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Continuity of Formal Power Series Products in Nonlinear Control Theory

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Abstract

Formal power series products appear in nonlinear control theory when systems modeled by Chen–Fliess series are interconnected to form new systems. In fields like adaptive control and learning systems, the coefficients of these formal power series are estimated sequentially with real-time data. The main goal is to prove the continuity and analyticity of such products with respect to several natural (locally convex) topologies on spaces of locally convergent formal power series in order to establish foundational properties behind these technologies. In addition, it is shown that a transformation group central to describing the output feedback connection is in fact an analytic Lie group in this setting with certain regularity properties.

Keywords Nonlinear control systems · Chen–Fliess series · System interconnection · Silva space · Real analytic · Locally convex Lie group · Regularity of Lie groups

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1 Introduction

The interconnection of simple input-output systems to form more complex and useful systems is commonplace in science and engineering. When each component is a nonlinear dynamical system, a weighted infinite sum of iterated integrals known as a *Chen–Fliess series*, F_c, provides a convenient way to represent its local behavior [15–17, 35, 43]. When this series converges on some set of admissible inputs, F_c defines a so called *Fliess operator*. It is uniquely specified by a formal power series c in $\mathbb{K}\langle\langle X\rangle\rangle$, known as its *generating series*, where X is a finite set of indeterminates, and $\mathbb{K}$ is a suitable field. An interconnection of two Chen–Fliess series F_c and F_d, represented by $F_c \square F_d$, induces a corresponding algebra $(\mathbb{K}\langle\langle X\rangle\rangle, \square')$ so that $F_c \square F_d = F_{c\square' d}$ [13–15, 25, 29]. Algebras defined in this manner provide computational frameworks for explicitly computing the generating series of interconnected systems for the purposes of analysis and design, especially in the field of nonlinear control theory. Historically, the coefficients of c have been determined by direct calculations using state space models derived from physical laws and other first principles [33, 38]. But with the growth of adaptive control and new types of learning based technologies, there is increasing interest in estimating these coefficients using real-time data and numerical methods from the field of system identification [23, 39]. Assuming that a given sequence of estimates asymptotically approaches its true value as more data are collected, a difficult problem in its own right, there is a fundamental question regarding continuity.

Consider a sequence of generating series c_i, $i \geq 1$ known to produce a sequence of corresponding Fliess operators F_{c_i}, $i \geq 1$. If $c_i \to c$ in some manner, is it also true that F_{c_i}, $i \geq 1$ converges to a well-defined Fliess operator F_c, i.e., does the limit point c ensure a convergent Chen–Fliess series? The answer, of course, depends directly on the ambient sets and the assumed topologies. For example, in [9, 44] the claim is shown to be false on the subset of *locally convergent* series in $\mathbb{K}\langle\langle X\rangle\rangle$, (i.e., a set of generating series under which their corresponding Fliess operators are known to converge for sufficiently small arguments in an L^p norm sense.) endowed with the ultrametric topology and where the operator space has an L^p type topology. As the ultrametric topology mirrors the algebra but provides almost no information on the analytic behavior of the series, this outcome is not surprising. On the other hand, in [9] the claim is shown to be true when the ultrametric topology is replaced with a certain Banach topology on a subspace. Ultimately, the question boils down to identifying topological vectors spaces contained in $\mathbb{K}\langle\langle X\rangle\rangle$ which ensure that every limit point is a generating series with a well-defined Fliess operator in some sense.

The main goal of this paper is to address a natural follow-up question: Suppose $c_i, d_i \in \mathbb{K}\langle\langle X\rangle\rangle$, $i \geq 1$ are two sequences of generating series converging to c and d, respectively, such that F_{c_i} and F_{d_i}, $i \geq 1$ are well defined Fliess operators as are their limits. Assuming that $c\Box' d$ has the same convergence properties as c and d (this theory is well understood, see [30, 42, 44]), under which conditions does $c_i\Box' d_i \mapsto c\Box' d$? Is it even possible to identify infinite-dimensional spaces with respect to which the products are smooth or even analytic?

Three formal power series products will be considered: the shuffle product, which models a type of parallel connection [15]; a composition product modeling series connections [13, 14, 29]; and a group product for a transformation group known to model dynamic output feedback, a central object of study in control theory [25]. In addition, the continuity of the shuffle inverse will be addressed. (A preliminary version of this analysis was presented in [40].) The shuffle group appears in the context of feedback linearization [26, 27]. In each case continuity will be considered in both the Fréchet and Silva topologies. In addition, analyticity of these products will be characterized. It should be noted that the Fréchet topology was used in [44] to show that the shuffle and composition products preserve a type of *global convergence*. Continuity issues in this setting are beyond the scope of the present paper. However, the Fréchet topology is employed as a natural (locally convex) topology on the space of all power series. Convergence in this topology does not preserve growth bounds. Thus, it is necessary to endow the space of locally convergent series with the finer Silva topology.

Next it will be shown that the output feedback transformation group is a *locally convex* Lie group (see [37] for a survey on (infinite-dimensional) Lie theory). This result builds on the development of a pre-Lie algebra presented in [12, 18]. Lie groups have a long history in feedback control theory originating with the work of Brockett in [7]. More recent applications in this context have appeared in [27, 28], albeit only in the formal case where an explicit differential structure is not specified. The present work will provide a means to fill this gap. Finally, the regularity of these Lie groups is investigated. Roughly speaking, regularity of a Lie group asks for the existence and smooth parameter dependence of certain ordinary differential equations on the Lie group. Note that since the Lie groups at hand are not modeled on Banach spaces, the usual theory for existence and uniqueness of ordinary differential equations does not apply. However, it is shown that the Fréchet Lie groups are regular. While some progress on the regularity problem is made for the Silva Lie groups, their regularity largely remains an open problem that the authors plan to pursue in future work.

2 Preliminaries

Throughout this paper let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, namely either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. It will be essential to admit complex coefficients in order to discuss analyticity of mappings on infinite-dimensional spaces. Note that the continuity results are unaffected by this choice. Refer to Appendix A for more information regarding calculus on infinite-dimensional spaces and “Appendix B” for a summary of the primary notation used in this paper.

2.1 Chen–Fliess Series

An *alphabet* $X = \{x_0, x_1, \dots, x_m\}$ is any non-empty and finite set of symbols referred to as *letters*. A *word* $\eta = x_{i_1} \cdots x_{i_k}$ is a finite sequence of letters from X. The number of letters in a word η, written as $|\eta|$, is called its *length*. The empty word, $\emptyset$, is taken to have length zero. The number of times the letter x_i appears in a word η is written as $|\eta|_{x_i}$. The collection of all words having length k is denoted by X^k. Define the set of all words $X^* = \bigcup_{k\geq 0} X^k$, which constitutes a non-commutative monoid under the concatenation product. Let ℓ be a natural number. Any mapping $c\colon X^* \to \mathbb{K}^\ell$ is called a *formal power series*. Often c is written as the formal sum $c = \sum_{\eta\in X^*}(c,\eta)\eta$, where the *coefficient* (c,η) is the image of $\eta \in X^*$ under c. The *support* of c, supp(c), is the set of all words having nonzero coefficients. A series c is said to be *proper* when $\emptyset \notin \mathrm{supp}(c)$. The set of all non-commutative formal power series over the alphabet X is denoted by $\mathbb{K}^\ell\langle\langle X\rangle\rangle$. The subset of series with finite support, i.e., polynomials, is represented by $\mathbb{K}^\ell\langle X\rangle$. If multiplication on $\mathbb{K}^\ell$ is defined componentwise, then each set is an associative $\mathbb{K}$-algebra under the concatenation product and an associative and commutative $\mathbb{K}$-algebra under the *shuffle product*, that is, the bilinear product uniquely specified by the shuffle product of two words

$$(x_i\eta) ⧢ (x_j\xi) = x_i(\eta ⧢ (x_j\xi)) + x_j((x_i\eta) ⧢ \xi),$$

where $x_i, x_j \in X$, $\eta, \xi \in X^*$ and with $\eta ⧢ \emptyset = \emptyset ⧢ \eta = \eta$ [15].

$\mathbb{K}^\ell\langle\langle X\rangle\rangle$ can also be viewed as a locally convex space whose topology is briefly described next. First note that identifying a formal power series with the sequence of its coefficients defines an isomorphism of vector spaces $\mathbb{K}^\ell\langle\langle X\rangle\rangle \cong \prod_{\eta\in X^*}\mathbb{K}^\ell$. The space on the right-hand side is a countable product of Banach spaces, hence a complete metrizable locally convex vector space (i.e., a Fréchet space). Thus, $\mathbb{K}^\ell\langle\langle X\rangle\rangle$ inherits a canonical Fréchet space structure. By construction the evaluation functionals $a_\eta\colon \mathbb{K}^\ell\langle\langle X\rangle\rangle \to \mathbb{K}^\ell$, $c \mapsto (c,\eta)$ are continuous. Therefore, convergence in this topology is equivalent to separate convergence of all coefficients of a series toward the corresponding coefficients of the limit series. Moreover, the Fréchet topology is initial with respect to the point evaluations, i.e., a map f to $\mathbb{K}^\ell\langle\langle X\rangle\rangle$ is continuous if and only if $a_\eta \circ f$ is continuous for every word $\eta \in X^*$.

Given any $c \in \mathbb{K}^\ell\langle\langle X\rangle\rangle$ one can associate a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{K}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable $\mathbb{K}$-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm. The closed ball of radius $R > 0$ is $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$.

Define inductively for each $\eta \in X^*$ the map $E_\eta : L_1^m[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The *Chen–Fliess series* corresponding to c is

$$y(t) = F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0) \tag{1}$$

[15, 16]. It can be shown that if there exists real numbers $K, M \geq 0$ such that

$$|(c, \eta)| \leq K M^{|\eta|}\, |\eta|!, \quad \forall \eta \in X^* \tag{2}$$

($|z| := \max_i |z_i|$ when $z \in \mathbb{K}^\ell$), then the series defining F_c converges absolutely and uniformly for sufficiently small $R, T > 0$ and constitutes a well-defined mapping from $B_1^m(R)[t_0, t_0 + T]$ into $B_\infty^\ell(S)[t_0, t_0 + T]$ for some $S > 0$. Any such mapping is called a *locally convergent Fliess operator*. Here, $\mathbb{K}_{LC}^\ell\langle\langle X\rangle\rangle$ will denote the set of all such *locally convergent* generating series, i.e., those series satisfying growth condition (2). Given any smooth state space realization of $y = F_c[u]$,

$$\dot{z} = g_0(z) + \sum_{i=1}^{m} g_i(z) u_i, \quad z(0) = z_0, \quad y = h(z),$$

it is known that the generating series c is determined by

$$(c_j, \eta) = L_{g_{i_1}} \cdots L_{g_{i_k}} h_j(z_0), \quad \eta = x_{i_k} \cdots x_{i_1} \in X^*, \quad j = 1, 2, \ldots, \ell \tag{3}$$

where $L_{g_i} h_j$ is the Lie derivative of h_j with respect to g_i.

2.2 Formal Power Series Products Induced by System Interconnection

Given Fliess operators F_c and F_d, where $c, d \in \mathbb{K}_{LC}^\ell\langle\langle X\rangle\rangle$, the parallel and product connections satisfy $F_c + F_d = F_{c+d}$ and $F_c F_d = F_{c \sqcup\!\sqcup d}$, respectively [15]. When Fliess operators F_c and F_d with $c \in \mathbb{K}_{LC}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{K}_{LC}^m\langle\langle X\rangle\rangle$ are interconnected in a cascade fashion, the composite system $F_c \circ F_d$ has the Fliess operator representation $F_{c \circ d}$, where the *composition product* of c and d is given by

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, \psi_d(\eta)(\mathbf{1}) \tag{4}$$

[13, 14]. Here, $\mathbf{1}$ denotes the monomial $1\emptyset$, and ψ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{K}\langle\langle X\rangle\rangle$ to the set $\mathrm{End}(\mathbb{K}\langle\langle X\rangle\rangle)$ of vector space endomorphisms on $\mathbb{K}\langle\langle X\rangle\rangle$ uniquely specified by $\psi_d(x_i \eta) = \psi_d(x_i) \circ \psi_d(\eta)$ with

$\psi_d(x_i)(e) = x_0(d[i] \sqcup\!\sqcup e)$, $i = 0, 1, \ldots, m$ for any $e \in \mathbb{K}\langle\langle X\rangle\rangle$, and where $d[i]$ is the i-th component series of d ($d[0] := \mathbf{1}$). By definition, $\psi_d(\emptyset)$ is the identity map on $\mathbb{K}\langle\langle X\rangle\rangle$.

