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Algorithms for rainbow vertex colouring diametral path graphs

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Abstract

Given a graph and a colouring of its vertices, a rainbow vertex path is a path between two vertices such that all the internal nodes of the path are coloured distinctly. A graph is rainbow vertex-connected if between every pair of vertices in the graph there exists a rainbow vertex path. In the RAINBOW VERTEX COLOURING PROBLEM we decide whether a given graph can be coloured using k or less colours such that it is rainbow vertex-connected. The STRONG RAINBOW VERTEX COLOURING problem concerns whether a graph can be coloured with k or less colours so that between every pair of vertices there exists a shortest path which is a rainbow path. The purpose of this thesis is to explore whether diametral path graphs can be rainbow vertex-coloured efficiently.

We prove that for graphs that have a dominating diametral path and one of the following properties: chordal, bipartite or claw-free, an optimal rainbow vertex-colouring can be found in polynomial time. We also provide an algorithm to find an optimal strong rainbow vertex-colouring for interval graphs.

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`Jakob Dyrseth`
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Chapter 1

Introduction

A common way to model problems in computer science is by using *graphs*. They consist of nodes and edges, where an edge is what connects two nodes. In general one could say a graph is a way of specifying the relation between a collection of items. Graph colouring is a fundamental computational problem in the field of graph theory. Its essential problem concerns properly colouring graphs, which is achieved when no edge of the graph has two endpoints sharing a colour. Another heavily researched field in graph theory is graph connectivity. This issue concerns how different nodes in a graph are connected to each other. The concepts of colouring and connectivity meet in rainbow colouring, a problem first introduced by Chartrand et al. [2] in 2008. In the RAINBOW COLOURING (RC) problem we want to edge colour a graph in such a way that between every pair of vertices there exists a *rainbow path*, which is when no two edges of a path share a colour. If this property is obtained for every pair of vertices we say that the graph is *rainbow connected*.

This topic garnered a lot of attention at the time and therefore led Krivelevich and Yuster [9] to define a variation which applied to vertex colourings, instead of edge colourings. For this variant we are concerned with *rainbow vertex paths*, which are paths between two vertices such that all the internal nodes of that path are coloured distinctly. We say a graph is *rainbow vertex connected* if between every pair of vertices there exists a rainbow vertex path. Resulting from this we have the decision problem RAINBOW VERTEX COLOURING (RVC) which takes as input a graph G and some integer k with the task to decide if G can be coloured using k or less colours such that it is rainbow vertex-connected. The minimum number of colours needed to make a graph G rainbow vertex-connected is denoted by $\mathbf{rvc}(G)$, and it is called the rainbow vertex-connection number of G.

A stronger variant of the RVC problem was later introduced by Li et al. [12]. For this variant we say a graph is *strongly rainbow vertex-connected* if between every pair of vertices of the graph there exists a shortest path which is also rainbow vertex path. The resulting decision problem Strong rainbow vertex colouring (SRVC) takes as input a graph G and an integer k and the task is to decide whether G has a colouring using k or less colours which is strongly rainbow vertex-connected. By $\mathbf{srvc}(G)$ we denote the *strong vertex connection number* of G which is the minimum number of colours needed to strongly rainbow vertex-colour G.

As it turns out both RVC and SRVC are NP-complete for $k \geq 2$ [3, 4, 7]. There are multiple ways of dealing with a problem which has been deemed NP-complete. One method is by designing parameterized algorithms. For these types of algorithms the running time is not only expressed through the input size, but also by an input parameter. Another method for dealing with intractability, which prioritises speed over the exactness of the solution, is by designing *approximation algorithms*. The technique we are going to use to cope with NP-completeness in this text is by restricting the problem to only certain types of input, that we call a graph class. This strategy has already been used on multiple graph classes for rainbow vertex colouring, with the following results.

To place the results we will achieve in this text within a larger context we will present some earlier known results on rainbow vertex-colouring and its stronger variant. For bipartite graphs RVC is NP-complete when $k \geq 3$ [8, 11]. On split graphs RVC is NP-complete for every $k \geq 2$ [8]. This, in turn, means that RVC is NP-complete for every $k \geq 2$ on chordal graphs, as split graphs are a well known subclass of chordal graphs. The result on split graphs is interesting as it is an instance where a result on a graph class differs between the vertex and edge variant of problem. RC on split graphs is in fact polynomial time solvable for every $k \geq 4$ [1]. In terms of positive results it has been shown that both RVC and SRVC are linear time solvable on planar graphs for fixed k [10]. The same has been shown for bipartite permutation graphs [8], although this result was later improved upon in [13], showing that for permutation graphs an optimal rainbow vertex colouring can be computed in polynomial time. It has also been proven that if G is a interval graph then an optimal rainbow vertex colouring can be computed in linear time [8]. These results were the fundamentals of my research as I tried to see if similar techniques to those in the different proofs could be applied to even broader graph classes. We especially drew inspiration from the techniques used in the paper of Heggernes et al. [8], which had the positive results on bipartite permutation graphs and interval graphs.

The graph classes we will mainly focus on are graphs which feature a dominating diametral path. A dominating set of a graph is a subset vertices of the graph, such that all vertices of the graph either are in the subset — or are adjacent to a vertex in that subset. A path is diametral if it is a shortest path whose length is equal to the diameter of the graph. There are many reasons these types of graph classes are interesting when it comes to rainbow colouring. As just mentioned there has been positive results shown on both bipartite permutation graphs and interval graphs. Graphs featuring dominating diametral paths generalise these graph classes. Another reason these graphs are compelling for our purposes lies in the dominating path and its potential for rainbow colouring, which we will discuss even further in the main section of this text. An example of a graph class which contains a dominating diametral path is that of diametral path graphs. A graph is a diametral path graph if every connected induced subgraph has a dominating diametral path. They were defined while investigating models of networks which would be more resilient in an hostile environment [5]. It was believed that these types of graphs were beneficial in terms of designing resilient and secure networks. This belief was based on how each node of the graph interacted with the dominating diametral path. Since every node is connected to the diametral path the integrity of the network can be increased by having reliable edges and nodes in the diametral path. As the graph scales it will still be maintainable as the length of the diametral path will still be relatively low compared to the number of edges and nodes in the graph. This security aspect of graphs with diametral paths is a fitting parallel to the security applications of rainbow colouring which we will discuss further now.

Although rainbow vertex colouring may seem abstract and almost purely theoretical it does have some quite interesting applications to real world problems especially in terms of encryption and data security. One example of this mentioned in [6] can be seen in onion routing, a technique for anonymously browsing online. In onion routing the goal is to prevent an adversarial from knowing what sites you are connecting to. The way this is achieved is by sending the message through a path of intermediaries before accessing the server you requested. This message will be sent using multiple layers of encryption [14], where each node of the path can only decrypt a single layer of this encryption. Assigning decryption keys to the network draws parallels to rainbow colouring.

