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The direct monodromy problem and isomonodromic deformations for the Rabi model.

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Abstract

We discuss the local and global solutions of the Rabi model in Garnier form, a linear system of first order differential equations, with complex rational coefficients. The analytic continuation of the local solutions are described by a monodromy group, which gives a matrix representation of the fundamental group of the punctured Riemann sphere. A detailed geometric description of linear systems of first order differential equations is given, in terms of a local family of connection forms on a principal bundle. The geometric description reveals the Frobenius integrability conditions, which are used to obtain necessary and sufficient conditions for an isomonodromic deformation of the Rabi model.

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Introduction

The Rabi model [Rab36], describes a light-atom interaction in quantum physics, where the frequency of the light is very close to the natural frequency of the atom. It is a relatively simple, but analytically solvable model. The Rabi model has recently attracted attention due to experimental and mathematical reasons [Bra11], with applications in quantum optics [Ved05] and quantum computing [Pel et al.95]. The Rabi model is described by a certain Hamiltonian function. The eigenvalue problem for this Hamiltonian can be reformulated as a linear system of first order differential equations, that are dealt with in this thesis (see [CAQ15]), the so-called Rabi model in Garnier form [Iwa et al.91]. In the thesis we focus on studying the integrability properties of the Rabi model, by applying the isomonodromic approach.

Hilbert's twenty-first problem in his celebrated list put forth in 1900 (1902 in English) [Hil02], can be roughly stated as to show that there always exists a linear second order differential equation of the Fuchsian class (see Definition 1.2.2), with given singular points and monodromy group (see Section 1.4.4). This problem was for a long time thought to be solved by Plemelj in 1908 [Ple64], however as late as in 1990, a paper given by Bolibrukh [Bol90] not only proclaimed an error in Plemelj's proof, but also gave a counterexample to the problem stated by Hilbert. A more general converse problem is the *direct monodromy problem*. Its objective is to find a monodromy group, given a linear second order differential equation with both Fuchsian and non-Fuchsian singular points. Further, if such a monodromy group is found, the *isomonodromic problem* concerns with finding a family of linear second order differential equations, all sharing the same monodromy group and singular points.

The direct monodromy problem is a construction problem, and the challenges mostly lie in dealing with the Stokes phenomenon at non-Fuchsian singular points, and computation of analytic continuation.

The isomonodromic problem have been studied since the early 20th century. The Fuchsian case, when the singularities of the differential equation are only simple poles, has as integrability conditions the classical Schlesinger equations [Sch12]. The necessary and sufficient condition for the solution of an isomonodromic problem in the Fuchsian case, was formulated first in [Sch12], and are called the Schlesinger equation for the integrability condition. Later, the term "Schlesinger equations" was adapted to any isomonodromic problem, and we follow this convention in the thesis.

The largest milestone in the modern development of isomonodromic deformations, is due to Jimbo, Miwa and Ueno in [JMU81], [JM81a], [JM81b]. In these three influential papers they show necessary and sufficient conditions for isomonodromic deformation in the case of poles of an arbitrary order.

All the above problems can be dealt with by using second order linear scalar equations, or first order linear systems of two equations. We will be considering the latter.

The Schlesinger integrability conditions for an isomonodromic problem, can be rewritten in the form of non-linear second order scalar differential equations, in certain cases this non-linear equation is one of the so called Painlevé equations, see [Fok et al.06]. There are six Painlevé equations, and their solutions can be regarded as a generalization of classical special functions, usually called the "Painlevé transcendents". They are an important tool for studying the isomonodromic problem, see [Con et al.99].

This thesis will regard the direct monodromy problem, and isomonodromy problem, posed for the Rabi model in Garnier form, in the paper [CAQ15]:

$$\frac{d\Phi}{dz} \cdot \Phi(z)^{-1} = \frac{\sigma_3}{2} + \frac{A_0}{z} + \frac{A_t}{z-t} = \mathcal{A}(z), \quad \mathcal{A} : \mathbb{S} \setminus \{ 0, t, \infty \} \to M_2(\mathbb{C}), \tag{1}$$

where σ_3 is the third Pauli matrix, A_0 and A_t are constant matrices, and the function $\mathcal{A}$ has poles at $z = 0$, $z = t$ and $z = \infty$. The scheme is to solve the direct monodromy problem for equation (1) with fixed t, and then solve the isomonodromy problem by imposing conditions on (1) such that the obtained monodromy group stay fixed, while varying the parameter $t \in \mathbb{S} \setminus \{ 0, \infty \}$.

In Chapter 1 we start with constructing local solutions of (1), in particular, substantial effort is made to describe the Stokes phenomenon around the non-Fuchsian point at $z = \infty$, see Section 1.3. The universal covering space of the domain $\mathbb{S} \setminus \{ 0, t, \infty \}$, is constructed, and the local solutions are analytically continued into a global solution on this universal cover. We then give an introduction to monodromy theory, and derive expressions for the canonical monodromy group (see Definition 1.4.7), by using the constructed global solution.

Chapter 2 gives a geometric description of first order linear system of differential equations. First we give an introduction to principal bundles with a connection, then we explain how a first order linear system fits into the theory. We then give an explicit construction of such a principal bundle, and relate the meaning of a solution to the differential equation, with a horizontal section of the principal bundle. Finally we give existence and uniqueness results through Frobenius integrability, and show how this infers solutions of the differential equation. The geometric language allows us to make a bridge between the integrability condition for the isomonodromic problem for (1), and the classical Frobenius Theorem from differential geometry.

Chapter 3 deals with the isomonodromic problem related to (1). After an introduction giving the relevant Definitions, we give motivation to why one might expect isomonodromic deformations for (1). We then follow [JMU81], and derive necessary conditions for isomonodromic deformations. Finally we were able to give the exact equations governing the isomonodromic flow, from different perspectives.

To facilitate the reading of the thesis, the Appendix gives a summary of useful facts and constructions on complex holomorphic manifolds. The proofs of the statements are collected from known sources, and is often a combination of several results in order to adapt the statement to our needs. It is included to make the thesis self sufficient, but is not to be regarded as an achievement of the thesis. In particular, we show how an n-dimensional complex holomorphic manifold, has a complex n-dimensional vector space as tangent space, called the holomorphic tangent space, see Definition A.1.6. The Appendix also contains a detailed description of analytic continuation on a Riemann surface and the construction of the universal covering space of a Riemann surface, together with an inherited manifold structure.

Table 1: Comparison of terminology in mathematical and physical gauge theory. The table has been provided through the courtesy of [Wik22]. We do not touch on all of these subjects in the thesis.

Mathematics	**Physics**
Principal bundle	Instanton sector or charge sector
Structure group	Gauge group or local gauge group
Gauge group	Group of global gauge transformations or global gauge group
Gauge transformation	Gauge transformation or gauge symmetry
Change of local trivialisation	Local gauge transformation
Local trivialisation	Gauge
Choice of local trivialisation	Fixing a gauge
Functional defined on the space of connections	Lagrangian of gauge theory
Object does not change under the effects of a gauge transformation	Gauge invariance
Gauge transformations that are covariantly constant with respect to the connection	Global gauge symmetry
Gauge transformations which are not covariantly constant with respect to the connection	Local gauge symmetry
Connection	Gauge field or gauge potential
Curvature	Gauge field strength or field strength
Induced connection/covariant derivative on associated bundle	Minimal coupling
Section of associated vector bundle	Matter field
Term in Lagrangian functional involving multiple different quantities (e.g. the covariant derivative applied to a section of an associated bundle, or a multiplication of two terms)	Interaction
Section of real or complex (usually trivial) line bundle	(Real or complex) Scalar field

Chapter 1

The direct monodromy problem

1.1 Context for solving the first order linear system

Definition 1.1.1 Linear system of differential equations on Riemann surface.
Let M be a Riemann surface and let $U \subset M$ an open set. A linear system of 2 *differential equations on U is an equation*

$$\frac{d\Phi}{dz} \cdot \Phi^{-1} = \mathcal{A}, \quad \textit{where } \mathcal{A} : U \to M_2(\mathbb{C}) \tag{1.1}$$

is given, and where we intend to find a function $\Phi : U \to GL_2(\mathbb{C})$ that satisfy the equation.

Definition 1.1.2 Fundamental solution of linear system.
Let M be a Riemann surface and let $U \subset M$ be an open connected set. If a function

$$\Phi : U \to GL_2(\mathbb{C}) \subset M_2(\mathbb{C})$$

satisfies the differential equation (1.1) *in U, then this solution will be called a fundamental solution of the equation in U.*

Given a fundamental solution of equation (1.1) in an open set $U \subset M$, any other solution of equation (1.1) in U is equal to Φ up to right multiplication by a constant invertible matrix C, where their domains coincide (see Lemma 1.2.2). Thus if the fundamental solution has a prescribed value at a point $z_0 \in U$, it is the unique solution in U which satisfy the differential equation, and has this initial value.

We will now foreshadow the geometric description in Chapter 2, and compare it with the analytic viewpoint. The differential equation gives rise to the construction of a trivial principal bundle:

$$Q\left((\mathbb{S} \setminus \{\, z_j \,\}_{j=1}^{m}, GL_2(\mathbb{C}), \pi\right) \simeq \mathbb{S} \setminus \{\, z_j \,\}_{j=1}^{m} \times GL_2(\mathbb{C}),$$

see Definition 2.1.1 and Corollary 2.3.1. Each point $z \in \mathbb{S} \setminus \{\, z_j \,\}_{j=1}^{m}$, is the projection of a fiber $\pi^{-1}(z) = \{\, [\alpha, z, b] \mid b \in GL_2(\mathbb{C}) \,\}$, in Q, where the fiber is isomorphic to $GL_2(\mathbb{C})$. And to each point $p \in Q$, there exists an unique integrable submanifold $S \subset Q$, with horizontal tangent space by Theorem 2.4.1. By Theorem 2.4.2, this is means that we have local existence of fundamental solutions to (1.1), unique up to an initial condition. The right multiplication of Φ by constant invertible matrices, is linked to the right action of $GL_2(\mathbb{C})$ on Q. The analogous description on the principal bundle, is that given a horizontal section $\tilde{\Phi} : U \subset \mathbb{S} \setminus \{\, z_j \,\}_{j=1}^{m} \to \pi^{-1}(U)$, (which is equivalent to a local fundamental solution Φ_α in U, by Theorem 2.3.1), the right action of $GL_2(\mathbb{C})$ on Q, induces a right action on sections:

$$\tilde{\Phi}(z) \,.\, C = [\alpha, z, \Phi_\alpha(z) \cdot C] \,.$$

Moreover, if $q = p\,.\,C$, where $q, p \in \pi^{-1}(z)$ and $C \in GL_2(\mathbb{C})$, then the right action of $GL_2(\mathbb{C})$, moves the unique submanifold through p to the unique submanifold through q, that is, moves the unique local fundamental solution Φ_α, with $[\alpha, z, \Phi_\alpha(z)] = p$ into the unique local fundamental solution $\Phi_\alpha \cdot C$, with $[\alpha, z, \Phi_\alpha(z) \cdot C] = q$.

The differential equation we are considering in this thesis is the so-called Rabi model in the standard Garnier form:

$$\frac{d\Phi}{dz} \cdot \Phi(z)^{-1} = \frac{\sigma_3}{2} + \frac{A_0}{z} + \frac{A_t}{z-t} = \mathcal{A}(z), \tag{1.2}$$

where

- $\mathcal{A} : \mathbb{S} \setminus \{\, 0, t, \infty \,\} \to M_2(\mathbb{C})$,
- $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the famous third Pauli matrix,
- the matrices A_0 and A_t are constant in z, and are diagonalizable:

  $$A_0 = P^{(0)} \Lambda_0^{(0)} \big(P^{(0)}\big)^{-1}, \qquad A_t = P^{(1)} \Lambda_0^{(1)} \big(P^{(1)}\big)^{-1}$$

  with eigenvalues $(\Lambda_0^{(j)})_{11}$, $(\Lambda_0^{(j)})_{22}$ such that $(\Lambda_0^{(j)})_{11} - (\Lambda_0^{(j)})_{22} \notin \mathbb{Z} \setminus \{\, 0 \,\}$, for $j = 0, 1$.

It is clear that this is a first order linear system, with rational coefficient functions. The primary goal is to describe a fundamental solution of (1.2)

$$\Phi : \quad \mathbb{S} \setminus \{\, 0, t, \infty \,\} \quad \to \quad GL_2(\mathbb{C}),$$

that is defined at every point of $\mathbb{S} \setminus \{\, 0, t, \infty \,\}$. This is not possible, as Φ will in general be a multivalued function. To work around this problem, we will in Chapter 1 first spend considerable effort in solving the system locally. In particular the solutions in the punctured neighbourhoods around the singularities $0, t, \infty$ requires us to be extra careful.

1.2 Local solutions of Rabi-model in Garnier form

1.2.1 Classifying the singular points

We consider the Rabi model in the standard Garnier form from equation (1.2). By following the general approach introduced in [Fok et al.06], we can find local solutions of such equations. The interesting and important information is the behaviour of the system around the singular points $\{\, 0, t, \infty \,\}$ of (1.2).

Definition 1.2.1 Singularities of a function with values in $\mathbf{M_2(\mathbb{C})}$.
Let M be a Riemann surface and $U \subset M$ an open subset of M. Let $f : U \setminus \{\, z_0 \,\} \to M_2(\mathbb{C})$ be holomorphic on $U \setminus \{\, z_0 \,\} \subset M$. Then

- *the point z_0 is a removable singularity of f if there exists a holomorphic function $g : U \to M_2(\mathbb{C})$, such that $g(z) = f(z)$ for all $z \in U \setminus \{\, z_0 \,\}$, i.e. g is a continuous extension of f.*
- *the point z_0 is a pole of f if there exists a holomorphic function $g : U \to M_2(\mathbb{C})$, such that $g(z_0) \neq 0$ and*

  $$g(z) = (z - z_0)^n f(z)$$

  for all $z \in U \setminus \{\, z_0 \,\}$, for some $n \in \mathbb{N}_1$. If such a function exist, then the smallest n such that the condition holds is called the order of the pole at z_0.

- *the point z_0 is an essensial singularity of f, if it is not a removable singularity or a pole of f.*

The above Definition gives names to singularities for a function. When the function $\mathcal{A}$ defines a differential equation like equation 1.1, we will give the singularities related to the differential equation different names.

Definition 1.2.2 Singularities of differential equation.
Let M be a Riemann surface and $U \subset M$ an open subset of M. Consider the linear first order differential equation (1.1)

$$\frac{d\Phi}{dz} \cdot \Phi^{-1} = \mathcal{A}.$$

Let $\mathcal{A} : U \setminus \{ z_0 \} \to M_2(\mathbb{C})$ be holomorphic on $U \setminus \{ z_0 \}$, then

- *if $\mathcal{A}$ has a removable singularity at z_0, we will say that z_0 is a regular point of the system, and that the system is regular at z_0. The system will also be called regular at any point in $U \setminus \{ z_0 \}$.*

- *if $\mathcal{A}$ has a pole at z_0 of order 1, we will say that z_0 is a Fuchsian singular point at z_0.*

- *if $\mathcal{A}$ has a pole at z_0 of order $n > 1$, we will say that z_0 is a non-Fuchsian singular point at z_0.*

- *If $\mathcal{A}$ has a pole of order n at z_0, the number $r = (n-1)$ is called the Poincaré rank of the singularity of $\mathcal{A}$ at z_0. Hence if $\mathcal{A}$ has a pole of order 2 at a point z_0, then the corresponding system $\mathcal{A}$ has a non-Fuchsian singularity at z_0 of Poincaré rank 1.*

We start by classifying the singularities of (1.2). Evidently we have a Fuchsian point at $z_0 = 0$ and another Fuchsian point at $z_1 = t$. We look for singularities at $z = \infty$. We introduce the chart ϕ_∞ on $\mathbb{S} \setminus \{ 0, t, \infty \}$:

$$\begin{array}{cccc} \phi_\infty : & \mathbb{S} \setminus \{ 0, t, \infty \} & \to & \mathbb{C} \setminus \left\{ 0, \frac{1}{t} \right\} \\ & z & \mapsto & \frac{1}{z} = \xi \end{array} \tag{1.3}$$

and substitute into equation (1.2).

$$\frac{d\Phi}{dz} \Phi \left(\frac{1}{z}\right)^{-1} = \frac{d\Phi}{d\xi}\frac{d\xi}{dz} \Phi \left(\frac{1}{z}\right)^{-1} = -\xi^2 \frac{d\Phi}{d\xi} \Phi\left(\xi\right)^{-1} = \frac{\sigma_3}{2} + A_0 \xi + \frac{A_t}{\frac{1}{\xi} - t}$$

$$\implies \frac{d\Phi}{d\xi} \Phi(\xi)^{-1} = -\frac{\sigma_3}{2\xi^2} - \frac{A_0}{\xi} - \frac{A_t}{\xi(1 - t\xi)}$$

We do a partial fraction decomposition on the rightmost term and thus for $|z| > t$, equation (1.2) under the transformation (1.3) takes the form

$$\frac{d\Phi}{d\xi} \Phi(\xi)^{-1} = -\frac{\sigma_3}{2\xi^2} - \frac{A_0 + A_t}{\xi} - \sum_{k=0}^{\infty} A_t t^{k+1} \xi^k \tag{1.4}$$

It is now clear that the system (1.2) has a non-Fuchsian singular point at $z_2 = \infty$ of Poincaré rank $r = 1$.

1.2.2 Transformations of the differential equation

If we precompose $\mathcal{A} : \mathbb{S} \setminus \{ 0, t, \infty \} \to M_2(\mathbb{C})$ with a Möbius transformation, the differential equation still contains exactly the same information, but in a new coordinate. We will regard Möbius transformations as "allowable" transformations of the differential equation (1.1). However, the transformation through a Möbius transformation, should be distinguished contextually from using a chart on $\mathbb{S} \setminus \{ 0, t, \infty \}$, regarding it as a manifold, like in (1.3).

We classified two Fuchsian points and one non-Fuchsian point of Poincaré rank 1, of $\mathcal{A}$. We note that by using a Möbius transformation of the Riemann sphere, the three points $\{ 0, t, \infty \}$ can be moved to arbitrary points on $\mathbb{S}$ by a conformal map.

As a motivation for the next Definition, we will again foreshadow the geometric description in Chapter 2. In particular, we have that the function $\mathcal{A}$, from the differential equation (1.2), is, up to the sign, the coefficient function of a Lie algebra valued 1-form A_α on $\mathbb{S} \setminus \{ 0, t, \infty \}$. By Definition 2.3.2, this Lie algebra valued 1-form, gives rise to a family of local connection forms

$$\{A_\beta : U_\beta \to T^*U_\beta \otimes \mathfrak{gl}_2(\mathbb{C})\}_{\beta \in J},$$

on $\mathbb{S} \setminus \{ 0, t, \infty \}$. The members A_β of a family of local connection forms are related by

$$A_\beta = \mathrm{Ad}(g_{\beta\alpha}) \circ A_\alpha + \left(g_{\beta\alpha}^{-1}\right)^* \theta,$$

where $g_{\beta\alpha} = f_\beta$, is a transition function on the principal bundle. The set of transition functions with right hand side index α, is defined to be the indexed set

$$\{ f_\beta : U_\beta \to GL_2(\mathbb{C}) \}_{\beta \in J}$$

of every $GL_2(\mathbb{C})$ valued, holomorphic functions locally defined on M. Written in matrix notation, and recalling that $A_\alpha = -\mathcal{A}\, dz$, we write out the expression using the coefficient functions and obtain by Proposition 2.2.2:

$$A_\beta \left(\frac{d}{dz}\right) = -\mathcal{B} = g_{\beta\alpha} \cdot \big(- \mathcal{A}\big) \cdot g_{\beta\alpha}^{-1} - \frac{dg_{\beta\alpha}}{dz} \cdot g_{\beta\alpha}^{-1}.$$

Definition 1.2.3 Gauge equivalent systems of differential equations. *Let M be a Riemann surface and consider two linear systems of differential equations defined on an open subset $U \subset M$:*

$$\frac{d\Phi}{dz}\Phi^{-1} = \mathcal{A} : U \to GL_2(\mathbb{C}), \quad \frac{d\Psi}{dz}\Psi(z)^{-1} = \mathcal{B} : U \to GL_2(\mathbb{C}).$$

The two systems are called gauge equivalent on U if there exists a holomorphic function $g : U \to GL_2(\mathbb{C})$ such that

$$\mathcal{B} = g \cdot \mathcal{A} \cdot g^{-1} + \frac{dg}{dz} \cdot g^{-1}, \quad \textit{on } U.$$

In terms of the solutions of the differential equations: $\Psi = g \cdot \Phi$ on U.

Thus solving the differential equation in equation (1.2), is the same as solving a gauge equivalent system. As long as you know the transition function g, you change between the equations and thus also between the solutions.

We will show that any solution to an equation of the form (1.1) have to satisfy the famous Liouville formula.

Lemma 1.2.1 [Tes12] Liouville formula.
Let M be a Riemann surface and $U \subset M$ a open subset of M. Consider a differential equation

$$\frac{d\Phi}{dz}\Phi^{-1} = \mathcal{A}, \qquad \mathcal{A} : U \to M_2(\mathbb{C}).$$

Then any solution $\Phi : U \to GL_2(\mathbb{C})$ to this differential equation satisfies

$$\left(\frac{d}{dz}\det(\Phi)\right)\frac{1}{\det(\Phi)} = \operatorname{trace}(\mathcal{A}).$$

In particular

$$\det(\Phi) = \textit{constant} \iff \operatorname{trace}(\mathcal{A}) = 0.$$

We can use the Liouville Lemma to impose a traceless property up to gauge equivalence on the differential equation. This will be useful in Chapter 3.

Proposition 1.2.1 Traceless $\mathcal{A}$ up to gauge equivalence.
Let M be a Riemann surface and $U \subset M$ an open, connected, simply connected subset of M. Let $\mathcal{A}$ be the coefficient function of a differential equation

$$\frac{d\Phi}{dz}\cdot\Phi^{-1} = \mathcal{A}, \qquad \mathcal{A} : U \to M_2(\mathbb{C}).$$

Consider the family of gauge equivalent systems of differential equations related to $\mathcal{A}$:

$$\left\{\mathcal{B} = \frac{d\Phi}{dz}\cdot\Phi(z)^{-1} \,\middle|\, \mathcal{B} = g\cdot\mathcal{A}\cdot g^{-1} + \frac{dg}{dz}\cdot g^{-1},\ g : U \to GL_2(\mathbb{C}) \text{ holomorphic}\right\}$$

There exists an element $\mathcal{B}$ of the family with $\operatorname{trace}(\mathcal{B}) = 0$.

Proof. Consider $\mathcal{A} = \frac{d\Phi}{dz}\Phi^{-1}$. Let $z_0 \in U$ be a fixed point, define the function

$$g(z) = \exp\left(-\frac{1}{2}\int_{z_0}^{z}\operatorname{trace}(\mathcal{A}(\omega))\,d\omega\right)I.$$

This function is well defined since: U is open and connected, thus path connected; U is simply connected, so the integral does not depend on the path of integration. The function is obviously holomorphic. Notice that

$$\frac{dg}{dz} = -\frac{1}{2}\operatorname{trace}(\mathcal{A})g, \quad \text{and } g^{-1}(z) = \exp\left(\frac{1}{2}\int_{z_0}^{z}\operatorname{trace}(\mathcal{A}(\omega))d\omega\right)I.$$

Then

$$\mathcal{B} = g\mathcal{A}g^{-1} + \frac{dg}{dz}g^{-1} = \mathcal{A} - \frac{1}{2}\operatorname{trace}(\mathcal{A})I$$

is gauge equivalent to $\mathcal{A}$, here we used that the scalar part of g commutes with $\mathcal{A}$. Further

$$\operatorname{trace}(\mathcal{B}) = \operatorname{trace}\left(\mathcal{A} - \frac{1}{2}\operatorname{trace}(\mathcal{A})I\right) = \operatorname{trace}(\mathcal{A}) - \frac{1}{2}\operatorname{trace}(\mathcal{A})\operatorname{trace}(I) = 0.$$

■

The following simple Lemma will be essential when we argue for uniqueness of the local fundamental solutions we find for our system. It also makes analytic continuation effortless.

Lemma 1.2.2 Constant matrix relation.
Consider the differential equation

$$\frac{d\Phi}{dz}\Phi^{-1} = \mathcal{A}, \qquad \mathcal{A} : M \to M_2(\mathbb{C})$$

where M is a Riemann surface. Let $\Phi^1 : U_1 \to GL_2(\mathbb{C})$ and $\Phi^2 : U_2 \to GL_2(\mathbb{C})$ be two solutions of the system on open sets $U_1, U_2 \subset M$ with $U_1 \cap U_2 \neq \emptyset$. Then $\Phi^2 = \Phi^1 C_k$ in each connected component V_k of $U_1 \cap U_2$, where C_k is a constant non-singular matrix.

Proof. Consider the matrix ratio $C = (\Phi^1)^{-1}\Phi^2$ defined on $U_1 \cap U_2$. We compute the derivative of C w.r.t. $z \in V \subset U_1 \cap U_2$. Consider a chart $\phi : V \to \phi(V) \subset \mathbb{C}$, then

$$\frac{dC}{dz} = \frac{d}{d\omega}\left(C \circ \phi^{-1}(\omega)\right)\Big|_{\omega=\phi(z)} = \frac{d}{d\omega}\left(\left(\Phi^1 \circ \phi^{-1}(\omega)\right)^{-1}\Phi^2 \circ \phi^{-1}(\omega)\right)\Big|_{\omega=\phi(z)}$$

Differentiating using the Leibniz rule and the derivative of the inverse of a matrix:

$$\begin{aligned} = -\left(\Phi^1 \circ \phi^{-1}(\omega)\right)^{-1}\left(\frac{d}{d\omega}(\Phi^1 \circ \phi^{-1}(\omega))\right)\left(\Phi^1 \circ \phi^{-1}(\omega)\right)^{-1}\left(\Phi^2 \circ \phi^{-1}(\omega)\right)\Big|_{\omega=\phi(z)} \\ + \left(\Phi^1 \circ \phi^{-1}(\omega)\right)^{-1}\left(\frac{d}{d\omega}(\Phi^2 \circ \phi^{-1}(\omega))\right)\Big|_{\omega=\phi(z)}. \end{aligned}$$

By using the fact that Φ^1 and Φ^2 solve the same differential equation we obtain, and writing $\phi^{-1}(\omega) = z$

$$= -\Phi^1(z)^{-1}\mathcal{A}(z)\Phi^1(z)\Phi^1(z)^{-1}\Phi^2(z) + \Phi^1(z)^{-1}\mathcal{A}(z)\Phi^2(z) = 0$$

Thus the derivative of $C : U_1 \cap U_2 \to GL_2(\mathbb{C})$ is identically zero. Hence by Lemma A.1.2, on each connected component V_k of $U_1 \cap U_2$, $C(z) = C_k$, a constant matrix. C_k is non-singular since Φ_1 and Φ_2 is non-singular. ■

1.2.3 Fundamental solutions around regular points

Let $a \in \mathbb{S} \setminus \{0, t, \infty\}$. Then a is a regular point of

$$\frac{d\Phi}{dz}\Phi(z)^{-1} = \frac{\sigma_3}{2} + \frac{A_0}{z} + \frac{A_t}{z-t} = \mathcal{A}(z).$$

We start by rewriting the expression for $\mathcal{A}$ into a series expression in $(z-a)$. The terms $\frac{A_0}{z}$ and $\frac{A_t}{z-t}$ are both devolved using the geometric series:

$$\frac{A_0}{z} = \frac{A_0}{a\left(1 - \frac{z-a}{-a}\right)} = A_0\sum_{k=0}^{\infty}(-1)^k\frac{(z-a)^k}{a^{k+1}}, \quad \text{for } |z-a| < |a|$$

$$\frac{A_t}{z-t} = \frac{A_t}{(a-t)\left(1 - \frac{z-a}{t-a}\right)} = -A_t\sum_{k=0}^{\infty}\frac{(z-a)^k}{(t-a)^{k+1}}, \quad \text{for } |z-a| < |a-t|$$

Hence we obtain the expression

$$\frac{d\Phi}{dz}\Phi(z)^{-1} = \mathcal{A}(z) = \sum_{k=0}^{\infty} A_{k+1}^{(a)}(z-a)^k, \tag{1.5}$$

$$A_1^{(a)} = \frac{\sigma_3}{2} - \frac{A_0}{a^2} - \frac{A_t}{(t-a)^2}, \quad A_{k+1}^{(a)} = (-1)^k\frac{A_0}{a^{k+1}} - \frac{A_t}{(t-a)^{k+1}} \tag{1.6}$$

We expect a holomorphic solution to this problem, and propose the following ansatz:

$$\Phi^{(a)}(z) = \sum_{k=0}^{\infty} \Psi_k^{(a)}(z-a)^k, \quad \Psi_k^{(a)} \in M_2(\mathbb{C}), \quad \Psi_0^{(a)} := I$$

We put the ansatz into equation (1.5) and obtain

$$\sum_{k=0}^{\infty}(k+1)\Psi_{k+1}^{(a)}(z-a)^k = \left(\sum_{k=0}^{\infty} A_{k+1}^{(a)}(z-a)^k\right)\left(\sum_{k=0}^{\infty} \Psi_k^{(a)}(z-a)^k\right)$$
$$= \sum_{k=0}^{\infty}\sum_{l=0}^{k} A_{k+1-l}^{(a)}\Psi_l^{(a)}(z-a)^k$$

Equating the coefficients we obtain

$$\Psi_{k+1}^{(a)} = \frac{1}{k+1}\sum_{l=0}^{k} A_{k+1-l}^{(a)}\Psi_l^{(a)}, \quad \Psi_0^{(a)} := I$$

Hence we obtain a formula determining all the coefficients uniquely. The following Theorem gives the convergence radius of the series solution.

Theorem 1.2.1 [[Sib90], T.1.8.2, T.1.8.3] Existence of fundamental solution and radius of convergence.
Consider the system $\frac{d\Phi}{dz} = \mathcal{A}(z)\Phi(z)$ in a series representation around a regular point a. Consider a formal series solution centred at the regular point a:

$$\Phi^{(a)}(z) = \sum_{k=0}^{\infty} \Psi_k^{(a)}(z-a)^k.$$

If the minimum radius of convergence of all entries $(\mathcal{A})_{ij}$ is R, then the radius of convergence of the series solution $\Phi^{(a)}$ is also R.
Further if $\Psi_0^{(a)} = \Phi^{(a)}(a) \in GL_2(\mathbb{C})$ then $\Phi^{(a)} \in GL_2(\mathbb{C})$.

Hence we can conclude that we have found a fundamental solution (see Definition 1.1.2)

$$\Phi^{(a)}(z) = \sum_{k=0}^{\infty} \Psi_k^{(a)}(z-a)^k$$

where

$$\Psi_{k+1}^{(a)} = \frac{1}{k+1}\sum_{l=0}^{k} A_{k+1-l}^{(a)}\Psi_l^{(a)}, \quad \Psi_0^{(a)} := I$$

determine the coefficients uniquely. The series converges in the disc

$$B(a,R) = \{\, z \in \mathbb{S} \setminus \{\, 0, t, \infty \,\} \mid |\, z-a| < R = \min\{\ |\, a-t|, |a|\,\}\,\}\,.$$

We chose $\Phi^{(a)}(0) = \Psi_0^{(a)} = I$, to obtain one specific solution. Any other solution to (1.2) in the disk $B(a,R)$, can be obtained by right multiplication by a constant matrix.

1.2.4 Fundamental solution around the Fuchsian singular point at the origin

We start by finding the local solution of the equation around the point $z_0 = 0$. We introduce similar notation as [Fok et al.06] and write equation (1.2) as

$$\begin{aligned}\frac{d\Phi}{dz}\Phi(z)^{-1} &= \frac{\sigma_3}{2} + \frac{A_0}{z} + \frac{A_t}{z-t} = \frac{A_0}{z} + \frac{\sigma_3}{2} - \sum_{k=0}^{\infty}\frac{A_t}{t^{k+1}}z^k \\ &= \frac{A_0^{(0)}}{z} + \sum_{k=0}^{\infty} A_{k+1}^{(0)} z^k, \\ A_0^{(0)} = A_0, \quad & A_1^{(0)} = \frac{\sigma_3}{2} - \frac{A_t}{t}, \quad A_{k+1}^{(0)} = -\frac{A_t}{t^{k+1}}\end{aligned} \tag{1.7}$$

Here the subscripted 0 is an index and the superscripted (0) relates the matrix to the pole at $z_0 = 0$, note that $A_0^{(0)} = A_0$. We use the diagonalization of the matrix $A_0^{(0)}$ to be able to define an ansatz:

$$A_0^{(0)} = P^{(0)}\Lambda_0^{(0)}P^{(0)-1}$$

Consider the following ansatz:

$$\Phi^{(0)}(z) = P^{(0)}\left(\sum_{k=0}^{\infty}\Psi_k^{(0)}z^k\right)\exp\left(\Lambda_0^{(0)}\log(z)\right), \quad \Psi_0^0 = I \tag{1.8}$$

Here $\Psi_k^{(0)}$ are complex matrices to be determined. The function exp is the matrix exponential function, that is to be distinguished from the scalar exponential function, $z \mapsto e^z$. Also the branch of the logarithm is yet to be determined and will be chosen when doing an analytic continuation

We differentiate the ansatz

$$\begin{aligned}\frac{d\Phi^{(0)}}{dz} = P^{(0)}&\left(\sum_{k=0}^{\infty}\Psi_{k+1}^{(0)}(k+1)z^k\right)\exp\left(\Lambda_0^{(0)}\log(z)\right) \\ &+ P^{(0)}\left(\sum_{k=0}^{\infty}\Psi_k^{(0)}z^k\right)\frac{\Lambda_0^{(0)}}{z}\exp\left(\Lambda_0^{(0)}\log(z)\right)\end{aligned}$$

and multiply $\mathcal{A}(z)$ with the ansatz (1.8)

$$\begin{aligned}\mathcal{A}(z)\Phi^{(0)}(z) = \frac{A_0^{(0)}}{z}P^{(0)}&\left(\sum_{k=0}^{\infty}\Psi_k^{(0)}z^k\right)\exp\left(\Lambda_0^{(0)}\log(z)\right) \\ &+ \left(\sum_{k=0}^{\infty}A_{k+1}^{(0)}z^k\right)P^{(0)}\left(\sum_{k=0}^{\infty}\Psi_k^{(0)}z^k\right)\exp\left(\Lambda_0^{(0)}\log(z)\right)\end{aligned}$$

equating the two expressions through the ODE, left multiplying by $P^{(0)-1}$ and cancelling the exponential factor we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\Psi_{k+1}^{(0)}(k+1)z^k &+ \sum_{k=0}^{\infty}\Psi_{k+1}^{(0)}\Lambda_0^0 z^k \\ &= \sum_{k=0}^{\infty}P^{(0)-1}A_0^{(0)}P^{(0)}\Psi_{k+1}^{(0)} + \sum_{k=0}^{\infty}\sum_{l=0}^{k}P^{(0)-1}A_{k+1-l}^{(0)}P^{(0)}\Psi_l^{(0)}\end{aligned}$$

By the above diagonalization $P^{(0)-1}A_0^{(0)}P^{(0)} = \Lambda_0^{(0)}$. We equate the coefficients of z^k and obtain the following recursive formulas for the coefficients $\Psi_{k+1}^{(0)}$

$$\Psi_{k+1}^{(0)}(k+1) + \left[\Psi_{k+1}^{(0)}, \Lambda_0^{(0)}\right] = \sum_{l=0}^{k} P^{(0)-1}A_{k+1-l}^{(0)}P^{(0)}\Psi_l^0, \quad k \geq 0$$
$$\left[\Psi_0^{(0)}, \Lambda_0^{(0)}\right] = 0$$

We can solve the first formula for the coefficients in the matrix $\Psi_{k+1}^{(0)}$ and obtain the following explicit formula

$$\left(\Psi_{k+1}^{(0)}\right)_{ij} = \frac{\sum_{l=0}^{k} P^{(0)-1}A_{k+1-l}^{0}P^{(0)}\Psi_l^0}{k+1+(\Lambda_0^{(0)})_{jj} - (\Lambda_0^{(0)})_{ii}}, \quad k \geq 0$$
$$\Psi_0^{(0)} = I$$

This formula determines the coefficients $\Psi_k^{(0)}$ uniquely, so we conclude that (1.8) is a formal solution. The series will converge in a neighbourhood of $z_0 = 0$ by Theorem 5 in [Bal00].

To summarize we found the local fundamental solution (see Definition 1.1.2)

$$\Phi^{(0)}(z) = P^{(0)}\left(\sum_{k=0}^{\infty}\Psi_k^{(0)}z^k\right)\exp\left(\Lambda_0^{(0)}\log_{\alpha_0}(z)\right), \tag{1.9}$$

in the branched neighbourhood $z \in B(0, R_0) \setminus \{\, re^{i\alpha_0} \mid r \geq 0 \,\}$ of $z = 0$. The series in the solution converges by Theorem 5 in [Bal00], and the branch of the logarithm in the expression will be chosen later, when we do an analytic continuation of the solution.

Definition 1.2.4 Canonical fundamental solution in a branched neighbourhood of a Fuchsian point.
We define

$$\Phi^{(j)}(z) = P^{(j)}\left(\sum_{k=0}^{\infty}\Psi_k^{(k)}(z-z_j)^k\right)\exp\left(\Lambda_0^{(j)}\log_{\alpha}(z-z_j)\right),$$

$$\textit{for } z \in B(z_j, R) \setminus \{\, z_j + re^{i\alpha} \mid r \geq 0 \,\},$$

where: $P^{(j)}$ is orthonormal with eigenvalues in descending order and $\Psi_0^{(j)} = I$, to be the canonical fundamental solution of equation (1.2) in the branched neighbourhood $B(z_j, R)\setminus \{\, z_j + re^{i\alpha} \mid r \geq 0 \,\}$ of the Fuchsian point z_j.

1.2.5 Fundamental solution around the Fuchsian singular point at t

We now similarly find a fundamental solution around the Fuchsian singular point $z_1 = t$. We do a change of coordinates in (1.2) for ease of notation and expand the remaining expressions into a Taylor series of $z - t = \eta$ so we obtain

$$\frac{d\Phi}{dz}\Phi(z)^{-1} = \frac{\sigma_3}{2} + \frac{A_0}{z} + \frac{A_t}{z-t} = \frac{A_t}{z-t} + \left(\frac{\sigma_3 t + 2A_0}{2t}\right) + \sum_{k=1}^{\infty}\frac{A_0}{t^{k+1}}(z-t)^k \tag{1.10}$$
$$\frac{d\Phi}{d\eta}\Phi(\eta)^{-1} = \frac{A_0^{(1)}}{\eta} + \sum_{k=0}^{\infty}A_{k+1}^{(1)}\eta^k$$

Here again we have introduced a similar notation as in [Fok et al.06].
We use the diagonalization

$$A_0^{(1)} = P^{(1)}\Lambda_0^{(1)}(P^{(1)})^{-1}, \tag{1.11}$$

and propose the following ansatz:

$$\Phi^{(1)}(\eta) = P^{(1)}\left(\sum_{k=0}^{\infty}\Psi_k^{(1)}\eta^k\right)\exp\left(\Lambda_0^{(1)}\log(\eta)\right) \tag{1.12}$$

Putting the ansatz into (1.12) we obtain

$$\begin{aligned}\frac{d\Phi^{(1)}}{d\eta} = P^{(1)}&\left(\sum_{k=0}^{\infty}\Psi_{k+1}^{(1)}(k+1)\eta^k\right)\exp\left(\Lambda_0^{(1)}\log(\eta)\right)\\ &+\left(P^{(1)}\sum_{k=0}^{\infty}\Psi_k^{(1)}\eta^k\right)\frac{\Lambda_0^{(1)}}{\eta}\exp\left(\Lambda_0^{(1)}\log(\eta)\right)\\ &=\left(\frac{A_0^{(1)}}{\eta}+\sum_{k=0}^{\infty}A_{k+1}^{(1)}\eta^k\right)\left(P^{(1)}\sum_{k=0}^{\infty}\Psi_k^{(1)}\eta^k\right)\exp\left(\Lambda_0^{(1)}\log(\eta)\right).\end{aligned}$$

We multiply on the left by $P^{(0)-1}$, cancel the exponential factor and equate the coefficients of η^k

$$\Psi_{k+1}^{(1)}(k+1)+\left[\Psi_{k+1}^{(1)},\Lambda_0^{(1)}\right] = \sum_{l=0}^{k}P^{(1)-1}A_{k+1-l}^{(1)}P^{(1)}\Psi_l^{(1)}, \quad k\geq 0 \tag{1.13}$$

the same formula as for the pole $z_0 = 0$ up to the eigenvalues of $A_0^{(i)}$.

We obtain an explicit formula for the coefficients of $\Psi_k^{(1)}$

$$\begin{aligned}\left(\Psi_{k+1}^{(1)}\right)_{ij} &= \frac{\sum_{l=0}^{k}P^{(1)-1}A_{k+1-l}^{(1)}P^{(1)}\Psi_l^{(1)}}{k+1+(\Lambda_0^{(1)})_{jj}-(\Lambda_0^{(1)})_{ii}}, \quad k\geq 0\\ \Psi_0^{(1)} &= I\end{aligned}$$

Thus the series in the ansatz is determined uniquely by this formula. We have found a canonical fundamental solution (see Definition 1.2.4) in a branched neighbourhood of the Fuchsian point $z_1 = t$ given by

$$\Phi^{(1)}(z) = P^{(1)}\left(\sum_{k=0}^{\infty}\Psi_k^{(1)}(z-t)^k\right)\exp\left(\Lambda_0^{(1)}\log_{\alpha_1}(z-t)\right), \tag{1.14}$$
$$z\in B(t,R_1)\setminus\{\, re^{i\alpha_1}\mid r\geq 0\,\}.$$

The series in the solution converges by Theorem 5 in [Bal00], and the branch of the logarithm in the expression, will be chosen later, when we do an analytic continuation of the solution.

1.2.6 Formal solution around the non-Fuchsian singular point

Lastly we find the solution of the system around the non-Fuchsian singular point $z_2 = \infty$. Previously we derived the form of the equation under the transformation $z \mapsto \frac{1}{z} = \xi$, (1.4). We write it out using the notation in [Fok et al.06].

$$(1.15)\qquad \frac{d\Phi}{d\xi}\Phi(\xi)^{-1} = -\frac{\sigma_3}{2\xi^2} - \frac{A_0+A_t}{\xi} - \sum_{k=0}^{\infty} A_t t^{k+1}\xi^k = \frac{A_{-1}^{(\infty)}}{\xi^2} + \frac{A_0^{(\infty)}}{\xi} + \sum_{k=0}^{\infty} A_{k+1}^{(\infty)}\xi^k$$

The coefficient matrix $A_{-1}^{(\infty)}$ is diagonal. To be consistent with the notation we write

$$A_{-1}^{(\infty)} = P^{(\infty)}\Lambda_{-1}^{(\infty)}P^{(\infty)-1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = -\frac{1}{2}\sigma_3,$$

where σ_3 is the famous third Pauli-matrix. We propose the following ansatz:

$$(1.16)\qquad \Phi^{(\infty)}(\xi) = P^{(\infty)}\left(\sum_{k=0}^{\infty} Y_k\xi^k\right)\exp\left(-\frac{\Lambda_{-1}^{(\infty)}}{\xi} + \Lambda_0^{(\infty)}\log_{\alpha_2}(\xi) + \sum_{k=1}^{\infty}\frac{\Lambda_k^{(\infty)}}{k}\xi^k\right),$$

where $Y_0 = I$, Y_k is off-diagonal and $\Lambda_k^{(\infty)}$ are all diagonal and to be determined. Putting the ansatz into equation (1.15) and immediately cancelling the exponential terms we obtain

$$\begin{aligned} P^{(\infty)}\sum_{k=0}^{\infty} Y_{k+1}(k+1)\xi^k + P^{(\infty)}\left(\sum_{k=0}^{\infty} Y_k\xi^k\right)\left(\frac{\Lambda_{-1}^{(\infty)}}{\xi^2} + \frac{\Lambda_0^{(\infty)}}{\xi} + \sum_{k=0}^{\infty}\Lambda_{k+1}^{(\infty)}\xi^k\right)& \\ = \left(\frac{A_{-1}^{(\infty)}}{\xi^2} + \frac{A_0^{(\infty)}}{\xi} + \sum_{k=0}^{\infty} A_{k+1}^{(\infty)}\xi^k\right)P^{(\infty)}\left(\sum_{k=0}^{\infty} Y_k\xi^k\right)& \end{aligned}$$

Multiplying out and equating the coefficients of ξ^k we obtain

$$\begin{aligned} Y_{k+1}(k+1) + Y_{k+2}\Lambda_{-1}^{(\infty)} + Y_{k+1}\Lambda_0^{(\infty)} + \sum_{l=0}^{k} Y_l\Lambda_{k+1-l}^{(\infty)}& \\ = \Lambda_{-1}^{(\infty)}Y_{k+2} + P^{(\infty)-1}A_0^{(\infty)}P^{(\infty)}Y_{k+1} + \sum_{l=0}^{k} P^{(\infty)-1}A_{k+1-l}^{(\infty)}P^{(\infty)}Y_l \quad k\geq 1& \end{aligned}$$

and for $k=0$:

$$Y_1\Lambda_{-1}^{(\infty)} + Y_0\Lambda_0^{(\infty)} = \Lambda_{-1}^{(\infty)}Y_1 + P^{(\infty)-1}A_0^{(\infty)}P^{(\infty)}Y_0$$

Recalling that $Y_0 = I$ and including the terms with Y_{k+1} in the sums we obtain

$$\begin{aligned} [Y_{k+2}, \Lambda_{-1}^{(\infty)}] + \Lambda_{k+1}^{(\infty)} = P^{(\infty)-1}A_{k+1}^{(\infty)}P^{(\infty)}& \\ + \sum_{l=1}^{k+1}\left(P^{(\infty)-1}A_{k+1-l}^{(\infty)}P^{(\infty)}Y_l - Y_l\Lambda_{k+1-l}^{(\infty)}\right) - (k+1)Y_{k+1}, \quad k\geq 1& \end{aligned}$$

and for $k=0$:

$$[Y_1, \Lambda_{-1}^{(\infty)}] + \Lambda_0^{(\infty)} = P^{(\infty)-1}A_0^{(\infty)}P^{(\infty)}$$

These formulas determine Y_k uniquely as off-diagonal matrices, and $\Lambda_k^{(\infty)}$ as diagonal matrices. Indeed we have the explicit formulas for $k\geq 0$

$$(Y_1)_{ij} = \begin{cases} 0 & \text{for } i = j \\ \frac{P^{(\infty)-1} A_0^{(\infty)} P^{(\infty)}}{(\Lambda_{-1}^{(\infty)})_{jj} - (\Lambda_{-1}^{(\infty)})_{ii}} & \text{for } i \neq j \end{cases} \tag{1.17}$$

$$(\Lambda_0^{(\infty)})_{ij} = \begin{cases} (P^{(\infty)-1} A_0^{(\infty)} P^{(\infty)})_{ij} & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{1.18}$$

$$(Y_{k+2})_{ij} = \begin{cases} 0 & \text{for } i = j \\ \frac{\left(P^{(\infty)-1} A_{k+1}^{(\infty)} P^{(\infty)} + \sum_{l=1}^{k+1} \left(P^{(\infty)-1} A_{k+1-l}^{(\infty)} P^{(\infty)} Y_l - Y_l \Lambda_{k+1-l}^{(\infty)}\right) - (k+1) Y_{k+1}\right)_{ij}}{(\Lambda_{-1}^{(\infty)})_{jj} - (\Lambda_{-1}^{(\infty)})_{ii}} & \text{for } i \neq j \end{cases} \tag{1.19}$$

$$(\Lambda_{k+1}^{(\infty)})_{ij} = \begin{cases} \left(P^{(\infty)-1} A_{k+1}^{(\infty)} P^{(\infty)} + \sum_{l=1}^{k+1} \left(P^{(\infty)-1} A_{k+1-l}^{(\infty)} P^{(\infty)} Y_l - Y_l \Lambda_{k+1-l}^{(\infty)}\right) - (k+1) Y_{k+1}\right)_{ij} & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{1.20}$$

It is possible to rewrite the solution (1.16) to the form

$$\Phi^{(\infty)}(\xi) = P^{(\infty)} \left(\sum_{k=0}^{\infty} \Psi_k^{(\infty)} \xi^k \right) \exp\left(-\frac{\Lambda_{-1}^{(\infty)}}{\xi} + \Lambda_0^{(\infty)} \log_{\alpha_2}(\xi) \right)$$

Indeed we equate the proposed expression with the solution (1.16) and note that the matrices in the exponential are diagonal, hence they commute with each other. We can thus rewrite the sum in the exponential as the product of the matrix exponentials and right multiply by

$$\exp\left(\frac{\Lambda_{-1}^{(\infty)}}{\xi} - \Lambda_0^{(\infty)} \log_{\alpha_2}(\xi) \right)$$

to obtain

$$\left(\sum_{k=0}^{\infty} \Psi_k^{(\infty)} \xi^k \right) = \left(\sum_{k=0}^{\infty} Y_k \xi^k \right) \exp\left(\sum_{k=1}^{\infty} \frac{\Lambda_k^{(\infty)}}{k} \xi^k \right) = \left(\sum_{k=0}^{\infty} Y_k \xi^k \right) \sum_{l=0}^{\infty} \frac{\left(\sum_{k=1}^{\infty} \frac{\Lambda_k^{(\infty)}}{k} \xi^k \right)^l}{l!}$$

We write out the first terms in the rightmost expression

$$I + \Lambda_1^{(\infty)} \xi + \frac{1}{2}(\Lambda_2^{(\infty)} + (\Lambda_1^{\infty})^2) \xi^2 + \left(\frac{\Lambda_3^{(\infty)}}{3} + \frac{\Lambda_1^{(\infty)} \Lambda_2^{(\infty)}}{2} + \frac{(\Lambda_1^{(\infty)})^3}{6} \right) \xi^3 \ldots =: \sum_{k=0}^{\infty} D_k \xi^k$$

Note that the coefficient matrices D_k are all diagonal. We then carry out the multiplication of the two series and equate the coefficients of ξ^k

$$\Psi_k^{(\infty)} = \sum_{l=0}^{k} Y_l D_{k-l}.$$

We conclude that we have found a formal solution of (1.15)

$$\begin{aligned} \Phi^{(\infty)}(\xi) &= P^{(\infty)} \left(\sum_{k=0}^{\infty} \Psi_k^{(\infty)} \xi^k \right) \exp\left(-\frac{\Lambda_{-1}^{(\infty)}}{\xi} + \Lambda_0^{(\infty)} \log_{\alpha_2}(\xi) \right) \\ &= P^{(\infty)} \left(\sum_{k=0}^{\infty} \Psi_k^{(\infty)} \xi^k \right) \exp\left(\Lambda^{(\infty)}(\xi) \right), \end{aligned} \tag{1.21}$$

in a branched neighbourhood of $\xi = 0$. We have found formulas that determine the coefficient matrices uniquely. In general the series in the above solution does not converge, hence this is only a formal solution, see [Fok et al.06] and [Bal00]. The exponential, and logarithmic terms are of course well defined for

$$\xi \in B(0, R_2) \setminus \{ re^{i\alpha_2} \mid r \geq 0 \} ,$$

where α_2 is the angle of the branch cut.

We will interpret the solution as an *asymptotic expansion* of a fundamental solution as $z \to \infty$, see Definition 1.3.1. In the next Section we show how this interpretation leads to the Stokes phenomenon, which further gives a nice description of how the solution behaves in sectors centered at the non-Fuchsian singular point.

1.3 The Stokes phenomenon for the non-Fuchsian singular point

In this Section we will in great detail describe the solution around the non-Fuchsian point $z_2 = \infty$. In Section 1.2.6 we found a formal solution, valid in a branched neighbourhood around $z_2 = \infty$. It is a formal solution, since the series in the expression is a formal series. In the following Section we will show that on sectorial domains, centered at $\xi = 0 \iff z_2 = \infty$, it is possible to find holomorphic solutions to equation (1.15). Finally, we can analytically continue these solutions along loops encircling $z_2 = \infty$.

1.3.1 Asymptotic expansions and existence Theorem in sectorial domains

First we define what we mean by an asymptotic expansion.

Definition 1.3.1 Asymptotic expansion of a function.
Given a function $f : \Sigma \to M_{2\times 2}$, defined in the sectorial domain

$$\Sigma = \{ z \in \mathbb{C} \mid 0 < |z| < R,\ \alpha < \mathrm{Arg}(z) < \beta \} .$$

If to every closed subsector $\overline{S} \subset \Sigma$ and for each fixed $N \in \mathbb{N}$ we have:

$$\left\| f(z) - \sum_{k=0}^{N} a_k \varphi_k(z) \right\| = O(\varphi_{N+1}), \ \textit{as } z \to 0 \textit{ in } \overline{S},$$

then we say that the series $\sum_{k=0}^{\infty} a_k \varphi_k(z)$ is an asymptotic expansion of the function f in Σ.

We remark that the asymptotic expansion is in general a divergent series. However the Definition state that: each truncation of the series converge to the function f as $z \to 0$, where f in general has a non-removable singular point at 0. In this sense, even though the series diverge, it describes well the behaviour of the function f around its singular point.

In our case the sequence of functions in the series are $\varphi_k(\xi) = \xi^k$. We will find that the solution obtained in equation (1.21) is an asymptotic expansion as $\xi \to 0$ of an actually existing solution Φ, of the equation (1.15).

The following Theorem give us fundamental solutions in an open sector with a certain bounded central angle. The midline of the sector can point in arbitrary directions.

Theorem 1.3.1 [T.12.3 in [Was87]] Existence of fundamental solutions of non–fuchsian system in sectors around a singular point.
Consider the system

$$\frac{d\Phi}{d\xi}\Phi(\xi)^{-1} = \mathcal{A}(\xi) = \sum_{k=-r-1}^{\infty} A_{k+1}\xi^k$$

where $\mathcal{A}$ is a $N \times N$ matrix valued function, holomorphic in the sector Σ,

$$\Sigma = \{\, \xi \in \mathbb{S} \mid 0 < \xi < R,\ \theta_1 < \operatorname{Arg}(\xi) < \theta_2,\ 0 < \theta_2 - \theta_1 < \frac{\pi}{r} \,\}$$

where 0 is a singularity of Poincaré rank r of the system. Suppose the matrix A_{-r} is diagonalisable, such that $A_{-r} = P\Lambda_{-r}P^{-1}$, where the eigenvalues of A_{-r} are all distinct.

Suppose there exists a formal solution of the system in a neighbourhood of 0, in the form

$$\Phi(\xi) = P\left(\sum_{k=0}^{\infty} \Psi_k \xi^k\right) \exp\left(\sum_{k=-r}^{-1} \frac{\Lambda_k}{k}\xi^k + \Lambda_0 \log_\alpha(\xi)\right) =: P\left(\sum_{k=0}^{\infty} \Psi_k \xi^k\right) \exp(\Lambda_\alpha(\xi)) \tag{1.22}$$

where Λ_k is diagonal for all $k \in \{-r, -r+1, \ldots, 0\}$. The series in this solution is only formal. Also α is a branch cut of the logarithm chosen outside the sector Σ. Then there exists a holomorphic function

$$\hat{\Psi}_\Sigma : \Sigma \to GL_2(\mathbb{C})$$

such that the system possesses a fundamental solution of the form

$$\Phi_\Sigma(\xi) = P\hat{\Psi}_\Sigma(\xi)\exp(\Lambda_\alpha(\xi)).$$

Also $\hat{\Psi}_\Sigma$ has the formal series as an asymptotic expansion, that is

$$\left\| \hat{\Psi}_\Sigma(\xi) - \sum_{k=0}^{N} \Psi_k \xi^k \right\| = O(\xi^{N+1}),\quad \xi \to 0,\ \xi \in \Sigma. \tag{1.23}$$

Thus for any ray out from 0, we can find a sector symmetric around the ray, with central angle less than $\frac{\pi}{r}$, where we have a fundamental solution. The condition on the central angle to be less than $\frac{\pi}{r}$ is not in general sharp. We will soon discover that we sometimes can extend the domain of the fundamental solution given by Theorem 1.3.1 into larger sectors.

Remark.
If the function $\hat{\Psi}_\Sigma$ can be analytically continued to a single valued function in a punctured neighbourhood U^* of $\xi = 0$, then 0 is a removable singularity. Indeed, since the asymptotic expansion hold in the whole punctured neighbourhood, we have

$$\lim_{\xi \to 0} \hat{\Psi}_\Sigma(\xi) = \lim_{\xi \to 0} \sum_{k=0}^{\infty} \Psi_k \xi^k = \Psi_0.$$

Thus $\xi = 0$ is a removable singularity of $\hat{\Psi}_\Sigma$, and it is holomorphic in the whole neighbourhood. Then since the asymptotic expansion of $\hat{\Psi}_\Sigma$, i.e. the formal series $\sum_{k=0}^{\infty} \Psi_k \xi^k$, holds in U^*, the series converges, and we do not need Theorem 1.3.1 to obtain a solution. Thus in general we expect the sectorial domain of a solution given by Theorem 1.3.1 to have a central angle less than 2π.