When two Fliess operators F_c and F_d are interconnected to form a feedback system with F_c in the forward path and F_d in the feedback path, the generating series of the closed-loop system is denoted by the *feedback product* $c@d$. It can be computed explicitly using the Hopf algebra of coordinate functions associated with the underlying *output feedback group* [25]. Specifically, in the single-input, single-output case where $X = \{x_0, x_1\}$ and $\ell = 1$, define the set of *unital* Fliess operators $\mathscr{F}_\delta = \{I + F_c \ : \ c \in \mathbb{K}_{LC}\langle\langle X\rangle\rangle\}$, where I denotes the identity map. It is convenient to introduce the symbol δ as the (fictitious) generating series for the identity map. That is, $F_\delta := I$ such that $I + F_c := F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$. The set of all such generating series for $\mathscr{F}_\delta$ will be denoted by $\delta + \mathbb{K}_{LC}\langle\langle X\rangle\rangle$. The central idea is that $(\mathscr{F}_\delta, \circ, I)$ forms a group of operators under the composition

$$F_{c_\delta} \circ F_{d_\delta} = (I + F_c) \circ (I + F_d) = F_{c_\delta \circ d_\delta},$$

where $c_\delta \circ d_\delta := \delta + c \circledcirc d$, $c \circledcirc d := d + c \,\tilde{\circ}\, d_\delta$, and $\tilde{\circ}$ denotes the *mixed* composition product. That is, the product

$$c \,\tilde{\circ}\, d_\delta = \sum_{\eta \in X^*} (c, \eta)\, \phi_d(\eta)(\mathbf{1}), \tag{5}$$

where ϕ_d is analogous to ψ_d in (4) except here $\phi_d(x_i)(e) = x_i e + x_0(d[i] \sqcup\!\sqcup e)$ with $d[0] := 0$ [29]. The set of unital generating series $\delta + \mathbb{K}\langle\langle X\rangle\rangle$ (not necessarily locally convergent) forms a group $(\delta + \mathbb{K}\langle\langle X\rangle\rangle, \circ, \delta)$. The restriction to the set of locally convergent series defines the subgroup $\delta + \mathbb{K}_{LC}\langle\langle X\rangle\rangle$. The mixed composition product can be viewed as a right action of $\delta + \mathbb{K}\langle\langle X\rangle\rangle$ acting freely on $\mathbb{K}\langle\langle X\rangle\rangle$ [24]. The corresponding Hopf algebra H is the free algebra generated by the coordinate maps

$$a_\eta : \delta + \mathbb{K}\langle\langle X\rangle\rangle \to \mathbb{K}, c_\delta \mapsto (c, \eta), \quad \eta \in X^*$$

under the commutative product

$$\mu : a_\eta \otimes a_\xi \mapsto a_\eta a_\xi,$$

where the unit $\mathbf{1}_\delta$ is defined to map every c_δ to one. Let V be the $\mathbb{K}$-vector space of coordinate functions. If the *degree* of a_η is defined as $\deg(a_\eta) = 2\,|\eta|_{x_0} + |\eta|_{x_1} + 1$, then both V and the algebra H are graded and connected with $V = \bigoplus_{n\geq 0} V_n$ and $H = \bigoplus_{n\geq 0} H_n$, where V_n and H_n are sets containing all the degree n elements, and $V_0 = H_0 = \mathbb{K}\mathbf{1}_\delta$. The coproduct Δ is defined so that

$$\Delta a_\eta(c_\delta, d_\delta) = a_\eta(c_\delta \circ d_\delta) = (c_\delta \circ d_\delta, \eta).$$

Of primary importance is the following lemma which describes how the group inverse $c_\delta^{\circ -1} := \delta + c^{\circ -1}$ is computed.

Lemma 2.1 [25] *The Hopf algebra* (H, μ, Δ) *has an antipode* S *satisfying* $a_\eta(c_\delta^{\circ -1}) = (Sa_\eta)(c_\delta)$ *for all* $\eta \in X^*$ *and* $c_\delta \in \delta + \mathbb{K}\langle\langle X \rangle\rangle$.

With this concept, the generating series for the feedback connection, $c@d$, can be computed explicitly as described in the next theorem. It states that feedback in the present context can be viewed in terms of the group $(\delta + \mathbb{K}\langle\langle X \rangle\rangle, \circ, \delta)$ acting on $\mathbb{K}\langle\langle X \rangle\rangle$ in a specific manner.

Theorem 2.1 [25] *For any* $c, d \in \mathbb{K}\langle\langle X \rangle\rangle$ *it follows that*

$$c@d = c \,\tilde{\circ}\, (-d \circ c)_\delta^{\circ -1}.$$

In addition to the elementary system interconnections described above, there is the quotient connection that is useful in the context of system inversion [26]. This is a type of parallel connection where the quotient of the subsystems' outputs is computed. In terms of generating series, the quotient is realized using the shuffle inverse as described next. Division by zero is avoided by requiring the divisor series to be non-proper.

Theorem 2.2 [26] *The set of non-proper series in* $\mathbb{K}\langle\langle X \rangle\rangle$ *is a group under the shuffle product. In particular, the shuffle inverse of any such series* c *is*

$$c^{\sqcup\!\sqcup -1} = ((c, \emptyset)(1 - c'))^{\sqcup\!\sqcup -1} = (c, \emptyset)^{-1}(c')^{\sqcup\!\sqcup *},$$

where $c' := \mathbf{1} - c/(c, \emptyset)$ *is proper, and* $(c')^{\sqcup\!\sqcup *} := \sum_{k \geq 0} (c')^{\sqcup\!\sqcup k}$.

Theorem 2.3 [26] *For* $c, d \in \mathbb{K}_{LC}\langle\langle X \rangle\rangle$, *the quotient connection* F_c/F_d *has a Fliess operator representation if and only if* d *is non-proper. In particular,* $F_c/F_d = F_{c/d}$, *where* $c/d := c \sqcup\!\sqcup d^{\sqcup\!\sqcup -1}$.

3 Continuity of Formal Power Series Products

In this section, the continuity of the various products modeling system interconnections described in the previous section is proved. The main goal is to establish continuity on spaces of locally convergent series. In [9], the authors described the space of locally convergent Chen–Fliess series as a locally convex space carrying a Silva space topology. That construction is summarized first, and then the continuity results are presented.

Fix $M > 0$ and define

$$\|c\|_{\ell_{\infty,M}} := \sup \left\{ \frac{|(c, \eta)|}{M^{|\eta|} \, |\eta|!} : \eta \in X^* \right\} \in [0, \infty]$$

for each $c \in \mathbb{K}^\ell\langle\langle X \rangle\rangle$. The set of all c with $\|c\|_{\ell_{\infty,M}} < \infty$ is denoted by $\ell_{\infty,M}(X^*, \mathbb{K}^\ell)$. It is straightforward to check that $\ell_{\infty,M}(X^*, \mathbb{K}^\ell)$ is a vector subspace of $\mathbb{K}^\ell\langle\langle X \rangle\rangle$ and that the assignment $\|\cdot\|_{\ell_{\infty,M}}$ is a norm on $\ell_{\infty,M}(X^*, \mathbb{K}^\ell)$. This space is a Banach space as it is isometrically isomorphic to the Banach space of all bounded functions

$\ell_\infty(X^*, \mathbb{K}^\ell) := \{c\colon X^* \to \mathbb{K}^\ell : \sup_\eta |(c, \eta)| < \infty\}$. The Banach space of generating series bounded with respect to the constant M obviously does not capture all locally convergent series. Indeed for larger M one obtains series which converge only on a smaller disc. To capture all locally convergent series in one space, it is necessary to pass to the limit of these Banach spaces as described next.

Definition 3.1 (Locally convergent series as a Silva space) Consider the union

$$\mathbb{K}^\ell_{LC}\langle\langle X\rangle\rangle = \bigcup_{M>0} \ell_{\infty,M}(X^*, \mathbb{K}^\ell).$$

Topologize this space as the locally convex inductive limit of the system $\left(\ell_{\infty,M}(X^*, \mathbb{K}^\ell)\right)_{M>0}$.

One can show that the inclusion mappings in this sequence are compact operators, hence the resulting space is a *Silva space* [6, 10]. Since the sequence $M_k = k$, $k \in \mathbb{N}$ is cofinal, one can always find an $M \in \mathbb{N}$ for which $\|c\|_{\ell_{\infty,M}} < \infty$. Thus, one could equivalently work only with $M \in \mathbb{N}$. Though the Silva space topology is more complicated than the Banach spaces from which it was built, some of its properties make it very amenable for the applications considered here. The most important properties are summarized in the next lemma. Refer to [45] for proofs and more information about Silva spaces.

Lemma 3.1 (Properties of Silva spaces)

(1) *A sequence converges in $\mathbb{K}^\ell_{LC}\langle\langle X\rangle\rangle$ if and only if there exists $M > 0$ such that the sequence is contained and converges in the Banach space $\ell_{\infty,M}(X^*, \mathbb{K}^\ell)$.*
(2) *Silva spaces are sequential, meaning that a map from a Silva space into an arbitrary topological space is continuous if and only if it is sequentially continuous. Moreover, Silva spaces are separable and finite products of Silva spaces are again Silva spaces.*
(3) *A mapping $f\colon \mathbb{K}^\ell_{LC}\langle\langle X\rangle\rangle \to E$ into a locally convex space is continuous (differentiable) if and only if for every $M > 0$ the induced mapping*

$$f_M := f|_{\ell_{\infty,M}(X^*,\mathbb{K}^\ell)}\colon \ell_{\infty,M}(X^*, \mathbb{K}^\ell) \to E$$

is continuous (differentiable).

Perhaps the most striking property of the Silva topology is that one can address continuity and differentiability questions in the Banach spaces from which the Silva space is built. This will be demonstrated in the next section addressing the continuity of formal power series products.

3.1 Continuity of Shuffle Product and Shuffle Inverse

The following lemma is a prerequisite for proving continuity of the shuffle product.

Lemma 3.2 *Fix $M > 0$ and let $M_\epsilon = M(1+\epsilon)$, $\epsilon > 0$. If $c, d \in \ell_{\infty,M}(X^*, \mathbb{K}^\ell)$, then $c \sqcup\!\sqcup\, d \in \ell_{\infty,M_\epsilon}(X^*, \mathbb{K}^\ell)$ for all $\epsilon > 0$. Moreover,*

$$\|c \sqcup\!\sqcup\, d\|_{\ell_\infty,M_\epsilon} \leq K_\epsilon \, \|c\|_{\ell_\infty,M} \|d\|_{\ell_\infty,M},$$

where $K_\epsilon = \sup_{\eta \in X^*}(|\eta|+1)/(1+\epsilon)^{|\eta|} \leq \hat{K}_\epsilon := e^{-1}(1+\epsilon)/(\log(1+\epsilon))$.

