



## Dispersive Estimates for Full Dispersion KP Equations

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**Abstract.** We prove several dispersive estimates for the linear part of the Full Dispersion Kadomtsev–Petviashvili introduced by David Lannes to overcome some shortcomings of the classical Kadomtsev–Petviashvili equations. The proof of these estimates combines the stationary phase method with sharp asymptotics on *asymmetric Bessel functions*, which may be of independent interest. As a consequence, we prove that the initial value problem associated to the Full Dispersion Kadomtsev–Petviashvili is locally well-posed in  $H^s(\mathbb{R}^2)$ , for  $s > \frac{7}{4}$ , in the capillary-gravity setting.

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### 1. Introduction

#### 1.1. Introduction of the Model and Physical Motivation

The classical Kadomtsev–Petviashvili equation (KP)

$$\partial_t u + \partial_{x_1} u + u \partial_{x_1} u + \partial_{x_1}^3 u \pm \partial_{x_1}^{-1} \partial_{x_2}^2 u = 0, \quad (1.1)$$

where  $+$  corresponds to KP-II and  $-$  to KP-I, was introduced in the pioneering paper [9] in order to investigate the stability properties of the KdV soliton with respect to long wave perturbations in the transverse direction. We are here in a long wave regime, that is the wavelengths in  $x_1$  and  $x_2$  are large, those in  $x_2$  being larger.

Actually the derivation in [9] was formal and concerned only the linear transport part of equation (1.1), in particular it is independent of the dispersive and nonlinear terms. It is only related to the finite propagation speed properties of the transport operator  $M = \partial_t + \partial_{x_1}$ .

Recall that  $M$  gives rise to one-directional waves moving to the right with speed one; i.e., a profile  $\varphi(x_1)$  evolves under the flow of  $M$  as  $\varphi(x_1 - t)$ . A weak transverse perturbation of  $\varphi(x_1)$  is a two-dimensional function  $\psi(x_1, x_2)$  close to  $\varphi(x_1)$ , localised in the frequency region  $|\frac{\xi_2}{\xi_1}| \ll 1$ , where  $\xi_1$  and  $\xi_2$  are the Fourier modes corresponding to  $x_1$  and  $x_2$ , respectively. We look for a two-dimensional perturbation

$$\widetilde{M} = \partial_t + \partial_{x_1} + \omega(D_1, D_2)$$

of  $M$  such that, similarly to above, the profile of  $\psi(x_1, x_2)$  does not change much when evolving under the flow of  $\widetilde{M}$ . Here  $\omega(D_1, D_2)$  denotes the Fourier multiplier with symbol the real function  $\omega(\xi_1, \xi_2)$ . Natural generalizations of the flow of  $M$  in two dimensions are the flows of the wave operators  $\partial_t \pm \sqrt{-\Delta}$  which enjoy the finite propagation speed property. Since

$$\sqrt{\xi_1^2 + \xi_2^2} \sim \pm \left( \xi_1 + \frac{1}{2} \xi_1^{-1} \xi_2^2 \right), \quad \text{when } |\xi_1|, \left| \frac{\xi_2}{\xi_1} \right| \ll 1,$$

we deduce the approximation in this regime

$$\partial_t + \partial_{x_1} + \frac{1}{2} \partial_{x_1}^{-1} \partial_{x_2}^2 \sim \partial_t \pm \sqrt{-\Delta},$$

which leads to the correction  $\omega(D_1, D_2) = \frac{1}{2} \partial_{x_1}^{-1} \partial_{x_2}^2$ .

Of course when the transverse effects are two-dimensional, the correction is  $\frac{1}{2} \partial_{x_1}^{-1} \Delta_{\perp}$ , where  $\Delta_{\perp} = \partial_{x_2}^2 + \partial_{x_3}^2$ .

Note that the term  $\frac{1}{2} \partial_{x_1}^{-1} \partial_{x_2}^2$  leads to a singularity at  $\xi_1 = 0$  in Fourier space which is not present in the original physical context where the KdV equation was derived and there is a price to pay for that, various shortcomings of the KP equation that we will describe now.

The first one concerns the accuracy of the KP approximation as a water wave model. As aforementioned, the derivation in [9] did not refer to a specific physical content. Its formal derivation in the context of water waves was done in [1] but a rigorous derivation, including error estimates, was only achieved in [19]. It is shown there that the error estimate between solutions of the full water waves system and solutions of the KP-II equation has the form, in suitable Sobolev norms, and for a fixed time interval,

$$\|U_{WW} - U_{KP}\| = o(1),$$

while the corresponding error in the Boussinesq (KdV) regime is  $O(\epsilon^2 t)$  where the small parameter  $\epsilon$  measures the comparable effects of shallowness and nonlinearity.

Another shortcoming of the KP equation is the (unphysical) constraint implied by the  $\partial_{x_1}^{-1} \partial_{x_2}^2 u$  term. Actually, in order to make sense,  $u$  should satisfy the constraint  $\hat{u}(0, \xi_2) = 0, \forall \xi_2 \in \mathbb{R}$ , or alternatively  $\int_{\mathbb{R}} u(x_1, x_2) dx_1 = 0, \forall x_2 \in \mathbb{R}$ , that makes no sense for real waves. We refer to [23] for further comments and results on this “constraint problem.”

Another drawback is a singularity at time  $t = 0$  that is already present at the linear level. Denoting by  $S_{\pm}(t)$  the (unitary in  $L^2$ ) linear group of the KP equations, one can express the linear solution corresponding to any  $L^2$  initial data  $u_0$  (without any constraint) in Fourier variables by

$$S_{KP\pm}(t)\widehat{u}_0(\xi_1, \xi_2) = \hat{u}(\xi_1, \xi_2, t) = \exp\left\{it\left(\xi_1^3 \pm \frac{\xi_2^2}{\xi_1}\right)\right\}\widehat{u}_0(\xi_1, \xi_2),$$

which defines of course a unitary group in any Sobolev space  $H^s(\mathbb{R}^2)$ . On the other hand, even for smooth initial data, say in the Schwartz class, the relation

$$u_{x_1 t} = u_{t x_1}$$

holds true only in a very weak sense, *e.g.* in  $S'(\mathbb{R}^2)$ , if  $u_0$  does not satisfy the constraint  $\hat{u}_0(0, \xi_2) = 0$  for any  $\xi_2 \in \mathbb{R}$ , or equivalently  $\int_{-\infty}^{\infty} u_0(x_1, x_2) dx_2 = 0$  for any  $\xi_2 \in \mathbb{R}$ .

In particular, even for smooth localised  $u_0$ , the mapping

$$\hat{u}_0 \mapsto \partial_t \hat{u} = i\left(\xi_1^3 \pm \frac{\xi_2^2}{\xi_1}\right) \exp\left\{it\left(\xi_1^3 \pm \frac{\xi_2^2}{\xi_1}\right)\right\} \hat{u}_0(\xi)$$

cannot be defined with values in a Sobolev space if  $u_0$  does not satisfy the zero mass constraint. In particular, if  $u_0$  is a gaussian,  $\partial_t u$  is not even in  $L^2$ .

Those shortcomings have led Lannes [20], see also [23], to introduce in the KP regime a full dispersion counterpart of the KP equation that would not suffer of such defects or at least at a lower level,<sup>1</sup>

This full dispersion KP equation (FDKP) reads

$$\partial_t u + L_{\beta, \epsilon}(D) \left(1 + \epsilon \frac{D_2^2}{D_1^2}\right)^{\frac{1}{2}} \partial_{x_1} u + 3\epsilon \partial_{x_1}(u^2) = 0, \tag{1.2}$$

where  $u = u(x_1, x_2, t)$  is a real-valued function,  $(D_1, D_2) = (-i\partial_{x_1}, -i\partial_{x_2})$ ,  $D^\epsilon = (D_1, \sqrt{\epsilon}D_2)$ , hence

$$|D^\epsilon| = \sqrt{D_1^2 + \epsilon D_2^2},$$

<sup>1</sup>We refer to [22] for another approach for an asymptotic water model in the KP regime leading to a local *weakly transverse Boussinesq system* leading to the optimal error estimate with the solutions of the full water waves system.

and  $L_{\beta,\epsilon}$  is a non-local operator defined by

$$L_{\beta,\epsilon}(D) = (1 + \beta\epsilon|D^{\epsilon}|^2)^{\frac{1}{2}} \left( \frac{\tanh(\sqrt{\epsilon}|D^{\epsilon}|)}{\sqrt{\epsilon}|D^{\epsilon}|} \right)^{1/2}.$$

Here  $\beta \geq 0$  is a dimensionless coefficient measuring the surface tension effects and  $\epsilon > 0$  is the shallowness parameter which is proportional to the ratio of the amplitude of the wave to the mean depth of the fluid.

In the case of purely gravity waves ( $\beta = 0$ ), the symbol of (1.2) writes

$$p(\xi_1, \xi_2) = \frac{1}{\epsilon^{1/4}} \left( \tanh[\sqrt{\epsilon}(\xi_1^2 + \epsilon\xi_2^2)^{\frac{1}{2}}] \right)^{\frac{1}{2}} (\xi_1^2 + \epsilon\xi_2^2)^{\frac{1}{4}} \operatorname{sgn} \xi_1, \tag{1.3}$$

while in the case of gravity-capillary waves ( $\beta > 0$ ), the symbol is

$$\tilde{p}(\xi_1, \xi_2) = (1 + \beta\epsilon(\xi_1^2 + \epsilon\xi_2^2))^{1/2} p(\xi_1, \xi_2). \tag{1.4}$$

The symbols  $p$  and  $\tilde{p}$  being real, it is clear that the linearized equations define unitary groups in all Sobolev spaces  $H^s(\mathbb{R}^2)$ ,  $s \in \mathbb{R}$ .

Contrary to the KP case,  $p$  and  $\tilde{p}$  are locally bounded on  $\mathbb{R}^2$ . However they are not continuous on the line  $\{(0, \xi_2), \xi_2 \neq 0\}$ , but they do not have the singularity  $\frac{1}{\xi_1}$  of the KP equations symbols.

We now describe some links between the FDKP equation and related nonlocal dispersive equations. We first observe that for waves depending only on  $x_1$ , the FDKP equation when  $\beta = 0$  reduces to the so-called Whitham equation, see [14],

$$\partial_t u + \left( \frac{\tanh(\sqrt{\epsilon}|D_1|)}{\sqrt{\epsilon}|D_1|} \right)^{1/2} \partial_{x_1} u + \epsilon \frac{3}{2} u \partial_{x_1} u = 0, \tag{1.5}$$

and when  $\beta > 0$ , it reduces to the Whitham equation with surface tension

$$\partial_t u + (1 + \beta\epsilon D_1^2)^{1/2} \left( \frac{\tanh(\sqrt{\epsilon}|D_1|)}{\sqrt{\epsilon}|D_1|} \right)^{1/2} \partial_{x_1} u + \epsilon \frac{3}{2} u \partial_{x_1} u = 0. \tag{1.6}$$

Note that the Whitham equations can be seen for large frequencies as perturbations of the fractional KdV (fKdV) equations

$$\partial_t u + \partial_{x_1} u + \beta^{1/2} \epsilon^{1/4} |D_1|^{1/2} \partial_{x_1} u + \epsilon \frac{3}{2} u \partial_{x_1} u = 0, \tag{1.7}$$

when  $\beta > 0$ , and

$$\partial_t u + \partial_{x_1} u + \epsilon^{-1/4} |D_1|^{-1/2} \partial_{x_1} u + \epsilon \frac{3}{2} u \partial_{x_1} u = 0, \tag{1.8}$$

when  $\beta = 0$ .

The FDKP equation may be therefore seen as a natural (weakly transverse) two-dimensional version of the Whitham equation, with and without surface tension.

On the other hand, those fKdV equations have KP versions, namely the fractional KP equations (fKP), see [18], which write for a general nonlocal operator  $D_1^\alpha$  :

$$\partial_t u + u \partial_{x_1} u - D_1^\alpha \partial_{x_1} u \pm \partial_{x_1}^{-1} \partial_{x_2}^2 u = 0, \quad -1 < \alpha < 1 \tag{1.9}$$

in its two versions, the fKP-II (+ sign) and the fKP-I version (− sign).