1.3.2 Stokes rays and Stokes sectors

Definition 1.3.2.
Consider the system from Theorem 1.3.1 with the formal solution

$$\Phi(\xi) = P\left(\sum_{k=0}^{\infty} \Psi_k \xi^k\right) \exp\left(\sum_{k=-r}^{-1} \frac{\Lambda_k}{k}\xi^k + \Lambda_0 \log_\alpha(\xi)\right)$$

in a punctured neighbourhood of $\xi = 0$. The rays in the complex plane defined by the condition

$$Re\left(\frac{(\Lambda_{-r})_{22} - (\Lambda_{-r})_{11}}{\xi^r}\right) = 0$$
$$\iff \operatorname{Arg}(\xi) = \frac{\operatorname{Arg}\left((\Lambda_{-r})_{22} - (\Lambda_{-r})_{11}\right)}{r} + \frac{\pi}{2r}(2n+1) =: \theta_n \quad n \in \mathbb{Z}$$

are called Stokes rays and are denoted by l_n. The angle that defines a Stokes ray l_n, is denoted by θ_n.

If we consider the formal solution (1.21) we found for the non-Fuchsian singular point we obtain two Stokes rays. Indeed, we have Poincaré rank $r = 1$ and by (1.15)

$$\Lambda_{-1}^{(\infty)} = -\frac{1}{2}\sigma_3 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

So Definition (1.3.2) gives

$$\theta_n = \operatorname{Arg}(1) + \frac{(2n-1)\pi}{2} = \frac{(2n-1)\pi}{2}, \quad n \in \mathbb{Z}$$

Hence we obtain 2 distinct Stokes rays

$$l_{2n+1} = l_1 = \left\{\xi \in \mathbb{C} \mid \ 0 \leq |\xi|, \ \operatorname{Arg}(\xi) = -\frac{\pi}{2} = \theta_1\right\} \tag{1.24}$$

$$l_{2n} = l_2 = \left\{\xi \in \mathbb{C} \mid \ 0 \leq |\xi|, \ \operatorname{Arg}(\xi) = \ \frac{\pi}{2} = \theta_2\right\} \tag{1.25}$$

Proposition 1.3.1.
Consider a system as in Theorem 1.3.1, with a non-Fuchsian singular point of Poincaré rank 1.

$$\frac{d\Phi}{d\xi}\Phi(\xi)^{-1} = \sum_{k=-2}^{\infty} A_{k+1}\xi^k,$$

with a formal solution given by

$$\Phi(\xi) = P\left(\sum_{k=0}^{\infty}\Psi_k\xi^k\right)\exp\left(-\frac{\Lambda_{-1}}{\xi} + \Lambda_0 \log_\alpha(\xi)\right)$$

Let Σ_1, Σ_2 be open sectors at 0 such that their intersection $\Sigma_1 \cap \Sigma_2$ contains exactly one Stokes ray l_n, in the direction θ_n. Let

$$\Phi_1 : \Sigma_1 \to GL_2(\mathbb{C}), \qquad \Phi_2 : \Sigma_2 \to GL_2(\mathbb{C})$$

where

$$\Phi_1(\xi) := P\hat{\Psi}_1(\xi)\exp\left(-\frac{\Lambda_{-1}}{\xi} + \Lambda_0 \log_\alpha(\xi)\right)$$
$$\Phi_2(\xi) := P\hat{\Psi}_2(\xi)\exp\left(-\frac{\Lambda_{-1}}{\xi} + \Lambda_0 \log_\alpha(\xi)\right)$$

be fundamental solutions as given by Theorem 1.3.1, in particular they both satisfy the asymptotic condition (1.23)

$$\left\|\hat{\Psi}_j(\xi) - \sum_{k=0}^{N}\Psi_k\xi^k\right\| = O(\xi^{N+1}), \ \ \xi \to 0, \ \xi \in \Sigma_j,$$

where the truncated series $\sum_{k=0}^{N}\Psi_k\xi^k$ is the one from the formal solution of the system.

Then $\Phi_1 = \Phi_2$ on $\Sigma_1 \cap \Sigma_2$, in particular the holomorphic functions $\hat{\Psi}_1$ and $\hat{\Psi}_2$ agree on a subsector, hence we can regard them as a direct analytic continuation of each other.

Proof. Let Φ_1 and Φ_2 be fundamental solutions in sectors Σ_1 and Σ_2 respectively. Assume that they satisfy the asymptotic condition in $\Sigma_1 \cap \Sigma_2$. By Lemma 1.2.2 $C = \Phi_1^{-1}\Phi_2$ is constant on $\Sigma_1 \cap \Sigma_2$. We will show that $C = I$. We evaluate the limit

$$\lim_{\xi \to 0} C = C = \lim_{\xi \to 0} (\Phi_1)^{-1}\Phi_2$$

By the asymptotic condition we have

$$\begin{aligned}\lim_{\xi\to 0}(\Phi_1)^{-1}\Phi_2 &= \lim_{\xi\to 0}\left(P\hat{\Psi}_1(\xi)\exp\left(\Lambda(\xi)\right)\right)^{-1}\left(P\hat{\Psi}_2(\xi)\exp\left(\Lambda(\xi)\right)\right)\\ &= \lim_{\xi\to 0}\exp\left(-\Lambda(\xi)\right)\left(I + O(\xi)\right)\exp\left(\Lambda(\xi)\right)\end{aligned}$$

Since $\Lambda(\xi)$ is diagonal we can write the matrix out with the scalar exponential function of its eigenvalues. We obtain

$$\lim_{\xi\to 0}\begin{pmatrix} e^{-(\Lambda(\xi))_{11}} & 0 \\ 0 & e^{-(\Lambda(\xi))_{22}}\end{pmatrix}\begin{pmatrix} 1+O(\xi) & O(\xi) \\ O(\xi) & 1+O(\xi)\end{pmatrix}\begin{pmatrix} e^{(\Lambda(\xi))_{11}} & 0 \\ 0 & e^{(\Lambda(\xi))_{22}}\end{pmatrix}$$

multiplying out we get

$$C = \lim_{\xi\to 0}\begin{pmatrix} 1+O(\xi) & e^{(\Lambda(\xi))_{22}-(\Lambda(\xi))_{11}}O(\xi) \\ e^{(\Lambda(\xi))_{11}-(\Lambda(\xi))_{22}}O(\xi) & 1+O(\xi)\end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}\end{pmatrix}. \tag{1.26}$$

Hence to conclude we need that the real parts of the exponential functions are negative in some subsector of $\Sigma_1 \cap \Sigma_2$. The exponent in the exponential can be written out

$$\begin{aligned}(\Lambda(\xi))_{jj} - (\Lambda(\xi))_{ii} &= \left(-\frac{\Lambda_{-1}}{\xi} + \Lambda_0^{(\infty)}\log(\xi)\right)_{jj} - \left(-\frac{\Lambda_{-1}}{\xi} + \Lambda_0^{(\infty)}\log(\xi)\right)_{ii}\\ &= -\frac{(\Lambda_{-1})_{jj} - (\Lambda_{-1})_{ii}}{\xi} + \left((\Lambda_0^{(\infty)})_{jj} - (\Lambda_0^{\infty})_{ii}\right)\log(\xi)\end{aligned}$$

When taking the limit $\xi \to 0$, the sign of

$$\mathrm{Re}\left(-\frac{(\Lambda_{-1})_{jj} - (\Lambda_{-1})_{ii}}{\xi}\right), \quad \sigma_{ij} := \mathrm{sign}\left(\mathrm{Re}\left(-\frac{(\Lambda_{-1})_{jj} - (\Lambda_{-1})_{ii}}{\xi}\right)\right)$$

will decide whether the limit exists or not. We find its zeroes. Let $(\Lambda_{-1})_{jj} - (\Lambda_{-1})_{ii} = r_\Lambda e^{i\,\mathrm{Arg}((\Lambda_{-1})_{jj} - (\Lambda_{-1})_{ii})}$ and $\xi = r_\xi e^{i\,\mathrm{Arg}(\xi)}$

$$0 = \mathrm{Re}\left(-\frac{(\Lambda_{-1})_{jj} - (\Lambda_{-1})_{ii}}{\xi}\right) = \mathrm{Re}\left(-\frac{r_\Lambda}{r_\xi}e^{i(\mathrm{Arg}((\Lambda_{-1})_{jj} - (\Lambda_{-1})_{ii}) - \mathrm{Arg}(\xi))}\right)$$

Since the eigenvalues are distinct and $r_\Lambda > 0$, this happens if and only if

$$\cos\left(\mathrm{Arg}\left((\Lambda_{-1})_{jj} - (\Lambda_{-1})_{ii}\right) - \mathrm{Arg}(\xi)\right) = 0$$

$$\iff \mathrm{Arg}(\xi) = \mathrm{Arg}\left((\Lambda_{-1})_{jj} - (\Lambda_{-1})_{ii}\right) + \frac{(2n+1)\pi}{2} \quad n \in \mathbb{Z}.$$

Finally since cosine changes sign at each zero, we have that σ_{ij} changes exactly when ξ pass the rays defined by

$$\mathrm{Arg}(\xi) = \mathrm{Arg}\left((\Lambda_{-1})_{jj} - (\Lambda_{-1})_{ii}\right) + \frac{(2n-1)\pi}{2} \quad n \in \mathbb{Z}. \tag{1.27}$$

Since the sector $\Sigma_1 \cap \Sigma_2$ contains exactly one such ray, the sector contains a subsector where $\sigma_{12} = -1$ and on the other side of the ray $\sigma_{21} = -1$. Hence in the respective subsectors the component C_{12} will have a real negative exponential in the limit and in the other subsector C_{21} will have a real negative exponential in the limit. Then we have that

$$C = \begin{pmatrix} 1 & \lim_{\xi\to 0} e^{(\Lambda(\xi))_{22}-(\Lambda(\xi))_{11}} O(\xi) \\ \lim_{\xi\to 0} e^{(\Lambda(\xi))_{11}-(\Lambda(\xi))_{22}} O(\xi) & 1 \end{pmatrix} = I$$

We can choose to evaluate the two limits in different subsectors of $\Sigma_1 \cap \Sigma_2$ since we know the limit exist, since C is constant. Thus the path we evaluate the limit along does not change the value of the limit. ∎

We now make one final key observation. By Theorem 1.3.1 we have existence of a fundamental solution in an arbitrary open sector with central angle less than $\frac{\pi}{r}$. If two such solutions overlap on one Stokes ray, then the solutions will be identical where they coincide. Hence we can extend our solution. The angle between consecutive Stokes rays are

$$\theta_{n+1} - \theta_n = \frac{\pi}{2r}(2n+1-(2n-1)) = \frac{\pi}{r},$$

hence if two sectors, Σ_n^+, Σ_n^- overlap in a sector with central angle $\delta > 0$, and a Stokes ray is exactly in the sector $\Sigma_n^+ \cap \Sigma_n^-$, then we will have one solution defined in the sector $\Sigma_n^+ \cup \Sigma_n^-$, with central angle $\frac{2\pi}{r} - \delta$.

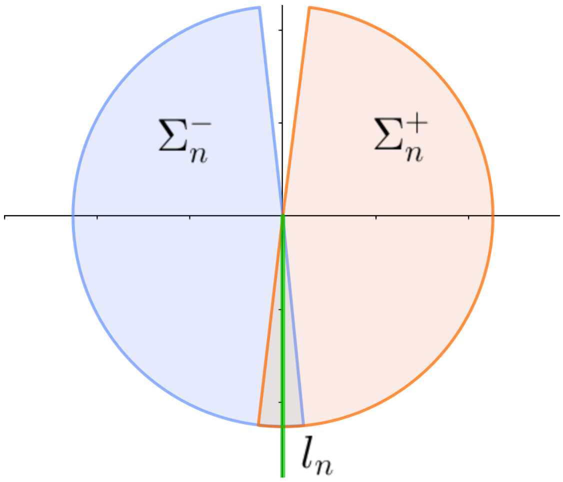


Figure 1.1: Here two sectors Σ_n^+ and Σ_n^- overlap in a sector with central angle $\delta > 0$. Also there is a Stokes ray l_n located in this subsector.

Definition 1.3.3.
We fix a $\delta > 0$ and consider a Stokes ray l_n directed in the angle θ_n. Consider the sectors

$$\Sigma_n^- = \left\{ \xi \in \mathbb{C} \mid 0 < |\xi| < R_n,\ \theta_{n-1} + \frac{\delta}{2} < \operatorname{Arg}(\xi) < \theta_n + \frac{\delta}{2} \right\}$$

$$\Sigma_n^+ = \left\{ \xi \in \mathbb{C} \mid 0 < |\xi| < R_n,\ \theta_n - \frac{\delta}{2} < \operatorname{Arg}(\xi) < \theta_{n+1} - \frac{\delta}{2} \right\}$$

$\Sigma_n := \Sigma_n^+ \cup \Sigma_n^-$ *is called a Stokes sector associated to the Stokes ray l_n.*

$$\Sigma_n = \left\{ \xi \in \mathbb{C} \mid 0 < |\xi| \le R_n,\ \theta_{n-1} + \frac{\delta}{2} < \operatorname{Arg}(\xi) < \theta_{n+1} - \frac{\delta}{2} \right\}$$

Remark that in a Stokes sector Σ_n, by Theorem 1.3.1 and Proposition 1.3.1 we have an unique solution of the form

$$\Phi_{\Sigma_n}(\xi) = P\hat{\Psi}(\xi)\exp(\Lambda_\alpha(\xi))$$

such that $\hat{\Psi}$ is an asymptotic expansion of the formal solution

$$\Phi(\xi) = P\left(\sum_{k=0}^{\infty} \Psi_k \xi^k\right)\exp(\Lambda_\alpha(\xi)).$$

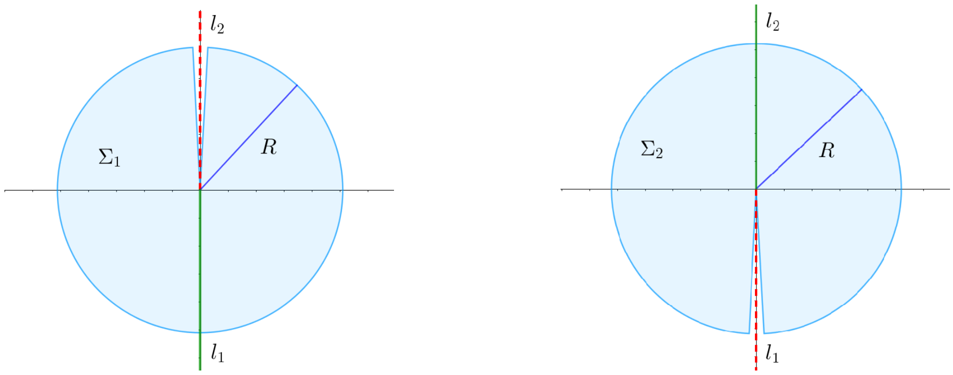
Thus the Stokes sectors are exactly the largest sectors where there exist a unique solution, with a given asymptotic expansion.

If we consider the Stokes rays obtained in (1.24), we obtain two different Stokes sectors for our system:

$$\Sigma_{2n+1} = \Sigma_1 = \left\{ \xi \in \mathbb{C} \mid 0 < |\xi| < R,\ -\frac{3\pi}{2} + \frac{\delta}{2} < \operatorname{Arg}(\xi) < \frac{\pi}{2} - \frac{\delta}{2} \right\} \tag{1.28}$$

$$\Sigma_{2n} = \Sigma_2 = \left\{ \xi \in \mathbb{C} \mid 0 < |\xi| < R,\ -\frac{\pi}{2} + \frac{\delta}{2} < \operatorname{Arg}(\xi) < \frac{3\pi}{2} - \frac{\delta}{2} \right\} \tag{1.29}$$

each containing exactly one Stokes ray, see Figure 1.2.



(a) Σ_1, the first Stokes sector, symmetric around the Stokes ray l_1 in the direction $\theta_1 = -\frac{\pi}{2}$.

(b) Σ_2, the second Stokes sector, symmetric around the Stokes ray l_2 in the direction $\theta_2 = \frac{\pi}{2}$.

Figure 1.2

1.3.3 Fundamental solutions in the Stokes sectors

We will now combine the results from the two previous sections, to obtain fundamental solutions in the sectors around the non-Fuchsian point, $z_2 = \infty$.

In order to be precise in the following construction, we first define the logarithm centered at a point z_j with an arbitrary branch cut $\alpha_j \neq 0 \mod 2\pi i$. Let $\lfloor\ \rfloor$ denote the floor function and find the whole number $n = \left\lfloor \frac{\alpha_j}{2\pi} \right\rfloor$. We round towards zero if $\alpha_j < 0$. Let $0 < x < R_j$ and denoting the branch at z_j with angle α_j by $b_{\alpha_j}^{(j)}$, we define ζ_z to be any path from $z_j + x$ to $z \in B(z_j, R_j) \setminus b_{\alpha_j}^{(j)}$ such that $\zeta_z \subset B(z_j, R_j) \setminus b_{\alpha_j}^{(j)}$, see Figure 1.3. Note that $B(z_j, R_j) \setminus b_{\alpha_j}^{(j)}$ is simply connected, so when we integrate along ζ_z, the integral is invariant of the choice of path between $z_j + x$ and z.

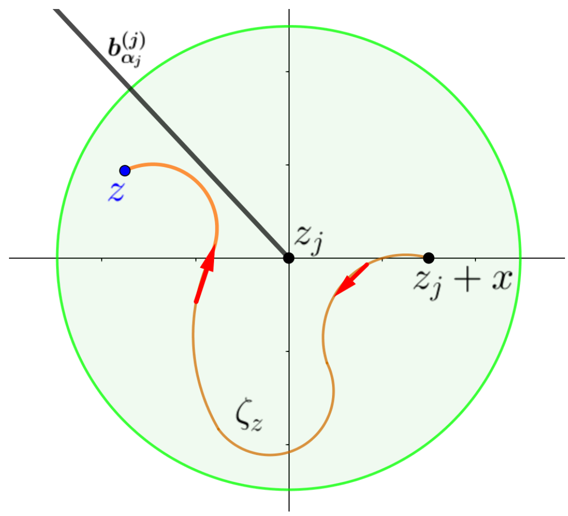


Figure 1.3: Definition of the logarithm with singularity at z_j, in an arbitrary branched neighbourhoods around z_j.

Definition 1.3.4 Logarithm with branch $\boldsymbol{\alpha_j}$ at $\mathbf{z_j}$.
Let $n = \left\lfloor \frac{\alpha_j}{2\pi} \right\rfloor$ rounded towards zero, and ζ_z be defined as above, see Figure 1.3. The logarithm with branch $\alpha_j \neq 0 \mod 2\pi i$, at z_j is defined by

$$\log_{\alpha_j} : \quad B(z_j, R_j) \setminus b_{\alpha_j}^{(j)} \quad \to \quad \mathbb{C}$$

$$\log_{\alpha_j}(z - z_j) := \int_{\zeta_z} \frac{1}{\omega - z_j} d\omega + \ln(x) + 2\pi i n$$

If we analytically continue the logarithm with branch cut α_j at z_j along a path η which encircles z_j once counter-clockwise, we obtain:

$$\big(\log_\alpha\big)_\eta(z) = \int_{\zeta_z * \eta} \frac{1}{\omega - z_j} d\omega + \ln(x) + 2\pi i n = \log_\alpha(z) + 2\pi i. \tag{1.30}$$

Hence the analytic continuation $\big(\log_\alpha\big)_\eta$ of the logarithm $\log_\alpha$, along a path η that encloses 0 is not a function in $B(0, R) \setminus \{0\}$.

We previously obtained the following formal solution in the whole punctured neighbourhood of $z_2 = \infty$, $z = \frac{1}{\xi}$:

$$\Phi^{(\infty)}(\xi) = P^{(\infty)} \left(\sum_{k=0}^{\infty} \Psi_k^{(\infty)} \xi^k \right) \exp\left(-\frac{\Lambda_{-1}^{(\infty)}}{\xi} + \Lambda_0^{(\infty)} \log(\xi) \right)$$

We consider the Stokes sector

$$\Sigma_1 = \left\{ \xi \in \mathbb{C} \mid 0 < |\xi| < R,\ -\frac{3\pi}{2} + \frac{\delta}{2} < \operatorname{Arg}(\xi) < \frac{\pi}{2} - \frac{\delta}{2} \right\}.$$

From the Definition of Σ_1, we have the subsectors Σ_1^- and Σ_1^+ which both has central angle less than $\pi/1 = \pi$. Thus by Theorem 1.3.1 we obtain solutions

$$\Phi_{\Sigma_1^-}^{(\infty)} = P^{(\infty)} \hat{\Psi}_1^- \exp\left(-\frac{\Lambda_{-1}^{(\infty)}}{\xi} + \Lambda_0^{(\infty)} \log_{\frac{\pi}{2}}(\xi) \right)$$

$$\Phi_{\Sigma_1^+}^{(\infty)} = P^{(\infty)} \hat{\Psi}_1^+ \exp\left(-\frac{\Lambda_{-1}^{(\infty)}}{\xi} + \Lambda_0^{(\infty)} \log_{\frac{\pi}{2}}(\xi) \right)$$

in the respective sectors. Notice that we now use Definition 1.3.4, and choose the branch cut $\alpha = \frac{\pi}{2}$, since this branch is outside the domain of both the functions. This coincides with the principal logarithm when the argument is less than $\frac{\pi}{2}$.

By Proposition 1.3.1 we have that the solutions $\Phi_{\Sigma_1^-}^{(\infty)}$ and $\Phi_{\Sigma_1^+}^{(\infty)}$ coincide in the subsector

$$\Sigma_1^- \cap \Sigma_1^+ = \{ \xi \in \mathbb{C} \mid 0 < |\, \xi < R|,\ \frac{-\pi - \delta}{2} < \operatorname{Arg}(\xi) < \frac{-\pi + \delta}{2} \},$$

hence we obtain that $\Phi_{\Sigma_1^+}^{(\infty)}$ is a direct analytic continuation of $\Phi_{\Sigma_1^-}^{(\infty)}$, and we can regard them as one solution defined in $\Sigma_1^- \cup \Sigma_1^+ = \Sigma_1$

$$\Phi_{\Sigma_1}^{(\infty)}(\xi) = P^{(\infty)} \hat{\Psi}_1(\xi) \exp\left(-\frac{\Lambda_{-1}^{(\infty)}}{\xi} + \Lambda_0^{(\infty)} \log_{\frac{\pi}{2}}(\xi) \right), \quad \xi \in \Sigma_1 \tag{1.31}$$

$$\Sigma_1 = \left\{ \xi \in \mathbb{C} \mid 0 < |\xi| < R,\ -\frac{3\pi}{2} + \frac{\delta}{2} < \operatorname{Arg}(\xi) < \frac{\pi}{2} - \frac{\delta}{2} \right\}.$$

We emphasis that

- $P^{(\infty)}$ is a constant matrix that diagonalizes $A_{-1}^{(\infty)}$. In our case the gauge representation (Definition 1.2.3) of the system (1.1) is chosen such that $P^{(\infty)} = I$. However, it is important for the analysis that we keep track of the matrices $P^{(i)}$, $i \in \{0, 1, \infty\}$.

- $\hat{\Psi}_1 : \Sigma_1 \to GL_2(\mathbb{C})$ is a holomorphic function, with asymptotic expansion equal to the series expression in the formal solution (1.21). Remark that similarly to the logarithmic function, the function $\hat{\Psi}_1 : \Sigma_1 \to GL_2(\mathbb{C})$ is defined on a branched cut neighbourhood of $\xi = 0$, a singular point of the function. Indeed, more hard analysis work can show that $\hat{\Psi}_1$ can be analytically extended as we let $\delta \to 0$, in Definition 1.3.3 of the Stokes sector. Thus obtaining a branched neighbourhood domain. As we will show in Section 1.3.4, the function can be analytically continued around $\xi = 0$, to a multivalued function. The graph of its analytic continuation, is a Riemann surface $\Gamma \subset \Sigma_1 \times GL_2(\mathbb{C})$, with similar traits as the Riemann surface of the logarithm, see Figure 1.7.

- The logarithm in the exponential term has a branch cut chosen to be $\alpha = \frac{\pi}{2}$ which is outside the domain. Hence the logarithm is defined in the whole domain of the function $\Phi^{(\infty)}_{\Sigma_1}$.

- The solution $\Phi^{(\infty)}_{\Sigma_1}$ is the unique fundamental solution to the system (1.15) in the Stokes sector Σ_1 with asymptotic expansion equal to the formal series solution (1.21) and logarithm branch $\frac{\pi}{2}$. In the formula for the coefficients of the formal series, there is a choice of the matrix $\Psi^{(\infty)}_0 = I$. Different choices gives different series expressions, however they are all in the end related by right multiplication of constant matrix, by Lemma 1.2.2.

Similarly we find a fundamental solution in the Stokes sector

$$\Sigma_2 = \left\{ \xi \in \mathbb{C} \mid 0 < |\xi| < R,\ -\frac{\pi}{2} + \frac{\delta}{2} < \mathrm{Arg}(\xi) < \frac{3\pi}{2} - \frac{\delta}{2} \right\}$$

$$\Phi^{(\infty)}_{\Sigma_2} = P^{(\infty)} \hat{\Psi}_2(\xi) \exp\left(-\frac{\Lambda^{(\infty)}_{-1}}{\xi} + \Lambda^{(\infty)}_0 \log_{\frac{3\pi}{2}}(\xi) \right), \quad \xi \in \Sigma_2 \tag{1.32}$$

The solution $\Phi^{(\infty)}_{\Sigma_2}$ is the unique fundamental solution to the system (1.15) in the Stokes sector Σ_2 with asymptotic expansion equal to the formal series solution (1.21) and logarithm branch $\frac{3\pi}{2}$.

Together the two solutions $\Phi^{(\infty)}_{\Sigma_1}, \Phi^{(\infty)}_{\Sigma_2}$ in the respective sectors give solutions at every point in the punctured neighbourhood of $\xi = 0$. We notice that the differences in the expressions for the functions are the holomorphic functions, $\hat{\Psi}_1$ and $\hat{\Psi}_2$, obtained from Theorem 1.3.1, and the branch of the logarithm.

1.3.4 Analytic continuation around a non-Fuchsian singular point

The goal is to obtain an analytic continuation of the solution $\Phi^{(\infty)}_{\Sigma_1}$ along paths encircling the singular point $\xi = 0$. This will in general not give us a function, since if a path ν_∞ goes once counter-clockwise around $\xi = 0$, the analytic continuation of the logarithm stops being a function, see equation (1.30).

Consider the solutions $\Phi^{(\infty)}_{\Sigma_1}$ and $\Phi^{(\infty)}_{\Sigma_2}$ obtained in the Stokes sectors Σ_1 and Σ_2 respectively, see Section 1.3.3. The Stokes sectors intersect into two connected components $U_1 = \Sigma_1 \cap \Sigma_2 \cap \{\, \mathrm{Re}(\xi) > 0 \,\}$ and $U_2 = \Sigma_1 \cap \Sigma_2 \cap \{\, \mathrm{Re}(\xi) < 0 \,\}$:

$$U_1 = \left\{ \xi \in \mathbb{C} \,\middle|\, 0 < |\xi| < R,\ -\frac{\pi}{2} + \frac{\delta}{2} < \mathrm{Arg}(\xi) < \frac{\pi}{2} - \frac{\delta}{2} \right\}, \tag{1.33}$$

$$U_2 = \left\{ \xi \in \mathbb{C} \,\middle|\, 0 < |\xi| < R,\ \frac{\pi}{2} + \frac{\delta}{2} < \mathrm{Arg}(\xi) < \frac{3\pi}{2} - \frac{\delta}{2} \right\}, \tag{1.34}$$

see Figure 1.4.

We will now analytically continue the solution $\Phi_{\Sigma_1}^{(\infty)}$ along the loop ν_∞, encircling $\xi = 0$ once counter-clockwise. At first we are interested in the sector U_1, since the path ν_∞ starts at $x = \nu_\infty(0)$ and moves counter-clockwise around $\xi = 0$. We expect in general the two solution $\Phi_{\Sigma_1}^{(\infty)}$, $\Phi_{\Sigma_2}^{(\infty)}$ to be different, see the Remark following Theorem 1.3.1. But the solutions solve the same linear differential equation in a common domain, so by Lemma 1.2.2 we can find a constant invertible matrix S_1 such that

$$\Phi_{\Sigma_2}^{(\infty)} = \Phi_{\Sigma_1}^{(\infty)} S_1, \quad \text{in } U_1.$$

The branch-cuts of the logarithms in $\Phi_{\Sigma_1}^{(\infty)}$ and $\Phi_{\Sigma_2}^{(\infty)}$ were chosen outside the respective sector domains. Thus the branch cuts are not in U_1 either. Actually we notice that in U_1 the logarithms $\log_{\frac{\pi}{2}}$ and $\log_{\frac{3\pi}{2}}$ coincide (see Definition 1.3.4):

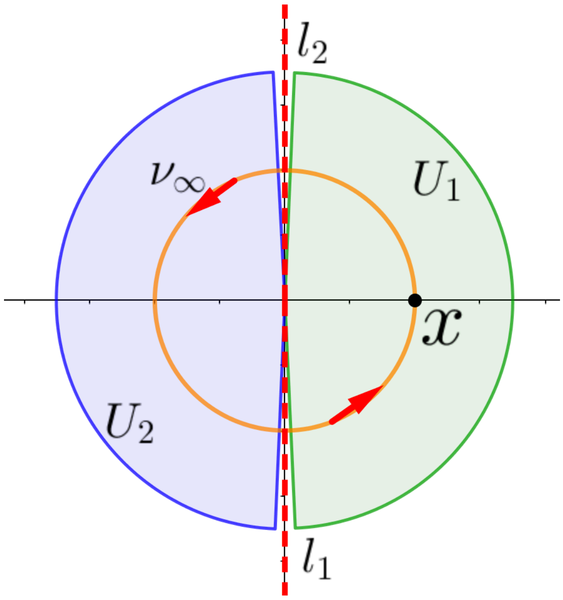


Figure 1.4: The two connected components U_1 and U_2 of the intersection $\Sigma_1 \cap \Sigma_2$ and the loop ν_∞.

$$\log_{\frac{\pi}{2}}(\xi) := \int_x^\xi \frac{1}{\omega} d\omega + \ln(x), \quad \xi \in \Sigma_1$$

$$\log_{\frac{3\pi}{2}}(\xi) := \int_x^\xi \frac{1}{\omega} d\omega + \ln(x), \quad \xi \in \Sigma_2$$

where the integrals can be taken along any path between x and ξ since the sectors are simply connected.

We can now construct an analytic continuation of the function $\Phi_{\Sigma_1}^{(\infty)}$ along ν_∞, where the first step involves that $\Phi_{\Sigma_2}^{(\infty)} S_1^{-1}$ is a direct analytic continuation of $\Phi_{\Sigma_1}^{(\infty)}$ since they coincide in the sector U_1. See Appendix B for details on analytic continuation. We remark that the analytic continuation is now defined in Σ_1 and Σ_2. In $U_1 \subset \Sigma_1 \cap \Sigma_2$ the two expressions coincide, but in U_2 they will in general not coincide, if they did, that would again make $\xi = 0$ into a removable singularity.

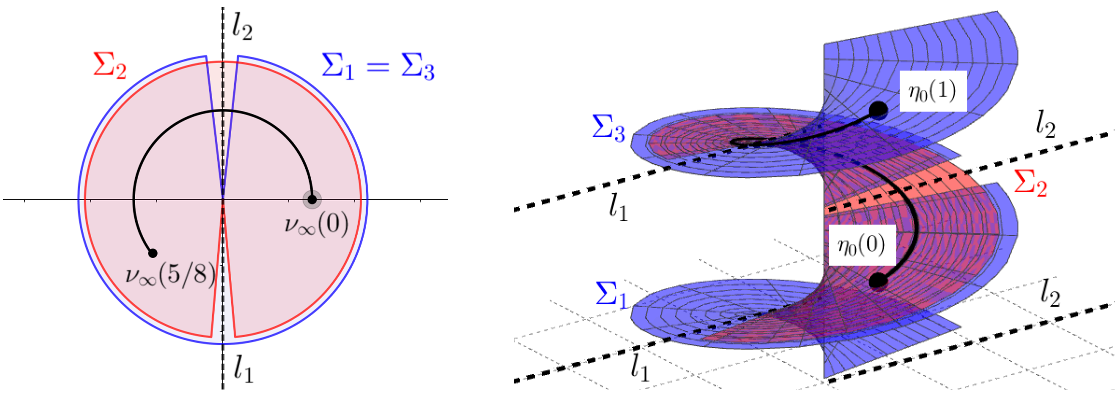


Figure 1.5: Here we see the Stokes sectors Σ_1, Σ_2 and Σ_3 and the loop ν_∞ from two different perspectives. In the right illustration, the multivalued argument function is plotted on the vertical axis.

If we want to complete the analytic continuation along ν_∞ we need to find a function defined in a sector that intersects Σ_2 and contains the Stokes ray $l_3 = l_1$, pointed in the direction $\theta_3 = \frac{3\pi}{2}$. Also the function need to coincide with $\Phi^{(\infty)}_{\Sigma_2}$ in a common subsector. The obvious choice for the sector is Σ_3, see Figure 1.5. In particular the second connected component U_2 will be the common subsector of Σ_2 and Σ_3. However the function $\Phi^{(\infty)}_{\Sigma_1}$ will obviously not work, since the analytic continuation of the logarithm (outlined above) has a term $+2\pi i$ added when continued along ν_∞ back into Σ_1. We thus try the function

$$\Phi^{(\infty)}_{\Sigma_3}(\xi) := P^{(\infty)}\hat{\Psi}_1(\xi)\exp\left(-\frac{\Lambda^{(\infty)}_{-1}}{\xi} + \Lambda^{(\infty)}_0(\log_{\frac{\pi}{2}}(\xi) + 2\pi i)\right), \quad \xi \in \Sigma_3 \tag{1.35}$$

$$\Sigma_3 = \left\{\xi \in \mathbb{C} \mid 0 < |\xi| < R,\ \frac{\pi}{2} + \frac{\delta}{2} < \operatorname{Arg}(\xi) < \frac{5\pi}{2} - \frac{\delta}{2}\right\} = \Sigma_1$$

We see that this function is related to $\Phi^{(\infty)}_{\Sigma_1}$ simply by

$$\Phi^{(\infty)}_{\Sigma_3} = \Phi^{(\infty)}_{\Sigma_1}\exp\left(2\pi i\Lambda^{(\infty)}_0\right). \tag{1.36}$$

This shows that $\Phi^{(\infty)}_{\Sigma_3}$ solves (1.15), since it is a right multiplication by a constant invertible matrix of a known solution. By Proposition 1.3.1 we actually obtain immediately that this is the unique solution in Σ_3 that has the asymptotic expansion

$$\Phi^{(\infty)}(\xi) = P^{(\infty)}\left(\sum_{k=0}^{\infty}\Psi^{(\infty)}_k\xi^k\right)\exp\left(-\frac{\Lambda^{(\infty)}_{-1}}{\xi} + \Lambda^{(\infty)}_0\left(\log_{\frac{\pi}{2}}(\xi) + 2\pi i\right)\right).$$

Hence since $\Phi^{(\infty)}_{\Sigma_2}$ and $\Phi^{(\infty)}_{\Sigma_3}$ both solve the same differential equation in U_2 (1.34), we have by Lemma 1.2.2 that there exists a constant invertible matrix S_2 such that

$$\Phi^{(\infty)}_{\Sigma_3} = \Phi^{(\infty)}_{\Sigma_2}S_2, \quad \text{in } U_2.$$

Hence $\Phi^{(\infty)}_{\Sigma_3}S_2^{-1}$ is a direct analytic continuation of $\Phi^{(\infty)}_{\Sigma_2}$, and we thus obtain that the analytic continuation of $\Phi^{(\infty)}_{\Sigma_1}$ along ν_∞ (see Definition B.1.5) is

$$\left\{\left(\Sigma_1, \Phi^{(\infty)}_{\Sigma_1}\right), \left(\Sigma_2, \Phi^{(\infty)}_{\Sigma_2}S_1^{-1}\right), \left(\Sigma_3, \Phi^{(\infty)}_{\Sigma_1}\exp\left(2\pi i\Lambda^{(\infty)}_0\right)S_2^{-1}S_1^{-1}\right)\right\}. \tag{1.37}$$

In particular, in Σ_3 we have

$$\left(\Phi^{(\infty)}_{\Sigma_1}\right)_{\nu_\infty} = \Phi^{(\infty)}_{\Sigma_3}S_2^{-1}S_1^{-1} = \Phi^{(\infty)}_{\Sigma_1}\exp\left(2\pi i\Lambda^{(\infty)}_0\right)S_2^{-1}S_1^{-1} \tag{1.38}$$

in the neighbourhood Σ_3 of $\nu_\infty(1)$. This function does again solve the differential equation (1.15) in Σ_3 since it is a right multiplication by a constant matrix of a known solution.

Remark.
It is pedagogical to denote the sector domain of $\Phi^{(\infty)}_{\Sigma_3}$ by Σ_3 and the angles defining the sector by the angles one would use to describe the path starting at $\xi = 1 = e^{2\pi i 0}$, encircling $\xi = 0$ and ending at $\xi = 1 = e^{2\pi i}$. This is because later, the domain $\mathbb{S} \setminus \{0, t, \infty\}$ will be exchanged with its universal covering space, constructed using the path-space construction in Theorem B.2.1.

The construction of $\Phi^{(\infty)}_{\Sigma_3}$ in the Stokes sector Σ_3 can be generalized to all Stokes sectors Σ_n, summarized in the following Theorem:

Theorem 1.3.2 Unique fundamental solutions in every Stokes sector. *Consider the system* (1.15) *with the formal solution* (1.21)*:*

$$\Phi^{(\infty)}(\xi) = P^{(\infty)}\left(\sum_{k=0}^{\infty}\Psi_k^{(\infty)}\xi^k\right)\exp\left(-\frac{\Lambda_{-1}^{(\infty)}}{\xi} + \Lambda_0^{(\infty)}\log_\alpha(\xi)\right).$$

In the odd numbered Stokes sectors (1.28)

$$\Sigma_{2n+1} = \left\{\xi\in\mathbb{C} \mid 0<|\xi|<R,\ \frac{(4n-3)\pi}{2}+\frac{\delta}{2}<\operatorname{Arg}(\xi)<\frac{(4n+1)\pi}{2}-\frac{\delta}{2}\right\}$$

there exists a unique fundamental solution to (1.15) *with the asymptotic expansion equal to the formal solution with logarithm* $\log_{\frac{\pi}{2}}(\xi)+2\pi i n$*. The solutions are given by*

$$\Phi_{\Sigma_{2n+1}}^{(\infty)} = \Phi_{\Sigma_1}^{(\infty)}\exp\left(2\pi i n\Lambda_0^{(\infty)}\right),\quad \text{in } \Sigma_{2n+1},\ n\in\mathbb{Z} \tag{1.39}$$

$$\Phi_{\Sigma_{2n+1}}^{(\infty)}(\xi) = P^{(\infty)}\hat{\Psi}_1(\xi)\exp\left(-\frac{\Lambda_{-1}^{(\infty)}}{\xi}+\Lambda_0^{(\infty)}\left(\log_{\frac{\pi}{2}}(\xi)+2\pi i n\right)\right), \xi\in\Sigma_{2n+1}, n\in\mathbb{Z}$$

In the even numbered Stokes sectors (1.29)

$$\Sigma_{2n+2} = \left\{\xi\in\mathbb{C} \mid 0<|\xi|<R,\ \frac{(4n-1)\pi}{2}+\frac{\delta}{2}<\operatorname{Arg}(\xi)<\frac{(4n+3)\pi}{2}-\frac{\delta}{2}\right\}$$

there exists a unique fundamental solution to (1.15) *with the asymptotic expansion equal to the formal solution with logarithm* $\log_{\frac{3\pi}{2}}(\xi)+2\pi i n$*. The solutions are given by*

$$\Phi_{\Sigma_{2n+2}}^{(\infty)} = \Phi_{\Sigma_2}^{(\infty)}\exp\left(2\pi i n\Lambda_0^{(\infty)}\right),\quad \text{in } \Sigma_{2n+2},\ n\in\mathbb{Z} \tag{1.40}$$

$$\Phi_{\Sigma_{2n+2}}^{(\infty)}(\xi) = P^{(\infty)}\hat{\Psi}_2(\xi)\exp\left(-\frac{\Lambda_{-1}^{(\infty)}}{\xi}+\Lambda_0^{(\infty)}\left(\log_{\frac{3\pi}{2}}(\xi)+2\pi i n\right)\right), \xi\in\Sigma_{2n+2}, n\in\mathbb{Z}.$$

Moreover in any two consecutive Stokes sectors Σ_m *and* Σ_{m+1}*, the intersection is given by two connected components* U_1 *and* U_2 *(see equation* (1.33), (1.34) *and Figure 1.4):*

$$U_1 = \Sigma_m\cap\Sigma_{m+1}\cap\{\operatorname{Re}(\xi)>0\},\qquad U_2 = \Sigma_m\cap\Sigma_{m+1}\cap\{\operatorname{Re}(\xi)<0\}$$

- *If* $m=2n+1$ *is odd we have*

  $$\Phi_{\Sigma_{2n+2}}^{(\infty)} = \Phi_{\Sigma_{2n+1}}^{(\infty)}S_{2n+1}\ \text{in } U_1.$$

  where S_{2n+1} *is a constant invertible matrix given by the formula*

  $$S_{2n+1} = \exp\left(-2\pi i n\Lambda_0^{(\infty)}\right)S_1\exp\left(2\pi i n\Lambda_0^{(\infty)}\right)$$

  and

  $$S_1 := \left(\Phi_{\Sigma_1}^{(\infty)}\right)^{-1}\Phi_{\Sigma_2}^{(\infty)},\quad \text{in } U_1.$$

- *If* $m=2n+2$ *is even we have*

  $$\Phi_{\Sigma_{2n+3}}^{(\infty)} = \Phi_{\Sigma_{2n+2}}^{(\infty)}S_{2n+2}\ \text{in } U_2,$$

  where S_{2n+2} *is a constant invertible matrix given by the formula*

  $$S_{2n+2} = \exp\left(-2\pi i n\Lambda_0^{(\infty)}\right)S_2\exp\left(2\pi i n\Lambda_0^{(\infty)}\right)$$

  and

  $$S_2 := \left(\Phi_{\Sigma_2}^{(\infty)}\right)^{-1}\Phi_{\Sigma_3}^{(\infty)} = \left(\Phi_{\Sigma_2}^{(\infty)}\right)^{-1}\Phi_{\Sigma_1}^{(\infty)}\exp\left(2\pi i\Lambda_0^{(\infty)}\right),\quad \text{in } U_2.$$

Proof. We present the proof assuming $m = 2n+1$ is odd, similar arguments hold when m is even.

The existence of a fundamental solution with the given asymptotic expansion in a sector $\Sigma_m = \Sigma_{2n+1}$ follows from the fact that: $\Phi^{(\infty)}_{\Sigma_{2n+1}} := \Phi^{(\infty)}_{\Sigma_1} \exp\left(2\pi i n \Lambda_0^{(\infty)}\right)$ has the given asymptotic expansion, and it also solves the differential equation (1.15) since it is a right multiplication by a constant matrix of a known solution, see (1.31). The uniqueness is given by Proposition 1.3.1.

The fact that

$$\Phi^{(\infty)}_{\Sigma_{2n+2}} = \Phi^{(\infty)}_{\Sigma_{2n+1}} S_{2n+1} \text{ in } U_1,$$

for some $S_{2n+1} \in GL_2(\mathbb{C})$ is just Lemma 1.2.2. To obtain the formula we remark that in U_1 the following equations hold:

$$\Phi^{(\infty)}_{\Sigma_{2n+2}} = \Phi^{(\infty)}_{\Sigma_{2n+1}} S_{2n+1} = \Phi^{(\infty)}_{\Sigma_1} \exp\left(2\pi i n \Lambda_0^{(\infty)}\right) S_{2n+1}$$

$$\Phi^{(\infty)}_{\Sigma_{2n+2}} = \Phi^{(\infty)}_{\Sigma_2} \exp\left(2\pi i n \Lambda_0^{(\infty)}\right) = \Phi^{(\infty)}_{\Sigma_1} S_1 \exp\left(2\pi i n \Lambda_0^{(\infty)}\right)$$

We equate the expressions and cancel the fundamental solution $\Phi^{(\infty)}_{\Sigma_1}$. Here it is important that it is a fundamental solution, in the sense that it is an invertible matrix. Solving for S_{2n+1} we obtain $S_{2n+1} = \exp\left(-2\pi i n \Lambda_0^{(\infty)}\right) S_1 \exp\left(2\pi i n \Lambda_0^{(\infty)}\right)$. ∎

1.3.5 Stokes matrices and Stokes parameters

Definition 1.3.5 Stokes matrices.
The constant matrices S_m, relating the solution between consecutive Stokes sectors Σ_m and Σ_{m+1} in Theorem 1.3.2 are called Stokes matrices.

Proposition 1.3.2 Form of Stokes matrices.
A Stokes matrix S_m is always in triangular form

$$S_m = \begin{pmatrix} 1 & s_m \\ 0 & 1 \end{pmatrix} \text{ or } S_m = \begin{pmatrix} 1 & 0 \\ s_m & 1 \end{pmatrix}$$

If S_{2n+1} is upper triangular then S_{2n+2} will be lower triangular. Similarly if S_{2n+2} is upper triangular then S_{2n+1} will be lower triangular.

Fixing a choice of the diagonalizing matrix $P^{(\infty)}$, fixes the Stokes matrices.Permuting the eigenvectors in $P^{(\infty)}$ changes S_m from upper to lower triangular (respectively from lower to upper triangular).

Remark.
The statement of this Proposition is slightly different for Stokes matrices of dimension higher then 2. Then we may need to permute the eigenvalues of the matrix $\Lambda_{-1}^{(\infty)}$ and accordingly permute the eigenvectors in the matrix $P^{(\infty)}$, in order to put the Stokes matrix into a triangular form. See [Fok et al.06, p.56-57] for details.

Proof. The proof has a similar setup as Proposition 1.3.1. By Theorem 1.3.2 we have that the solutions are related by

$$\Phi^{(\infty)}_{\Sigma_{m+1}} = \Phi^{(\infty)}_{\Sigma_m} S_m$$

in the subsector $U = U_1$ or $U = U_2$ (see equation (1.33) and (1.34) and Figure 1.4) depending on whether m is odd or even. We also know that the logarithms in $\Phi^{(\infty)}_{\Sigma_{2n+1}}$ and $\Phi^{(\infty)}_{\Sigma_{2n+2}}$ coincide in U_1 and $\Phi^{(\infty)}_{\Sigma_{2n+2}}$ and $\Phi^{(\infty)}_{\Sigma_{2n+3}}$ coincide in U_2.

The solutions also share the same asymptotic expansion given by the formal solution:

$$\Phi^{(\infty)}(\xi) = P^{(\infty)} \left(\sum_{k=0}^{\infty} \Psi_k^{(\infty)} \xi^k \right) \exp\left(-\frac{\Lambda_{-1}^{(\infty)}}{\xi} + \Lambda_0^{(\infty)} \log_{\alpha_m}(\xi) \right).$$

We consider the limit $S_m = \lim_{\xi \to 0} S_m = \lim_{\xi \to 0} \left(\Phi_{\Sigma_m}^{(\infty)} \right)^{-1} \Phi_{\Sigma_{m+1}}^{(\infty)}$ where ξ is kept in U. The left equality comes from the fact that S_m is a constant matrix, see Lemma 1.2.2. We do the same calculation as in Proposition 1.3.1, and obtain equation (1.26):

$$S_m = \lim_{\xi \to 0} \begin{pmatrix} 1 + O(\xi) & e^{(\Lambda(\xi))_{22} - (\Lambda(\xi))_{11}} O(\xi) \\ e^{(\Lambda(\xi))_{11} - (\Lambda(\xi))_{22}} O(\xi) & 1 + O(\xi) \end{pmatrix} = \begin{pmatrix} 1 & S_{12} \\ S_{21} & 1 \end{pmatrix}. \tag{1.41}$$

Again by the same argument as in Proposition 1.3.1, the terms S_{12} and S_{21} are zero if and only if the exponential function has non-positive exponent. That is,

$$\lim_{\xi \to 0} (S_m)_{12} = 0 \iff \operatorname{Re}\left(-\frac{\left(\Lambda_{-1}^{(\infty)}\right)_{22} - \left(\Lambda_{-1}^{(\infty)}\right)_{11}}{\xi} \right) \leq 0$$

Since we do not have any Stokes rays in U, we know that by the Definition 1.3.2 of Stokes rays,

$$\operatorname{Re}\left(-\frac{\left(\Lambda_{-1}^{(\infty)}\right)_{22} - \left(\Lambda_{-1}^{(\infty)}\right)_{11}}{\xi} \right) = a_{12}(\xi) \neq 0.$$

Hence we have by continuity of a_{12}, connectivity of U and the intermediate value Theorem that either $a_{12}(\xi) < 0$, in U or $a_{12} > 0$, in U. If we compare with $(S_m)_{21}$ we get the same relation, but with opposite sign dependency:

$$\lim_{\xi \to 0} (S_m)_{21} = 0 \iff \operatorname{Re}\left(-\frac{\left(\Lambda_{-1}^{(\infty)}\right)_{11} - \left(\Lambda_{-1}^{(\infty)}\right)_{22}}{\xi} \right) < 0.$$

We arrive at the following conclusion:

$$a_{12}(\xi) = \operatorname{Re}\left(-\frac{\left(\Lambda_{-1}^{(\infty)}\right)_{22} - \left(\Lambda_{-1}^{(\infty)}\right)_{11}}{\xi} \right) < 0 \tag{1.42}$$

at one point in U (and consequently at every point in U), if and only if S_m have the form: $S_m = \begin{pmatrix} 1 & 0 \\ s_m & 1 \end{pmatrix}$. If not then

$$-a_{12}(\xi) = \operatorname{Re}\left(-\frac{\left(\Lambda_{-1}^{(\infty)}\right)_{11} - \left(\Lambda_{-1}^{(\infty)}\right)_{22}}{\xi} \right) < 0, \text{ in } U$$

and $S_m = \begin{pmatrix} 1 & s_m \\ 0 & 1 \end{pmatrix}$. Notice that the above condition (1.42) on a_{12} does not depend on m. This proves that we always have a triangular form.

We show the second statement of Theorem. If S_m is upper triangular and m is odd, then $U = U_1 = \Sigma_m \cap \Sigma_{m+1} \cap \{ \operatorname{Re}(\xi) > 0 \}$ and $-a_{12} < 0$ in U_1. Then $m+1$ is even so the formula

$$S_{m+1} = \left(\Phi_{\Sigma_{m+1}}^{(\infty)} \right)^{-1} \Phi_{\Sigma_{m+2}}^{(\infty)}$$

holds in $U_2 = \Sigma_m \cap \Sigma_{m+1} \cap \{\operatorname{Re}(\xi) < 0\}$, and we will get the same condition (1.42), but for S_{m+1}. In U_2, $a_{12} < 0$, since a_{12} will change sign exactly when going from U_1 to U_2, this follows directly from Definition 1.3.2 and since the Stokes rays are "between" U_1 and U_2, see Figure 1.4. Hence $a_{12} < 0$ in U_2 and S_{m+1} will be lower triangular. The result now follows by induction.

The dependency on $P^{(\infty)}$ follows from the fact that $A_{-1}^{(\infty)} = P^{(\infty)}\Lambda_{-1}^{(\infty)}(P^{(\infty)})^{-1}$. So changing the order of the columns in $P^{(\infty)}$, changes the index of the eigenvalues $(\Lambda_{-1}^{(\infty)})_{jj}$. ■

Definition 1.3.6 Stokes parameters.
The complex number s_m, determining the Stokes matrix

$$S_m = \begin{pmatrix} 1 & 0 \\ s_m & 1 \end{pmatrix} \text{ or } S_m = \begin{pmatrix} 1 & s_m \\ 0 & 1 \end{pmatrix}$$

is called a Stokes parameter.

Using the expression $\Lambda_{-1}^{(\infty)} = -\frac{1}{2}\sigma_3 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ into Proposition 1.3.2, we obtain the Stokes matrices S_1 and S_2 as:

$$S_1 = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix} \tag{1.43}$$

The other Stokes matrices are obtained through the relations from Theorem 1.3.2:

$$S_{2n+1} = \exp\left(-2\pi i n \Lambda_0^{(\infty)}\right) S_1 \exp\left(2\pi i n \Lambda_0^{(\infty)}\right), \quad n \in \mathbb{Z}$$

$$S_{2n+2} = \exp\left(-2\pi i n \Lambda_0^{(\infty)}\right) S_2 \exp\left(2\pi i n \Lambda_0^{(\infty)}\right), \quad n \in \mathbb{Z}.$$

By computing the above matrix products we obtain similar relations for the Stokes parameters:

$$s_{2n+1} = s_1 e^{2\pi i n\left((\Lambda_0^{(\infty)})_{11} - (\Lambda_0^{(\infty)})_{22}\right)} \qquad s_{2n+2} = s_2 e^{2\pi i n\left((\Lambda_0^{(\infty)})_{22} - (\Lambda_0^{(\infty)})_{11}\right)} \tag{1.44}$$

We see by the above discussion that we only need two consecutive Stokes parameters, e.g. s_1 and s_2, to determine all the Stokes matrices, and thus glue together all the solutions from Theorem 1.3.2 in the Stokes sectors around $\xi = 0$.

1.3.6 Summary of solutions around the non-Fuchsian singular point

The goal was to find a local solution of the first order linear system of differential equations:

$$\frac{d\Phi}{d\xi}\Phi(\xi)^{-1} = \frac{A_{-1}^{(\infty)}}{\xi^2} + \frac{A_0^{(\infty)}}{\xi} + \sum_{k=0}^{\infty} A_{k+1}^{(\infty)}\xi^k \tag{1.45}$$

around $\xi = 0$.

$$A_{-1}^{(\infty)} = P^{(\infty)}\Lambda_{-1}^{(\infty)}P^{(\infty)-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{2}\sigma_3$$

where σ_3 is the famous third Pauli-matrix.

In Section 1.2.6 we found a formal solution (1.21) of the system (1.45) in a branched neighbourhood of $\xi = \frac{1}{z} = 0$.

$$\Phi^{(\infty)}(\xi) = P^{(\infty)}\left(I + \sum_{k=1}^{\infty}\Psi_k^{(\infty)}\xi^k\right)\exp\left(-\frac{\Lambda_{-1}^{(\infty)}}{\xi} + \Lambda_0^{(\infty)}\log_\alpha(\xi)\right)$$

- The formal solution involves a choice of the eigenvector matrix $P^{(\infty)}$ which diagonalizes the matrix $A_{-1}^{(\infty)} = P^{(\infty)}\Lambda_{-1}^{(\infty)}P^{(\infty)-1}$. We chose $P^{(\infty)} = I$.

- The formal solution involves a formal series, which does not converge. The coefficients of the series is uniquely determined by the recurrent formulas in Section 1.2.6, up to the choice of the first matrix $\Psi_0^{(\infty)}$, which we chose as $\Psi_0^{(\infty)} = I$. If we change this matrix, the recurrent formulas generate a different series (still divergent), and the two solutions are related by right multiplication by a constant matrix by Lemma 1.2.2.

- Finally there is a choice of the branch cut for the logarithm which appears in the formula. When finding the solutions of (1.45) which actually exist, we use several different branches to obtain several distinct solutions in sectors covering the punctured neighbourhood of $\xi = 0$.

Definition 1.3.7 Canonical formal solution.
The formal solution outlined above, found in Section 1.2.6 is called the canonical formal solution of system (1.45).

Theorem 1.3.1 gives us the existence of a fundamental solution in sectors of central angle less than π, such that the formal solution is an asymptotic expansion of this fundamental solution. Further Proposition 1.3.1 says that if there is exactly one Stokes ray in the domain of the solution given by Theorem 1.3.1, then this solution is the only solution in its domain with the formal solution as an asymptotic expansion. Remark that here we fix a branch cut for the logarithm in the formal solution.

The Stokes sectors are defined with the goal of being the largest sectorial domain where Theorem 1.3.1 give a unique solution, given a formal solution with a fixed logarithmic branch.

Definition 1.3.8 Canonical fundamental solution in Stokes sectors.
The solutions

$$\Phi_{\Sigma_{2n+1}}^{(\infty)} = P^{(\infty)}\hat{\Psi}_1(\xi)\exp\left(-\frac{\Lambda_{-1}^{(\infty)}}{\xi} + \Lambda_0^{(\infty)}\left(\log_{\frac{\pi}{2}}(\xi) + 2\pi i n\right)\right), \quad \xi \in \Sigma_{2n+1},\ n \in \mathbb{Z}$$

$$\Phi_{\Sigma_{2n+2}}^{(\infty)} = P^{(\infty)}\hat{\Psi}_2(\xi)\exp\left(-\frac{\Lambda_{-1}^{(\infty)}}{\xi} + \Lambda_0^{(\infty)}\left(\log_{\frac{3\pi}{2}}(\xi) + 2\pi i n\right)\right), \ \xi \in \Sigma_{2n+2},\ n \in \mathbb{Z}.$$

defined in the Stokes sectors

$$\Sigma_{2n+1} = \Sigma_1 = \left\{\xi \in \mathbb{C} \mid 0 < |\xi| < R,\ -\frac{3\pi}{2} + \frac{\delta}{2} < \operatorname{Arg}(\xi) < \frac{\pi}{2} - \frac{\delta}{2}\right\}.$$

$$\Sigma_{2n+2} = \Sigma_2 = \left\{\xi \in \mathbb{C} \mid 0 < |\xi| < R,\ -\frac{\pi}{2} + \frac{\delta}{2} < \operatorname{Arg}(\xi) < \frac{3\pi}{2} - \frac{\delta}{2}\right\}$$

are called the canonical fundamental solutions in the Stokes sectors Σ_{2n+1} and Σ_{2n+2} respectively.