We conclude the introduction by giving a brief overview of how the rest of this thesis will be structured. The text is divided into four chapters. The second chapter is the preliminary chapter. Here we define terms which will be used throughout the text and show how certain concepts will be denoted. We give a brief overview on the graph classes that will be discussed during the text, and how they are related to each other. In addition we give a overview on the rainbow vertex-connection number and which bounds have been achieved on it. Following this there is chapter three, which is the main chapter of the thesis. Here we begin by presenting basic properties of graphs with a dominating diametral path. After this, there are four sections, all of which describe a proof for a certain graph class. We conclude the text with chapter four by presenting related open problems which remain unsolved, and our thoughts on these particular issues.

Chapter 2

Preliminaries

2.1 Graph terminology

For purposes of this thesis we are working with undirected graphs. Such graphs are denoted $G = (V, E)$, where V is the set of vertices of G, while E is the set of edges. We let n denote $|V(G)|$ i.e. the number of vertices in the graph. We will denote an edge in a graph (a, b), where a and b are the two endpoints of that edge. We say a and b are *adjacent* if $(a, b) \in E(G)$. We say two vertices a and b are *neighbours* if they are adjacent. For some vertex $x \in V(G)$ we let $N(x)$ denote the neighbours of x while $\deg(x)$ represents how many neighbours x has. A *path* between two vertices is sequence of vertices connected by a sequence of edges where no two vertices of the path are equal to each other. By $p_1p_2p_3...p_t$ we denote the path from p_1 to p_t where $(p_1, p_2), (p_2, p_3), ..., (p_{t-1}, p_t) \in E(G)$. The *length* of a path is the number of edges in that path. The *distance* between a pair of vertices u and v we denote by $\text{dist}(u, v)$, and refers to the length of a shortest path between u and v in G. The notation $\text{diam}(G)$ describes the *diameter* of G, which is the longest distance between a pair of vertices in the graph. A set S is said to be a *dominating set* of G if every vertex in $V \setminus S$ is a neighbour of S. We call a set of vertices S a *clique* if every pair of vertices of S are adjacent. A set of vertices is called an *independent set* if no pair of vertices in the set are adjacent. A *cycle* is a path such that the first and last vertex of the path is the same, while all other vertices of the path are distinct from each other. We denote cycles C_k, where k is the number of edges in that cycle. A graph H is a *subgraph* of G if $V(H)$ is a subset of $V(G)$ and $E(H)$ is a subset of $E(G)$. Let $S \subset V(G)$, we denote by $G[S]$ the *induced subgraph* of G on the subset S. The induced subgraph consists of all vertices in S and all edges in G for which both endpoints are in S.

From this point on we will denote a set of consecutive integers from 1 to k as $[k]$. A *k-colouring* of G is a function $c : V \to [k]$. We say a colouring is *proper* if for every edge

$(a, b) \in E(G)$, $c(a) \neq c(b)$. Given some path $P = ux_1x_2...x_mv$ the vertices $x_1x_2...x_m$ are called the *internal vertices* of P. A path is a rainbow vertex path if for every pair of vertices x_i, x_j of the internal path $c(x_i) \neq c(x_j)$. For some graph G if $\text{diam}(G) = 2$, then $\mathbf{rvc}(G) = \mathbf{srvc}(G) = 1$. The reason each node of the G can have the same colour is that between every pair of vertices there will always exist some path such that there is only one internal node.

Breadth-first search (BFS) is an algorithm for traversing graph structures. The algorithm works by exploring the graph in layers, starting from some arbitrary root node v_0. The layers are indicated L_i, where L_0 is the layer of the root vertex. Thus all vertices in layer $v_i \in L_i$ of the BFS-structure have a distance of i to v_0. Throughout this thesis we will often, for the sake of convenience, indicate a vertex v belonging to some layer L_i of a BFS-structure with v_i.

A *diametral path* is a shortest path whose length is equal to $\text{diam}(G)$. A dominating diametral path is a path $P = x...y$ such that $\text{dist}(x, y) = \text{diam}(G)$ and the set $V(P) = \{x, ..., y\}$ is a dominating set of G. Throughout this text we will often run a BFS-search on one end of the dominating diametral path. We denote the vertices of the diametral path p_i, where p_i belongs to layer L_i of the BFS-tree. We will also often refer to vertices as being *dominated to the right*, *dominated to the left* or *dominated in layer*. For some vertex $v \in L_i$ we say it is dominated to the right if $(v, p_{i+1}) \in E(G)$, we say it is dominated to the left if $(v, p_{i-1}) \in E(G)$ and we say it is dominated in layer if $(v, p_i) \in E(G)$. For this thesis there will be a lot of focus on graphs with dominating diametral paths in conjunction with some other well known graph class. We will present some of these graph classes now, in addition we will show how these graph classes are related to other known graph classes.

2.2 Graph classes

We say a graph is a *diametral path graph* if every connected induced subgraph has a dominating diametral path. It has been shown that the existence of a dominating diametral path in a graph can be checked and the path itself can be computed in $\mathcal{O}(n^3m)$ time [5]. Throughout this text we will often run a BFS-search on one end of the dominating diametral path. We denote the vertices of the diametral path p_i, where p_i belongs to layer L_i of the BFS-tree. We will also often refer to vertices as being *dominated to the right*, *dominated to the left* or *dominated in layer*. For some vertex $v \in L_i$ we say it is dominated

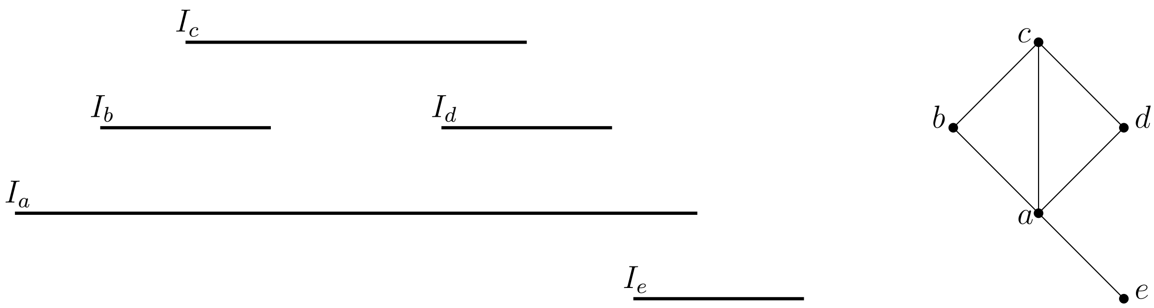

Figure 2.1: To the left is the interval model, while on the right is the corresponding graph.

to the right if $(v, p_{i+1}) \in E(G)$, we say it is dominated to the left if $(v, p_{i-1}) \in E(G)$ and we say it is dominated in layer if $(v, p_i) \in E(G)$.