Proof For any $\eta \in X^*$

$$\begin{aligned}
|(c \sqcup\!\sqcup\, d, \eta)| &= \left| \sum_{k=0}^{|\eta|} \sum_{\substack{\nu \in X^k \\ \xi \in X^{|\eta|-k}}} (c, \nu)(d, \xi)(\nu \sqcup\!\sqcup\, \xi, \eta) \right| \\
&\leq \sum_{k=0}^{|\eta|} \sum_{\substack{\nu \in X^k \\ \xi \in X^{|\eta|-k}}} \|c\|_{\ell_\infty,M} M^k k! \; \|d\|_{\ell_\infty,M} M^{|\eta|-k} (|\eta|-k)! \; (\nu \sqcup\!\sqcup\, \xi, \eta) \\
&= \|c\|_{\ell_\infty,M} \|d\|_{\ell_\infty,M} M^{|\eta|} \sum_{k=0}^{|\eta|} k! \; (|\eta|-k)! \binom{|\eta|}{k} \\
&= \|c\|_{\ell_\infty,M} \|d\|_{\ell_\infty,M} M^{|\eta|} \sum_{k=0}^{|\eta|} |\eta|! \\
&= \|c\|_{\ell_\infty,M} \|d\|_{\ell_\infty,M} M^{|\eta|} (|\eta|+1)!.
\end{aligned}$$

Note that this bound is achievable when $c = \sum_{\eta \in X^*} K_c M^{|\eta|} |\eta|! \, \eta$ and $d = \sum_{\eta \in X^*} K_d M^{|\eta|} |\eta|! \, \eta$ for any $K_c, K_d \geq 0$. Now define $M_\epsilon = M(1+\epsilon)$ with $\epsilon > 0$ and rewrite the final inequality above as

$$\frac{|(c \sqcup\!\sqcup\, d, \eta)|}{M_\epsilon^{|\eta|} \, |\eta|!} \leq \|c\|_{\ell_\infty,M} \|d\|_{\ell_\infty,M} \frac{|\eta|+1}{(1+\epsilon)^{|\eta|}}, \quad \forall \eta \in X^*.$$

Taking the supremum over $\eta \in X^*$ gives

$$\|c \sqcup\!\sqcup\, d\|_{\ell_\infty,M_\epsilon} \leq K_\epsilon \, \|c\|_{\ell_\infty,M} \|d\|_{\ell_\infty,M}, \quad \forall \eta \in X^*,$$

where $K_\epsilon = \sup_{\eta \in X^*}(|\eta|+1)/(1+\epsilon)^{|\eta|}$. The upper bound for K_ϵ is found by showing that $f_\epsilon(x) = (x+1)/(1+\epsilon)^x$ has a single maximum at $x_\epsilon^* = (1/\log(1+\epsilon)) - 1 > 0$ when $0 < \epsilon \leq e - 1$, and $\hat{K}_\epsilon = f_\epsilon(x_\epsilon^*) = e^{-1}(1+\epsilon)/(\log(1+\epsilon))$. In this case, the upper bound is tight (see Fig. 1). For $\epsilon > e - 1$, $K_\epsilon = 1$ and $\hat{K}_\epsilon > 1$, and thus this upper bound is conservative.

Theorem 3.1 *The shuffle product is continuous on $\mathbb{K}\langle\langle X \rangle\rangle$ and $\mathbb{K}_{LC}\langle\langle X \rangle\rangle$ with respect to the Fréchet and the Silva topology, respectively.*

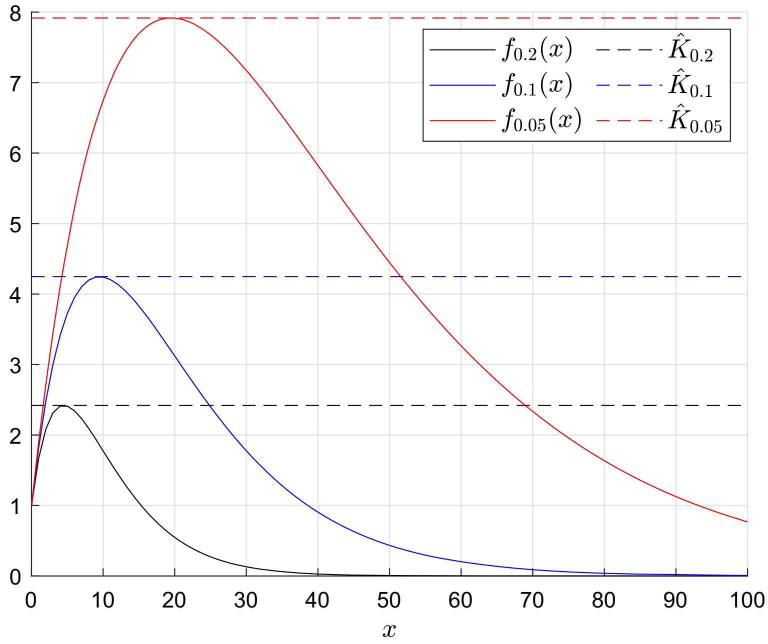

Fig. 1 Sample plots of $f_\epsilon(x)$ and $\hat{K}_\epsilon$ in Lemma 3.2

Proof Consider the shuffle product on $\mathbb{K}\langle\langle X\rangle\rangle$. Since the topology of $\mathbb{K}\langle\langle X\rangle\rangle$ is initial with respect to the coordinate functions $a_\eta\colon \mathbb{K}\langle\langle X\rangle\rangle \to \mathbb{K}$, $c \mapsto (c, \eta)$, it suffices to prove that $a_\eta \circ ⧢$ is continuous for each $\eta \in X^*$. However, as was seen in the proof of Lemma 3.2 for $\eta \in X^*$, it follows that

$$\begin{aligned} a_\eta \circ ⧢ (c, d) = (c ⧢ d, \eta) &= \sum_{k=0}^{|\eta|} \sum_{\substack{\nu \in X^k \\ \xi \in X^{|\eta|-k}}} (c, \nu)(d, \xi)(\nu ⧢ \xi, \eta) \\ &= \sum_{k=0}^{|\eta|} \sum_{\substack{\nu \in X^k \\ \xi \in X^{|\eta|-k}}} a_\nu(c) a_\xi(d)(\nu ⧢ \xi, \eta). \end{aligned}$$

This shows that $a_\eta \circ ⧢ (c, d)$ is a polynomial in the variables $a_\nu(c)$, $a_\xi(d)$. Since the coordinate functions are continuous in the series c and d, it is clear that the shuffle product is continuous. Thus, the shuffle product is continuous on $\mathbb{K}\langle\langle X\rangle\rangle$. For the corresponding result on $\mathbb{K}_{LC}\langle\langle X\rangle\rangle$, apply Lemma 3.2 for any $\epsilon > 0$:

$$\begin{aligned} \|(c ⧢ d) - (c_j ⧢ d_j)\|_{\ell_\infty, M_\epsilon} &= \|(c - c_j) ⧢ d + c_j ⧢ (d - d_j)\|_{\ell_\infty, M_\epsilon} \\ &\le \|(c - c_j) ⧢ d\|_{\ell_\infty, M_\epsilon} + \|c_j ⧢ (d - d_j)\|_{\ell_\infty, M_\epsilon} \\ &\le K_\epsilon \|(c - c_j)\|_{\ell_\infty, M} \|d\|_{\ell_\infty, M} + K_\epsilon \|c_j\|_{\ell_\infty, M} \|(d - d_j)\|_{\ell_\infty, M}. \end{aligned}$$

Thus, $\lim_{j\to\infty} \|(c \sqcup\!\sqcup d) - (c_j \sqcup\!\sqcup d_j)\|_{\ell_\infty, M_\epsilon} = 0$, proving the second part of the theorem.

The next lemma will be needed for proving continuity of the shuffle inverse and composition product.

Lemma 3.3 *If c and c_j, $j \geq 1$ are proper series in $\ell_{\infty,M}(X^*, \mathbb{K})$ for some $M \in \mathbb{N}$, and $\|c - c_j\|_{\ell_\infty, M} \to 0$ as $j \to \infty$, then for $N \in \mathbb{N}$ sufficiently large it follows that*

$$\sum_{n=1}^{\infty} \|c^{\sqcup\!\sqcup n} - c_j^{\sqcup\!\sqcup n}\|_{\ell_\infty, N} \to 0$$

as $j \to \infty$.

Proof It is first shown that the sum is uniformly bounded for some large enough $N \in \mathbb{N}$. The fact that $c_j \to c$ in $\ell_{\infty,M}$ and that $\|\cdot\|_{\ell_\infty,N} < \|\cdot\|_{\ell_\infty,M}$ for nonzero proper elements whenever $N > M$, implies that one can choose $N \in \mathbb{N}$ so that $\|c\|_{\ell_\infty,N} + \sup_{j\in\mathbb{N}} \|c_j\|_{\ell_\infty,N} \leq \frac{1}{2}$. Define a proper series $d \in \ell_{\infty,N}(X^*, \mathbb{K})$ by

$$(d, \eta) := (\|c\|_{\ell_\infty,N} + \sup_{j\in\mathbb{N}} \|c_j\|_{\ell_\infty,N})\, N^{|\eta|}\, |\eta|!, \quad \eta \neq \emptyset.$$

Then, $(d, \eta) \geq |(c, \eta)| + |(c_j, \eta)|$ for any $\eta \in X^*$ and all $j \in \mathbb{N}$. In fact, $(d^{\sqcup\!\sqcup n}, \eta) \geq |(c^{\sqcup\!\sqcup n}, \eta)| + |(c_j^{\sqcup\!\sqcup n}, \eta)|$ for any $n \geq 1$ and all $j \in \mathbb{N}$ by a standard induction argument. In particular, if $N' \geq N$, then

$$\|d^{\sqcup\!\sqcup n}\|_{\ell_\infty,N'} \geq \|c^{\sqcup\!\sqcup n} - c_j^{\sqcup\!\sqcup n}\|_{\ell_\infty,N'}.$$

Now observe that

$$\begin{aligned}
(d^{\sqcup\!\sqcup n}, \eta) &\leq (\|c\|_{\ell_\infty,N} + \sup_{j\in\mathbb{N}} \|c_j\|_{\ell_\infty,N})^n\, N^{|\eta|} \binom{(n-1)+|\eta|}{n-1} |\eta|! \\
&\leq \frac{1}{2^n} N^{|\eta|} \binom{(n-1)+|\eta|}{n-1} |\eta|! \\
&= \frac{1}{2^n} (4N)^{|\eta|}\, |\eta|! \binom{(n-1)+|\eta|}{n-1} \frac{1}{4^{|\eta|}}.
\end{aligned}$$

The first inequality can be shown via induction. Moreover, one can show the existence of a positive constant K for which

$$\binom{(n-1)+|\eta|}{n-1} \frac{1}{4^{|\eta|}} \leq K, \quad \forall n \in \mathbb{N},\ |\eta| \geq n.$$

As d is proper, $(d^{\sqcup\!\sqcup n}, \eta) = 0$ for all words $|\eta| < n$. Therefore,

$$(d^{\sqcup\!\sqcup n}, \eta) \leq \frac{1}{2^n} K (4N)^{|\eta|}\, |\eta|!, \quad \forall \eta \in X^*,$$

and so $\|d^{\sqcup\!\sqcup n}\|_{\ell_\infty,4N} \leq K/2^n$. From this one can see that

$$\sum_{n=1}^{\infty} \|c^{\sqcup\!\sqcup n} - c_j^{\sqcup\!\sqcup n}\|_{\ell_\infty,4N} \leq \sum_{n=1}^{\infty} \|d^{\sqcup\!\sqcup n}\|_{\ell_\infty,4N} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} K = K < \infty.$$

Having shown that the sum is uniformly bounded, it is now claimed that for each $n \geq 1$, $\lim_{j\to\infty} \|c^{\sqcup\!\sqcup n} - c_j^{\sqcup\!\sqcup n}\|_{4N} = 0$. If this holds, then

$$\lim_{j\to\infty} \sum_{n=1}^{\infty} \|c^{\sqcup\!\sqcup n} - c_j^{\sqcup\!\sqcup n}\|_{\ell_\infty,4N} = \sum_{n=1}^{\infty} \lim_{j\to\infty} \|c^{\sqcup\!\sqcup n} - c_j^{\sqcup\!\sqcup n}\|_{\ell_\infty,4N} = 0.$$

To prove the claim, define $N_n := N(1+(1-\frac{1}{n}))$ for $n \geq 1$ so that $2N > N_n > N_{n-1} > \cdots > N_1 = N$. It is shown by induction on $n \geq 1$ that $\|c^{\sqcup\!\sqcup n} - c_j^{\sqcup\!\sqcup n}\|_{\ell_\infty,N_n} \to 0$ as $j \to \infty$. The case $n = 1$ follows immediately as $N_1 = N$ and $\|c - c_j\|_{\ell_\infty,N} \leq \|c - c_j\|_{\ell_\infty,M}$. Let $n > 1$. Using the bilinearity of the shuffle product, it follows that

$$\begin{aligned}
\|c^{\sqcup\!\sqcup n} - c_j^{\sqcup\!\sqcup n}\|_{\ell_\infty,N_n} &= \|(c - c_j) \sqcup\!\sqcup c^{\sqcup\!\sqcup (n-1)} + c_j \sqcup\!\sqcup (c^{\sqcup\!\sqcup (n-1)} - c_j^{\sqcup\!\sqcup (n-1)})\|_{\ell_\infty,N_n} \\
&\leq K_n \|c - c_j\|_{\ell_\infty,N_{n-1}} \|c^{\sqcup\!\sqcup (n-1)}\|_{\ell_\infty,N_{n-1}} \\
&\quad + K_n \|c_j\|_{\ell_\infty,N_{n-1}} \|c^{\sqcup\!\sqcup (n-1)} - c_j^{\sqcup\!\sqcup (n-1)}\|_{\ell_\infty,N_{n-1}},
\end{aligned}$$

where as in Lemma 3.2 the $K_n > 0$ are the constants corresponding to the $\epsilon_n > 0$ for which $N_n = (1 + \epsilon_n)N_{n-1}$. By the induction hypothesis, the latter expression tends to zero, implying the same for the former. This proves the claim since $\|c^{\sqcup\!\sqcup n} - c_j^{\sqcup\!\sqcup n}\|_{\ell_\infty,4N} \leq \|c^{\sqcup\!\sqcup n} - c_j^{\sqcup\!\sqcup n}\|_{\ell_\infty,N_n}, \quad \forall n \geq 1$.