The fKP equation has several motivations. First, when  $\alpha = 1$  the fKP-II equation is the relevant version of the Benjamin-Ono equation. For general values of  $\alpha$  the fKP equation is the KP version of the fractional KdV equation (fKdV), which in turn is a useful toy model to understand the effect of a “weak” dispersion on the dynamics of the inviscid Burgers equation. When  $-1 < \alpha < 0$ , both equations are mainly “hyperbolic”, with the possibility of shocks (but a dispersive effect leading to the possibility of global existence and scattering of small solutions) while when  $0 < \alpha < 1$  the dispersive effects are strong enough to prevent the appearance of shocks for instance. We refer to [7, 14, 15, 17, 18] for results and numerical simulations on those equations.

The fKP equation can also be seen as a weakly dispersive perturbation of the (inviscid) Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation (see [24]) which has a “hyperbolic” character with the possible appearance of shocks.

Those fKP equations are not directly connected to the FDKP equation but they share the property of being two dimensional nonlocal dispersive perturbations of the Burgers equation.

Some of the properties of the FDKP equations are displayed in [22]. In particular, it is easy to check by viewing it as a skew-adjoint perturbation of the Burgers equation, that the Cauchy problem is locally well-posed, without need of any constraint, in  $H^s(\mathbb{R}^2)$ ,  $s > 2$ . Note that this result does not use any dispersive property of the linear group.

The natural energy space associated to the FDKP equation is, in the case without surface tension  $\beta = 0$ ,

$$E = \left\{ \mathbf{u} \in L^2(\mathbb{R}^2) \cap L^3(\mathbb{R}^2) : |D^\epsilon|^{1/4} |D_1|^{-1/2} \mathbf{u}, |D^\epsilon|^{1/2} |D_1|^{-1/2} \mathbf{u} \in L^2(\mathbb{R}^2) \right\}.$$

This space is associated to a natural Hamiltonian. In fact, as for the classical KP I/II equations, the  $L^2$  norm is formally conserved by the flow of (1.2), and so is the Hamiltonian

$$\mathfrak{H}_\epsilon(\mathbf{u}) = \frac{1}{2} \int_{\mathbb{R}^2} |H_\epsilon(D)\mathbf{u}|^2 + \frac{\epsilon}{4} \int_{\mathbb{R}^2} \mathbf{u}^3, \tag{1.10}$$

where

$$\begin{aligned} H_\epsilon(D) &= \left( \frac{(1 + \sigma\epsilon|D^\epsilon|^2) \tanh(\sqrt{\epsilon}|D^\epsilon|)}{\epsilon^{1/2}|D_\epsilon|} \right)^{1/4} \left( 1 + \epsilon \frac{D_2^2}{D_1^2} \right)^{1/4} \\ &= \left( \frac{(1 + \sigma\epsilon|D^\epsilon|^2) \tanh(\sqrt{\epsilon}|D^\epsilon|)}{\epsilon^{1/2}} \right)^{1/4} \frac{|D^\epsilon|^{1/4}}{|D_1|^{1/2}}. \end{aligned} \tag{1.11}$$

The cases  $\beta = 0$  and  $\beta > 0$  correspond respectively to purely gravity waves and capillary-gravity waves.

One finds the standard KP I/II Hamiltonians by expanding formally  $H_\epsilon(D)$  in powers of  $\epsilon$ , namely

$$H_\epsilon(D)(\mathbf{u}) = \frac{\epsilon}{4} \int_{\mathbb{R}^2} [|\partial_{x_2} \partial_{x_1}^{-1} \mathbf{u}|^2 + (\beta - \frac{1}{3}) |\partial_{x_1} \mathbf{u}|^2 + \mathbf{u}^3] dx_1 dx_2 + o(\epsilon).$$

Contrary to the Cauchy problem which can be solved without constraint, the Hamiltonian for the FDKP equation is well defined (and conserved by the flow) provided  $\mathbf{u}$  satisfies a constraint, weaker however than that of the classical KP equations.<sup>2</sup>

Finally, as noticed in [15] by considering the solution of the linear KP I/II equations,

$$\hat{\mathbf{u}}(\xi_1, \xi_2, t) = \hat{\mathbf{u}}_0(\xi_1, \xi_2) \exp \left( it \left( \xi_1^3 \pm \frac{\xi_2^2}{\xi_1} \right) \right),$$

the singularity  $\frac{\xi_2^2}{\xi_1}$  implies that a strong decay of the initial data is not preserved by the linear flow,<sup>3</sup> for instance the solution corresponding to a gaussian initial data cannot decay faster than  $1/(x_1^2 + x_2^2)$  at infinity. In fact, the Riemann–Lebesgue theorem implies that  $\mathbf{u}(\cdot, t) \notin L^1(\mathbb{R}^2)$  for any  $t \neq 0$ . The same conclusion holds of course even if  $\mathbf{u}_0$  satisfies the zero-mass constraint, *e.g.*  $\mathbf{u}_0 \in \partial_x \mathcal{S}(\mathbb{R}^2)$  and also for the nonlinear problem as shows the Duhamel representation of the solution, see [15].

A similar obstruction holds for the FDKP equations. In particular, the localised solitary waves solutions found in [6] cannot decay fast at infinity.

<sup>2</sup>In the sense that the order of vanishing of the Fourier transform at the frequency  $\xi_1 = 0$  is weaker than the corresponding one for the KP equations.

<sup>3</sup>and also actually for the nonlinear flow.

### 1.2. Presentation of the Results

In order to study the Cauchy problem associated to FDKP in spaces larger than the “hyperbolic space”  $H^s(\mathbb{R}^2)$ ,  $s > 2$ , and to investigate the scattering of small solutions, we will focus in this paper on the derivation of dispersive estimates on the linear group. This is not a simple matter since the symbol  $p$  and  $\tilde{p}$  defined in (1.3) and (1.4) are non-homogeneous and also non-polynomial. Similar difficulties occur for other non-standard dispersive equation such as the Novikov-Veselov equation [10] or a higher dimensional version of the Benjamin-Ono equation [8].

In the rest of the paper, we work with  $\epsilon = 1$ . Based on the identity

$$\left(1 + \frac{D_2^2}{D_1^2}\right)^{\frac{1}{2}} \partial_{x_1} = \frac{iD_1}{|D_1|} |D|,$$

we rewrite (1.2) as

$$\partial_t u + \tilde{L}_\beta(D)u + 3\partial_{x_1}(u^2) = 0, \tag{1.12}$$

where

$$\tilde{L}_\beta(D) = \frac{iD_1}{|D_1|} |D| (1 + \beta|D|^2)^{\frac{1}{2}} \left(\frac{\tanh(\sqrt{|D|})}{\sqrt{|D|}}\right)^{1/2}.$$

The solution propagator for the linear equation is given by

$$[S_{m_\beta}(t)f](x) := \int_{\mathbb{R}^2} e^{ix \cdot \xi + it \operatorname{sgn}(\xi_1) m_\beta(|\xi|)} \hat{f}(\xi) d\xi, \tag{1.13}$$

where

$$m_\beta(r) = r (1 + \beta r^2)^{\frac{1}{2}} \left(\frac{\tanh(r)}{r}\right)^{\frac{1}{2}} \tag{1.14}$$

and  $|\xi| = \sqrt{\xi_1^2 + \xi_2^2}$ .

Our first result is a  $L^1 - L^\infty$  decay estimate for the linear propagator associated to (1.2). Since the symbol  $m_\beta$  is non-homogeneous, we will derive our estimate for frequency localised functions. For a dyadic number  $\Lambda \in 2^{\mathbb{Z}}$ , let  $P_\Lambda$  denote the Littlewood–Paley projector localising the frequency around the dyadic number  $\Lambda$  (a more precise definition of  $P_\Lambda$  will be given in the notations below).

**Theorem 1.1.** (Localised dispersive estimate) *Let  $\beta \in \{0, 1\}$ . Then, there exists a positive constant  $c_\beta$  such that*

$$\|S_{m_\beta}(t)P_\Lambda f\|_{L_x^\infty(\mathbb{R}^2)} \leq c_\beta \langle \sqrt{\beta}\Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{3}{2}} |t|^{-1} \|P_\Lambda f\|_{L_x^1(\mathbb{R}^2)} \tag{1.15}$$

for all  $\Lambda \in 2^{\mathbb{Z}}$  and  $f \in \mathcal{S}(\mathbb{R}^2)$ , and where  $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$ .

By a standard argument, the proof reduces to proving a uniform bound for the two dimensional oscillatory integral

$$I_{\Lambda,t}(x) = \int_{\mathbb{R}^2} e^{ix \cdot \xi + it \operatorname{sgn}(\xi_1) m_\beta(|\xi|)} \rho(\Lambda^{-1}|\xi|) d\xi, \tag{1.16}$$

where  $\rho$  is a smooth function whose compact support is localised around 1. Observe that in the KP and fractional KP cases, the corresponding oscillatory integral

$$\int_{\mathbb{R}^2} e^{it\left(\varphi(\xi_1) + \frac{\xi_2^2}{\xi_1}\right) + ix \cdot \xi} d\xi$$

can be reduced to a one-dimensional integral by integrating in  $\xi_2$  and using the explicit representation of the linear Schrödinger propagator (see [17, 23, 25]).

This is not the case anymore for the oscillatory integral (1.16) and for this reason we need to employ 2-dimensional methods. After passing to polar coordinates, we write

$$I_{\Lambda, t}(x) = \Lambda^2 \int_0^\infty \left[ e^{itm_\beta(\Lambda r)} J_+(\Lambda r x) + e^{-itm_\beta(\Lambda r)} J_-(\Lambda r x) \right] r \rho(r) dr, \tag{1.17}$$

where

$$J_\pm(x) = \int_{\omega \in S^1} \mathbb{1}_{\{\pm \omega_1 > 0\}} e^{ix \cdot \omega} d\sigma(\omega) \tag{1.18}$$

are *asymmetric Bessel functions*. Then, by using complex integration, we derive sharp asymptotics for these asymmetric Bessel functions which may be of independent interest (see Proposition 2.1). With these asymptotics in hand, we can conclude the proof of Theorem 1.1 by combining the stationary phase method with careful estimates on the symbol  $m_\beta$  and its derivatives both in the case  $\beta = 0$  and  $\beta = 1$ .

Once Theorem 1.1 is proved, the corresponding Strichartz estimates are deduced from a classical  $\Pi^*$  argument.

**Theorem 1.2.** (Localised Strichartz estimates) *Let  $\beta \in \{0, 1\}$ . Assume that  $q, r$  satisfy*

$$2 < q \leq \infty, \quad 2 \leq r < \infty \quad \text{and} \quad \frac{1}{r} + \frac{1}{q} = \frac{1}{2}. \tag{1.19}$$

*Then, there exists a positive constant  $c_\beta$  such that*

$$\|S_{m_\beta}(t) P_\Lambda f\|_{L_t^q L_x^r(\mathbb{R}^{2+1})} \leq c_\beta \left[ \langle \sqrt{\beta} \Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{3}{2}} \right]^{\frac{1}{2} - \frac{1}{r}} \|P_\Lambda f\|_{L_x^2(\mathbb{R}^2)} \tag{1.20}$$

*for all  $\Lambda \in 2^{\mathbb{Z}}$  and all  $f \in \mathcal{S}(\mathbb{R}^2)$ , and where  $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$ .*

*Remark 1.3.* Estimate (1.20) in the case  $\beta = 1$  only requires a loss slightly smaller than 1/4 derivative close to the end point  $(q, r) = (2, \infty)$ . This is better than the corresponding<sup>4</sup> Strichartz estimate for the fractional KP equation with  $\alpha = \frac{1}{2}$  where the loss is slightly smaller than 3/8 derivatives (see Proposition 4.9 in [18]).

As an application of these Strichartz estimates, we are able to improve the standard well-posedness result  $H^s(\mathbb{R}^2)$ ,  $s > 2$ , for FDKP in the case of capillary-gravity waves ( $\beta > 0$ ).