A graph is *chordal* if it contains no induced cycle of a length which is greater than three. A well known subclass of the chordal graph is that of *interval graphs.* A graph G is an interval graph if and only if each of its vertices can be correlated to some interval on the real line, in a particular way. We call intervals on the real line the *interval model* of the graph and it is denoted by I. Two vertices are adjacent in the graph if and only if their corresponding intervals intersect in the interval model. An example of an interval model and its corresponding interval graph can be seen in Figure 2.1. Interval graphs is a subclass of the chordal graphs with a dominating diametral path. We see this by first observing all the intervals on the real line. We denote by u the vertex of the interval model with its right endpoint leftmost. We denote by v the vertex of the interval model with its left endpoint rightmost. Notice that the shortest path between u and v is a dominating path [8]. It is known that a graph having a dominating shortest path will also have dominating diametral path [5]. Therefore interval graphs are diametral path graphs. Combine this with the impossibility of constructing a cycle of size greater than or equal to four using intervals on the real line means the graph is also chordal. We therefore know that interval graphs are a subclass of the chordal graphs with a dominating diametral path.

If the vertices of graph can be divided into two disjoint subsets such that each of the subsets are independent it is said to be a *bipartite graph.* Notice that if G is a bipartite graph and $\text{diam}(G) = 3$, then $\mathbf{rvc}(G) = \mathbf{srvc}(G) = 2$. The way we would colour the graph in this instance is by colouring the vertices of one of the independent sets 1, while the vertices of the other independent set is coloured 2. Thus, since the diameter is three, any path between two vertices will contain at most two internal vertices. The internal vertices of a path cannot share a colour as they must belong to different independent sets.

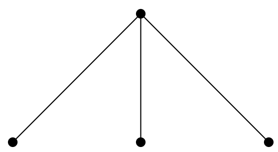

Figure 2.2: A claw

A known subclass of the bipartite graph with a dominating diametral path is *bipartite permutation graphs*. A graph G is a permutation graph if its vertices can be correlated with elements of a permutation of integers σ from $1...n$ such that vertices u and v are adjacent in G if and only if their corresponding elements are inverted by σ. Thus a bipartite permutation graph is permutation graph which is also bipartite. When performing a BFS-search on one of the end points of the diametral path on a bipartite permutation graph each layer will consist of some vertex a_i such that $L_{i+1} \subset N(a_i)$ [15]. The path consisting of all a_i's in the resulting BFS-tree will therefore be a dominating shortest path. As we have already established this means the graph also has a dominating diametral path. Therefore bipartite permutation graphs is a subclass of bipartite graphs with a dominating diametral path.

A graph is *claw-free* if it contains no induced *claw*. Claws are denoted $K_{1,3}$ and can be seen in Figure 2.2. When referring to claws throughout this text we will write them $\{abcd\}$ such that $\{b, c, d\} \subset N(a)$, or said in another way the center of the claw will be written first.

Unit interval graphs are similar to interval graphs as they also have a corresponding interval model. The difference between the two graph classes is that for unit interval graphs all intervals on the real line have to be of the same length. Therefore every unit interval graph is also an interval graph. We can see the relationship between claw-free and unit interval graphs by observing that, if all intervals have length one, there is no way for a single interval to intersect three other intervals such that the three other intervals do not intersect each other. Hence unit interval graphs are a subclass of claw-free graphs. How all the graph classes are related is shown in Figure 2.3. In the figure we shorten dominating diametral path with the abbreviation DDP.

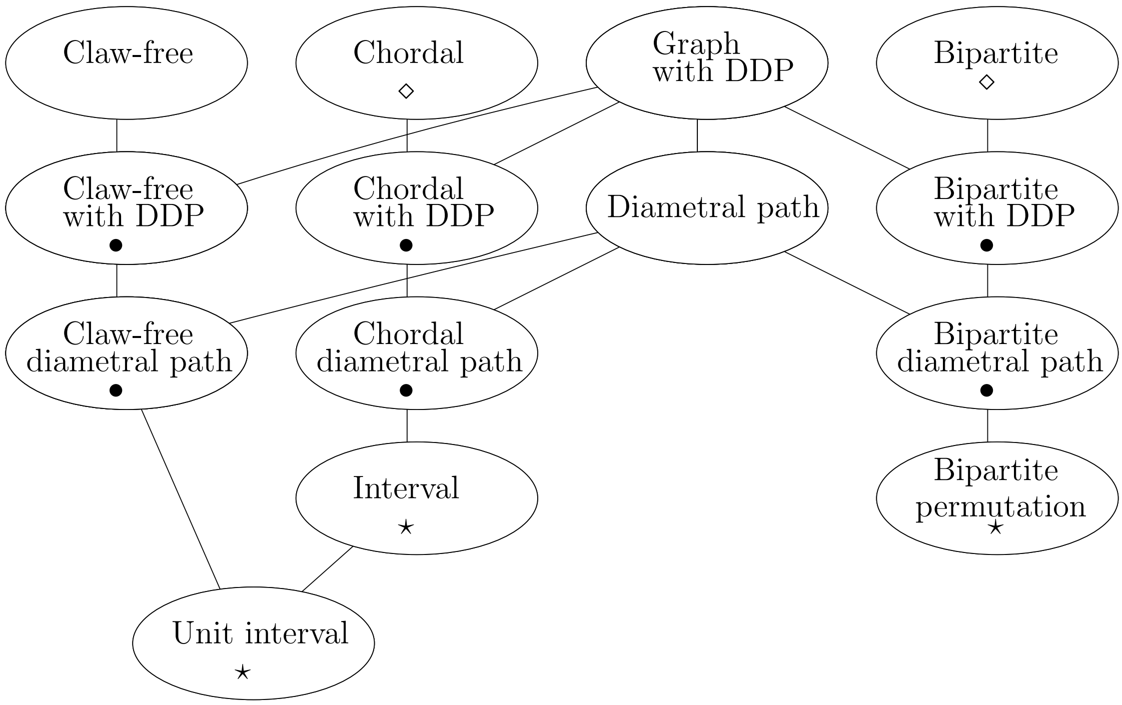

Figure 2.3: Classes marked with $\diamond$ are NP-complete. Those marked with $\star$ have already been proven to have polynomial time algorithms and those marked with $\bullet$ are proven to have polynomial time algorithms in this text.

2.3 Lower and upper bounds on the rainbow vertex-connection number

An important subject in rainbow colouring is finding lower and upper bounds for the rainbow vertex-connection number. We let $G = (V, E)$ be a graph with a diameter greater than two. Firstly we start with a simple observation. If $k \geq n$ then we achieve a rainbow vertex colouring just by giving each vertex of the graph a different colour. Therefore we have the following observation.