The canonical fundamental solutions are of course, as discussed above, uniquely determined by the canonical formal solution. By Theorem 1.3.2, these two solutions can be used to generate all fundamental solutions in sectorial domains at $\xi = 0$ such that the formal solution is its asymptotic expansion, with the correct branch cut for the logarithm.

Further, given a solution in a Stokes sector,we can do an analytic continuation along any path in the punctured neighbourhood $B(0, R)\setminus\{0\}$, by utilizing the Stokes matrices which connect the solutions:

- If $m = 2n+1$ is odd we have

$$\Phi^{(\infty)}_{\Sigma_{2n+2}} = \Phi^{(\infty)}_{\Sigma_{2n+1}} S_{2n+1}, \qquad \text{in } U_1.$$

where S_{2n+1} is a constant invertible matrix given by the formula

$$S_{2n+1} = \exp\left(-2\pi i n \Lambda_0^{(\infty)}\right) S_1 \exp\left(2\pi i n \Lambda_0^{(\infty)}\right), \quad S_1 = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}$$

- If $m = 2n+2$ is even we have

$$\Phi^{(\infty)}_{\Sigma_{2n+3}} = \Phi^{(\infty)}_{\Sigma_{2n+2}} S_{2n+2}, \qquad \text{in } U_2$$

where S_{2n+2} is a constant invertible matrix given by the formula

$$S_{2n+2} = \exp\left(-2\pi i n \Lambda_0^{(\infty)}\right) S_2 \exp\left(2\pi i n \Lambda_0^{(\infty)}\right), \quad S_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}$$

1.3.7 The Stokes phenomenon

The Stokes phenomenon is the situation where two different solutions of a differential equation admit the same asymptotic expansion [Fok et al.06].

In our case we showed that each connected component of the intersection between Stokes sectors, is a sector without any Stokes rays. As Proposition 1.3.2 shows, the Stokes matrices which relate the solutions in this sector is in general not the identity. Following [Fok et al.06], we make the following Definition:

Definition 1.3.9 Stokes phenomenon data.
Given the canonical formal solution of (1.1) around a non-Fuchsian singular point z_0 of Poincaré rank $r > 0$, and the canonical fundamental solutions in the Stokes sectors. The set

$$\mathbf{SPh}_{\mathbf{z_0}} := \{\, \Lambda^{(z_0)}_{-r}, \ldots, \Lambda^{(z_0)}_{-1}, \Lambda^{(z_0)}_{0}; S_1, S_2, \ldots, S_{2r} \,\}$$

is called the Stokes phenomenon data corresponding to a non-Fuchsian singular point z_0.

In the case of (1.45)

$$\mathbf{SPh}_\infty = \{\, \Lambda^{(\infty)}_{-1}, \Lambda^{(\infty)}_{0}; S_1, S_2 \,\} \tag{1.46}$$

Proposition 1.3.3 [Fok et al.06, Th. 1.1.5].
Consider two systems of differential equations

$$\frac{d\Phi}{dz} = \mathcal{A}(z)\Phi(z), \qquad \frac{d\Upsilon}{dz} = \mathcal{B}(z)\Upsilon(z)$$

that have the same point $z_0 \in \mathbb{S}$ as a irregular singular point, both of Poincaré rank $r > 0$. Moreover, suppose that

$$\mathbf{SPh}_{\mathbf{z_0}}(\mathcal{A}) = \mathbf{SPh}_{\mathbf{z_0}}(\mathcal{B})$$

Then the above systems are locally gauge equivalent, i.e.,

$$\mathcal{A} = g\mathcal{B}g^{-1} + \frac{dg}{dz}g^{-1}, \quad \textit{in } B(z_0, R).$$

The Theorem is stated in a more general setting than we need. Details on the canonical solutions in the general setting can be found in [Fok et al.06].

Proof. Let $\{ \Phi_n \}_{n=1}^{2r+1}$ and $\{ \Upsilon_n \}_{n=1}^{2r+1}$ denote the sets of canonical solutions of the two systems defined in the same Stokes sectors Σ_n. Recall that the Stokes sectors are determined by the Stokes rays which in turn is determined by Λ_{-r}. The following equation shown in (1.35) hold in general:

$$\Phi_{2r+1}(z) = \Phi_1(z) \exp\left(2\pi i \Lambda_0^{(z_0)}\right)$$

We will prove the existence of $g : B(0, R) \to GL_2(\mathbb{C})$ such that $\Phi_n(z) = g(z)\Upsilon_n(z)$ and thus by Definition 1.2.3 the two systems will be gauge equivalent.

Define

$$g(z) = \Phi_1(z)\Upsilon_1(z)^{-1}, \quad z \in \Sigma_1$$

Here the inverse means the inverse matrix operation. This makes g into a holomorphic $GL_2(\mathbb{C})$ valued function in Σ_1, since the canonical solution Φ_n and Υ_n are fundamental, holomorphic solutions. We now analytically continue g around the punctured disk $B(z_0, R) \setminus \{ 0 \}$.

We know by the construction in the previous sections that starting in Σ_1 we can analytically continue Φ_1 and Υ_1 along

$$\begin{array}{cccc} \nu_0 : & [0, 2\pi] & \to & \mathbb{S} \\ & t & \mapsto & z_0 + \frac{R}{2}e^{it} \end{array}$$

a path looping counter-clockwise around z_0. Thus for $z \in \Sigma_n$, we can restrict ν_0 such that $\nu_0|_{[0,t_n]}(t_n) \in \Sigma_n$. Then the following equations hold:

$$\Phi_{\nu_0|_{[0,t_n]}}(z) = \Phi_n(z)S_{n-1}^{-1}S_{n-2}^{-1}\dots S_1^{-1}$$
$$\Upsilon_{\nu_0|_{[0,t_n]}}(z) = \Upsilon_n(z)S_{n-1}^{-1}S_{n-2}^{-1}\dots S_1^{-1}$$

Hence g also admits an analytic continuation along ν_0 :

$$g_{\nu_0|_{[0,t_n]}}(z) = \Phi_n(z)\Upsilon_n^{-1}(z)$$

Hence we have that in each Stokes sector $g_{\nu_0|_{[0,t_n]}}$ is an invertible, holomorphic function. Further in the Stokes sector Σ_{2r+1} :

$$g_{\nu_0|_{[0,t_{2r+1}]}}(z) = g_{\nu_0}(z) = \Phi_{2r+1}(z)\Upsilon_{2r+1}^{-1}(z)$$
$$= \Phi_1(z) \exp\left(2\pi i \Lambda_o^{(z_0)}\right) \exp\left(-2\pi i \Lambda_0^{(z_0)}\right) \Upsilon_1(z) = g(z)$$

Hence g is analytically continued to the whole punctured disc, and thus z_0 is a removable singularity of g. Thus g is holomorphic in $B(z_0, R)$. ∎

Proposition 1.3.3 shows that the Stokes phenomenon data, together with the canonical formal and fundamental solutions, uniquely describes the local behaviour around a non-Fuchsian singular point, of course up to right multiplication by a constant matrix.

1.4 A fundamental solution on the universal cover of the punctured Riemann sphere

In this Section we will describe a global solution of the differential equation (1.2). From the local solutions found in Section 1.2 and 1.3.6, it is clear that a "global solution" defined on $M = \mathbb{S} \setminus \{ 0, t, \infty \}$ would be a multi valued function. The main goal in Chapter 1 is to describe this multi valued function, in an effective and detailed manner. This is done by utilizing the monodromy theory, see Section 1.4.4. In order to describe and understand the monodromy theory, it is wise to construct the solution on the universal covering space $\tilde{M}$, of $M = \mathbb{S} \setminus \{ 0, t, \infty \}$, as a single valued globally defined function. This requires some tedious calculations, and a lot of compositions with charts and projections, but has the benefit of giving a clear meaning and explanation to all the different terms that show up along the way. E.g. connections matrices (Definition 1.4.3).

1.4.1 The universal holomorphic cover of the punctured Riemann sphere

In this thesis the model space is $M = \mathbb{S} \setminus \{0, t, \infty\}$. This is a connected complex holomorphic manifold of dimension one, i.e. a Riemann surface, see Example A.1.1. The Definition and construction of an universal holomorphic covering can be found in Appendix B.2.

Corollary 1.4.1 of Theorem B.2.1.
Let $M = \mathbb{S} \setminus \{0, t, \infty\}$. There exists a simply-connected Riemann surface $\tilde{M}$ and a holomorphic map $p : \tilde{M} \to M$ such that $(\tilde{M}, M, p)$ is a universal holomorphic covering, see Definition B.2.1.

The universal cover $\tilde{M}$ consists of homotopy classes of curves ζ, from a basepoint $z_b \in M$, to any other point on M. We will use the notations $[z, \zeta]$ and $\tilde{z}$ for points on $\tilde{M}$, the latter when we don't need to be specific about where $\tilde{z}$ is in the fiber $\pi^{-1}(z)$.

The covering map $p : \tilde{M} \to M$ takes a neighbourhood $\tilde{U}_{[z,\zeta]}$, and projects it to the neighbourhood U of $z \in M$. The charts on $\tilde{M}$ is characterized by first a projection $p|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to U_\alpha$ then followed by either $\phi_0(z) = z$ or $\phi_\infty(z) = \frac{1}{z}$, see Example A.1.1.

The fundamental group of $M = \mathbb{S} \setminus \{0, t, \infty\}$ can be identified. Consider the following sequence of homeomorphisms:

$$\mathbb{S} \setminus \{0, t, \infty\} \simeq \mathbb{C} \setminus \{0, 1\} \simeq \mathbb{R}^2 \setminus \{0, 1\} \simeq \text{“figure-eight-space”}.$$

where the last homeomorphism is obtained by a *deformation retract*, see [Mun00] for a more detailed description. The homeomorphism $\mathbb{S} \setminus \{0, t, \infty\} \to$ “figure-eight-space”, induces an isomorphism of fundamental groups. It is well known that the fundamental group of the “figure-eight-space” is isomorphic to the *Free group on two elements*, F_2. This group has an easy description: Let the elements a and b be the “generating” elements of the group, and e the identity element, where we always write “ae” as “a”, etc. Then

$$F_2 = \{\text{all words made by the letters } a, a^{-1}, b \text{ and } b^{-1} \mid e = aa^{-1}, a^{-1}a, bb^{-1}, b^{-1}b\}.$$

There should exist two elements that can generate all the elements of the group $\pi_1(M, z_b) = \pi_1(\mathbb{S} \setminus \{0, t, \infty\}, z_b)$. We will represent these two elements by homotopy classes $[\gamma_0], [\gamma_1] \in \pi_1(M, z_b)$. A representative of $[\gamma_0]$ is a loop starting at z_b going close to 0, looping 0 once counter-clockwise and then going back to z_b. $[\gamma_0]^{-1}$ is of course the homotopy class represented by γ_0^{-1}, which is γ_0 traversed clockwise, similarly for $[\gamma_1]^{-1}$. Obviously there is not only one such homotopy class that can be used to represent the element going around 0. But we can make the choice clear by first removing some lines from $M = \mathbb{S} \setminus \{0, t, \infty\}$.

Consider the following lines on the sphere $\mathbb{S}$, see Figure 1.6:

$$\begin{aligned} L_0 &= \{\text{the short segment of the great circle connecting } z_b \text{ and } 0\} \\ L_1 &= \{\text{the short segment of the great circle connecting } z_b \text{ and } t\} \\ L_\infty &= \{\text{the short segment of the great circle connecting } z_b \text{ and } \infty\} \end{aligned}$$

if we define

$$\hat{M} := (\mathbb{S} \setminus \{L_0, L_1, L_\infty\}) \cup \{z_b\} \subset \mathbb{S} \setminus \{0, t, \infty\}, \tag{1.47}$$

then evidently $\hat{M}$ is a simply connected subset of M, see Figure 1.6. On $\hat{M}$ there is up to homotopy a unique choice of a loop γ_0 going from z_b, counter-clockwise around 0 and back to z_b, without enclosing any of the other punctured points. Similarly for t and ∞.

It is clear from Figure 1.6 and a topological mind-argument that the path $\gamma_0 * \gamma_1$ can be stretched around the sphere to prove that its homotopic to the loop γ_∞^{-1}, from z_b clockwise around ∞. We formulate the result as a Lemma.

Lemma 1.4.1 Homotopy relation in fundamental group of $\mathbb{S}\setminus\{\,\mathbf{0},\mathbf{t},\infty\,\}$. *Consider the elements $[\gamma_0],[\gamma_1],[\gamma_\infty]\in\pi_1(\mathbb{S}\setminus\{\,0,t,\infty\,\},z_b)$, represented by the loops in Figure 1.6. The following relation holds:*

$$[\gamma_0]*[\gamma_1]*[\gamma_\infty]=[z_b].$$

The group $\pi_1(M,z_b)$ is isomorphic to F_2, and we can thus generate the group using the two elements $[\gamma_0],[\gamma_1]$ as generators.

1.4.2 Analytic continuation of the solutions around Fuchsian points

In Section 1.2 we found local solutions of the system in equation (1.2) around every point in $\mathbb{S}\setminus\{\,0,t,\infty\,\}$. In particular the solutions $\Phi^{(0)}$ in equation (1.9) and $\Phi^{(1)}$ in equation (1.14) involve a logarithmic term which cannot be defined in a punctured neighbourhood of $z_0=0$ or $z_1=t$ respectively. However, when we change the domain to $\tilde{M}$, the universal cover of $M=\mathbb{S}\setminus\{\,0,t,\infty\,\}$, we can take advantage of the fact that $\tilde{M}$ is simply connected and thus define the logarithm everywhere in $\tilde{M}$. We already did a similar construction when we constructed the solutions $\Phi^{(\infty)}_{\Sigma_{2n+1}}$ and $\Phi^{(\infty)}_{\Sigma_{2n+2}}$ in Section 1.3.4.

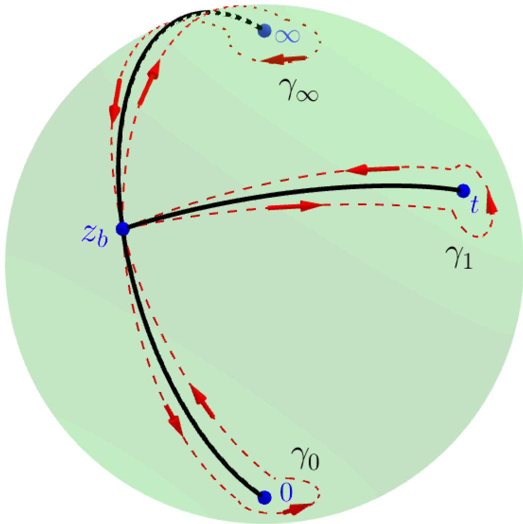


Figure 1.6: $\mathbb{S}\setminus\{\,0,t,\infty\,\}$ with the cuts L_0,L_1 and L_∞. The indicated red paths are representatives of the homotopy classes $[\gamma_0]$, $[\gamma_1]$ and $[\gamma_\infty]$

The local solutions $\Phi^{(0)}$ and $\Phi^{(1)}$, of the differential equation (1.2) around the Fuchsian points $z_0=0$ and $z_1=t$, respectively, are given by

$$\begin{aligned}\Phi^{(0)}(z)=&P^{(0)}\left(\sum_{k=0}^{\infty}\Psi_k^{(0)}z^k\right)\exp\left(\Lambda_0^{(0)}\log_{\alpha_0}(z)\right),\quad z\in B(0,R_0)\setminus b_{\alpha_0}^{(0)},\\ \Phi^{(1)}(z)=&P^{(1)}\left(\sum_{k=0}^{\infty}\Psi_k^{(1)}(z-t)^k\right)\exp\left(\Lambda_0^{(1)}\log_{\alpha_1}(z-t)\right),\quad z\in B(t,R_1)\setminus b_{\alpha_1}^{(1)},\end{aligned}$$

where $b_{\alpha_j}^{(j)}=\{\,z_j+re^{i\alpha_j}\in\mathbb{S}\setminus\{\,0,t,\infty\,\}\mid r\geq 0\,\}$ with $z_0=0$ and $z_1=t$. We will now do an analytic continuation of the solutions along loops around the singularities. Since the solutions are similar we can do the construction for the solution of the form

$$\Phi^{(j)}(z)=P^{(j)}\left(\sum_{k=0}^{\infty}\Psi_k^{(j)}(z-z_j)^k\right)\exp\left(\Lambda_0^{(j)}\log_{\alpha_j}(z-z_j)\right),\tag{1.48}$$

$z\in B(z_j,R_j)\setminus b_{\alpha_j}^{(j)}$, $j=0,1$. Let $0<x<R_j$ and consider a loop $\nu_j:I\to B(z_j,R_j)$, starting at z_j+x, going once counter-clockwise around z_j. We want to continue analytically $\Phi^{(j)}$ along ν_j and obtain a function $(\Phi^{(j)})_{\nu_j}$, defined in a neighbourhood of z_j+x. However, this is easy, since the series

$$P^{(j)}\sum_{k=0}^{\infty}\Psi_k^{(j)}(z-z_j)^k,$$

converges in $B(z_j,R_j)$, and is thus not changed under the analytic continuation. hence we only need to analytically continue the logarithmic term, see Figure 1.7. By equation

(1.30) we obtain:

$$\left(\Phi^{(j)}\right)_{\nu_j}(z) = P^{(j)}\left(\sum_{k=0}^{\infty}\Psi_k^{(j)}(z-z_j)^k\right)\exp\left(\Lambda_0^{(j)}\left(\log_{\alpha_j}(z-z_j)+2\pi i\right)\right) = \Phi^{(j)}(z)\exp\left(2\pi i\Lambda_0^{(j)}\right) \tag{1.49}$$

1.4.3 Construction of a fundamental solution on the universal cover of a punctured Riemann sphere

The goal is to give a fundamental solution to the system of equations:

$$\frac{d\Phi}{dz}\Phi(z)^{-1} = \frac{\sigma_3}{2} + \frac{A_0}{z} + \frac{A_t}{z-t} = \mathcal{A}(z) \tag{1.50}$$

As we have seen, when analytically continuing the local solutions around the singularities, one obtains multivalued functions. We will thus change the domain M of the differential equation (1.50) into $\tilde{M}$, the universal covering space of M. We can then construct a global holomorphic solution on $\tilde{M}$, using Corollary B.1.1, where we exploit the fact that $\tilde{M}$ is path connected and simply connected. We first define the differential equation (1.50) on the universal cover $\tilde{M}$ of M.

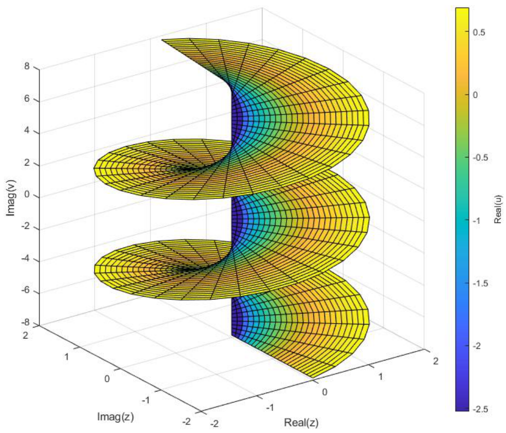


Figure 1.7: The Riemann surface of the logarithm, obtained as a graph of the analytic continuation of the logarithm in $\mathbb{C} \setminus \{0\}$. The vertical axis represent the imaginary part of the logarithm, and the colour gradient the real part.

Definition 1.4.1 Differential equation on a holomorphic covering space. *Let $(\tilde{M}, M, p)$ be a holomorphic covering. Let D be a differential operator acting on functions defined on M. We define a differential operator $\tilde{D}$ on $\tilde{M}$ by $\tilde{D} := D \circ p$.*

By Definition 1.4.1 we have that

$$\frac{d\tilde{\Phi}}{d\tilde{z}}\tilde{\Phi}(\tilde{z})^{-1} = \mathcal{A}\circ p(\tilde{z}) = \frac{\sigma_3}{2} + \frac{A_0}{p(\tilde{z})} + \frac{A_t}{p(\tilde{z})-t} = \frac{\sigma_3}{2} + \frac{A_0}{z} + \frac{A_t}{z-t} \tag{1.51}$$

is a differential equation defined on $\tilde{M}$. We note that by Corollary B.2.1, the function $\mathcal{A}\circ p : \tilde{M} \to M_2(\mathbb{C})$ is holomorphic. The main goal is to prove the existence of $\tilde{\Phi} : \tilde{M} \to GL_2(\mathbb{C})$, solving equation (1.51). We give a summary of the tools we have available:

- In Sections 1.2 and 1.3 we found local solutions for (1.50) at every point $z \in \mathbb{S} \setminus \{0, t, \infty\}$. We also found solutions around the singular points. These solutions depend, in particular, on a choice of a branch for the logarithmic expressions appearing in their formulas. By lifting the problem into $\tilde{M}$ this choice will no longer be needed, as the logarithm can be defined everywhere in $\tilde{M}$.

- In Section 1.4.2 we found an explicit formula for the analytic continuation of the local solutions around the Fuchsian singular points $z_0 = 0$ and $z_1 = t$, along a path encircling the point z_j once.

- Around the non-Fuchsian singular point $z_2 = \infty$ the Stokes phenomenon appears, and the solutions are pretty complicated as shown in Section 1.3. Nevertheless, given the Stokes sector Σ_1 at ∞, and a solution $\Phi^{(\infty)}_{\Sigma_1}$ in the given sector, it is possible to continue analytically this function around the non-Fuchsian singular point $z_2 = \infty$, using the solutions in the different Stokes sectors, as shown in (1.37).

- In Section 1.4.1 we constructed the universal holomorphic covering space $\tilde{M}$ of M. This is a simply connected Riemann surface. In Appendix B we prove the monodromy Theorem B.1.1, and its important Corollary B.1.1, on a Riemann surface. The Corollary states that if we have a function $\tilde{\Phi} : \tilde{U} \to GL_2(\mathbb{C})$, locally defined on $\tilde{M}$, then if $\tilde{\Phi}$ can be analytically continued along ny path on $\tilde{M}$, we can extend $\tilde{\Phi}$ into an unique globally defined holomorphic function $\tilde{\Phi} : \tilde{M} \to GL_2(\mathbb{C})$.

- In Section A.1 we have defined the meaning of $\frac{d\tilde{\Phi}}{d\tilde{z}}$ at a point $\tilde{z} \in \tilde{M}$. From Section 1.4.1 we know that we have two types of charts on $\tilde{M}$, the type $\tilde{\phi}_0 = \phi_0 \circ p|_{\tilde{U}}$ and $\tilde{\phi}_\infty = \phi_\infty \circ p|_{\tilde{U}}$. To be consistent we will only use $\tilde{\phi}_0$. Let $\tilde{\phi}_0 = \phi_0 \circ p|_{\tilde{U}} : \tilde{U} \to U \subset \mathbb{C} \setminus \{ 0, t \}$. Since $\phi_0(z) = z$, we can identify the chart $\tilde{\phi}_0$ with $p|_{\tilde{U}}$.

$$\left.\frac{d\tilde{\Phi}}{d\tilde{z}}\right|_{\tilde{z}} := \frac{d}{d\omega}\left(\tilde{\Phi} \circ \tilde{\phi}_0^{-1}(\omega)\right)\bigg|_{\tilde{\phi}_0(\tilde{z})=z} = \frac{d}{d\omega}\left(\tilde{\Phi} \circ p|_{\tilde{U}}^{-1}(\omega)\right)\bigg|_{p(\tilde{z})=z}.$$

We will now construct a global solution of (1.51) on $\tilde{M}$, by analytically continuing a local solution defined in a neighbourhood of a particular point z_b, the basepoint from the construction of the universal cover $\tilde{M}$. We will follow the convention in [Fok et al.06] and choose the basepoint z_b for the universal cover $\tilde{M}$, in the Stokes sector Σ_1 at $z = \infty \iff \xi = \frac{1}{z} = 0$. To be consistent we will now denote points on $\mathbb{S} \setminus \{ 0, t, \infty \}$ only by the identity chart $\phi_0(z) = z$. Changing between the charts ϕ_0 and ϕ_∞ on $\mathbb{S} \setminus \{ 0, t, \infty \}$ the Stokes sector Σ_1 transforms as:

$$\Sigma_1 = \left\{ \xi \in \mathbb{C} \setminus \left\{ 0, \frac{1}{t} \right\} \,\middle|\, 0 < |\xi| < R,\ -\frac{3\pi}{2} + \frac{\delta}{2} < \operatorname{Arg}(\xi) < \frac{\pi}{2} - \frac{\delta}{2} \right\} \tag{1.52}$$

$$\phi_\infty^{-1}(\Sigma_1) = \left\{ z \in \mathbb{S} \setminus \{ 0, t, \infty \} \,\middle|\, \frac{1}{R} < |z|,\ -\frac{\pi}{2} + \frac{\delta}{2} < \operatorname{Arg}(\xi) < \frac{3\pi}{2} - \frac{\delta}{2} \right\} \tag{1.53}$$

We can choose a condition $\tilde{\Phi}_{[z_b, z_b]} \in GL_2(\mathbb{C})$ for the solution to satisfy at $[z_b, z_b]$, the point in the fiber above z_b reached by the constant loop. This is the *initial value* of the system. Recall the canonical solution $\Phi^{(\infty)}_{\Sigma_1}$ in the Stokes sector Σ_1, given in Definition 1.3.8,

$$\Phi^{(\infty)}_{\Sigma_1}(\xi) = P^{(\infty)} \hat{\Psi}_1(\xi) \exp\left(-\frac{\Lambda^{(\infty)}_{-1}}{\xi} + \Lambda^{(\infty)}_0 \log_{\frac{\pi}{2}}(\xi) \right), \quad \xi \in \Sigma_1$$

and the chart

$$\begin{array}{rccc} \phi_\infty : & \mathbb{S} \setminus \{0, t, \infty\} & \to & \mathbb{C} \setminus \{ 0, \frac{1}{t} \} \\ & z & \mapsto & \frac{1}{z} = \xi \end{array}$$

We note that $\phi_\infty^{-1}(\Sigma_1)$ is evenly covered by p, since it is open, connected and simply-connected, see the construction in Theorem B.2.1.

Remark.
The logarithm with branch $0 < \alpha_\infty < 2\pi$ in the chart $\phi_\infty(z) = \frac{1}{z} = \xi$ centered at $\xi = 0$, is defined in Definition 1.3.4 by

$$\log_{\alpha_\infty} : B(0, R_\infty) \setminus \{ b^{(\infty)}_{\alpha_\infty} \} \to \mathbb{C},$$

$$\log_{\alpha_\infty}(\xi) := \int_{\zeta_\xi} \frac{1}{\omega} d\omega + \ln(x_\xi)$$

We insert $\xi = \phi_\infty(z) = \frac{1}{z}$, and thus define a function in a branched neighbourhood of $z_2 = \infty$ on $\mathbb{S} \setminus \{0, t, \infty\}$. The branched neighbourhood $B(0, R_\infty) \setminus \{ b^{(\infty)}_{\alpha_\infty} \}$ is mapped bi-holomorphically through $\phi_\infty^{-1}(\xi) = \frac{1}{\xi}$ to the branched neighbourhood

$$\Big(\mathbb{S} \setminus \{0, t, \infty\} \setminus \mathrm{cl}\big(B\big(0, 1/R_\infty\big)\big)\Big) \setminus \left\{re^{-i\alpha_\infty} \mid r \geq 1/R_\infty\right\}.$$

In particular, $\phi_\infty^{-1}(x_\xi) =: x_z$, the path ζ_ξ is mapped to $\eta_z = \phi_\infty^{-1}(\zeta_\xi)$, and the derivative of this reparametrization is given by

$$\frac{d\phi_\infty}{dz} = \frac{d\xi}{dz} = -\frac{1}{z^2}.$$

Using the integration variable ω for ξ and s for z, we compute

$$\begin{aligned}(1.54)\quad \log_{\alpha_\infty}(\phi_\infty(z)) &= \int_{\zeta_\xi} \frac{1}{\omega} d\omega + \ln(x_\xi) \\ &= \int_{\eta_z} s\left(-\frac{1}{s^2}\right) ds + \ln\left(\frac{1}{x_z}\right) = -\int_{\eta_z} \frac{1}{s} ds - \ln(x_z) = -\log_{-\alpha_\infty}(z).\end{aligned}$$

We will use this formula to express the logarithm centered at $z_2 = \infty$, when using the chart ϕ_0 on $\mathbb{S} \setminus \{0, t, \infty\}$.

Let $p|_{\tilde{\Sigma}[z_b]}$ denote the restriction of p to the sheet above $\phi_\infty^{-1}(\Sigma_1)$ containing $[z_b, z_b]$. We now define a holomorphic function in the sector

$$\begin{aligned}(1.55)\quad \tilde{\Sigma}[z_b] &= p|^{-1}_{\tilde{\Sigma}[z_b]} \circ \phi_\infty^{-1}(\Sigma_1) \\ &= \Big\{[x, \eta] \in \tilde{M} \mid x \in \phi_\infty^{-1}(\Sigma_1),\ \ \eta \sim \zeta_{z_b,x},\ \ \zeta_{z_b,x} \subset \phi_\infty^{-1}(\Sigma_1)\Big\} \subset \tilde{M},\end{aligned}$$

where for any $x \in \phi_\infty^{-1}$, $\zeta_{z_b,x}$ is a path between z_b and x, contained in $\phi_\infty^{-1}(\Sigma_1)$, and η is homotopic to $\zeta_{z_b,x}$.

Proposition 1.4.1 Defining a function seed.
The function

$$(1.56)\qquad \begin{array}{rccc}\tilde{\Phi}: & \tilde{\Sigma}[z_b] \subset \tilde{M} & \to & GL_2(\mathbb{C}) \\ & [z, \zeta] & \mapsto & \Phi^{(\infty)}_{\Sigma_1} \circ \phi_\infty(z) E^{(\infty)}\end{array} \qquad \tilde{\Phi}([z_b, z_b]) := \tilde{\Phi}_{[z_b, z_b]}$$

$$\begin{aligned}\tilde{\Phi}([z, \zeta]) &= \Phi^{(\infty)}_{\Sigma_1} \circ \phi_\infty \circ p([z, \zeta]) E^{(\infty)} \\ &= P^{(\infty)} \hat{\Psi}_1 \circ \phi_\infty(z) \exp\left(-z\Lambda^{(\infty)}_{-1} - \Lambda^{(\infty)}_0 \log_{-\frac{\pi}{2}}(z)\right) E^{(\infty)}\end{aligned}$$

is a holomorphic function defined in the neighbourhood $\tilde{\Sigma}[z_b] \subset \tilde{M}$ of $[z_b, z_b]$, with the constant matrix $E^{(\infty)}$ determined by

$$E^{(\infty)} := \left(\Phi^{(\infty)}_{\Sigma_1} \circ \phi_\infty(z_b)\right)^{-1} \tilde{\Phi}_{[z_b, z_b]}.$$

It is the unique solution to equation (1.51) in $\tilde{\Sigma}[z_b]$ (1.55) which equals $\tilde{\Phi}_{[z_b, z_b]}$ at $[z_b, z_b]$.

Proof. To show that the function is well defined, the only thing to check is that we can define $E^{(\infty)}$ as promised. Indeed since $\Phi_{\Sigma_1}^{(\infty)}$ is a fundamental solution, see Definition 1.1.2, it is an invertible matrix at each point $\xi = \frac{1}{z}$. Hence we can define $E^{(\infty)}$ by:

$$E^{(\infty)} := \left(\Phi_{\Sigma_1}^{(\infty)} \circ \phi_\infty \left(z_b \right) \right)^{-1} \tilde{\Phi}_{[z_b, z_b]}.$$

we note that at $[z_b, z_b]$

$$\tilde{\Phi}([z_b, z_b]) = \Phi_{\Sigma_1}^{(\infty)} \circ \phi_\infty \circ p([z_b, z_b]) E^{(\infty)} = \tilde{\Phi}_{[z_b, z_b]}.$$

The function is holomorphic since it is a composition of holomorphic functions:

$$\tilde{\Phi}([z, \zeta]) = \Phi_{\Sigma_1}^{(\infty)} \circ \phi_\infty \circ p([z, \zeta]) : \tilde{\Sigma}[z_b] \xrightarrow{p} \phi_\infty^{-1}\left(\Sigma_1\right) \xrightarrow{\phi_\infty} \Sigma_1 \xrightarrow{\Phi_{\Sigma_1}^{(\infty)}} GL_2(\mathbb{C})$$

p being holomorphic by Corollary 1.4.1, ϕ_∞ since it is a chart and $\Phi_{\Sigma_1}^{(\infty)}$ by its construction in Section 1.3.3.

We show that $\tilde{\Phi}$ solves equation (1.51) by calculating $\frac{d\tilde{\Phi}}{d\tilde{z}} \tilde{\Phi}^{-1}$ in the chart

$$\tilde{\phi} = \phi_0 \circ p\big|_{\tilde{\Sigma}[z_b]} : \tilde{\Sigma}[z_b] \to \mathbb{C} \setminus \{ 0, t \}.$$

First we compute the function composed with the inverse chart

$$\tilde{\Phi} \circ \tilde{\phi}^{-1}(\omega) = \Phi_{\Sigma_1}^{(\infty)} \circ \phi_\infty \circ p \circ p\big|_{\tilde{\Sigma}[z_b]}^{-1} \circ \phi_0^{-1}(\omega) = \Phi_{\Sigma_1}^{(\infty)} \circ \phi_\infty(\omega) = \Phi_{\Sigma_1}^{(\infty)} \left(\frac{1}{\omega} \right) = \Phi_{\Sigma_1}^{(\infty)}(\xi).$$

Then the result is trivial since

$$\frac{d(\tilde{\Phi} \circ \tilde{\phi}^{-1}(\omega))}{d\omega} \left(\tilde{\Phi} \circ \tilde{\phi}^{-1}(\omega) \right)^{-1} = \frac{d\Phi_{\Sigma_1}^{(\infty)}(\xi)}{d\xi} \frac{d\xi}{d\omega} \left(\Phi_{\Sigma_1}^{(\infty)}(\xi) \right)^{-1},$$

which solves (1.50) by what we showed in Section 1.3.3. The uniqueness follows from the fact that any other solution to (1.51) in $\tilde{\Sigma}[z_b]$ is equal to $\tilde{\Phi}$ up to right multiplication by a constant matrix, see Lemma 1.2.2. However since the functions agree at $[z_b, z_b]$, this constant matrix is the identity. ∎

The constant matrix $E^{(\infty)}$ depends on the initial condition $\tilde{\Phi}_{[z_b, z_b]}$. The usual convention is to choose the initial condition $\tilde{\Phi}_{[z_b, z_b]}$ such that $E^{(\infty)} = I$, see [Fok et al.06] and [JMU81]. We will follow this convention and thus we give the following definition.

Definition 1.4.2 Connection matrix related to $z_2 = \infty$.
The constant matrix $E^{(\infty)}$ defined in Proposition 1.4.1 is called the connection matrix related to $z_2 = \infty$. The initial condition in $\tilde{\Phi}$ can be chosen as

$$\tilde{\Phi}_{[z_b, z_b]} = \Phi_{\Sigma_1}^{(\infty)} \circ \phi_\infty(z_b),$$

and thus

$$E^{(\infty)} = \left(\Phi_{\Sigma_1}^{(\infty)} \circ \phi_\infty(z_b) \right)^{-1} \tilde{\Phi}_{[z_b, z_b]} = I$$

The function $\tilde{\Phi} : \tilde{\Sigma}[z_b] \to GL_2(\mathbb{C})$ from Proposition 1.4.1 will now be continued analytically using Corollary B.1.1. In order to apply the Corollary we need to show that $\tilde{\Phi}$ can be continued analytically along any path $\tilde{\zeta}$ in $\tilde{M}$.

Proposition 1.4.2 Analytic continuation of $\tilde{\Phi}$ along any path in $\tilde{M}$.
The holomorphic function

$$\begin{aligned}\tilde{\Phi}: \quad \tilde{\Sigma}[z_b] &\to \quad GL_2(\mathbb{C}) \\ [z,\zeta] &\mapsto \Phi^{(\infty)}_{\Sigma_1} \circ \phi_\infty(z) E^{(\infty)}\end{aligned} \qquad \tilde{\Phi}([z_b, z_b]) := \tilde{\Phi}_{[z_b,z_b]}$$

can be continued analytically along any path in $\tilde{M}$.

Proof. Let $\tilde{\zeta} : I \to \tilde{M}$ be a path in $\tilde{M}$ starting at $[z_b, z_b]$, ending at $[z, \zeta]$. By the construction of the universal covering $(\tilde{M}, M, p)$ in Section B.2 we know that the path $\tilde{\zeta}$ projects to the path $\zeta = p \circ \tilde{\zeta}$ in M through the covering map p. Since $\tilde{M}$ is simply connected, an analytic continuation along $\tilde{\zeta}$ is invariant up to the homotopy-class of $\tilde{\zeta}$. Since the image $\tilde{\zeta}(I)$ is compact, we can cover it by a finite set $\{ \tilde{D}_k \}_{k=0}^n$ of open sets such that:

- The functions $p|_{\tilde{D}_k} : \tilde{D}_k \to D_k \subset M$, is bi-holomorphic, and $\{ D_k \}_{k=0}^n$ are open discs in $\mathbb{S} \setminus \{ 0, t, \infty \}$.
- There exists a partition
$$\{ 0 = a_0, a_1, \ldots, a_n, a_{n+1} = 1 \}$$
of I such that $\tilde{\zeta}([a_k, a_{k+1}]) \subset \tilde{D}_k$.
- The sets $\tilde{D}_k$ is chosen small enough so that in $D_k \subset M$ there exists a solution $\Phi_k := \Phi^{(a)} : D_k \to GL_2(\mathbb{C})$ of (1.50) by using one of the local solutions $\Phi^{(a)}$, constructed in Section 1.2. Φ_0 is chosen to be $\Phi^{(\infty)}_{\Sigma_1} \circ \phi_\infty|_{D_0}$.
- By Corollary B.2.1 in each $\tilde{D}_k$ we can define the holomorphic function
$$\Phi_k \circ p : \tilde{D}_k \to GL_2(\mathbb{C}).$$
Adapting to the proof in Proposition 1.4.1, it is easy to show that $\Phi_k \circ p$ solves (1.51) in $\tilde{D}_k$.

Compare with Definition B.1.5 we see that we need to figure out how to glue together solutions $\Phi_k \circ p$ and $\Phi_{k+1} \circ p$ in $\tilde{D}_k \cap \tilde{D}_{k+1}$. This is dealt with by once again appealing to Lemma 1.2.2. We let $C_0 = E^{(\infty)}$. Inductively, having defined C_k, we let

$$\begin{aligned} C_{k+1} &:= \left(\Phi_{k+1} \circ p \circ \tilde{\zeta}(a_{k+1})\right)^{-1} \left(\Phi_k \circ p \circ \tilde{\zeta}(a_{k+1})\right) C_k \\ &= \left(\Phi_{k+1} \circ \zeta(a_{k+1})\right)^{-1} \left(\Phi_k \circ \zeta(a_{k+1})\right) C_k \end{aligned}$$

Then we define the functions

$$\begin{aligned}\tilde{\Phi}_k: \quad \tilde{D}_k &\to \quad GL_2(\mathbb{C}) \\ [z,\zeta] &\mapsto \left(\Phi_k \circ p([z,\zeta])\right) C_k\end{aligned} \tag{1.57}$$

that still solve (1.51) since it is a constant matrix times a known solution. We thus obtain that

$$\tilde{\Phi}_0 = (\Phi_k \circ p([z,\zeta])) C_0 = \left(\Phi^{(\infty)}_{\Sigma_1} \circ \phi_\infty \circ p|_{\tilde{D}_0}\right) E^{(\infty)} = \tilde{\Phi}|_{\tilde{D}_0},$$

and

$$\tilde{\Phi}_k \circ \tilde{\zeta}(a_{k+1}) = \left(\Phi_k \circ \zeta(a_{k+1})\right) C_k = \left(\Phi_{k+1} \circ \zeta(a_{k+1})\right) C_{k+1} = \tilde{\Phi}_{k+1} \circ \tilde{\zeta}(a_{k+1}),$$

by the Definition of C_{k+1}. The latter implies that $\tilde{\Phi}_k = \tilde{\Phi}_{k+1}$ in $\tilde{D}_k \cap \tilde{D}_{k+1}$ since they differ only by a constant matrix, i.e. the identity matrix. We conclude that $(\tilde{\Phi}_k, \tilde{D}_k)_{k=0}^n$ is an analytic continuation of $\tilde{\Phi}$ along $\tilde{\zeta}$. ∎

Definition 1.4.3 Connection matrices of an analytic continuation along a path. *Let $(\tilde{\Phi}_k, \tilde{D}_k)_{k=1}^n$ be an analytic continuation of $\tilde{\Phi}$, as defined in Proposition 1.4.2, along a path $\tilde{\zeta}$, where $\tilde{\Phi}_k = (\Phi_k \circ p)C_k$. The matrices $\{ C_k \}_{k=0}^n$ defined by*

$$C_0 = E^{(\infty)} := \left(\Phi_{\Sigma_1}^{(\infty)} \circ \phi_\infty(z_b)\right)^{-1} \tilde{\Phi}_{[z_b,z_b]}$$

$$C_{k+1} := \left(\Phi_{k+1} \circ \zeta(a_{k+1})\right)^{-1} \left(\Phi_k \circ \zeta(a_{k+1})\right) C_k$$

are called the connection matrices of the analytic continuation.

We will now highlight some very significant connection matrices. Recall the subset $\hat{M} \subset M = \mathbb{S} \setminus \{ 0, t, \infty \}$ from (1.47), see Figure 1.8. $\hat{M}$ is actually evenly covered by p, since it is open, connected and simply connected. Define the paths ζ_0 and ζ_1 in $\hat{M}$ to start at z_b and end in $B(0, R_0) \setminus L_0 \subset \hat{M}$ and $B(t, R_1) \setminus L_1 \subset \hat{M}$ respectively. Since $\hat{M}$ is simply connected, there is up to homotopy classes only one choice of these paths once the endpoints are fixed and we force the paths to begin on a counter-clockwise journey around z_j, see Figure 1.8. The specific position of the endpoint does not matter for the following discussion.

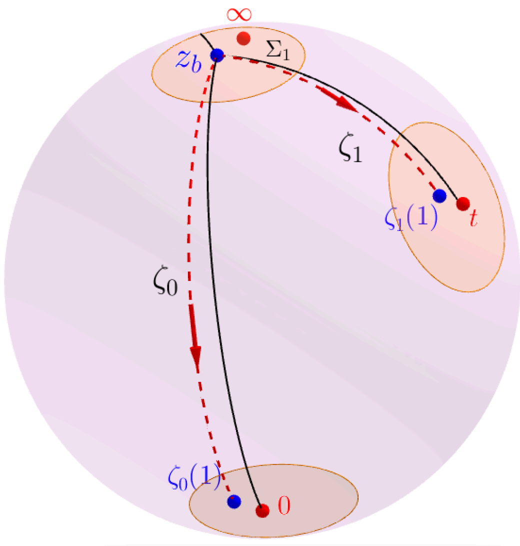


Figure 1.8: Paths from z_b to the Fuchsian singular points.

Having defined ζ_0 and ζ_1 we can by Lemma B.2.1 uniquely lift the paths to $\tilde{\zeta}_0 : I \to \tilde{M}$ and $\tilde{\zeta}_1 : I \to \tilde{M}$, to paths starting at $[z_b, z_b] \in \tilde{M}$. Also these paths lie entirely in the sheet above $\hat{M}$ containing $[z_b, z_b]$.

The open, connected, simply connected sets $B(0, R_0) \setminus L_0$ and $B(t, R_1) \setminus L_1$ are both evenly covered by p. Let $\tilde{U}^{(0)}$ and $\tilde{U}^{(1)}$ denote the respective sheets over $B(0, R_0) \setminus L_0$ and $B(t, R_1) \setminus L_1$, with $\tilde{\zeta}_0(1) \in \tilde{U}^{(0)}$ and $\tilde{\zeta}_1(1) \in \tilde{U}^{(1)}$. By using Proposition 1.4.2 to continue analytically $\tilde{\Phi}$ along $\tilde{\zeta}_0$ and $\tilde{\zeta}_1$ we obtain analytic continuations of $\tilde{\Phi}$ into $\tilde{U}^{(0)}$ and $\tilde{U}^{(1)}$, which we will denote as $(\tilde{\Phi})_{\tilde{\zeta}_0}$ and $(\tilde{\Phi})_{\tilde{\zeta}_1}$.

Recall now the local solutions around 0 and t from equation (1.48)

$$\Phi^{(j)}(z) = P^{(j)} \left(\sum_{k=0}^{\infty} \Psi_k^{(j)} (z - z_j)^k \right) \exp\left(\Lambda_0^{(j)} \log_{\alpha_j}(z - z_j) \right), \; z \in B(z_j, R_j) \setminus b_{\alpha_j}^{(j)}$$

with, $j \in \{ 0, 1 \}$. We choose the branch cuts α_0 and α_1 in $(0, 2\pi]$ and to align with the cuts L_0 and L_1 in $M = \mathbb{S} \setminus \{ 0, t, \infty \}$. We define the holomorphic functions

$$\Phi^{(j)} \circ p : \quad \tilde{U}^{(j)} \;\to\; GL_2(\mathbb{C}), \quad j \in \{ 0, 1 \}. \tag{1.58}$$

Obviously these functions solve the differential equation (1.51), and is thus by Lemma 1.2.2 related to the analytic continuations $(\tilde{\Phi})_{\tilde{\zeta}_j}$ by constant matrices, which importance is emphasized by giving them their own Definition.

Definition 1.4.4 Connection matrices related to $z_0 = 0$ and $z_1 = t$. *Consider the analytic continuations $(\tilde{\Phi})_{\tilde{\zeta}_0}$ and $(\tilde{\Phi})_{\tilde{\zeta}_1}$ of $\tilde{\Phi}$ into the neighbourhoods $\tilde{U}^{(0)}$ and $\tilde{U}^{(1)}$ of $\tilde{\zeta}_0$ and $\tilde{\zeta}_1$ respectively. Consider also the local solutions $\Phi^{(j)} \circ p$, $\; j \in \{ 0, 1 \}$,* *defined in equation* (1.58)*. The matrices* $E^{(0)}$ *and* $E^{(1)}$ *defined by*

$$E^{(0)} := \left(\Phi^{(0)}_{\alpha_0} \circ p([\zeta_0(1), \zeta_0])\right)^{-1} (\tilde{\Phi})_{\tilde{\zeta}_0}([\zeta_0(1), \zeta_0])$$

$$E^{(1)} := \left(\Phi^{(1)}_{\alpha_1} \circ p([\zeta_1(1), \zeta_1])\right)^{-1} (\tilde{\Phi})_{\tilde{\zeta}_1}([\zeta_1(1), \zeta_1])$$

are respectively called the connection matrix related to $z_0 = 0$ *and the connection matrix related to* $z_1 = t$*.*

Note that these matrices are just the last connection matrices of the analytic continuation of $\tilde{\Phi}$ along $\tilde{\zeta}_0$ and $\tilde{\zeta}_1$ respectively, see Definition 1.4.3. They are also independent of how we analytically continue $\tilde{\Phi}$ along $\tilde{\zeta}_0$, respectively $\tilde{\zeta}_1$, and at what point in $U^{(0)}$ respectively $U^{(1)}$, we use to define them, by Lemma B.1.1 and Corollary B.1.1.

We conclude this Section with a Theorem summarizing the results.

Theorem 1.4.1 Global fundamental solution on $\tilde{M}$.
Consider the differential equation (1.51) *defined on the Riemann surface* $\tilde{M}$*. There exists a holomorphic function*

$$\tilde{\Phi}: \quad \tilde{M} \quad \to \quad GL_2(\mathbb{C}), \quad \tilde{\Phi}([z_b, z_b]) = \tilde{\Phi}_{[z_b, z_b]},$$

that is a global solution to equation (1.51)*. This is the unique solution with* $\tilde{\Phi}([z_b, z_b]) = \tilde{\Phi}_{[z_b, z_b]}$*.*

Moreover consider the neighbourhoods $\tilde{\Sigma}[z_b]$*,* $\tilde{U}^{(0)}$ *and* $\tilde{U}^{(1)}$ *of* $[z_b, z_b], [\zeta_0(1), \zeta_0]$ *and* $[\zeta_1(1), \zeta_1]$*, respectively. In these respective neighbourhoods* $\tilde{\Phi}$ *is given by*

$$\begin{aligned}\tilde{\Phi}([z,\zeta]) &= \Phi^{(\infty)}_{\Sigma_1} \circ \phi_\infty \circ p([z,\zeta]) E^{(\infty)} \\ &= P^{(\infty)} \hat{\Psi}_1 \circ \phi_\infty(z) \exp\left(-z\Lambda^{(\infty)}_{-1} - \Lambda^{(\infty)}_0 \log_{-\frac{\pi}{2}}(z)\right) E^{(\infty)},\end{aligned}$$

$$\tilde{\Phi}([z,\zeta]) = \Phi^{(0)} \circ p([z,\zeta]) E^{(0)} = P^{(0)} \left(\sum_{k=0}^{\infty} \Psi^{(0)}_k z^k\right) \exp\left(\Lambda^{(0)}_0 \log_{\alpha_0}(z)\right) E^{(0)},$$

$$\tilde{\Phi}([z,\zeta]) = \Phi^{(1)} \circ p([z,\zeta]) E^{(1)} = P^{(1)} \left(\sum_{k=0}^{\infty} \Psi^{(1)}_k (z-t)^k\right) \exp\left(\Lambda^{(1)}_0 \log_{\alpha_1}(z-t)\right) E^{(1)},$$

where $E^{(\infty)}, E^{(0)}$ *and* $E^{(1)}$ *are the connection matrices related to the singular points* $z_2 = \infty, z_0 = 1$ *and* $z_1 = t$*, respectively.*
Letting $\tilde{\Phi}_{[z_b, z_b]} = \Phi^{(\infty)}_{\Sigma_1} \circ \phi_\infty(z_b)$*, gives* $E^{(\infty)} = I$*, and thus* $\tilde{\Phi}$ *with* $E^{(\infty)} = I$ *is uniquely determined by the canonical formal solution, see Definition 1.3.7.*

Definition 1.4.5 Canonical global fundamental solution.
The solution $\tilde{\Phi} : \tilde{M} \to GL_2(\mathbb{C})$ *to* (1.51) *with* $E^{(\infty)} = I$ *and the canonical formal solution (see Definition 1.3.7) as asymptotic expansion, is called the canonical global fundamental solution of* (1.51)*.*

1.4.4 Monodromy theory

In Theorem 1.4.1 we obtained a global fundamental solution to the differential equation in equation (1.51). However, the solution is obtained through analytic continuations of the locally constructed solutions from Section 1.2 and 1.3. Such a solution is hard to work with, and impractical for calculations. To solve this problem there is developed a monodromy theory, which describes how a local solution on M changes when analytically continued along loops from the fundamental group of M. The goal is to represent the fundamental group $\pi_1(M, z_b)$ by constant matrices in $GL_2(\mathbb{C})$, such that analytic continuation along a loop $[\gamma]$, is obtained simply by right multiplication by a constant matrix.

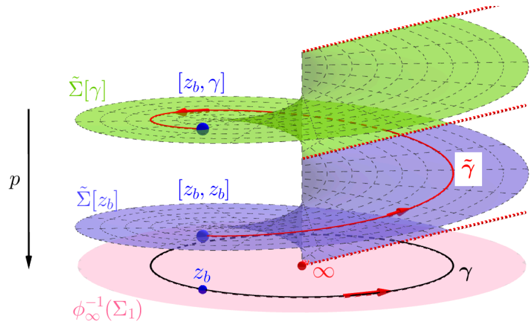


Figure 1.9: Sheets above $\phi_\infty^{-1}(\Sigma_1)$. If γ encircles other singularities than $z_2 = \infty$, the visualization is more complicated. The red dashed line is the Stokes ray $\phi_\infty^{-1}(l_2)$, see equation (1.28) and Figure 1.2.

Consider a fundamental solution

$$\tilde{\Phi} : \tilde{M} \to GL_2(\mathbb{C}), \quad \tilde{\Phi}([z_b, z_b]) = \tilde{\Phi}_{[z_b,z_b]},$$

of (1.51), as given by Theorem 1.4.1, not necessarily with $E^{(\infty)} = I$. Recall the Stokes sector $\phi_\infty^{-1}(\Sigma_1)$ from equation (1.53), which is evenly covered by $p : \tilde{M} \to M$, that is, the preimage of $\phi_\infty^{-1}(\Sigma_1)$ by the projection p is given by:

$$p^{-1}(\phi_\infty^{-1}(\Sigma_1)) = \coprod_{[\gamma]\in\pi_1(M,z_b)} \tilde{\Sigma}[\gamma],$$

$$= \coprod_{[\gamma]\in\pi_1(M,z_b)} \left\{[x, \eta] \in \tilde{M} \mid x \in \phi_\infty^{-1}(\Sigma_1),\ \eta \sim \gamma * \zeta_{z_b,x},\ \zeta_{z_b,x} \subset \phi_\infty^{-1}(\Sigma_1)\right\}$$

see Figure 1.9 and Theorem B.2.1. In this disjoint union, the sheet $\tilde{\Sigma}[\gamma]$ is the unique sheet above $\phi_\infty^{-1}(\Sigma_1)$ containing $\tilde{\gamma}(1) = [z, \gamma]$. Here $\tilde{\gamma} : I \to \tilde{M}$, is the unique lift of $\gamma : I \to M$, such that $\tilde{\gamma}$ starts at $[z_b, z_b]$.

By projecting the solution $\tilde{\Phi}$ down to M, we will for each $[\gamma] \in \pi_1(M, z_b)$ obtain a local solution to

$$\frac{d\Phi}{dz} \cdot \Phi(z)^{-1} = \frac{\sigma_3}{2} + \frac{A_0}{z} + \frac{A_t}{z-t}, \tag{1.59}$$

given by

$$\Phi[\gamma] : \phi_\infty^{-1}(\Sigma_1) \to GL_2(\mathbb{C}), \qquad \Phi[\gamma](z) := \tilde{\Phi} \circ \left(p|_{\tilde{\Sigma}[\gamma]}\right)^{-1}(z). \tag{1.60}$$

Lemma 1.4.2 Properties of projected solution.

i. *For each $[\gamma] \in \pi_1(M, z_b)$, the function $\Phi[\gamma]$ solves equation (1.59), and describes the global solution $\tilde{\Phi}$ in the sense that*

$$\tilde{\Phi}|_{\tilde{\Sigma}[\gamma]} = \Phi[\gamma] \circ p$$

ii. *The analytic continuation of $\Phi[\gamma]$ along $[\eta] \in \pi_1(M, z_b)$ is given by*

$$\left(\Phi[\gamma]\right)_\eta = \Phi[\gamma * \eta]$$

iii. *Given two elements $[\gamma], [\eta] \in \pi_1(M, z_b)$, the functions $\Phi[\gamma]$, and $\Phi[\gamma * \eta]$ are related by a constant matrix $m_\gamma^\eta \in GL_2(\mathbb{C})$:*

$$\Phi[\gamma * \eta] = \Phi[\gamma] m_\eta^\gamma$$

Proof. We first prove that $\Phi[\gamma]$ solve (1.59). By how we constructed $\tilde{\Phi}$, we have by (1.57):

$$\tilde{\Phi}([z,\zeta]) = (\Phi_k \circ p([z,\zeta]))\, C_k = \left(\Phi^{(\infty)}_{\Sigma_{2n+1}} \circ \phi_\infty \circ p([z,\zeta])\right) C_k,$$

where

$$\Phi^{(\infty)}_{\Sigma_{2n+1}} \circ \phi_\infty : \phi_\infty^{-1}(\Sigma_1) \to GL_2(\mathbb{C})$$

is a canonical solution in the Stokes sector $\phi_\infty^{-1}(\Sigma_{2n+1}) = \phi_\infty^{-1}(\Sigma_1)$, for some $n \in \mathbb{Z}$, see Definition 1.3.8. Hence

$$\Phi[\gamma](z) = \left(\Phi^{(\infty)}_{\Sigma_{2n+1}} \circ \phi_\infty \circ p \circ \left(p\big|_{\tilde{U}_{[z,\gamma]}}\right)^{-1}(z)\right) C_k = \left(\Phi^{(\infty)}_{\Sigma_{2n+1}} \circ \phi_\infty(z)\right) C_k.$$

$\Phi[\gamma]$ is a right multiplication multiplication by a constant invertible matrix, of a known solution. We conclude that $\Phi[\gamma]$ solve (1.59). By the Definition of $\Phi[\gamma]$:

$$\tilde{\Phi}\big|_{\tilde{\Sigma}[\gamma]} = \tilde{\Phi} \circ \left(p\big|_{\tilde{\Sigma}[\gamma]}\right)^{-1} \circ p = \Phi[\gamma] \circ p,$$

thus finishing the proof of property *i*.