**Theorem 1.4.** *Assume that  $\beta = \epsilon = 1$  and  $s > \frac{7}{4}$ . Then, for any  $u_0 \in H^s(\mathbb{R}^2)$ , there exist a positive time  $T = T(\|u_0\|_{H^s})$  (which can be chosen as a nonincreasing function of its argument) and a unique solution  $u$  to the IVP associated to the FDKP equation (1.2) in the class*

$$C([0, T] : H^s(\mathbb{R}^2)) \cap L^1((0, T) : W^{1,\infty}(\mathbb{R}^2)), \tag{1.21}$$

*satisfying  $u(\cdot, 0) = u_0$ .*

*Moreover, for any  $0 < T' < T$ , there exists a neighborhood  $\mathcal{U}$  of  $u_0$  in  $H^s(\mathbb{R}^2)$  such that the flow map data-to-solution*

$$\mathcal{U} \rightarrow C([0, T'] : H^s(\mathbb{R}^2)), \quad v_0 \mapsto v,$$

*is continuous.*

We now comment on the main ingredients in the proof of Theorem 1.4. A standard energy estimate combined with the Kato–Ponce commutator estimate yields

$$\sup_{t \in [0, T]} \|u(\cdot, t)\|_{H_x^s}^2 \leq \|u(0)\|_{H_x^s}^2 + c \left( \int_0^T \|\nabla u(\cdot, t)\|_{L_x^\infty} dt \right) \sup_{t \in [0, T]} \|u(\cdot, t)\|_{H_x^s}^2, \quad s > 0. \tag{1.22}$$

Therefore, the main difficulty is to control the term  $\|\nabla u\|_{L_t^1 L_x^\infty}$ . This can be done easily using the Sobolev embedding at the “hyperbolic” regularity  $s > 2$ . To lower this threshold, we use a refined Strichartz estimate on the linear non-homogeneous version of (1.2) (see Lemma 4.1). More precisely, after performing

<sup>4</sup>For high frequency, the dispersive symbol  $\tilde{p}$  of FDKP defined in (1.4) satisfies  $|\tilde{p}(\xi)| \sim |\xi|^{\frac{3}{2}}$ .

a Littlewood–Paley decomposition on the function  $\mathbf{u}$ , we chop the time interval  $[0, T]$  into small intervals whose size is inversely proportional to the frequency of the Littlewood–Paley projector. Then, we apply our frequency localised Strichartz estimate (Theorem 1.2) to each of these pieces and sum up to get the result. This estimate allows to control  $\|\nabla \mathbf{u}\|_{L^1_t L^\infty_x}$  at the regularity level  $s > \frac{7}{4}$ . Note that similar estimates have already been used for nonlinear dispersive equations (see for instance to [2, 4, 8, 12, 13, 16–18, 28]).

This estimate combined with the energy estimate (1.22) provides an *a priori* bound on smooth solutions of (1.2). The existence of solutions in Theorem 1.4 is then deduced by using compactness methods, while the uniqueness follows from an energy estimate for the difference of two solutions in  $L^2$  combined with Gronwall’s inequality. Finally, to prove the persistence property and the continuity of the flow, we use the Bona-Smith argument.

The paper is organized as follows: in Sect. 2, we prove the sharp asymptotics for the asymmetric Bessel functions, which will be used to prove Theorem 1.1 and 1.2 in Sect. 3. Section 4 is devoted to the proof of the local well-posedness result. Finally, we derive some useful estimates on the derivatives of the symbol  $m_\beta$  in the appendix.

**Notation.** For any positive numbers  $a$  and  $b$ , the notation  $a \lesssim b$  stands for  $a \leq cb$ , where  $c$  is a positive constant that may change from line to line. Moreover, we denote  $a \sim b$  when  $a \lesssim b$  and  $b \lesssim a$ .

We also set  $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$ .

For  $x = (x_1, x_2)$ ,  $\mathbf{u} = \mathbf{u}(x) \in \mathcal{S}'(\mathbb{R}^2)$ ,  $\mathcal{F}\mathbf{u} = \hat{\mathbf{u}}$  will denote the Fourier transform of  $\mathbf{u}$ . For  $s \in \mathbb{R}$ , we define the Bessel potential of order  $-s$ ,  $J^s$  by

$$J^s \mathbf{u} = \mathcal{F}^{-1}(\langle \xi \rangle^s \mathcal{F}\mathbf{u}).$$

Throughout the paper, we fix a smooth cutoff function  $\chi$  such that

$$\chi \in C_0^\infty(\mathbb{R}), \quad 0 \leq \chi \leq 1, \quad \chi|_{[-1,1]} = 1 \quad \text{and} \quad \text{supp}(\chi) \subset [-2, 2]. \tag{1.23}$$

We set

$$\rho(s) = \chi(s) - \chi(2s).$$

Thus,  $\text{supp} \rho = \{s \in \mathbb{R} : 1/2 \leq |s| \leq 2\}$ . For  $\Lambda \in 2^{\mathbb{Z}}$  we set  $\rho_\Lambda(s) := \rho(s/\Lambda)$  and define the frequency projection  $P_\Lambda$  by

$$\widehat{P_\Lambda f}(\xi) = \rho_\Lambda(|\xi|) \widehat{f}(\xi).$$

Any summations over capitalized variables such as  $\Lambda$  or  $\Gamma$  are presumed to be over dyadic numbers. We also define

$$P_{\leq \Lambda} = \sum_{\Gamma \leq \Lambda} P_\Gamma \quad \text{and} \quad P_{> \Lambda} = 1 - P_{\leq \Lambda} = \sum_{\Gamma > \Lambda} P_\Gamma.$$

We sometimes write  $f_\Lambda := P_\Lambda f$ , so that

$$f = \sum_{\Lambda \in 2^{\mathbb{Z}}} f_\Lambda = P_{\leq 1} f + \sum_{\Lambda > 1} f_\Lambda.$$

For  $1 \leq p \leq \infty$ ,  $L^p(\mathbb{R}^2)$  denotes the usual Lebesgue space and for  $s \in \mathbb{R}$ ,  $H^s(\mathbb{R}^2)$  is the  $L^2$ -based Sobolev space with norm  $\|f\|_{H^s} = \|J^s f\|_{L^2}$ . If  $B$  is a space of functions on  $\mathbb{R}^2$ ,  $T > 0$  and  $1 \leq p \leq \infty$ , we define the spaces  $L^p((0, T) : B)$  and  $L^p(\mathbb{R} : B)$  respectively through the norms

$$\|f\|_{L^p_T B_x} = \left( \int_0^T \|f(\cdot, t)\|_B^p dt \right)^{\frac{1}{p}} \quad \text{and} \quad \|f\|_{L^p_t B_x} = \left( \int_{\mathbb{R}} \|f(\cdot, t)\|_B^p dt \right)^{\frac{1}{p}},$$

when  $1 \leq p < \infty$ , with the usual modifications when  $p = +\infty$ .

## 2. Identities and Decay for the Asymmetric Bessel Functions

**Proposition 2.1.** (*Identities and decay for  $J_+$* ) Let  $x = (x_1, x_2)$ ,  $s_1 = \text{sgn}(x_1)$  and  $x'_2 = x_2/|x|$ . Define

$$J_+(x) = \int_{-\pi/2}^{\pi/2} e^{ix \cdot \omega(\theta)} d\theta,$$

where  $\omega(\theta) = (\cos \theta, \sin \theta)$ . Then we have the following:

(i).  $J_+$  can be written as

$$J_+(x) = F(|x|, |x'_2|) + F^{s_1}(|x|, |x'_2|), \tag{2.1}$$

where

$$F(r, a) = \int_{-a}^a e^{irs} \frac{ds}{\sqrt{1-s^2}},$$

$$F^\pm(r, a) = 2 \int_a^1 e^{\pm irs} \frac{ds}{\sqrt{1-s^2}}$$

for  $a \in [0, 1]$ .

(ii). The functions  $F$  and  $F^\pm$  can be written as

$$F(r, a) = e^{iar} f_a^+(r) + e^{-iar} f_a^-(r), \tag{2.2}$$

$$F^\pm(r, a) = 2e^{\pm ir} f_a^\pm(r) - 2e^{\pm iar} f_a^\pm(r), \tag{2.3}$$

where

$$f_a^\pm(r) = \mp i \int_0^\infty e^{-rs} (s^2 + 1 - a^2 \mp 2ais)^{-1/2} ds. \tag{2.4}$$

(iii). Moreover, the functions  $f_a^\pm$  and their derivatives satisfy the decay estimates

$$|\partial_r^j f_a^\pm(r)| \leq Cr^{-j-1/2} \quad (j = 0, 1) \tag{2.5}$$

for all  $r \geq 1$  and  $a \in [0, 1]$ .

The proof of Proposition 2.1 is given in the following subsections.

### 2.1. Proof of Proposition 2.1(i)

Writing  $x = |x|\omega(\alpha)$ , where  $\alpha \in [0, 2\pi)$ , we have

$$J_+(x) = \int_{-\pi/2}^{\pi/2} e^{i|x|\omega(\alpha) \cdot \omega(\theta)} d\theta = \int_{-\pi/2}^{\pi/2} e^{i|x|\cos(\theta-\alpha)} d\theta$$

$$= \int_{\alpha-\pi/2}^{\alpha+\pi/2} e^{i|x|\cos(\theta)} d\theta.$$

We shall use the following change of variables:

$$s = \cos \theta \Rightarrow \sin \theta = \pm \sqrt{1-s^2}, \quad ds = -\sin \theta d\theta.$$

Now if  $\alpha \in [0, \pi/2]$ , i.e.,  $x_1 \geq 0$  and  $x_2 \geq 0$ , we write

$$J_+(x) = \left( \int_{\alpha-\pi/2}^0 + \int_0^{\alpha+\pi/2} \right) e^{i|x|\cos(\theta)} d\theta$$

$$= \left( \int_{\sin \alpha}^1 + \int_{-\sin \alpha}^1 \right) e^{i|x|s} \frac{ds}{\sqrt{1-s^2}}$$

$$= \left( \int_{-\sin \alpha}^{\sin \alpha} + 2 \int_{\sin \alpha}^1 \right) e^{i|x|s} \frac{ds}{\sqrt{1-s^2}}$$

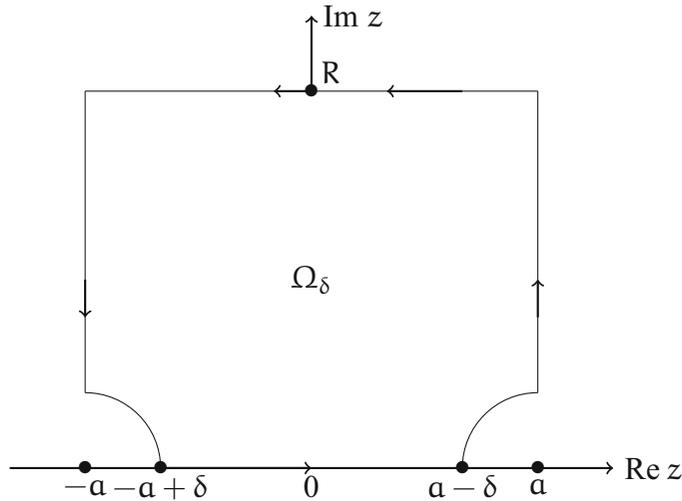


FIG. 1. The region  $\Omega_\delta(a, R)$  with  $0 \leq a \leq 1$ ,  $0 < \delta \ll 1$  and  $R \gg 1$

$$= F(|x|, |x'_2|) + F^+(|x|, |x'_2|),$$

where we used the fact that  $x'_2 = x_2/|x| = \sin \alpha > 0$  and  $s_1 = \operatorname{sgn}(x_1) = +$ .