Observation 2.1. *If G is a connected graph, then* $\mathbf{rvc}(G) \leq n$.

A more interesting upper bound is found by looking at connected dominating sets. The minimum size of a connected dominating set of some graph G is denoted by $\gamma_C(G)$.

Observation 2.2. *[9] If G is a connected graph, then* $\mathbf{rvc}(G) \leq \gamma_C(G)$.

Proof. By colouring each vertex of the dominating set a distinct colour, and colouring the rest of the graph arbitrarily with those same colours, then this graph will be rainbow vertex-connected as each rainbow path can be found traversing the dominating set. ⌟

Another simple observation we make is on how the rainbow vertex-connection number relates to the strong rainbow vertex-connection number. If we have a strong rainbow vertex colouring of a graph, then this colouring must also necessarily be a rainbow vertex colouring for the graph. We thus end up with the following relation.

Observation 2.3. *If G is a connected graph, then* $\mathbf{rvc}(G) \leq \mathbf{srvc}(G)$.

In terms of lower bounds, one which is of particular interest for this thesis concerns the diameter of the graph. The diameter of a graph is the length of the longest shortest path in the graph. This means that there is a pair of vertices in the graph such that any path between them has at least $\mathrm{diam}(G)-1$ internal nodes. For the graph to be rainbow vertex-connected, we must therefore use at least $\mathrm{diam}(G) - 1$ colours. We thus have the following lower bound.

Observation 2.4. *If G is a connected graph, then* $\mathrm{diam}(G) - 1 \leq \mathbf{rvc}(G)$.

What is especially nice about this lower bound is that if one achieves a correct colouring using $\mathrm{diam}(G) - 1$ colours, then this colouring must be optimal for the graph. We have thus arrived at the following bounds.

Claim 2.5. *If G is a connected graph, then*

$$\mathrm{diam}(G) - 1 \leq \mathbf{rvc}(G) \leq \mathbf{srvc}(G) \leq \gamma_C(G)$$

.

Chapter 3

Diametral path graphs

We begin this main chapter by referring to the conjecture which concludes the paper of Heggernes et al. stating the following:

Conjecture 3.1. *[8] Let G be a diametral path graph. Then* $\mathbf{rvc}(G) = \mathrm{diam}(G) - 1$.

We believe there were two main reasons leading to the conjecture. One is to do with the notion of how powerful a dominating diametral path is in terms of rainbow vertex colouring a graph. Consider the upper bound mentioned in Observation 2.2 of the preliminaries. What this upper bound states is that a distinctly coloured dominating set rainbow vertex colours a graph. Since we know a diametral path graph contains a connected dominating set of size $\mathrm{diam}(G) + 1$, it seems intuitive that one would manage to colour the graph using only two fewer colours. This intuition is actually so strong that for graphs only containing a dominating diametral path this conjecture has already, falsely I might add, been proven true in [10].

The other reason we believe the conjecture was made is due to results on relevant graph classes. To support their conjecture the authors in [8] provided examples of subclasses of diametral path graphs for which the conjecture is true, such as interval graphs and bipartite permutation graphs. We will start this chapter by demonstrating that the presence of such a diametral path will not always be enough to have a rainbow vertex coloured graph using $\mathrm{diam}(G) - 1$ colours. This will be shown using the counterexample in Figure 3.1 featuring a graph with a dominating diametral path of length five. It should be noted that this does not disprove the conjecture as the example graph in Figure 3.1 is not a diametral path graph. It is rather a graph containing a dominating diametral path. The distinction can be seen by observing that the graph in the figure contains two induced cycles of a length greater than six, which is a forbidden structure in diametral path graphs [5].

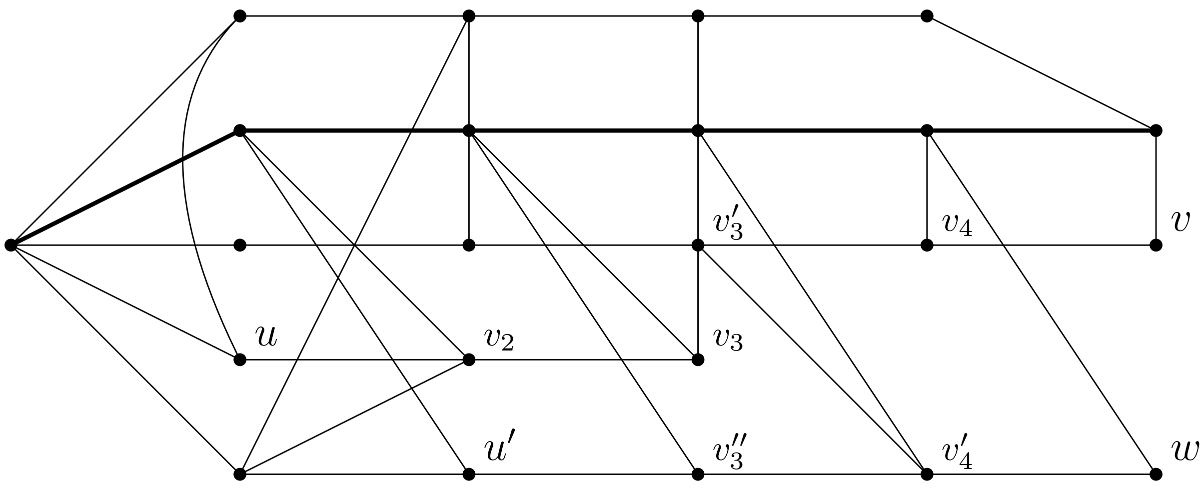

Figure 3.1: The graph has a diameter of 5, while also needing 5 colours to be rainbow vertex-coloured. The dominating diametral path is highlighted.

To see why the graph in Figure 3.1 must use at least 5 colours to be rainbow vertex coloured we will try giving the graph a colouring using 4 colours. First we highlight the three paths of interest in the graph: $uv_2v_3v_3'v_4v$, $uv_2v_3v_3'v_4'w$ and $u'v_3''v_4'v_3'v_4$. What makes these paths interesting for our purposes is that they are the only paths of length less than or equal to the diameter between their two endpoints. Thus, if there exists a rainbow vertex colouring for the graph using 4 colours all of these paths must be rainbow coloured simultaneously. We begin by giving the uv-path an arbitrary rainbow colouring, the internal vertices $v_2v_3v_3'v_4$ receive the colouring $1, 2, 3, 4$ such that $c(v_2) = 1$, $c(v_3) = 2$, $c(v_3') = 3$ and $c(v_4) = 4$. For the uw-path we see that v_4' must be coloured 4 for it to be rainbow, but this means that the $u'v$-path will not be rainbow as there are two nodes in its internal path coloured 4. Thus we see that there is no way for all of these paths to be rainbow at the same time. This proves that there exists a graph G which contain a diametral path and where $\mathbf{rvc}(G) \neq \mathrm{diam}(G) - 1$.