We show property *ii*. Recall how we continue $\tilde{\Phi}$ analytically in the proof of Proposition 1.4.2. We do it exactly by constructing an analytical continuation $(D_k, \Phi_k)_{k=0}^n$ in M along η, and then lifting the pairs $(D_k, \Phi_k)_{k=0}^n$ into $(\tilde{D}_k, \tilde{\Phi}_k)_{k=0}^n$ by demanding that D_k is evenly covered (open, connected and simply connected). Remark that $\tilde{D}_n \subset \tilde{\Sigma}[\gamma * \eta]$, hence for $[z,\zeta] \in \tilde{D}_n$ we have that:

$$\begin{aligned}\left(\Phi[\gamma]\right)_\eta(z) = \left(\Phi[\gamma]\right)_\eta \circ p([z,\zeta]) = \Phi_n \circ p([z,\zeta])& \\ = \tilde{\Phi}\big|_{\tilde{\Sigma}[\gamma*\eta]}([z,\zeta]) = \Phi[\gamma * \eta] \circ p([z,\zeta]) = \Phi[\gamma * \eta](z)&,\end{aligned}$$

by using property *i*.

Property *iii*. is just Lemma 1.2.2, once we know that $\Phi[\gamma]$ and $\Phi[\gamma * \eta]$ solve (1.59), which they do by property *i*. ∎

We can now define the general monodromy map of $\pi_1(M, z_b)$.

Definition 1.4.6 The general monodromy map.
Let

$$\mathfrak{m} = \left\{ m_\gamma^\eta \in GL_2(\mathbb{C}) \;\middle|\; \left(\Phi[\gamma]\right)_\eta = \Phi[\gamma * \eta] = \Phi[\gamma] m_\gamma^\eta,\ [\gamma], [\eta] \in \pi_1(M, z_b) \right\}$$

be the constant matrices defined by Lemma 1.4.2. The map

$$\begin{array}{rccc} \varphi : & \pi_1(M, z_b) \times \pi_1(M, z_b) & \to & \mathfrak{m} \subset GL_2(\mathbb{C}) \\ & \left([\gamma], [\eta]\right) & \mapsto & m_\gamma^\eta \end{array}$$

is called the general monodromy map related to $\tilde{\Phi}$ (with a prescribed initial condition).

By Property *i*. of Lemma 1.4.2, the general monodromy map related to a function $\tilde{\Phi}$ tells you how the solution through a point $[z_b, \gamma]$ in the fiber $p^{-1}(z_b)$, changes when analytically continued along $[\eta] \in \pi_1(M, z_b)$.

We will need the following Lemma, which will also be useful later, in order to prove some distinguishing properties of φ.

Lemma 1.4.3 Analytic continuation of constant multiplication.
Let $f : U \subset M \to GL_2(\mathbb{C})$ be an analytic function defined in an open set U on a Riemann surface M. Let be $A \in GL_2(\mathbb{C})$ a constant matrix. If f can be analytically continued along $\eta : I \to M$, then

$$(f \cdot A)_\eta = (f)_\eta \cdot A,$$

that is, the analytic continuation of $f \cdot A$ along η, is equal to the analytic continuation of f along η, multiplied by A.

Proof. From Definition B.1.5 there exists an analytic continuation (f_k, D_k) of f along η. $g = f \cdot A : U \to GL_2(\mathbb{C})$ is analytic and it obviously also has an analytic continuation along η, given by $(g_k = f_k \cdot A, D_k)$. It is then clear by induction and extensive use of Lemma B.1.1, that since $f_0 \cdot A = g_0$ in D_0, then we have $f_n \cdot A = g_n$ in D_n. Hence by definition, $(f)_\eta \cdot A = (g)_\eta$. ■

Proposition 1.4.3 Properties of the general monodromy map.
Consider the general monodromy map related to $\tilde{\Phi}$:

$$\begin{array}{rccl} \varphi : & \pi_1(M, z_b) \times \pi_1(M, z_b) & \to & \mathfrak{m} \subset GL_2(\mathbb{C}) \\ & ([\gamma], [\eta]) & \mapsto & m_\gamma^\eta \end{array}.$$

- *φ depends on the initial value $\tilde{\Phi}_{[z_b, z_b]}$ of $\tilde{\Phi}$: If $\tilde{\Phi}^1, \tilde{\Phi^2} : \tilde{M} \to GL_2(\mathbb{C})$, and $\tilde{\Phi}^2_{[z_b,z_b]} = \tilde{\Phi}^1_{[z_b,z_b]} C$, then*
$$\varphi^2 = C^{-1} \varphi^1 \; C.$$

- *φ has the following properties:*

$$\varphi\big([\gamma], [\eta_1] * [\eta_2]\big) = \varphi\big([\gamma], [\eta_2]\big) \; \varphi\big([\gamma], [\eta_1]\big) \tag{1.61}$$
$$\varphi\big([\gamma], [\eta_1] * [\eta_2]\big) = \varphi\big([\gamma], [\eta_1]\big) \; \varphi\big([\gamma] * [\eta_1], [\eta_2]\big) \tag{1.62}$$
$$\varphi\big([\gamma_1] * [\gamma_2], [\eta]\big) = \big(\varphi\big([\gamma_1], [\gamma_2]\big)\big)^{-1} \varphi\big([\gamma_1], [\eta]\big) \; \varphi\big([\gamma_1], [\gamma_2]\big) \tag{1.63}$$
$$\varphi\big([\gamma] * [\eta], [\eta^{-1}]\big) = \varphi\big([\gamma], [\eta]\big)^{-1} = \varphi\big([\gamma], [\eta^{-1}]\big), \tag{1.64}$$

for any $[\gamma], [\gamma_1], [\gamma_2], [\eta], [\eta_1], [\eta_2] \in \pi_1(M, z_b)$, where the products between the φ's are matrix products.

Proof. To prove the first statement, we consider two functions $\tilde{\Phi}^1, \tilde{\Phi^2} : \tilde{M} \to GL_2(\mathbb{C})$, such that $\tilde{\Phi}^2_{[z_b,z_b]} = \tilde{\Phi}^1_{[z_b,z_b]} C$. We use Lemma 1.4.2 and compute

$$\begin{aligned} \varphi^2\big([\gamma], [\eta]\big) &= \big(\Phi^2[\gamma](z_b)\big)^{-1} \Phi^2[\gamma * \eta](z_b) = \Big(\tilde{\Phi}^1([z_b, \gamma]) C\Big)^{-1} \tilde{\Phi}^1([z_b, \gamma * \eta]) C \\ &= C^{-1} \left(\Phi^1[\gamma](z_b)\right)^{-1} \tilde{\Phi}^1(z_b) C = C^{-1} \varphi^1\big([\gamma], [\eta]\big) C \end{aligned}$$

We now prove the relations with φ. Relation (1.62) is easy to prove using Lemma 1.4.2. Let $[\gamma], [\eta_1], [\eta_2] \in \pi_1(M, z_b)$, then we have

$$\begin{aligned} \varphi\big([\gamma], [\eta_1] * [\eta_2]\big) &= \varphi\big([\gamma], [\eta_1 * \eta_2]\big) = \left(\Phi[\gamma](z_b)\right)^{-1} \Phi[\gamma * \eta_1 * \eta_2](z_b) \\ &= \left(\Phi[\gamma](z_b)\right)^{-1} \Phi[\gamma * \eta_1](z_b) m^{\eta_2}_{\gamma * \eta_1} = m^{\eta_1}_\gamma m^{\eta_2}_{\gamma * \eta_1} = \varphi\big([\gamma], [\eta_1]\big) \; \varphi\big([\gamma] * [\eta_1], [\eta_2]\big) \end{aligned}$$

To prove relation (1.61), we again use Lemma 1.4.2 and also need to use Lemma 1.4.3. Starting exactly as in the previous equation we obtain

$$\varphi\big([\gamma], [\eta_1] * [\eta_2]\big) = \left(\Phi[\gamma](z_b)\right)^{-1} \Phi[\gamma * \eta_1](z_b) m^{\eta_2}_{\gamma * \eta_1} = \Big(\tilde{\Phi}[\gamma]\Big)^{-1} \Big(\tilde{\Phi}[\gamma] m^{\eta_1}_\gamma\Big)_{\eta_2}$$

$$= \left(\Phi[\gamma]\right)^{-1} \Big(\big(\Phi[\gamma]\big)_{\eta_2}\Big) m^{\eta_1}_\gamma = \left(\Phi[\gamma]\right)^{-1} \Phi[\gamma] m^{\eta_2}_\gamma m^{\eta_1}_\gamma = \varphi\big([\gamma], [\eta_2]\big) \; \varphi\big([\gamma], [\eta_1]\big).$$

Relation (1.63) is just a combination of the two first, where

$$[\gamma] \mapsto [\gamma_1], \quad [\eta_1] \mapsto [\gamma_2], \quad [\eta_2] \mapsto [\eta],$$

and then solving for $\varphi\big([\gamma_1] * [\gamma_2], [\eta]\big)$.

For relation (1.64) we first prove

$$\varphi\big([\gamma]*[\eta],[\eta^{-1}]\big) = \big(\Phi[\gamma]m_\gamma^\eta\big)^{-1}\,\Phi[\gamma*\eta*\eta^{-1}] \tag{1.65}$$
$$= \big(m_\gamma^\eta\big)^{-1}\,(\Phi[\gamma])^{-1}\,\Phi[\gamma] = \varphi\big([\gamma],[\eta]\big)^{-1}.$$

Then using relation (1.63) together with (1.65):

$$\varphi\big([\gamma]*[\eta],[\eta^{-1}]\big) = \varphi\big([\gamma],[\eta]\big)^{-1}\varphi\big([\gamma],[\eta^{-1}]\big)\varphi\big([\gamma],[\eta]\big)$$
$$\iff I = \varphi\big([\gamma],[\eta^{-1}]\big)\varphi\big([\gamma],[\eta]\big).$$

■

Some authors, e.g. [Fok et al.06], define "the" monodromy representation as the map $\varphi([z_b],\cdot):\pi_1(M,z_b)\to\mathfrak{m}$. This

- depends on the choice of the initial value $\tilde{\Phi}_{[z_b,z_b]}$ of the function $\tilde{\Phi}_{\tilde{\Sigma}[z_b]}$
- depends on the choice of the local solution $\Phi[z_b]$, that all the monodromy matrices are calculated relative to
- does not show how the two above choices affect the chosen monodromy representation.

Definition 1.4.7 Monodromy representations.
Consider the general monodromy map related to $\tilde{\Phi}$:

$$\begin{array}{rccc} \varphi: & \pi_1(M,z_b)\times\pi_1(M,z_b) & \to & \mathfrak{m}\subset GL_2(\mathbb{C}) \\ & \big([\gamma],[\eta]\big) & \mapsto & m_\gamma^\eta \end{array}.$$

For each $[\gamma]\in\pi_1(M,z_b)$, we define an anti representation of $\pi_1(M,z_b)$ into the subgroup $\mathfrak{m}[\gamma]\subset\mathfrak{m}\subset GL_2(\mathbb{C})$ by

$$\begin{array}{rcccl} \varphi([\gamma],\cdot): & \pi_1(M,z_b) & \to & \mathfrak{m}[\gamma] & \subset\mathfrak{m}\subset GL_2(\mathbb{C}) \\ & [\eta] & \mapsto & m_\gamma^\eta & \end{array}$$

called the monodromy representation related to $[\gamma]\in\pi_1(M,z_b)$. The subgroup $\mathfrak{m}[\gamma]$ is called the monodromy group related to $[\gamma]$. It is an anti representation in the sense that

$$\varphi\big([\gamma],[\eta_1]*[\eta_2]\big) = \varphi\big([\gamma],[\eta_2]\big)\;\varphi\big([\gamma],[\eta_1]\big),$$

which is (1.61) in Proposition 1.4.3.

The monodromy representation related to $[z_b]\in\pi_1(M,z_b)$:

$$\begin{array}{rcccl} \varphi([z_b],\cdot): & \pi_1(M,z_b) & \to & \mathfrak{m}[z_b] & \subset\mathfrak{m}\subset GL_2(\mathbb{C}) \\ & [\eta] & \mapsto & m_{z_b}^\eta & \end{array},$$

is called the canonical monodromy representation. The subgroup $\mathfrak{m}[z_b]$ is called the canonical monodromy group. The elements $m_{z_b}^\gamma$ of the canonical monodromy group $\mathfrak{m}[z_b]$,will from now on be denoted by m^γ. The particular elements m^{γ_0}, m^{γ_1} and m^{γ_∞}, the image of the loops $[\gamma_0]$, $[\gamma_1]$ and $[\gamma_\infty]$ see Section 1.4.1, will be denoted by $m^{(0)}$, $m^{(1)}$ and $m^{(\infty)}$ respectively.

Corollary 1.4.2 Representation conjugation relation.
The monodromy representation related to $[\gamma]$, is related to the canonical monodromy representation by conjugation:

$$m_\gamma^\eta = (m^\gamma)^{-1}\,m^\eta m^\gamma,$$

for every $[\eta]\in\pi_1(M,z_b)$. Thus $\mathfrak{m}[z_b]=\mathfrak{m}$.

Proof. Apply the third relation, equation (1.63) from Proposition 1.4.3 to $\varphi([z_b]*[\gamma],[\eta]) = m^{\eta}_{\gamma}$. ■

Corollary 1.4.3 Free group relation in the canonical monodromy group.
The group $\mathfrak{m}$ is generated by $m^{(0)}$ and $m^{(1)}$. Further the following relation holds

$$m^{(\infty)}m^{(1)}m^{(0)} = I$$

Proof. Both statements are a direct consequence from the fact that $\pi_1(M, z_b)$ is generated by $[\gamma_0], [\gamma_1]$ and that the canonical monodromy representation is an anti homomorphism. Thus the subgroup $\mathfrak{m}[z_b]$ is generated by $m^{(0)}$ and $m^{(1)}$, and by Corollary 1.4.2 the entire monodromy group $\mathfrak{m}$ is generated by $m^{(0)}$ and $m^{(1)}$.

The free group relation from Lemma 1.4.1 reads

$$[\gamma_0] * [\gamma_1] * [\gamma_\infty] = [z_b] \iff [\gamma_0] * [\gamma_1] = [\gamma_\infty]^{-1}.$$

Hence by using (1.61) from Proposition 1.4.3, we obtain

$$\varphi\big([z_b], [\gamma_0] * [\gamma_1]\big) = \varphi\big([z_b], [\gamma_1]\big)\varphi\big([z_b], [\gamma_0]\big) = m^{(1)}m^{(0)} = (m^{(\infty)})^{-1}.$$

■

The canonical monodromy representation is uniquely determined by the canonical global solution $\tilde{\Phi}$ from Definition 1.4.5. As Corollary 1.4.2 show, given a global solution $\tilde{\Phi}$, the canonical representation can be used to express any other representation. And choosing two initial values A, B for $\tilde{\Phi}_{[z_b,z_b]}$, the canonical monodromy representations are related by $\varphi^A = A^{-1}B\varphi^B B^{-1}A$. However by Proposition 1.4.1 and Definition 1.4.2, choosing $\tilde{\Phi}_{[z_b,z_b]}$ is equivalent to choosing $E^{(\infty)}$, and to simplify the canonical monodromy representation we can always let $E^{(\infty)} = I$, as in Definition 1.4.2. We summarize this Section on monodromy theory with a Theorem stating the relevant results.

Theorem 1.4.2 Unique canonical monodromy representation.
Given the differential equation

$$\frac{d\tilde{\Phi}}{d\tilde{z}}\tilde{\Phi}(\tilde{z})^{-1} = \frac{\sigma_3}{2} + \frac{A_0}{z} + \frac{A_t}{z-t}$$

with the canonical global solution $\tilde{\Phi} : \tilde{M} \to GL_2(\mathbb{C})$, as in Definition 1.4.5, uniquely determined by the canonical solution in the Stokes sector Σ_1 and $E^{(\infty)} = I$. Then there exists a unique canonical monodromy representation

$$\begin{array}{rccc} \varphi([z_b], \cdot) : & \pi_1(M, z_b) & \to & \mathfrak{m} \subset GL_2(\mathbb{C}) \\ & [\eta] & \mapsto & m^{\eta} \end{array},$$

into the monodromy group $\mathfrak{m}$. The group $\mathfrak{m}$ is generated by the two elements $m^{(0)}$ and $m^{(1)}$. Moreover, if $\zeta_{z_b,z} : I \to \phi_\infty^{-1}(\Sigma_1)$, and $[\gamma], [\eta_1], [\eta_2] \in \pi_1(M, z_b)$, then the following relations hold:

$$\tilde{\Phi}([z, \gamma * \zeta_{z_b,z}]) = \Phi[z_b](z)m^{\gamma} = \left(\Phi^{(\infty)}_{\Sigma_1} \circ \phi_\infty^{-1}(z)\right) m^{\gamma} \tag{1.66}$$

$$m^{(\infty)}m^{(1)}m^{(0)} = I \tag{1.67}$$

$$(m^{\gamma})^{-1} = m^{(\gamma^{-1})} \tag{1.68}$$

$$\varphi\big(z_b, [\eta_1] * [\eta_2]\big) = m^{\eta_2}m^{\eta_1}. \tag{1.69}$$

1.4.5 Generators for the canonical monodromy group

In Section 1.4.1 we described the holomorphic covering $(\tilde{M}, M, p)$ and found that the fundamental group of $M = \mathbb{S} \setminus \{0, t, \infty\}$ is isomorphic to the free group generated by two elements. We constructed two elements $[\gamma_0]$ and $[\gamma_1]$ which we can use to represent the fundamental group $\pi_1(M, z_b)$, and we derived the free group relation in Lemma 1.4.1. In Section 1.4.4 we described the monodromy theory, and how to obtain representations of the fundamental group, as matrix groups acting on solutions to the differential equation in equation (1.51). Corollary 1.4.3 shows that the canonical monodromy group is also generated by two elements $m^{(0)}$ and $m^{(1)}$, and how the free group relation translates through the canonical monodromy representation. If we can find formulas for the generators $m^{(0)}, m^{(1)}$ together with $m^{(\infty)}$, we can use Corollary 1.4.3 to obtain relations between the elements of $m^{(0)}$, $m^{(1)}$ and $m^{(\infty)}$.

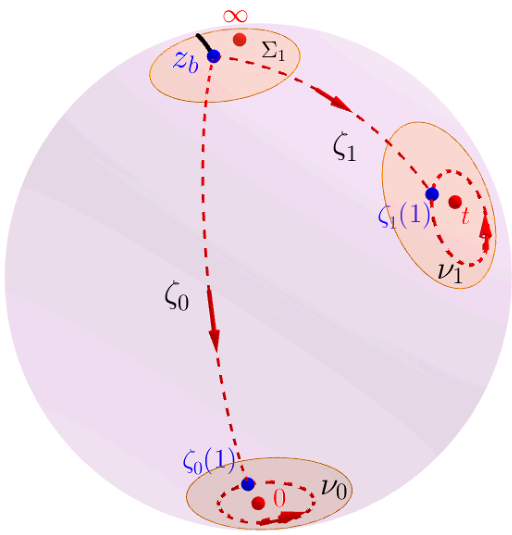


Figure 1.10: The paths ζ_0 and ζ_1, goes from z_b, into a neighbourhood of the Fuchsian singular points $z_0 = 0$ and $z_1 = t$, respectively, and are used to calculate the monodromy generators. $\zeta_0 * \nu_0 * \zeta_0^{-1}$ and $\zeta_1 * \nu_1 * \zeta_1^{-1}$ are homotopic to γ_0 and γ_1, respectively (see Figure 1.6).

We now want to find formulas for

$$m^{(0)} = \varphi\big([z_b], [\gamma_0]\big), \quad m^{(1)} = \varphi\big([z_b], [\gamma_1]\big), \quad m^{(\infty)} = \varphi\big([z_b], [\gamma_\infty]\big). \tag{1.70}$$

The procedure for finding the formulas for $m^{(0)}$ and $m^{(1)}$ are identical, and will be handled together using the usual notation $m^{(j)} = m^{(0)}, m^{(1)}$, $j \in \{0, 1\}$. In M, the path γ_j (defined in Section 1.4.1, see Figure 1.6), is homotopic to the concatenation of the paths ζ_j, ν_j, and ζ_j^{-1}, see Figure 1.10.

$$\gamma_j \sim \zeta_j * \nu_j * \zeta_j^{-1}. \tag{1.71}$$

Here ζ_j is the path defined in Section 1.4.3 that goes from $z_b \in \phi_\infty^{-1}(\Sigma_1)$ into the branched neighbourhood $B(z_j, R_j) \setminus b^{(j)}_{\alpha_j}$ of z_j, such that $\zeta_j \subset \hat{M} \subset M$. And ν_j is a loop at $\zeta_j(1)$, going once around the singular point z_j, see Figure 1.10.

We will obtain the formulas by using the analytic continuation already calculated in Section 1.4.2 and 1.4.3. We start of with considering the analytic continuation of $\Phi[z_b]$ along ζ_j. This we already did in Section 1.4.3. Recall that $\tilde{U}^{(j)} \subset \tilde{M}$ is the sheet above the branched neighbourhood $B(z_j, R_j) \setminus L_j$ of z_j, with $\zeta_j(1) \in \tilde{U}^{(j)}$, defined right before Definition 1.4.4. Then by Theorem 1.4.1:

$$\big(\Phi[z_b]\big)_{\zeta_j} = \tilde{\Phi} \circ \left(p|_{\tilde{U}^{(j)}}\right)^{-1} = \Phi^{(j)} \circ\ p \circ \left(p|_{\tilde{U}^{(j)}}\right)^{-1} E^{(j)} = \Phi^{(j)} E^{(j)} \tag{1.72}$$

Now we can analytically continue the function in equation (1.72) along the loop ν_j, that goes once counter-clockwise around the Fuchsian singular point z_j. In equation (1.49) we obtained that

$$\left(\Phi^{(j)}\right)_{\nu_j}(z) = \Phi^{(j)}(z) \exp\left(2\pi i \Lambda_0^{(j)}\right), \quad z \in B(z_j, R_j) \setminus b^{(j)}_{\alpha_j}.$$

We can then use Lemma 1.4.3 and obtain:

$$\big(\Phi[z_b]\big)_{\zeta_j * \nu_j} = \left(\Phi^{(j)} E^{(j)}\right)_{\nu_j} = \left(\Phi^{(j)}\right)_{\nu_j} E^{(j)} = \Phi^{(j)} \exp\left(2\pi i \Lambda_0^{(j)}\right) E^{(j)} \tag{1.73}$$

The last step is to analytically continue the function in equation (1.73) along $\tilde{\zeta}_j^{-1}$. Using equation (1.49), Lemma 1.4.3 and noting $\zeta_j * \zeta_j^{-1}$ is homotopic to the constant path at z_b, we obtain

$$(\Phi[z_b])_{\zeta_j * \nu_j * \zeta_j^{-1}}(z) = \left(\Phi^{(j)}(z)\exp\left(2\pi i \Lambda_0^{(j)}\right)E^{(j)}\right)_{\zeta_j^{-1}}(z)$$
$$= \left(\Phi^{(j)}\right)_{\zeta_j^{-1}}(z)\exp\left(2\pi i \Lambda_0^{(j)}\right)E^{(j)} = \Phi[z_b]\big(E^{(j)}\big)^{-1}\exp\left(2\pi i \Lambda_0^{(j)}\right)E^{(j)}.$$

We summarize the result in the following Proposition

Proposition 1.4.4 Formulas for the generators of the canonical monodromy group.
The analytic continuation of $\Phi[z_b] = \left(\Phi^{(\infty)} \circ \phi_\infty\right)E^{(\infty)}$ *along the loop* γ_j *at* z_b *is given by:*

$$\left(\Phi[z_b]\right)_{\gamma_j} = \left(\Phi^{(\infty)} \circ \phi_\infty\right)E^{(\infty)}\big(E^{(j)}\big)^{-1}\exp\left(2\pi i \Lambda_0^{(j)}\right)E^{(j)}$$

Thus the generators $m^{(0)}$ *and* $m^{(1)}$ *of the canonical monodromy group* $\mathfrak{m}[z_b]$*, is given by*

$$m^{(0)} = \left(E^{(0)}\right)^{-1}\exp\left(2\pi i \Lambda_0^{(0)}\right)E^{(0)}, \quad m^{(1)} = \left(E^{(1)}\right)^{-1}\exp\left(2\pi i \Lambda_0^{(1)}\right)E^{(1)}$$

where $\Lambda_0^{(j)}$ *is diagonal with diagonalization*

$$\Lambda_0^{(0)} = \left(P^{(0)}\right)^{-1}A_0P^{(0)}, \quad \Lambda_0^{(1)} = \left(P^{(1)}\right)^{-1}A_tP^{(1)}.$$

We now find a formula for $m^{(\infty)} = \varphi\big([z_b],[\gamma_\infty]\big)$. We will analytically continue

$$\Phi[z_b] = \left(\Phi_{\Sigma_1}^{(\infty)} \circ \phi_\infty\right)E^{(\infty)},$$

along γ_∞, the loop going once counter-clockwise around the non-Fuchsian point $z_2 = \infty$. Notice from Figure 1.10, that this path is entirely contained in the two Stokes sectors $\Sigma_1 \cup \Sigma_2$. From equation (1.38) and Lemma 1.4.3 we obtain

$$\left(\Phi[z_b]\right)_{\gamma_\infty} = \left(\left(\Phi_{\Sigma_1}^{(\infty)} \circ \phi_\infty\right)E^{(\infty)}\right)_{\gamma_\infty} = \left(\Phi_{\Sigma_1}^{(\infty)} \circ \phi_\infty\right)_{\gamma_\infty}E^{(\infty)}$$
$$= \left(\Phi_{\Sigma_1}^{(\infty)} \circ \phi_\infty\right)\exp\left(2\pi i \Lambda_0^{(\infty)}\right)S_2^{-1}S_1^{-1}E^{(\infty)}.$$

We summarize the result in the following Proposition.

Proposition 1.4.5 Formula for the monodromy matrix related to $z_2 = \infty$.
The analytic continuation of

$$\Phi[z_b] = \left(\Phi_{\Sigma_1}^{(\infty)} \circ \phi_\infty\right)E^{(\infty)}$$

along the loop γ_∞ *at* z_b *is given by:*

$$\left(\Phi[z_b]\right)_{\gamma_\infty} = \left(\Phi_{\Sigma_1}^{(\infty)} \circ \phi_\infty\right)\exp\left(2\pi i \Lambda_0^{(\infty)}\right)S_2^{-1}S_1^{-1}E^{(\infty)}$$

Thus the monodromy matrix $m^{(\infty)}$ *is given by*

$$m^{(\infty)} = \left(E^{(\infty)}\right)^{-1}\exp\left(2\pi i \Lambda_0^{(\infty)}\right)S_2^{-1}S_1^{-1}E^{(\infty)} \overset{\dagger}{=} \exp\left(2\pi i \Lambda_0^{(\infty)}\right)S_2^{-1}S_1^{-1},$$

where $\Lambda_0^{(\infty)}$ *is a diagonal matrix given by*

$$\Lambda_0^{(\infty)} = -diag\left(\left(P^{(\infty)}\right)^{-1}(A_0 + A_t)P^{(\infty)}\right) \overset{\dagger\dagger}{=} -diag\left((A_0 + A_t)\right).$$

$\dagger$: *If* $E^{(\infty)} = I$.
$\dagger\dagger$: *If* $P^{(\infty)} = I$.

Chapter 2

Geometric description

2.1 A connection on a principal bundle

In this Section we will define a principal bundle and three equivalent notions of a connection on a principal bundle. See [KN63] for more theory on differential geometry.

Definition 2.1.1 Principal bundle.
Let M be a complex manifold and G a complex Lie group (see Definition A.2.1). A (holomorphic) principal bundle over M with structure group G consists of a complex manifold P and an action

$$\begin{array}{rccc} \mu : & P \times G & \to & P \\ & (p, a) & \mapsto & p\,.\,a \end{array}$$

of G on P such that

i. G acts freely on P from the right, that is:

- *action: $p\,.\,e = p$ and $\big((p\,.\,a)\,.\,b\big) = p\,.\,(a \cdot b)$, for every $a, b \in G$ and $p \in P$, where $a \cdot b$ denotes the product in G.*
- *free: for $a, b \in G$, if there exists an element $p \in P$ such that $p\,.\,a = p\,.\,b$, then $a = b$.*

ii. M is the quotient space of P by the equivalence relation induced by G, $M = P/G$. That is, if $p, q \in P$, then $p \sim q$ if and only if $q = p\,.\,a$ for some $a \in G$. Further, the associated projection $\pi : P \to M$ is holomorphic.

iii. P is locally trivial, that is every point $z \in M$ has a neighbourhood U such that $\pi^{-1}(U)$ is isomorphic with $U \times G$. More specifically there exists a bi-holomorphic map: $\varphi : \pi^{-1}(U) \to U \times G$, of manifolds, called a local trivialization of P, such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times G \arrow[dl, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] \\ U & G \end{tikzcd}$$

Note that we can write $\varphi = \pi \times g$, where we define $g : \pi^{-1}(U) \to G$ by $g = \mathrm{pr}_2 \circ \varphi$. Additionally we require from φ that

$$g(p\,.\,a) = g(p) \cdot a$$

Thus if $q = p\,.\,a$, we have

$$\varphi(q) = \big(\pi(p\,.\,a), g(p\,.\,a)\big) = \big(\pi(p), g(p) \cdot a\big)$$

A principal fiber bundle will be denoted by $P(M, G, \pi)$ where P is called the total space, M is the base space, G the structure group and π is the projection.

Notational note: the right action $\mu : P \times G \to P$, induces two holomorphic maps

$$\begin{array}{rccc} \mu_\alpha : & P & \to & P \\ & p & \mapsto & p \,.\, a \end{array} \quad \text{and} \quad \begin{array}{rccc} \mu_p : & G & \to & P \\ & a & \mapsto & p \,.\, a \end{array}$$

by keeping the right or respectively left, coordinate constant. In particular the map μ_a is frequently denoted by R_a in other literature. We will reserve the notation R_a for the map induced by the multiplication in G.

In the Definition we required the action to be free. Together with Property *ii.*, the action is transitive (only) on the fibers $\pi^{-1}(z)$, thus G acts regularly on the fibers of P. Indeed, let $p, q \in P$. Then, $p, q \in \pi^{-1}(z) \iff$ there exists an $a \in G$ such that $p \,.\, a = q$.

Definition 2.1.2 Isomorphism of principal bundles.
The principal bundles $P(M, G, \pi)$ and $P'(M', G', \pi')$ are isomorphic if there exists a bi-holomorphic map $\Psi : P \to P'$ and an isomorphism of Lie groups $\psi : G \to G'$ such that

$$\Psi(p \,.\, a) = \Psi(p) \,.\, \psi(a) \tag{2.1}$$

Such an isomorphism induces a map from M to M'. Indeed, if $z \in M$, the fiber $\pi^{-1}(z)$ is mapped by Ψ into a single fiber of P' by equation (2.1). Thus the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "\Psi"] \arrow[d, "\pi"] & P' \arrow[d, "\pi'"] \\ M \arrow[r, dashed] & M' \end{tikzcd}$$

We say that a principal bundle $P(M, G, \pi)$ is trivial if P is isomorphic (in the category of principal bundles) to $M \times G$.

Proposition 2.1.1 Characterization of trivial bundles.
A principal fiber bundle $P(M, G, \pi)$ is trivial if and only if there exists a global holomorphic section $s : M \to P$ of the projection map π.

Proof. If such a section exists, we define the map $\Psi : M \times G \to P$ by
$\Psi = \mu \circ \big((s \circ \mathrm{pr}_1) \times \mathrm{pr}_2\big)$, $(z, a) \mapsto s(z) \,.\, a$. Evidently this map is a bi-holomorphic map between manifolds, since for a fixed z, $\Psi(z, \cdot)$ is a Lie group isomorphism from G onto the fiber $\pi^{-1}(z)$.

Conversely, if $M \times G$ is isomorphic to $P(M, G, \pi)$ under Ψ, we can define $s = \Psi(\cdot, e) : M \to P$. ■

We will familiarize ourself with the local trivialization of a principal bundle $P(M, G, \pi)$, in particular how to change between the local trivializations, Property *iii.* in Definition 2.1.1. On a non-empty intersection $U_\alpha \cap U_\beta \subset M$, consider the trivializations φ_α and φ_β such that:

$$\begin{tikzcd} U_\alpha \cap U_\beta \times G \arrow[d, "\mathrm{pr}_2"'] \arrow[drr, "\mathrm{pr}_1"'] & & \pi^{-1}\big(U_\alpha \cap U_\beta\big) \arrow[ll, "\varphi_\beta"'] \arrow[rr, "\varphi_\alpha"] \arrow[d, "\pi"] & & U_\alpha \cap U_\beta \times G \arrow[d, "\mathrm{pr}_2"] \arrow[dll, "\mathrm{pr}_1"] \\ G & & U_\alpha \cap U_\beta & & G \end{tikzcd}$$

Let $g_\alpha = \mathrm{pr}_2 \circ \varphi_\alpha$ and $g_\beta = \mathrm{pr}_2 \circ \varphi_\beta$. Define the function $\tilde{g}_{\alpha\beta} : \pi^{-1}\big(U_\alpha \cap U_\beta\big) \to G$ by

$$\tilde{g}_{\alpha\beta}(p) = g_\alpha(p) \cdot (g_\beta(p))^{-1}.$$

This function is constant on each fibre, since

$$\tilde{g}_{\alpha\beta}(p\,.\,a) = g_\alpha(p\,.\,a)\cdot(g_\beta(p\,.\,a))^{-1} = g_\alpha(p)\cdot a\cdot a^{-1}\cdot(g_\beta(p))^{-1} = g_\alpha(p)\cdot(g_\beta(p))^{-1} = \tilde{g}_{\alpha\beta}(p).$$

Definition 2.1.3 Transition functions on M.
The function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ defined such that $g_{\alpha\beta} \circ \pi = \tilde{g}_{\alpha\beta}$, is called the transition function on M between the trivializations φ_α and φ_β.

Proposition 2.1.2 Transitive relation of transition functions.
Let $g_{\alpha\beta}$, $g_{\beta\gamma}$ and $g_{\alpha\gamma}$ be the transition functions between the trivializations $\varphi_\alpha, \varphi_\beta$ and φ_γ, where $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. Then if e is the identity element of G:

i. $g_{\alpha\beta} \cdot g_{\beta\alpha} = e$

ii. $g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$

Proof.

i. $g_{\alpha\beta} \circ \pi \cdot g_{\beta\alpha} \circ \pi = \tilde{g}_{\alpha\beta} \cdot \tilde{g}_{\beta\alpha} = g_\alpha \cdot (g_\beta)^{-1} \cdot g_\beta \cdot (g_\alpha)^{-1} = e$

ii. $g_{\alpha\beta} \circ \pi \cdot g_{\beta\gamma} \circ \pi = \tilde{g}_{\alpha\beta} \cdot \tilde{g}_{\beta\gamma} = g_\alpha \cdot (g_\beta)^{-1} \cdot g_\beta \cdot (g_\gamma)^{-1} = g_\alpha \cdot (g_\gamma)^{-1} = \tilde{g}_{\alpha\gamma} = g_{\alpha\gamma} \circ \pi.$

■

Even though there in general does not exists global sections $s : M \to P$ of a principal bundle, there exists local sections. It will be useful to define a local section related to each trivialization.

Definition 2.1.4 Trivial section of a trivialization.
Consider a principal bundle $P(M, G, \pi)$ and a trivialization $\varphi_\alpha = \pi \times g_\alpha$. Define the trivial section $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$, by $s_\alpha(z) = \varphi_\alpha^{-1}(z, e)$.

Proposition 2.1.3 Relations with trivial sections.
Consider the trivializations $\varphi_\alpha = \pi \times g_\alpha$ and $\varphi_\beta = \pi \times g_\beta$, where $U_\alpha \cap U_\beta \neq \emptyset$, the associated transition function $g_{\alpha\beta}$ and the trivial sections s_α and s_β. Then

i. $s_\beta = s_\alpha \cdot g_{\alpha\beta}$

ii. $g_\alpha \circ s_\alpha = e$

iii. $g_{\alpha\beta} = (g_\beta \circ s_\alpha)^{-1} = g_\alpha \circ s_\beta$

Proof.

i. $\varphi_\alpha \circ s_\beta(z) = \big(z, g_\alpha \circ s_\beta(z)\big) \overset{iii.}{=} \big(z, g_{\alpha\beta}(z)\big) = \varphi_\alpha\big(s_\alpha(z) \cdot g_{\alpha\beta}(z)\big) \implies s_\beta = s_\alpha \cdot g_{\alpha\beta}$

ii. $g_\alpha(s_\alpha(z)) = g_\alpha \circ \varphi_\alpha^{-1}(z, e) = e$

iii. $g_{\alpha\beta} = g_{\alpha\beta} \circ \pi \circ s_\alpha = (g_\alpha \cdot (g_\beta)^{-1}) \circ s_\alpha \overset{ii.}{=} (g_\beta \circ s_\alpha)^{-1}$
$g_{\alpha\beta} = g_{\alpha\beta} \circ \pi \circ s_\beta = (g_\alpha \cdot (g_\beta)^{-1}) \circ s_\beta \overset{ii.}{=} g_\alpha \circ s_\beta$

■

Proposition 2.1.4 [Prop 5.2, Ch.1 in [KN63]] Construction of a principal bundle.
Let M be a complex manifold, $\{U_\alpha\}_\alpha$ an open covering of M and G a complex Lie group. Given a holomorphic function $f_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, for every non-empty $U_\alpha \cap U_\beta$, in such a way that the relations in Proposition 2.1.2 are satisfied. Then we can construct a principal bundle $P(M, G, \pi)$, where the transition functions will be given by $\{f_{\alpha\beta}\}$.

We will now turn our focus to the tangent space of P. In Section A.1 we give a detailed description of the (holomorphic) tangent space of a complex manifold. At each point p of a n-dimensional complex manifold P, there is a related n-dimensional complex vector space which we call the tangent space of P at p. The tangent space consist of all $\mathbb{C}$-linear derivations of holomorphic function germs at the point p. On a principal bundle $P(M,G,\pi)$, we have some additional structure than on just a manifold. In particular, the projection $\pi : P \to M$ induces linear maps from the tangent space of P at p, to the tangent space of M at $\pi(p)$, for each $p \in P$.

$$\pi : \quad P \quad \to \quad M, \qquad \pi_{*,p} : \quad T_pP \quad \to \quad T_{\pi(p)}M.$$

The kernel of the map $\pi_{*,z}$, is a linear subspace of T_pP for each $p \in P$. The dimension of the subspace will equal the dimension of the Lie group G, since the right action of G preserves the fibers of P.

Definition 2.1.5 Vertical tangent space of a principal bundle.
Given a principal bundle $P(M,G,\pi)$, the kernel V_p of the differential of the projection, $\ker(\pi_{*,p})$ is a subspace of T_pP, for each $p \in P$. V_p is called the vertical subspace of T_pP. It is also called the vertical tangent space of P at p. A vector field $v \in \mathfrak{X}(P)$ is called vertical, if $v(p) \in V_p$, for all $p \in P$. The vertical bundle is the distribution $V \subset TP$, consisting of all vertical vectors.

Proposition 2.1.5 Vertical tangent space and fundamental vector fields.
Consider a principal bundle $P(M,G,\pi)$ with vertical tangent space V_p at $p \in P$. Let $\mathfrak{X}(P)$ denote the vector fields on P. Then

i. V is involutive, that is if $v,w \in \mathfrak{X}(P)$ are two vertical vector fields of P, such that $v_p, w_p \in V_p$, then also $[v,w]_p \in V_p$ for each p. By the Frobenius Theorem 2.4.1, V is an integrable distribution (see Definition 2.4.1).

ii. The distribution $V \subset TP$ is G-invariant, in the sense that $(\mu_a)_* V_p = V_{p\,.\,a}$, where $\mu_a : P \to P$, such that $p \mapsto p\,.\,a$, the right action of G on P, with a fixed group element $a \in G$.

iii. There exists a Lie algebra homomorphism

$$\sigma : \quad \mathfrak{g} \quad \to \quad \mathfrak{X}(P),$$

that maps $X \in \mathfrak{g}$ into a vector field σX, called a fundamental vector field on P. Pointwise it is defined by

$$(\sigma X)_p = \sigma_p X := (\mu_p)_{*,e} X,$$

where $X \in \mathfrak{g}$ and $\mu_p : G \to P$, is the right action of G on P with a fixed point $p \in P$. It is a Lie algebra homomorphism in the sense that

$$\sigma[X,Y]_{\mathfrak{g}} = [\sigma X, \sigma Y]_{\mathfrak{X}(P)}, \quad \textit{for any } X, Y \in \mathfrak{g}.$$

For each $p \in P$, $\sigma_p := (\mu_p)_{*,e} : \mathfrak{g} \to V_p \subset T_pP$ is a vector space isomorphism. So the fundamental vector fields are all vertical.

iv. Locally, if $\varphi_\alpha = \pi \times g_\alpha$ trivialises $P(M,G,\pi)$ in $\pi^{-1}(U_\alpha)$ and $\theta : G \to T^*G \times \mathfrak{g}$ is the Maurer-Cartan form on G (Definition A.2.5), then

$$\sigma \circ (g_\alpha^* \theta) = I_V, \quad (g_\alpha^* \theta) \circ \sigma = I_{\mathfrak{g}}.$$

v. $(\mu_a)_{*,p} \sigma_p X = \sigma_{p\,.\,a}\left(\mathrm{Ad}\left(a^{-1}\right) X\right)$, where $X \in \mathfrak{g}$ and $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is the adjoint representation of G in $\mathfrak{g}$, see Definition A.2.4.

Proof. Proof given in the Appendix, Proposition A.3.1. ■

In the absence of any additional structure, there is no standard complement to $V_p \subset T_pP$. The next Definition formalises the wanted properties of such a complementary subspace.

Definition 2.1.6 Principal connection.
Consider a principal bundle $P(M, G, \pi)$ with vertical tangent space V_p at each $p \in P$. A principal connection on P is a holomorphic choice of subspaces $H_p \subset T_pP$ at each $p \in P$, called the horizontal subspaces, such that :

i. $T_p = V_p \oplus H_p$

ii. $(\mu_a)_{*,p}H_p = H_{p\,.\,a}$

The horizontal bundle is the distribution $H \subset TP$, consisting of all horizontal vectors. A section $s : U \subset M \to \pi^{-1}(U) \subset P$ is called a horizontal section, if it is a holomorphic section with $s_{*,z}T_zU \subset H_{s(z)}$.

Thus, a principal connection is a $G-$invariant distribution $H \subset TP$, complementary to $V \subset TP$. The dimension of the horizontal subspace is equal to the dimension of M. Indeed, the projection map induces a vector space isomorphism $\pi_{*,p}|_{H_p} : H_p \to T_{\pi(p)}M$.

Lemma 2.1.1 [Prop 1.2, Ch.1 in [KN63]] Horizontal lift.
Given a principal connection $H \subset TP$ on a principal bundle $P(M, G, \pi)$. The projection $\pi : P \to M$ induces a vector space isomorphism $\pi_{*,p}|_{H_p} : H_p \to T_{\pi(p)}M$. For every vector field Z on M, there is a unique horizontal lift $\tilde{Z}$ of Z, where $\tilde{Z}$ is a horizontal vector field on P. The lift $\tilde{Z}$ is invariant by G, in the sense that

$$(\mu_a)_{*,p}\tilde{Z}_p = \tilde{Z}_{p\,.\,a}.$$

Conversely, every horizontal vector field $\tilde{Z}$ on P, invariant by G, is the unique horizontal lift of a vector field Z on M.

Proof. The fact that $\pi_{*,p}\big|_{H_p}$ is an isomorphism follows directly from the fact that $\ker(\pi_{*,p}) = V_p \cap H_p = \{0\}$. Since it is an isomorphism, given a vector field Z on M, the existence and uniqueness of $\tilde{Z}$ is clear. The fact that $\tilde{Z}$ is holomorphic, follows from the fact that if we consider a trivialization $\varphi : \pi^{-1}(U) \to U \times G$, and consider any vector field Z' on $U \times G$, such that $(\mathrm{pr}_1)_* = Z$, then $(\varphi^{-1})_*Z'$ is a holomorphic vector field on $\pi^{-1}(U)$. Its horizontal component is $\tilde{Z}$. Finally

$$\pi_{*,p\,.\,a} \circ (\mu_a)_*\tilde{Z}_p = \pi_{*,p}\tilde{Z}_p = Z_{\pi(p)}.$$

Here we used the G invariance of π. But also

$$\pi_{*,p\,.\,a}\tilde{Z}_{p\,.\,a} = Z_{\pi(p\,.\,a)} = Z_{\pi(p)}.$$

Conversely, let $\tilde{Z}$ be a horizontal vector field on P, invariant by G. For every $z \in M$, pick a $p \in P$ such that $\pi(p) = z$. Then define $Z_z := \pi_{*,p}\tilde{Z}_p$. This construction is independent of which p we chose in the fiber over z. Indeed if $q = p\,.\,a$, then $\pi_{*,q}\tilde{Z}_q = \pi_{*,p\,.\,a} \circ (\mu_a)_*\tilde{Z}_p = \pi_{*,p}\tilde{Z}_p = Z_{\pi(p)}$, by the G invariance of $\tilde{Z}$. It is obvious that the lift of Z is $\tilde{Z}$. ■

The following diagram gives an overview of some of the maps we have defined on T_pP. Here we use a trivialization $\varphi_\alpha = \pi \times g_\alpha$ with $p \in \pi^{-1}(U_\alpha)$. Let $\varphi_\alpha(p) = (z, a)$ then

$$
\begin{tikzcd}
H_p \arrow[r, hook, "\iota_{H_p}"] & T_pP \arrow[dl, "\pi_{*,p}"'] \arrow[d, "(g_\alpha)_{*,p}"] & V_p \arrow[l, hook', "\iota_{V_p}"'] \\
T_zM \arrow[u, "\text{horizontal lift}", "\wr"'] & T_aG \arrow[r, "\sim", "\theta_a"'] & \mathfrak{g} \arrow[u, "\sigma_p"', "\wr"]
\end{tikzcd} \tag{2.2}
$$

Definition 2.1.7 Chart basis for the horizontal tangent space.
Let a principal bundle $P(M, G, \pi)$ with a principal connection $H \subset TP$ be given. The chart basis for the horizontal tangent space H_p is given by the horizontal lift of the basis for $T_{\pi(p)}M$:

$$\left\{\widetilde{\frac{\partial}{\partial z^k}}\right\}_{k=1}^{n} = \left\{\widetilde{\frac{\partial}{\partial z^1}}, \widetilde{\frac{\partial}{\partial z^2}}, \ldots, \widetilde{\frac{\partial}{\partial z^n}}\right\},$$

where $\left\{\frac{\partial}{\partial z^k}\right\}_{k=1}^{n}$ is the basis for $T_{\pi(p)}M$.

The vertical tangent space V_p is defined by the means of the kernel of a linear map, namely the tangential map of the projection π from P to M. Similarly to V_p, we can give an equivalent characterization of a principal connection on P, by the means of the kernel of a linear map.

Definition 2.1.8 Connection form.
Consider a principal bundle $P(M, G, \pi)$. A function $\omega : P \to T^*P \otimes \mathfrak{g}$, on P, is called a connection form on P if it satisfies:

i. $\omega(\sigma X) = X$, *for every fundamental vector field $\sigma X \in \mathfrak{X}(P)$ and every $X \in \mathfrak{g}$.*

ii. $(\mu_a)^*\omega = \mathrm{Ad}(a^{-1}) \circ \omega$, *that is,* $\omega_{p\,.\,a}\big((\mu_a)_{*,p}v_p\big) = \mathrm{Ad}(a^{-1}) \circ \omega_p(v_p)$, *for every tangent vector $v_p \in T_pP$, every $a \in G$ and every $p \in P$.*

A map from P to $T^*P \otimes \mathfrak{g}$ is called a Lie algebra valued 1-form on P.

Proposition 2.1.6 Characterization of a principal connection by a connection form.
Consider a principal bundle $P(M, G, \pi)$. Given a principal connection $H \subset TP$ we define a connection form ω on P by:

$$\omega : \quad P \quad \to \quad T^*P \otimes \mathfrak{g}, \qquad \left| \qquad \begin{array}{rcl} \omega_p : \quad T_pP & \to & \mathfrak{g} \\ v_p & \mapsto & \omega_p(v_p) = \begin{cases} X, & \text{if } v_p = \sigma X \\ 0, & \text{if } v_p \in H_p \end{cases} \end{array} \right.$$

And given a connection form ω on P, we define a principal connection $H \subset TP$ by

$$H_p := \ker(\omega_p).$$

Proof. Proof given in Appendix, Proposition A.3.2. ■

We will make one more characterization of a principal connection. We will define a family of Lie algebra valued 1-forms, each locally defined on M, the base space. Together they contain the same information as the connection form on P. In the diagram in equation (2.2), we see that $g_\alpha^*\theta : P \to T^*P \otimes \mathfrak{g}$ defines how the connection form should deal with vertical tangent vectors. Since $\dim(V_p) = \dim(G)$ and $\dim(H_p) = \dim(M)$, we need $\dim(M)$ number of equations to define the horizontal tangent space at $p \in P$. Thus 1-forms on M is able to capture the information that defines the horizontal tangent spaces.

Definition 2.1.9 Family of local connection forms.
Consider a principal bundle $P(M,G,\pi)$ with local trivializations

$$\{\, \varphi_\alpha = \pi \times g_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G \,\}_\alpha$$

and transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ (see Definition 2.1.3). A family of local connection forms on M, is a family of Lie algebra valued 1-forms $\{A_\alpha : U_\alpha \to T^*U_\alpha \otimes \mathfrak{g}\}_\alpha$ on M, where $\{\, U_\alpha \,\}_\alpha$ gives an open cover of M, and whenever $U_\alpha \cap U_\beta \neq \emptyset$:

$$A_\beta = \operatorname{Ad}(g_{\beta\alpha}) \circ A_\alpha + g^*_{\alpha\beta}\theta.$$

Proposition 2.1.7 Characterization of a connection form by a family of local connection forms.
Consider a principal bundle $P(M,G,\pi)$ with local trivializations

$$\{\, \varphi_\alpha = \pi \times g_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G \,\}_\alpha$$

and transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. Recall the trivial sections $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ from Definition 2.1.4. Given a connection form ω on P, we define a local family of connection forms on M by

$$A_\alpha := s^*_\alpha \omega.$$

And given a local family of connection forms $\{\, A_\alpha \,\}_\alpha$ on M, we define a connection form ω on P by in each trivialised set $\pi^{-1}(U_\alpha)$ defining

$$\omega_\alpha := \operatorname{Ad}\left((g_\alpha)^{-1}\right) \circ \pi^* A_\alpha + g^*_\alpha \theta.$$

Then any pair ω_α and ω_β agree on $\pi^{-1}(U_\alpha \cap U_\beta)$, and $\{\, \omega_\alpha \,\}_\alpha$ defines a connection form ω on P.

Proof. Proof given in Appendix, Proposition A.3.3. ∎

In summary we have given three equivalent descriptions of a “connection” on P:

1. a principal connection, which is a G-invariant horizontal distribution $H \subset TP$,
2. a connection form ω on P, which is a Lie algebra valued 1-form on P, satisfying $\omega(\sigma X) = X$ and $\mu^*_a \omega = \operatorname{Ad}\left(a^{-1}\right) \circ \omega$,
3. a family of local connection forms $\{\, A_\alpha \,\}_\alpha$ on M, which is a family of Lie algebra valued 1-forms on M, such that $A_\beta = \operatorname{Ad}(g_{\beta\alpha}) \circ A_\alpha + g^*_{\alpha\beta}\theta$.

Definition 2.1.10 A connection on a principal bundle.
Let $P(M,G,\pi)$ be a principal bundle. If P possesses either a principal connection, a connection form or a family of local connection forms, then we simply say that P possesses a connection.

By the means of Proposition 2.1.6 and Proposition 2.1.7 we will switch between the different perspectives when convenient.

2.2 Motivation for the geometric description

The main problem of this thesis is to describe the solutions of the linear system of first order differential equations of the form

$$\mathcal{A}(z) = \frac{d\Phi}{dz} \cdot \Phi(z)^{-1}, \quad \mathcal{A} : \mathbb{S} \setminus \{\, z_j \,\}_{j=1}^m \to M_2(\mathbb{C}), \tag{2.3}$$

where $U \subset \mathbb{S} \setminus \{ z_j \}_{j=1}^m = M$ is an open subset of a punctured Riemann sphere. We want to show the existence of functions $\Phi : U \subset \mathbb{S} \setminus \{ z_j \}_{j=1}^m \to GL_2(\mathbb{C})$, which locally can solve the differential equation.

We give some motivation for trying to formulate the differential equation using differential geometry. For comparison, we restate the relation that we called gauge equivalence, see Definition 1.2.3.

Proposition 2.2.1.
Given a differential equation

$$\mathcal{A} = \frac{d\Phi}{dz} \cdot \Phi^{-1}, \quad \mathcal{A} : \mathbb{S} \setminus \{ z_j \}_{j=1}^m \to M_2(\mathbb{C}),$$

which is solved locally by $\Phi_\alpha : U_\alpha \to GL_2(\mathbb{C})$. Let $f_\beta : U_\beta \to GL_2(\mathbb{C})$ be a holomorphic function on M such that $U_\alpha \cap U_\beta \neq \emptyset$. Then the function $\Phi_\beta := f_\beta \cdot \Phi_\alpha$ solves the differential equation

$$\mathcal{B} = f_\beta \cdot \mathcal{A} \cdot f_\beta^{-1} + \frac{df_\beta}{dz} \cdot f_\beta^{-1} = \frac{d\Phi}{dz} \cdot \Phi^{-1}.$$

Proof. We compute $\frac{d\Phi_\beta}{dz}$:

$$\frac{d\Phi_\beta}{dz} = \frac{df_\beta}{dz} \cdot \Phi_\alpha + f_\beta \cdot \frac{d\Phi_\alpha}{dz} = \frac{df_\beta}{dz} \cdot \Phi_\alpha + f_\beta \cdot \mathcal{A} \cdot \Phi_\alpha.$$

Then right multiplying by $\Phi_\beta(z)^{-1} = \Phi_\alpha(z)^{-1} \cdot f_\beta(z)^{-1}$

$$\frac{d\Phi_\beta}{dz} \cdot \Phi_\beta(z)^{-1} = \frac{df_\beta}{dz} \cdot f_\beta(z)^{-1} + f_\beta(z) \cdot \mathcal{A}(z) \cdot (f_\beta(z))^{-1}.$$

■

The relation between the functions $\mathcal{A}$ and $\mathcal{B}$ in Proposition 2.2.1 resembles the relations between the forms $\{ A_\alpha \}$, in a family of local connection forms, see Definition 2.1.9.

We will now explain in a geometric language how $\mathcal{A}$ is the coordinate function of a Lie algebra valued 1-form on M. Given a holomorphic function $\Phi : U \subset M \to GL_2(\mathbb{C})$. We differentiate the function, and write the equation

$$(A)_z = -\frac{d\Phi}{dz} \cdot \Phi^{-1},$$

where $(A)_z : U \to M_2(\mathbb{C})$ denotes the function defined by this expression. The presence of the minus sign is explained in the proof of Proposition 2.2.2. The function Φ is a map between the complex manifold $U \subset \mathbb{S} \setminus \{ z_j \}_{j=1}^m$ and the complex Lie group $GL_2(\mathbb{C})$. So its tangential map is

$$\Phi_{*,z} : T_z U \to T_{\Phi(z)} G, \quad \Phi_{*,z} \left(\frac{d}{dz} \right) = \frac{d\Phi}{dz}.$$

The right multiplication of the group element $\Phi(z)^{-1} \in GL_2(\mathbb{C})$ is thus interpreted as $(R_{\Phi(z)^{-1}})_{*,\Phi(z)}$. A more geometric notation is

$$(A)_z = -\frac{d\Phi}{dz} \cdot \Phi^{-1} = -(R_{\Phi(z)^{-1}})_{*,\Phi(z)} \circ (\Phi)_{*,z} \frac{d}{dz}$$

$$= -(L_{\Phi(z)})_{*,\Phi(z)^{-1}} \circ (R_{\Phi(z)^{-1}})_{*,e} \circ (L_{\Phi(z)^{-1}})_{*,\Phi(z)} \circ (\Phi)_{*,z} \frac{d}{dz}$$

$$= - \operatorname{Ad}\left(\Phi(z)\right) \circ \left(\Phi^*\theta\right)\left(\frac{d}{dz}\right),$$

where

$$\theta : GL_2(\mathbb{C}) \to T^*GL_2(\mathbb{C}) \otimes \mathfrak{gl}_2(\mathbb{C})$$

is the Maurer-Cartan form on $GL_2(\mathbb{C})$, see Definition A.2.5 and $\operatorname{Ad} : GL_2(\mathbb{C}) \to \operatorname{Aut}(\mathfrak{gl}_2(\mathbb{C}))$ is the adjoint representation of $GL_2(\mathbb{C})$ in $\mathfrak{gl}_2(\mathbb{C})$, see Definition A.2.4. We conclude that

$$A = (A)_z\ dz = - \operatorname{Ad}(\Phi) \circ \Phi^*\theta : \quad \begin{tikzcd} & U \arrow[r, "\Phi"] & GL_2(\mathbb{C}) \arrow[dll, rounded corners, "\theta"'] \\ T^*GL_2(\mathbb{C}) \otimes \mathfrak{gl}_2(\mathbb{C}) \arrow[r, "\Phi^*"] & T^*U \otimes \mathfrak{gl}_2(\mathbb{C}) \arrow[r, "\operatorname{Ad}(\Phi)"] & T^*U \otimes \mathfrak{gl}_2(\mathbb{C}) \end{tikzcd} \tag{2.4}$$

is indeed a Lie algebra valued 1-form on U, where we identify the Lie algebra $\mathfrak{gl}_2(\mathbb{C})$ of $GL_2(\mathbb{C})$ with $M_2(\mathbb{C})$.