If  $\alpha \in [\pi/2, \pi)$ , i.e.,  $x_1 \leq 0$  and  $x_2 > 0$ , we split the integral over  $[\alpha - \pi/2, \pi]$  and  $[\pi, \alpha + \pi/2]$ , and write

$$\begin{aligned} J_+(x) &= \left( \int_{-\sin \alpha}^{\sin \alpha} + 2 \int_{-1}^{-\sin \alpha} \right) e^{i|x|s} \frac{ds}{\sqrt{1-s^2}} \\ &= F(|x|, |x'_2|) + F^-(|x|, |x'_2|). \end{aligned}$$

The remaining cases can be established similarly. In fact, if  $\alpha \in [\pi, 3\pi/2)$ , i.e.,  $x_1 < 0$  and  $x_2 \leq 0$ , we split the integral over  $[\alpha - \pi/2, \pi]$  and  $[\pi, \alpha + \pi/2]$  whereas if  $\alpha \in [3\pi/2, 2\pi)$ , i.e.,  $x_1 \geq 0$  and  $x_2 < 0$ , we split the integral over  $[\alpha - \pi/2, 2\pi]$  and  $[2\pi, \alpha + \pi/2]$  to obtain the desired identities.

## 2.2. Proof of Proposition 2.1(ii)

We follow [27, Chapter 4, Lemma 3.11].

For fixed  $0 < \delta \ll 1$  and  $R \gg 1$ , let  $\Omega_\delta(a, R)$  be the region in the complex plane obtained from the rectangle with vertices at points  $(-a, 0)$ ,  $(a, 0)$ ,  $(a, R)$  and  $(-a, R)$ , by removing two quarter circles of radius  $\delta$  and centered at  $(a, 0)$  and  $(-a, 0)$ , denoted  $C_\delta(a)$  and  $C_\delta(-a)$ , respectively; see Fig. 1 below.

The functions

$$h^\pm(z) = e^{\pm irz} (1 - z^2)^{-1/2}$$

have no poles in  $\Omega_\delta$ . So by Cauchy's theorem we have

$$\begin{aligned} 0 &= \int_{\partial \Omega_\delta} h^+(z) dz \\ &= \int_{-a+\delta}^{a-\delta} h^+(s) ds + i \int_\delta^R h^+(a + is) ds - i \int_\delta^R h^+(-a + is) ds + \mathcal{E}_\delta(a, R), \end{aligned}$$

where

$$\mathcal{E}_\delta(a, R) = \int_{C_\delta(a)} h^+(z) dz + \int_{C_\delta(-a)} h^+(z) dz - \int_{-a}^a h^+(s + iR) ds.$$

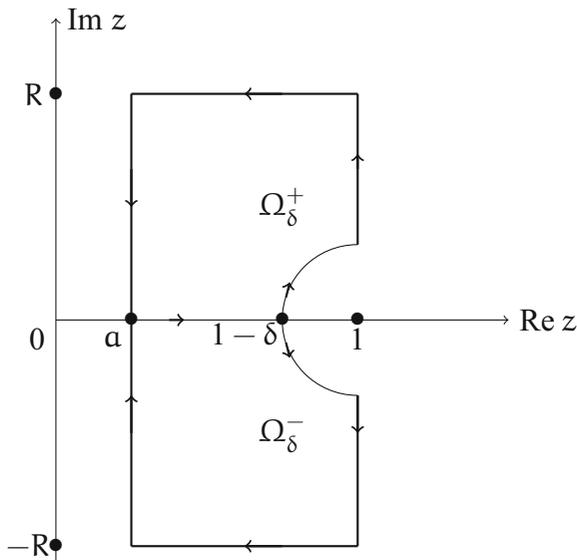


FIG. 2. The regions  $\Omega_\delta^+(\mathbf{a}, \mathbf{R})$  and  $\Omega_\delta^-(\mathbf{a}, \mathbf{R})$  with  $0 \leq \mathbf{a} \leq 1$ ,  $0 < \delta \ll 1$  and  $\mathbf{R} \gg 1$

Now letting  $\delta \rightarrow 0$  and  $\mathbf{R} \rightarrow \infty$ , one can show that  $\mathcal{E}(\delta, \mathbf{R}) \rightarrow 0$ , and hence

$$\begin{aligned} F(r, \mathbf{a}) &= \int_{-\mathbf{a}}^{\mathbf{a}} e^{irs} (1 - s^2)^{-1/2} ds = ie^{-air} \int_0^\infty e^{-rs} (s^2 + 1 - \mathbf{a}^2 + 2ais)^{-1/2} ds \\ &\quad - ie^{iar} \int_0^\infty e^{-rs} (s^2 + 1 - \mathbf{a}^2 - 2ais)^{-1/2} ds \\ &= e^{iar} f_{\mathbf{a}}^+(r) + e^{-iar} f_{\mathbf{a}}^-(r) \end{aligned}$$

which proves (2.2).

The identity (2.3) is proved in a similar way. Indeed, let  $\Omega_\delta^\pm(\mathbf{a}, \mathbf{R})$  be the region in the complex plane obtained from the rectangle with vertices at points  $(\mathbf{a}, 0)$ ,  $(1, 0)$ ,  $(1, \pm R)$  and  $(\mathbf{a}, \pm R)$ , by removing the quarter circle of radius  $\delta$  and centered at  $(1, 0)$ , denoted  $C_\delta$ ; see Fig. 2 below.

Again, the functions  $h^\pm(z)$  have no poles in  $\Omega_\delta^\pm$ . So by Cauchy’s theorem we have

$$\begin{aligned} 0 &= \int_{\partial\Omega_\delta^+} h^+(z) dz \\ &= \int_{\mathbf{a}}^{1-\delta} h^+(s) ds + i \int_\delta^{\mathbf{R}} h^+(1 + is) ds - i \int_0^{\mathbf{R}} h^+(\mathbf{a} + is) ds + \mathcal{E}_\delta^+(\mathbf{a}, \mathbf{R}), \end{aligned}$$

where

$$\mathcal{E}_\delta^+(\mathbf{a}, \mathbf{R}) = \int_{C_\delta} h^+(z) dz - \int_{\mathbf{a}}^1 h^+(s + i\mathbf{R}) ds.$$

Letting  $\delta \rightarrow 0$  and  $\mathbf{R} \rightarrow \infty$ , one can show that  $\mathcal{E}_\delta^+(\mathbf{a}, \mathbf{R}) \rightarrow 0$ , and hence

$$\begin{aligned} F^+(r, \mathbf{a}) &= 2 \int_{\mathbf{a}}^1 e^{irs} (1 - s^2)^{-1/2} ds \\ &= 2ie^{air} \int_0^\infty e^{-rs} (s^2 + 1 - \mathbf{a}^2 - 2ais)^{-1/2} ds \end{aligned}$$

$$\begin{aligned} & -2ie^{ir} \int_0^\infty e^{-rs} (s^2 - 2is)^{-1/2} ds \\ & = 2e^{ir} f_1^+(r) - 2e^{air} f_a^+(r). \end{aligned}$$

Similarly, integrating  $h^-(z)$  over  $\partial\Omega_s^-$ , one can show

$$F^-(r, a) = 2e^{-ir} f_1^-(r) - 2e^{-air} f_a^-(r).$$

**2.3. Proof of Proposition 2.1(iii)**

Observe that the following estimate holds for all  $s \geq 0$  and  $0 \leq a \leq 1$ :

$$\left| (s^2 - a^2 + 1 \pm 2ais)^{-\frac{1}{2}} \right| \leq (\max(s^2 + 1 - a^2, 2as))^{-\frac{1}{2}}.$$

We treat the cases  $0 \leq a \leq 1/\sqrt{2}$  and  $1/\sqrt{2} \leq a \leq 1$  separately.

**Case 1:**  $0 \leq a \leq 1/\sqrt{2}$ . In this case we have  $2as \leq s^2 + 1 - a^2$ , and hence

$$\begin{aligned} |f_a^\pm(r)| & \leq \int_0^\infty e^{-rs} (s^2 + 1 - a^2)^{-\frac{1}{2}} ds \\ & \lesssim \frac{1}{\sqrt{1-a^2}} \int_0^{\sqrt{1-a^2}} e^{-rs} ds + \int_{\sqrt{1-a^2}}^\infty e^{-rs} s^{-1} ds \end{aligned}$$

Now using the fact that  $1/2 \leq 1 - a^2 \leq 1$  and  $r \geq 1$ , we bound the first integral on the right by

$$\frac{1}{\sqrt{2}} \int_0^1 e^{-rs} ds = \frac{1 - e^{-r}}{\sqrt{2}r} \sim r^{-1}.$$

Similarly, the second integral on the right is bounded by

$$\frac{1}{\sqrt{2}} \int_{1/\sqrt{2}}^\infty e^{-rs} ds \lesssim r^{-1} e^{-r/\sqrt{2}}.$$

Thus,  $|f_a^\pm(r)| \lesssim r^{-1}$ .

For the derivative, we simply estimate

$$\begin{aligned} |\partial_r f_a^\pm(r)| & \leq \int_0^\infty e^{-rs} s (s^2 + 1 - a^2)^{-\frac{1}{2}} ds \\ & \lesssim \frac{1}{\sqrt{1-a^2}} \int_0^{\sqrt{1-a^2}} e^{-rs} s ds + \int_{\sqrt{1-a^2}}^\infty e^{-rs} ds \\ & \lesssim r^{-2} + r^{-1} \int_{r/\sqrt{2}}^\infty e^{-s} ds \\ & \lesssim r^{-2} + r^{-1} e^{-r/\sqrt{2}} \lesssim r^{-2}. \end{aligned}$$

**Case 2:**  $1/\sqrt{2} \leq a \leq 1$ . We split

$$f_a^\pm(r) = i \left( \int_0^a + \int_a^\infty \right) e^{-rs} (s^2 + 1 - a^2 \mp 2ais)^{-1/2} ds.$$

Then the first integral on the right-hand side is bounded by

$$\begin{aligned} \int_0^a e^{-rs} (2as)^{-\frac{1}{2}} ds & = \frac{1}{\sqrt{2ar}} \int_0^{ar} e^{-s} s^{-\frac{1}{2}} ds \\ & \leq \frac{1}{\sqrt{2ar}} \left( \int_0^a s^{-\frac{1}{2}} ds + a^{-\frac{1}{2}} \int_a^{ar} e^{-s} ds \right) \lesssim r^{-\frac{1}{2}} \end{aligned}$$

and the second integral on the right is bounded by

$$\begin{aligned} \int_a^\infty e^{-rs} (s^2 + 1 - a^2)^{-\frac{1}{2}} ds &\leq \int_a^\infty e^{-rs} s^{-1} ds \\ &= \int_{ar}^\infty e^{-s} s^{-1} ds \leq (ar)^{-1} e^{-ar} \lesssim r^{-1}. \end{aligned}$$

Thus,  $|f_a^\pm(r)| \lesssim r^{-\frac{1}{2}}$ .

For the derivative, we simply estimate

$$\begin{aligned} |\partial_r f_a^\pm(r)| &\leq \int_0^\infty e^{-rs} s(2as)^{-\frac{1}{2}} ds = \frac{r^{-\frac{3}{2}}}{\sqrt{2a}} \int_0^\infty e^{-s} s^{\frac{1}{2}} ds \\ &\lesssim r^{-\frac{3}{2}} \int_0^\infty e^{-s/2} ds \lesssim r^{-\frac{3}{2}}. \end{aligned}$$

### 3. Localised Dispersive and Strichartz Estimates

Before proving Theorem 1.1, we state two useful lemmas. The first one is the classical Van der Corput’s Lemma, whose proof can be found in [26].

**Lemma 3.1.** (Van der Corput’s Lemma) *Assume  $g \in C^1(a, b)$ ,  $\psi \in C^2(a, b)$  and  $|\psi''(r)| \geq A$  for all  $r \in (a, b)$ . Then*

$$\left| \int_a^b e^{it\psi(r)} g(r) dr \right| \leq C(At)^{-1/2} \left[ |g(b)| + \int_a^b |g'(r)| dr \right], \tag{3.1}$$

for some constant  $C > 0$  that is independent of  $a, b$  and  $t$ .

The second one states some pointwise estimates on the first and second order derivatives of the function  $m_\beta$ , for  $\beta = 0, 1$ , defined in (1.14). Its proof, which is an adaptation of the one of the corresponding estimates for  $m_0$ , derived recently by the last two authors in [5], will be given in the appendix.