Although dominating diametral paths are not quite as powerful as perhaps first assumed they will prove sufficient when combined with some other well known property in terms of achieving a colouring with $\mathrm{diam}(G) - 1$ colours. This will be demonstrated for multiple graph classes in the following sections.

3.1 Basic properties of diametral path graphs

Let $G = (V, E)$ be a graph containing a dominating diametral path to which a BFS-search is performed on one end of the diametral path. We let $k = \mathrm{diam}(G)$ thus the BFS-tree

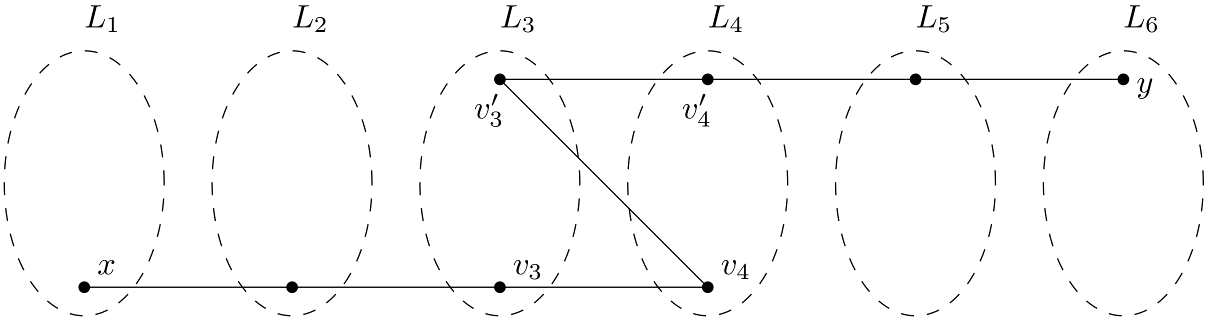

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Figure 3.2: The example demonstrates a path zig-zagging in layer 4 between vertices x and y. We denote this path $x...v_3v_4v_3'v_4'...y$.

will consist of $k+1$ layers. For each layer L_i, for $0 \leq i \leq k$, we let p_i indicate the vertex of the domintaing diametral path in layer i, where p_0 is the root vertex. For this section we will present some generalities and proofs which apply to graphs G. We will also conclude this section by providing a polynomial time algorithm for colouring graphs with a dominating diametral path using $\text{diam}(G)$ colours.

Observation 3.1. *A vertex $x \in L_i$ can only have neighbours in L_{i-1}, L_i and L_{i+1}.*

Proof. For the sake of contradiction let us assume there is a $x \in L_i$ which is neighbours with some $y \in L_j$ where $|i-j| > 1$. If x is encountered first when constructing the BFS-tree then y either is in L_i or L_m, for $m > i$. Since we are assuming $|i-j| > 1$ we know that $y \in L_m$. When the BFS algorithm is in L_i, since x is neighbours with y, y is added to the queue which means y will be in L_{i+1}. The same argument applies if y is the vertex encountered first. ⌟

We will now introduce two new definitions which are used repeatedly throughout the text that hopefully are quite intuitive based on their names, but which are important to present in a formal way as to prevent any vagueness and uncertainties for the rest of the thesis. Recall that for convenience of notation we indicate vertices by the layer they belong to, i.e. a vertex $v \in L_i$ is denoted v_i.

Definition 3.2. *We say a path between two vertices $x \in L_i$ and $y \in L_j$ is a direct path if $\text{dist}(x,y) = j - i$, that is, it intersects every layer from L_i to L_j only once.*

Definition 3.3. *We say a path between two vertices $x \in L_i$ and $y \in L_j$ zig-zags in L_l if it is of the following structure: $x...v_{l-1}v_lv_{l-1}'v_l'...y$, where $v_{i-1} \neq v_{i-1}'$, $v_i \neq v_i'$ and the paths x to v_l and v_l' to y are direct paths.*

As our strategy is largely based on colouring layers distinctly this zig-zag structure is often what can create issues in terms of our colouring. For an example of what a zig-zagging path looks like see Figure 3.2. This next observation comes naturally as a result of being the root vertex of a tree.

Observation 3.4. *The root vertex p_0 of the BFS-tree will have a direct path to every vertex in the tree.*

Claim 3.5. *Let $c : V \to [k-1]$ be the colouring for our graph G. The colouring is given accordingly, each vertex p_i of the diametral path is coloured $c(p_i) = i$, for $1 \leq i < k$. The vertices p_0 and p_k are coloured $k-1$ and 1 respectively and the vertices not in the diametral path are coloured with arbitrary colours in $[k-1]$. For pairs of vertices $x \in L_i$ and $y \in L_j$, where $0 \leq i < j \leq k$, if $i > 2$ a rainbow path can always be found between x and y traversing the diametral path.*

Proof. By Observation 3.1 we know x is either dominated by some p_l where $l \in [i-1, i+1]$, or is equal to p_i. In a similar manner we know either y is dominated by some p_m where $m \in [j-1, j+1]$, or is equal to p_j. We thus have four possible paths:

1. $xp_l...p_my$. None are in the dominating path.

2. $xp_l...p_j = y$. Only y is in the dominating path.

3. $x = p_i...p_my$. Only x is in the dominating path.

4. $x = p_i...p_j = y$. Both x and y are in the dominating path.

In all four possible paths the internal nodes of the path intersect any layer only once, and belong to the diametral path. By our colouring c we know that the vertices in the diametral path from p_2 to p_k are all uniquely coloured. Therefore, since $i, j, l, m \in [2, k]$, the colouring of all the paths will be rainbow. ⌟

The consequence of this claim is that for every diametral path graph, when looking at pairs of vertices $u \in L_i$ and $v \in L_j$, where $0 \leq i < j \leq k$ if the colouring of the diametral path is equal to c then we only have to examine cases when $i < 3$. This next claim ensures that when $i = j$ for two vertices we only have to examine the case when the diameter of the graph is three.

Claim 3.6. *Let $c : V \to [k-1]$ be the colouring for our graph G. The colouring is given accordingly, each vertex p_i of the diametral path is coloured $c(p_i) = i$, for $1 \leq i < k$. The vertices p_0 and p_k are coloured $k-1$ and 1 respectively and the vertices not in the diametral path are coloured with arbitrary colours in $[k-1]$. For a pair of vertices $x, y \in L_i$ in G there will always be a rainbow path connecting them, unless* $\text{diam}(G) = 3$.