Proposition 3.1 *Denote by* $(\mathbb{K}_{LC}\langle\langle X\rangle\rangle)^\times$ *the set of invertible elements of the algebra* $(\mathbb{K}_{LC}\langle\langle X\rangle\rangle, \sqcup\!\sqcup)$. *The shuffle inverse*

$$\sqcup\!\sqcup^{-1} : (\mathbb{K}_{LC}\langle\langle X\rangle\rangle)^\times \to (\mathbb{K}_{LC}\langle\langle X\rangle\rangle)^\times\ ,\ c \mapsto (c, \emptyset)^{-1} \sum_{k\geq 0} (c')^{\sqcup\!\sqcup k},$$

where $c' = \mathbf{1} - c/(c, \emptyset)$, *is well defined and continuous.*

Proof Well definedness follows from [26, Theorem 5]. To show continuity, first observe that $(\mathbb{K}_{LC}\langle\langle X\rangle\rangle)^\times$ is an open subset of $\mathbb{K}_{LC}\langle\langle X\rangle\rangle$. Indeed it is easily verified that for any $\eta \in X^*$ the evaluation map a_η is continuous on the Silva space $\mathbb{K}_{LC}\langle\langle X\rangle\rangle$. In particular, $(\mathbb{K}_{LC}\langle\langle X\rangle\rangle)^\times = a_\emptyset^{-1}(\mathbb{K} \setminus \{0\})$ is open. Since $\mathbb{K}_{LC}\langle\langle X\rangle\rangle$ is sequential, the same is true for the open subset $(\mathbb{K}_{LC}\langle\langle X\rangle\rangle)^\times$, and consequently it suffices to test continuity of $\sqcup\!\sqcup^{-1}$ via sequences. With this in mind suppose $c_j \to c$ for elements $c_j, c \in (\mathbb{K}_{LC}\langle\langle X\rangle\rangle)^\times$, say $\|c - c_j\|_{\ell_\infty,M} \to 0$ for some $M > 0$. Then also $\|c' - c'_j\|_{\ell_\infty,M} \to 0$. Since c' and c'_j are proper series, applying Lemma 3.3 gives

$$\left\| \sum_{k=0}^{\infty} (c')^{\sqcup\!\sqcup k} - (c'_j)^{\sqcup\!\sqcup k} \right\|_{\ell_\infty,N} = \left\| \sum_{k=1}^{\infty} (c')^{\sqcup\!\sqcup k} - (c'_j)^{\sqcup\!\sqcup k} \right\|_{\ell_\infty,N} \to 0$$

for some $N > M$. Hence,

$$\|\sqcup\!\sqcup^{-1}(c) - \sqcup\!\sqcup^{-1}(c_j)\|_{\ell_{\infty,N}} = \left\| (c,\emptyset)^{-1} \sum_{k=0}^{\infty} (c')^{\sqcup\!\sqcup k} - (c_j,\emptyset)^{-1} \sum_{k=0}^{\infty} (c_j')^{\sqcup\!\sqcup k} \right\|_{\ell_{\infty,N}} \to 0,$$

or in other words, $\sqcup\!\sqcup^{-1}(c_j) \to \sqcup\!\sqcup^{-1}(c)$ in $\mathbb{K}_{LC}\langle\langle X\rangle\rangle$.

3.2 Continuity of the Composition Product

In addition to Lemma 3.3, the next result is needed in order to address the continuity of the composition product.

Lemma 3.4 *Fix $M > 0$. If $c \in \ell_{\infty,M}(X^*, \mathbb{K}^\ell)$ and $d \in \ell_{\infty,M}(X^*, \mathbb{K}^m)$, then $c \circ d \in \ell_{\infty,M_\epsilon}(X^*, \mathbb{K}^\ell)$ for any $M_\epsilon = M(1+\epsilon)$, $\epsilon > \phi(m\|d\|_{\ell_{\infty,M}})$ with $\phi(x) = x/2 + \sqrt{x^2/4 + x}$ and*

$$\|c \circ d\|_{\ell_{\infty,M_\epsilon}} \leq \|c\|_{\ell_{\infty,M}} (K_\epsilon \circ \phi)(m\|d\|_{\ell_{\infty,M}}),$$

where $K_\epsilon(a) = \sup_{\eta \in X^*} (|\eta|+1)(1+a)^{|\eta|}/(1+\epsilon)^{|\eta|}$.

Proof It was shown in [29] that under the stated conditions

$$|(c \circ d, \eta)| \leq \|c\|_{\ell_{\infty,M}} ((1+\phi(m\|d\|_{\ell_{\infty,M}}))M)^{|\eta|} (|\eta|+1)!, \ \ \forall \eta \in X^*.$$

Therefore,

$$\frac{|(c \circ d, \eta)|}{M_\epsilon\ |\eta|!} \leq \|c\|_{\ell_{\infty,M}} (1+\phi(m\|d\|_{\ell_{\infty,M}}))^{|\eta|} \frac{(|\eta|+1)}{(1+\epsilon)^{|\eta|}}, \ \ \forall \eta \in X^*.$$

Taking the supremum over $\eta \in X^*$ proves the lemma.

Theorem 3.2 *The composition product on $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ is continuous in the Silva topology.*

Proof Left and right continuity of the composition product is first proved, beginning with left continuity. Let $M > 0$ be fixed. Let $c, d \in \ell_{\infty,M}(X^*, \mathbb{K}^m)$, and assume c_j, $j \geq 1$ is a sequence in $\ell_{\infty,M}(X^*, \mathbb{K}^m)$ converging to c. Applying Lemma 3.4 gives

$$\begin{aligned}\|(c \circ d) - (c_j \circ d)\|_{\ell_{\infty,M_\epsilon}} &= \|(c - c_j) \circ d\|_{\ell_{\infty,M_\epsilon}} \\ &\leq \|c - c_j\|_{\ell_{\infty,M}} (K_\epsilon \circ \phi)(m\|d\|_{\ell_{\infty,M}}).\end{aligned}$$

Thus, $\lim_{j\to\infty} \|(c \circ d) - (c_j \circ d)\|_{\ell_{\infty,M_\epsilon}} = 0$.

Right continuity is addressed next. It is more complicated given the nonlinearity in the right argument of the product. Let $c, d \in \ell_{\infty,M}(X^*, \mathbb{K}^m)$ and assume d_j, $j \geq 1$ is a sequence in $\ell_{\infty,M}(X^*, \mathbb{K}^m)$ converging to d. For a fixed $\xi \in X^*$, observe that

$$\begin{aligned}
|((c \circ d) - (c \circ d_j), \xi)| &= \left| \sum_{\eta \in X^*} (c, \eta)(\eta \circ d - \eta \circ d_j, \xi) \right| \\
&\leq \sum_{n=0}^{\infty} \|c\|_{\ell_{\infty,M}} M^n\, n! \left| \sum_{\eta \in X^n} (\eta \circ d - \eta \circ d_j, \xi) \right| \\
&= \|c\|_{\ell_{\infty,M}} \sum_{n=0}^{\infty} M^n\, n! \left| \sum_{\substack{r_0 \geq 0, \ldots, r_m \geq 0 \\ r_0 + \cdots + r_m = n}} ((x_0^{r_0} ⧢ \cdots ⧢ x_m^{r_m}) \circ d \right. \\
&\quad \left. -(x_0^{r_0} ⧢ \cdots ⧢ x_m^{r_m}) \circ d_j, \xi) \right|.
\end{aligned}$$

Applying the identities $x_i^{⧢ n} = n!\, x_i^n$, $n \geq 0$ and $(c ⧢ d) \circ e = (c \circ e) ⧢ (d \circ e)$ gives

$$\begin{aligned}
&|((c \circ d) - (c \circ d_j), \xi)| \\
&\leq \|c\|_{\ell_{\infty,M}} \sum_{n=0}^{\infty} M^n \left| \sum_{\substack{r_0 \geq 0, \ldots, r_m \geq 0 \\ r_0 + \cdots + r_m = n}} \binom{n}{r_0 \cdots r_m} ((x_0^{⧢ r_0} ⧢ \cdots ⧢ x_m^{⧢ r_m}) \circ d \right. \\
&\quad \left. -(x_0^{⧢ r_0} ⧢ \cdots ⧢ x_m^{⧢ r_m}) \circ d_j, \xi) \right|. \\
&= \|c\|_{\ell_{\infty,M}} \sum_{n=0}^{\infty} M^n \left| \left(\left(\sum_{k=0}^{m} x_k \circ d \right)^{⧢ n} - \left(\sum_{k=0}^{m} x_k \circ d_j \right)^{⧢ n}, \xi \right) \right| \\
&= \|c\|_{\ell_{\infty,M}} \sum_{n=0}^{\infty} \left| \left(\bar{d}^{⧢ n} - \bar{d}_j^{⧢ n}, \xi \right) \right|,
\end{aligned}$$

where $\bar{d} := M x_0 \sum_{k=0}^{m} d[k]$ and $\bar{d}_j := M x_0 \sum_{k=0}^{m} d_j[k]$ are proper series in $\mathbb{K}\langle\langle X \rangle\rangle$. Here, $d[k]$ denotes the k-th component series of d. It is clear that $\bar{d} \in \ell_{\infty,M}(X^*, \mathbb{K})$, and $\bar{d}_j$ is a sequence in $\ell_{\infty,M}(X^*, \mathbb{K})$. Furthermore, $\bar{d}_j \to \bar{d}$ as $j \to \infty$ since

$$
\begin{aligned}
\|\bar{d}-\bar{d}_j\|_{\infty,M} &= \sup_{\eta\in X^*}\frac{|(\bar{d}-\bar{d}_j,\eta)|}{M^{|\eta|}\,|\eta|!}\\
&\le \sum_{k=1}^{m}\sup_{\eta\in X^*}\frac{M|(x_0(d[l]-d_j[k]),\eta)|}{M^{|\eta|}\,|\eta|!}\\
&= \sum_{k=1}^{m}\sup_{x_0\eta\in X^*}\frac{|(d[k]-d_j[k],\eta)|}{M^{|\eta|}(|\eta|+1)!}\\
&= \sum_{k=1}^{m}\sup_{\eta\in X^*}\frac{|(d[k]-d_j[k],\eta)|}{M^{|\eta|}\,|\eta|!(|\eta|+1)}\\
&\le m\|d-d_j\|_{\ell_\infty,M}.
\end{aligned}
$$

Finally, right continuity follows by applying Lemma 3.3 with $N > M$ sufficiently large so that

$$
\|(c\circ d)-(c\circ d_j)\|_{\ell_\infty,N} \le \|c\|_{\ell_\infty,M}\sum_{n=0}^{\infty}\|\bar{d}^{\,\sqcup\!\sqcup\, n}-\bar{d}_j^{\,\sqcup\!\sqcup\, n}\|_{\ell_\infty,N}\to 0. \qquad (6)
$$

Note that the estimates for left and right continuity imply joint continuity of the composition product due to the following simple observation that for N as above

$$
\begin{aligned}
\|(c\circ d)-(c_j\circ d_j)\|_{\ell_\infty,N} &\le \|(c\circ d)-(c_j\circ d)\|_{\ell_\infty,N}+\|(c_j\circ d)-(c_j\circ d_j)\|_{\ell_\infty,N}\\
&\overset{(6)}{\le} \|(c-c_j)\circ d\|_{\ell_\infty,N}+\|c_j\|_{\ell_\infty,M}\sum_{n=0}^{\infty}\|\bar{d}^{\,\sqcup\!\sqcup\, n}-\bar{d}_j^{\,\sqcup\!\sqcup\, n}\|_{\ell_\infty,N},
\end{aligned}
$$

where the last inequality is a direct consequence of (6).