**Lemma 3.2.** *Let  $\beta \in \{0, 1\}$ . Then for all  $r > 0$  we have*

$$0 < m'_\beta(r) \sim \langle \sqrt{\beta}r \rangle (r)^{-1/2} \tag{3.2}$$

$$|m''_\beta(r)| \sim r \langle \sqrt{\beta}r \rangle (r)^{-5/2} \tag{3.3}$$

We are now in position to give the proof of Theorem 1.1.

*Proof of Theorem 1.1.* We may assume  $t > 0$ . Now recalling the definition of  $S_{m_\beta}$  in (1.13)–(1.14), we can write

$$[S_{m_\beta}(t)f_\Lambda](x) = (I_{\Lambda,t} * f)(x),$$

where  $I_{\Lambda,t}$  is as in (1.17)–(1.18), i.e.,

$$I_{\Lambda,t}(x) = \Lambda^2 \int_0^\infty \left[ e^{itm_\beta(\Lambda r)} J_+(\Lambda r x) + e^{-itm_\beta(\Lambda r)} J_-(\Lambda r x) \right] \tilde{\rho}(r) dr,$$

with  $\tilde{\rho}(r) = r\rho(r)$  and

$$J_\pm(x) = \int_{\omega \in S^1} \mathbb{1}_{\{\pm\omega_1 > 0\}} e^{ix \cdot \omega} d\sigma(\omega).$$

By Young’s inequality

$$\|S_{m_\beta}(t)f_\Lambda\|_{L^\infty_x(\mathbb{R}^2)} \leq \|I_{\Lambda,t}\|_{L^\infty_x(\mathbb{R}^2)} \|f\|_{L^1_x(\mathbb{R}^2)} \tag{3.4}$$

Thus, (1.15) reduces to

$$\|I_{\Lambda,t}\|_{L^\infty_x(\mathbb{R}^2)} \lesssim \langle \sqrt{\beta}\Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{3}{2}} t^{-1}. \tag{3.5}$$

Using the parametrization  $\omega(\theta) = (\cos \theta, \sin \theta)$ , we can write

$$J_+(x) = \int_{-\pi/2}^{\pi/2} e^{ix \cdot \omega(\theta)} d\theta$$

and

$$J_-(x) = \int_{\pi/2}^{3\pi/2} e^{ix \cdot \omega(\theta)} d\theta = \int_{-\pi/2}^{\pi/2} e^{-ix \cdot \omega(\theta)} d\theta = \overline{J_+(x)}.$$

Thus,

$$I_{\Lambda,t}(x) = 2\Lambda^2 \operatorname{Re} \tilde{I}_{\Lambda,t}(x),$$

where

$$\tilde{I}_{\Lambda,t}(x) = \int_{1/2}^2 e^{itm_\beta(\Lambda r)} J_+(\Lambda r x) \tilde{\rho}(r) dr. \quad (3.6)$$

So (3.5) reduces to proving

$$\left\| \tilde{I}_{\Lambda,t} \right\|_{L^\infty(\mathbb{R}^2)} \lesssim \Lambda^{-2} \langle \sqrt{\beta} \Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{3}{2}} t^{-1}. \quad (3.7)$$

We treat the cases  $|x| \lesssim \Lambda^{-1}$  and  $|x| \gg \Lambda^{-1}$  separately. First assume  $|x| \lesssim \Lambda^{-1}$ . Then for all  $r \in (1/2, 2)$  and  $k = 0, 1$ , we have

$$\left| \partial_r^k J_+(\Lambda r x) \right| \leq (\Lambda |x|)^k \lesssim 1. \quad (3.8)$$

Integration by parts yields

$$\begin{aligned} \tilde{I}_{\Lambda,t}(x) &= -i(\Lambda t)^{-1} \int_{1/2}^2 \frac{d}{dr} \left\{ e^{itm_\beta(\Lambda r)} \right\} [m'_\beta(\Lambda r)]^{-1} J_+(\Lambda r x) \tilde{\rho}(r) dr \\ &= i(\Lambda t)^{-1} \int_{1/2}^2 e^{itm_\beta(\Lambda r)} [m'_\beta(\Lambda r)]^{-1} \partial_r [J(\Lambda r x) \tilde{\rho}(r)] dr \\ &\quad - i(\Lambda t)^{-1} \int_{1/2}^2 e^{itm_\beta(\Lambda r)} [m'_\beta(\Lambda r)]^{-2} \Lambda m''_\beta(\Lambda r) J(\Lambda r x) \tilde{\rho}(r) dr. \end{aligned}$$

Now applying Lemma 3.2 and (3.8) we obtain

$$\begin{aligned} \left| \tilde{I}_{\Lambda,t}(x) \right| &\lesssim (\Lambda t)^{-1} \langle \sqrt{\beta} \Lambda \rangle^{-1} \left( \langle \Lambda \rangle^{1/2} + \Lambda^2 \langle \Lambda \rangle^{-\frac{3}{2}} \right) \\ &\lesssim \Lambda^{-1} \langle \sqrt{\beta} \Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{1}{2}} t^{-1} \\ &\lesssim \Lambda^{-2} \langle \sqrt{\beta} \Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{3}{2}} t^{-1}. \end{aligned}$$

So from now on we assume  $|x| \gg \Lambda^{-1}$ . Let  $x = (x_1, x_2)$ ,  $s_1 = \operatorname{sgn}(x_1)$  and  $x'_2 = x_2/|x|$ . By Proposition 2.1(i)–(ii) we can write

$$\begin{aligned} J_+(\Lambda r x) &= 2e^{is_1 \Lambda |x| r} f_1^{s_1}(\Lambda |x| r) - s_1 e^{i\Lambda |x_2| r} f_{|x'_2|}^+(\Lambda r |x|) \\ &\quad + s_1 e^{-i\Lambda |x_2| r} f_{|x'_2|}^-(\Lambda |x| r), \end{aligned} \quad (3.9)$$

whereas by Proposition 2.1(iii) the functions  $f_\alpha^\pm$ , with  $\alpha = 1$  or  $|x'_2|$ , satisfy the estimates

$$\left| \partial_r^j [f_\alpha^\pm(\Lambda r |x|)] \right| \lesssim (\Lambda |x|)^{-\frac{1}{2}} \quad (j = 0, 1) \quad (3.10)$$

for all  $r \in [1/2, 2]$ . Set

$$\begin{aligned} G_\Lambda^\pm(x, r) &:= f_1^\pm(\Lambda r |x|) \tilde{\rho}(r), \\ H_\Lambda^\pm(x, r) &:= f_{|x'_2|}^\pm(\Lambda r |x|) \tilde{\rho}(r). \end{aligned} \quad (3.11)$$

Then by (3.10) we have

$$|\partial_r^j G_\Lambda^\pm(x, r)| + |\partial_r^j H_\Lambda^\pm(x, r)| \lesssim (\Lambda |x|)^{-\frac{1}{2}} \quad (j = 0, 1). \quad (3.12)$$

Now using (3.9) and (3.11) in (3.6) we can write

$$\tilde{I}_{\Lambda,t}(x) = 2\mathcal{J}_{\Lambda,t}^{s_1}(x) - s_1\mathcal{J}_{\Lambda,t}^+(x) + s_1\mathcal{J}_{\Lambda,t}^-(x), \tag{3.13}$$

where

$$\begin{aligned} \mathcal{J}_{\Lambda,t}^\pm(x) &= \int_{1/2}^2 e^{it\phi_\Lambda^\pm(x,r)} G_\Lambda^\pm(x,r) \, dr \\ \mathcal{J}_{\Lambda,t}^\pm(x) &= \int_{1/2}^2 e^{it\psi_\Lambda^\pm(x,r)} H_\Lambda^\pm(x,r) \, dr \end{aligned}$$

with

$$\begin{aligned} \phi_\Lambda^\pm(x,r) &= m_\beta(\Lambda r) \pm \Lambda|x|r/t, \\ \psi_\Lambda^\pm(x,r) &= m_\beta(\Lambda r) \pm \Lambda|x_2|r/t. \end{aligned}$$

Observe that

$$\begin{aligned} \partial_r \phi_\Lambda^\pm(x,r) &= \Lambda [m'_\beta(\Lambda r) \pm |x|/t], & \partial_r^2 \phi_\Lambda^\pm(x,r) &= \Lambda^2 m''_\beta(\Lambda r), \\ \partial_r \psi_\Lambda^\pm(x,r) &= \Lambda [m'_\beta(\Lambda r) \pm |x_2|/t], & \partial_r^2 \psi_\Lambda^\pm(x,r) &= \Lambda^2 m''_\beta(\Lambda r), \end{aligned}$$

By Lemma 3.2, we have

$$|\partial_r^2 \phi_\Lambda^\pm(x,r)| \sim |\partial_r^2 \psi_\Lambda^\pm(x,r)| \sim \Lambda^3 \langle \sqrt{\beta}\Lambda \rangle \langle \Lambda \rangle^{-\frac{5}{2}} \tag{3.14}$$

for all  $r \in (1/2, 2)$ .

First we estimate  $\mathcal{J}_{\Lambda,t}^+(x)$  and  $\mathcal{J}_{\Lambda,t}^-(x)$ . The same argument works for  $\mathcal{J}_{\Lambda,t}^\pm(x)$ , and we shall comment on this below.

*Estimate for  $\mathcal{J}_{\Lambda,t}^+(x)$ .* By Lemma 3.2, we have

$$|\partial_r \psi_\Lambda^+(x,r)| \gtrsim \Lambda \langle \sqrt{\beta}\Lambda \rangle \langle \Lambda \rangle^{-\frac{1}{2}} \tag{3.15}$$

where to obtain this lower bound we also used the fact that  $m'_\beta$  is positive.

Integration by parts yields

$$\begin{aligned} \mathcal{J}_{\Lambda,t}^+(x) &= -it^{-1} \int_{1/2}^2 \partial_r \left[ e^{it\psi_\Lambda^+(x,r)} \right] [\partial_r \psi_\Lambda^+(x,r)]^{-1} H_\Lambda^+(x,r) \, dr \\ &= it^{-1} \int_{1/2}^2 e^{it\psi_\Lambda^+(x,r)} \left\{ \frac{\partial_r H_\Lambda^+(x,r)}{[\partial_r \psi_\Lambda^+(x,r)]} - \frac{\partial_r^2 \psi_\Lambda^+(x,r) H_\Lambda^+(x,r)}{[\partial_r \psi_\Lambda^+(x,r)]^2} \right\} \, dr. \end{aligned}$$

Then using (3.12), (3.14) and (3.15) we obtain

$$\begin{aligned} |\mathcal{J}_{\Lambda,t}^+(x)| &\lesssim t^{-1} \langle \sqrt{\beta}\Lambda \rangle^{-1} \left\{ \Lambda^{-1} \langle \Lambda \rangle^{\frac{1}{2}} + \Lambda \langle \Lambda \rangle^{-\frac{3}{2}} \right\} (\Lambda|x|)^{-\frac{1}{2}} \\ &\lesssim \Lambda^{-1} \langle \sqrt{\beta}\Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{1}{2}} t^{-1} \\ &\lesssim \Lambda^{-2} \langle \sqrt{\beta}\Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{3}{2}} t^{-1}. \end{aligned}$$

*Estimate for  $\mathcal{J}_{\Lambda,t}^-(x)$ .* We treat the non-stationary case

$$|x_2| \ll \langle \sqrt{\beta}\Lambda \rangle \langle \Lambda \rangle^{-\frac{1}{2}} t \quad \text{or} \quad |x_2| \gg \langle \sqrt{\beta}\Lambda \rangle \langle \Lambda \rangle^{-\frac{1}{2}} t \tag{3.16}$$

and the stationary case

$$|x_2| \sim \langle \sqrt{\beta}\Lambda \rangle \langle \Lambda \rangle^{-\frac{1}{2}} t \tag{3.17}$$

separately.