Proof. If either $x = p_i$ or $y = p_i$ then $\text{dist}(x, y) \leq 2$, thus let us assume both x and y are vertices not in the dominating path. By Observation 3.1 we know x is dominated by some p_l, where $l \in [i-1, i+1]$, and y is dominated by some p_m, where $m \in [i-1, i+1]$. If $p_l = p_m$ then the path between x and y has a single internal node and thus is rainbow. If $p_l \neq p_m$ we have some path: $P = xp_l...p_my$. For every case of G, unless $\text{diam}(G) \leq 3$, three consecutive dominating vertices will be uniquely coloured - and since $p_l...p_m$ is at most three vertices P will be rainbow. ⌟

To round off this section we present a colouring which proves that graphs with a dominating diametral path need no more than $\text{diam}(G)$ colours to be rainbow vertex-coloured.

Theorem 3.7. *If G is a graph with a dominating diametral path, then* $\mathbf{rvc}(G) \leq \text{diam}(G)$.

Proof. To construct the colouring $c : V \to [k]$, we run a BFS on some end-point of the diametral path. We let k equal the number of layers of the tree excluding p_0 and let p_i denote the vertex of the diametral in layer L_i. We assign the colouring for $v \in L_i$:

$$c(v) = \begin{cases} k & \text{if } i = 0 \text{ or } \quad i = k \\ i & \text{otherwise.} \end{cases}$$

To see that G is rainbow coloured under c let us consider an arbitrary pair of vertices $u, v \in V$, where $u \in L_i$ and $v \in L_j$. We begin by considering cases when $i = j$. If either $u = p_i$ or $v = p_i$ then $\text{dist}(u, v) \leq 2$ and any shortest path between them will be a rainbow path under any colourinig. We therefore assume neither vertex is of the diametral path. Since there is a diametral path we know u is dominated by some p_l, and v is dominated by some p_m where $l, m \in [i-1, i+1]$. The path $P = up_l...p_mv$ will have at most three internal vertices, all belonging to the diametral path. Assuming $\text{diam}(G) > 2$ we know three consecutive vertices in the diametral path will always be uniquely colored, thus P is rainbow.

From this point on we examine cases of u and v where $0 \leq i < j \leq k$. For cases when $i \geq 2$ we can use the diametral path for all cases of v. The proof of this is almost identical to that of Claim 3.5. The difference is that with the addition of an extra colour the range of uniquely coloured vertices in the diametral path is extended by one, from p_1 to p_k. Thus even if u is in L_2 there exists a rainbow path for any case of v using the diametral path.

When $i = 0$, $u = p_0$ and thus the diametral path is rainbow for any case of v. Therefore the only case we have to analyze is when $i = 1$. If u is dominated in layer or to the right we can once again use the diametral path for every case of v, thus let us assume u is only dominated to the left. There are exactly two cases of v we cannot use the dominating path:

1. $j = k - 1$, while v is only dominated to the right. Observation 3.4 tells us there is a direct path from p_0 to v, thus we have the following rainbow path: $up_0v_1...v$.
2. $j = k$, while v is only dominated in layer. Again we utilize the fact that there must be some direct path from p_0 to v, and although the length of the path is greater than k, the internal nodes of the path are uniquely coloured, and thus also rainbow coloured.

This proves that c is a rainbow vertex colouring for G using $\text{diam}(G)$ colours. □

3.2 Bipartite diametral path graphs

For this first proof we investigate the complexity of RVC on bipartite graphs with a dominating diametral path. Remember that for bipartite graphs RVC has already been proven to be NP-complete for $k \geq 3$. The result we achieve for this graph class is really nice as it serves as a direct improvement on an earlier result [8] which stated that a polynomial time algorithm exists to give a optimal rainbow vertex colouring to *bipartite permutation graphs*. Bipartite permutation graphs are as mentioned in the preliminaries a known subclass of the bipartite graphs with a dominating diametral path.

Theorem 3.8. *If G is a bipartite graph with a dominating diametral path, then* $\textbf{rvc}(G) = \text{diam}(G) - 1$ *and the corresponding rainbow vertex colouring can be found in time that is polynomial in the size of G.*

Proof. Let $G = (V, E)$ be a bipartite graph with a dominating diametral path. We run a BFS search on one of the endpoints of the path which will be our root p_0. We let k equal the number of layers in the tree excluding p_0, and denote each vertex of the diametral path p_i for $1 \leq i \leq k$, where p_i is the vertex of the diametral path belonging to layer L_i. To construct a colouring $c : V \to [k-1]$ for G we assign for each $v \in L_i$ the colouring:

$$c(v) = \begin{cases} k-1 & \text{if } i = 0 \\ 1 & \text{if } i = k \\ i & \text{otherwise.} \end{cases}$$

We will also for this proof introduce the sets $A, B \subset V(G)$. These sets become necessary as we need to recolour some vertices in the graph. The vertices in A will retain their colour while the vertices in B will need to be recoloured. For some vertex in layer L_i if it belongs to B we will denote it b_i, if it belongs to A we denote it a_i and if we do not know we denote it v_i. We will now define which vertices are added to B.

To find vertices which need to be recoloured we root a directed graph D_v for each vertex v belonging to L_2 which is only dominated by p_1. The digraph is constructed by performing a BFS where we only traverse edges forward. To be precise, when the search is in some layer L_i of the original BFS structure we only travel edges that go from L_i to L_{i+1}. The search does not visit vertices of the diametral path. When the digraph is created we note the direction each vertex of D_v is dominated in, from layers 2 to and including $k-2$. If none of the vertices are dominated to the right we add the vertices of D_v to the set B. Thus after this operation is performed on all v's in L_2, B will be the union of all the vertices of the digraphs which satisfy our condition about only containing vertices dominated to the left for layers L_2 to L_{k-2}. The rest of the graph which is not in B is in A. For vertices $v \in L_i$ belonging to B we assign the colouring:

$$c(v) = \begin{cases} k-2 & \text{if } i = 2 \\ k-4 & \text{if } i = 3 \text{ and for all } x \in (N(v) \cap L_2) : \operatorname{dist}(x, p_k) = k-2 \\ k-1 & \text{if } i = k-2 \\ i-2 & \text{otherwise.} \end{cases}$$

When $k = 5$ there might be a conflict in the colouring if a vertex satisfies the second line of our colouring. In this scenario there are two colourings for a vertex in layer 3. For this case the third line has precedence i.e. $c(v) = k-1$ for $v \in L_3$ when $k = 5$. It must also be mentioned that when $k \leq 4$ the recolouring will not apply.

Before proving the correctness of our colouring let us establish some claims and observations which will prove useful for the rest of this proof. This first claim exploits a special property of bipartite graphs after a BFS-search is performed on them. It is well known that the resulting BFS-tree consists of layers all of which are independent sets, meaning no edges exist within a layer. The result of this is that we can rule out many potential paths between vertices as shown in the following claim.