4 Analyticity of the Composition and Shuffle Product

In this section, it is proved that the formal power series products and inverse presented in the previous sections are not only continuous but also analytic. Note that on the infinite-dimensional spaces involved, both complex and real analyticity make sense, cf. Appendix A. For real analyticity, one needs only to identify the complexification of the spaces $\mathbb{R}\langle\langle X\rangle\rangle$ and $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$.

As locally convex spaces, the complexification of $\mathbb{R}\langle\langle X\rangle\rangle$ is $\mathbb{C}\langle\langle X\rangle\rangle$. This is clear on the level of vector spaces, and for the topology simply note that as topological vector spaces $\mathbb{C}\langle\langle X\rangle\rangle = \mathbb{R}\langle\langle X\rangle\rangle \oplus i\mathbb{R}\langle\langle X\rangle\rangle$. Similarly, the complexification of the Silva space $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$ is $\mathbb{C}_{LC}\langle\langle X\rangle\rangle$. Again this is clear on the level of vector spaces but more complicated on the level of the vector space topologies. However, also the vector space topologies coincide as it is easy to see that for every $M > 0$ the Banach space $\ell_{\infty,M}(X^*,\mathbb{C}^\ell)$ is the complexification of $\ell_{\infty,M}(X^*,\mathbb{R}^\ell)$, and the inductive limit of a sequence of compact operators between Banach spaces commutes with the formation of complexifications [32, Theorem 3.4].

Having identified the complexification of the infinite-dimensional spaces, observe that the shuffle product, the composition product and the shuffle inverse are all well defined on both the complexification and on the real space. Hence, if it can be proved that these mappings are holomorphic on the complexification, then real analyticity is obtained for the corresponding mappings on the real space. Before continuing with the shuffle product and the shuffle inverse, it is helpful to recall a special type of locally convex algebra.

Definition 4.1 Let (A, β) be an associative unital locally convex algebra, i.e., A is a locally convex space such that the bilinear map β is continuous and admits a unit $\mathbf{1}$ with $\beta(\mathbf{1}, x) = x = \beta(x, \mathbf{1})$. Then, A is called a ***continuous inverse algebra*** (CIA) if the unit group $A^\times$ is an open subset of A, and inversion $\iota\colon A^\times \to A^\times$ is continuous.

Proposition 4.1 *The algebras* $(\mathbb{K}\langle\langle X\rangle\rangle, ⧢)$ *and* $(\mathbb{K}_{LC}\langle\langle X\rangle\rangle, ⧢)$ *are continuous inverse algebras with respect to their natural topologies.*

Proof It was shown in Theorem 3.1 that the bilinear shuffle product is continuous with respect to the Silva and the Fréchet topology. Furthermore, the non-proper series are precisely the invertible elements with respect to the shuffle product. By definition of a non-proper series it is evident that if A is either the algebra $\mathbb{K}\langle\langle X\rangle\rangle$ or the algebra $\mathbb{K}_{LC}\langle\langle X\rangle\rangle$, then $A^\times = a_\emptyset^{-1}(\mathbb{C} \setminus \{0\})$ is open as the preimage of an open set under a continuous map. Continuity of the shuffle inverse for the Silva topology on $\mathbb{K}_{LC}\langle\langle X\rangle\rangle$ was established in Proposition 3.1. To see that the shuffle inverse is also continuous on $(\mathbb{K}\langle\langle X\rangle\rangle)^\times$ it suffices to test continuity of the composition $a_\eta \circ ⧢^{-1}$ for every $\eta \in X^*$. However, due to the definition of the shuffle inverse, it is clear that $a_\eta \circ ⧢^{-1}(c)$ is a polynomial in finitely many evaluations of the series c. Therefore, $a_\eta \circ ⧢^{-1}$ is continuous, and hence the shuffle inverse is continuous in the Fréchet topology.

It is well known that the unit group of a CIA is an infinite-dimensional Lie group. Before stating the next result, recall some standard notation. For a differentiable map $f\colon M \to N$ between manifolds, one denotes by $Tf\colon TM \to TN$ the tangent map. The tangent map generalizes the differential of a map, i.e., locally in charts or on open subsets of a vector space, the tangent Tf is given as (f, df), where df is the differential. With this notation in place, recall the following standard notions from infinite-dimensional Lie theory stated below.

Definition 4.2 Consider a Lie group G with unit 1 and write $\mathbf{L}(G)$ for the Lie algebra of G (identified with the tangent space T_1G at the identity). Let $\lambda_g\colon G \to G$, $\lambda_g(h) = gh$ be the left multiplication with a fixed element $g \in G$. Then G is called C^r*-semiregular*, $r \in \mathbb{N}_0 \cup \{\infty\}$, if for each C^r-curve $u\colon [0, 1] \to \mathbf{L}(G)$ the initial value problem

$$\begin{cases} \dot{\gamma}(t) = \gamma(t).u(t) := T\lambda_{\gamma(t)}(u(t)) \\ \gamma(0) = 1 \end{cases}$$

has a (necessarily unique) C^{r+1}-solution $\mathrm{Evol}(u) := \gamma\colon [0, 1] \to G$. If in addition the map

$$\mathrm{evol}\colon C^r([0, 1], \mathbf{L}(G)) \to G, \quad u \mapsto \mathrm{Evol}(u)(1)$$

is smooth, G is called a *C^r-regular Lie group* (or just C^r-regular).[1] A C^∞-regular Lie group G is called *regular (in the sense of Milnor)*.

Every Banach Lie group is C^0-regular (cf. [37]). Several important results in infinite-dimensional Lie theory are only available for regular Lie groups. For example, the interplay between Lie algebra and Lie group hinges on regularity as this property guarantees existence of a smooth Lie group exponential function. Moreover, if one wants to lift morphisms of Lie algebras to the Lie group by integration, this requires the group to be regular, cf. [34].

Proposition 4.2 *The group* $((\mathbb{K}\langle\langle X\rangle\rangle)^\times, ⧢)$ *with the Fréchet topology and the group* $((\mathbb{K}_{LC}\langle\langle X\rangle\rangle)^\times, ⧢)$ *with the Silva topology are* C^0*-regular analytic Lie groups.*

Proof It was established that the groups are unit groups of continuous inverse algebras; hence, they are infinite-dimensional analytic Lie groups by [19, Theorem 5.6]. Moreover, since the shuffle product is abelian, and $\mathbb{K}\langle\langle X\rangle\rangle$ and $\mathbb{K}_{LC}\langle\langle X\rangle\rangle$ are both complete locally convex spaces, an application of [22, p. 3 Corollary and Proposition 3.4 (a)] shows that the Lie groups $\mathbb{K}\langle\langle X\rangle\rangle$ and $\mathbb{K}_{LC}\langle\langle X\rangle\rangle$ are C^0-regular (even with analytic evolution map evol).

Remark 4.1 In [22, Lemma 2.2] it was proved that the solution to the initial value problem for regularity in the unit group of a CIA is given by the Volterra series

$$\gamma(t) = 1 + \sum_{n=1}^{\infty} \int_0^t \int_0^{t_{n-1}} \cdots \int_0^{t_2} \eta(t_1)\cdots\eta(t_n)\mathrm{d}t_1 \ldots \mathrm{d}t_n. \tag{7}$$

Hence, the Volterra series describes both the solution of the initial value problem in $\mathbb{K}\langle\langle X\rangle\rangle$ and the subgroup $\delta + \mathbb{K}_{LC}\langle\langle X\rangle\rangle$.

Proposition 4.3 *The composition product on* $\mathbb{K}\langle\langle X\rangle\rangle$ *and on* $\mathbb{K}_{LC}\langle\langle X\rangle\rangle$ *is analytic.*

Proof In light of the previous observations regarding the complexifications, it suffices to prove the statement for the case $\mathbb{K} = \mathbb{C}$. Fix $\eta \in X^*$. By definition of the composition product, an induction argument shows that $a_\eta(c \circ d)$ is a polynomial in finitely many $a_\gamma(c)$ and $a_\rho(d)$ for words such that $|\gamma|, |\rho| \leq |\eta|$ (for a detailed proof see [40, Lemma 83]). As the coordinate functions are continuous linear (thus holomorphic) in the Fréchet topology on $\mathbb{K}\langle\langle X\rangle\rangle$ and in the Silva topology on $\mathbb{K}_{LC}\langle\langle X\rangle\rangle$, one can deduce the following:

(1) the composition product $\circ\colon \mathbb{K}\langle\langle X\rangle\rangle^2 \to \mathbb{K}\langle\langle X\rangle\rangle$ is continuous with respect to the Fréchet topology (which is initial with respect to the a_η);
(2) for every $\eta \in X^*$ the map $(c, d) \mapsto a_\eta(c \circ d)$ is holomorphic both on $\mathbb{K}\langle\langle X\rangle\rangle^2$ and on $\mathbb{K}_{LC}\langle\langle X\rangle\rangle^2$.

[1] The function space $C^r([0, 1], \mathbf{L}(G))$ is endowed with the compact open C^r-topology (controlling a function and its derivatives on compact subsets). With this topology and pointwise addition and scalar multiplication $C^r([0, 1], \mathbf{L}(G))$ is a locally convex space. Thus, it makes sense to define smooth mappings on this space, cf. Appendix A.

Furthermore, the coordinate functions a_η, $\eta \in X^*$ separate the points on $\mathbb{C}\langle\langle X\rangle\rangle$ and on $\mathbb{C}_{LC}\langle\langle X\rangle\rangle$. Now apply Lemma A.1. Since the composition product is continuous on $\mathbb{C}\langle\langle X\rangle\rangle$ and analytic after composition with a_η, $\eta \in X^*$, the composition product is analytic as a mapping on $\mathbb{C}\langle\langle X\rangle\rangle$. A similar argument holds for the composition product on $\mathbb{C}_{LC}\langle\langle X\rangle\rangle$ as continuity for this product was established in Theorem 3.2.

5 The Lie Group $(\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle, \circ, \delta)$

A Lie group structure on the group $(\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle, \circ, \delta)$ is presented in this section. This group is known to have an associated graded and connected Hopf algebra (H, μ, Δ) as described in Sect. 2.2 and more completely in [25, Section 3]. This structure will play an important role in the proof of the Lie group property. Note, however, that $(\delta + \mathbb{K}^m\langle\langle X\rangle\rangle, \circ, \delta)$ is *not* the character group of said Hopf algebra, and thus the Lie theory for such groups from [5, 10] is not directly applicable. The main claim, as stated below, is established from first principles.

Theorem 5.1 *The group $(\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle, \circ, \delta)$ is an analytic Lie group under the Silva topology.*

Proof The proof is carried out in four main steps.

