)_0+(S\mathbf{v}^R)_0$ and $(S\mathbf{v}^L)_0-(S\mathbf{v}^R)_0$ are non-zero and non-vanishing.) However, a bound on v^R_0 or v^L_0, which is needed for stand-alone well-posedness of the two problems, is not obtained from the interface stability. (The corresponding eigenvalue is zero on p. 359.) To recover the necessary estimates on v^R_0 or v^L_0, we proceed via the energy estimate of (5) which gives the bounds

$$\begin{aligned} &\sup_{t\in\{0,T\}} \|\mathbf{v}^L\| \le \mathcal{C}, \qquad \int_0^T (\mathbf{v}^L)A\mathbf{v}^L\,dt \le \mathcal{C}, \\ &\sup_{t\in\{0,T\}} \|\mathbf{v}^R\| \le \mathcal{C}, \qquad \int_0^T (\mathbf{v}^R)A\mathbf{v}^R\,dt \le \mathcal{C}. \end{aligned} \tag{7}$$

Since the estimates obtained with the A-matrix above correspond to L^2-estimates of the gradients, we intend to use Sobolev embedding to bound $\mathbf{v}^L$ and $\mathbf{v}^R$ in $L^2(0,T;L^\infty(\Omega_{L,R}))$. To demonstrate this, we define the $N\times N$ matrix

$$D = h^{-1}\begin{pmatrix} -1 & 1 & 0 & \dots & \\ 0 & -1 & 1 & 0 & \dots \\ & & & \ddots & \\ & & & -1 & 1 \\ & & & 0 & 0 \end{pmatrix},$$

such that $D\mathbf{v}$ is an approximation of the derivative in the domain and $\|D\mathbf{v}\|_h^2 = h\mathbf{v}^T D^T D\mathbf{v}$ bounds the L^2-norm of the derivative of $\mathbf{v}$ (interpreted as a piecewise linear function).

Lemma 3.1. *For any grid function* $\mathbf{v}$, $\mathbf{v}^T A\mathbf{v} > c\|D\mathbf{v}\|_h^2$ *for some* $c > 0$.

Proof. By assumption of nullspace consistency, A has one zero eigenvalue associated with its rows summing to zero. (By symmetry its columns therefore also sum to zero.) Furthermore, its components are $\mathcal{O}(h^{-1})$ and we assume that A is $N \times N$.

A Cholesky factorisation is given as $U^T U = A$. Since A is semi-definite, it will result in one zero on the diagonal of the upper-triangular matrix U. Furthermore, the banded structure of A is preserved in U. Since $A\mathbf{1} = \mathbf{0}$, we must have $U\mathbf{1} = \mathbf{0}$. Hence, the element in the lower right corner has to be 0.

Since the rows of U sum to zero, we can write $U = RD$, where R is an $N \times N$ matrix with the last row being zero. (The last column is also taken to be zero since it does not contribute anyway due to the band structure.) Furthermore, no other row of R sum to zero, since (most) rows of $A = D^T R^T RD$ are second derivative approximations and the two derivatives are accounted for by D^T and D. Hence, $R^T R$ is symmetric positive semi-definite with one zero eigenvalue associated with the last row and column. Moreover, we denote the upper-left $(N-1) \times (N-1)$ submatrix of $R^T R$ as M, which is then symmetric positive definite and has elements of $\mathcal{O}(h)$. Furthermore, since R is banded, $M \geq chI$ for some $c > 0$ where I is the $(N-1)^2$-identity matrix. Consequently, $(D\mathbf{v})^T R^T R(D\mathbf{v}) \geq c\|D\mathbf{v}\|_h^2$, since the zero in the last row of $R^T R$ coincide with the zero in $D\mathbf{v}$ and M defines a norm. □

By (7) and Lemma 3.1, we have $D\mathbf{v}^{L,R} \in L^2(0,T; L^2(\Omega_{L,R}))$ which, together with the L^2 bounds in (7) on $\mathbf{v}^{L,R}$ and Sobolev embedding, implies $v_i^{L,R} \in L^2(0,T; L^\infty(\Omega_{L,R}))$ for all i. With these a priori estimates, and since we already know from the stability proof that $(S\mathbf{v}^{L,R})_0 \in L^2(0,T)$, the right problem bounds the "data" f in (6) for the left problem (and vice versa for the right problem).

Remark. *Above, we used the bound on* $(S\mathbf{v}^{L,R})_0$ *obtained from the stability proof. However, since only* $h(S\mathbf{v}^{L,R})_0 \in L^2(0,T)$ *was needed, we could also have obtained that directly from the* $L^2(0,T; L^2(\Omega_{L,R}))$ *bounds on* $D\mathbf{v}^{L,R}$, *which gives* $\sqrt{h}D\mathbf{v}^{L,R} \in L^2(0,T; L^\infty(\Omega_{L,R}))$. *Seeing that* $(S\mathbf{v}^{L,R})_0$ *are linear combinations of* $(D\mathbf{v}^{L,R})_i$, *the* $L^2(0,T)$ *bounds on* $\sqrt{h}(S\mathbf{v}^L)_0$ *and* $\sqrt{h}(S\mathbf{v}^R)_0$ *follow.*

We end this section with a few remarks. The same reasoning applies to parabolic systems since they too yield $L^2(0,T)$ bounds on both the variables and their gradients, which is the only requirement needed to apply the theory in [7]. Furthermore, by mirroring one domain and rewriting a parabolic equation (scalar or system) such that the interface turns into a reflective boundary, the convergence rate of the latter follows immediately.

4. Multi-dimensions

A possible problem with the previous procedure is that L^∞ is not embedded in H^1 in multiple dimensions. However, in multiple dimensions the boundedness requirements are suitably milder. Consider the 2D advection-diffusion equation, $u_t + au_x + bu_y = \epsilon(u_{xx} + u_{yy})$, with periodicity in the y-direction (and N points).

Using von Neumann analysis in the y-direction reduces the problem to $k = 1...N$ 1-D problems, one for each mode, $\hat{\mathbf{v}}_k^{L,R}$ in the y-direction. The y-derivative is reduced to a low-order term and can be ignored. The H^1 embedding implies an $L^2(L^\infty)$ bound on $\hat{\mathbf{v}}_k$, for all $k = 1...N$. Each mode converges with optimal rate and by Parseval's relation, so does $\mathbf{v}^{L,R}$.

This reasoning shows that a bound on $\int_0^T \int_{\partial\Omega_{L,R}} (u^{L,R})^2\, ds\, dt$ is sufficient in the periodic (in the extra dimensions) case, and given by the H^1 embedding. Theory for the non-periodic case is lacking, but it appears reasonable that the same bound would be sufficient.

CRediT authorship contribution statement

Magnus Svärd: Writing original draft. Jan Nordström: Reviewing and editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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