In the non-stationary case (3.16), we have

$$|\partial_r \psi_\Lambda^-(x,r)| \gtrsim \Lambda \langle \sqrt{\beta}\Lambda \rangle \langle \Lambda \rangle^{-\frac{1}{2}}.$$

Hence  $\mathcal{J}_{\Lambda,t}^-(x)$  can be estimated in exactly the same way as  $\mathcal{J}_{\Lambda,t}^+(x)$ , and satisfy the same bound.

So it remains to treat the stationary case. In this case, we use Lemma 3.1, (3.14), (3.12) and (3.17) to obtain

$$\begin{aligned}
 |\mathcal{J}_{\Lambda,t}^-(x)| &= \left| \int_{1/2}^2 e^{it\psi_{\Lambda}^-(x,r)} H_{\Lambda}^-(x,r) \, dr \right| \\
 &\lesssim t^{-\frac{1}{2}} \Lambda^{-\frac{3}{2}} \langle \sqrt{\beta} \Lambda \rangle^{-\frac{1}{2}} \langle \Lambda \rangle^{\frac{5}{4}} \left[ |H_{\Lambda}^-(x,2)| + \int_{1/2}^2 |\partial_r H_{\Lambda}^-(x,r)| \, dr \right] \\
 &\lesssim t^{-\frac{1}{2}} \Lambda^{-\frac{3}{2}} \langle \sqrt{\beta} \Lambda \rangle^{-\frac{1}{2}} \langle \Lambda \rangle^{\frac{5}{4}} \cdot (\Lambda|x|)^{-\frac{1}{2}} \\
 &\lesssim \Lambda^{-2} \langle \sqrt{\beta} \Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{3}{2}} t^{-1},
 \end{aligned} \tag{3.18}$$

where we also used the fact that  $H_{\Lambda}^-(x,2) = 0$ , and (3.17) which also implies

$$|x| \geq |x_2| \sim \langle \sqrt{\beta} \Lambda \rangle \langle \Lambda \rangle^{-\frac{1}{2}} t.$$

*Estimate for  $\mathcal{J}_{\Lambda,t}^+(x)$ .* By Lemma 3.2 we have

$$|\partial_r \phi_{\Lambda}^+(x,r)| \gtrsim \Lambda \langle \sqrt{\beta} \Lambda \rangle \langle \Lambda \rangle^{-\frac{1}{2}}.$$

Then integrating by parts and using the estimates (3.12) and (3.14) we obtain

$$|\mathcal{J}_{\Lambda,t}^+(x)| \lesssim \Lambda^{-2} \langle \sqrt{\beta} \Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{3}{2}} t^{-1}.$$

*Estimate for  $\mathcal{J}_{\Lambda,t}^-(x)$ .* In the non-stationary case

$$|x| \ll \langle \sqrt{\beta} \Lambda \rangle \langle \Lambda \rangle^{-\frac{1}{2}} t \quad \text{or} \quad |x| \gg \langle \sqrt{\beta} \Lambda \rangle \langle \Lambda \rangle^{-\frac{1}{2}} t$$

we have

$$|\partial_r \phi_{\Lambda}^-(x,r)| \sim \Lambda \langle \sqrt{\beta} \Lambda \rangle \langle \Lambda \rangle^{-\frac{1}{2}}.$$

Hence combining this estimate with (3.12) and (3.14) we see that the integration by parts argument goes through.

In the stationary case,

$$|x| \sim \langle \sqrt{\beta} \Lambda \rangle \langle \Lambda \rangle^{-\frac{1}{2}} t \tag{3.19}$$

we use Lemma 3.1, (3.12) and (3.14), as in (3.18), to obtain the desired estimate.  $\square$

*Proof of Theorem 1.2.* We shall use the Hardy-Littlewood-Sobolev inequality which asserts that

$$\| |\cdot|^{-\alpha} * f \|_{L^a(\mathbb{R})} \lesssim \| f \|_{L^b(\mathbb{R})} \tag{3.20}$$

whenever  $1 < b < a < \infty$  and  $0 < \alpha < 1$  obey the scaling condition

$$\frac{1}{b} = \frac{1}{a} + 1 - \alpha.$$

First note that (1.20) holds true for the pair  $(q, r) = (\infty, 2)$  as this is just the energy inequality. So we may assume  $q \in (2, \infty)$ .

Let  $q'$  and  $r'$  be the conjugates of  $q$  and  $r$ , respectively, i.e.,  $q' = \frac{q}{q-1}$  and  $r' = \frac{r}{r-1}$ . By the standard TT\*-argument, (1.20) is equivalent to the estimate

$$\| \text{TT}^* F \|_{L_t^q L_x^r(\mathbb{R}^{2+1})} \lesssim \left[ \langle \sqrt{\beta} \Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{3}{2}} \right]^{1-\frac{2}{r}} \| F \|_{L_t^{q'} L_x^{r'}(\mathbb{R}^{2+1})}, \tag{3.21}$$

where

$$\begin{aligned}
 \text{TT}^* F(x,t) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{ix \cdot \xi + i(t-s) \text{sgn}(\xi_1) m_{\beta}(\xi)} \rho_{\Lambda}^2(\xi) \widehat{F}(\xi, s) \, ds d\xi \\
 &= \int_{\mathbb{R}} K_{\Lambda,t-s} * F(\cdot, s) \, ds,
 \end{aligned} \tag{3.22}$$

with

$$K_{\Lambda,t}(x) = \int_{\mathbb{R}^d} e^{ix \cdot \xi + it \text{sgn}(\xi_1) m_{\beta}(\xi)} \rho_{\Lambda}^2(\xi) \, d\xi.$$

Observe that

$$K_{\Lambda,t} * g(x) = S_{m_\beta}(t)P_\Lambda g_\Lambda(x).$$

So it follows from (1.15) that

$$\|K_{\Lambda,t} * g\|_{L_x^\infty(\mathbb{R}^2)} \lesssim \langle \sqrt{\beta}\Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{3}{2}} |t|^{-1} \|g\|_{L_x^1(\mathbb{R}^2)}. \tag{3.23}$$

On the other hand, we have by Plancherel

$$\|K_{\Lambda,t} * g\|_{L_x^2(\mathbb{R}^2)} \lesssim \|g\|_{L_x^2(\mathbb{R}^2)}. \tag{3.24}$$

So interpolation between (3.23) and (3.24) yields

$$\|K_{\Lambda,t} * g\|_{L_x^r(\mathbb{R}^2)} \lesssim \left[ \langle \sqrt{\beta}\Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{3}{2}} \right]^{1-\frac{2}{r}} |t|^{-(1-\frac{2}{r})} \|g\|_{L_x^{q'}(\mathbb{R}^2)}. \tag{3.25}$$

for all  $r \in [2, \infty]$ .

Applying Minkowski's inequality to (3.22), and then (3.25) and (3.20) with  $(a, b) = (q, q')$  and  $\alpha = 1 - 2/r$ , we obtain

$$\begin{aligned} \|\mathbb{T}\mathbb{T}^*F\|_{L_t^q L_x^r(\mathbb{R}^{2+1})} &\leq \left\| \int_{\mathbb{R}} \|K_{\Lambda,t-s} * F(s, \cdot)\|_{L_x^r(\mathbb{R}^2)} ds \right\|_{L_t^q(\mathbb{R})} \\ &\lesssim \left[ \langle \sqrt{\beta}\Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{3}{2}} \right]^{1-\frac{2}{r}} \left\| \int_{\mathbb{R}} |t-s|^{-(1-\frac{2}{r})} \|F(s, \cdot)\|_{L_x^{q'}(\mathbb{R}^2)} ds \right\|_{L_t^q(\mathbb{R})} \\ &\lesssim \left[ \langle \sqrt{\beta}\Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{3}{2}} \right]^{1-\frac{2}{r}} \left\| \|F\|_{L_x^{q'}(\mathbb{R}^2)} \right\|_{L_t^{q'}(\mathbb{R})} \\ &= \left[ \langle \sqrt{\beta}\Lambda \rangle^{-1} \langle \Lambda \rangle^{\frac{3}{2}} \right]^{1-\frac{2}{r}} \|F\|_{L_t^{q'} L_x^{q'}(\mathbb{R}^{2+1})}, \end{aligned}$$

which is the desired estimate (3.21). □

### 4. Proof of Theorem 1.4

We recall that  $\epsilon = \beta = 1$  in this section.

#### 4.1. Refined Strichartz Estimate

As in [4, 8, 12, 13, 16–18] the main ingredient in our analysis is a refined Strichartz estimates for solutions of the non-homogeneous linear equation

$$\partial_t w + L_{1,1}(D) \left( 1 + \frac{D_2^2}{D_1^2} \right)^{\frac{1}{2}} \partial_{x_1} w = F. \tag{4.1}$$

**Lemma 4.1.** *Let  $s > \frac{7}{4}$  and  $0 < T$ . Suppose that  $w$  is a solution of the linear problem (4.1). Then,*

$$\|\nabla P_{>1} w\|_{L_T^2 L_x^\infty} \lesssim T^{\frac{1}{2}} \|J^s w\|_{L_T^\infty L_x^2} + \|J^{s-1} F\|_{L_{T,x}^2}. \tag{4.2}$$

*Proof.* To establish the estimate (4.2) we follow the argument in [16].

Let  $w_\Lambda := P_\Lambda w$  and  $F_\Lambda := P_\Lambda F$ . It is enough to prove that for any dyadic number  $\Lambda > 1$  and for any small real number  $\theta > 0$ ,

$$\|\nabla w_\Lambda\|_{L_T^2 L_x^\infty} \lesssim T^{\frac{1}{2}} \Lambda^{\frac{7}{4}+\theta} \|w_\Lambda\|_{L_T^\infty L_x^2} + \Lambda^{\frac{3}{4}+\theta} \|F_\Lambda\|_{L_{T,x}^2}. \tag{4.3}$$

Indeed, then we would have, by choosing  $\theta > 0$  such that  $s > \theta + \frac{7}{4}$  and using Cauchy-Schwarz in  $\Lambda$ ,

$$\|\nabla P_{>1} w\|_{L^2_T L^\infty_x} \leq \sum_{\Lambda > 1} \|\nabla w_\Lambda\|_{L^2_T L^\infty_x} \lesssim T^{\frac{1}{2}} \left( \sum_{\Lambda > 1} \Lambda^{2s} \|w_\Lambda\|_{L^2_T L^2_x} \right)^{\frac{1}{2}} + \left( \sum_{\Lambda > 1} \Lambda^{2(s-1)} \|F_\Lambda\|_{L^2_{T,x}} \right)^{\frac{1}{2}},$$

which implies (4.2).

Now we prove estimate (4.3). To do so, we split the interval  $[0, T]$  into small intervals  $I_j$  of size  $\Lambda^{-1}$ . In other words, we have  $[0, T] = \cup_{j \in J} I_j$ , where  $I_j = [a_j, b_j]$ ,  $|I_j| \sim \Lambda^{-1}$  and  $\#J \sim \Lambda T$ . Observe from Bernstein's inequality that

$$\|\nabla w_\Lambda\|_{L^2_T L^\infty_x} \lesssim \Lambda^{1+\frac{2}{r}} \|w_\Lambda\|_{L^2_T L^r_x},$$

for any  $2 < r < \infty$ . Thus it follows applying Hölder's inequality in time that

$$\|\nabla w_\Lambda\|_{L^2_T L^\infty_x} \lesssim \Lambda^{1+\frac{2}{r}} \left( \sum_j \|w_\Lambda\|_{L^2_{I_j} L^r_x} \right)^{\frac{1}{2}} \lesssim \Lambda^{1+\frac{1}{r}} \left( \sum_j \|w_\Lambda\|_{L^q_{I_j} L^r_x} \right)^{\frac{1}{2}},$$

where  $(q, r)$  is an admissible pair satisfying condition (1.19).