Claim 3.9. *For two vertices $x \in L_i$ and $y \in L_j$ the distance between the vertices must be of the form* $\operatorname{dist}(x, y) = j - i + 2m$, *where* $0 \leq m \leq 2$.

Proof. Because of the diametral path we know that $\operatorname{dist}(u, v)$ is at most $j-i+4$. In cases where $\operatorname{dist}(u, v) = j - i + 1$ this implies a path which reapeats only once in some layer.

The only way this can be achieved is by using some edge within a layer, but as already noted no such edge exists within bipartite graphs on which BFS has been performed. There are two ways for a path to be of length $j - i + 3$. One is if the paths zig-zags and repeats a layer, but repetition in a layer is not allowed so we can rule this case out. The other is if the path repeats three times in some layer, but again this contradicts with our graph being bipartite. We can therefore conclude that the length of a shortest path between two vertices is either $j - i$, $j - i + 2$ or $j - i + 4$. ⌟

We make the following two simple observations.

Observation 3.10. *A vertex a_2 is either dominated to the right or must have a direct path to some a_j such that $2 \leq j < k - 1$, where a_j is dominated by p_{j+1}.*

Proof. Assuming, for the sake of contradiction, some a_2 is neither dominated to the right nor has a direct path to some a_j, $i < j < k - 1$, which is dominated to the right. Then this, according to our colouring c, is the exact definition of a root vertex in a recoloured digraph. Therefore a_2 must be in this case in B, a contradiction. ⌟

Observation 3.11. *No vertex b_i belonging to some recoloured digraph can have a direct path to some a_j where $i < j < k - 1$.*

Proof. For the sake of contradiction let us assume that there is some $b_i \in B$ which has a direct path to some a_j where $i < j < k - 1$. If a_j is dominated to the right this would contradict our colouring c. This is because the digraph which b_i belongs to would now consist of some vertex in L_j, where $j < k - 1$, which is dominated to the right. This would mean $b_i \in A$. We, as a result, assume a_j is only dominated to the left. Since we are assuming b_i is in B and a_j belongs to this digraph while dominated to the left, this would mean $a_j \in B$ and there is a direct path between b_i and a_j. ⌟

To see why the colouring is correct we will look at a pair of arbitrary vertices $u \in L_i$ and $v \in L_j$. We begin by examining the cases when $i = j$. Claim 3.6 states that there always is a rainbow path between vertices of the same layer, unless $\text{diam}(G) = 3$. For this diameter we can see by using Claim 3.9 that to not exceed the diameter of the graph all paths between vertices in the same layer must be of length 2, and therefore are rainbow for any colouring.

For the rest of the proof we will look at cases of u and v where $0 \leq i < j \leq k$, and show that there will always exist a rainbow coloured path between them. For cases when $i > 2$ we already know by Claim 3.5 there is a rainbow path between u and v, so we therefore examine all the remaining cases of i in detail. Note that since G is bipartite, every vertex in L_k is dominated to the left thereby reducing the number of cases we must check. We will also finish each sub case by showing why the colouring is correct for when $k \leq 4$, as for these cases the recolouring does not apply.

Case 1. $i = 0$.
The only case we cannot use the diametral path is when $j = k - 1$ while v is only dominated to the right. By Observation 3.4 we know there must be some direct path $P = p_0 v_1 v_2 ... v_{k-2} v$. If $v_{k-2} \in A$ we know by Observation 3.11 that no vertex in B can have a direct path to v_{k-2}, thus all of P is in A which means it is rainbow. If $v_{k-2} \in B$ we know it is dominated to the left. We thus have the path: $p_0 p_1 ... p_{k-3} v_{k-2} v$, and since v_{k-2} is coloured $k - 1$ this path is also rainbow. When $k \leq 4$ there is a direct path from u to v.

Case 2. $i = 1$.
If u is dominated to the right any case of j will be rainbow coloured using the diametral path, therefore let us assume u is only dominated by p_0. There are three cases of j we cannot arbitrarily use the diametral path. They are the following:

1. $j = k$: There must be some path such that $\text{dist}(u, v) \leq k$ and by Claim 3.9 we thus know that $\text{dist}(u, v) = k - 1$, therefore there must be some direct path $P = uv_2v_3...v_{k-1}v$ between u and v. If $v_2 \in A$, by Observation 3.10, we know that either v_2 is dominated to the right giving us the rainbow path $uv_2p_3...p_{k-1}v$ or it has a direct path to some a_j, where $2 \leq j \leq k - 2$, and a_j is dominated to the right. We would thus have the rainbow path: $ua_2...a_jp_{j+1}...p_{k-1}v$. If $v_2 \in B$ we know by Observation 3.11 that all of v_2 to v_{k-1} are also in B. In either case of $c(v_3)$ the internal nodes will be rainbow coloured. If $c(v_3) = 1$ the internal nodes are coloured $k - 2, 1, 2, ..., k - 5, k - 1, k - 3$, while if $c(v_3) = k - 4$ the internal nodes of the path has the colouring: $k - 2, k - 4, 2, ..., k - 5, k - 1, k - 3$. When $k \leq 4$ there is a direct path from u to v, using the same reasoning as for when $k > 4$.

2. $j = k - 1$: We are assuming v is only dominated to the right, as otherwise we could just arbitrarily use the diametral path. By using Claim 3.9 we know there must be a direct path $P = uv_2v_3...v_{k-1}p_k$ between u and p_k. If $v_2 \in A$, by Observation 3.10, either v_2 is dominated to the right or it has a direct path to some a_j, for $2 < j < k - 2$, which is dominated to the right. In the former case there is the rainbow path: $ua_2p_3...p_kv$, while in the latter case the path: $ua_2...a_jp_{j+1}...p_kv$ is rainbow coloured. If $v_2 \in B$ we know by Observation 3.11 that the rest of P is also in B which means we have the path: $ub_2b_3...b_{k-1}p_kv$. Since b_3 has a direct path to p_k all its neighbours

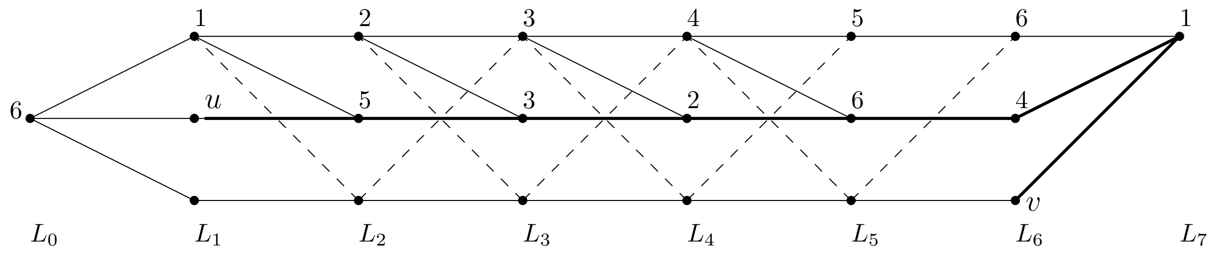

Figure 3.3: A graph with diameter 7. The path highlighted goes through B and is a rainbow-vertex coloured path between u and v.

in L_2 must have a direct path to p_k which means that $c(v_3) = k - 4$. Thus v_3 and p_k do not share a colour thereby implying the path is rainbow coloured. Figure 3.3 demonstrates a specific case of this sub case, when the graph has a diameter of 7. When $k \leq 4$ there is direct path from u to p_k and thus we have the rainbow path $u...p_k v$.