By the discussion above, the function $\mathcal{A} : M \to M_2(\mathbb{C})$ on the RHS. of differential equation (2.3) has the interpretation as the coefficient function of a Lie algebra valued 1-form.

$$A_\alpha = -\mathcal{A}\ dz : M \to T^*M \otimes \mathfrak{gl}_2(\mathbb{C}). \tag{2.5}$$

The following Proposition is formulated and proved by the author, and to the authors knowledge, not found in other literature in this formulation.

Proposition 2.2.2.
Consider the Lie algebra valued 1-form

$$A_\alpha = -\mathcal{A}\ dz : M \to T^*M \otimes \mathfrak{gl}_2(\mathbb{C}),$$

and the differential equation

$$\mathcal{A} = \frac{d\Phi}{dz} \cdot \Phi(z)^{-1}$$

which is solved locally by $\Phi_\alpha : U_\alpha \to GL_2(\mathbb{C})$. Let $f_\beta : U_\beta \to GL_2(\mathbb{C})$ be a holomorphic function on M such that $U_\alpha \cap U_\beta \neq \emptyset$. Then the function $\Phi_\beta := f_\beta \cdot \Phi_\alpha$ defines the Lie algebra valued 1-form:

$$A_\beta = - \operatorname{Ad}(\Phi_\beta) \circ \Phi_\beta^*\theta,$$

such that

$$A_\beta = \operatorname{Ad}(f_\beta) \circ A_\alpha + (f_\beta^{-1})^*\theta, \qquad \text{on } U_\alpha \cap U_\beta.$$

Or using the basis $\left\{\frac{d}{dz}\right\}$ for T_zM, and when $G = GL_2(\mathbb{C})$, then the function $\Phi_\beta : f_\beta \cdot \Phi_\alpha$ defines the coefficient function:

$$A_\beta\left(\frac{d}{dz}\right) = -\mathcal{B} = -\frac{\partial \Phi_\beta}{dz} \cdot \Phi_\beta(z)^{-1},$$

such that

$$A_\beta\left(\frac{d}{dz}\right) = -\mathcal{B} = f_\beta \cdot \big(-\mathcal{A}\big) \cdot f_\beta^{-1} - \frac{df_\beta}{dz} \cdot f_\beta^{-1}, \text{ on } U_\alpha \cap U_\beta.$$

Compare Proposition 2.2.2 with Proposition 2.2.1. Observe how the gauge equivalence (see Definition 1.2.3), is exactly the change of local trivialization for the local family of connection forms, when the dimension of the base space is 1, and the structure group is $GL_2(\mathbb{C})$.

Proof. The fact that Φ_β defines the Lie algebra valued 1-form

$$A_\beta : U_\beta \to T^*U_\beta \otimes \mathfrak{gl}_2(\mathbb{C}),$$

is clear from what we showed leading up to Diagram (2.4).

We show that

$$A_\beta = \mathrm{Ad}(f_\beta) \circ A_\alpha + (f_\beta^{-1})^*\theta, \qquad \text{on } U_\alpha \cap U_\beta.$$

Starting with the Definition of A_β, we insert $\Phi_\beta = f_\beta \cdot \Phi_\alpha$:

$$A_\beta = -\,\mathrm{Ad}(f_\beta \cdot \Phi_\alpha) \circ (f_\beta \cdot \Phi_\alpha)^*\theta. \tag{2.6}$$

we compute:

$$(f_\beta \cdot \Phi_\alpha)_* = (L_{f_\beta})_* \circ (\Phi_\alpha)_* + (R_{\Phi_\alpha})_* \circ (f_\beta)_*.$$

Inserting this into (2.6) we obtain

$$A_\beta = -\,\mathrm{Ad}(f_\beta \cdot \Phi_\alpha) \circ \theta\Big((L_{f_\beta})_* \circ (\Phi_\alpha)_* + (R_{\Phi_\alpha})_* \circ (f_\beta)_*\Big)$$

Writing out every term we obtain

$$\begin{aligned}
= -(L_{f_\beta})_* \circ (L_{\Phi_\alpha})_* \circ (R_{f_\beta^{-1}})_* \circ (R_{\Phi_\alpha^{-1}})_* \circ (L_{\Phi_\alpha^{-1}})_* \circ (L_{f_\beta^{-1}})_* \circ (L_{f_\beta})_* \circ (\Phi_\alpha)_* \\
- (L_{f_\beta})_* \circ (L_{\Phi_\alpha})_* \circ (R_{f_\beta^{-1}})_* \circ (R_{\Phi_\alpha^{-1}})_* \circ (L_{\Phi_\alpha^{-1}})_* \circ (L_{f_\beta^{-1}})_* \circ (R_{\Phi_\alpha})_* \circ (f_\beta)_*
\end{aligned}$$

cancelling expressions, we obtain

$$\begin{aligned}
&= -\,\mathrm{Ad}(f_\beta) \circ (R_{\Phi_\alpha^{-1}})_* \circ (\Phi_\alpha)_* - (R_{f_\beta^{-1}})_*(f_\beta)_* \\
&= -\,\mathrm{Ad}(f_\beta) \circ \mathrm{Ad}(\Phi_\alpha) \circ (L_{\Phi_\alpha^{-1}})_* \circ (\Phi_\alpha)_* - \mathrm{Ad}(f_\beta) \circ (L_{f_\beta^{-1}})_* \circ (f_\beta)_* \\
&= \mathrm{Ad}(f_\beta) \circ A_\alpha - \mathrm{Ad}(f_\beta) \circ (L_{f_\beta^{-1}}) \circ \big(-(L_{f_\beta})_* \circ (R_{f_\beta})_* \circ (f_\beta^{-1})_*\big) \\
&= \mathrm{Ad}(f_\beta) \circ A_\alpha + \mathrm{Ad}(f_\beta) \circ \mathrm{Ad}(f_\beta^{-1}) \circ (L_{f_\beta})_* \circ (f_\beta^{-1})_* \\
&= \mathrm{Ad}(f_\beta) \circ A_\alpha + (f_\beta^{-1})^*\theta.
\end{aligned}$$

We notice that changing the sign of

$$A_\alpha = -\,\mathrm{Ad}(\Phi_\alpha) \circ \Phi_\alpha^*\theta,$$

changes the sign in front of $(f_\beta^{-1})^*\theta$, and would give us a formula which did not coincide with the one in Definition 2.1.9. ■

2.3 Construction of a principal bundle with a connection, from a first order linear system of differential equations.

The scheme of the following construction is to construct a principal bundle $Q(M, GL_2(\mathbb{C}), \pi)$ with base space $M = \mathbb{S} \setminus \{\, z_j \,\}_{j=1}^m$, structure group $GL_2(\mathbb{C})$, and transition functions $g_{\alpha\beta}$ which represent every $GL_2(\mathbb{C})$ valued, locally defined holomorphic function on M. Then using the Lie algebra valued 1-form A_α in equation (2.5), we construct a family of local connection forms on M.

We begin by defining the functions used to construct a principal bundle. These functions will become the transition functions.

Definition 2.3.1 Transition functions for the principal bundle.
Consider the indexed set $\{ f_\beta : U_\beta \to GL_2(\mathbb{C}) \}_{\beta \in J}$ *of every* $GL_2(\mathbb{C})$ *valued, holomorphic functions locally defined on* M. *For each* β, *we define the functions*

$$g_{\beta\alpha} : U_\beta \to GL_2(\mathbb{C}), \qquad g_{\beta\alpha}(z) = f_\beta(z) \cdot e^{-1} = f_\beta(z),$$

where $e \in GL_2(\mathbb{C})$ *is the identity matrix in* $GL_2(\mathbb{C})$. *And for each pair* $\beta, \kappa \in J$ *we define*

$$g_{\beta\kappa} : U_\beta \cap U_\kappa \to GL_2(\mathbb{C}), \qquad g_{\beta\kappa}(z) = f_\beta(z) \cdot f_\kappa(z)^{-1}.$$

Remark.

- The choice of functions in Definition 2.3.1 includes the constant function $g_{\alpha\alpha} = e : M \to G$, equal to the identity in $GL_2(\mathbb{C})$. This will make the principal bundle $Q(M, GL_2(\mathbb{C}), \pi)$ we construct, trivial, i.e. $Q \simeq M \times G$, see Corollary 2.3.1. However, still working with the principal bundle using different trivializations has the benefit of giving us an equivalence class of differential equation. Then solving one of them, we obtain solutions for the entire family by left multiplying with transition functions, see Proposition 2.2.1. This makes the comparison with the general theory of principal bundles and connection forms clearer.

- Even though the principal bundle will be trivial, the horizontal bundle coming from the local family of connection forms will not be trivial, in the sense that $s_\alpha\big(T_zM\big) \neq H_{s_\alpha(z)}$, where $s_\alpha(z) = \varphi_\alpha^{-1}(z, e)$, and $s_\alpha(M) \simeq M \times \{ e \}$.

- The functions $g_{\beta\alpha} : U_\beta \to GL_2(\mathbb{C})$ with one of the indices equal to α, are simply special cases of $g_{\beta\kappa} : U_\beta \cap U_\kappa \to GL_2(\mathbb{C})$. These transition functions connects the trivialization $\varphi_\alpha : Q \to M \times G$, to any of the other local trivializations.

It is easy to show that the relations in Proposition 2.1.2 are satisfied. Indeed

$$g_{\beta\kappa} \cdot g_{\kappa\beta} = f_\beta \cdot f_\kappa^{-1} \cdot f_\kappa \cdot f_\beta^{-1} = e \in GL_2(\mathbb{C}).$$

$$g_{\beta\kappa} \cdot g_{\kappa\gamma} = f_\beta \cdot f_\kappa^{-1} \cdot f_\kappa \cdot f_\gamma^{-1} = f_\beta \cdot f_\gamma^{-1} = g_{\beta\gamma}.$$

We will repeat the construction done in the proof of Proposition 2.1.4 for this special case. The general proof is given in Proposition 5.2, Ch.1 in [KN63].

Corollary 2.3.1 Construction of the principal bundle $Q(M, GL_2(\mathbb{C}), \pi)$.
Let $M = \mathbb{S} \setminus \{ 0, t, \infty \}$ *be regarded as a complex manifold, with the open cover* $\{ U_\beta \}$ *such that for any* $U_\beta \cap U_k \neq \emptyset$, *we have the function* $g_{\beta\kappa} : U_\beta \cap U_\kappa \to GL_2(\mathbb{C})$ *from Definition 2.3.1 satisfying the properties in Proposition 2.1.2. Then we can construct a principal bundle* $Q(M, GL_2(\mathbb{C}), \pi)$, *where the transition functions are given by* $\{ g_{\beta\kappa} \}$.

Proof. Let $X_\beta = U_\beta \times GL_2(\mathbb{C})$ for each $\beta \in J$, and let $X = \coprod_{\beta \in J} X_\beta$ be the disjoint union of the X_β's, such that each element of X is a triple (β, z, b), where β is the index, $z \in M$ and $b \in GL_2(\mathbb{C})$. Evidently X is a complex manifold, since each X_β is a complex manifold. We introduce an equivalence relation on X:

$$(\beta, z, b) \sim (\kappa, z', c) \iff z = z' \quad \& \quad b = g_{\beta\kappa}(z) \cdot c.$$

Evidently this is an equivalence relation, in particular transitivity utilizes the property $g_{\beta\kappa} \cdot g_{\kappa\gamma} = g_{\beta\gamma}$ from Proposition 2.1.2. We define the topological space

$$Q = X/\sim = \coprod_{\beta \in J} (U_\beta \times GL_2(\mathbb{C}))/\sim,$$

where we give Q the quotient topology. Thus we have the projection

$$\begin{array}{rccc} \pi: & Q & \to & M \\ & [\beta, z, b] & \mapsto & z \end{array}$$

We now define the right action of $GL_2(\mathbb{C})$ on Q,

$$\begin{array}{rccc} \mu: & Q \times GL_2(\mathbb{C}) & \to & Q \\ & \big([\beta, z, b],\ d\big) & \mapsto & [\beta, z, b \cdot d] \end{array}$$

this is well defined since if $(\kappa, z, c) \sim (\beta, z, b)$, then

$$\mu\big([\kappa, z, c], d\big) = [\kappa, z, c \cdot d] = [\beta, z, g_{\beta\kappa}(z) \cdot c \cdot d] = [\beta, z, b \cdot d] = \mu\big([\beta, z, b],\ d\big).$$

We check Property i. of Definition 2.1.1. The function is a right action, since if $p = [\beta, z, b] \in Q$

$$\mu(p, e) = \mu\big([\beta, z, b],\ e\big) = [\beta, z, b \cdot e] = [\beta, z, b] = p,$$

$$\begin{aligned} \mu(p, b \cdot d) = \mu\big([\kappa, z, c], b \cdot d\big) &= [\kappa, z, c \cdot b \cdot d] = \mu\big([\kappa, z, c \cdot b],\ d\big) \\ &= \mu\left(\mu\big([\kappa, z, c],\ b\big),\ d\right) = \mu\big(\mu(p, b),\ d\big). \end{aligned}$$

And it is free since: if $d, h \in G$ and there exists an element $p \in P$ such that

$$\mu(p, d) = [\beta, z, b \cdot d] = [\beta, z, b \cdot h] = \mu(p, h),$$

then

$$b \cdot d = g(\beta\beta)(z) \cdot b \cdot h = b \cdot h \iff d = h.$$

We will from now denote the product in $GL_2(\mathbb{C})$ by "$\cdot$" and the right action of $GL_2(\mathbb{C})$ on P by "$.$".

Property ii. follows from the fact that $\pi(p) = \pi(q) \iff q = p\,.\,d$ for some $d \in GL_2(\mathbb{C})$. Indeed this is true since if $\pi(p) = \pi(q)$, we have $q = [\beta, z, b]$ and $p = [\kappa, z, c]$. By i. there exists a $d \in GL_2(\mathbb{C})$ such that $b = g_{\beta\kappa}(z) \cdot c \cdot d$, then $p\,.\,d = [\kappa, z, c \cdot d] = [\beta, z, g_{\beta\kappa}(z) \cdot c \cdot d] = [\beta, z, b] = q$. Conversely if $q = p\,.\,d$, then $\pi(p) = \pi(q)$ by the Definition of $\sim$.

To make P into a complex manifold, we first notice that, by the quotient map $X \to Q = X/\sim$, each $X_\beta = U_\beta \times GL_2(\mathbb{C})$ is homeomorphically mapped onto $\pi^{-1}(U_\beta)$. We make Q into a manifold by requiring that $\pi^{-1}(U_\beta)$ is an open submanifold of Q and that the mapping $X \to Q = X/\sim$ is a bi-holomorphic map between $X_\beta = U_\beta \times GL_2(\mathbb{C})$ and $\pi^{-1}(U_\beta)$. This is well defined since the identification of $[\beta, z, b]$ with $[\kappa, z, g_{\kappa\beta}(z) \cdot b]$ is made by means of a holomorphic map $g_{\kappa\beta}$. The fact that Q is a complex manifold and that the right action is holomorphic readily follows.

Finally we give a description of property iii., the local trivialization. By construction we have for each U_β the following commutative diagram:

$$\begin{tikzcd} \pi^{-1}(U_\beta) \arrow[r, "\varphi_\beta"] \arrow[d, "\pi"] & U_\beta \times GL_2(\mathbb{C}) \arrow[dl, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] \\ U_\beta & GL_2(\mathbb{C}) \end{tikzcd}$$

where $\varphi_\beta : \pi^{-1}(U_\beta) \to U_\beta \times GL_2(\mathbb{C})$ is bi-holomorphic by how we defined the manifold structure on Q. The maps φ_β and g_β are explicitly given by

$$\varphi_\beta\big([\beta, z, b]\big) = (z, b), \quad g_\beta([\beta, z, b]) = \mathrm{pr}_2 \circ \varphi_\beta = b.$$

Thus

$$g_\beta([\beta, z, b] \,.\, c) = g_\beta([\beta, z, b \cdot c]) = b \cdot c = g_\beta([\beta, z, b]) \cdot c.$$

So finally

$$g_\beta\big([\beta, z, b]\big) \cdot \big(g_\kappa\big([\beta, z, b]\big)\big)^{-1} = g_\beta\big([\beta, z, b]\big) \cdot \big(g_\kappa\big([\kappa, z, g_{\kappa\beta}(z) \cdot b]\big)\big)^{-1} = b \cdot b^{-1} g_{\beta\kappa}(z),$$

so the transition function will exactly be given by $g_{\beta\kappa}$. We also have the useful formulas:

$$\varphi_\beta\big([\kappa, z, c]\big) = (z, g_{\beta\kappa}(z) \cdot c), \quad g_\beta\big([\kappa, z, c]\big) = g_{\beta\kappa}(z) \cdot c.$$

The principal bundle $Q(M, GL_2(\mathbb{C}), \pi)$ is trivial. Indeed, among the functions $g_{\beta\kappa}$, we have the function $g_{\alpha\alpha} = e : M \to GL_2(\mathbb{C})$, such that the local trivialization $\varphi_\alpha : \pi^{-1}(M) \to M \times GL_2(\mathbb{C})$, is in fact global. For any $p \in Q$, we have $p = [\alpha, z, a]$, for some $z \in M$, $a \in GL_2(\mathbb{C})$. Then

$$\varphi_\alpha(p) = \varphi_\alpha\big([\alpha, z, a]\big) = (z, a).$$

■

Now that we have defined a principal bundle $Q(M, GL_2(\mathbb{C}), \pi)$, we will construct a family of local connection forms on M. The one form A_α from equation (2.5) is defined on the whole of M. We will define the rest of the family such that it satisfies the formula in Definition 2.1.9.

Definition 2.3.2 A family of local connection forms on M.
Consider the principal bundle $Q(M, GL_2(\mathbb{C}), \pi)$ from Corollary 2.3.1. Using the Lie algebra valued 1-form A_α from equation (2.5), we define a family of local connection forms

$$\{A_\beta : U_\beta \to T^*U_\beta \otimes \mathfrak{gl}_2(\mathbb{C})\}_{\beta \in J},$$

on M, given by

$$A_\beta := \mathrm{Ad}(g_{\beta\alpha}) \circ A_\alpha + g^*_{\alpha\beta}\theta,$$

for each $\beta \in J$, that is, one for each transition function $g_{\beta\alpha} = f_\beta$.

Lemma 2.3.1.
The family of Lie algebra valued 1-forms defined in Definition 2.3.2, is a family of local connection forms. That is, for $\beta, \kappa \in J$ such that $U_\beta \cap U_\kappa \neq \emptyset$

$$A_\kappa = \mathrm{Ad}(g_{\kappa\beta}) \circ A_\beta + g^*_{\beta\kappa}\theta.$$

Proof. Let $\beta, \kappa \in J$ such that $U_\beta \cap U_\kappa \neq \emptyset$. Then

$$\mathrm{Ad}(g_{\kappa\beta}) \circ A_\beta + g^*_{\beta\kappa}\theta = \mathrm{Ad}(g_{\kappa\beta}) \circ \big(\mathrm{Ad}(g_{\beta\alpha}) \circ A_\alpha + g^*_{\alpha\beta}\theta\big) + g^*_{\beta\kappa}\theta$$

$$= \mathrm{Ad}(g_{\kappa\alpha}) \circ A_\alpha + \mathrm{Ad}(g_{\kappa\beta}) \circ g^*_{\alpha\beta}\theta + g^*_{\beta\kappa}\theta$$

To conclude we need to show that

$$\mathrm{Ad}(g_{\kappa\beta}) \circ g^*_{\alpha\beta}\theta + g^*_{\beta\kappa}\theta = g^*_{\alpha\kappa}\theta. \tag{2.7}$$

We compute the term $g^*_{\beta\kappa}\theta = (g_{\beta\alpha} \cdot g_{\alpha\kappa})^*\theta$, by using the Leibniz rule

$$(g_{\beta\alpha} \cdot g_{\alpha\kappa})^*\theta = \theta\big((g_{\beta\alpha} \cdot g_{\alpha\kappa})_*\big) = \theta\big((L_{g_{\beta\alpha}})_* \circ (g_{\alpha\kappa})_* + (R_{g_{\alpha\kappa}})_* \circ (g_{\beta\alpha})_*\big)$$

$$= (L_{g_{\kappa\alpha}})_* \circ (L_{g_{\alpha\beta}})_* \circ (L_{g_{\beta\alpha}})_* \circ (g_{\alpha\kappa})_* + (L_{g_{\kappa\alpha}})_* \circ (L_{g_{\alpha\beta}})_* \circ (R_{g_{\alpha\kappa}})_* \circ (g_{\beta\alpha})_*$$

$$\implies g^*_{\beta\kappa}\theta = g^*_{\alpha\kappa}\theta + \mathrm{Ad}(g_{\kappa\alpha}) \circ g^*_{\beta\alpha}\theta. \tag{2.8}$$

Finally we compute $g^*_{\beta\alpha}\theta$ in terms of $g_{\alpha\beta}$. We differentiate the constant map $e = g_{\alpha\beta} \cdot g_{\beta\alpha}$

$$0 = (L_{g_{\alpha\beta}})_* \circ (g_{\beta\alpha})_* + (R_{g_{\beta\alpha}})_* \circ (g_{\alpha\beta})_* \iff (g_{\beta\alpha})_* = -(L_{g_{\beta\alpha}})_* \circ (R_{g_{\beta\alpha}})_* \circ (g_{\alpha\beta})_*$$

Applying θ on both sides:

$$g^*_{\beta\alpha}\theta = -(L_{g_{\alpha\beta}})_* \circ (L_{g_{\beta\alpha}})_* \circ (R_{g_{\beta\alpha}})_* \circ (g_{\alpha\beta})_* = -\operatorname{Ad}(g_{\alpha\beta}) \circ g^*_{\alpha\beta}\theta \tag{2.9}$$

Finally, we can conclude by inserting (2.9) into (2.8):

$$g^*_{\beta\kappa}\theta = g^*_{\alpha\kappa}\theta - \operatorname{Ad}(g_{\kappa\alpha}) \circ \operatorname{Ad}(g_{\alpha\beta}) \circ g^*_{\alpha\beta}\theta \iff \operatorname{Ad}(g_{\kappa\beta}) \circ g^*_{\alpha\beta}\theta + g^*_{\beta\kappa}\theta = g^*_{\alpha\kappa}\theta.$$

which is exactly (2.7). ■

Corollary 2.3.2 Connection form and principal connection on $Q(M, GL_2(\mathbb{C}), \pi)$.
Consider the principal bundle $Q(M, GL_2(\mathbb{C}), \pi)$ constructed in Corollary 2.3.1, together with the family of local connection forms from Definition 2.3.2. The associated connection form ω on Q, is in each local trivialization

$$\varphi_\beta = \pi \times g_\beta : \pi^{-1}(U_\beta) \to U_\beta \times G,$$

given by

$$\omega_\beta = \operatorname{Ad}\left(g_\beta^{-1}\right) \circ \pi^* A_\beta + g^*_\beta\theta.$$

Moreover, using the trivialization

$$\varphi_\alpha : \pi \times g_\alpha : Q \to M \times G,$$

ω is globally given by

$$\omega = \operatorname{Ad}\left(g_\alpha^{-1}\right) \circ \pi^* A_\alpha + g^*_\alpha\theta.$$

The horizontal bundle $H \subset TP$ is given by

$$H = \ker(\omega) = \ker\left(\operatorname{Ad}\left(g_\alpha^{-1}\right) \circ \pi^* A_\alpha + g^*_\alpha\theta\right).$$

Proof. This is just direct applications of Proposition 2.1.6 and Proposition 2.1.7. ■

As a summary of the above construction, we took every $GL_2(\mathbb{C})$ valued, holomorphic function $f_\beta : U_\beta \to GL_2(\mathbb{C})$, locally defined on $U_\beta \subset M = \mathbb{S} \setminus \{ z_j \}_{j=1}^m$, and defined the functions $g_{\beta\kappa} = f_\beta \cdot f_\kappa^{-1}$. We then defined a principal bundle $Q(M, GL_2(\mathbb{C}), \pi)$ with base space M and structure group $GL_2(\mathbb{C})$, with the given functions $g_{\beta\kappa}$ as transition functions. The Lie algebra valued 1-form A_α on M, was then used to construct a family of local connection forms on M. The family of local connection forms $\{ A_\beta \}_{\beta \in J}$ on M, give a connection form ω on Q and a horizontal bundle $H \subset TP$.

The following Lemma and Theorem is formulated and proved by the author, and to the authors knowledge, not found in other literature using this formulation. Remark that the results are stated and proved for an arbitrary principal bundle possessing a connection.

Lemma 2.3.2 Horizontal section formulated with different notion of a connection.
Consider a principal bundle $P(N, G, \pi)$, with principal connection $H \subset TP$, connection form ω, and family of local connection forms $\{ A_\kappa \}_{\kappa \in J}$. Let $\hat{\Phi} : U \to \pi^{-1}(U) \subset P$, be a holomorphic section of P. The following statements are equivalent:

i. $\hat{\Phi}$ is horizontal (see Definition 2.1.6).

ii. $\hat{\Phi}^*\omega = 0$.

iii. *For each trivialization* $\varphi_\kappa = \pi \times g_\kappa : \pi^{-1}(U_\kappa) \to U_\kappa \times G$, *such that* $U_\kappa \cap U \neq \emptyset$, *the following relation holds:*

$$A_\kappa = -\operatorname{Ad}\left(g_\kappa \circ \hat{\Phi}\right) \circ \left(g_\kappa \circ \hat{\Phi}\right)^* \theta, \quad \text{in } U_\kappa \cap U.$$

Proof. First, *i.* $\iff$ *ii.* is obvious, since by Definition 2.1.6 of a horizontal section, $\hat{\Phi}_{*,z}(T_z U) = H_{\hat{\Phi}(z)}$ and by Proposition 2.1.6, $H_{\hat{\Phi}(z)} = \ker(\omega_{\hat{\Phi}(z)})$.

We show that *ii.* $\iff$ *iii.*
Given a trivialization $\varphi_\kappa = \pi \times g_\kappa : \pi^{-1}(U_\kappa) \to U_\kappa \times G$, such that $U_\kappa \cap U \neq \emptyset$, we use Proposition 2.1.7 and compute

$$\hat{\Phi}^*\omega = \hat{\Phi}^*\left(\operatorname{Ad}(g_\kappa^{-1}) \circ \pi^* A_\kappa + g_\kappa^*\theta\right) = \operatorname{Ad}\left(\left(g_\kappa \circ \hat{\Phi}\right)^{-1}\right) \circ \hat{\Phi}^*\pi^* A_\kappa + \hat{\Phi}^* g_\kappa^*\theta,$$

using the co-functorial property of the pull-back and the fact that $\hat{\Phi}$ is a section:

$$\hat{\Phi}^*\omega = \operatorname{Ad}\left(\left(g_\kappa \circ \hat{\Phi}\right)^{-1}\right) \circ A_\kappa + \left(g_\kappa \circ \hat{\Phi}\right)^*\theta. \tag{2.10}$$

Solving (2.10) for A_κ, we obtain:

$$A_\kappa = \operatorname{Ad}\left(g_\kappa \circ \hat{\Phi}\right) \circ \left(\hat{\Phi}^*\omega - \left(g_\kappa \circ \hat{\Phi}\right)^* \theta\right).$$

Now, it is obvious that *iii.* holds if and only if $\hat{\Phi}^*\omega = 0$.

■

Theorem 2.3.1 Characterization of horizontal sections of a principal bundle. *Consider a principal bundle* $P(N, G, \pi)$, *with a family of local connection forms* $\{ A_\beta \}_{\beta \in J}$. *Let* $\beta \in J$. *The following statements are equivalent:*

i. *There exists a horizontal section* $\hat{\Phi} : U_\beta \to \pi^{-1}(U_\beta) \subset P$.

ii. *There exists a holomorphic function* $\Phi_\beta : U_\beta \to G$, *such that*

$$A_\beta = -\operatorname{Ad}(\Phi_\beta) \circ \Phi_\beta^*\theta, \quad \text{in } U_\beta \subset N.$$

iii. *For each* $\kappa \in J$ *such that* $U_\kappa \cap U_\beta \neq \emptyset$, *there exists a holomorphic function* $\Phi_\kappa : U_\kappa \cap U_\beta \to G$, *such that*

$$A_\kappa = -\operatorname{Ad}(\Phi_\kappa) \circ \Phi_\kappa^*\theta, \quad \text{in } U_\kappa \cap U_\beta \subset N.$$

Moreover, if $U_\kappa \cap U_\gamma \cap U_\beta \neq \emptyset$, *we have the relation* $\Phi_k = g_{\kappa\gamma} \cdot \Phi_\gamma$.

Remark.
The fact that a function $\Phi_\beta : U_\beta \to G$, satisfies

$$A_\beta = -\operatorname{Ad}(\Phi_\beta) \circ \Phi_\beta^*\theta, \quad \text{in } U_\beta \subset N,$$

can be written in terms of the coordinate functions related to the basis $\{\frac{\partial}{\partial z^k}\}$ for $T_z N$. If we consider

$$A_\beta = (A_\beta)_k \, dz^k,$$

then

$$(A_\beta)_k = A_\beta\left(\frac{\partial}{\partial z^k}\right) = -\operatorname{Ad}\left(\Phi_\beta(z)\right) \circ (\Phi_\beta^*\theta)_z \left(\frac{\partial}{\partial z^k}\right),$$

in $U_\beta \subset N$, for $k \in \{1, \dots, \dim(N)\}$. And if $G = \mathrm{GL}_n(\mathbb{C})$:

$$(A_\beta)_k = A_\beta\left(\frac{\partial}{\partial z^k}\right) = -\frac{\partial \Phi_\beta}{\partial z^k} \cdot \Phi_\beta(z)^{-1},$$

in $U_\beta \subset N$, for $k \in \{1, \dots, \dim(N)\}$.

Proof. We will show that $i. \implies iii. \implies ii. \implies i.$.
We first show that $i \implies iii$. This is easy, since by Lemma 2.3.2 we have that $\hat{\Phi}$ is a horizontal section if and only if for each trivialization $\varphi_\kappa = \pi \times g_\kappa : \pi^{-1}(U_\kappa) \to U_\kappa \times G$, such that $U_\kappa \cap U_\beta \neq \emptyset$:

$$A_\kappa = -\operatorname{Ad}\left(g_\kappa \circ \hat{\Phi}\right) \circ \left(g_\kappa \circ \hat{\Phi}\right)^* \theta, \quad \text{in } U_\kappa \cap U_\beta.$$

We define the holomorphic function

$$\Phi_\kappa : U_\kappa \cap U_\beta \to G, \quad \Phi_\kappa := g_\kappa \circ \hat{\Phi}.$$

If $U_\kappa \cap U_\gamma \cap U_\beta \neq \emptyset$, then by Definition 2.1.3

$$\Phi_\kappa(z) = g_\kappa \circ \hat{\Phi}(z) = g_{\kappa\gamma} \circ \pi \circ \hat{\Phi}(z) \cdot g_\gamma \circ \hat{\Phi}(z) = g_{\kappa\gamma}(z) \cdot \Phi_\gamma(z)$$

Hence $i. \implies iii.$

We show that $iii. \implies ii$. Assume that for each $\kappa \in J$, such that $U_\kappa \cap U_\beta \neq \emptyset$, there exists a holomorphic function $\Phi_k : U_\kappa \cap U_\beta \to G$ with

$$A_\kappa = -\operatorname{Ad}(\Phi_\kappa) \circ \Phi_\kappa^* \theta, \quad \text{in } U_\kappa \cap U_\beta \subset N.$$

Then in particular, for $\kappa = \beta \in J$, we have the function $\Phi_\beta : U_\beta \to G$, with

$$A_\beta = -\operatorname{Ad}(\Phi_\beta) \circ \Phi_\beta^* \theta, \quad \text{in } U_\beta \subset N.$$

We show that $ii. \implies i$. Assume there exists a holomorphic function $\Phi_\beta : U_\beta \to G$, such that

$$A_\beta = -\operatorname{Ad}(\Phi_\beta) \circ \Phi_\beta^* \theta, \quad \text{in } U_\beta \subset N.$$

Using the local trivialization

$$\varphi_\beta = \pi \times g_\beta : \pi^{-1}(U_\beta) \to U_\beta \times G,$$

we define the section

$$\hat{\Phi} : U_\beta \to \pi^{-1}(U_\beta), \quad \hat{\Phi}(z) := \varphi_\beta^{-1}\big(z, \Phi_\beta(z)\big).$$

Obviously it is holomorphic since it is a composition of holomorphic functions. Notice that

$$g_\beta \circ \hat{\Phi} = g_\beta \circ \varphi_\beta^{-1}\big(z, \Phi_\beta(z)\big) = \mathrm{pr}_2 \circ \varphi_\beta \circ \varphi_\beta^{-1} \circ \left(I_{U_\beta} \times \Phi_\beta\right) = \Phi_\beta,$$

thus $\hat{\Phi}$ is horizontal by Lemma 2.3.2, since

$$A_\beta = -\operatorname{Ad}(\Phi_\beta) \circ \Phi_\beta^* \theta = -\operatorname{Ad}\left(g_\beta \circ \hat{\Phi}\right) \circ \left(g_\beta \circ \hat{\Phi}\right)^* \theta, \quad \text{in } U_\beta \subset N.$$

■

Relating this geometric construction to the problem of solving a differential equation reveals that the given differential equation is just one of a family of differential equation. If we find a solution Φ_β of one of the equations A_β, we can construct a solution for any A_κ in the family such that $U_\beta \cap U_\kappa \neq 0$, simply by multiplying by the transition function $\Phi_\kappa = g_{\kappa\beta} \cdot \Phi_\beta$. Theorem 2.3.1, shows that in this geometric setting, we can solve the given differential equation in equation (2.3), by finding horizontal sections of Q.

2.4 Frobenius integrability of horizontal distributions

In this Section we will use the knowledge of a connection on a principal bundle, to construct a Lie algebra valued 2-form on P, called the curvature form. The curvature form measures how far the horizontal distribution $H \subset TP$ is from being involutive. If the curvature form is zero, H will be integrable by the Frobenius Theorem 2.4.1. This means that at any point $p \in P$ there is an unique submanifold S of P, such that $T_pS = H_p \subset T_pP$. We will use this to induce a horizontal section of P. Thus by Theorem 2.3.1, finding a solution to the differential equation (2.3).

We start of with giving the preliminary definitions, and the statement of the Frobenius Theorem. The Frobenius Theorem is an essential tool in differential geometry. It uses knowledge of a subbundle of the tangent bundle on a manifold P, with commuting vector fields, in order to infer the existence of submanifolds of P. In order to present it, we need some definitions.

Definition 2.4.1 Distributions, involutive and integrable.

- *Let P be an n-dimensional complex manifold. An $r-$dimensional distribution on P, is a collection $\mathcal{D} = \{ D_p \mid p \in P \}$ of $r-$dimensional subspaces $D_p \subset T_pP$, one for each $p \in P$, that are holomorphic in the sense that they are locally described by the span of r-holomorphic vector fields on P.*
- *A distribution is integrable if for any point $p \in P$ there is an unique submanifold S of P, such that the tangent space $T_pS = D_p \subset T_pP$.*
- *A distribution is said to be involutive if for any sections $v, w : \tilde{U} \subset P \to \mathcal{D}$, then the Lie bracket $[v, w]$ is also a section of $\mathcal{D}$. That is, if $\mathfrak{X}(P; \mathcal{D})$ denotes the subspace of $\mathfrak{X}(P)$ consisting of sections on P with range in $\mathcal{D}$, then $\mathcal{D}$ is involutive if $\mathfrak{X}(P; \mathcal{D})$ is also a Lie subalgebra.*

Theorem 2.4.1 [T.4.1, [Sha97]] The Frobenius Theorem.
Let P be an n-dimensional complex manifold with an $r-$dimensional distribution $\mathcal{D} \subset TP$. Then $\mathcal{D}$ is integrable if and only if it is involutive.

We will now create tools in order to apply the Frobenius Theorem to a horizontal distribution on a principal bundle. First we define the horizontal projection.

Definition 2.4.2 Horizontal projection.
Given a principal bundle $P(M, G, \pi)$ with a connection $H \subset TP$. We define the horizontal projection $h : TP \to H$ by for each $p \in P$:

$$\begin{array}{rccl} h_p : & T_pP & \to & H_p \\ & v & \mapsto & h_pv = \begin{cases} v, & \text{if } v \in H_p \\ 0, & \text{if } v \in V_p \end{cases} \end{array}$$

and then extending by linearity. We will let $h_p^* : T_p^*P \to H_p^*$ denote the dual maps of the horizontal projection.

Remark that h^* is not the pull-back by a holomorphic map. In particular, h^* does not commute with the exterior derivative on P.

Definition 2.4.3 Curvature form.
Consider a principal bundle $P(M, G, \pi)$ and a connection $H \subset TP$, such that $H = \ker(\omega)$, where ω is the connection form. We define the curvature form to be the Lie

algebra valued 2-form on P given by

$$\Omega = h^* d\omega : P \to \bigwedge^2 T^* P \otimes \mathfrak{g}$$

$$\Omega(u,v) = \big(h^* d\omega\big)(u,v) = d\omega(hu, hv),$$

where d is the exterior derivative.

Using the coordinate independent formula for the exterior derivative of a 1-form, we can give another expression for the curvature form. If $u, v \in T_pP$:

$$\begin{aligned}\Omega(u,v) &= \big(h^* d\omega\big)(u,v) = d\omega\big(hu, hv\big) \\ &= hu\ \omega(hv) - hv\ \omega(hu) - \omega[hu, hv] = -\omega[hu, hv],\end{aligned} \tag{2.11}$$

here we used that $hu \in \ker(\omega)$ since it is horizontal. We see that $\Omega = 0$ if and only if $[hu, hv]$ is horizontal, for any $u, v \in T_pP$. Thus H is an involutive distribution if and only if Ω is zero. By the Frobenius Theorem 2.4.1, H is an integrable distribution if an only if $\Omega = 0$.

We will derive another formula for Ω, one that resembles the structure equation for the Maurer-Cartan form, see Proposition A.2.2. As the construction from the previous Section might have revealed, a connection form on a principal bundle is a generalization of the Maurer-Cartan form on a Lie group.

Proposition 2.4.1 Structure equation for the curvature form.
Given a principal bundle $P(M, G, \pi)$ with a connection form ω, and curvature form Ω. Then

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega],$$

that is, for any $u, v \in T_pP$

$$\Omega(u,v) = d\omega(u,v) + \frac{1}{2}[\omega \wedge \omega](u,v) = d\omega(u,v) + [\omega(u), \omega(v)] \overset{(2.11)}{=} -\omega[hu, hv].$$

The notation $[\omega \wedge \omega]$ is explained in Proposition A.2.2.

Proof. Proof given in [KN63]. ∎

Analogous to the family of local connection forms $\{ A_\beta \}_\beta$ on M, induced by a connection form ω on P, see Proposition 2.1.7, we can induce a family of local curvature forms $\{ F_\beta \}_\beta$ on M, by the curvature form Ω on P.

Definition 2.4.4 Family of local curvature forms.
Consider a principal bundle $P(M, G, \pi)$ with a connection form ω and curvature form Ω. Locally on M, we define a family of Lie algebra valued 2-forms, $\{ F_\beta \}_\beta$,

$$F_\beta : U_\beta \to \bigwedge^2 T^* U_\beta \otimes \mathfrak{g},$$

defined by

$$F_\beta := s_\beta^* \Omega,$$

where $s_\beta : U_\beta \to \pi^{-1}(U_\beta)$ is the trivial sections of the local trivializations of P, see Definition 2.1.4.

Proposition 2.4.2 Structure equation for local curvature forms.
Let $P(M, G, \pi)$ be a principal bundle with a connection form ω and curvature form Ω. Let $\{ A_\beta \}_\beta$ be the family of local connections forms on M. Consider a family of local curvature forms on M, induced by the curvature form Ω. Then

$$F_\beta = dA_\beta + \frac{1}{2}[A_\beta \wedge A_\beta].$$

Proof. We use the structure equation for the curvature form in Proposition 2.4.1, and compute

$$F_\beta = s_\beta^*\Omega = s_\beta^*\left(d\omega + \frac{1}{2}[\omega\wedge\omega]\right) = d(s_\beta^*\omega) + \frac{1}{2}s_\beta^*[\omega\wedge\omega] = dA_\beta + \frac{1}{2}[A_\beta\wedge A_\beta].$$

■

Corollary 2.4.1 Change of trivialization for local curvature forms.
A family of local curvature forms $\{F_\beta\}_{\beta\in J}$ *are related by*

$$F_\kappa = \mathrm{Ad}(g_{\kappa\beta})\circ F_\beta, \qquad \textit{for } U_\kappa\cap U_\beta\neq\emptyset$$

Proof. We let $U_\kappa\cap U_\beta\neq\emptyset$, and use Proposition 2.4.1 combined with Definition 2.1.9

$$F_\kappa = dA_\kappa + \frac{1}{2}[A_\kappa\wedge A_\kappa]$$

$$= d\left(\mathrm{Ad}(g_{\kappa\beta})\circ A_\beta + g_{\beta\kappa}^*\theta\right) + \frac{1}{2}\left[\mathrm{Ad}(g_{\kappa\beta})\circ A_\beta + g_{\beta\kappa}^*\theta,\ \mathrm{Ad}(g_{\kappa\beta})\circ A_\beta + g_{\beta\kappa}^*\theta\right]$$

$$= \mathrm{Ad}(g_{\kappa\beta})\circ\left(dA_\beta + \frac{1}{2}[A_\beta\wedge A_\beta]\right) + d(g_{\beta\kappa}^*\theta)$$
$$+ \frac{1}{2}\Big([\mathrm{Ad}(g_{\kappa\beta})\circ A_\beta,\ g_{\beta\kappa}^*\theta] + [g_{\beta\kappa}^*\theta,\ \mathrm{Ad}(g_{\kappa\beta}\circ A_\beta)] + [g_{\beta\kappa}^*\theta, g_{\beta\kappa}^*\theta]\Big)$$
$$= \mathrm{Ad}(g_{\kappa\beta})\circ F_\beta + d(g_{\beta\kappa}^*\theta) + \frac{1}{2}[g_{\beta\kappa}^*\theta, g_{\beta\kappa}^*\theta].$$

To conclude, we need to show that $d(g_{\beta\kappa}^*\theta) + \frac{1}{2}[g_{\beta\kappa}^*\theta, g_{\beta\kappa}^*\theta] = 0$.

$$d(g_{\beta\kappa}^*\theta) + \frac{1}{2}\left[g_{\beta\kappa}^*\theta,\ g_{\beta\kappa}^*\theta\right] = g_{\beta\kappa}^*\left(d\theta + \frac{1}{2}[\theta\wedge\theta]\right) = 0$$

where the last equality follows from Proposition A.2.1. ■

We now return to the constructed principal bundle $Q(M, GL_2(\mathbb{C}), \pi)$ from Corollary 2.3.1. Even without computing the terms in the structure equations in Proposition 2.4.1 or Proposition 2.4.2, we know that $\Omega = 0$. This is simply due to a dimension argument. Since the dimension of M is 1, the dimension of the horizontal tangent space is also 1. Hence the bracket of horizontal vector fields is always zero. The Frobenius Theorem thus apply, and at each $p\in P$, there exists a unique submanifold $S\subset P$, such that $T_pS = H_p\subset T_pP$.

The two following Corollaries are proved for an arbitrary principal bundle, with an integrable horizontal bundle. They are formulated and proved by the author, and to the authors knowledge, not found in other literature in this formulation.

Corollary 2.4.2 Horizontal distributions induces sections from M to P.
Let $P(M, G, \pi)$ be a principal bundle, with a connection $H\subset TP$. Assume H is integrable, equivalently involutive, equivalently that the curvature form Ω is zero. Let $S\subset P$ be the unique submanifold through the point $p\in P$, such that $T_pS = H_p\subset T_pP$. Then in a neighbourhood $U\subset M$ of $\pi(p) = z$, there exists a horizontal section $\hat{\Phi}: U\to S$, such that, $\hat{\Phi}(z) = p$.
The section is unique in the sense that: if there exists another horizontal section $\hat{\Psi}: U'\to S$, with $\hat{\Psi}(z) = p$, then $\hat{\Phi} = \hat{\Psi}$ on $U\cap U'$.

Proof. Since S is a submanifold there exists a chart $\psi : S \to \mathbb{C}^n$, where $n = \dim(H_p) = \dim(M)$, with $\psi^{-1}(0) = p$. Since $T_pS = H_p$,

$$\left\{ (\psi^{-1})_{*,0} \frac{\partial}{\partial x^j} \right\}_{j=1}^{n}$$

is a basis for H_p. We have that

$$\pi \circ \psi^{-1} : \quad B(0,\epsilon) \subset \mathbb{C}^n \quad \to \quad U' \subset M$$

is a holomorphic function between two complex manifolds of dimension n. We will show that $\pi \circ \psi^{-1}$ is locally invertible. By the (complex) inverse function Theorem [FG02][p.33] we need to show that the Jacobian matrix of $\pi \circ \psi^{-1}$ is non-zero at $0 \in \mathbb{C}^n$. Let $(U, z^1, \ldots, z^n)$ be a chart around $\pi \circ \psi^{-1}(0) = \pi(p)$ in M. We recall the chart basis $\left\{ \widetilde{\frac{\partial}{\partial z^k}} \right\}$ for H_p (see Definition 2.1.7). For clarity we write S instead of $\psi^{-1}(B(0,\epsilon))$ etc. The Jacobian is computed by considering

$$\pi \circ (\psi^{-1}) : \mathbb{C}^n \to S \to M.$$

and mapping the basis for $T_0\mathbb{C}^n$ through its tangential map:

$$(\pi \circ \psi^{-1})_{*,0} : T_0\mathbb{C}^n \to H_p \to T_{\pi(p)}M$$

$$(\pi \circ \psi^{-1})_{*,0} \left(\frac{\partial}{\partial x^k} \right) = \pi_{*,p} \circ (\psi^{-1})_{*,0} \left(\frac{\partial}{\partial x^k} \right)$$

$$= \pi_{*,p} \left(\left(\frac{\partial(\psi^{-1})}{\partial x^j} \right)^k \widetilde{\frac{\partial}{\partial z^k}} \right) = \left(\frac{\partial(\psi^{-1})}{\partial x^j} \right)^k \frac{\partial}{\partial z^k}.$$

Thus we need to show that the matrix $\left(\frac{\partial(\psi^{-1})}{\partial x^j} \right)^k$ is invertible. However this is exactly the condition that $\left\{ (\psi^{-1})_{*,0} \frac{\partial}{\partial x^j} \right\}_{j=1}^{n}$ is a basis for H_p. Thus, we conclude that by the invertible function Theorem, that in a neighbourhood U of $\pi \circ \psi^{-1}(0) \in M$, there exists a holomorphic function $y : U \to \mathbb{C}^n$, such that $\pi \circ \psi^{-1} \circ y(z) = z$ and $y \circ \pi \circ \psi^{-1}(x) = x$. We define the holomorphic section

$$\hat{\Phi} : U \to \psi^{-1}\big(y(U)\big) \subset S, \qquad z \mapsto \psi^{-1} \circ y(z) = [\alpha,\ \pi \circ \psi^{-1} \circ y(z),\ g_\alpha \circ \psi^{-1} \circ y(z)] = [\alpha, z, \Phi_\alpha(z)].$$

Note that $\hat{\Phi}$ is actually a bi-holomorphic function. The fact that $\hat{\Phi}_{*,z} \left(\frac{\partial}{\partial z^k} \right)$ is horizontal, is obvious since

$$\hat{\Phi}_{*,z} \left(\frac{\partial}{\partial z^k} \right) = (\psi^{-1})_{*,0} \left(y_{*,z} \frac{\partial}{\partial z^k} \right)$$

and the image of $(\psi^{-1})_*$ is horizontal, see Figure 2.1 for an illustration.

We now prove the uniqueness. Let $q \in \hat{\Phi}(U) \subset S$, $\pi(q) = z_q \in U$. Since $\hat{\Phi}$ is a bi-holomorphic section from U to $\hat{\Phi}(U) = \tilde{U}$, it has inverse function $\pi\big|_{\tilde{U}}$. Hence there is only one point in S above $z_q \in M$. Thus any other section $\hat{\Psi}' : U' \to S$, with $\hat{\Psi}(z) = p$, $\hat{\Psi}_{*,z} T_z U' = H_p$, equals $\hat{\Phi}$ on $U \cap U'$. ∎

Corollary 2.4.3 Relation between sections.
Consider a principal bundle $P(M, G, \pi)$, with a connection $H \subset TP$. Let $p, q \in P$ with $\pi(p) = z_0 = \pi(q)$, and $q = p \,.\, a$. Apply Corollary 2.4.2 to obtain two unique sections $\hat{\Phi}^p : U \to S^p$ and $\hat{\Phi}^q : U \to S^q$, into two distinct submanifolds S^p and S^q of P, in a common domain $U \subset M$. Then $\hat{\Phi}^q(z) = \hat{\Phi}^p(z) \,.\, a$.

Proof. By the uniqueness of Corollary 2.4.2, there is only one section $\hat{\Phi}^q : U \to S^q$, with $\hat{\Phi}^q(z_0) = q$ and $\hat{\Phi}^q_{*,z}(T_{z_0}U) = H_q$. Consider the section

$$\hat{\Phi}^p \,.\, a : U \to \pi^{-1}(U).$$

Obviously, $\hat{\Phi}^p(z_0) \,.\, a = p \,.\, a = q$. Also,

$$\left(\hat{\Phi}^p \,.\, a\right)_{*,z_0} (T_{z_0}U) = (\mu_a)_{*,p} \circ \hat{\Phi}^p_{*,z_0}(T_{z_0}U) = (\mu_a)_{*,p} H_p = H_q,$$

by Property ii. of a principal connection, see Definition 2.1.6. The image of $\hat{\Phi}^p \,.\, a$ induces a submanifold through $p \,.\, a = q \in P$, with tangent space H_q. By the uniqueness of S^q in the Frobenius Theorem, this submanifold is S^q (or at least a subset of S^q). Hence $\hat{\Phi}^p \,.\, a : U \to S^q$, and thus $\hat{\Phi}^p \,.\, a = \hat{\Phi}^q$. ∎

Theorem 2.4.2 Existence and uniqueness of local fundamental solutions to (2.3).
Consider the differential equation (2.3):

$$\mathcal{A}(z) = \frac{d\Phi}{dz} \cdot \Phi(z)^{-1}, \quad \mathcal{A} : \mathbb{S} \setminus \{\, z_j \,\}_{j=1}^m \to M_2(\mathbb{C})$$

i. Let $a_0 \in GL_2(\mathbb{C})$, and $z_0 \in \mathbb{S} \setminus \{\, z_j \,\}_{j=1}^m = M$. Then there is a neighbourhood $U \subset M$ of z_0, where there exists a unique fundamental solution $\Phi_\alpha : U \to GL_2(\mathbb{C})$, such that $\Phi_\alpha(z_0) = a_0$.

ii. If $a_0, b_0 \in GL_2(\mathbb{C})$, and the functions $\Phi_\alpha : U \to GL_2(\mathbb{C})$ and $\Phi'_\alpha : U' \to GL_2(\mathbb{C})$ solves the differential equation with $\Phi_\alpha(z_0) = a_0$, $\Phi'_\alpha(z_0) = b_0$. Then $\Phi'_\alpha = \Phi_\alpha \cdot a_0^{-1} \cdot b_0$ in $U \cap U'$.

Compare this result with the findings of Section 1.2.3, and Lemma 1.2.2.

Proof. We will give a guide on how to apply the above results in order to prove the existence and uniqueness. First, the differential equation is rewritten in geometrical terms,

$$A_\alpha = -\mathcal{A}\; dz = -\operatorname{Ad}(\Phi) \circ \Phi^*\theta,$$

and the function $-\mathcal{A}$ is interpreted as the coordinate function of a Lie algebra valued 1-form A_α, see equation (2.5). We then construct the principal bundle $Q(M, GL_2(\mathbb{C}), \pi)$ in Corollary 2.3.1. A_α defines a family of local connection forms on M in Definition 2.3.2, and thus induces a connection on $Q(M, GL_2(\mathbb{C}), \pi)$, see Corollary 2.3.2. By Theorem 2.3.1, solving the differential equation

$$\mathcal{A} = \frac{d\Phi}{dz} \cdot \Phi(z)^{-1} \tag{2.12}$$

locally, is now equivalent to finding a horizontal section $\hat{\Phi} : U \to \pi^{-1}(U) \subset Q$.

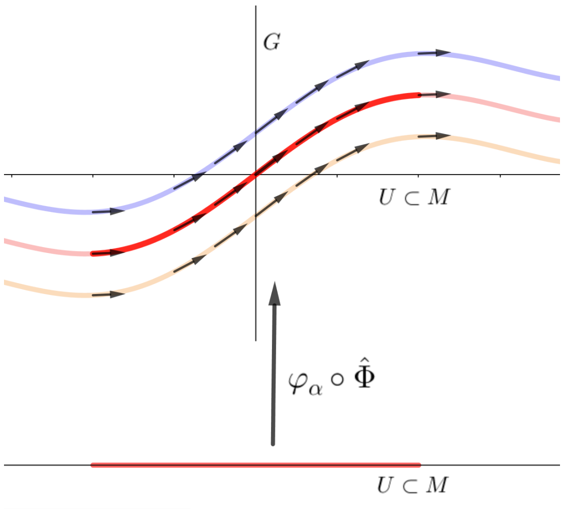


Figure 2.1: The figure shows a section $\varphi_\alpha \circ \hat{\Phi} : U \to U \times G$. The arrows illustrate the horizontal tangent vectors, and the red, blue and orange lines are the submanifolds induced by the Frobenius Theorem. The word "integrable", in integrable distribution, refers to the vector field(s) which is "integrated" to obtain the function $\hat{\Phi}$.

By Definition 2.4.3 of the curvature form of the connection, $\Omega = 0$, since M has dimension 1. Thus we can apply the Frobenius Theorem 2.4.1, which in each point $p \in Q$, gives a unique submanifold S, with $T_pS = H_p$. Now let $z_0 \in M$, $a_0 \in G$. We then obtain a unique point $p = [\alpha, z_0, a_0] \in Q$, using the trivialization φ_α. Applying Corollary 2.4.2, we obtain a unique horizontal section $\hat{\Phi} : U \to S$, defined in some neighbourhood U of z_0, with $\hat{\Phi}(z_0) = [\alpha, z_0, a_0]$ and $\hat{\Phi}_{*,z_0}(T_{z_0}U) = H_p$. Finally, Proposition 2.3.1 implies that $\Phi_\alpha = g_\alpha \circ \hat{\Phi} : U \to GL_2(\mathbb{C})$, is the desired solution.

The second statement is a direct consequence of Corollary 2.4.3. Indeed, if we let $p = [\alpha, z_0, a_0]$ and $q = [\alpha, z_0, b_0]$, we have

$$\Phi'_\alpha(z) = g_\alpha \circ \hat{\Phi}^q(z) = g_\alpha \left(\hat{\Phi}^p(z) \,.\, a_0^{-1} \cdot b_0\right) = \left(g_\alpha \circ \hat{\Phi}^p(z)\right) \cdot a_0^{-1} \cdot b_0 = \Phi_\alpha(z) \cdot a_0^{-1} \cdot b_0.$$

∎

2.5 Necessary and sufficient conditions for holomorphic deformations

Consider the differential equation

$$\mathcal{A}(z) = \frac{d\Phi}{dz} \cdot \Phi(z)^{-1}, \quad z \in \mathbb{S} \setminus \{\, z_j \,\}_{j=1}^m = M.$$

In Theorem 2.4.2, we showed that at each point of M we can find a unique local solution of the equation, satisfying an initial condition at that point. The idea is now to extend the problem, with an additional parameter $t \in \mathbb{C}$, such that

$$\mathcal{A}(z) = \mathcal{A}_z(z, t),$$

and each solution on M can be deformed holomorphically by varying t in $W \subset \mathbb{C}$.