Next, employing the Duhamel formula of (4.1) in each  $I_j$  and recalling the definition of  $S_{m_1}$  in (1.13), we have for  $t \in I_j$ ,

$$w_\Lambda(t) = S_{m_1}(t - a_j) w_\Lambda(a_j) + \int_{a_j}^t S_{m_1}(t - t') F_\Lambda(t') dt',$$

so that  $\|\nabla w_\Lambda\|_{L^2_T L^\infty_x}$  is bounded by

$$\Lambda^{1+\frac{1}{r}} \left( \sum_j \|S_{m_1}(t - a_j) w_\Lambda(a_j)\|_{L^q_{I_j} L^r_x}^2 + \sum_j \left( \int_{I_j} \|S_{m_1}(t - t') F_\Lambda(t')\|_{L^q_{I_j} L^r_x} dt' \right)^2 \right)^{\frac{1}{2}}.$$

Thus it follows from Theorem 1.2 that

$$\begin{aligned} \|\nabla w_\Lambda\|_{L^2_T L^\infty_x} &\lesssim \Lambda^{\frac{5}{4}+\frac{1}{2r}} \left( \sum_j \|w_\Lambda(a_j)\|_{L^2_x}^2 + \sum_j \left( \int_{I_j} \|F_\Lambda(t')\|_{L^2_x} dt' \right)^2 \right)^{\frac{1}{2}} \\ &\lesssim T^{\frac{1}{2}} \Lambda^{\frac{7}{4}+\frac{1}{2r}} \|w_\Lambda\|_{L^\infty_T L^2_x} + \Lambda^{\frac{3}{4}+\frac{1}{2r}} \|F_\Lambda\|_{L^2_{T,x}}, \end{aligned}$$

which implies (4.3) by choosing  $2 < r < \infty$  such that  $\frac{1}{2r} < \theta$ .  $\square$

## 4.2. Energy Estimate

We begin by deriving a classical energy estimate on smooth solutions of (1.2).

**Lemma 4.2.** *Let  $s > 1$  and  $T > 0$ . There exists  $c_{1,s} > 0$  such that for any smooth solution of (1.2), we have*

$$\|\mathbf{u}\|_{L^\infty_T H^s_x}^2 \leq \|\mathbf{u}(0)\|_{H^s}^2 + c_{1,s} \|\nabla \mathbf{u}\|_{L^1_T L^\infty_x} \|\mathbf{u}\|_{L^\infty_T H^s_x}^2. \quad (4.4)$$

The proof relies on the Kato–Ponce commutator estimate (see [11]).

**Lemma 4.3.** *For  $s > 1$ , we denote  $[J^s, f]g = J^s(fg) - fJ^s g$ . Then,*

$$\|[J^s, f]g\|_{L^2} \lesssim \|\nabla f\|_{L^\infty} \|J^{s-1} g\|_{L^2} + \|J^s f\|_{L^2} \|g\|_{L^\infty}. \quad (4.5)$$

*Proof of Lemma 4.2.* Applying  $J^s$  to (1.2), multiplying by  $J^s u$  and integrating in space leads to

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (J^s u)^2 dx = -6 \int_{\mathbb{R}^2} [J^s, u] \partial_{x_1} u J^s u dx - 6 \int_{\mathbb{R}^2} u J^s \partial_{x_1} u J^s u dx.$$

We use the Cauchy-Schwarz inequality and the commutator estimate (4.5) to deal the first term on the right-hand side and integrate by parts in  $x_1$  and use Hölder’s inequality to deal with the second term. This implies that

$$\frac{d}{dt} \|J^s u\|_{L_x^2}^2 \lesssim \|\nabla u\|_{L_x^\infty} \|J^s u\|_{L_x^2}^2.$$

Estimate (4.4) follows then by integrating the former estimate between 0 and  $T$  and applying Hölder’s inequality in time in the nonlinear term.  $\square$

Now we use the refined Strichartz estimate to control the term  $\|\nabla u\|_{L_T^1 L_x^\infty}$ .

**Lemma 4.4.** *Let  $s > \frac{7}{4}$  and  $T > 0$ . Then, there exists  $c_{2,s} > 0$  such that for any solution of (1.2), we have*

$$\|\nabla u\|_{L_T^1 L_x^\infty} \leq c_{2,s} T (1 + \|u\|_{L_T^\infty H_x^s}) \|u\|_{L_T^\infty H_x^s}. \tag{4.6}$$

*Proof.* First, we deduce from the Cauchy-Schwarz inequality in time that

$$\|\nabla u\|_{L_T^1 L_x^\infty} \leq T^{\frac{1}{2}} \|\nabla u\|_{L_T^2 L_x^\infty} \leq T^{\frac{1}{2}} \|P_{\leq 1} \nabla u\|_{L_T^2 L_x^\infty} + T^{\frac{1}{2}} \|P_{>1} \nabla u\|_{L_T^2 L_x^\infty}. \tag{4.7}$$

The first term on the right-hand side of (4.7) is controlled by using Bernstein’s inequality,

$$T^{\frac{1}{2}} \|P_{\leq 1} \nabla u\|_{L_T^2 L_x^\infty} \lesssim T \|u\|_{L_T^\infty L_x^2}. \tag{4.8}$$

To estimate the second term on the right-hand side of (4.7) we use the refined Strichartz estimate (4.2). It follows that

$$T^{\frac{1}{2}} \|P_{>1} \nabla u\|_{L_T^2 L_x^\infty} \lesssim T \|u\|_{L_T^\infty H_x^s} + T^{\frac{1}{2}} \|J^{s-1} \partial_{x_1} (u^2)\|_{L_{T,x}^2} \lesssim T \|u\|_{L_T^\infty H_x^s} + T^{\frac{1}{2}} \|u^2\|_{L_T^2 H_x^s}.$$

Hence, we deduce since  $H^s(\mathbb{R}^2)$  is a Banach algebra for  $s > 1$  and Hölder’s inequality in time that

$$T^{\frac{1}{2}} \|P_{>1} \nabla u\|_{L_T^2 L_x^\infty} \lesssim T \|u\|_{L_T^\infty H_x^s} + T \|u\|_{L_T^\infty H_x^s}^2. \tag{4.9}$$

We conclude the proof of (4.6) gathering (4.7), (4.8) and (4.9).  $\square$

### 4.3. Uniqueness and $L^2$ -Lipschitz Bound of the Flow

Let  $u_1$  and  $u_2$  be two solutions of the equation in (1.2) in the class (1.21) for some positive  $T$  with respective initial data  $u_1(\cdot, 0) = \varphi_1$  and  $u_2(\cdot, 0) = \varphi_2$ . We define the positive number  $K$  by

$$K = \max \{ \|\nabla u_1\|_{L_T^1 L_x^\infty}, \|\nabla u_2\|_{L_T^1 L_x^\infty} \}. \tag{4.10}$$

We set  $v = u_1 - u_2$ . Then  $v$  satisfies

$$\partial_t v + L_{1,1}(D) \left( 1 + \epsilon \frac{D_2^2}{D_1^2} \right)^{\frac{1}{2}} \partial_{x_1} v + 3 \partial_{x_1} ((u_1 + u_2)v) = 0, \tag{4.11}$$

with initial datum  $v(\cdot, 0) = \varphi_1 - \varphi_2$ .

We want to estimate  $v$  in  $L^2(\mathbb{R}^2)$ . We multiply (4.11) by  $v$ , integrate in space and integrate by parts in  $x_1$  to deduce that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} v^2 dx = -3 \int_{\mathbb{R}^2} \partial_{x_1} ((u_1 + u_2)v) v dx = -\frac{3}{2} \int_{\mathbb{R}^2} \partial_{x_1} (u_1 + u_2) v^2 dx.$$

This implies from Hölder’s inequality that

$$\frac{d}{dt} \|v\|_{L_x^2}^2 \lesssim (\|\partial_{x_1} u_1\|_{L_x^\infty} + \|\partial_{x_1} u_2\|_{L_x^\infty}) \|v\|_{L_x^2}^2.$$

Therefore, it follows from Gronwall’s inequality that

$$\sup_{t \in [0, T]} \|v(\cdot, t)\|_{L^2_x} = \sup_{t \in [0, T]} \|\mathbf{u}_1(\cdot, t) - \mathbf{u}_2(\cdot, t)\|_{L^2_x} \leq e^{cK} \|\varphi_1 - \varphi_2\|_{L^2}. \tag{4.12}$$

Estimate (4.12) provides the uniqueness result in Theorem 1.4 by choosing  $\varphi_1 = \varphi_2 = \mathbf{u}_0$ .

#### 4.4. A Priori Estimate

When  $s > 2$ , Theorem 1.4 follows from a standard parabolic regularization method. The argument also yields a blow-up criterion.

**Proposition 4.5.** *Let  $s > 2$ . Then, for any  $\mathbf{u}_0 \in H^s(\mathbb{R}^2)$ , there exist a positive time  $T(\|\mathbf{u}_0\|_{H^s})$  and a unique maximal solution  $\mathbf{u}$  of (1.2) in  $C^0([0, T^*] : H^s(\mathbb{R}^2))$  with  $T^* > T(\|\mathbf{u}_0\|_{H^s})$ . Moreover, if the maximal time of existence  $T^*$  is finite, then*

$$\lim_{t \nearrow T^*} \|\mathbf{u}(t)\|_{H^s} = +\infty$$

and the flow map  $\mathbf{u}_0 \mapsto \mathbf{u}(t)$  is continuous from  $H^s(\mathbb{R}^2)$  to  $H^s(\mathbb{R}^2)$ .

Let  $\mathbf{u}_0 \in H^\infty(\mathbb{R}^2)$ . From the above result, there exists a solution  $\mathbf{u} \in C([0, T^*] : H^\infty(\mathbb{R}^2))$  to (1.2), where  $T^*$  is the maximal time of existence of  $\mathbf{u}$  satisfying  $T^* \geq T(\|\mathbf{u}_0\|_{H^s})$  and we have the blow-up alternative

$$\lim_{t \nearrow T^*} \|\mathbf{u}(t)\|_{H^3_x} = +\infty \quad \text{if } T^* < +\infty. \tag{4.13}$$

Then, by using a bootstrap argument, we prove that the solution satisfies a suitable *a priori* estimate on positive time interval depending only on the  $H^s$  norm the initial datum.

**Lemma 4.6.** *Let  $\frac{7}{4} < s \leq 2$ . There exist  $K_s > 0$  and  $A_s > 0$  such that  $T^* > (A_s \|\mathbf{u}_0\|_{H^s} + 1)^{-2}$ ,*

$$\|\mathbf{u}\|_{L^\infty_T H^s} \leq 2\|\mathbf{u}_0\|_{H^s_x} \quad \text{and} \quad \|\nabla \mathbf{u}\|_{L^1_T L^\infty_x} \leq K_s \quad \text{with } T = (A_s \|\mathbf{u}_0\|_{H^s} + 1)^{-2}. \tag{4.14}$$

*Proof.* For  $\frac{7}{4} < s \leq 2$ , let us define

$$T_0 := \sup \left\{ T \in (0, T^*) : \|\mathbf{u}\|_{L^\infty_T H^s_x} \leq 2\|\mathbf{u}_0\|_{H^s_x} \right\}.$$

Note that the above set is nonempty since  $\mathbf{u} \in C([0, T^*] : H^\infty(\mathbb{R}^2))$ , so that  $T_0$  is well-defined. We argue by contradiction assuming that  $0 < T_0 < (A_s \|\mathbf{u}_0\|_{H^s} + 1)^{-2} \leq 1$  for  $A_s = 8(1 + c_{1,s} + c_{1,3})(1 + c_{2,s})$  (where  $c_{1,s}$  and  $c_{2,s}$  are respectively defined in Lemmas 4.2 and 4.4).

Let  $0 < T_1 < T_0$ . We have from the definition of  $T_0$  that  $\|\mathbf{u}\|_{L^\infty_{T_1} H^s} \leq 4\|\mathbf{u}_0\|_{H^s}^2$ . Then estimate (4.6) yields

$$\|\nabla \mathbf{u}\|_{L^1_{T_1} L^\infty_x} \leq 2c_{2,s} T_1 (1 + 2\|\mathbf{u}_0\|_{H^s}) \|\mathbf{u}_0\|_{H^s} \leq \frac{1}{4(1 + c_{1,s} + c_{1,3})}.$$

Thus, we deduce by using the energy estimate (4.4) with  $s = 3$  that

$$\|\mathbf{u}\|_{L^\infty_{T_1} H^3_x}^2 \leq \frac{4}{3} \|\mathbf{u}_0\|_{H^3}^2, \quad \forall 0 < T_1 < T_0.$$

This implies in view of the blow-up alternative (4.13) that  $T_0 < T^*$ .