Step 1: *The group product is continuous in the Silva topology*. Fix $M \geq 0$ and let $M_\epsilon = M(1+\epsilon)$. If $c, d \in \ell_{\infty,M}(X^*, \mathbb{K}^m)$, then the proof of Theorem 3.2 can be easily modified to show that $c \,\tilde{\circ}\, d_\delta$ is continuous in the Silva topology. Specifically, the only change is in the definition of $\bar{d}$ and $\bar{d}_j$. For example, $\bar{d} = M(\sum_{k=0}^m x_k + x_0 d[k])$, in which case, it follows directly that $c_\delta \circ d_\delta = \delta + d + c \,\tilde{\circ}\, d_\delta$ is continuous in both its left and right arguments in the Banach space $\ell_{\infty,M_\epsilon}(X^*, \mathbb{K}^m)$. Joint continuity follows then verbatim as in the proof of Theorem 3.2.

Step 2: *The group inverse is degreewise a polynomial*. Assume without loss of generality that $m = 1$. Let $c_j \to c$ in $\ell_{\infty,M}(X^*, \mathbb{K})$. It was shown in [25] that the composition inverse preserves local convergence. Thus, there exists an $M_1 > 0$ such that $c_\delta^{\circ -1} \in \delta + \ell_{\infty,M_1}(X^*, \mathbb{K})$ and $(c_{\delta,j})^{\circ -1} \in \delta + \ell_{\infty,M_1}(X^*, \mathbb{K})$ for every $j \geq 1$. Set $M_2 = \max(M, M_1)$. Since H is graded and connected with respect to the degree grading, it follows from Lemma 2.1 (cf. [36]) that

$$
\begin{aligned}
(c_\delta^{\circ -1}, \eta) =& S(a_\eta)(c) = -a_\eta(c) - \sum S(a'_{(\eta 1)})(c) a'_{(\eta 2)}(c) \\
=& - a_\eta(c) + \sum_{k=1}^{\deg(a_\eta)} (-1)^{k+1} \mu_k \circ \Delta'_k(a_\eta)(c),
\end{aligned}
\tag{8}
$$

where $\Delta' a = \Delta a - a \otimes \mathbf{1}_\delta - \mathbf{1}_\delta \otimes a = \sum a'_{(\eta 1)} \otimes a'_{(\eta 2)}$ is the reduced coproduct in the notation of Sweedler[2], $\Delta'_k = \Delta'_{k-1} \otimes \mathrm{id}$ is defined inductively, and μ_k is the k-fold multiplication in the target algebra. In particular, $a'_{(\eta 1)} \in V_{n_1}$ and $a'_{(\eta 2)} \in H_{n_2}$ with $n_1, n_2 < n$. As the summation in (8) is always finite, the η component of $c_\delta^{\circ -1}$ is a polynomial in the variables $\{a_\xi(c) : \deg(a_\xi) \leq \deg(a_\eta)\}$. This implies immediately

[2] Given the bijection between $\delta + \mathbb{K}\langle\langle X\rangle\rangle$ and $\mathbb{K}\langle\langle X\rangle\rangle$, for brevity $a_\eta(c_\delta)$ will be written as $a_\eta(c)$.

that inversion is continuous (and analytic) in the Fréchet space $\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$. However, this does not yet yield continuity with respect to the Silva space topology on $\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$.

Step 3: *Continuity of the group inverse in the Silva topology*. It is first proved that inversion is continuous at the unit δ. It is again assumed without loss of generality that $m = 1$. Recalling that $c_\delta := \delta + c$, the series $c_{\delta,j} = \delta + c_j$, $j \in \mathbb{N}$ converges to δ in the Silva topology if and only if the series c_j converges to 0 in $\ell_{\infty,M}(X^*, \mathbb{K})$ for some $M > 0$. Fix $c \in \ell_{\infty,M}(X^*, \mathbb{K})$ and define $\bar{c} = \sum_{\eta\in X^*} KM^{|\eta|}\,|\eta|!\,\eta$ with $K = \|c\|_{\ell_{\infty,M}}$ so that $|(c,\eta)| \leq (\bar{c},\eta)$, $\forall \eta \in X^*$. It can be verified directly that $y = F_{\bar{c}_\delta}[u] = u + F_{\bar{c}}[u]$ has the state space realization

$$\dot{z} = \frac{M}{K}(1+u), \quad z(0) = K, \quad y = z + u.$$

Therefore, $y = F_{\bar{c}_\delta^{\circ -1}}[u] = u + F_{\bar{c}^{\circ -1}}[u]$ has the realization

$$\dot{z} = \frac{M}{K}(z^2 - z^3) + z^2 u, \quad z(0) = K, \quad y = -z + u. \tag{9}$$

It is shown in [25, Theorem 6] that $c^{\circ -1} = (-c)@\delta$, where the right-hand side denotes the generating series for the unity feedback system $v \mapsto y$ defined by $y = F_{-c}[u]$ and $u = v + y$. Combining this fact with a minor extension of [42, Lemma 10], it follows that the condition $|(c,\eta)| \leq (\bar{c},\eta)$ implies $\left|(c^{\circ -1},\eta)\right| \leq \left|(\bar{c}^{\circ -1},\eta)\right|$, $\forall \eta \in X^*$. The fastest growing coefficients of $\bar{c}^{\circ -1}$ have been shown to be the sequence $(\bar{c}^{\circ -1}, x_0^k)$, $k \geq 0$ [42, Lemma 7]. Therefore, for any word $\eta \in X^*$ of length k

$$\left|(c^{\circ -1},\eta)\right| \leq \left|(\bar{c}^{\circ -1},\eta)\right| \leq \left|(\bar{c}^{\circ -1},x_0^k)\right| = \left|L_{g_0}^k h(z_0)\right|,$$

where the right-most inequality follows from (3) with $g_0(z) = (M/K)(z^2 - z^3)$, $h(z) = -z$, and $z_0 = K$ as derived in (9). A direct calculation gives

$$(\bar{c}^{\circ -1}, x_0^k) = b_k(K)KM^k, \quad k \geq 0, \tag{10}$$

where the first few polynomials $b_k(K)$ are:

$$\begin{aligned}
b_0(K) &= -1\\
b_1(K) &= -1 + K\\
b_2(K) &= -2 + 5K - 3K^2\\
b_3(K) &= -6 + 26K - 35K^2 + 15K^3\\
b_4(K) &= -24 + 154K - 340K^2 + 315K^3 - 105K^4\\
b_5(K) &= -120 + 1044K - 3304K^2 + 4900K^3 - 3465K^4 + 945K^5\\
b_6(K) &= -720 + 8028K - 33740K^2 + 70532K^3 - 78750K^4 + 45045K^5 - 10395K^6\\
b_7(K) &= -5040 + 69264K - 367884K^2 + 1008980K^3 - 1571570K^4 + 1406790K^5
\end{aligned}$$

$$- 675675K^6 + 135135K^7$$

$$\vdots$$

As c_j, $j \in \mathbb{N}$ converges to $0 \in \ell_{\infty,M}(X^*, \mathbb{K})$, one can discard finitely many initial terms and thus assume without loss of generality that $\|c_j\|_{\ell_{\infty,M}} \leq K \leq 1$. However, when $K \leq 1$ it is known that $b_k(K) \leq \bar{b}_k$, where $\bar{b}_k$, $k \geq 0$ is the integer sequence A112487 in [41], namely 1, 2, 10, 82, 938, 13778, 247210, ... Its exponential generating function is the real analytic function

$$G(x) = \frac{-1}{1 + W(-2\exp(x-2))},$$

where W is the Lambert W-function (see [42, Example 5]), in which case, there exists growth constants $\bar{K}, \bar{M} > 0$ such that $\bar{b}_k \leq \bar{K}\bar{M}^k k!$, $k \geq 0$. Combining this inequality with (10) gives

$$\left|(c^{\circ -1}, \eta)\right| \leq \|c\|_{\ell_{\infty,M}} \bar{K}(M\bar{M})^{|\eta|}\, |\eta|!, \quad \forall \eta \in X^*.$$

Hence, if $c_{\delta,j} \to \delta$ in $\mathbb{K} \times \ell_{\infty,M}(X^*, \mathbb{K})$, then $c_{\delta,j}^{\circ-1} \to \delta$ in $\ell_{\infty,M\overline{M}}(X^*, \mathbb{K})$. Therefore, inversion is continuous at the unit with respect to the Silva topology. Exploiting the fact that inversion is a group antimorphism, this implies that inversion is continuous everywhere on $\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ in the Silva topology.[3]

Step 4: *Group product and inverse are analytic.* Since the complexification of $\delta + \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ is $\delta + \mathbb{C}^m_{LC}\langle\langle X\rangle\rangle$, it suffices to consider the complex case. In view of Lemma A.1 and Step 1, all one needs to prove is that for every $\eta \in X^*$ the mappings $(c_\delta, d_\delta) \mapsto a_\eta(c_\delta \circ d_\delta)$ and $c_\delta \mapsto a_\eta(c_\delta^{\circ-1})$ are holomorphic. Regarding the composition product recall that $(\delta + c) \circ (\delta + d) = \delta + d + c\tilde{\circ}d_\delta$. Now for the mixed composition $\tilde{\circ}$ it was shown in the proof of Proposition 4.3 that $a_\eta(c\tilde{\circ}d_\delta)$ is given by a polynomial in finitely many of the variables $a_\xi(c)$ and $a_\nu(d)$. Hence, this part of the product is analytic on $\delta + \mathbb{C}^m_{LC}\langle\langle X\rangle\rangle$, and therefore the composition product is analytic. Similarly, for the inversion ι, Step 2 shows that $a_\eta \circ \iota(c)$ is given as a polynomial in finitely many evaluations of c. As before, the coordinate functions are holomorphic, and this implies that $a_\eta \circ \iota$ is holomorphic on $\delta + \mathbb{C}^m_{LC}\langle\langle X\rangle\rangle$. Hence, the inversion is also holomorphic.

The argument for the Lie group structure on subsets of locally convergent series can be adapted almost verbatim to the case where no convergence of the series is assumed.

Corollary 5.1 *The group $(\delta + \mathbb{K}^m\langle\langle X\rangle\rangle, \circ, \delta)$ is an analytic Lie group.*

Proof Again it suffices to prove the case where $\mathbb{K} = \mathbb{C}$. In Step 2 of the proof for Theorem 5.1 it was shown that after composition with a coordinate function a_η both the composition and the inversion in the group are given by a polynomial in finitely many coordinate functions applied to the arguments. Since the Fréchet topology

[3] Alternatively, continuity can be deduced from a more general criterion, see [1, Lemma 1.3].

is initial with respect to the coordinate functions, it follows directly that the group operations are continuous. Applying Lemma A.1 gives immediately that the group operations are also analytic.

While the Fréchet Lie group $\delta+\mathbb{K}^m\langle\langle X\rangle\rangle$ is much simpler (topologically speaking) than the Silva group $\delta+\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$, it supplies a useful template for the Lie theoretic arguments considered next, namely identifying the Lie algebra and proving that a Lie group is regular in the sense of Milnor. The first goal is to establish these properties for the simpler FréchetLie group. Subsequently, it is shown that these results then imply corresponding properties for the Silva Lie group. However, it is first necessary to introduce a new structure which will yield a convenient description of the Lie bracket. This structure is the so called *pre-Lie product*, which was developed in [18] for the case where $m=1$ and generalized in [12, Section 3.2] for the case where $m\geq 1$.

Definition 5.1 Let $X=\{x_0,x_1,\dots,x_m\}$ and denote by $d[i]$ the i-th component of a series $d\in\mathbb{K}^m\langle\langle X\rangle\rangle$. The pre-Lie product is the bilinear product on $\mathbb{K}^m\langle\langle X\rangle\rangle\times\mathbb{K}^m\langle\langle X\rangle\rangle$

$$c\lhd d=\sum_{\eta\in X^*}(c,\eta)\,\eta\lhd d,$$

where $\eta\lhd d$ is defined inductively by

$$\begin{aligned}(x_0\eta)\lhd d&=x_0(\eta\lhd d)\\(x_j\eta)\lhd d&=x_j(\eta\lhd d)+x_0(\eta\sqcup\!\sqcup\, d[j]),\quad j=1,2,\dots,m\end{aligned}$$

and $\emptyset\lhd d=0$.