Definition 2.5.1 Holomorphic deformation.
Consider the differential equation

$$\mathcal{B}(z, t) = \frac{\partial\Phi}{\partial z} \cdot \Phi(z)^{-1}, \qquad \mathcal{B} : \left(\mathbb{S} \setminus \{\, z_j \,\}_{j=1}^m\right) \times W \to M_2(\mathbb{C}), \tag{2.13}$$

where $W \subset \mathbb{C}$ denotes an open subset of $\mathbb{C}$ and $\mathcal{B}$ depend holomorphically on $z \in \mathbb{S} \setminus \{\, z_j \,\}_{j=1}^m$. If $\mathcal{B}$ depends holomorphically on t, we call the family $\{\, B(\cdot, t) \mid t \in W \,\}$ a holomorphic deformation of the differential equation (2.13)

We need some conditions on a solution $\Phi : U \subset M \times W \to GL_2(\mathbb{C})$, which should come in form of a differential equation

$$\mathcal{A}_t(z, t) = \frac{\partial\Phi}{\partial t} \cdot \Phi(z, t)^{-1}.$$

We want to employ the same ideas, and construct a principal bundle $\widetilde{Q}(\widetilde{M}, GL_2(\mathbb{C}), \pi)$ similar to $Q\,(M, GL_2(\mathbb{C}), \pi)$, but where the basespace will be have an additional complex parameter $t \in W$, for some $W \subset C$ open and connected, that is:

$$\widetilde{M} = \mathbb{S} \setminus \{\, z_j \,\}_{j=1}^m \times W = M \times W \ni (z, t).$$

If we then can find a family of local connection forms $\{\, \widetilde{A}_\beta \,\}_\beta$ on $\widetilde{M}$, with vanishing curvature form $\tilde{\Omega}$ on $\widetilde{Q}$, we can once again apply the Frobenius Theorem 2.4.1, Corollary 2.4.2 and Theorem 2.3.1. In particular, by Theorem 2.3.1, a solution $\Phi : U \subset \widetilde{M} \to GL_2(\mathbb{C})$ of the two parameter problem, should satisfy

$$\widetilde{A}_\alpha = -\operatorname{Ad}(\Phi) \circ \Phi^*\theta = (\widetilde{A}_\alpha)_z \, dz + (\widetilde{A}_\alpha)_t \, dt$$

$$-\mathcal{A}_z\ dz - \mathcal{A}_t\ dt = -\operatorname{Ad}(\Phi)\circ\Phi^*\theta\left(\frac{\partial}{\partial z}\right)dz - \operatorname{Ad}(\Phi)\circ\Phi^*\theta\left(\frac{\partial}{\partial t}\right)dt,$$

where $\left\{\frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right\}$ is the basis for the tangent space $T_{(z,t)}\widetilde{M} \simeq T_zM \times T_t\mathbb{C}$ at $(z,t) \in \widetilde{M}$. Or written in matrix group notation:

$$\widetilde{A}_\alpha = -\mathcal{A}_z\ dz - \mathcal{A}_t\ dt = -\frac{\partial\Phi}{\partial z}\cdot\Phi(z,t)^{-1}\ dz - \frac{\partial\Phi}{\partial t}\cdot\Phi(z,t)^{-1}\ dt.$$

So, to employ the theory developed in Section 2.4 we need to find a coefficient function

$$(\widetilde{A}_\alpha)_t = -\mathcal{A}_t : \widetilde{M} \to M_2(\mathbb{C}).$$

If we can find a suitable $-\mathcal{A}_t$, we can define a family of local connection forms

$$\left\{\widetilde{A}_\beta = \operatorname{Ad}(g_{\beta\alpha})\circ\widetilde{A}_\alpha + g_{\alpha\beta}^*\theta\right\}$$

on $\widetilde{M}$. This induces a connection on $\widetilde{Q}$, which by the Frobenius Theorem 2.4.1 gives an integrable distribution $\tilde{H} \subset T\widetilde{Q}$ if and only if the associated curvature form $\tilde{\Omega}$ vanishes. Using Definition 2.4.4, this translates down to $\widetilde{M}$ by requiring that the family of local curvature forms

$$\left\{\tilde{F}_\beta = s_\beta^*\tilde{\Omega} : U_\beta \to \bigwedge^2 T^*U_\beta \otimes \mathfrak{g}\right\}$$

all vanish.

Lemma 2.5.1 Characterization of vanishing curvature form.
Given a principal bundle $P(N,G,\pi)$, with a connection form ω on P, and associated curvature form Ω on P. Then

$$\Omega = 0 \iff F_\beta = s_\beta^*\Omega = 0,\ \textit{for every } \beta \in J.$$

Proof. It is clear that if $\Omega = 0$, then for any β, $F_\beta = s_\beta^*\Omega = 0$.

Conversely, if $F_\beta = 0$ for every $\beta \in J$, that means that for any $p \in P$, with $\pi(p) = z$ and for any $u, v \in\in T_zM$:

$$F_\beta(u,v) = \big(s_\beta^*\Omega\big)(u,v) = -\omega\big[h\circ(s_\beta)_*u,\ h\circ(s_\beta)_*v\big] = 0.$$

From this we want to infer that for any horizontal tangent vectors $\tilde{u}$, $\tilde{v} \in H_p$: $-\omega[\tilde{u},\tilde{v}] = 0$. Thus it suffices to show that $h\circ(s_\beta)_* : T_zM \to H_p$ is surjective. Recall that $(s_\beta)_*$ has rank $\dim(M)$ and that $\pi_*\circ(s_\beta)_* = 0$. Thus $(s_\beta)_*u$ has non-vanishing horizontal component for any $u \in T_zM$. If not then $\dim(\ker((s_\beta)_*)) > \dim(G) = \dim(P) - \dim(M)$, a contradiction. Thus we conclude that $h\circ(s_\beta)_* : T_zM \to H_p$ is surjective, and that $\Omega = 0$. ∎

If we manage to define $\widetilde{A}_\alpha$ globally on $\widetilde{M}$, then $\widetilde{F}_\alpha$ will also be globally defined. By Corollary 2.4.1 and Lemma 2.5.1, we have that

$$\tilde{F}_\alpha = 0 \iff \tilde{F}_\beta = 0,\ \text{for every } \beta \in J \iff \tilde{\Omega} = 0.$$

Thus by the structure equation for local curvature forms (Proposition 2.4.2), it is necessary and sufficient to show that

$$\widetilde{F}_\alpha = d\widetilde{A}_\alpha + \frac{1}{2}\left[\widetilde{A}_\alpha \wedge \widetilde{A}_\alpha\right] = 0.$$

Theorem 2.5.1 Integrability conditions of horizontal distribution.
Consider the principal bundle $\widetilde{Q}(\widetilde{M}, GL_2(\mathbb{C}), \pi)$*, where*

$$\widetilde{M} = \mathbb{S} \setminus \{ z_j \}_{j=1}^{m} \times W = M \times W \ni (z,t).$$

A local connection form, globally defined on M*,*

$$\widetilde{A}_\alpha = (\widetilde{A}_\alpha)_z \ dz + (\widetilde{A}_\alpha)_t \ dt = -\mathcal{A}_z \ dt - \mathcal{A}_t \ dt,$$

gives rise to an integrable horizontal distribution H *on* $\widetilde{Q}$ *if and only if*

$$\widetilde{F}_\alpha = d\widetilde{A}_\alpha + \frac{1}{2}\left[\widetilde{A}_\alpha \wedge \widetilde{A}_\alpha\right] = 0$$

$$\Longleftrightarrow \ \frac{\partial(\widetilde{A}_\alpha)_t}{\partial z} - \frac{\partial(\widetilde{A}_\alpha)_z}{\partial t} + \left[(\widetilde{A}_\alpha)_z, (\widetilde{A}_\alpha)_t\right] = 0$$

Proof. The discussion above, proves the statement of the Theorem. All that is left, is to prove the last formula. Since $\widetilde{F}_\alpha$ is a 2-form on a 2 dimensional manifold, it has a single form in its basis, namely $dz \wedge dt$. Thus $\widetilde{F}_\alpha$ vanish if and only if the following expression vanish:

$$\left(d\widetilde{A}_\alpha + \frac{1}{2}\left[\widetilde{A}_\alpha \wedge \widetilde{A}_\alpha\right]\right)\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right).$$

By computing we get

$$\begin{aligned} d((\widetilde{A}_\alpha)_z dz + (\widetilde{A}_\alpha)_t dt) &= \frac{\partial(\widetilde{A}_\alpha)_z}{\partial t} dt \wedge dz + \frac{\partial(\widetilde{A}_\alpha)_t}{\partial z} dz \wedge dt \\ &= \left(\frac{\partial(\widetilde{A}_\alpha)_t}{\partial z} - \frac{\partial(\widetilde{A}_\alpha)_z}{\partial t}\right) dz \wedge dt. \end{aligned} \tag{2.14}$$

Also

$$\begin{aligned} &\left[((\widetilde{A}_\alpha)_z dz + (\widetilde{A}_\alpha)_t dt)\left(\frac{\partial}{\partial z}\right), \ \ ((\widetilde{A}_\alpha)_z dz + (\widetilde{A}_\alpha)_t dt)\left(\frac{\partial}{\partial t}\right)\right] \\ &\qquad = \left[(\widetilde{A}_\alpha)_z + 0 \ , \ \ 0 + (\widetilde{A}_\alpha)_t\right] \end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15) we obtain

$$\begin{aligned} &\left(d\widetilde{A}_\alpha + \frac{1}{2}\left[\widetilde{A}_\alpha \wedge \widetilde{A}_\alpha\right]\right)\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right) \\ &= \left(\frac{\partial(\widetilde{A}_\alpha)_t}{\partial z} - \frac{\partial(\widetilde{A}_\alpha)_z}{\partial t}\right)(dz \wedge dt)\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right) + \left[(\widetilde{A}_\alpha)_z, \ \ (\widetilde{A}_\alpha)_t\right] \\ &= \frac{\partial(\widetilde{A}_\alpha)_t}{\partial z} - \frac{\partial(\widetilde{A}_\alpha)_z}{\partial t} + \left[(\widetilde{A}_\alpha)_z, \ \ (\widetilde{A}_\alpha)_t\right]. \end{aligned}$$

■

We summarize the discussion of this Section in a Theorem.

Theorem 2.5.2 Integrability condition of differential equations.
The system of differential equations

$$\begin{cases} \mathcal{A}_z(z,t) = \frac{\partial \Phi}{\partial z} \cdot \Phi(z,t)^{-1} \\ \mathcal{A}_t(z,t) = \frac{\partial \Phi}{\partial t} \cdot \Phi(z,t)^{-1} \end{cases} \qquad A_z, A_t : \widetilde{M} \to M_2(\mathbb{C})$$

has local fundamental solutions $\Phi : U \to GL_2(\mathbb{C})$ *if and only if*

$$\frac{\partial \mathcal{A}_z}{\partial t} - \frac{\partial \mathcal{A}_t}{\partial z} + [\mathcal{A}_z, \mathcal{A}_t] = 0.$$

Moreover, if such solutions exist, they satisfy:

i. If $a_0 \in GL_2(\mathbb{C})$, and $(z_0, t_0) \in \widetilde{M}$. Then there is a neighbourhood $U \subset \widetilde{M}$ of (z_0, t_0) where there exists a unique fundamental solution $\Phi_\alpha : U \to GL_2(\mathbb{C})$, such that $\Phi_\alpha(z_0, t_0) = a_0$.

ii. If $a_0, b_0 \in GL_2(\mathbb{C})$, and the functions $\Phi_\alpha : U \to GL_2(\mathbb{C})$ and $\Phi'_\alpha : U' \to GL_2(\mathbb{C})$ solves the differential equation with $\Phi_\alpha(z_0, t_0) = a_0$, $\Phi'_\alpha(z_0, t_0) = b_0$. Then $\Phi'_\alpha = \Phi_\alpha \cdot a_0^{-1} \cdot b_0$ in $U \cap U'$.

Proof. We give a guide on how to apply the geometric construction i order to infer the result. A principal bundle $\widetilde{Q}(\widetilde{M}, GL_2(\mathbb{C}), \pi)$ is constructed, similarly to Corollary 2.3.1. On $\widetilde{M}$ we have the Lie algebra valued 1-form

$$\widetilde{A}_\alpha = -\mathcal{A}_z \, dz - \mathcal{A}_t \, dt,$$

defined globally. Using the transition function on $g_{\beta\alpha}$, we define a family of local connection forms

$$\left\{\widetilde{A}_\beta = \operatorname{Ad}(g_{\beta\alpha}) \circ \widetilde{A}_\alpha + g^*_{\alpha\beta}\theta\right\}, \qquad \text{on } \widetilde{M}.$$

By Theorem 2.5.1, this family of local connection forms give rise to an integrable horizontal distribution H on $\widetilde{Q}$ if and only if

$$\widetilde{F}_\alpha = d\widetilde{A}_\alpha + \frac{1}{2}\left[\widetilde{A}_\alpha \wedge \widetilde{A}_\alpha\right] = 0.$$

Inserting

$$(\widetilde{A}_\alpha)_z = -\mathcal{A}_z, \qquad (\widetilde{A}_\alpha)_t = -\mathcal{A}_t$$

we obtain the condition

$$\frac{\partial \mathcal{A}_z}{\partial t} - \frac{\partial \mathcal{A}_t}{\partial z} + [\mathcal{A}_z, \mathcal{A}_t] = 0.$$

Then using Corollary 2.4.2, we infer that for the unique submanifold S through a point $q \in \widetilde{Q}$, there exists a neighbourhood $U \subset \widetilde{M}$ of $\pi(q)$ and a horizontal section $\hat{\Phi} : U \to S$, unique in U. Remark that if for each point $q \in \tilde{Q}$, there exists such a horizontal section, then these sections give unique submanifolds at each $q \in \widetilde{Q}$. Further, by Theorem 2.3.1, this horizontal section is equivalent to the existence of a holomorphic function

$$\Phi_\alpha = g_\alpha \circ \hat{\Phi} : U \to GL_2(\mathbb{C}),$$

such that

$$\widetilde{A}_\alpha = -\operatorname{Ad}(\Phi_\alpha) \circ \Phi^*_\alpha\theta,$$

that is

$$-\mathcal{A}_z \, dz - \mathcal{A}_t \, dt = -\frac{\partial \Phi_\alpha}{\partial z} \cdot \Phi(z,t)^{-1} \, dz - \frac{\partial \Phi_\alpha}{\partial t} \cdot \Phi(z,t)^{-1} \, dt,$$

hence establishing the equivalence.

We show the first property of the local solutions: Given $a_0 \in GL_2(\mathbb{C})$ and $(z_0, t_0) \in \widetilde{M}$. Then this gives a unique point $q = [\alpha, (z_0, t_0), a_0] \in \widetilde{Q}$. Through this point there exists a unique submanifold S^q. By the arguments from the above statement, there exists a neighbourhood $U \subset \tilde{M}$ and a holomorphic function $\Phi_\alpha : U \to GL_2(\mathbb{C})$, that solves the differential equations. Also Φ_α is the only such function in U, by Corollary 2.4.2.

The second statement is a direct consequence of Corollary 2.4.3. Indeed, if we let $p = [\alpha, (z_0, t_0), a_0]$ and $q = [\alpha, (z_0, t_0), b_0]$, we have

$$\begin{aligned}\Phi'_\alpha(z,t) &= g_\alpha \circ \hat{\Phi}^q(z,t) = g_\alpha\left(\hat{\Phi}^p(z,t) \,.\, a_0^{-1} \cdot b_0\right) \\ &= \left(g_\alpha \circ \hat{\Phi}^p(z,t)\right) \cdot a_0^{-1} \cdot b_0 = \Phi_\alpha(z,t) \cdot a_0^{-1} \cdot b_0.\end{aligned}$$

∎

Chapter 3

Isomonodromic deformation

3.1 Introduction to isomonodromic deformations

In Chapter 1 we found local fundamental solutions of the differential equation

$$\frac{d\Phi}{dz}\Phi(z)^{-1} = \frac{\sigma_3}{2} + \frac{A_0}{z} + \frac{A_t}{z-t} = \mathcal{A}(z), \tag{3.1}$$

and then used these local solutions to construct a global fundamental solution

$$\tilde{\Phi} : \tilde{M} \to GL_2(\mathbb{C})$$

on the universal covering space $\tilde{M}$ of $M = \mathbb{S} \setminus \{0, t, \infty\}$. The function $\tilde{\Phi}$ is the function one obtains from taking a local solution of the above differential equation and analytically continuing the solution along every path in $M = \mathbb{S} \setminus \{0, t, \infty\}$. The function is defined on the universal covering space of M in order to obtain a well defined function (not multivalued).

In Section 1.4.4 we described the monodromy theory, a theory developed to ease the work of calculating analytic continuation of local solutions of differential equations. Further in Section 1.4.5 we derived formulas for the generators $m^{(0)}$ and $m^{(1)}$ of the monodromy group $\mathfrak{m}$, and also a formula for the monodromy matrix $m^{(\infty)}$ related to $z_2 = \infty$.

$$m^{(0)} = \left(E^{(0)}\right)^{-1} \exp\left(2\pi i \Lambda_0^{(0)}\right) E^{(0)}, \tag{3.2}$$

$$m^{(1)} = \left(E^{(1)}\right)^{-1} \exp\left(2\pi i \Lambda_0^{(1)}\right) E^{(1)}, \tag{3.3}$$

$$m^{(\infty)} = \left(E^{(\infty)}\right)^{-1} \exp\left(2\pi i \Lambda_0^{(\infty)}\right) S_2^{-1} S_1^{-1} E^{(\infty)} \tag{3.4}$$

Where

- $E^{(0)}, E^{(1)}$ and $E^{(\infty)} = I$ are the connection matrices related to the respective Fuchsian singular points $z_0 = 0$, and $z_1 = t$ see Definition 1.4.4, and the non-Fuchsian singular point $z_2 = \infty$ see Definition 1.4.2.
- $\Lambda_0^{(0)}$, $\Lambda_0^{(1)}$ and $\Lambda_0^{(\infty)}$ are the diagonal coefficient matrix of the logarithmic term, in the respective local solutions around $z_0 = 0$, $z_1 = t$ and $z_2 = \infty$.
- $S_1 = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}$ and $S_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}$ are the Stokes matrices, see Definition 1.3.5.

These formulas show explicitly what information one needs from the solutions of equation $\mathcal{A}$ in order to represent the monodromy group $\mathfrak{m}$. Corollary 1.4.3 also give a constraint on this information.

$$m^{(\infty)} m^{(1)} m^{(0)} = I$$

Definition 3.1.1 Monodromy data.
The monodromy data of equation (3.1) is the set

$$\mathfrak{M} = \left\{ \Lambda_0^{(0)}, \Lambda_0^{(1)}, \Lambda_0^{(\infty)}; s_1, s_2; E^{(0)}, E^{(1)}, E^{(\infty)} = I \right\}.$$

The monodromy data is exactly the information needed in order to construct the canonical monodromy representation of the fundamental group.

The objective of this chapter is to show the existence of, and describe, isomonodromic deformations of the system of differential equations (3.1). That is, we want to introduce an additional parameter $t \in \mathbb{C}$ in the coefficient matrix $\mathcal{A}(z)$ in (3.1), and find a one parameter family $\mathcal{A}(z,t)$ of coefficient matrices, all sharing the same set of monodromy data. We will closely follow the reasoning in [Fok et al.06, Ch. 4].

Definition 3.1.2 Admissible deformations.
Consider the differential equation

$$\mathcal{B}(z,t) = \frac{\partial \Phi}{\partial z} \cdot \Phi(z)^{-1}, \quad \mathcal{B} : \left(\mathbb{S} \setminus \{ z_j \}_{j=1}^{m} \right) \times W \to M_2(\mathbb{C})$$

where, $W \subset \mathbb{C}$ denotes an open subset of $\mathbb{C}$ and $\mathcal{B}$ depend holomorphically on $z \in \mathbb{S} \setminus \{ z_j \}_{j=1}^{m}$ and $t \in W \subset \mathbb{C}$.
We call the holomorphic family $\{ B(\cdot, t) \mid t \in W \}$, an admissible deformation of the differential equation, if the following conditions are satisfied:

i. The number m of singular points in the variable z, does not depend on t, however, the position of the singularities may very well vary with t.

ii. Around each singular point $z_j(t)$ there exists a disk $D_j \subset \mathbb{S}$ such that $z_j(t) \in D_j$ for any $t \in W$ and $D_j \cap D_k = \emptyset$, for any $j, k \in \{ 1, \ldots, m \}$.

iii. The Poincaré rank (Definition 1.2.2) of each singular point does not depend on t.

iv. The leading coefficient of the Laurent series of the matrices $\mathcal{A}(z,t)$ at each Fuchsian singular point, is diagonalizable with eigenvalues $(\Lambda_0^{(j)})_{11}$, $(\Lambda_0^{(j)})_{22}$ such that $(\Lambda_0^{(j)})_{11} - (\Lambda_0^{(j)})_{22} \notin \mathbb{Z} \setminus \{ 0 \}$, for $j = 0, 1$, where $z_j(t)$ is Fuchsian, for all $t \in W$. At non-Fuchsian singular points, the leading coefficient in the Laurent series should be diagonalizable with distinct eigenvalues, for all $t \in W$.

v. At the non-Fuchsian singular points z_j, the Stoke sectors $\Sigma_n^{(j)}$ (Definition 1.3.3) can be chosen in such a way that they do not change under the map $z \mapsto z - z_j(t)$, for every $t \in W$.

vi. Canonical solutions (see Definition 1.2.4 and 1.3.8) can be chosen in such a way that they are holomorphic with respect to t, and for the canonical solutions near an irregular singular point (see Definition 1.3.8), the asymptotic conditions (see Definition 1.3.1) hold uniformly with respect to $t \in W$.

It follows from Definition 3.1.2 and the general constructions of canonical solutions in [Fok et al.06, Ch. 1] that if $\mathcal{B}(z,t)$ is an admissible deformation of

$$\mathcal{B}(z,t) = \frac{\partial \Phi}{\partial z} \cdot \Phi(z)^{-1},$$

then one can define the set of monodromy data in such a (not unique) way that all the Stokes matrices and connection matrices are holomorphic functions of $t \in W$.

Definition 3.1.3 Isomonodromic deformation.
Consider the differential equation

$$\mathcal{B}(z,t) = \frac{\partial \Phi}{\partial z} \cdot \Phi(z)^{-1}, \qquad \text{where } \mathcal{B} : \left(\mathbb{S} \setminus \{ z_j \}_{j=1}^m \right) \times W \to M_2(\mathbb{C})$$

depend holomorphically on $z \in \mathbb{S} \setminus \{ z_j \}_{j=1}^m$ and $t \in W \subset \mathbb{C}$. We call the family $\{ B(\cdot,t) \mid t \in W \}$ an isomonodromic deformation of the differential equation, if in addition to being an admissible deformation (Definition 3.1.2), one can construct a set of canonical solutions such that the set

$$\left\{ \Lambda_0^{(1)}, \ldots, \Lambda_0^{(m)}; S_1^{(1)}, \ldots, S_{k_1}^{(1)}, \ldots, S_1^{(m)}, \ldots, S_{k_m}^{(m)}; E^{(1)}, \ldots, E^{(m)} \right\}$$

does not depend on $t \in W$.
Notational note: if z_j is Fuchsian, the Stokes matrices $S_n^{(j)}$ is not defined, and thus not included. The index k_j counts the different Stokes matrices for the non-Fuchsian singular point z_j.

Returning to our system, we find that we can introduce the variable $t \in W \subset \mathbb{S} \setminus \{ 0, \infty \}$, in such a way that the conditions in Definition 3.1.2 are satisfied. Fix $1 > \epsilon > 0$, and let $B(0,\epsilon)$ be the ball around $0 \in \mathbb{C}$ of radius ϵ, and let $B(0, \frac{1}{\epsilon})$ be the ball around $0 \in \mathbb{C}$ of radius $\frac{1}{\epsilon}$. Then we can define

$$W_\epsilon = B\left(0, \frac{1}{\epsilon}\right) \setminus B(0,\epsilon). \tag{3.5}$$

We consider again the linear differential equation (3.1), but now we let $t \in W_\epsilon$, and also let A_0 and A_t depend on t.

Proposition 3.1.1 An (almost) admissible deformation.
Consider the linear system of partial differential equations

$$\frac{\partial \Phi}{\partial z} \cdot \Phi(z,t)^{-1} = \frac{\sigma_3}{2} + \frac{A_0(t)}{z} + \frac{A_t(t)}{z-t} = \mathcal{A}_z(z,t) \tag{3.6}$$

$$\mathcal{A}_z : \mathbb{S} \setminus \{ 0, t, \infty \} \times W_\epsilon \to M_2(\mathbb{C}), \tag{3.7}$$

where we assume that $A_0(t)$ and $A_t(t)$ is diagonalizable with eigenvalues $(\Lambda_0^{(j)})_{11}$, $(\Lambda_0^{(j)})_{22}$ such that $(\Lambda_0^{(j)})_{11} - (\Lambda_0^{(j)})_{22} \notin \mathbb{Z} \setminus \{ 0 \}$, for $j = 0, 1$. Then $\{ \mathcal{A}_z(\cdot,t) \mid t \in W \}$ is an admissible deformation of equation (3.1) if and only if condition vi. of Definition 3.1.2 is satisfied. That is, if and only if canonical solutions can be chosen in such a way that they are holomorphic with respect to t, and for the canonical solutions near the non-Fuchsian point $z_2 = \infty$, the asymptotic expansion (Definition 1.3.7) hold uniformly with respect to $t \in W$.

Proof. We verify the conditions of an admissible deformation. Condition *i.* is satisfied, since $z_1 = t$ cannot merge with any of the other singularities, since $W_\epsilon = B\left(0, \frac{1}{\epsilon}\right) \setminus B(0,\epsilon)$. Also A_0 and A_t does not depend on z, so the singularities cannot cancel. Condition *ii.* is satisfied by how we defined W_ϵ. Condition *iii.*: The Laurent expansion around $z_2 = \infty$ of $\mathcal{A}_z$ is given by

$$\mathcal{A}_z(z,t) = -\frac{\sigma_3}{2\xi^2} - \frac{A_0 + A_t}{\xi} + \sum_{k=0}^{\infty} A_{k+1}^{(\infty)} \xi^k,$$

hence the Poincaré index at infinity is constant in t. Condition *iv.* is verified by the hypothesis on the diagonalization of $A_0(t)$ and $A_t(t)$ together with he fact that the leading term in the Laurent expansion around infinity $A_{-1}^{(\infty)} = -\frac{\sigma_3}{2}$, is diagonal with distinct eigenvalues and independent of t. ■

In the next Section we will explain why we expect to obtain an isomonodromic deformation, from the deformation in Proposition 3.1.1.

3.2 Motivation for expecting isomonodromic deformations

In order to infer why we expect the deformation in Proposition 3.1.1 to be isomonodromic, under possibly more restrictive conditions, we will do a simple counting argument. We will count the number of undetermined parameters in the coefficient function of equation (3.1), and count the number of parameters in the monodromy data in Definition 3.1.1 and compare. If there are more undetermined coefficients in the coefficient function, that suggests that it may be possible to parametrize the coefficient function in such a way that the monodromy data stays fixed.

We define the singularity data of a first order linear system of differential equations.

Definition 3.2.1 Singular data.
Consider a first order linear system of differential equations with rational coefficients

$$\mathcal{B} = \frac{d\Phi}{dz} \cdot \Phi(z)^{-1}.$$

If $\{ z_j \}_{j=1}^{m}$ *denotes the singular points of* $\mathcal{B}$*, consider the Laurent expansions*

$$\mathcal{B}(z) = \sum_{k=-r^j}^{\infty} A_k^{(j)} (z - z_j)^k$$

of $\mathcal{B}$ *centred at* z_j *for* $j = 1, \ldots, m$*. We define the singular data of* $\mathcal{B}$ *as the set*

$$\left\{ \{ z_j \}_{j=1,\ldots m} ; \{ A_{-r_j}^{(j)}, \ldots, A_0^{(j)} \}_{j=1,\ldots,m} ; \{ P^{(j)} \}_{j=1,\ldots,m} \right\}$$

This Definition is made only to be consistent with the notations and Definitions from [JMU81]. We will use the paper [JMU81] later, in order to state the necessary and sufficient conditions for $\mathcal{A}_z$ from Proposition 3.1.1 to be an isomonodromic deformation.

Lemma 3.2.1 Undetermined parameters of singular data.
Consider the linear system of differential equations

$$\mathcal{A} = \frac{\sigma_3}{2} + \frac{A_0}{z} + \frac{A_t}{z-t} = \frac{d\Phi}{dz} \cdot \Phi(z)^{-1}, \qquad \mathcal{A} : \mathbb{S} \setminus \{ 0, t, \infty \} \to M_2(\mathbb{C}). \tag{3.8}$$

Up to Möbius transformations of $\mathbb{S} \setminus \{ 0, t, \infty \}$*, and gauge equivalence of* (3.8) *(see Definition 1.2.3), the singular data of* (3.8) *has 7 undetermined complex numbers.*

Proof. It is clear that the information in the singular data is in the case of $\mathcal{B} = \mathcal{A}$ contained in $\left\{ \{ 0, t, \infty \} ; \{ \frac{\sigma_3}{2}, A_0, A_t \} \right\}$. Further the matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is completely determined. The singular points $z_0 = 0$ and $z_2 = \infty$ are also fixed. We are left with the eight parameters in $A_0, A_t \in M_2(\mathbb{C})$ and the movable singularity at $z_1 = t$. Thus nine undetermined parameters.

We will now induce conditions on A_0 and A_t, to remove 2 parameters in total. By Proposition 1.2.1 we can infer that there exists a gauge transformation of $\mathcal{A}$, such that $\operatorname{trace}(\mathcal{A}) = 0$. Thus we can assume that $\mathcal{A}$ is traceless. Moreover, if we compute:

$$0 = \operatorname{trace}(\mathcal{A}) = \operatorname{trace}\left(\frac{\sigma_3}{2} + \frac{A_0}{z} + \frac{A_t}{z-t} \right) = \frac{\operatorname{trace}(A_0)}{z} + \frac{\operatorname{trace}(A_t)}{z-t}.$$

It is clear that since A_0 and A_t is independent of z, the above expression vanishes if and only

$$\operatorname{trace}(A_0) = 0 = \operatorname{trace}(A_t). \tag{3.9}$$

It is clear that (3.9) gives 2 linearly independent equations on the 8 parameters in A_0 and A_t. Thus the total number of unknowns in A_0 and A_t is 6, and in $\mathcal{A}$ when we include $t \in W_\epsilon$, there are 7 unknowns. Hence the singular data of $\mathcal{A}$ also has 7 unknowns. ■

Lemma 3.2.2 Undetermined parameters of monodromy data.
Consider the monodromy data (Definition 3.1.1) of equation (3.1)

$$\mathfrak{M} = \left\{\Lambda_0^{(0)}, \Lambda_0^{(1)}, \Lambda_0^{(\infty)}; s_1, s_2; E^{(0)}, E^{(1)}, E^{(\infty)}\right\}.$$

This set contains 6 undetermined complex numbers.

Proof. Similarly to the proof of Lemma 3.2.1, it is easier to work with the generators $m^{(0)}, m^{(1)}, m^{(\infty)}$ of the monodromy group, when counting the parameters. This is obviously equivalent to the monodromy data set in terms of undetermined parameters, since the monodromy data is exactly the set needed to calculate the generators, see equation (3.2), (3.3) and (3.4). We also remark that $m^{(\infty)}$ is strictly not needed in order to generate the monodromy group, but is included to utilize the free group relation in Corollary 1.4.3.

In $m^{(0)}, m^{(1)}, m^{(\infty)}$ there are a total of 12 parameters. First, we will induce that the determinant of the matrices $m^{(0)}$ and $m^{(1)}$, equals unity. This gives 2 independent constrains.

Indeed, if $j = 0, 1$

$$\det\left(m^{(j)}\right) = \det\left((E^{(j)})^{-1} \exp\left(2\pi i \Lambda_0^{(j)}\right) E^{(j)}\right) = \det\left(\exp\left(2\pi i \Lambda_0^{(j)}\right)\right).$$

Now using the well known formula for matrix exponentials that

$$\det\left(\exp\left(A\right)\right) = \exp\left(\operatorname{trace}\left(A\right)\right),$$

see [Bel97], we obtain

$$\det\left(\exp\left(2\pi i \Lambda_0^{(j)}\right)\right) = \exp\left(2\pi i \operatorname{trace}\left(\Lambda_0^{(j)}\right)\right) = \exp\left(0\right) = 1,$$

by using the traceless property of $A_j = P^{(j)}\Lambda_0^{(j)}(P^{(j)})^{-1}$ from the proof of Lemma 3.2.1.

The determinants

$$\det(m^{(0)}) = 1, \quad \det(m^{(1)}) = 1$$

give 2 equations on the parameters. Additionally, we have the free group relation

$$m^{(\infty)} m^{(1)} m^{(0)} = I$$

which give an additional 4 equations. In total we have 12 parameters, and 6 constraining equations, leaving us with 6 undetermined parameters. ∎

Remark.
In the above proofs, it is of course important that the constraints are independent, in the sense that two equations do not contain the same information encoded differently. E.g. notice that the determinant of $m^{(\infty)}$ can also be shown to be unity. However, this is done by using the free group relation, hence we do not get a constraint that is independent from the previous information.

By comparing Lemma 3.2.1 and Lemma 3.2.2 , we can conclude that that the singular data contains an extra undetermined parameter, compared to the monodromy data. This simple observation gives a relatively strong indication that it should be possible to parametrise the singular data by a complex parameter, while keeping the monodromy data fixed. This is exactly the scheme of isomonodromic deformations.

3.3 Necessary and sufficient conditions for isomonodromic deformations

From Theorem 2.5.2, we know that a system of differential equations

$$\begin{cases} \mathcal{A}_z(z,t) = \frac{\partial \Phi}{\partial z} \cdot \Phi(z,t)^{-1} \\ \mathcal{A}_t(z,t) = \frac{\partial \Phi}{\partial t} \cdot \Phi(z,t)^{-1} \end{cases} \qquad A_z, A_t : \mathbb{S} \setminus \{ 0, t, \infty \} \times W_\epsilon \to M_2(\mathbb{C})$$

has local solutions $\Phi : U \subset \mathbb{S} \setminus \{ 0, t, \infty \} \times W_\epsilon \to GL_2(\mathbb{C})$ if and only if

$$\frac{\partial \mathcal{A}_z}{\partial t} - \frac{\partial \mathcal{A}_t}{\partial z} + [\mathcal{A}_z, \mathcal{A}_t] = 0. \tag{3.10}$$

We want to determine the coefficient function $\mathcal{A}_t$, such that the integrability condition (3.10) is satisfied, and such that

$$\{\mathcal{A}(\cdot,t) \mid t \in U_t\} = \left\{ \frac{\sigma_3}{2} + \frac{A_0(t)}{z} + \frac{A_t(t)}{z-t} \;\middle|\; t \in U_t \right\}$$

is an isomonodromic deformation. We will derive the necessary conditions, and these will turn out to actually be sufficient.

Denote $\mathbb{S} \setminus \{ 0, t, \infty \}$ by M, and the universal covering space of M by $\tilde{M}$, see Corollary 1.4.1. Recall the projection

$$p : \begin{array}{ccc} \tilde{M} & \to & M \\ \tilde{z} = [z, \gamma_z] & \mapsto & z \end{array}.$$

It is clear by Theorem 1.4.1, that for each $t \in W_\epsilon$, there exists an unique function $\tilde{\Phi}(\cdot,t) : \tilde{M} \to GL_2(\mathbb{C})$, such that $\tilde{\Phi}(\cdot,t)$ is the unique canonical fundamental solution to the differential equation

$$\frac{\partial \tilde{\Phi}}{\partial \tilde{z}} \cdot \tilde{\Phi}(\tilde{z},t)^{-1} = \frac{\sigma_3}{2} + \frac{A_0(t)}{p(\tilde{z})} + \frac{A_t(t)}{p(\tilde{z}) - t}, \quad \tilde{z} \in \tilde{M}, \tag{3.11}$$

with $E^{(\infty)} = I$.

Let U_t be some open subset of W_ϵ. We will assume now that $\{ \mathcal{A}(\cdot,t) \mid t \in U_t \}$ is an isomonodromic deformation. In particular, we assume that the function

$$\tilde{\Phi} : \tilde{M} \times U_t \to GL_2(\mathbb{C}),$$

is holomorphic in t, and that the asymptotic conditions around the non-Fuchsian point $z_2 = \infty$ holds uniformly in t, see Definition 1.3.7. This is condition *vi.* of Definition 3.1.2.

Theorem 3.3.1 Necessary conditions for isomonodromic deformation. *Assume that*

$$\{\mathcal{A}(\cdot,t) \mid t \in U_t\} = \left\{ \frac{\sigma_3}{2} + \frac{A_0(t)}{z} + \frac{A_t(t)}{z-t} \;\middle|\; t \in U_t \right\}$$

is an isomonodromic deformation, with diagonalization matrices

$$P^{(0)}(t), \ P^{(1)}(t) \ \text{and} \ P^{(\infty)}(t) = I,$$

related to the canonical solutions

$$\Phi^{(0)}(z,t) = P^{(0)}(t) \left(I + \sum_{k=1}^{\infty} \Psi_k^{(0)}(t) z^k \right) \exp\left(\Lambda_0^{(0)} \log_{\alpha_0}(z) \right), \tag{3.12}$$

$$\Phi^{(1)}(z,t) = P^{(1)}(t) \left(I + \sum_{k=1}^{\infty} \Psi_k^{(1)}(t) (z-t)^k \right) \exp\left(\Lambda_0^{(1)} \log_{\alpha_1}(z-t) \right), \tag{3.13}$$

$$\Phi^{(\infty)}(z,t) = P^{(\infty)}(t) \left(I + \sum_{k=1}^{\infty} \Psi_k^{(\infty)}(t) \frac{1}{z^k} \right) \exp\left(-\Lambda_{-1}^{(\infty)}(t) z - \Lambda_0^{(\infty)} \log_{\alpha_\infty}(z) \right), \tag{3.14}$$

where (3.12) *and* (3.13) *comes from Definition 1.2.4, and* (3.14) *is the canonical formal solution in Definition 1.3.7.*
If $\Phi : U_z \times U_t \to GL_2(\mathbb{C})$ *is a local solution of the isomonodromic deformation, then* Φ *also solves*

$$\mathcal{A}_t = -\frac{A_t}{z-t} = \frac{\partial \Phi}{\partial t} \cdot \Phi(z,t)^{-1},$$

where $\mathcal{A}_t : \mathbb{S} \setminus \{0,t,\infty\} \times U_t \to M_2(\mathbb{C})$ *satisfy*

$$\frac{\partial \mathcal{A}_z}{\partial t} - \frac{\partial \mathcal{A}_t}{\partial z} + [\mathcal{A}_z, \mathcal{A}_t] = 0.$$

Moreover $P^{(0)}$ *and* $P^{(1)}$ *satisfy the equations*

$$\frac{dP^{(0)}}{dt} \cdot P^{(0)}(t)^{-1} = \frac{A_t(t)}{t}, \qquad \frac{dP^{(1)}}{dt} \cdot P^{(1)}(t)^{-1} = \frac{\sigma_3}{2} + \frac{A_0(t)}{t}.$$

Proof. Consider the given local solution $\Phi : U_z \times U_t \to GL_2(\mathbb{C})$ of the isomonodromic deformation. As a function of $z \in U_z$, Φ can be related to the unique global solution $\tilde{\Phi}(\tilde{z}, t)$ defined on the universal cover $\tilde{M}$ of $M = \mathbb{S} \setminus \{0,t,\infty\}$, such that $E^{(\infty)} = I$:

$$\tilde{\Phi}(\tilde{z}, t) = \Phi \circ p(\tilde{z}) \cdot C,$$

for some $C \in GL_2(\mathbb{C})$.

We use $\tilde{\Phi}$ to define the coefficient function $\widetilde{\mathcal{A}}_t : \tilde{M} \times U_t \to M_2(\mathbb{C})$:

$$\widetilde{\mathcal{A}}_t(\tilde{z}, t) := \frac{d\tilde{\Phi}}{\partial t} \cdot \tilde{\Phi}(\tilde{z}, t)^{-1} \tag{3.15}$$

Actually, $\widetilde{\mathcal{A}}_t$ is constant on each fiber $\pi^{-1}(z) \subset \tilde{M}$. Indeed, we consider $\breve{g}[z, \gamma * \zeta_{z_b,z}] \in \tilde{M}$, where $[\gamma] \in \pi_1(M, z_b)$ and $\zeta_{z_b,z} \subset \hat{M}$, is the path (unique up to homotopy) between z_b and z, contained in $\hat{M}$. We have by Lemma 1.4.3, Theorem 1.4.2 and by the construction in Section 1.4.3:

$$\tilde{\Phi}([z, \gamma * \zeta_{z_b,z}],\, t) = \left(\tilde{\Phi}([z,\gamma],\, t)\right)_{\zeta_{z_b,z}}$$

$$= \left(\Phi[z_b](\cdot, t) m^{\gamma}\right)_{\zeta_{z_b,z}} = \left(\Phi[z_b](\cdot,\, t)\right)_{\zeta_{z_b,z}} m^{\gamma} = \tilde{\Phi}([z, \zeta_{z_b,z}],\, t) m^{\gamma}.$$

Since the monodromy data is independent of t:

$$\frac{d}{dt} m^{\gamma} = 0,$$

we obtain:

$$\widetilde{\mathcal{A}}_t([z, \gamma * \zeta_{z_b,z}],\, t) = \frac{\partial\left(\tilde{\Phi}([z, \zeta_{z_b,z}],\, t) m^{\gamma}\right)}{\partial t} \cdot \left(\tilde{\Phi}([z, \zeta_{z_b,z}],\, t) m^{\gamma}\right)^{-1}$$

$$= \frac{\partial \tilde{\Phi}([z, \zeta_{z_b,z}],\, t)}{\partial t} \cdot m^{\eta}_{\gamma} \cdot (m^{\eta}_{\gamma})^{-1} \cdot \left(\tilde{\Phi}([z, \zeta_{z_b,z}],\, t)\right)^{-1} = \widetilde{\mathcal{A}}_t([z, \zeta_{z_b,z}],\, t).$$

Thus we can define a coefficient function $\mathcal{A}_t : \mathbb{S} \setminus \{0,t,\infty\} \times U_t \to M_2(\mathbb{C})$ by

$$\mathcal{A}_t\big(p(\tilde{z}),\, t\big) := \widetilde{\mathcal{A}}_t(\tilde{z}, t). \tag{3.16}$$

It is clear that close to the Fuchsian singular points, $\mathcal{A}_t$ can be described by

$$\mathcal{A}_t(z,t) = \frac{\partial \Phi^{(j)}}{\partial t} \cdot \Phi^{(j)}(z,t)^{-1}, \quad j = 0, 1.$$

And close to the non-Fuchsian point $z_2 = \infty$, we have the asymptotic behaviour:

$$\lim_{z \to \infty} \mathcal{A}_t(z,t) = \lim_{z \to \infty} \frac{\partial \Phi^{(\infty)} \circ \phi_\infty}{\partial t} \cdot \left(\Phi^{(\infty)} \circ \phi_\infty(z,t) \right)^{-1}.$$

We will now use the fact that the canonical solutions around the Fuchsian and non-Fuchsian points are assumed to be holomorphic with respect to t. In the following calculations we will do operations on the formal series in (3.14), as if it was a formal series. When working with asymptotic expansions, this makes perfect sense. See [Was87] for details. Also we will extensively use Lemma A.1.3.

Claim 1: Close to the Fuchsian point $z_0 = 0$, $\mathcal{A}_t$ is given by

$$\mathcal{A}_t(z,t) = \frac{dP^{(0)}}{dt} \cdot P^{(0)}(t)^{-1} + O(z).$$

Indeed, we simplify the expression $\mathcal{A}_t(z,t) = \frac{\partial \Phi^{(0)}}{\partial t} \cdot \Phi^{(0)}(z,t)^{-1}$ with (3.12). Now we suppress the t dependence to simplify the notation.

$$\frac{\partial \Phi^{(0)}}{\partial t} \cdot \Phi^{(0)}(z,t)^{-1} = \frac{\partial}{\partial t} \left(P^{(0)} \left(I + \sum_{k=1}^{\infty} \Psi_k^{(0)} z^k \right) \exp\left(\Lambda_0^{(0)} \log_{\alpha_0}(z) \right) \right) \cdot \Phi^{(0)}(z,t)^{-1}$$
$$= \frac{\partial P^{(0)}}{\partial t} \cdot (P^{(0)})^{-1} + P^{(0)} \left(\frac{d\Psi_1^{(0)}}{dt} z + O(z^2) \right) (P^{(0)})^{-1} = \frac{dP^{(0)}}{dt} \cdot P^{(0)}(t)^{-1} + O(z).$$

Here we used the fact that $\frac{d}{dt}\Lambda_0^{(0)} = 0$. This proves the claim.

Claim 2: Close to the Fuchsian point $z_1 = t$, $\mathcal{A}_t$ is given by

$$\mathcal{A}_t(z,t) = \frac{dP^{(1)}}{dt} \cdot P^{(1)}(t)^{-1} - \frac{A_t(t)}{z-t} - \left(\frac{\sigma_3}{2} + \frac{A_0(t)}{t} \right) + O(z-t).$$

Indeed, we simplify the expression $\mathcal{A}_t(z,t) = \frac{\partial \Phi^{(1)}}{\partial t} \cdot \Phi^{(1)}(z,t)^{-1}$ with (3.13).

$$\frac{\partial \Phi^{(1)}}{\partial t} \cdot \Phi^{(1)}(z,t)^{-1}$$
$$= \frac{\partial}{\partial t} \left(P^{(1)} \left(I + \sum_{k=1}^{\infty} \Psi_k^{(1)} (z-t)^k \right) \exp\left(\Lambda_0^{(1)} \log_{\alpha_1}(z) \right) \right) \cdot \Phi^{(1)}(z,t)^{-1}$$

$$= \frac{\partial P^{(1)}}{\partial t} \cdot (P^{(1)})^{-1}$$
$$+ P^{(1)} \left(\sum_{k=1}^{\infty} \left(\frac{d\Psi_k^{(1)}}{dt} - (k+1)\Psi_{k+1}^{(1)} \right) (z-t)^k - \Psi_1^{(1)} \right) (I + O(z-t)) (P^{(1)})^{-1}$$
$$- P^{(1)} \left(I + \Psi_1^{(1)}(z-t) + O(z-t)^2 \right) \left(\frac{\Lambda_0^{(1)}}{z-t} \right) \left(I - \Psi_1^{(1)}(z-t) + O(z-t)^2 \right)$$

Here we used the fact that $\frac{d}{dt}\Lambda_0^{(1)} = 0$. Keeping the terms of negative and zero power, we are left with:

$$\mathcal{A}_t(z,t)$$
$$= \frac{dP^{(1)}}{dt} \cdot P^{(1)}(t)^{-1} - \frac{P^{(1)}\Lambda_0^{(1)}(P^{(1)})^{-1}}{z-t} + P^{(1)} \left([\Lambda_0^{(1)}, \Psi_1^{(1)}] - \Psi_1^{(1)} \right) (P^{(1)})^{-1} + O(z-t).$$

Now we observe that from equation (1.11),

$$P^{(1)}\Lambda_0^{(1)}(P^{(0)})^{-1} = A_0^{(1)} = A_t.$$

And from equation (1.13) with $k = 0$,

$$[\Lambda_0^{(1)}, \Psi_1^{(1)}] - \Psi_1^{(1)} = -\left(\Psi_1^{(1)} + [\Psi_1^{(1)}, \Lambda_0^{(1)}]\right) = -(P^{(1)})^{-1}A_1^{(1)}P^{(1)} \cdot I$$

$$= -(P^{(1)})^{-1}\left(\frac{\sigma_3}{2} + \frac{A_0}{t}\right)(P^{(1)}).$$

The claim follows.

Claim 3: The asymptotic expansion of $\mathcal{A}_t$ around the non-Fuchsian point $z_2 = \infty$ is given by:

$$(\mathcal{A}_t)_{asymp} = -\frac{d\Lambda_{-1}^{(\infty)}}{dt}z + \left[\frac{d\Lambda_{-1}^{(\infty)}}{dt},\ \Psi_1^{(\infty)}\right] + O\left(\frac{1}{z}\right) = O\left(\frac{1}{z}\right).$$

We use that $P^{(\infty)} = I$ for every t, and compute the asymptotic expansion:

$$(\mathcal{A}_t)_{asymp} = \frac{\partial \Phi^{(\infty)} \circ \phi_\infty}{\partial t} \cdot \left(\Phi^{(\infty)} \circ \phi_\infty(z,t)\right)^{-1}$$

$$= \frac{\partial}{\partial t}\left(\left(I + \sum_{k=1}^{\infty} \Psi_k^{(\infty)} \frac{1}{z^k}\right) \exp\left(-\Lambda_{-1}^{(\infty)} z - \Lambda_0^{(\infty)} \log_\alpha(z)\right)\right)$$
$$\cdot \exp\left(\Lambda_{-1}^{(\infty)} z + \Lambda_0^{(\infty)} \log_\alpha(z)\right)\left(I - \Psi_1^{(\infty)}\frac{1}{z} + O\left(\frac{1}{z^2}\right)\right)$$

$$= \left(\sum_{k=1}^{\infty} \frac{d\Psi_k^{(\infty)}}{dt}\frac{1}{z^k}\right)\left(I - \Psi_1^{(\infty)}\frac{1}{z} + O\left(\frac{1}{z^2}\right)\right)$$
$$+ \left(I + \Psi_1^{(\infty)}\frac{1}{z} + O\left(\frac{1}{z^2}\right)\right)\left(-\frac{d\Lambda_{-1}^{(\infty)}}{dt}z\right)\left(I - \Psi_1^{(\infty)}\frac{1}{z} + O\left(\frac{1}{z}\right)\right)$$

Keeping the terms of negative and zero power, we are left with:

$$= -\frac{d\Lambda_{-1}^{(\infty)}}{dt}z + \left[\frac{d\Lambda_{-1}^{(\infty)}}{dt},\ \Psi_1^{(\infty)}\right] + O\left(\frac{1}{z}\right).$$

However, from equation (1.15)

$$\Lambda_{-1}^{(\infty)} = (P^{(\infty)})^{-1}A_{-1}^{(\infty)}P^{(\infty)} = A_{-1}^{(\infty)} = -\frac{\sigma_3}{2},$$

so it is independent of t, proving the claim.

Using the claims, we will now give several arguments which will determine $\mathcal{A}_t$ uniquely. We remark that by how $\mathcal{A}_t(\cdot, t)$ is defined in equation (3.16), is at most singular $z_0 = 0$, $z_1 = t$ or $z_2 = \infty$.

- from Claim 1, we see that $\mathcal{A}_t(\cdot, t)$ has a removable singularity at $z_0 = 0$, with

$$\mathcal{A}_t(0,t) = \frac{dP^{(0)}}{dt} \cdot P^{(0)}(t)^{-1}.$$

- Claim 3 shows that there also at $z_2 = \infty$, is a removable singularity in z, and that $\mathcal{A}_t$ vanishes at $z_2 = \infty$. Hence there can be no terms with zero or positive powers, except in a series with finite radius of convergence.
- Claim 2 gives a singular term $-\frac{A_t}{z-t}$, with a simple pole at $z_1 = t$. If there where other negative power terms of $(z-t)$, they would appear in this expansion. Since there can be no constant terms,

$$\frac{dP^{(1)}}{dt} \cdot P^{(1)}(t)^{-1} = \frac{\sigma_3}{2} + \frac{A_0(t)}{t}.$$

- The $O(z-t)$ term in Claim 2, could be a holomorphic function g, represented by a series centered at $z_1 = t$. However, since $\mathcal{A}_t$, has no other singularities than in $z_1 = t$, the function g would be entire, and by Claim 3, it would be bounded. Hence by the Liouville Theorem (see [Rud87]), it is constant. However, the expression does not have a constant term. We conclude that the term $O(z-t) \equiv 0$.
- The term $O(z)$ from Claim 1, is exactly the series expansion of $-\frac{A_t}{z-t}$ around $z_0 = 0$, excluding the constant term.

$$-\frac{A_t}{z-t} = \frac{A_t}{t}\sum_{k=1}^{\infty}\frac{z^k}{t^k}, \quad |z| < t.$$

We can conclude that

$$\mathcal{A}_t(z,t) = -\frac{A_t}{z-t}, \tag{3.17}$$

and that

$$\frac{dP^{(0)}}{dt} \cdot P^{(0)}(t)^{-1} = \lim_{z\to t} -\frac{A_t}{z-t} = \frac{A_t}{t}, \quad \frac{dP^{(1)}}{dt} \cdot P^{(1)}(t)^{-1} = \frac{\sigma_3}{2} + \frac{A_0}{t}. \tag{3.18}$$

By Theorem 2.5.2, the deformation is holomorphic precisely when:

$$\frac{\partial \mathcal{A}_z}{\partial t} - \frac{\partial \mathcal{A}_t}{\partial z} + [\mathcal{A}_z, \mathcal{A}_t] = 0.$$

■

Remarkably, the necessary conditions from Theorem 3.3.1 are actually also sufficient. The proof of this statement however, out of the scope of this thesis. The sufficiency was first proven in the influential paper [JMU81] in great generality.

Theorem 3.3.2 [Th. 4.1 [JMU81]] Sufficient conditions for isomonodromic deformations.
If $\Phi : U_z \times U_t \subset \mathbb{S} \setminus \{0, t, \infty\} \times W_\epsilon \to GL_2(\mathbb{C})$ solve the differential equations

$$\mathcal{A}_z(z,t) = \frac{\sigma_3}{2} + \frac{A_0(t)}{z} + \frac{A_t(t)}{z-t} = \frac{\partial \Phi}{\partial z} \cdot \Phi(z,t)^{-1},$$

$$\mathcal{A}_t(z,t) = -\frac{A_t(t)}{z-t} = \frac{\partial \Phi}{\partial t} \cdot \Phi(z,t)^{-1},$$

the coefficient matrices satisfy the integrability condition

$$\frac{\partial \mathcal{A}_z}{\partial t} - \frac{\partial \mathcal{A}_t}{\partial z} + [\mathcal{A}_z, \mathcal{A}_t] = 0.$$

and the diagonalization matrices $P^{(0)}(t)$ and $P^{(1)}(t)$ satisfy

$$\frac{dP^{(0)}}{dt} \cdot P^{(0)}(t)^{-1} = \frac{A_t(t)}{t}, \qquad \frac{dP^{(1)}}{dt} \cdot P^{(1)}(t)^{-1} = \frac{\sigma_3}{2} + \frac{A_0(t)}{t},$$

then

$$\{\mathcal{A}(\cdot,t) \mid t \in U_t\} = \left\{ \frac{\sigma_3}{2} + \frac{A_0(t)}{z} + \frac{A_t(t)}{z-t} \;\middle|\; t \in U_t \right\}$$

is an isomonodromic deformation, in particular, the monodromy data (Definition 3.1.1)

$$\mathfrak{M} = \left\{ \Lambda_0^{(0)}, \Lambda_0^{(1)}, \Lambda_0^{(\infty)}; s_1, s_2; E^{(0)}, E^{(1)}, E^{(\infty)} = I \right\}.$$

is independent of t.

3.4 Schlesinger equations and the Painlevé V equation

In the previous Section we showed that

$$\{\mathcal{A}(\cdot,t) \mid t \in U_t\} = \left\{ \frac{\sigma_3}{2} + \frac{A_0(t)}{z} + \frac{A_t(t)}{z-t} \;\middle|\; t \in U_t \right\}$$

is an isomonodromic deformation exactly when it is part of the total differential equation

$$\mathcal{A}_z(z,t) = \frac{\sigma_3}{2} + \frac{A_0(t)}{z} + \frac{A_t(t)}{z-t} = \frac{\partial \Phi}{\partial z} \cdot \Phi(z,t)^{-1},$$

$$\mathcal{A}_t(z,t) = -\frac{A_t(t)}{z-t} = \frac{\partial \Phi}{\partial t} \cdot \Phi(z,t)^{-1},$$

which satisfy the integrability condition

$$\frac{\partial \mathcal{A}_z}{\partial t} - \frac{\partial \mathcal{A}_t}{\partial z} + [\mathcal{A}_z, \mathcal{A}_t] = 0,$$

and where the diagonalization matrices satisfy

$$\frac{dP^{(0)}}{dt} \cdot P^{(0)}(t)^{-1} = \frac{A_t(t)}{t}, \qquad \frac{dP^{(1)}}{dt} \cdot P^{(1)}(t)^{-1} = \frac{\sigma_3}{2} + \frac{A_0(t)}{t}.$$

Among these conditions, the integrability condition is the only condition constraining the coefficients $A_0(t)$ and $A_t(t)$ in the function $\mathcal{A}_z$. In this Section we will take a closer look on these conditions. Remark that the integrability conditions gives a differential equation in the variable t, for the matrices A_0, A_t. Moreover, as is shown in Corollary 3.4.1, the integrability condition is equivalent to a non-linear system of differential equations, called the Schlesinger equations. The Schlesinger equations are named after German mathematician Ludwig Schlesinger, who did pioneering work in the field of linear differential equations. In particular, he showed that the isomonodromic deformation of any (generic) linear differential equations, all satisfy the (general form of the) Schlesinger equations, in which Corollary 3.4.1 is a special case of.

Corollary 3.4.1 Schlesinger equations.
Consider the system of differential equations

$$\mathcal{A}_z(z,t) = \frac{\sigma_3}{2} + \frac{A_0(t)}{z} + \frac{A_t(t)}{z-t} = \frac{\partial \Phi}{\partial z} \cdot \Phi(z,t)^{-1},$$

$$\mathcal{A}_t(z,t) = -\frac{A_t(t)}{z-t} = \frac{\partial \Phi}{\partial t} \cdot \Phi(z,t)^{-1}.$$

The integrability condition

$$\frac{\partial \mathcal{A}_z}{\partial t} - \frac{\partial \mathcal{A}_t}{\partial z} + [\mathcal{A}_z, \mathcal{A}_t] = 0,$$

is equivalent to

$$\frac{dA_0}{dt} = \left[\frac{A_t}{t},\ A_0\right] \qquad = \left[\frac{dP^{(0)}}{dt}\cdot\left(P^{(0)}\right)^{-1},\ A_0\right] \tag{3.19}$$

$$\frac{dA_t}{dt} = \left[\frac{\sigma_3}{2} + \frac{A_0}{t}, A_t\right] = \left[\frac{dP^{(1)}}{dt}\cdot\left(P^{(1)}\right)^{-1},\ A_t\right] \tag{3.20}$$

These equations are called the Schlesinger equations. The equations give two Lax pair (L_0, M_0) *and* (L_1, M_1)*, where*

$$L_0 = A_0 \qquad\qquad L_1 = A_1$$

$$M_0 = \frac{A_t}{t} = \frac{dP^{(0)}}{dt}\cdot P^{(0)}(t)^{-1} \qquad\qquad M_1 = \frac{\sigma_3}{2} + \frac{A_0}{t} = \frac{dP^{(1)}}{dt}\cdot P^{(1)}(t)^{-1}$$

such that

$$\frac{dL_0}{dt} = [M_0, L_0] \qquad\qquad \frac{dL_1}{dt} = [M_1, L_1]$$

$$L_0(t) = P^{(0)}(t)\cdot\Lambda_0^{(0)}\cdot P^{(0)}(t)^{-1} \qquad\qquad L_1(t) = P^{(1)}(t)\cdot\Lambda_0^{(1)}\cdot P^{(1)}(t)^{-1}$$

see [BBT03] for more on Lax pairs.