Now, the energy estimate (4.4) at the level  $s$  yields  $\|\mathbf{u}\|_{L^\infty_{T_0} H^s_x} \leq \frac{4}{3} \|\mathbf{u}_0\|_{H^s}^2$ , so that by continuity,  $\|\mathbf{u}\|_{L^\infty_{T_2} H^s} \leq \frac{5}{3} \|\mathbf{u}_0\|_{H^s}^2$  for some  $T_0 < T_2 < T^*$ . This contradicts the definition of  $T_0$ .

Therefore  $T_0 \geq T := (A_s \|\mathbf{u}_0\|_{H^s} + 1)^{-2}$  and we argue as above to get the bound for  $\|\nabla \mathbf{u}\|_{L^1_T L^\infty_x}$ . This concludes the proof of Lemma 4.6.  $\square$

### 4.5. Existence, Persistence and Continuous Dependence

Let fix  $\frac{7}{4} < s \leq 2$  and  $\mathbf{u}_0 \in H^s(\mathbb{R}^2)$ . We regularize the initial datum as follows. Let  $\chi$  be the cut-off function defined in (1.23), then we define

$$\mathbf{u}_{0,n} = P_{\leq n} \mathbf{u}_0 = (\chi(|\xi|/n) \widehat{\mathbf{u}}_0(\xi))^\vee,$$

for any  $n \in \mathbb{N}$ ,  $n \geq 1$ .

Then, the following estimates are well-known (see for example Lemma 5.4 in [18]).

**Lemma 4.7.** (i) Let  $\sigma \geq 0$  and  $n \geq 1$ . Then,

$$\|\mathbf{u}_{0,n}\|_{H^{s+\sigma}} \lesssim n^\sigma \|\mathbf{u}_0\|_{H^s}, \tag{4.15}$$

(ii) Let  $0 \leq \sigma \leq s$  and  $m \geq n \geq 1$ . Then,

$$\|\mathbf{u}_{0,n} - \mathbf{u}_{0,m}\|_{H^{s-\sigma}} \underset{n \rightarrow +\infty}{=} o(n^{-\sigma}) \tag{4.16}$$

Now, for each  $n \in \mathbb{N}$ ,  $n \geq 1$ , we consider the solution  $\mathbf{u}_n$  emanating from  $\mathbf{u}_{0,n}$  defined on their maximal time interval  $[0, T_n^*)$ . In other words,  $\mathbf{u}_n$  is a solution to the Cauchy problem

$$\begin{cases} \partial_t \mathbf{u}_n + L_{1,1}(D) \left(1 + \frac{D_2^2}{D_1^2}\right)^{\frac{1}{2}} \partial_{x_1} \mathbf{u}_n + 3\partial_{x_1}(\mathbf{u}_n^2) = 0, & x \in \mathbb{R}^2, 0 < t < T_n^*, \\ \mathbf{u}_n(x, 0) = \mathbf{u}_{0,n}(x) = P_{\leq n} \mathbf{u}_0(x), & x \in \mathbb{R}^2. \end{cases} \tag{4.17}$$

From Lemmas 4.6 and 4.7 (i), there exists a positive time

$$T = (A_s \|\mathbf{u}_0\|_{H^s} + 1)^{-2}, \tag{4.18}$$

(where  $A_s$  is a positive constant), independent of  $n$ , such that  $\mathbf{u}_n \in C([0, T] : H^\infty(\mathbb{R}^2))$  is defined on the time interval  $[0, T]$  and satisfies

$$\|\mathbf{u}_n\|_{L_T^\infty H_x^s} \leq 2\|\mathbf{u}_0\|_{H^s} \tag{4.19}$$

and

$$K := \sup_{n \geq 1} \{\|\nabla \mathbf{u}_n\|_{L_T^1 L_x^\infty}\} < +\infty. \tag{4.20}$$

Let  $m \geq n \geq 1$ . We set  $v_{n,m} := \mathbf{u}_n - \mathbf{u}_m$ . Then,  $v_{n,m}$  satisfies

$$\partial_t v_{n,m} + L_{1,1}(D) \left(1 + \frac{D_2^2}{D_1^2}\right)^{\frac{1}{2}} \partial_{x_1} v_{n,m} + 3\partial_{x_1}((\mathbf{u}_n + \mathbf{u}_m)v_{n,m}) = 0, \tag{4.21}$$

with initial datum  $v_{n,m}(\cdot, 0) = \mathbf{u}_{0,n} - \mathbf{u}_{0,m}$ .

Arguing as in Sect. 4.3, we see from Gronwall's inequality and (4.16) with  $\sigma = s$  that

$$\|v_{n,m}\|_{L_T^\infty L_x^2} \leq e^{cK} \|\mathbf{u}_{0,n} - \mathbf{u}_{0,m}\|_{L^2} \underset{n \rightarrow +\infty}{=} o(n^{-s}) \tag{4.22}$$

which implies interpolating with (4.19) that

$$\|v_{n,m}\|_{L_T^\infty H_x^\sigma} \leq \|v_{n,m}\|_{L_T^\infty H_x^s}^{\frac{\sigma}{s}} \|v_{n,m}\|_{L_T^\infty L_x^2}^{1-\frac{\sigma}{s}} \underset{n \rightarrow +\infty}{=} o(n^{-(s-\sigma)}), \tag{4.23}$$

for all  $0 \leq \sigma < s$ .

Therefore, we deduce that  $\{\mathbf{u}_n\}$  is a Cauchy sequence in  $L^\infty([0, T] : H^\sigma(\mathbb{R}^2))$ , for any  $0 \leq \sigma < s$ . Hence, it is not difficult to verify passing to the limit as  $n \rightarrow +\infty$  that  $\mathbf{u} = \lim_{n \rightarrow +\infty} \mathbf{u}_n$  is a weak solution to (1.2) in the class  $C([0, T] : H^\sigma(\mathbb{R}^2))$ , for any  $0 \leq \sigma < s$ .

Finally, the proof that  $\mathbf{u}$  belongs to the class (1.21) and of the continuous dependence of the flow follows from the Bona-Smith argument [3]. Since it is a classical argument, we skip the proof and refer the readers to [8, 18] for more details in this setting.

### 5. Appendix

This appendix is devoted to the proof of Lemma 3.2.

*Proof of Lemma 3.2.* Let

$$T(r) = \tanh(r), \quad S(r) = \operatorname{sech}(r), \quad K(r) = \sqrt{T(r)/r}.$$

Then

$$m_\beta(r) = r\langle\sqrt{\beta r}\rangle K(r).$$

First we prove (3.2). Since

$$K' = \frac{rS^2 - T}{2r^2K} = \frac{1}{2r} (K^{-1}S^2 - K)$$

we have

$$\begin{aligned} m'_\beta &= \langle\sqrt{\beta r}\rangle(K + rK') + \beta r\langle\sqrt{\beta r}\rangle^{-1}(rK) \\ &= \frac{1}{2}\langle\sqrt{\beta r}\rangle (K + K^{-1}S^2) + \beta r^2\langle\sqrt{\beta r}\rangle^{-1}K \\ &\sim \langle\sqrt{\beta r}\rangle r^{-1/2}, \end{aligned}$$

where in the last line we used the fact that

$$K(r) \sim \langle r \rangle^{-1/2} \quad \text{and} \quad S(r) \sim e^{-r}. \tag{5.1}$$

Next we prove (3.3). We have

$$m''_\beta = (2K' + rK'') \langle\sqrt{\beta r}\rangle + 2(K + rK') \langle\sqrt{\beta r}\rangle' + r\langle\sqrt{\beta r}\rangle''K.$$

Now we can write

$$K'' = -\frac{TS^2}{rK} - \frac{(rS^2 - T)}{r^3K} - \frac{(rS^2 - T)^2}{4r^4K^3}$$

which in turn implies

$$\begin{aligned} 2K' + rK'' &= -\frac{TS^2}{K} - \frac{(rS^2 - T)^2}{4r^3K^3} \\ &= -\frac{rK^{-3}}{4} \left[ 4K^4S^2 + ((K^2 - S^2)/r)^2 \right]. \end{aligned}$$

We write

$$(K^2 - S^2)/r = ES^2,$$

where

$$E(r) = \frac{e^{2r} - e^{-2r} - 4r}{4r^2}.$$

We estimate  $E(r)$  as follows: If  $0 < r < 1$  we write

$$E(r) = \frac{1}{2r^2} \int_0^r (e^{2s} + e^{-2s} - 2) \, ds = \frac{2}{r^2} \int_0^r \int_0^s \int_0^t (e^{2x} + e^{-2x}) \, dx \, dt \, ds.$$

Then since  $e^{2x} + e^{-2x} \sim 1$  for  $0 < x < r < 1$ , we have

$$E(r) \sim \frac{2}{r^2} \int_0^r \int_0^s \int_0^t 1 \, dx \, dt \, ds \sim r.$$

On the other hand, if  $r \geq 1$ , we simply have

$$E(r) = \frac{e^{2r}}{4r^2} [1 - e^{-4r} - 4re^{-2r}] \sim \langle r \rangle^{-2} e^{2r}.$$

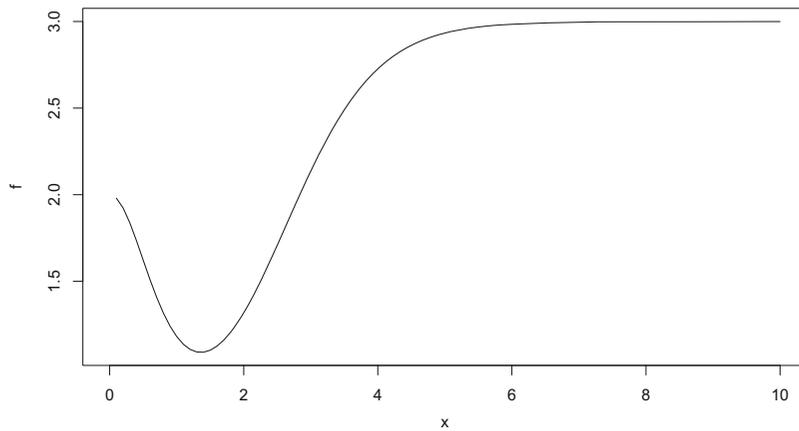


FIG. 3. The graph of  $f_1(x) = 4A(x)/B(x) - 1$  for  $x \geq 0$ . It satisfies  $1 < f(x) \leq 3$

Therefore,

$$E(r) \sim r\langle r \rangle^{-3} e^{2r} \quad \text{for all } r > 0. \tag{5.2}$$

Now letting

$$A_\beta(r) = \langle \sqrt{\beta}r \rangle^{-2} \left[ 1 + K^{-2}S^2 + \langle \sqrt{\beta}r \rangle^{-2} \right],$$

$$B(r) = 4S^2 + K^{-4}E^2S^4, \quad f_\beta(r) = 4\beta \frac{A_\beta(r)}{B(r)} - 1$$

we can write

$$m_\beta''(r) = 4^{-1}r\langle \sqrt{\beta}r \rangle KB(r)f_\beta(r).$$

By (5.1) and (5.2) we have

$$B(r) \sim e^{-2r} + \langle r \rangle^2 \cdot r^2 \langle r \rangle^{-6} e^{4r} \cdot e^{-4r} \sim \langle r \rangle^{-2},$$

and hence

$$|m_\beta''(r)| \sim r\langle \sqrt{\beta}r \rangle \langle r \rangle^{-\frac{3}{2}} |f_\beta(r)|.$$

So (3.3) reduces to proving for all  $r \geq 0$

$$|f_\beta(r)| \sim 1 \quad \text{for } \beta \in \{0, 1\}. \tag{5.3}$$

Clearly,  $|f_0(r)| = 1$ . On the other hand, it can be easily checked that  $|f_1(r)| \sim 1$  for  $r \geq 0$  (see Fig. 3 below). □

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**Compliance with ethical standards**

**Conflicts of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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