This product can be viewed as the linear part of the group product, that is,

$$c_\delta\circ d_\delta=\delta+d+c\,\tilde{\circ}\,d_\delta=\delta+c+d+c\lhd d+\mathcal{O}(c,d^2),\tag{11}$$

where $\mathcal{O}(c,d^2)$ denotes all terms depending linearly on c and on higher powers of d. One can show that the pre-Lie product preserves the length of words in the sense that $(\eta\lhd\xi,\nu)=0$ when $|\eta|+|\xi|\neq|\nu|$. Therefore, the product is well defined as it is locally finite. Moreover, defining $d[0]=0$, the recursive formulas reduce to a single expression

$$(x_j\eta)\lhd d=x_j(\eta\lhd d)+x_0(\eta\sqcup\!\sqcup\, d[j]),\quad j\in\{0,1,\dots,m\}.\tag{12}$$

Example 5.1 Consider the computation of the pre-Lie product for a few words of short length. For example, if $c=x_0^n$, $n\in\mathbb{N}$ and $d\in\mathbb{K}^m\langle\langle X\rangle\rangle$, then $x_0^n\lhd d=x_0^n(\emptyset\lhd d)=0$. For any $x_k\in X$ with $k\neq 0$,

$$x_k\lhd d=x_k(\emptyset\lhd d)+x_0(\emptyset\sqcup\!\sqcup\, d[k])=x_0d[k],\quad k\in\{1,\dots,m\}.\tag{13}$$

Observe $a_{x_k}(x_k \lhd d) = 0$ as every word in the support of $x_k \lhd d$ must have the prefix x_0. Furthermore, it is clear from (13) that the length of the words in $\text{supp}(x_k \lhd d)$ coincide with the length of those in $\text{supp}(d)$ except incremented by one. On the other hand, if $d = \mathbf{e}_k\eta$ (where $\mathbf{e}_k \in \mathbb{R}^m$ is the k-th unit vector), then

$$\deg(a_{x_k \lhd d}) = 2 + \deg(a_\eta) \geq \deg(a_{x_k}) + \deg(a_\eta), \quad k \in \{1, 2, \ldots, m\}.$$

Indeed, one always obtains $|\eta \lhd d| + |\eta \lhd d|_{x_0} \geq |\eta| + |\eta|_{x_0} + |d| + |d|_{x_0}$ (where the length of a sum of words is defined as the maximum of the length of the words). Consider next $c = x_j x_k$ where both j and k are not zero. Applying the definition gives

$$\begin{aligned} x_j x_k \lhd d &= x_j(x_k \lhd d) + x_0(x_k ⧢ d[j]) \\ &= x_j(x_k(\emptyset \lhd d) + x_0 d[k]) + x_0(x_k ⧢ d[j]) \\ &= x_j x_0 d[k] + x_0(x_k ⧢ d[j]). \end{aligned} \tag{14}$$

For comparison, it follows from (5) that

$$\begin{aligned} x_j x_k \,\tilde{\circ}\, d_\delta &= \phi_d(x_j x_k)(\mathbf{1}) \\ &= \phi_d(x_j) \circ \phi(x_k)(\mathbf{1}) \\ &= \phi_d(x_j)(x_k + x_0 d[k]) \\ &= x_j(x_k + x_0 d[k]) + x_0(d[j] ⧢ (x_k + x_0 d[k])) \\ &= x_j x_k + x_j x_0 d[k] + x_0(d[j] ⧢ x_k) + x_0(d[j] ⧢ (x_0 d[k])) \\ &= x_j x_k + x_j x_k \lhd d + x_0(d[j] ⧢ (x_0 d[k])), \end{aligned}$$

which is consistent with (11). Applying now the coordinate function $a_{x_j x_k}$ to (14) gives $a_{x_j x_k}(x_j x_k \lhd d) = 0$ for any series d. A trivial induction shows that

$$a_\eta(\eta \lhd d) = 0, \ \forall \eta \in X^*, \ d \in \mathbb{K}^m\langle\langle X \rangle\rangle.$$

Finally, consider a word η with $|\eta|_{x_0} = 0$. Observe $a_\eta(\rho \lhd d) = 0$ because every word in the support of $\rho \lhd d$ must contain at least one x_0 and $|\eta|_{x_0} = 0$.

Proposition 5.1 *The Lie algebra of $\delta + \mathbb{K}^m\langle\langle X \rangle\rangle$ is the space $\mathbb{K}^m\langle\langle X \rangle\rangle$ with the Lie bracket given by the formula*

$$[c, d] = c \lhd d - d \lhd c. \tag{15}$$

Proof The Lie bracket of the Lie algebra associated with the Lie group $\delta + \mathbb{K}^m\langle\langle X \rangle\rangle$ is given by evaluating the Lie bracket of left invariant vector fields on $\delta + \mathbb{K}^m\langle\langle X \rangle\rangle$ at the identity δ. Note that since $\delta + \mathbb{K}^m\langle\langle X \rangle\rangle$ is an affine subspace of $\delta + \mathbb{K}^m\langle\langle X \rangle\rangle$, it is easy to see that the left-invariant vector field associated with $c \in \mathbb{K}^m\langle\langle X \rangle\rangle$ is given by the formula $X^c(\delta + e) = c + e \lhd c$, hence

$$\begin{aligned} [c, d] &= [X^c, X^d](\delta) \\ &= (dX^d \circ X^c - dX^c \circ X^d)(\delta) \end{aligned}$$

$$= \lim_{t\to 0} t^{-1} \left(X^d(\delta + tc) - X^c(\delta) - X^c(\delta + td) + X^d(\delta)\right)$$
$$= \lim_{t\to 0} t^{-1} \left(d + tc \lhd d - d - c - td \lhd c + c\right)$$
$$= \lim_{t\to 0} t^{-1}\, t\, (c \lhd d - d \lhd c) = c \lhd d - d \lhd c.$$

Corollary 5.2 *The Lie algebra of* $(\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle, \circ, \delta)$ *is* $\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ *with bracket* (15).

Proof The canonical inclusion $\iota\colon \delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle \to \delta + \mathbb{K}\langle\langle X\rangle\rangle$ is the restriction of a continuous linear map to a closed (affine linear) subset, whence smooth. Obviously it is a Lie group morphism. Derivating the morphism at the identity δ yields a Lie group morphism

$$\mathbf{L}(\iota) := T_\delta \iota\colon T_\delta(\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle) \to T_\delta(\delta + \mathbb{K}\langle\langle X\rangle\rangle), \quad v \mapsto \iota(v)(= v).$$

Observe that the Lie bracket on $\mathbf{L}(\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle) = T_\delta(\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle) \cong \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ coincides (pointwise) with one on $\mathbf{L}(\delta + \mathbb{K}\langle\langle X\rangle\rangle)$, and the latter is (15).

Regularity of the Fréchet Lie group $\delta + \mathbb{K}\langle\langle X\rangle\rangle$ is investigated next. For a curve $\gamma_\delta(t) = (\delta + \gamma(t)) \in \delta + \mathbb{K}\langle\langle X\rangle\rangle$ consider the Lie type differential equation

$$\begin{cases} \dot{\gamma_\delta}(t) = \gamma_\delta(t).c(t) = c(t) + \gamma(t) \lhd c(t) \\ \gamma_\delta(0) = \delta, \end{cases} \tag{16}$$

where $c\colon [0,1] \to \mathbb{K}\langle\langle X\rangle\rangle$ is a continuous curve. For every $\eta \in X^*$ observe that $(\gamma_\delta(t), \eta) = (\gamma(t), \eta)$. Now since the coordinate functions are continuous and linear, a differential equation is obtained for every word $\eta \in X^*$:

$$\begin{aligned} (\dot{\gamma_\delta}(t), \eta) &= (c(t), \eta) + (\gamma(t) \lhd c(t), \eta) \\ &= (c(t), \eta) + \sum_{\rho\in X^*} (\gamma(t), \rho)(\rho \lhd c(t), \eta) \\ &= (c(t), \eta) + \sum_{1\leq|\rho|\leq|\eta|} (\gamma(t), \rho)(\rho \lhd c(t), \eta). \end{aligned} \tag{17}$$

The computations in Example 5.1 have been used above, and the products of elements in $\mathbb{K}^m$ are taken as componentwise products. Note now that the sum in (17) only appears if $|\eta|_{x_0} \neq 0$. Hence, if a word does not contain the letter x_0, then the differential equation (17) reduces to

$$(\gamma(t), \eta) = \int_0^t (c(s), \eta)\mathrm{d}s, \quad \forall \eta \in X^*, |\eta|_{x_0} = 0. \tag{18}$$

Since $(c(t), \eta)$ is a continuous $\mathbb{K}^m$-valued curve, one can solve the above equation for all $t \in [0,1]$. Now if η is a word with $|\eta|_{x_0} \neq 0$, observe that all elements in (17) appearing as coefficients of evaluations of γ are continuous $\mathbb{K}^m$-valued curves of the form $(c(t), \eta)$ or

$$C_{\rho,\eta}\colon [0,1] \to \mathbb{K}^m, \quad t \mapsto C_{\rho,\eta}(t) := (\rho \lhd c(t), \eta). \tag{19}$$

It is now proved via induction on the length of the words that equation (17) admits a solution on $[0, 1]$ for every word. Note first that for any word without an x_0 (such as the empty word, which is the only length zero element), the statement follows directly from the integral equation (18). If $|\eta| = n > 1$ assume that the statement is true for all words of lower length. If $|\eta|_{x_0} = 0$, the statement follows again from (18). To obtain solutions for the words of length n containing x_0, pick an enumeration $(\eta_i)_{i\in I_n}$ of words of length n. Using the enumeration and (19), define

$$\boldsymbol{v}_n(t) := \begin{bmatrix} (\gamma(t), \eta_1) \\ (\gamma(t), \eta_2) \\ \vdots \\ (\gamma(t), \eta_{|I_n|}) \end{bmatrix}, \quad C_n(t) := \begin{bmatrix} 0 & C_{\eta_1,\eta_2}(t) & \cdots & C_{\eta_1,\eta_{|I_n|}}(t) \\ C_{\eta_2,\eta_1}(t) & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & C_{\eta_{|I_n|-1},\eta_{|I_n|}}(t) \\ C_{\eta_{|I_n|},\eta_1}(t) & \cdots & C_{\eta_{|I_n|},\eta_{|I_n|-1}}(t) & 0 \end{bmatrix},$$

$$\boldsymbol{b}_n(t) := \sum_{|\rho|<n} \begin{bmatrix} (\gamma(t), \rho)(\rho \lhd c(t), \eta_1) \\ (\gamma(t), \rho)(\rho \lhd c(t), \eta_2) \\ \vdots \\ (\gamma(t), \rho)(\rho \lhd c(t), \eta_{|I_n|}) \end{bmatrix}$$

Then, (17) together with the observation that $(\eta \lhd c, \eta) = 0$ give rise to the following inhomogeneous system of linear differential equations on $(\mathbb{K}^m)^{|I_n|}$:

$$\dot{\boldsymbol{v}}_n(t) = C_n(t)\boldsymbol{v}_n(t) + \boldsymbol{b}_n(t), \qquad t \in [0, 1], \tag{20}$$

Now by the induction hypothesis the inhomogeneity $\boldsymbol{b}_n$ in (20) is already completely determined by the previous computations. Furthermore, the coefficient matrix C_n is determined by c and thus continuous in t. Hence, one can solve the system (20) and obtain a solution on $[0, 1]$ (via the usual solution theory for linear differential equations on finite-dimensional spaces). This completes the induction, and thus, one can iteratively solve the inhomogeneous linear system (20) for every $n \in \mathbb{N}_0$ with a unique solution on $[0, 1]$. Following [11, §6] (cf. also [2]), the solution to the Lie type equation (16) is the solution to the infinite system of differential equations (18) and

$$\dot{\boldsymbol{v}}_n(t) = C_n(t)\boldsymbol{v}_n(t) + \boldsymbol{b}_n(t), \quad n \in \mathbb{N}_0.$$

The earlier discussion has shown that this system is lower diagonal, i.e., the right-hand side of the equation in degree n depends only on the solutions up to degree n. One can now solve the differential equation on the Fréchet space by adapting the argument in [11, p. 79-80]: Lower diagonal systems can be solved iteratively component-by-component, if each solution exists on a time interval $[0, \varepsilon]$ for some fixed $\varepsilon > 0$. Choosing $\varepsilon = 1$, observe that the Lie type equation (16) admits a unique global solution which can be computed iteratively. Thus, the following result is evident.