Proof. We will calculate the left and the right hand side of the equation

$$\frac{\partial \mathcal{A}_z}{\partial t} - \frac{\partial \mathcal{A}_t}{\partial z} = [\mathcal{A}_t, \mathcal{A}_z].$$

Starting with the left hand side:

$$\frac{\partial \mathcal{A}_z}{\partial t} - \frac{\partial \mathcal{A}_t}{\partial z} = \frac{\frac{dA_0}{dt}}{z} + \frac{\frac{dA_t}{dt}}{z-t} + \frac{A_t}{(z-t)^2} - \frac{A_t}{(z-t)^2} = \frac{\frac{dA_0}{dt}}{z} + \frac{\frac{dA_t}{dt}}{z-t} \tag{3.21}$$

Then the right hand side

$$\begin{aligned} &\left[-\frac{A_t}{z-t}, \frac{\sigma_3}{2}\right] + \left[-\frac{A_t}{z-t}, \frac{A_0}{z}\right] + \left[-\frac{A_t}{z-t}, \frac{A_t}{z-t}\right] \\ &= \frac{1}{z-t}\left[\frac{\sigma_3}{2}, A_t\right] + \frac{1}{z(z-t)}[A_0, A_t] \\ &= \frac{1}{z-t}\left[\frac{\sigma_3}{2}, A_t\right] + \left(\frac{1}{t(z-t)} - \frac{1}{zt}\right)[A_0, A_t] \\ &= \frac{1}{z-t}\left[\frac{\sigma_3}{2} + \frac{A_0}{t}, A_t\right] + \frac{1}{z}\left[\frac{A_t}{t}, A_0\right] \end{aligned} \tag{3.22}$$

Equating the coefficient functions in (3.21) and (3.22), we obtain

$$\frac{dA_0}{dt} = \left[\frac{A_t}{t}, A_0\right]$$

and

$$\frac{dA_t}{dt} = \left[\frac{\sigma_3}{2} + \frac{A_0}{t}, A_t\right].$$

The statement with the Lax pairs are direct from the Definition of a Lax pair, see [BBT03]. ∎

Thus by Corollary 3.4.1, we can solve the Schlesinger equations (3.19) and (3.20), if we can solve the two first order systems of differential equations in t:

$$\frac{dP^{(0)}}{dt}\cdot P^{(0)}(t)^{-1}=\frac{A_t(t)}{t},\qquad \frac{dP^{(1)}}{dt}\cdot P^{(1)}(t)^{-1}=\frac{\sigma_3}{2}+\frac{A_0(t)}{t}.$$

In this thesis we have worked on the linear differential equation

$$\frac{d\Phi}{dz}\cdot\Phi(z)^{-1}=\frac{\sigma_3}{2}+\frac{A_0}{z}+\frac{A_t}{z-t}. \tag{3.23}$$

Under the Möbius transformation

$$z\mapsto\frac{z}{t}=w,$$

the equation transform into

$$\frac{d\Phi}{dw}\cdot\Phi(w)^{-1}=\frac{\sigma_3 t}{2}+\frac{A_0}{w}+\frac{A_t}{w-1}. \tag{3.24}$$

An equivalent treatment of (3.24) as we have presented for (3.23), gives the total system of differential equations

$$\frac{\partial\Phi}{\partial w}\cdot\Phi(w)^{-1}=\frac{\sigma_3 t}{2}+\frac{A_0(t)}{w}+\frac{A_t(t)}{w-1}, \tag{3.25}$$

$$\frac{\partial\Phi}{\partial t}\cdot\Phi(\omega)^{-1}=\frac{\sigma_3}{2}w+\frac{B}{t}, \tag{3.26}$$

where B depends on the matrices $\frac{\sigma_3}{2}t$, A_0 and A_t in a complicated way. The Schlesinger equations for this system are given by

$$\begin{aligned}\frac{dA_0}{dt}&=\left[\frac{B}{t},A_0\right]\\ \frac{dA_t}{dt}&=\left[\frac{\sigma_3}{2}+\frac{B}{t},A_t\right]\end{aligned} \tag{3.27}$$

It is shown in [Fok et al.06] and [AK00] that if one parametrize the matrices A_0, A_t and B in a certain way, then using the Schlesinger equations in (3.27), one of the parameters in A_t can be shown to satisfy the Painlevé V equation:

$$PV:\quad \frac{d^2u}{dx^2}=\left(\frac{1}{2u}\pm\frac{1}{u-1}\right)\left(\frac{du}{dx}\right)^2-\frac{1}{x}\frac{du}{dx}+\frac{(u-1)^2}{x^2}\left(\alpha u+\frac{\beta}{u}\right)+\frac{\gamma u}{x}+\frac{\delta u(u+1)}{u-1}, \tag{3.28}$$

where α,β,γ and δ are related to the eigenvalues of A_0, A_t. The Painlevé equations, are six non-linear second-order ordinary differential equations, with the "Painlevé' property": the only movable singularities are the poles [Pai00]. Actually it is shown that any second order ordinary differential equation of the form

$$u_{zz}=R(z,u,u_z),$$

meromorphic in z and rational in u and u_z, can be put into one of fifty "canonical" forms, see Painlevé's original work [Pai00], [Pai02] or a summary in [Inc56]. Forty-four of the fifty equations are reducible in the sense that they can be solved in terms of previously known functions. The six Painlevé equations are the exceptions, and their solutions, the Painlevé transcendents, give six new functions playing the same role in nonlinear mathematical physics that the classical special functions, such as Bessel functions and Airy functions play in linear physics, [Iwa et al.91], [Olv97].

3.5 Future projects and possible directions

In this Section we briefly discuss some possible future projects, continuing the work of the thesis.

The Painlevé V

The differential equation

$$\frac{d\Phi}{dw} \cdot \Phi(w)^{-1} = \frac{\sigma_3 t}{2} + \frac{A_0}{w} + \frac{A_t}{w-1}.$$

is equivalent to the differential equation treated in this thesis, by a simple Möbius transformation $\omega \mapsto zt$. However, the Schlesinger equations in Corollary 3.4.1 and equation (3.27), which are used to derive the Painlevé V in the latter case are different.

Conjecture 1.
Given the first order linear system of differential equations

$$\frac{d\Phi}{dz} \cdot \Phi(z)^{-1} = \frac{\sigma_3}{2} + \frac{A_0}{z} + \frac{A_t}{z-t}.$$

Then there exists a parametrization of A_0 and A_t, such that the Schlesinger equations (3.19) *and* (3.20) *implies that one of the parameters satisfy the Painlevé V equation* (3.28).

One possibility is to find the relation between the matrix B in (3.27) and A_t by "following" the Möbius transformation.

Continuing geometric description

There is a well developed theory of *Riemann-Hilbert correspondence*, which investigates the correspondence between regular singular flat connections on algebraic vector bundles and representations of the fundamental group, see [Kas84] and [Meb80]. This is exactly the problem treated in this thesis, but in the category of algebraic vector bundles, instead of complex principal bundles. It would be interesting to learn the techniques from algebraic geometry, and find corresponding treatments for the complex analytic situation.

Appendix A

Complex holomorphic manifolds

A.1 Complex manifolds and tangent spaces

Definition A.1.1 Holomorphic function of several variables .
Let $f : V \subset \mathbb{C}^n \to \mathbb{C}$ be a function. We say f is holomorphic if the function

$$f(z_1, \ldots, \omega, \ldots, z_n)$$

is holomorphic in ω, when the other variables are fixed. That is, if it is holomorphic in each variable separately. To denote the complex derivative w.r.t the complex variable z_i we write

$$\frac{\partial f}{\partial z^i}.$$

If $f : V \subset \mathbb{C}^n \to \mathbb{C}^n$, we say that f is holomorphic if for each $k \in \{ 1, \ldots, n \}$ the function

$$f^k := r^k \circ f : V \to \mathbb{C}$$

where $r^k : \mathbb{C}^n \to \mathbb{C}$ is the k-th coordinate projection in $\mathbb{C}^n$.

Evidently this is just an extension of the definitions given for smooth functions.

Definition A.1.2 Complex manifold.
A complex manifold M, of dimension n, is a smooth real manifold of dimension $2n$ admitting a holomorphic atlas. That is an atlas $\mathcal{U} = \{ (U_\alpha, \varphi_\alpha) \}$ containing charts $(U_\alpha, \varphi_\alpha)$ where $U_\alpha \subset M$ is open,

$$\varphi_\alpha : U_\alpha \to \varphi(U_\alpha) \subset \mathbb{C}^n$$

is a homeomorphism, and for every pair (U_α, φ_a), $(U_\beta, \varphi_b) \in \mathcal{U}$ the function

$$\varphi_\alpha \circ \varphi_b^{-1} : \varphi_b(U_a \cap U_\beta) \subset \mathbb{C}^n \to U_\alpha \cap U_\beta \to \varphi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$$

is a holomorphic function between domains in C^n.

Remark.
If M is a complex manifold with atlas $\mathcal{U} = \{ (U_\alpha, \varphi_\alpha) \}$, then in particular M is a smooth real manifold. A nice real atlas to use on M is the collection of real charts $\{ (U_\alpha, \tilde{\varphi}_\alpha) \}$ where

$$\tilde{\varphi}_\alpha(p) = (z^1(p), z^2(p), \ldots, z^n(p)) = (x^1(p), y^1(p), x^2(p), y^2(p), \ldots, x^n(p), y^n(p)) \in \mathbb{R}^{2n}$$

where $z^k(p) = x^k(p) + iy^k(p)$ is a coordinate function on U_α.

Definition A.1.3 Holomorphic functions defined on complex manifolds.
A function $f : U \subset M \to \mathbb{C}$ defined on an open set U of a complex manifold M is holomorphic if for each $p \in U$ there is a chart $(U_\alpha, \varphi_\alpha)$ such that $p \in U_\alpha \cap U = U'$ and $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U') \to \mathbb{C}$ is holomorphic.

The real manifold structure on a complex manifold M can also be used to define smooth complex valued functions on M.

Definition A.1.4 Smooth complex valued functions on a complex manifold.
Let M be a complex manifold and let $f = u + iv : U \to \mathbb{C}$ be a complex valued function on M. Consider the real charts $(U_\alpha, \tilde{\varphi}_\alpha)$ where $\tilde{\varphi}_\alpha : U \to \mathbb{R}^{2n}$. Then the function f is smooth if for each $p \in U$ there is a chart $(U_\alpha, \tilde{\varphi}_\alpha)$ where $p \in U_\alpha \cap U = U'$ and

$$u \circ \tilde{\varphi}_\alpha^{-1} : \tilde{\varphi}_\alpha(U') \subset \mathbb{R}^{2n} \to \mathbb{R}$$

$$v \circ \tilde{\varphi}_\alpha^{-1} : \tilde{\varphi}_\alpha(U') \subset \mathbb{R}^{2n} \to \mathbb{R}$$

are smooth functions.

Lastly we define holomorphic maps between complex manifolds.

Definition A.1.5 Holomorphic maps between complex manifolds.
Let M, N be complex manifolds with atlases $\mathcal{U} = \{ U_\alpha, \varphi_\alpha \}$ and $\mathcal{V} = \{ (V_\beta, \psi_\beta) \}$ respectively. A continuous function $f : M \to N$ is holomorphic if for each $p \in M$ there is a chart $(U_\alpha, \varphi_\alpha)$ and a chart (V_β, ψ_β) with $f(U_\alpha) \subset V_\beta$ such that the function

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \subset \mathbb{C}^m \to \psi_\beta(f(U_\alpha)) \subset \mathbb{C}^n$$

is holomorphic.

Definition A.1.6 Holomorphic tangent space of a complex manifold at point.
Let M be a complex manifold. Let $\mathcal{H}_{M,\mathbb{C}}(p)$ denote the $\mathbb{C}$-algebra of holomorphic function germs at $p \in M$. Consider the set

$$\mathrm{Der}\,(\mathcal{H}_{M,\mathbb{C}}(p) \to \mathbb{C})$$

of $\mathbb{C}$-linear complex derivations of the germs in $\mathcal{H}_{M,\mathbb{C}}(p)$. This is a complex vector space, which we will call the holomorphic tangent space of M at p and denote it by T_pM.

This Definition is a very natural extension of the similar Definition of the real tangent space of a smooth manifold. However it is for now unclear how to describe the elements of this vector space using the complex structure on M. We will now construct two equivalent viewpoints that justify the Definition.

Definition A.1.7 The real tangent space of a complex manifold.
Let M be a complex manifold. In particular it is a smooth real manifold. Let $\mathcal{C}^\infty_\mathbb{R}(p)$ denote the $\mathbb{R}$-algebra of smooth real valued function germs at p. Consider the set

$$\mathrm{Der}(\mathcal{C}^\infty_\mathbb{R}(p) \to \mathbb{R})$$

of $\mathbb{R}$-linear real derivations of the germs in $\mathcal{C}^\infty_\mathbb{R}(p)$. This is a real vector space which we will call the real tangent space of M at $p \in M$, and denote it by $T_pM_\mathbb{R}$.

Using the complex structure in the complex vector space $\mathbb{C}^n$, we can induce a complex structure on the real tangent space $T_pM_\mathbb{R}$.

Definition A.1.8 Complex structure map on a real vector space.
Let V be a real even dimensional vector space. A complex structure map on V is a endomorphism $J : V \to V$ such that $J \circ J = -I$.

Let M be a complex manifold and consider the chart $(U_\alpha, \tilde{\varphi}_\alpha)$, with $p \in U_\alpha$ and

$$\tilde{\varphi}_\alpha : U_\alpha \to \mathbb{R}^{2n}.$$

This is a diffeomorphism between real smooth manifolds, thus at each point $p \in U_\alpha$ it induces an isomorphism of their real tangent spaces :

$$(\tilde{\varphi}_\alpha)_{*,p} : T_pM_\mathbb{R} \to T_{\tilde{\varphi}_\alpha(p)}\mathbb{R}^{2n}$$

If we consider $T_{\tilde{\varphi}_\alpha(p)}\mathbb{R}^{2n}$ with basis

$$\left\{ \frac{\partial}{\partial x^k_\mathbb{R}}, \frac{\partial}{\partial y^k_\mathbb{R}} \right\}_{k=1}^n$$

there is a natural complex structure map $J_{\mathbb{R}}$ defined by

$$J_{\mathbb{R}}\left(\frac{\partial}{\partial x^k_{\mathbb{R}}}\right) = \frac{\partial}{\partial y^k_{\mathbb{R}}}$$
$$J_{\mathbb{R}}\left(\frac{\partial}{\partial y^k_{\mathbb{R}}}\right) = -\frac{\partial}{\partial x^k_{\mathbb{R}}}$$

for each k, and then extending by linearity. Using the map $J_{\mathbb{R}}$ we define a complex structure map on the real tangent space $T_pM_{\mathbb{R}}$ of M by

$$J = (\tilde{\varphi}_\alpha)^{-1}_{*,p} J_{\mathbb{R}} (\tilde{\varphi}_\alpha)_{*,p} \tag{A.1}$$

$$\begin{tikzcd}
T_pM_{\mathbb{R}} \arrow[r, dashed, "J"] \arrow[d, "(\tilde{\varphi}_\alpha)_{*,p}"'] & T_pM_{\mathbb{R}} \\
T_{\tilde{\varphi}_\alpha}\mathbb{R}^{2n} \arrow[r, "J_{\mathbb{R}}"] & T_{\tilde{\varphi}_\alpha}\mathbb{R}^{2n} \arrow[u, "(\tilde{\varphi}_\alpha)^{-1}_{*,p}"']
\end{tikzcd}$$

In particular, on the basis $\left\{ \frac{\partial}{\partial x^k}\big|_p, \frac{\partial}{\partial y^k}\big|_p \right\}$ for $T_pM_{\mathbb{R}}$, the complex structure map J is given by

$$J\left(\frac{\partial}{\partial x^k}\bigg|_p\right) = (\tilde{\varphi}_\alpha)^{-1}_{*,p} J_{\mathbb{R}}\left(\frac{\partial}{\partial x^k_{\mathbb{R}}}\right) = (\tilde{\varphi}_\alpha)^{-1}_{*,p}\left(\frac{\partial}{\partial y^k_{\mathbb{R}}}\right) = \frac{\partial}{\partial y^k}\bigg|_p$$
$$J\left(\frac{\partial}{\partial y^k}\bigg|_p\right) = (\tilde{\varphi}_\alpha)^{-1}_{*,p} J_{\mathbb{R}}\left(\frac{\partial}{\partial y^k_{\mathbb{R}}}\right) = (\tilde{\varphi}_\alpha)^{-1}_{*,p}\left(-\frac{\partial}{\partial x^k_{\mathbb{R}}}\right) = -\frac{\partial}{\partial x^k}\bigg|_p$$

The pair $(T_pM_{\mathbb{R}}, J)$ is now a real vector space of dimension $2n$ with a complex structure map, so in order to construct a complex vector space, we are missing multiplication by complex numbers. It is therefore natural to define the complexification of the vector space, $\mathbb{C} \otimes T_pM_{\mathbb{R}}$. The properties of the complexification of a vector space V together with a complex structure map J, is summarized in the following Proposition.

Proposition A.1.1 [Bog91][Ch.3.2]Complexification of a vector space with a complex structure map.
Let V be a real vector space of dimension $2n$ with a complex structure map J. Then

- *$\mathbb{C} \otimes V$ is a complex vector space of dimension $2n$*
- *J has an extension to $\mathbb{C} \otimes V$, $\tilde{J}$, given by*

  $$\tilde{J}(z \otimes v) := z \otimes Jv$$

  We will suppress the notation for the tensor product when it cannot lead to confusion, thus we can write the equation as

  $$\tilde{J}(zv) := zJ(v)$$

- *$\tilde{J} : \mathbb{C} \otimes V \to \mathbb{C} \otimes V$ has two eigenspaces $V^{1,0}$ and $V^{0,1}$, of dimension n, related to the eigenvalues $i, -i$ respectively, such that*

  $$\mathbb{C} \otimes V = V^{1,0} \oplus V^{0,1} = V^{1,0} \oplus \overline{V^{1,0}}.$$

- *Given any set of n linearly independent vectors $\{ v_j \}_{j=1}^n$ in V, then $\{ v_j, Jv_j \}_{j=1}^n$ is a basis for V. Further $\left\{\frac{1}{2}(v_j - iJv_j)\right\}_{j=1}^n$ is a basis for $V^{1,0}$ and $\left\{\frac{1}{2}(v_j + iJv_j)\right\}_{j=1}^n$ is a basis for $V^{0,1}$.*
- *The dual space of $\mathbb{C} \otimes V$ can be written*

  $$(\mathbb{C} \otimes V)^* = V^{1,0*} \otimes V^{0,1*} := V_{1,0} \otimes V_{0,1}$$

- *If $\{ \alpha^j \}_{j=1}^n$ are dual to $\{ v_j \}_{j=1}^n$ then the dual bases for the dual spaces $V_{1,0}$ and $V_{0,1}$ are given by $\{ \alpha^j - iJ^*\alpha^j \}_{j=1}^n$ and $\{ \alpha^j + iJ^*\alpha^j \}_{j=1}^n$ respectively, where J^* is the dual map of J.*

We can now use all this information to describe the complexification of the real tangent space of M, $\mathbb{C} \otimes T_pM_{\mathbb{R}}$. We summarize the properties in the following Proposition.

Proposition A.1.2 Complexified real tangent space of a complex manifold.
Let M be a complex manifold with its real tangent space at p, $T_pM_{\mathbb{R}}$ of real dimension $2n$, together with the complex structure map J, defined in (A.1). The complexified tangent space of M at p is the complex vector space $\mathbb{C}\otimes T_pM_{\mathbb{R}}$ of complex dimension $2n$, with the following properties, as a result of the previous Proposition.

- *J has an extension to $\mathbb{C}\otimes T_pM_{\mathbb{R}}$, $\tilde{J}$, given by*
$$\tilde{J}\left(z\frac{\partial}{\partial x^j}\bigg|_p\right) = z\frac{\partial}{\partial y^j}\bigg|_p, \quad \tilde{J}\left(z\frac{\partial}{\partial y^j}\bigg|_p\right) = -z\frac{\partial}{\partial x^j}\bigg|_p$$
- *$\mathbb{C}\otimes T_pM_{\mathbb{R}} = T_p^{1,0}M \oplus T_p^{0,1}M$, where $T_p^{1,0}M, T_p^{0,1}M$ is the eigenspaces of $\tilde{J}$ related to the eigenvalue i and $-i$ respectively.*
- *Since $\left\{\frac{\partial}{\partial x^j}\big|_p, \frac{\partial}{\partial y^j}\big|_p\right\}_{j=1}^n$ is a basis for $T_pM_{\mathbb{R}}$, we have that*
$$\left\{\frac{\partial}{\partial z^j}\bigg|_p\right\}_{j=1}^n := \left\{\frac{1}{2}\left(\frac{\partial}{\partial x^j}\bigg|_p - i\frac{\partial}{\partial y^j}\bigg|_p\right)\right\}_{j=1}^n, \quad \text{is a basis for } T_p^{1,0}M$$
$$\left\{\frac{\partial}{\partial \bar{z}^j}\bigg|_p\right\}_{j=1}^n := \left\{\frac{1}{2}\left(\frac{\partial}{\partial x^j}\bigg|_p + i\frac{\partial}{\partial y^j}\bigg|_p\right)\right\}_{j=1}^n, \quad \text{is a basis for } T_p^{0,1}M$$
- *The complexified cotangent space, $\mathbb{C}\otimes T_p^*M_{\mathbb{R}} = (\mathbb{C}\otimes T_pM_{\mathbb{R}})^*$, of $T_pM_{\mathbb{R}}$ can be written*
$$\mathbb{C}\otimes T_p^*M_{\mathbb{R}} = T_p^{1,0}M \oplus T_p^{0,1}M := T_{p,1,0}^*M \oplus T_{p,0,1}^*M$$
$$\{dz_p^j\}_{j=1}^n := \{dx_p^j + idy_p^j\}_{j=1}^n, \quad \text{is the dual basis of the basis for } T_p^{1,0}M$$
$$\{d\bar{z}_p^j\}_{j=1}^n := \{dx_p^j - idy_p^j\}_{j=1}^n, \quad \text{is the dual basis of the basis for } T_p^{0,1}M$$

The subspace $T_p^{1,0}M$ will be of special interest later. This is a n-dimensional complex vector space. We can find a real isomorphism from $T_p^{1,0}M$ to $T_pM_{\mathbb{R}}$, the real tangent space of M. Further using the complex structure map J on $T_pM_{\mathbb{R}}$, we loose no information of the complex structure of $T_p^{1,0}M$ going to $T_pM_{\mathbb{R}}$.

Note that in the proceeding text we use the Einstein summing convention:
$$\sum_{j=1}^n V^j\frac{\partial}{\partial z^j}\bigg|_p = V^j\frac{\partial}{\partial z^j}\bigg|_p$$

Corollary A.1.1 [Bog91].
Consider the $2n$ dimensional real tangent space $T_pM_{\mathbb{R}}$ of M and the complex structure map J on $T_pM_{\mathbb{R}}$ defined in (A.1). Consider also the subspace $T_p^{1,0}M$ of $\mathbb{C}\otimes T_pM_{\mathbb{R}}$ with the complex structure map $\tilde{J}$. Then there exists a real isomorphism
$$\begin{array}{cccc} \chi: & T_p^{1,0}M & \to & T_pM_{\mathbb{R}} \\ & V = V^j\frac{\partial}{\partial z^j}\big|_p & \mapsto & \mathrm{Re}(V^j)\frac{\partial}{\partial x^j}\big|_p + \mathrm{Im}(V^j)\frac{\partial}{\partial y^j}\big|_p \end{array}$$
such that
$$\chi(\tilde{J}(V)) = J(\chi(V)).$$

We now relate the derivation of smooth complex functions to derivation of smooth real valued functions. Let $\mathcal{C}_{M,\mathbb{C}}^\infty(p)$ be the $\mathbb{C}$-algebra of smooth complex valued function germs at p. Consider $\mathrm{Der}(\mathcal{C}_{M,\mathbb{C}}^\infty(p)\to\mathbb{C})$, the $\mathbb{C}$-linear derivations of the germs in $\mathcal{C}_{M,\mathbb{C}}^\infty(p)$.

Lemma A.1.1 [Bog91].
Given a complex manifold M. Then
$$\mathbb{C}\otimes T_pM_{\mathbb{R}} \simeq \mathrm{Der}(\mathcal{C}_{M,\mathbb{C}}^\infty(p)\to\mathbb{C})$$

The isomorphism Ξ from Lemma A.1.1, tells us how we should evaluate an element V of $T_pM_{\mathbb{R}}$ on an element $F = f + ig \in C^{\infty}_{M,\mathbb{C}}(p)$, i.e on smooth complex valued function germ at p, and that is by evaluating V on the real and complex part of $F = f + ig$ separately.

From now on we will identify the complexified tangent space $\mathbb{C} \otimes T_pM_{\mathbb{R}}$ with $\mathrm{Der}(\mathcal{C}^{\infty}_{M,\mathbb{C}}(p) \to \mathbb{C})$.

Corollary A.1.2.
Given a complex manifold M. Consider the complex vector space $\mathrm{Der}(\mathcal{C}^{\infty}_{M,\mathbb{C}}(p) \to \mathbb{C})$ containing the subspace $\mathrm{Der}(\mathcal{H}_{M,\mathbb{C}}(p) \to \mathbb{C}) = T_pM$ which is the holomorphic tangent space of M at p. A basis for $\mathrm{Der}(\mathcal{C}^{\infty}_{M,\mathbb{C}}(p) \to \mathbb{C})$ is given by

$$\left\{ \frac{\partial}{\partial z^j}\bigg|_p, \frac{\partial}{\partial \bar{z}^j}\bigg|_p \right\}_{j=1}^{n} := \left\{ \frac{1}{2}\left(\frac{\partial}{\partial x^j}\bigg|_p - i\frac{\partial}{\partial y^j}\bigg|_p \right), \frac{1}{2}\left(\frac{\partial}{\partial x^j}\bigg|_p + i\frac{\partial}{\partial y^j}\bigg|_p \right) \right\}_{j=1}^{n}$$

Hence

$$\mathrm{Der}(\mathcal{C}^{\infty}_{M,\mathbb{C}}(p) \to \mathbb{C}) = \mathrm{Der}(\mathcal{H}_{M,\mathbb{C}}(p) \to \mathbb{C}) \oplus \mathrm{Der}(\mathcal{AH}_{M,\mathbb{C}}(p) \to \mathbb{C}) = T_p^{1,0}M \oplus T_p^{0,1}M$$

Where $\mathrm{Der}(\mathcal{AH}_{M,\mathbb{C}}(p) \to \mathbb{C})$ are the C-linear derivations of the antiholomorphic function germs at p.

In particular a basis for T_pM is given by

$$\left\{ \frac{\partial}{\partial z^j}\bigg|_p \right\}_{j=1}^{n} := \left\{ \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j} \right) \right\}_{j=1}^{n}$$

Proof. Follows directly from Proposition A.1.2, Lemma A.1.1 and the Cauchy-Riemann equations. ■

To summarize we defined the holomorphic tangent space T_pM of M at p to be the complex vector space of $\mathbb{C}$-linear derivations of holomorphic function germs at p, $\mathrm{Der}(\mathcal{H}_{M,\mathbb{C}} \to \mathbb{C})$.

- The $2n$ dimensional real tangent space $T_pM_{\mathbb{R}}$ of M at p, that is obtained by regarding M as a smooth real manifold, is real isomorphic to T_pM (Corollary A.1.1, Corollary A.1.2).
- The $2n$ dimensional complexified real tangent space $\mathbb{C} \otimes T_pM_{\mathbb{R}}$ of M, is isomorphic to $\mathrm{Der}(\mathcal{C}^{\infty}_{M,\mathbb{C}}(p) \to \mathbb{C})$, the $2n$ dimensional complex vector space of $\mathbb{C}$-linear derivations of complex function germs at p. If we were interested in all smooth complex functions on a complex manifold, we would consider this space as the tangent space.
- T_pM is an n dimensional subspace of $\mathrm{Der}(C^{\infty}_{M,\mathbb{C}} \to \mathbb{C})$. Actually we showed that

  $$\mathrm{Der}(\mathcal{C}^{\infty}_{M,\mathbb{C}}(p) \to \mathbb{C}) = \mathrm{Der}(\mathcal{H}_{M,\mathbb{C}}(p) \to \mathbb{C}) \oplus \mathrm{Der}(\mathcal{AH}_{M,\mathbb{C}}(p) \to \mathbb{C}),$$

  where $\mathrm{Der}(\mathcal{AH}_{M,\mathbb{C}} \to \mathbb{C})$ is the complex vector space of $\mathbb{C}$-linear derivations of antiholomorphic function germs at p.

$$\begin{tikzcd}
\mathbb{C} \otimes T_pM_{\mathbb{R}} \arrow[r, equal] \arrow[d, "\pi_2"] & T_p^{1,0}M \oplus \overline{T_p^{1,0}M} \arrow[r, leftrightarrow, "\overset{\mathbb{C}}{\sim}"] \arrow[d, "\pi_1"'] & \mathrm{Der}(\mathcal{H}_{M,\mathbb{C}}(p) \to \mathbb{C}) \oplus \mathrm{Der}(\mathcal{AH}_{M,\mathbb{C}}(p) \to \mathbb{C}) \arrow[d, "\pi_1"] \\
T_pM_{\mathbb{R}} \arrow[r, leftrightarrow, "\overset{\mathbb{R}}{\sim}"] & T_p^{1,0}M \arrow[r, leftrightarrow, "\overset{\mathbb{C}}{\sim}"] & T_pM
\end{tikzcd}$$

The construction of the holomorphic tangent bundle and the holomorphic cotangent bundle over M is now completely analogous to the construction of the tangent bundle on a smooth real manifold, where we require the chart to be holomorphic instead of smooth.

Example A.1.1 The Riemann sphere.
The Riemann sphere $\mathbb{S}$, is the *one-point compactification* of the complex plane. That is $\mathbb{S} = \mathbb{C} \cup \{\infty\}$, where we define the topology to be

- If $U \subset \mathbb{S}$ and $\infty \notin U$ then U is open in $\mathbb{S}$ if it is open as a subset of $\mathbb{C}$.
- If $U \subset \mathbb{S}$ and $\infty \in U$, then U is open in $\mathbb{S}$ if $\mathbb{C} \setminus U$ is compact.

This space is compact by construction, for a more detailed description see [Mun00]. An equivalent construction can be done by the stereographic projection from the 2-sphere $S^2 \setminus \{ (0,0,1) \} \subset \mathbb{R}^3$, a real smooth manifold, to $\mathbb{C} \simeq \mathbb{R}^2$ where we map $(0,0,1)$ to ∞, or even as the complex projective line $\mathbb{P}^1\mathbb{C}$.

The Riemann sphere is a complex manifold. Indeed it is homeomorphic to the 2-sphere by the stereographic projection, hence it is has a smooth manifold structure. It can be given a complex atlas by the two charts

$$\begin{array}{rccc} \varphi_0 : & \mathbb{S} \setminus \{ \infty \} & \to & \mathbb{C} \\ & \omega & \mapsto & \omega \end{array}$$

when ∞ is not on the chart and as

$$\begin{array}{rccc} \varphi_\infty : & \mathbb{S} \setminus \{ 0 \} & \to & \mathbb{C} \\ & \omega & \mapsto & \frac{1}{\omega}, \end{array}$$

when 0 is not on the chart. Obviously the transition maps are holomorphic, and $\mathbb{S}$ is covered by the charts, hence we have a holomorphic atlas. The coordinate functions will also be denoted by $r^1 \circ \varphi_\alpha(\omega) = \mathrm{id} \circ \varphi_\alpha(\omega) = z(\omega)$

Since $\mathbb{S}$ is locally homeomorphic to $\mathbb{C}$, the holomorphic tangent space $T_\omega\mathbb{S}$ of $\mathbb{S}$ at p is a one dimensional complex vector space. In local coordinates it has the basis vector $\frac{\partial}{\partial z}\big|_\omega = \frac{d}{dz}\big|_\omega$.

Lemma A.1.2 [Rud87] Constant function criterion.
Let M, N be two complex manifolds. Let U be an open connected subset of M, and consider $f : U \to N$, a holomorphic function. If $f_{*,p} = 0$ for every $p \in U$, then f is a constant function on U.

When we have a matrix power series, it is useful with a criterion on when we can expect the series to be invertible as a matrix.

Lemma A.1.3 [Neu77] Neumann series inverse.
Consider the series

$$f(z) = \sum_{k=0}^{\infty} A_k (z-a)^k, \; A_k \in M_2(\mathbb{C}),$$

convergent in the disc $B(a, R)$. If $A_0 \in GL_2(\mathbb{C})$, then there exists a $0 < \delta \leq R$ such that $f(z) \in GL_2(\mathbb{C})$ and

$$(f(z))^{-1} = \sum_{n=0}^{\infty} (-1)^n \left(\sum_{k=1}^{\infty} A_0^{-1} A_k (z-a)^k \right)^n A_0^{-1}$$

for all $z \in B(a, \delta)$.

A.2 Complex Lie groups

Definition A.2.1 Complex Lie group.
Let G be a connected complex manifold. Then G is a complex Lie-group if it is also a group, such that the group multiplication

$$\begin{array}{rccc} \mu_G : & G \times G & \to & G \\ & (g,h) & \mapsto & gh \end{array}$$

and the inversion

$$\begin{array}{ccc} G & \to & G \\ g & \mapsto & g^{-1} \end{array}$$

are holomorphic maps between complex manifolds.

Definition A.2.2 Vector fields on complex manifolds and left invariance.
A holomorphic vector field X, on a complex manifold G, is a Section of the holomorphic tangent bundle TG, i.e. $X : M \to TM$ such that $\pi \circ X = \mathrm{id}_M$, is holomorphic. We will only deal with holomorphic vector fields and will therefore refer to them as simply vector fields. If G is a Lie group, and X is a vector field on G with the property that $(L_a)_{*,b} X_b = X_{a \cdot b}$, then X is called left invariant.

Definition A.2.3 Lie algebra.
Let G be a complex Lie-group. We define its Lie algebra, $\mathfrak{g}$, to be the vector space of left invariant vector fields

on G. On $\mathfrak{g}$ we define the Lie bracket $[-,-]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that for any $X, Y \in \mathfrak{g}$ and choosing $f \in \mathcal{H}$ holomorphic functions at a points on G:

$$[X, Y]_{\mathfrak{g}} f = X(Yf) - Y(Xf).$$

The tangent space of G at the identity T_eG can be identified with the Lie algebra $\mathfrak{g}$ of G. Indeed we have the $\mathbb{C}$-linear isomorphism

$$\begin{array}{ccc} \mathfrak{g} & \to & T_eG \\ X & \mapsto & X(e) = X_e \end{array}$$

by mapping a left invariant vector field X at any point $a \in G$ to T_eG by $(L_{a^{-1}})_{*,a} X_a = X_e$.

Definition A.2.4 The adjoint representation.
Given a Lie group G with Lie algebra $\mathfrak{g}$. Consider the Lie group homomorphism Con: $G \to \mathrm{Aut}(G)$ where

$$Con(a) = L_a \circ R_{a^{-1}} = R_{a^{-1}} \circ L_a : \begin{array}{ccc} G & \to & G \\ b & \mapsto & aba^{-1} \end{array}$$

Taking the differential at the identity $e \in G$, of $Con(a) : G \to G$ for each $a \in G$, we obtain a representation of G in $\mathfrak{g}$ called the adjoint representation.

$$\mathrm{Ad} : \quad G \quad \to \quad \mathrm{Aut}(\mathfrak{g})$$

where

$$\mathrm{Ad}(a) = Con(a)_{*,e} : \begin{array}{ccc} \mathfrak{g} & \to & \mathfrak{g} \\ X & \to & (L_a)_{*,a^{-1}} \circ (R_{a^{-1}})_{*,e} X \end{array}$$

In particular $\mathrm{Ad}(a)$ preserves the Lie-bracket in $\mathfrak{g}$.

Since $L_a : G \to G$ is a bi-holomorphic function, its push forward maps vector fields to vector fields: $X \mapsto (L_a)_* X$. This can be utilized to obtain an isomorphism between the tangent bundle TG, and $G \times \mathfrak{g}$, by mapping

$$\Upsilon : \begin{array}{ccc} TG & \to & G \times \mathfrak{g} \\ X_a & \mapsto & \left(a, (L_{a^{-1}})_* X_a\right) \end{array}$$

For each $a \in G$ we thus have $(L_{a^{-1}})_* \in \mathrm{Hom}(T_aG, \mathfrak{g}) \simeq T_a^*G \otimes \mathfrak{g}$. We want to isolate the part of Υ that acts on the tangent vectors.

Definition A.2.5 Maurer Cartan form.
Let G be a complex Lie-group with Lie algebra $\mathfrak{g}$. Consider

$$\theta : \begin{array}{ccc} G & \to & T^*G \otimes \mathfrak{g} \\ a & \mapsto & \theta_a \end{array}$$

defined by

$$\theta_a(X_a) = (L_{a^{-1}})_* X_a, \quad X_a \in T_aG$$

If $x^1, \dots, x^n$ are local coordinates in a neighbourhood U of $a \in G$, $\xi_1, \dots, \xi_n$ is a basis for $\mathfrak{g}$, and X is a vector field on U, we write

$$\theta(X) = \left(\theta_j^k(a) dx^j \otimes \xi_k\right)(X) := \theta_j^k(a) dx^j(X) \xi_k,$$

where $\theta_j^k : U \to \mathbb{C}$ are holomorphic functions for every $j, k \in \{1, \dots, n\}$. The map θ is called the left invariant Maurer-Cartan form.

Proposition A.2.1 [Sha97] Properties of the Maurer Cartan form .
Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let θ be the Maurer Cartan form on G. Then

1. *On $T_eG = \mathfrak{g}$, θ is the identity. $\theta_e(X_e) = X_e$.*

2. *θ is left invariant, in the sense that it is invariant by left translations: $(L_a)^*\theta = \theta$.*

3. *By right translations: $(R_a)^*\theta = \mathrm{Ad}(a^{-1}) \circ \theta$.*

The exterior derivative of the Maurer Cartan form has a particular nice form.

Proposition A.2.2 [Sha97] **Structure equation.**
Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let θ be the Maurer Cartan form on G. Then

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0.$$

Here $[-\wedge-]$ denotes the bracket of Lie algebra valued forms, that is if $\alpha, \beta : G \to T^*G \otimes \mathfrak{g}$ are Lie algebra valued 1-forms on G and $X, Y \in \mathfrak{X}(G)$ are vector fields on G, then $[\alpha \wedge \beta](X, Y) = ([\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)])$.

A.3 Proofs from Chapter 2

Proposition A.3.1 Vertical tangent space and fundamental vector fields.
Consider a principal bundle $P(M, G, \pi)$ with vertical tangent space V_p at $p \in P$. Let $\mathfrak{X}(P)$ denote the vector fields on P. Then

i. *V is involutive, that is if $v, w \in \mathfrak{X}(P)$ are two vertical vector fields of P, such that $v_p, w_p \in V_p$, then also $[v, w]_p \in V_p$ for each p. By the Frobenius theorem, Theorem 2.4.1, V is an integrable distribution (see Definition 2.4.1).*

ii. *The distribution $V \subset TP$ is G-invariant, in the sense that $(\mu_a)_* V_p = V_{p\,.\,a}$, where $\mu_a : P \to P$, such that $p \mapsto p\,.\,a$, the right action of G on P, with a fixed group element $a \in G$.*

iii. *There exists a Lie algebra homomorphism*

$$\sigma : \quad \mathfrak{g} \quad \to \quad \mathfrak{X}(P),$$

that maps $X \in \mathfrak{g}$ into a vector field σX, called a fundamental vector field on P. Pointwise it is defined by

$$(\sigma X)_p = \sigma_p X := (\mu_p)_{*,e} X,$$

where $X \in \mathfrak{g}$ and $\mu_p : G \to P$, is the right action of G on P with a fixed point $p \in P$. It is a Lie algebra homomorphism in the sense that

$$\sigma[X, Y]_{\mathfrak{g}} = [\sigma X, \sigma Y]_{\mathfrak{X}(P)}, \quad \textit{for any } X, Y \in \mathfrak{g}.$$

For each $p \in P$, $\sigma_p := (\mu_p)_{*,e} : \mathfrak{g} \to V_p \subset T_p P$ is a vector space isomorphism. So the fundamental vector fields are all vertical.

iv. *Locally, if $\varphi_\alpha = \pi \times g_\alpha$ trivialises $P(M, G, \pi)$ in $\pi^{-1}(U_\alpha)$ and $\theta : G \to T^*G \times \mathfrak{g}$ is the Maurer-Cartan form (Definition A.2.5), then*

$$\sigma \circ (g_\alpha^* \theta) = I_V, \quad (g_\alpha^* \theta) \circ \sigma = I_{\mathfrak{g}}.$$

v. *$(\mu_a)_{*,p} \sigma_p X = \sigma_{p\,.\,a}\left(\mathrm{Ad}\left(a^{-1}\right) X\right)$, where $X \in \mathfrak{g}$ and $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is the adjoint representation of G in $\mathfrak{g}$, see Definition A.2.4.*

Proof.

i. Let $v, w \in \mathfrak{X}(P)$ be two vertical vector fields on P. Then $\pi_* v = 0 = \pi_* w$. Let $f \in \mathcal{H}(z)$ be a holomorphic function at $z \in M$, then

$$\pi_*\big([v, w]\big) f = v(w(f \circ \pi)) - w(v(f \circ \pi)) = v(\pi_* w(f)) - w(\pi_* v(f)) = 0.$$

ii. Let $v_p \in V_p$, then $(\mu_a)_* v_p \in T_{p\,.\,a} P$ and $\pi_{*,p\,.\,a} \circ (\mu_a)_{*,p} v_p = (\pi \circ \mu_a)_{*,p}\, v_p = \pi_{*,p} v_p = 0$, since $\mu_a \circ \pi = \pi$.

iii. We consider the right action $\mu : P \times G \to P$, fixing a point $p \in P$, we obtain the holomorphic function $\mu_p : G \to P$. Taking the tangential map at $e \in G$, we define $\sigma_p := (\mu_p)_{*,e} : \mathfrak{g} \to T_p P$. We observe that if $X \in \mathfrak{g}$ and $f \in \mathcal{H}(z)$ is a holomorphic function at $z \in M$, then if we denote the points in a neighbourhood of e in G by x,

$$(\pi_* \circ \sigma_p X f)(x) = X f\,(\pi \circ \mu_p(x)) = X f\,(\pi(p \cdot x)) = X f(\pi(p)) = 0.$$

Thus $\sigma_p X \in V_p$ for every X, since the function $f \circ \pi \circ \mu_p$ is constant.

The fact that σ is a Lie algebra homomorphism, follows from Proposition 4.1 Ch.1 in [KN63]. The rest of iii. follows from iv.

iv. If $v_p \in V_p$ and $f \in \mathcal{H}(p)$ is a holomorphic function at $p \in P$, we let $c : B(0,\epsilon) \subset \mathbb{C} \to \pi^{-1}(U_\alpha)$ be a holomorphic function into $\pi^{-1}(U_\alpha)$ with $c'(0) = v_p$. By the trivialization we can write $c(t) = \varphi_\alpha^{-1}\big(z(t), a(t)\big)$, for two holomorphic functions $z : B(0,\epsilon) \to U_\alpha$ and $a : B(0,\epsilon) \to G$. We notice that $\varphi_\alpha^{-1}\big(z, a \cdot b\big) = \varphi_\alpha^{-1}\big(z, a\big) \,.\, b$. We thus compute:

$$(\sigma \circ (g_\alpha^* \theta)\, v_p)\, f = \big((\mu_p)_{*,e} \circ (L_{g_\alpha(p)^{-1}})_{*,g_\alpha(p)} \circ (g_\alpha)_{*,p} v_p\big)\, f = c'(0)\big(f(p \,.\, g_\alpha(p)^{-1} \,.\, g_\alpha)\big)$$

inserting the derivative of the holomorphic function:

$$= \frac{d}{dt}\bigg|_{t=0} f\left(c(0) \,.\, g_\alpha\big(c(0)\big)^{-1} \,.\, g_\alpha\big(c(t)\big)\right) = \frac{d}{dt}\bigg|_{t=0} f\left(\varphi_\alpha^{-1}\Big(z(0), a(0)\Big) \,.\, a(0)^{-1} \,.\, a(t)\right)$$

using $\varphi_\alpha^{-1}\big(z, a \cdot b\big) = \varphi_\alpha^{-1}\big(z, a\big) \,.\, b :$

$$= \frac{d}{dt}\bigg|_{t=0} f \circ \varphi_\alpha^{-1}\big(z(0), a(t)\big) + 0 \overset{\dagger}{=} \frac{d}{dt}\bigg|_{t=0} f \circ \varphi_\alpha^{-1}\big(z(0), a(t)\big) + \frac{d}{dt}\bigg|_{t=0} f \circ \varphi_\alpha^{-1}\big(z(t), a(0)\big)$$

$$\overset{\dagger\dagger}{=} \frac{d}{dt}\bigg|_{t=0} f \circ \varphi_\alpha^{-1}\big(z(t), a(t)\big) = \frac{d}{dt}\bigg|_{t=0} f \circ c(t) = v_p f.$$

$\dagger$: Here we use that $v_p = c'(0)$ is vertical, if $\tilde{f} = f \circ \varphi_\alpha^{-1}\big(\cdot, a(0)\big) : U_\alpha \to \mathbb{C}$:

$$0 = \pi_* v_p \tilde{f} = \frac{d}{dt}\bigg|_{t=0} \tilde{f} \circ \pi \circ c(t) = \frac{d}{dt}\bigg|_{t=0} \tilde{f} \circ z(t) = \frac{d}{dt}\bigg|_{t=0} f \circ \varphi_\alpha^{-1}\big(z(t), a(0)\big).$$

$\dagger\dagger$: This is just complex partial derivatives of a function $f \circ \varphi_\alpha^{-1} \circ (z \times a) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

Conversely, if $X \in \mathfrak{g}$, $f \in \mathcal{H}(e)$ is a holomorphic function at $e \in G$ and $x \in N(e)$, a neighbourhood of $e \in G$, then we compute

$$\big((g_\alpha \theta) \circ \sigma v_p\big) f(x) = \big((L_{g_a(p)^{-1}})_* \circ (g_\alpha)_* \circ (\mu_p)_{*,e} v_p\big) f(x) = v_p\left(f\Big(g_\alpha(p)^{-1} \cdot g_\alpha\big(\mu_p(x)\big)\Big)\right)$$

$$= v_p\left(f\Big(g_\alpha(p)^{-1} \cdot g_\alpha\big(p \,.\, x\big)\Big)\right) = v_p\left(f\Big(g_\alpha(p)^{-1} \cdot g_\alpha\big(p\big) \cdot x\Big)\right) = v_p\big(f(x)\big).$$

v. If $X \in \mathfrak{g}$, $f \in \mathcal{H}(p)$ is a holomorphic function at $p \in P$ and $x \in N(e) \subset G$, a point in a neighbourhood $N(e)$ of $e \in G$. Then:

$$\Big(\big((\mu_a)_{*,p} \circ \sigma_p X\big) f\Big)(x) = \Big(\big((\mu_a)_{*,p} \circ (\mu_p)_{*,e} X\big) f\Big)(x) = X\Big(f\big(\mu_a \circ \mu_p(x)\big)\Big)$$

$$= X\Big(f\big(p \,.\, x \,.\, a^{-1}\big)\Big) = X\Big(f\big(p \,.\, a \,.\, a^{-1} \cdot x \cdot a^{-1}\big)\Big) = X\Big(f\big(\mu_{p\,.\,a} \circ L_{a^{-1}} \circ R_a \circ (x)\big)\Big)$$

$$= \Big(\big(\sigma_{p\,.\,a} \circ \mathrm{Ad}\,\big(a^{-1}\big)\, X\big) f\Big)(x).$$

■

Proposition A.3.2 Characterization of a principal connection by a connection form.
Consider a principal bundle $P(M, G, \pi)$. Given a principal connection $H \subset TP$ we define a connection form ω by:

$$\omega : \; P \to T^*P \otimes \mathfrak{g}, \qquad \begin{array}{rcl} \omega_p : \; T_pP & \to & \mathfrak{g} \\ v_p & \mapsto & \omega_p(v_p) = \begin{cases} X, & \text{if } v_p = \sigma X \\ 0, & \text{if } v_p \in H_p \end{cases} \end{array}$$

And given a connection form ω on P we define a principal connection $H \subset TP$ by

$$H_p := \ker(\omega_p).$$

Proof. Define ω as in the statement of the Proposition. Given a trivialization $\varphi_\alpha = \pi \times g_\alpha$ of P, the inverse of σ can also be explicitly given by $g_\alpha^* \theta$ by Proposition 2.1.5. Hence in $\pi^{-1}(U_\alpha)$ the 1-form ω is given by

$$\begin{array}{rcl} \omega_p : \; T_pP & \to & \mathfrak{g} \\ v_p & \mapsto & \omega_p(v_p) = \begin{cases} (g_\alpha^* \theta)_p(v_p), & \text{if } v_p \in V_p \\ 0, & \text{if } v_p \in H_p \end{cases} \end{array}$$

We need to show that ω has the properties i. and ii. from Definition 2.1.8. Property i. is obvious from how we defined ω. To prove property ii., we consider the situation when v_p is horizontal and vertical separately. If v_p is horizontal, then $(\mu_a)_{*,p}v_p \in H_{p\,.\,a}$ is still horizontal by property ii. from Definition 2.1.6 of a principal connection, so $\omega\big((\mu_a)_* v_p\big) = 0$. Since $Ad(a^{-1})$ is a linear map, $\mathrm{Ad}(a^{-1}) \circ \omega_p(v_p) = 0$.

If v_p is vertical, we compute $\big(\mu_a^*\omega\big)(v_p) = \omega\big((\mu_a)_* v_p\big)$, and note that by the bundle properties, $(\mu_a)_* v_p \in T_{p\,.\,a}P$ is also vertical, thus

$$g_\alpha^*\theta_{p\,.\,a}\big((\mu_a)_{*,p}v_p\big) = \big(L_{g_\alpha(p\,.\,a)^{-1}}\big)_{*,(g_\alpha(p\,.\,a))} \circ (g_\alpha)_{*,p\,.\,a} \circ (\mu_a)_{*,p}v_p \tag{A.2}$$

We use the chain rule on the Maurer-Cartan form expression. The $G-$invariance of g_α, see Definition 2.1.1, ensures that

$$(g_\alpha)_{*,p\,.\,a} \circ (\mu_a)_{*,p} = (R_a)_{*,g_\alpha(p)} \circ (g_\alpha)_{*,p}.$$

Thus

$$\begin{aligned}(\text{A.2}) = \quad & (L_{a^{-1}})_{*,a} \circ \big(L_{g_\alpha(p)^{-1}}\big)_{*,g_a(p)\cdot a} \circ (R_a)_{*,g_\alpha(p)} \circ (g_\alpha)_{*,p}v_p \\ = \mathrm{Ad}(a^{-1}) \circ & \big(L_{g_\alpha(p)^{-1}}\big)_{*,g_\alpha(p)} \circ (g_\alpha)_{*,p}v_p = \mathrm{Ad}(a^{-1}) \circ (g_\alpha^*\theta)_p(v_p).\end{aligned}$$

Conversely, given a connection form ω on P we define a principal connection by for each $p \in P$

$$H_p := \ker(\omega_p).$$

Obviously this depends holomorphicly on p, since ω_p is holomorphic. The direct sum is verified by the fact that if $v_p \in \ker(\pi_{*,p}) \subset T_pP$, then there exists a unique $X \in \mathfrak{g}$ such that $\sigma_p X = v_p$. But then $\omega_p(v_p) = \omega_p(\sigma_p X) = X = 0$ if and only if $v_p = \sigma_p X = 0$, hence $\ker(\omega_p) \cap \ker(\pi_{*,p}) = \{\,0\,\}$. Since $\dim(\ker(\pi_{*,p})) = \dim(G)$, and $\dim(\ker(\omega_p)) = \dim(M)$, we can conclude that $T_pP = H_p \oplus V_p$. Finally by property ii. of a connection form,

$$\omega_{p\,.\,a}\big((\mu_a)_{*,p}v_p\big) = \mathrm{Ad}\left(a^{-1}\right) \circ \omega_p(v_p),$$

it is clear that $(\mu_a)_{*,p}H_p = H_{p\,.\,a}$. ■

Proposition A.3.3 Characterization of a connection form by a family of local connection forms. *Consider a principal bundle $P(M,G,\pi)$ with local trivializations $\{\,\varphi_\alpha = \pi \times g_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G\,\}_\alpha$ and transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. Recall the trivial sections $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ from Definition 2.1.4. Given a connection form ω on P, we define a local family of connection forms on M by*

$$A_\alpha := s_\alpha^*\omega.$$

And given a local family of connection forms $\{\,A_\alpha\,\}_\alpha$ on M, we define a connection form on P by in each trivialised set $\pi^{-1}(U_\alpha)$ defining

$$\omega_\alpha := \mathrm{Ad}\left((g_\alpha)^{-1}\right) \circ \pi^*A_\alpha + g_\alpha^*\theta.$$

Then any pair ω_α and ω_β agree on $\pi^{-1}\left(U_\alpha \cap U_\beta\right)$, and $\{\,\omega_\alpha\,\}_\alpha$ defines a connection form ω on P.

Proof. We first show that given a connection form ω on P, we can define a family of local connection forms on M. First, A_a is a well defined Lie algebra valued 1-form since if $s_\alpha(z) = p$, for $z \in U_\alpha$,

$$A_{\alpha,z} = (s_\alpha^*\omega)_z : T_zU_\alpha \xrightarrow{(s_\alpha)_{*,z}} T_p\pi^{-1}(U_\alpha) \xrightarrow{\ \omega_p\ } \mathfrak{g}$$

We choose the cover $\{\,U_\alpha\,\}_\alpha$ to be a trivialising cover. Let $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, and recall property i. from Proposition 2.1.3, $s_\beta = s_\alpha\,.\,g_{\alpha\beta}$. We want to show the change trivialization formula in Definition 2.1.9. We compute:

$$A_\beta = s_\beta^*\omega = (s_\alpha\,.\,g_{\alpha\beta})^*\omega.$$

For notation we let $z \in U_{\alpha\beta}$, $s_\alpha(z) = p \in P$, $g_{\alpha\beta}(z) = a \in G$ and $s_\beta(z) = p\,.\,a = s_\alpha(z)\,.\,g_{\alpha\beta(z)}$. To continue we need to calculate

$$\big(\mu(s_\alpha, g_{\alpha\beta})\big)_{*,z} : T_zU_{\alpha\beta} \xrightarrow{(s_\alpha)_{*,z} \times (g_{\alpha\beta})_{*,z}} T_p\pi^{-1}(U_{\alpha\beta}) \times T_aG \xrightarrow{\quad (\mu)_{*,(p,a)} \quad} T_{p\,.\,a}\pi^{-1}(U_{\alpha\beta})\ .$$

We need to use the "Leibniz rule" on the right action μ of G on P, see Proposition 1.4, Chapter I in [KN63]. If $u_z \in T_zU_\alpha$ is a tangent vector to U_α at $z \in U_\alpha$, $s_\alpha(z) = p \in P$ and $g_{\alpha\beta}(z) = a \in G$ we get:

$$\star = \big(\mu(s_\alpha, g_{\alpha\beta})\big)_{*,z}u_z = \big(\mu_a\big)_{*,p} \circ (s_\alpha)_{*,z}u_z + \big(\mu_p\big)_{*,a} \circ (g_{\alpha\beta})_{*,z}u_z$$

$$\overset{\dagger}{=} \left(\mu_a\right)_{*,p} \circ (s_\alpha)_{*,z} u_z + \sigma_{p\,.\,a} \circ \theta_\alpha \circ (g_{\alpha\beta})_{*,z} u_z.$$

$\dagger$: We used the fact that $\left(\mu_p\right)_{*,a} = \sigma_{p\,.\,a} \circ \theta_\alpha$

$$\begin{tikzcd}
T_aG \arrow[r, "\theta_a"] \arrow[dr, "\left(\mu_p\right)_{*,a}"'] & \mathfrak{g} \arrow[d, "\sigma_{p\,.\,a}"] \\
 & T_{p\,.\,a}P
\end{tikzcd}$$

Indeed, this is true since if $X_a \in T_aG$, and $f \in \mathcal{H}(p\,.\,a)$ is a holomorphic function at $p\,.\,a \in P$ with $x \in N(a)$ a neighbourhood of a in G:

$$\Big(\Big(\left(\mu_p\right)_{*,a} X_a\Big) f\Big)(x) = X_a\Big(f\big(\mu_p(x)\big)\Big) = X_a\big(f(p\,.\,x)\big) = X_a\big(f(p\,.\,a \cdot a^{-1} \cdot x)\big)$$

$$= X_a\Big(f\big(\mu_{p\,.\,a} \circ L_{a^{-1}}(x)\big)\Big) = \Big(\big(\sigma_{p\,.\,a} \circ (L_{a^{-1}})_{*,a} X_a\big) f\Big)(x) = \Big(\big(\sigma_{p\,.\,a} \circ \theta_a(X_a)\big) f\Big)(x).$$

We have just justified $\star$, and can now calculate

$$A_{\beta,z} = \Big(\big(\mu(s_\alpha, g_{\alpha\beta})\big)^* \omega\Big)_z \overset{\star}{=} \omega_{p\,.\,a}\Big(\left(\mu_a\right)_{*,p} \circ (s_\alpha)_{*,z} + \sigma_{p\,.\,a} \circ \theta_\alpha \circ (g_{\alpha\beta})_{*,z}\Big)$$

$$= \left(\mu_a^* \circ \omega_{p\,.\,a}\right)_p \circ (s_\alpha)_{*,z} + \theta_a \circ (g_{\alpha\beta})_{*,z}$$

Here we used the defining property of ω, that $\omega_{p\,.\,a} \circ \sigma_{p\,.\,a} X = X$, for $X \in \mathfrak{g}$. Now using the other defining property of ω, that $\left(\mu_a^* \omega_{p\,.\,a}\right)_p = \operatorname{Ad}\left(a^{-1}\right) \circ \omega_p$, we finally obtain

$$A_{\beta,z} = \operatorname{Ad}\left((g_{\alpha\beta}(z))^{-1}\right) \circ (s_\alpha^* \omega)_z + (g_{\alpha\beta}^* \theta)_z = \operatorname{Ad}\left(g_{\beta\alpha}(z)\right) \circ A_{\alpha,z} + (g_{\alpha\beta}^* \theta)_z.$$

Thus the family of Lie algebra valued 1-forms transforms in the correct way when changing charts, and are thus a family of local connection forms on M.