Proposition 5.2 *The Lie group $\delta + \mathbb{K}^m\langle\langle X\rangle\rangle$ is C^0-regular.*

Proof It was seen in the discussion above that the Fréchet Lie group $\delta + \mathbb{K}^m\langle\langle X\rangle\rangle$ is C^0-semiregular, i.e., for every continuous curve $\eta\colon [0,1] \to \mathbb{K}\langle\langle X\rangle\rangle$, the differential equation

$$\begin{cases} \dot{\gamma}(t) = \eta(t).\gamma(t) \\ \gamma(0) = \delta \end{cases}$$

admits a unique solution $\gamma\colon [0,1] \to \delta + \mathbb{K}\langle\langle X\rangle\rangle$. However, due to [31,Corollary D], every C^0-semiregular Lie group modeled on a Fréchet space is already C^0-regular.

Observe that one can leverage the regularity of the Fréchet Lie group in the investigation of the regularity for the Silva Lie group $\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$. The inclusion $\iota\colon \delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle \to \delta + \mathbb{K}^m\langle\langle X\rangle\rangle$ is a Lie group morphism which relates the solutions of the evolution equation on the Silva and the Fréchet Lie groups. Indeed, [21,1.16] shows that for a continuous curve $c\colon [0,1] \to \mathbf{L}(\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle) = \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ a solution to the evolution equation (16) in $\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ must satisfy

$$\iota \circ \mathrm{Evol}_{\delta+\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle}(c) = \mathrm{Evol}_{\delta+\mathbb{K}^m\langle\langle X\rangle\rangle}(\mathbf{L}(\iota)\circ c) = \mathrm{Evol}_{\delta+\mathbb{K}^m\langle\langle X\rangle\rangle}(c),$$

where c is interpreted canonically as a curve into $\mathbf{L}(\delta + \mathbb{K}^m\langle\langle X\rangle\rangle) = \mathbb{K}^m\langle\langle X\rangle\rangle$ via the natural inclusion. Hence, the Silva Lie group will be C^0-semiregular if and only if it can be proved that the solutions to the evolution equation on the Fréchet Lie group are bounded when the curve c is bounded. Unfortunately, at present it is not obvious how to bound these solutions to the evolution equation, which leads to the following.

Open problem: Is the Silva Lie group $\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ C^0-semiregular?

Remark 5.1 (1) Note that words which do not contain the letter x_0 already obey the necessary bound for the solution of the evolution equation as the differential equation reduces to the integral equation (18) for these words.

(2) For words which contain the letter x_0, the linear system (20) governs the evolution equation. A natural Ansatz for the problem would thus be to apply a Gronwall-type argument. Looking closer at the pre-Lie product, one easily sees that the top-level words (i.e., of length n when dealing with length n-words) only yield an exponential bound in the Gronwall argument. Unfortunately, there seems to be no clear way to bound the norm of the inhomogeneity $\mathbf{b}_n$ in (20).

(3) Observe that regularity of the Silva Lie group $\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ follows almost directly once C^0-semiregularity is known: Having the semiregularity in place, it is assumed that the estimates will directly yield that for every curve c taking values in

$$\mathbf{L}(\delta + \mathbb{K}^m\langle\langle X\rangle\rangle) \cap B_1^{\|\cdot\|_M}(0) = \{x \in \mathbb{K}^m\langle\langle X\rangle\rangle \mid \|x\|_M \leq 1\},\ M > 0,$$

the evolution $\mathrm{Evol}(c)$ is contained in $B_K^{\|\cdot\|_N}(0)$, $N, K > 0$ fixed (but depending on M). If this is true, C^1-regularity of $\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle$ follows from the arguments presented in the proof of [6,Theorem 4.3].

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Appendix A. Infinite-dimensional Calculus

In this appendix, some basic definitions are recalled concerning the infinite-dimensional calculus used throughout the article. For more information, the reader is referred to the presentations in [20, 37].

Definition A.1 Let $r \in \mathbb{N}_0 \cup \{\infty\}$ and E, F locally convex $\mathbb{K}$-vector spaces and $U \subseteq E$ open. A map $f : U \to F$ is called a $C^r_{\mathbb{K}}$-map if it is continuous and the iterated directional derivatives

$$d^k f(x, y_1, \ldots, y_k) := (D_{y_k} \cdots D_{y_1} f)(x)$$

exist for all $k \in \mathbb{N}_0$ with $k \leq r$ and $y_1, \ldots, y_k \in E$ and $x \in U$, and the mappings $d^k f : U \times E^k \to F$ so obtained are continuous. If f is $C^\infty_{\mathbb{R}}$, it is called *smooth*. If f is $C^\infty_{\mathbb{C}}$, it is said to be *complex analytic* or *holomorphic* and that f is of class $C^\omega_{\mathbb{C}}$.[4]

Definition A.2 (Complexification of a locally convex space) Let E be a real locally convex topological vector space. Endow the locally convex product $E_{\mathbb{C}} := E \times E$ with the following operation

$$(x, y).(u, v) := (xu - yv, xv + yu), \quad \forall x, y \in \mathbb{R},\ u, v \in E.$$

The complex vector space $E_{\mathbb{C}}$ is called the *complexification* of E. Identify E with the closed real subspace $E \times \{0\}$ of $E_{\mathbb{C}}$.

Definition A.3 Let E, F be real locally convex spaces and $f : U \to F$ defined on an open subset U. f is called *real analytic* (or $C^\omega_{\mathbb{R}}$) if f extends to a $C^\infty_{\mathbb{C}}$-map $\tilde{f} : \tilde{U} \to F_{\mathbb{C}}$ on an open neighborhood $\tilde{U}$ of U in the complexification $E_{\mathbb{C}}$.

For $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, being of class $C^r_{\mathbb{K}}$ is a local condition, i.e., if $f|_{U_\alpha}$ is $C^r_{\mathbb{K}}$ for every member of an open cover $(U_\alpha)_\alpha$ of its domain, then f is $C^r_{\mathbb{K}}$. (See [20, pp. 51–52] for the case of $C^\omega_{\mathbb{R}}$. The other cases are clear by definition.) In addition, the composition of $C^r_{\mathbb{K}}$-maps (if possible) is again a $C^r_{\mathbb{K}}$-map (cf. [20, Propositions 2.7 and 2.9]).

[4] Recall from [8,Proposition 1.1.16] that $C^\infty_{\mathbb{C}}$ functions are locally given by series of continuous homogeneous polynomials (cf. [3, 4]). This justifies the abuse of notation.

Definition A.4 ($C^r_{\mathbb{K}}$-manifolds and $C^r_{\mathbb{K}}$-mappings between them) For $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, manifolds modeled on a fixed locally convex space can be defined as usual. Direct products of locally convex manifolds, tangent spaces and tangent bundles as well as $C^r_{\mathbb{K}}$-maps between manifolds may be defined as in the finite-dimensional setting.

For $C^r_{\mathbb{K}}$-manifolds M, N the notation $C^r_{\mathbb{K}}(M, N)$ denotes the set of all $C^r_{\mathbb{K}}$-maps from M to N. Furthermore, for $s \in \{\infty, \omega\}$ define the *locally convex $C^s_{\mathbb{K}}$-Lie groups* as groups with a $C^s_{\mathbb{K}}$-manifold structure turning the group operations into $C^s_{\mathbb{K}}$-maps.

The following lemma seems to be part of the mathematical folklore, a proof can be found in [6, Lemma A.3].

Lemma A.1 *Let U be an open subset of a complex locally convex space E and F be a complex locally convex space which is sequentially complete. Consider a set $\Lambda \subseteq L(F, \mathbb{C})$ of complex linear functionals which separates the points on F.*[5] *If a map $f : U \to F$ is continuous and*

$$\lambda \circ f : U \to \mathbb{C}$$

is complex analytic for each $\lambda \in \Lambda$, then f is complex analytic.

Appendix B. Table of Notation

See Table 1.

Table 1 Table of primary notation and symbols

Object	Definition / Explanation
Symbols	
X^*	set of all words over the alphabet $X = \{x_0, x_1, \ldots, x_m\}$
F_c	causal m-input, ℓ-output operator associated with $c \in \mathbb{K}^\ell\langle\langle X\rangle\rangle$
δ	fictitious generating series for the identity operator
c_δ	$= \delta + c$
(c, η)	coefficient of the series c for the word $\eta \in X^*$
a_η	evaluation map sending a series c to its coefficient (c, η)
ψ_d, ϕ_d	algebra homomorphisms, see composition products
Spaces and groups	
$\mathbb{K}^\ell\langle\langle X\rangle\rangle$	vector space of all non-commutative formal power series over the alphabet X with coefficients in $\mathbb{K}^\ell$
$\mathbb{K}^\ell\langle X\rangle$	all series in $\mathbb{K}^\ell\langle\langle X\rangle\rangle$ with finite support, i.e., polynomials

[5] That is, for each $x \in F$ there is a $\lambda \in \Lambda$ with $\lambda(x) \neq 0$.

Table 1 continued

Object	Definition / Explanation
$\mathbb{K}^{\ell}_{LC}\langle\langle X\rangle\rangle$	space of locally convergent series, i.e., all series satisfying $\lvert(c,\eta)\rvert \leq KM^{\lvert\eta\rvert}\lvert\eta\rvert!$ for all $\eta \in X^*$ and $K, M \in]0,\infty[$. Coincides with the locally convex inductive limit $\bigcup_{M>0}\ell_{\infty,M}(X^*,\mathbb{K}^{\ell})$
$(\ell_{\infty,M}(X^*,\mathbb{K}^{\ell}), \lVert\cdot\rVert_{\ell_{\infty},M})$	Banach space of generating series bounded with respect to M with norm $\lVert c\rVert_{\ell^{\infty},M} = \sup_{\eta\in X^*}\frac{\lvert(c,\eta)\rvert}{M^{\lvert\eta\rvert}\lvert\eta\rvert!}$, this space is isomorphic as a Banach space to ℓ_{∞}
$L^m_{\mathfrak{p}}[t_0,t_1]$	Banach space of $\mathbb{K}^m$-valued $L^{\mathfrak{p}}$-functions on the interval $[t_0,t_1]$
$B^m_{\mathfrak{p}}(R)[t_0,t_1]$	closed ball $\{u \in L^m_{\mathfrak{p}}[t_0,t_1] \colon \lVert u\rVert_{\mathfrak{p}} \leq R\}$
$\mathbb{K}^m\langle\langle X\rangle\rangle^{\times}, \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle^{\times}$	unit group of the algebra $(\mathbb{K}^m\langle\langle X\rangle\rangle, ⧢)$ (resp. $(\mathbb{K}^m_{LC}\langle\langle X\rangle\rangle, ⧢)$
$(\delta + \mathbb{K}^m\langle\langle X\rangle\rangle, \circ, \delta)$	Lie group of power series consisting of $\mathbb{K}^{\ell}\langle\langle X\rangle\rangle$ together with δ (similar notation for $(\delta + \mathbb{K}^m_{LC}\langle\langle X\rangle\rangle, \circ, \delta)$)
Products	
⧢	shuffle product
$c \circ d$	composition product, (4): $\sum_{\eta\in X^*}(c,\eta)\psi_d(\eta)(\mathbf{1})$
$c\,\tilde{\circ}\, d_{\delta}$	mixed composition product, (5): $\sum_{\eta\in X^*}(c,\eta)\phi_d(\eta)(\mathbf{1})$
$c@d$	feedback product
$\triangleleft$	pre-Lie product, see Definition 5.1

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