Conversely, given a family of local connection forms on M, we want to deduce the expression for ω_α. We want to construct $\{\, \omega_\alpha \,\}_\alpha$ such that ω_α and ω_β coincide on $\pi^{-1}(U_{\alpha\beta})$. Also we expect that $A_\alpha = s_\alpha^* \omega_\alpha$. Combining these reasonable expectations, we can on $\pi^{-1}(U_{\alpha\beta})$ write:

$$\operatorname{Ad}\left(g_{\beta\alpha}(z)\right) \circ A_{\alpha,z} + \left(g_{\alpha\beta}^* \theta\right)_z = A_{\beta,z} = (s_\beta^* \omega_\beta)_z = (s_\beta^* \omega_\alpha)_z$$

Note that we use a subscripted z when a form on $U_{\alpha\beta}$ is evaluated at $z \in U_{\alpha\beta}$. Keeping the leftmost and rightmost expressions, and using property i. in Proposition 2.1.2 and property iii. in Proposition 2.1.3, we obtain

$$(s_\beta^* \omega_\alpha)_z = \operatorname{Ad}\left((g_{\alpha\beta}(z))^{-1}\right) \circ A_{\alpha,z} + (g_{\alpha\beta}^* \theta)_z = \operatorname{Ad}\left(\Big(g_\alpha\big(s_\beta(z)\big)\Big)^{-1}\right) \circ A_{\alpha,z} + \big((g_\alpha \circ s_\beta)^* \theta\big)_z$$

Now using that $Id = (\pi \circ s_\beta)^* = s_\beta^* \pi^* : T_z^* U_{\alpha\beta} \to T_z^* U_{\alpha\beta}$,

$$= \operatorname{Ad}\left(\Big(g_\alpha\big(s_\beta(z)\big)\Big)^{-1}\right) \circ \left(s_\beta^* \pi^* A_\alpha\right)_z + \left(s_\beta^* g_\alpha^* \theta\right)_z = \Big(s_\beta^*\big(\operatorname{Ad}\left(g_\alpha^{-1}\right) \circ \pi^* A_\alpha + g_\alpha^* \theta\big)\Big)_z.$$

Motivated by this calculation, we define

$$\omega_\alpha := \operatorname{Ad}\left(g_\alpha^{-1}\right) \circ \pi^* A_\alpha + g_\alpha^* \theta, \qquad \text{on } U_\alpha.$$

Reversing the above calculation, we can conclude that $\omega_\alpha = \omega_\beta$ on $s_\beta(U_{\alpha\beta})$. Using the trivialization map φ_β, $s_\beta(U_{\alpha\beta})$ is mapped to $U_{\alpha\beta} \times \{\, e \,\} \subset U_{\alpha\beta} \times G$. It follows that ω_α and ω_β agree on $\pi^{-1}(U_{\alpha\beta})$ once we show that both ω_α and ω_β translelates by $\omega_{p\,.\,a} = \operatorname{Ad}\left(a^{-1}\right) \circ \left(\mu_{a^{-1}}^* \omega_p\right)$, since the action of G on $\pi^{-1}(\pi(p))$ is regular, in particular transitive. And indeed we have that for $a^{-1} \in G$, and $\pi(p) = z$

$$\mu_{a^{-1}}^* \omega_{\alpha,p} = \mu_{a^{-1}}^* \Big(\operatorname{Ad}\left(g_\alpha^{-1}\right) \circ \pi^* A_\alpha + g_\alpha^* \theta\Big)_p$$

$$= \mathrm{Ad}\left(\left(g_\alpha(p) \cdot a \cdot a^{-1}\right)^{-1}\right) \circ A_\alpha\left(\pi_{*,p} \circ (\mu_{a^{-1}})_{*,p\,.\,a}\right) + \left(\mu_{a^{-1}}^* g_\alpha^* \theta\right)_{p\,.\,a}$$

Using the bundle properties $g_\alpha \circ \mu_{a^{-1}}(p) = R_{a^{-1}} \circ g_\alpha(p)$, and $\pi \circ \mu_{a^{-1}} = \pi$, we obtain

$$= \mathrm{Ad}\,(a) \circ \mathrm{Ad}\left(\left(g_\alpha(p\,.\,a)\right)^{-1}\right) \circ \left(\pi^* A_\alpha\right)_{p\,.\,a} + \left(g_\alpha^* R_{a^{-1}}^* \theta\right)_{p\,.\,a}.$$

By using Proposition A.2.1, we obtain

$$\mu_{a^{-1}}^* \omega_{\alpha,p} = \mathrm{Ad}\,(a) \circ \Big(\, \mathrm{Ad}\left(\left(g_\alpha(p\,.\,a)\right)^{-1}\right) \circ \left(\pi^* A_\alpha\right)_{p\,.\,a} + (g_\alpha^* \theta)_{p\,.\,a}\Big) = \mathrm{Ad}\,(a) \circ \omega_{\alpha,p\,.\,a},$$

and thus

$$\mathrm{Ad}\left(a^{-1}\right) \circ \mu_{a^{-1}}^* \omega_{\alpha,p} = \mathrm{Ad}\left(a^{-1}\right) \circ \mathrm{Ad}\,(a) \circ \omega_{\alpha,p\,.\,a} = \omega_{\alpha,p\,.\,a}.$$

We have thus showed ω that is a well defined Lie algebra valued 1-form on P, given in $\pi^{-1}(U_\alpha)$ by ω_α, and agreeing on overlaps. We have also showed that it satisfies property ii. in Definition 2.1.8. We have left to show that it satisfies property i. in its Definition. That is, $\omega\big(\sigma X\big) = X$ for every $X \in \mathfrak{g}$. This is easy since by property iii. in Proposition 2.1.5, $\sigma_p X$ is vertical for each $X \in \mathfrak{g}$. Hence $\sigma_p X \in \ker(\pi_{*,p})$. By using property iv. in Proposition 2.1.5:

$$\omega_{\alpha,p}\Big(\sigma_p X\Big) = \mathrm{Ad}\left(\left(g_\alpha(p)\right)^{-1}\right) \circ A_{\alpha,\pi(p)}\left(\pi_{*,p} \circ \sigma_p X\right) + \left(g_\alpha^* \theta\right)_p \circ \sigma_p X = \left(g_\alpha^* \theta\right)_p \circ \sigma_p X = X.$$

■

Appendix B

Constructions on Riemann surfaces

B.1 Analytic continuation on a Riemann surface

We recall the Definition of a complex manifold.

Definition B.1.1 Complex holomorphic manifold and Riemann surface.
A complex manifold M, of dimension n, is a smooth manifold of dimension $2n$ admitting a holomorphic atlas. That is an atlas $\mathcal{U} = \{ (U_\alpha, \varphi_\alpha) \}$ containing charts $(U_\alpha, \varphi_\alpha)$ where $U_\alpha \subset M$ and

$$\varphi_\alpha : U_\alpha \to \varphi(U_\alpha) \subset \mathbb{C}^n$$

and for every pair (U_α, φ_a), $(U_\beta, \varphi_b) \in \mathcal{U}$

$$\varphi_\alpha \circ \varphi_b^{-1} : \varphi_b(U_a \cap U_\beta) \subset \mathbb{C}^n \to U_\alpha \cap U_\beta \to \varphi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$$

is a holomorphic function between C^n and C^n. A Riemann surface is a connected a 1-dimensional complex manifold.

Definition B.1.2 Complex analytic function on a Riemann surface.
Let M be a Riemann surface, let (U_α, ϕ_α) denote the charts on M and let U be an open subset of M. A function $f : U \to GL_2(\mathbb{C})$ is called analytic if for every point $z \in U$ there exists a chart (U_α, ϕ_α) with $z \in U_\alpha \subset U$ such that $f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to GL_2(\mathbb{C})$ is an analytic function from $\mathbb{C}$ to $GL_2(\mathbb{C})$.

Analytic continuation is a technique from complex analysis relying on the following essential fact:

Lemma B.1.1 The identity Theorem for analytic functions on Riemann surfaces.
Let $f : U \to GL_2(\mathbb{C})$ and $g : V \to GL_2(\mathbb{C})$ be analytic functions on U, V open subsets of M, a Riemann surface. Assume also that $U \cap V$ is connected (if not one gets the result on each connected component of $U \cap V$). Then either $f = g$ on $U \cap V$ or $f = g$ only on a discrete subset of $U \cap V$.

Proof. We consider $u = f - g$, which is analytic on $U \cap V$. First we show that if $z_0 \in \mathcal{Z}(u)$ then z_0 is either an interior point or an isolated point. Let $u(z_0) = u \circ \phi_\alpha^{-1}(\omega_0) = 0$. By analyticity we have

$$u \circ \phi_\alpha^{-1}(\omega) = \sum_{n=1}^{\infty} A_n (\omega - \omega_0)^n, \ \ \omega \in B(\omega_0, \delta') \subset \mathbb{C}$$

for some $\delta' > 0$. Assume that there exists an $m > 0$ such that $\|A_m\| \neq 0$, and that A_m is the first such matrix in the series for $u \circ \phi_\alpha^{-1}$ at ω_0. Then

$$u \circ \phi_\alpha^{-1}(\omega) = \sum_{n=m}^{\infty} A_n (\omega - \omega_0)^n = (\omega - \omega_0)^m \left(A_m + \sum_{n=1}^{\infty} A_{n+m} (\omega - \omega_0)^n \right)$$

By continuity of the series on the right we have that there exists a $\delta'' > 0$ such that

$$\left\| \sum_{n=1}^{\infty} A_{n+m} (\omega - \omega_0)^n \right\| < \|A_m\|,$$

for $|\omega - \omega_0| < \delta''$. Hence we have that for $\delta = \min\{\delta', \delta''\}$ and $\omega \in B(\omega_0, \delta) \setminus \{\omega_0\}$: $(\omega - \omega_0)^n \neq 0$ and $A_m + \sum_{n=1}^{\infty} A_{n+m}(\omega - \omega_0)^n \neq 0$, hence $u(z) \neq 0$ in $\phi_\alpha^{-1}(B(\omega_0, \delta)) \setminus \{z_0\}$. This implies that z_0 is an isolated point in $\mathcal{Z}(u)$. However we assumed that there exists an $A_m \neq 0$. If there does not exists such an A_m then $u(z) = 0$ in $\phi_\alpha^{-1}(B(\omega_0, \delta'))$, which implies that z_0 is an interior point in $\mathcal{Z}(u)$.

We show that $\mathrm{int}(\mathcal{Z}(u))$ is closed. Indeed if $z_1 \in (\mathrm{int}(\mathcal{Z}(u)))^c$, then by what we just showed either $u(z_1) \neq 0$ or $u(z_1) = 0$ and z_1 is isolated. If $u(z_1) \neq 0$, then by continuity of $u \circ \phi_\alpha^{-1}$ there exists a $\delta > 0$ such that for $z \in \phi_\alpha^{-1}(B(\omega_1, \delta))$, $u(z) \neq 0$. Hence $\phi_\alpha^{-1}(B(\omega_1, \delta)) \subset \mathrm{int}(\mathcal{Z}(u))^c$. And if z_1 is isolated that implies that there exists a $\delta > 0$ such that $u(z) \neq 0$ for $z \in \phi_\alpha^{-1}(B(\omega_1, \delta))$. Thus we have $\phi_\alpha^{-1}(B(\omega_1, \delta)) \subset \mathrm{int}(\mathcal{Z}(u))^c$. In both cases we find that the complement of $\mathrm{int}(\mathcal{Z}(u))$ is open, hence $\mathrm{int}(\mathcal{Z}(u))$ is closed. Since $U \cap V$ is connected, either $\mathrm{int}(\mathcal{Z}(u)) = \emptyset$ or $\mathrm{int}(\mathcal{Z}(u)) = U \cap V$.

If $\mathrm{int}(\mathcal{Z}(u)) = U \cap V$, then $f = g$ on $U \cap V$. And if $\mathrm{int}(\mathcal{Z}(u)) = \emptyset$, then we know it has to be discrete, so $f = g$ only on a discrete subset of $U \cap V$. ■

The above Lemma shows that if the domain of an analytic function $f : U \to GL_2(\mathbb{C})$, is extended by another analytic function $g : V \to GL_2(\mathbb{C})$, such that they agree on a non-discrete set, then this is the only way to extend f analytically into V.

Definition B.1.3 Direct analytic continuation.
Let $U, V \subset M$, where M is a Riemann surface. If $f : U \to GL_2(\mathbb{C})$ and $g : V \to GL_2(\mathbb{C})$ are analytic functions, $U \cap V$ is non-discrete and $f = g$ on $U \cap V$ then we say that the pair (g, V) is a direct analytic continuation of (f, U).

The idea of analytic continuation is to take an analytic function $f : U \to GL_2(\mathbb{C})$, and try to extend its domain by using Lemma B.1.1.

Definition B.1.4 A path.
Let M be a topological space. A path in M is a continuous function $\gamma : [a, b] \to M$, from a closed interval of $\mathbb{R}$ into M. We say the path starts at $x \in M$ if $\gamma(a) = x$, and ends at $y \in M$ if $\gamma(b) = y$.

Definition B.1.5 Analytic continuation along a path.
Let $U \subset M$, where M is a Riemann surface. Let $\gamma : [a, b] \to M$ be a path and $f : U \to GL_2(\mathbb{C})$ an analytic function, such that $\gamma(a) \in U$. We define the analytic continuation of (f, U) along γ by a finite sequence $(f_k, D_k)_{k=0}^n$ with the following properties:

- *There is a collection of charts that can be numbered (possibly with repetition) (U_k, ϕ_k) such that*

$$D_k := \phi_k^{-1}(B(\phi_k(z_k), r_k)) = \{ z \in U_k \mid |\phi_k(z) - \phi_k(z_k)| < r_k \}$$

  where r_k is chosen such that $\overline{D_k} \subset U_k$, and $D_0 \cap U \neq \emptyset$.

- *There exists a partition $\{a = a_0, a_1, \ldots, a_n, a_{n+1} = b\}$, of $[a, b]$ such that $\gamma([a_k, a_{k+1}]) \subset D_k$.*

- *In each D_k there exists an analytic function $f_k : D_k \to GL_2(\mathbb{C})$ and in D_0: $f\big|_{D_0} = f_0$. $f_k = f_{k+1}$ in $D_k \cap D_{k+1}$.*

We denote $f_\gamma(z) = f_n(z)$ for $z \in D_n$. This is an analytic function in D_n.

Remark.
We can also restrict γ to $\gamma\big|_{[0,a_k]}$ and obtain analytic continuations into any of the other discs. Of course, given a function $f : U \to GL_2(\mathbb{C})$, such a setup may not exist. And if it exist, there is no reason for $g : \cup_{k=0}^n D_k \to GL_2(\mathbb{C})$, defined by $f_k = g$ in D_k to be a function. If for example f is the logarithm defined in $U = \mathbb{C} \setminus \{\mathrm{Re}(z) \leq 0\}$, and γ is the unit circle traversed counter-clockwise. Then $f_\gamma(z) = \log(z) + 2\pi i$. We will come back to this example.

Lemma B.1.2 Independence of partition, discs and intermediate functions.
Let M be a Riemann surface, $\gamma : [a, b] \to M$ a path, $(f_k, D_k)_{k=0}^n$ an analytic continuation of (f_0, D_0) along γ and $(g_j, E_j)_{j=0}^m$ an analytic continuation of (g_0, E_0) along γ. If $f_0 = g_0$ in a non-discrete subset of $D_0 \cap E_0$, then $f_n = g_m$ in $D_n \cap E_m \ni \gamma(b) = z_1$.

Proof. Consider the refined partition $\{ c_i \}_{i=0}^{p+1}$ of $[a,b]$ consisting of the union of the two partitions $P_f = \{ a = a_0, \ldots, a_{n+1} = b \}$, $P_g = \{ a = b_0, \ldots, b_{m+1} = b \}$, removing the duplicate endpoints, and then ordered. There may still be other duplicates. We will prove by induction that:

$$\text{For every } i,\ \gamma([c_i, c_{i+1}]) \subset D_k \cap E_j \text{ and } f_k = g_j \text{ in } D_k \cap E_j, \text{ for some } k, j.$$

- $i = 0$. Then $\gamma([c_0, c_1]) \subset D_0$ and $\gamma([c_0, c_1]) \subset E_0$ no matter if $c_1 = a_1$ or b_1. Also by hypothesis $f_0 = g_0$ in a non-discrete subset of $D_0 \cap E_0$, hence by Lemma B.1.1 on the whole $D_0 \cap E_0$.

- Assume the induction hypothesis for $i = l$. Then we know that: $\gamma([c_l, c_{l+1}]) \subset D_k \cap E_j$ and $f_k = g_j$ in $D_k \cap E_j$, for some k, j. Explicitly we have

  $$\gamma([c_l, c_{l+1}]) \subset D_k, \quad a_k \le c_l \le c_{l+1} \le a_{k+1}$$

  $$\gamma([c_l, c_{l+1}]) \subset E_j, \quad b_j \le c_l \le c_{l+1} \le b_{j+1}$$

  So either $c_{l+1} = a_{k+1}$, $c_{l+1} = b_{j+1}$ if not we have the trivial cases $c_{l+1} = a_k$ or $c_{l+1} = b_k$ which immediately imply $c_l = c_{l+1}$ and the statement is true.

  Assume without loss of generality that $c_{l+1} = a_{k+1}$. Then

  $$\gamma([c_{l+1}, c_{l+2}]) \subset D_{k+1}, \quad a_{k+1} = c_{l+1} \le c_{l+2} \le a_{k+2}$$

  $$\gamma([c_l, c_{l+2}]) \subset E_j, \quad b_j \le c_l \le c_{l+2} \le b_{j+1}$$

  and we obtain that $\gamma([c_{l+1}, c_{l+2}]) \subset D_{k+1} \cap E_j$. Also we have $\gamma(c_{l+1}) \in D_k \cap D_{k+1} \cap E_j \neq \emptyset$ and

  $$f_{k+1} = f_k = g_j, \quad \text{in } D_k \cap D_{k+1} \cap E_j$$

  and thus by Lemma B.1.1 we have $f_{k+1} = g_j$ in $D_{k+1} \cap E_j$.

This process will continue until $c_{p+1} = b$. Then $\gamma([c_p, c_{p+1}]) \subset D_k \cap E_j$ and $f_k = g_j$ in $D_k \cap E_j$, for some j, k. Then $c_p = a_n$ or $c_p = b_m$. Without loss of generality, assume the latter. Then j have to be m by the algorithm from above.

$$\gamma([c_p, c_{p+1}]) = \gamma([b_m, b_{m+1}]) \subset E_m = E_j$$

Also we have

$$a_k \le c_p < c_{p+1} = a_{n+1} \le a_{k+1}$$

Thus $a_{k+1} = a_{n+1}$ and $k = n$. We conclude that $k = n$ and $j = m$ so by the induction $f_n = g_m$ in $D_n \cap E_m \ni \gamma(b)$. ■

If we analytically continue (f, U) along two paths $\gamma : [a,b] \to M$ and $\eta : [c,d] \to M$ such that $\gamma(a) = \eta(c) = z_0$ and $\gamma(b) = \eta(d) = z_1$, we obtain to analytic functions f_γ and f_η defined in neighbourhoods of $\gamma(b) = \eta(d) = z_1$. Can we relate f_γ and f_η? The famous monodromy Theorem gives sufficient conditions for when $f_\gamma = f_\eta$.

Theorem B.1.1 The monodromy Theorem.
Let U be an open and connected subset of M, a Riemann surface. Let $f : V \to GL_2(\mathbb{C})$ be analytic with $z_0 \in V \subset U$. Assume f admits an analytic continuation along any path in U. Let γ and η begin at z_0 and end at $z_1 \in U$. If γ and η are homotopic, then $f_\gamma = f_\eta$ in a neighbourhood of z_1.

Proof. Let $H : [0,1] \times [0,1] \to U$ with $H(\tau, 0) = \gamma(\tau)$, $H(\tau, 1) = \eta(\tau)$ and $H(0, s) = z_0$, $H(1, s) = z_1$ be the fixed point homotopy. We will denote $H(\tau, s) = \gamma_s(\tau)$ for simplicity. Let

$$\mathcal{S} = \{ u \in [0,1] \mid f_\gamma = f_{\gamma_u} \text{ in a neighbourhood of } \gamma(1) = z_1 \}$$

If we show $\mathcal{S} = [0,1]$, then in particular $f_\gamma = f_{\gamma_0} = f_{\gamma_1} = f_\eta$ in a neighbourhood of z_1 and the proof is done. We will use a connectivity argument on $[0,1]$. First we note that $u = 0 \in \mathcal{S}$, so $\emptyset \neq \mathcal{S} \subset [0,1]$. $\mathcal{S}$ is open. Indeed let $u \in \mathcal{S}$. Then $\gamma_u(\tau) = H(\tau, u)$ and $f_{\gamma_u} = f_\gamma$ in a neighbourhood of z_1. We choose an analytic continuation, $(f_k, D_k)_{k=0}^n$, of (f, V) along γ_u, such that $\gamma_u([a_k, a_{k+1}]) \subset D_k$. Recall the chart (U_k, ϕ_k) with $\overline{D_k} \subset U_k$ and

$$D_k = \{ z \in U_k \mid |\, \phi_k(z) - \phi_k(z_k)| < r_k \}.$$

We want to show that for s close to u in $[0,1]$, we can use the same discs and functions as an analytic continuation along γ_s. By Definition B.1.5, the only thing to show is that $\gamma_s([a_k,a_{k+1}]) \subset D_k$ for each k, that is $\phi_k \circ \gamma_s([a_k,a_{k+1}]) \in B(\phi_k(z_k), r_k) \subset \mathbb{C}$. Since H is a homotopy it is continuous and thus each $\phi_k \circ \gamma_u$ is also a continuous function, defined on the compact set $[a_k,a_{k+1}]$. By the max-min Theorem, there exists an $\epsilon_k > 0$ such that

$$p_k(\tau) = |\phi_k \circ \gamma_u(\tau) - \phi_k(z_k)| \leq r_k - \epsilon_k,$$

where the equality holds for some $\tau_k \in [a_k,a_{k+1}]$. Now define $\epsilon := \min_k \{ \epsilon_k \}$. The function

$$\phi_k \circ H : H^{-1}(\overline{D_k}) \to \mathbb{C}$$

is continuous and defined on the compact set $H^{-1}(\overline{D_k})$, compact since it is closed and bounded in $I \times I$, which is a subspace of $\mathbb{R}^2$. Hence it is uniformly continuous and hence there exists a $\delta_k > 0$ such that if $\|(\tau,s) - (\tau',u)\|_2 < \delta_k$ then

$$|\phi_k \circ H(\tau,s) - \phi_k \circ H(\tau',u)| = |\phi_k \circ \gamma_s(\tau) - \phi_k \circ \gamma_u(\tau')| < \epsilon,$$

for any pair of points in $H^{-1}(\overline{D_k})$. Finally if we define $\delta := \min_k \{ \delta_k \}$ then for any $m \in \{ 0,1,\ldots,n \}$:

$$|\phi_m \circ \gamma_s(\tau) - \phi_m(z_m)| \leq |\phi_m \circ \gamma_s(\tau) - \gamma_u(\tau)| + |\gamma_u(\tau) - \phi_m(z_m)| < \epsilon + r_m - \epsilon_m \leq r_m$$

for $\tau \in [a_m,a_{m+1}]$ and $\|(\tau,s)-(\tau,u)\|_2 = |s-u| < \delta$. Thus for $|s-u| < \delta$ we have that $\gamma_s([a_k,a_{k+1}]) \subset D_k$, and we can conclude that

$$f_{\gamma_u} = f_n = f_{\gamma_s}, \quad \text{in the whole of } D_k.$$

Thus $\mathcal{S}$ is open.

$\mathcal{S}$ is also closed. Indeed if $s \in [0,1] \setminus \mathcal{S}$, then there does not exists a neighbourhood U_s of z_1 such that $f_\gamma = f_{\gamma_s}$ in U_s. Exactly as before, there exists a $\delta > 0$ such that we can use the analytic continuation $(f_k,D_k)_{k=0}^n$ of (f,V) along γ_s, to give an analytic continuation of (f,V) along γ_σ, where $|s-\sigma| < \delta$. Then evidently $f_{\gamma_\sigma} = f_{\gamma_s}$ in D_n. And we can conclude that for $\sigma \in [0,1]$ such that $|s-\sigma| < \delta$, there does not exists a neighbourhood U_σ of z_1 such that $f_\gamma = f_{\gamma_\sigma}$. Hence $B(s,\delta) \subset [0,1] \setminus \mathcal{S}$ and $\mathcal{S}$ is closed.

■

Corollary B.1.1 Analytic continuation into a function.
Let M be a Riemann surface and let $U \subset M$ be connected, simply connected and open, $z_0 \in U$ with $f : V \to GL_2(\mathbb{C})$ analytic in a neighbourhood V of z_0. Suppose f can be analytically continued along any path in γ in U, which starts at z_0. For any $z \in U$ define a path $\gamma_z : [0,1] \to U$ such that $\gamma_z(0) = z_0$ and $\gamma_z(1) = z$. Then the function $g(z) := f_{\gamma_z}(z)$ is analytic in U.

Proof. First since U is open and connected it is path connected, so we can always define γ_z. And since U is simply connected, any paths from z_0 to z will be homotopic. Hence g is well-defined since by the monodromy Theorem if γ_z and η_z are to paths from z_0 to z, then $f_{\gamma_z} = f_{\eta_z}$. We show g is analytic. If $z_1 \in U$ there is an analytic continuation $(f_k,D_k)_{k=0}^n$, of f along γ_{z_1}. We remark that the function $f_n : D_n \to GL_2(\mathbb{C})$ is analytic in D_n and that for each $z \in D_n$, g makes a new analytic continuation along γ_z in order to evaluate $g(z) = f_{\gamma_z}$. Lets show that $g = f_n$ in D_n. We can choose the path γ_z to be:

$$\gamma_z = \gamma_{z_1} * \gamma_{z_1,z}.$$

Here $\gamma_{z_1,z}$ is chosen to be inside D_n. We now observe that $g(z) = f_{\gamma_z}(z)$ is obtained by analytically continuing f along the path $\gamma_z = \gamma_{z_1} * \gamma_{z_1,z}$. When computing this analytic continuation we can use the analytic continuation $(f_k,D_k)_{k=0}^n$ that was used to analytically continue f along γ_{z_1}. We can do this since $\gamma_{z_1,z} \subset D_n$. Hence $g(z) = f_{\gamma_z}(z) = f_n(z)$. ■

We have now outlined a technique to extend the domain of an analytic function, given another function that coincide in some common domain with at least one limit point. When solving a differential equation a common situation is that one can solve the equation locally. But the main goal is to give a global solution. We can then apply analytic continuation, where the local solutions (f_k,U_k) play the role of the analytic continuation, see Definition B.1.5. If we additionally can find relations between the local solutions where their domains overlap, we can along any path starting at a point z_0, cover the path by local solutions (f_k,D_k) and glue them together.

If we are to use Corollary B.1.1, we need to assure that the domain of the differential equation is simply connected. This is a pretty big constraint, and in many situations, e.g. if the differential equations has singularities, we do not in general have a simply connected domain. But all is not lost, there are tricks to alter the domain of the problem into a simply-connected one.

B.2 The universal cover of a Riemann surface

Definition B.2.1 Holomorphic covering and universal covering.

i) *Let $p : \tilde{M} \to M$ be a continuous surjective map between two topological spaces. The open set $U \subset M$ is said to be evenly covered by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in $\tilde{M}$,*

$$p^{-1}(U) = \coprod_\alpha V_\alpha.$$

Also for each α, the restriction of p to V_α is a homeomorphism of V_α onto U.

ii) *A covering is a triple $(\tilde{M}, M, p)$, where $p : \tilde{M} \to M$ is a continuous and surjective function, and such that every point $z \in M$ has a neighbourhood U that is evenly covered by p. $\tilde{M}$ is called a covering space, M is called base space and p is called a covering map.*

iii) *Let $\tilde{M}$ and M be connected complex manifolds. A holomorphic covering, is a covering $(\tilde{M}, M, p)$ with the additional properties that the covering map p, is holomorphic as a map between manifolds and when it evenly covers a neighbourhood U of $z \in M$, $p|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to U$ is a biholomorphism.*

iv) *A universal covering is a covering $(\tilde{M}, M, p)$ where the covering space $\tilde{M}$ is simply connected.*

v) *Let $(\tilde{M}, M, p)$ be a covering and $f : N \to M$ a continuous function from some topological space N into M. A lifting of f to $\tilde{M}$ is a continuous function $\tilde{f} : N \to \tilde{M}$ such that $f = p \circ \tilde{f}$ i.e. the following diagram commutes:*

$$\begin{tikzcd} & \tilde{M} \arrow[d, "p"] \\ I \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & M \end{tikzcd}$$

We are going to need two topological lemmas to be able to lift paths from a base space to a covering space.

Lemma B.2.1 [[Mun00][L.54.1]] **.**
Let $(\tilde{M}, M, p)$ be a covering and $p(\tilde{z}_b) = z_b$. Any path $\zeta : I \to M$ beginning at $\zeta(0) = z_b$ has a unique lifting to a path $\tilde{\zeta} : I \to \tilde{M}$, beginning at $\tilde{z}_b$. I.e. such that $\zeta = p \circ \tilde{\zeta}$.

Lemma B.2.2 [[Mun00][T.54.3]]**.**
Let $(\tilde{M}, M, p)$ be a covering and $p(\tilde{z}_b) = z_b$. If $H : I \times I \to M$ is a homotopy between $H(\cdot, 0) = \zeta_0$ and $H(\cdot, 1) = \zeta_1$, there is a unique lifting of H to a continuous map $\tilde{H} : I \times I \to \tilde{M}$ such that

- *$\tilde{H}(0, 0) = \tilde{z}_b$.*
- *$\tilde{H}$ is a homotopy between $\tilde{H}(\cdot, 0) = \tilde{\zeta}_0$, the unique lift of ζ_0 starting at $\tilde{z}_b$, and $\tilde{H}(\cdot, 1) = \tilde{\zeta}_1$, the unique lifting of ζ_1 starting at $\tilde{z}_b$.*

Theorem B.2.1 [For81], [Lee03]**. The universal cover of a connected complex manifold.**
Let M be a connected complex manifold. Then there exists a connected, simply-connected complex manifold $\tilde{M}$ and a holomorphic map $p : \tilde{M} \to M$ such that $(\tilde{M}, M, p)$ is a universal holomorphic covering.

Proof. Fix $z_b \in M$. The construction will depend on this base-point, and a different choice of base-point gives a covering space which is biholomorphic to the first.

Consider the set

$$\mathcal{M} = \{\, (z, \zeta) \mid z \in M,\ \zeta \text{ is a path between } z_b \text{ and } z \,\},$$

and define the equivalence relation

$$(z, \zeta) \sim (z', \zeta') \iff z = z', \text{ and } \zeta \text{ is homotopic to } \zeta'.$$

We define

$$\tilde{M} := \mathcal{M}/\sim \qquad\qquad p := \pi_1 : \begin{array}{ccc} \tilde{M} & \to & M \\ [z, \zeta] & \mapsto & z \end{array} \tag{B.1}$$

and claim that $(\tilde{M}, M, p)$ is a universal holomorphic covering.

Step 1. We define a topology on $\tilde{M}$. Let $[z,\zeta] \in \tilde{M}$ and $U \subset M$ an open connected, simply-connected neighbourhood of $p([z,\zeta]) = z$, we will also refer to such a neighbourhood in M as admissible. Define a subset $\tilde{U}_{[z,\zeta]} \subset \tilde{M}$ by:

$$\tilde{U}_{[z,\zeta]} = \{\, [x,\eta] \in \tilde{M} \mid x \in U \text{ and } \eta \sim \zeta * \zeta_{z,x}, \text{ where } \zeta_{z,x} \text{ is a path from } z \text{ to } x,\ \ \zeta_{z,x} \subset U \,\}$$

Since U is simply connected all paths from z to x in U is homotopic. Let $\mathscr{B}$ be the family of all such $\tilde{U}$. We show $\mathscr{B}$ is a basis for a topology on $\tilde{U}$.

- If $[z,\zeta] \in \tilde{M}$ then $[z,\zeta] \in \tilde{U}_{[z,\zeta]}$.
- Let $[z,\zeta] \in U_{[x,\eta]} \cap V_{[y,\gamma]}$. Then $z \in U \cap V$ and there exists an admissible neighbourhood $W \subset U \cap V$ of z. Further
$$[z,\zeta] \in \tilde{W}_{[z,\zeta]} \subset U_{[x,\eta]} \cap V_{[y,\gamma]}$$
since every path $\mu \subset W$ is also in $U \cap V$.

Step 2. $p : \tilde{M} \to M$ is a covering map, so $(\tilde{M}, M, p)$ is a covering. To show this we note that since $p := \pi_1$ it is obviously surjective and continuous. If $z \in M$ then z is contained in an open connected, simply-connected neighbourhood U, and we have that

$$p|_{\tilde{U}_{[z,\zeta]}} : \tilde{U}_{[z,\zeta]} \to U$$

is a homeomorphism for any ζ, since π_1 is an open map. We want to show that

$$p^{-1}(U) = \coprod_{[\zeta]} \tilde{U}_{[z,\zeta]},$$

where the union is taken over all homotopy classes $[\zeta]$ of paths ζ going from z_b to z. We first show that any point in $p^{-1}(U)$ is in this union. Let $[x,\eta] \in p^{-1}(U)$, by Definition $x \in U$ and η is a path from z_b to x. Consider

$$\tilde{U}_{[z,\zeta]} = \{\, [y,\gamma] \mid y \in U \ \&\ \gamma \sim \zeta * \zeta_{z,y} \,\},$$

where $\zeta_{z,y}$ is a path in U between z and y, U being open, connected and simply-connected. We define

$$\zeta' := \eta * \zeta_{x,z},$$

where $\zeta_{x,z}$ is a path in U between x and z. Then we have that ζ' is a path from z_b to z, and its homotopy class is thus included in the union above. We also have that

$$\zeta' * \zeta_{z,x} = \eta * \zeta_{x,z} * \zeta_{z,x} \sim \eta$$

as any closed path in U is homotopic to a point. Hence

$$[x,\eta] \in \tilde{U}_{[z,\zeta']} \subset \bigcup_{[\zeta]} \tilde{U}_{[z,\zeta]}.$$

We show that the union is disjoint. Assume to get a contradiction that $[x,\eta] \in \tilde{U}_{[z,\zeta]} \cap \tilde{U}_{[z,\zeta']}$ where $\zeta \not\sim \zeta'$. Let $\zeta_{x,z}$ be a path in U from x to z, then

$$\zeta \sim \eta * \zeta_{x,z} \sim \zeta'$$

by Definition of $[x,\eta] \in \tilde{U}_{[z,\zeta]} \cap \tilde{U}_{[z,\zeta']}$. We thus obtain the desired conclusion.

We also remark that the above proof shows that if $[z,\zeta] \neq [z,\zeta']$ then if U is any admissible neighbourhood of z, then $\tilde{U}_{[z,\zeta]} \cap \tilde{U}_{[z,\zeta']} = \emptyset$. And if $[z,\zeta] \neq [x,\eta]$, then obviously there are admissible neighbourhoods $U, V \in M$ such that $\tilde{U}_{[z,\zeta]} \cap \tilde{V}_{[x,\eta]} = \emptyset$ by the Hausdorff property in M. We have thus shown that $\tilde{M}$ is Hausdorff.

Step 3. $\tilde{M}$ is connected and simply-connected. We will show that $\tilde{M}$ is path connected. Let $[z,\zeta] \in \tilde{M}$. By Lemma B.2.1, since $(\tilde{M}, M, p)$ is a covering there exists a path $\tilde{\zeta} : I \to \tilde{M}$ such that $\tilde{\zeta}$ starts at $\tilde{\zeta}(0) = [z_b, z_b]$ and ends at $\tilde{\zeta}(1) = [z,\zeta]$.

$$\begin{tikzcd} & \tilde{M} \arrow[d, "p"] \\ I \arrow[ur, "\tilde{\zeta}"] \arrow[r, "\zeta"'] & M \end{tikzcd}$$

Here $[z_b, z_b]$ denotes the equivalence class with basepoint z_b and homotopy class equal to the constant path at z_b. Thus any point of $\tilde{M}$ is path connected to $[z_b, z_b]$.

To show that $\tilde{M}$ is simply-connected, note that since p is continuous, $p : \tilde{M} \to M$ induces a homomorphism of fundamental groups, when fixing $p([z_b, z_b]) = z_b$:

$$\begin{array}{rccc} p_* : & \pi_1(\tilde{M}, [z_b, z_b]) & \to & \pi_1(M, z_b) \\ & [\tilde{\zeta}] & \mapsto & [p \circ \tilde{\zeta}] \end{array}$$

We will show that this homomorphism is trivial. Let $[\tilde{\gamma}]$ be any element of $\pi_1(\tilde{M}, [z_b, z_b])$. That is, $\tilde{\gamma}$ is a loop at $[z_b, z_b]$ in $\tilde{M}$. Since p is continuous, $\zeta := p \circ \tilde{\gamma}$ is a path in M with $\zeta(0) = p \circ \tilde{\gamma}(0) = z_b = \zeta(1)$. Thus $\zeta : I \to M$ has by Lemma B.2.1 a unique lift to a path $\tilde{\zeta}$ in $\tilde{M}$ such that $\tilde{\zeta}$ at $\tilde{\zeta}(0) = [z_b, z_b]$, thus $\tilde{\gamma} = \tilde{\zeta}$. We also know by the construction of the lifted path that $\tilde{\zeta}(1) = [z_b, \zeta] = [z_b, z_b]$, which implies that $\zeta \sim z_b$. Now since we know that the unique lift of the constant path at z_b in M is the constant path at $[z_b, z_b]$ in $\tilde{M}$, we can conclude by Lemma B.2.2 that

$$\tilde{\gamma} = \tilde{\zeta} \sim [z_b, z_b].$$

Thus p_* is a trivial map, and any loop in $\tilde{M}$ starting at $[z_b, z_b]$ is homotopic to the point $[z_b, z_b]$, which means that $\tilde{M}$ is simply-connected.

Step 4. $\tilde{M}$ is a complex manifold. We have already shown that $\tilde{M}$ is Hausdorff. The map $p : \tilde{M} \to M$ is a local homeomorphism, hence since M is locally euclidean by charts $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ we can compose these charts with p. We restrict U_α to a neighbourhood U'_α such that $p|_{\tilde{U}'_\alpha} : \tilde{U}'_\alpha \to U'_\alpha$ is a homeomorphism and $\tilde{\phi} := \phi \circ p : \tilde{U}'_\alpha \to \mathbb{C}^n$ is a chart on $\tilde{M}$.

To show that $\tilde{M}$ is second countable, we first prove that the fibers of p is countable, i.e. the set $p^{-1}(z_b)$ is countable. We have that for any $[\zeta] \in \pi_1(M, z_b)$, there is a representative $\zeta : I \to M$ such that by Lemma B.2.1, if we consider the unique lift $\tilde{\zeta}$ of ζ, starting at $\tilde{\zeta}(0) = [z_b, z_b]$, the path ends at $\tilde{\zeta}(1) \in \tilde{M}$. This endpoint is actually invariant of the representative chosen for $[\zeta]$ by Lemma B.2.2. Thus we can define the map

$$\begin{array}{rccc} L : & \pi_1(M, z_b) & \to & p^{-1}(z_b) \\ & [\zeta] & \mapsto & \tilde{\zeta}(1) \end{array}$$

We show that the fact that $\tilde{M}$ is path-connected leads to L being surjective and the fact that $\tilde{M}$ is simply connected leads to L being injective. Let $[z_b, \zeta] \in p^{-1}(z_b)$. Since $\tilde{M}$ is path connected there is a path $\tilde{\gamma}$ in $\tilde{M}$ from $[z_b, z_b]$ to z_b, ζ. Then $\gamma = p \circ \tilde{\gamma}$ is a loop in M at z_b, hence $[\gamma] \in \pi_1(M, z_b)$, and $L([\gamma]) = [z_b, \zeta]$ by Definition. For injectivity of L, let $L([\zeta]) = L([\gamma])$. Then consider the lifts $\tilde{\zeta}$ and $\tilde{\gamma}$, of the representatives ζ and γ using Lemma B.2.1. Then

$$L([\zeta]) = \tilde{\zeta}(1) = \tilde{\gamma}(1) = L([\gamma]),$$

so since $\tilde{M}$ is simply connected we can find a homotopy $\tilde{H}$ between $\tilde{\zeta}$ and $\tilde{\gamma}$ in $\tilde{M}$. This implies that $H := p \circ \tilde{H}$ is a homotopy between ζ and γ in M, thus $[\zeta] = [\gamma]$ and L is injective. Thus to show that p^{z_b} is countable we can show that the homotopy group of M is countable. But this is covered by the known fact that the fundamental group of a manifold is always countable, see [Lee11] Theorem 8.11.

We prove that $\tilde{M}$ is second countable. The set of all evenly covered open subset of M is an open cover for M, so we can choose a countable subcover $\{ U_n \}_{n=1}^{\infty}$. Then for each U_n we pick one basepoint $z_n \in U_n$,

$$p^{-1}(U_n) = \coprod_j (\tilde{U}_n)_{[z_n, \zeta_{nj}]}$$

where the union goes over all $[\zeta_{nj}] \in \pi_1(M, z_n)$, which is countable. Thus

$$\{ (\tilde{U}_i)_{[z_i, \zeta_{ij}]} \mid z_i \in U_i, \ [\zeta_{ij}] \in \pi_1(M, z_i) \}$$

is a countable open cover for $\tilde{M}$, and each set $(\tilde{U}_i)_{[z_i, \zeta_{ij}]}$ is second countable. Thus taking a countable basis for each such set, gives a countable union of countable basis elements, thus finally a countable basis for $\tilde{M}$.

To conclude that $\tilde{M}$ is a complex manifold, all that remains is to show that it has a holomorphic atlas. We know that M possesses a holomorphic atlas

$$\mathscr{U}_M = \{ (\phi_\alpha, U_\alpha) \mid \phi : U_\alpha \to \mathbb{C}^n \}, \ \forall \alpha, \beta \ \ \phi_\alpha \circ \phi_b^{-1} : \phi_b(U_\alpha \cap U_\beta) \subset \mathbb{C}^n \to \mathbb{C}^n, \text{ is holomorphic}$$

Consider

$$\mathscr{U}_{\tilde{M}} := \{ (\tilde{\phi}_\alpha, \tilde{U}_\alpha) \mid \tilde{\phi}_\alpha := \phi_\alpha \circ p \mid_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to \mathbb{C}^n \}$$

where U_α is shrunken accordingly so that $p|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to U_\alpha$ is a homeomorphism. For any α, β we have that:

$$\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1} = \phi_\alpha \circ p|_{\tilde{U}_\alpha} \circ p|_{\tilde{U}_\beta}^{-1} \circ \phi_\beta^{-1} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n \quad \to \quad \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$$

which is a biholomorphism by assumption.

Step 5. $(\tilde{M}, M, p)$ is a holomorphic covering. We need to show by Definition B.2.1 that p is holomorphic and that $p|_{\tilde{U}_\alpha} : \tilde{U}_\alpha :\to U$ is a biholomorphism. Let $[z, \zeta] \in \tilde{M}$ and let $\tilde{U}_{[z,\zeta]} = \tilde{U}$ be a neighbourhood of $[z, \zeta]$ such that $p|_{\tilde{U}}(\tilde{U}) = U$ is evenly covered, and such that we have charts

$$\tilde{\phi} = \phi \circ p|_{\tilde{U}} : \tilde{U} \to \mathbb{C}^n, \quad \phi : U \to \mathbb{C}^n.$$

Then the result is trivial as:

$$\phi \circ p_{\tilde{U}} \circ \tilde{\phi}^{-1} : \tilde{\phi}(\tilde{U}) \xrightarrow{\tilde{\phi}^{-1}} \tilde{U} \xrightarrow{p|_{\tilde{U}}} U \xrightarrow{\phi} \phi(U) \ , \qquad \phi \circ p|_{\tilde{U}} \circ \tilde{\phi}^{-1} = \phi \circ p|_{\tilde{U}} \circ p|_{\tilde{U}}^{-1} \circ \phi^{-1} = I$$

and similarly

$$\tilde{\phi} \circ p_{\tilde{U}}^{-1} \circ \phi^{-1} : \phi(U) \xrightarrow{\phi^{-1}} U \xrightarrow{p|_{\tilde{U}}^{-1}} \tilde{U} \xrightarrow{\tilde{\phi}} \tilde{\phi}(\tilde{U}) \ , \qquad \tilde{\phi} \circ p|_{\tilde{U}}^{-1} \circ \phi^{-1} = \phi \circ p|_{\tilde{U}} \circ p|_{\tilde{U}}^{-1} \circ \phi^{-1} = I$$

■

Corollary B.2.1.
Let $(\tilde{M}, M, p)$ be a holomorphic covering and $f : M \to N$ a map into some complex manifold N. Consider the function $\tilde{f}$ defined by

$$\begin{tikzcd} \tilde{M} \arrow[d, "p"] \arrow[dr, dashed, "\tilde{f}"] & \\ M \arrow[r, "f"'] & N \end{tikzcd} \qquad \tilde{f} := f \circ p$$

Then $\tilde{f}$ is holomorphic if and only if f is holomorphic.

Proof. If f is holomorphic, then $\tilde{f} = f \circ p$ is well defined since p is surjective, and is a composition of two holomorphic maps and is thus holomorphic.
If $\tilde{f}$ is holomorphic, let $z \in M$ and let U be an evenly covered neighbourhood of z, such that $p|_{\tilde{U}} : \tilde{U} \to U$ is a biholomorphism. Then locally we can write f as

$$f = \tilde{f} \circ p|_{\tilde{U}}^{-1} = (f \circ p) \circ p|_{\tilde{U}}^{-1}$$

which is a composition of holomorphic maps. Thus since f is holomorphic at any $z \in M$, it is holomorphic in M. ■

Bibliography

[AK00] F V Andreev and A V Kitaev. 'Connection formulae for asymptotics of the fifth Painlevé transcendent on the real axis'. In: *Nonlinearity* 13.5 (Aug. 2000), pp. 1801–1840. DOI: 10.1088/0951-7715/13/5/319. URL: https://doi.org/10.1088/0951-7715/13/5/319 (cit. on p. 89).

[Bal00] Werner Balser. *Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations*. Springer, 2000 (cit. on pp. 15, 16, 19).

[BBT03] Olivier Babelon, Dennis Bernard, and Michel Talon. *Introduction to Classical Integrable Systems*. Ed. by P. V. Landshoff, D. R. Nelson, and S. Weinberg. Cambridge University Press, 2003 (cit. on p. 88).

[Bel97] R. Bellman. *Introduction to Matrix Analysis: Second Edition*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 1997. ISBN: 9780898713992. URL: https://books.google.no/books?id=QVCflvTPYE8C (cit. on p. 81).

[Bog91] Albert Boggess. *CR Manifolds and the Tangential Cauchy-Riemann Complex*. CRC Press Inc., 1991. ISBN: 9780367450526 (cit. on pp. 93, 94).

[Bol90] Andrey A. Bolibrukh. 'The Riemann-Hilbert problem'. In: *Russian Mathematical Surveys* 45.2 (Apr. 1990), pp. 1–58. DOI: 10.1070/RM1990v045n02ABEH002350 (cit. on p. 4).

[Bra11] D. Braak. 'Integrability of the Rabi Model'. In: *Phys. Rev. Lett.* 107 (10 Aug. 2011), p. 100401. DOI: 10.1103/PhysRevLett.107.100401. URL: https://link.aps.org/doi/10.1103/PhysRevLett.107.100401 (cit. on p. 4).

[CAQ15] Bruno Carneiro da Cunha, Manuela Carvalho de Almeida, and Amilcar Rabelo de Queiroz. 'On the Existence of Monodromies for the Rabi model'. In: (2015). DOI: 10.48550/ARXIV.1508.01342 (cit. on pp. 4, 5).

[Con et al.99] Conte et al. *The Painlevé Property, One Century Later*. Ed. by Robert Conte. CRM Series in Mathematical Physics. Springer-Verlag, New York, Inc., 1999 (cit. on p. 5).

[FG02] Klaus Fritzsche and Hans Grauert. *From Holomorphic Functions to Complex Manifolds*. Springer, New York, NY, 2002 (cit. on p. 71).

[Fok et al.06] Fokas, Its, Kapaev, and Novokshenov. *Painlevé Transcendents: The Riemann-Hilbert Approach*. American Mathematical Society, 2006 (cit. on pp. 5, 8, 14, 16, 19, 30, 34, 39, 41, 48, 78, 89).

[For81] Otto Forster. *Lectures on Riemann Surfaces*. Ed. and trans. by Bruce Gilligan. Springer-Verlag New Yorc Inc., 1981. ISBN: 0-387-90617-7 (cit. on p. 107).

[Hil02] David Hilbert. 'Mathematical Problems'. In: *Bulletin of the American Mathematical Society*. American Mathematical Society, 1902 (cit. on p. 4).

[Inc56] Edward L. Ince. *Ordinary Differential Equations*. Dover, 1956 (cit. on p. 89).

[Iwa et al.91] Katsunori Iwasaki, Hironobu Kimura, Shun Shimomura, and Masaaki Yoshida. *From Gauss to Painlevé, A Modern Theory of Special Functions*. Vieweg+Teubner Verlag, 1991. DOI: 10.1007/978-3-322-90163-7 (cit. on pp. 4, 89).

[JM81a] Michio Jimbo and Tetsuji Miwa. 'Monodromy perserving deformation of linear ordinary differential equations with rational coefficients. II'. In: *Physica D: Nonlinear Phenomena* 2.3 (1981), pp. 407–448. ISSN: 0167-2789. DOI: https://doi.org/10.1016/0167-2789(81)90021-X. URL: https://www.sciencedirect.com/science/article/pii/016727898190021X (cit. on p. 4).

[JM81b] Michio Jimbo and Tetsuji Miwa. 'Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III'. In: *Physica D: Nonlinear Phenomena* 4.1 (1981), pp. 26–46. ISSN: 0167-2789. DOI: https://doi.org/10.1016/0167-2789(81)90003-8. URL: https://www.sciencedirect.com/science/article/pii/0167278981900038 (cit. on p. 4).

[JMU81] Michio Jimbo, Tetsuji Miwa, and Kimio Ueno. 'Monodromy preserving deformation of linear ordinary differential equations with rational coefficients: I. General theory and -function'. In: *Physica D: Nonlinear Phenomena* 2.2 (1981), pp. 306–352. ISSN: 0167-2789. DOI: https://doi.org/10.1016/0167-2789(81)90013-0. URL: https://www.sciencedirect.com/science/article/pii/0167278981900130 (cit. on pp. 4, 5, 41, 80, 86).

[Kas84] Masaki Kashiwara. 'The Riemann-Hilbert problem for holonomic systems'. In: *Publications of the Research Institute for Mathematical Sciences* 20.2 (1984), pp. 319–365. DOI: 10.2977/prims/1195181610 (cit. on p. 90).

[KN63] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry*. Vol. I. John Wiley & Sons, 1963 (cit. on pp. 52, 54, 56, 62, 69, 98, 100).

[Lee03] John M. Lee. *Introduction to Smooth Manifolds*. 2nd ed. Graduate Texts in Mathematics. Springer-Verlag New York, Inc., 2003. ISBN: 978-1-4419-9981-8 (cit. on p. 107).

[Lee11] John M. Lee. *Introduction to Topological Manifolds*. 2nd ed. Graduate Texts Mathematics. Springer-Verlag New York, Inc., 2011. ISBN: 978-1-4419-7940-7 (cit. on p. 109).

[Meb80] Zoghman Mebkhout. 'Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory'. In: ed. by D. Iagolnitzer. Springer Berlin Heidelberg, 1980. Chap. Sur le problème de Hilbert-Riemann, pp. 90–110. DOI: 10.1007/3-540-09996-4 (cit. on p. 90).

[Mun00] James R. Munkres. *Topology*. 2nd ed. Prentice Hall, Inc, 2000 (cit. on pp. 36, 96, 107).

[Neu77] Carl G. Neumann. 'Untersuchungen über das logarithmische und Newtonsche Potential'. In: *Teubner* (1877) (cit. on p. 96).

[Olv97] Frank Olver. *Asymptotics and Special Functions*. A K Peters and CRC Press, Jan. 1997. DOI: 10.1201/9781439864548 (cit. on p. 89).

[Pai00] Paul Painlevé. 'Mémoire sur les équations différentielles dont l'intégrale générale est uniforme'. fr. In: *Bulletin de la Société Mathématique de France* 28 (1900), pp. 201–261. DOI: 10.24033/bsmf.633. URL: http://www.numdam.org/articles/10.24033/bsmf.633/ (cit. on p. 89).

[Pai02] P. Painlevé. 'Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme'. In: *Acta Mathematica* 25.none (1902), pp. 1–85. DOI: 10.1007/BF02419020. URL: https://doi.org/10.1007/BF02419020 (cit. on p. 89).

[Pel et al.95] T. Pellizzari, S. A. Gardiner, J. I. Cirac, and P. Zoller. 'Decoherence, Continuous Observation, and Quantum Computing: A Cavity QED Model'. In: *Phys. Rev. Lett.* 75 (21 Nov. 1995), pp. 3788–3791. DOI: 10.1103/PhysRevLett.75.3788. URL: https://link.aps.org/doi/10.1103/PhysRevLett.75.3788 (cit. on p. 4).

[Ple64] Josip Plemelj. *Problems in the sense of Riemann and Klein*. Ed. by J R M Radok. New York, Interscience Publishers, 1964 (cit. on p. 4).

[Rab36] I. I. Rabi. 'On the Process of Space Quantization'. In: *Phys. Rev.* 49 (4 Feb. 1936), pp. 324–328. DOI: 10.1103/PhysRev.49.324. URL: https://link.aps.org/doi/10.1103/PhysRev.49.324 (cit. on p. 4).

[Rud87] Walter Rudin. *Real and Complex Analysis, 3rd Ed.* McGraw-Hill, Inc., 1987. ISBN: 0070542341 (cit. on pp. 86, 96).

[Sch12] Ludwig Schlesinger. 'Über eine Klasse von Differentialsystemen beliebiger Ordnung mit fetenkritischen Punkten'. In: *J. Reine Angew. Math.* 1912.141 (1912), pp. 96–145 (cit. on p. 4).

[Sha97] R.W. Sharpe. *Differential Geometry. Cartan's Generalization of Klein's Erlangen Program*. 1st ed. Springer-Verlag New York, 1997 (cit. on pp. 68, 97, 98).

[Sib90] Yasutaka Sibuja. *Linear Differential Equation in the Complex Domain*. American Mathematical Society, 1990 (cit. on p. 13).

[Tes12] Gerald Teschl. *Ordinary differential equations and dynamical systems*. English. Vol. 140. Grad. Stud. Math. Providence, RI: American Mathematical Society (AMS), 2012. ISBN: 978-0-8218-8328-0 (cit. on p. 11).

[Ved05] Vlatko Vedral. *Modern Foundations of Quantum Optics*. Imperial College Press, 2005 (cit. on p. 4).

[Was87] Wolfgang Wasow. *Asymptotic Expansions for Ordinary Differential Equations*. Dover Publications, Inc., 1987 (cit. on pp. 19, 84).

[Wik22] Wikipedia. *Gauge theory (mathematics)*. Apr. 2022. URL: https://en.wikipedia.org/wiki/Gauge_theory_(mathematics) (cit. on p. 6).