3. $j = k-2$: We again assume v is only dominated to the right. According to Claim 3.9 there exists two possible paths between u and v which we will now examine:

   (a) $\text{dist}(u, v) = k-3$. There is a direct path $P = ua_2...a_{k-3}v$ between the vertices. By Observation 3.11, since v is dominated to the right in L_{k-2}, none of a_2 to v in P can be in B and thus P is rainbow.

   (b) $\text{dist}(u, v) = k - 1$. There must, by Observation 3.4, exist some direct path $P = p_0 v_1 a_2...v$. By the same argument as above none of the vertices in P can be in B. Going from u to p_0 and from there traversing P will thus be a rainbow coloured path.

   When $k \leq 4$ we can just use the direct path from p_0 to v in conjunction with the fact that u and p_0 are adjacent resulting in the rainbow path $up_0...v$.

Case 3. $i = 2$.
For this case the only time we cannot use the diametral path is when $j = k - 1$, while v is only dominated to the right. To argue the correctness of the colouring we consider the two sub cases, either u is the root of some recoloured digraph, or it is not. In the latter case we know by Observation 3.10 that there is some direct path from u to some vertex a_i, $2 < i < k - 1$, where a_i is dominated to the right. Hence, we have the following path: $ua_3...a_i p_{i+1}...p_k v$ which is a rainbow.

If $u \in B$ there are two possible cases we have to examine, Claim 3.9 states that either $\text{dist}(u, v) = k - 3$ or $\text{dist}(u, v) = k - 1$. In the former case there is a direct path between u and v and we know by Observation 3.11 that all nodes of the path are in B, hence for either case of $c(v_3)$ it is rainbow. If $\text{dist}(u, v) = k - 1$ the path will zig-zag in some layer L_ℓ, for $2 \leq \ell \leq k$, giving us the following structure of the path: $u...b_\ell v_{\ell-1}...v_{k-2}v$. By Observation 3.11 we can deduce that if any of $v_{\ell-1}...v_{k-3}$ is in B, then v_{k-2} must also be in B giving us the rainbow path $up_1...p_{k-3}b_{k-2}v$ between u and v. We therefore assume all of $v_{\ell-1}...v$ is in A. Firstly, we observe that if $\text{dist}(u, p_k) = k - 2$ i.e. there is a direct path from u to p_k we have the path: $ub_3...b_{k-2}b_{k-1}p_kv$. According to our colouring $c(v_3) = k - 4$, since no neighbour of v_3 in L_2 can have a path to p_k of length k, thus the colouring of the internal path is $k - 4, 2, 3, ..., k - 1, k - 3, 1$ which is rainbow. We will therefore from this point on assume $\text{dist}(u, p_k) = k$. We examine the colouring of the path for every case of ℓ, assuming the zig-zag happens in L_ℓ:

1. $\ell = 2$. The path will be of the following structure: $ua_1a_2...a_{k-2}v$. The internal vertices are of non-intersecting layers and all belong to A thus this path is rainbow coloured.

2. $2 < \ell < k - 2$. The path will be of the following structure: $ub_3...b_\ell a_{\ell-1}a_\ell...v$. The vertices $b_3...b_\ell$ are coloured $1, 2, ..., \ell - 2$ while $a_{\ell-1}a_\ell...v$ are coloured according to their layer i.e. $\ell - 1, \ell, ..., k - 2$.

3. $\ell = k-2$. The path will be of the following structure: $ub_3...b_{k-3}b_{k-2}a_{k-3}a_{k-2}v$. The vertices $b_3...b_{k-2}$ are coloured $1, 2, ..., k - 5, k - 1$, while a_{k-3} and a_{k-2} are coloured $k - 3$ and $k - 2$, respectively.

4. $\ell = k - 1$. The path will be of the following structure: $ub_3...b_{k-2}b_{k-1}a_{k-2}v$. The vertices $b_3...b_{k-2}b_{k-1}$ are coloured $1, 2, ..., k - 5, k - 1, k - 3$ and a_{k-2} is coloured $k - 2$.

5. $\ell = k$. The path will be of the following structure: $ub_3...b_{k-1}b_kv$. The internal vertices all belong to B and are coloured $1, 2, ..., k - 5, k - 1, k - 3, k - 2$, which is also rainbow.

When $k \leq 4$, we can first discount cases when $k \leq 3$ as when $k = 3$ u and v are in the same layer, which we have already accounted for. When $k < 3$ the diameter is less than 3 which means it is rainbow coloured for any colouring. Therefore we examine the case when $k = 4$. Either u and v are adjacent or the path zig-zags. For the zig-zagging path there are three possible paths: uv_1v_2v, uv_3v_2v and uv_3v_4v, all of which are rainbow coloured under c.

Having exhaustively checked all possible cases of u and v we can conclude there will always exist some rainbow path between two vertices for the colouring c. We can therefore conclude that $\mathbf{rvc}(G) = \text{diam}(G) - 1$. Finding the root vertex takes polynomial time while creating all the digraphs takes $\mathcal{O}(n^2)$-time, and as a result the algorithm runs polynomial time. □

3.3 Chordal diametral path graphs

In a similar vein to the previous section we are now going to present a proof which serves as an improvement on earlier results in terms of optimally rainbow colouring a graph. This time we are looking at chordal graphs. As stated in the introduction it is already known that RVC is NP-complete on chordal graphs for $k \geq 2$. However Heggernes et al. [8] also presented a linear time algorithm which is optimal for interval graphs. Recall that interval graphs are a subclass of chordal graphs with a dominating diametral path. This result on chordal graphs with a dominating diametral path is therefore an improvement as it shows that **rvc** can be computed optimally on these graphs in polynomial time.

Theorem 3.12. *If G is a chordal graph with a dominating diametral path, then* $\mathbf{rvc}(G) = \text{diam}(G) - 1$ *and the corresponding rainbow vertex colouring can be found in time that is polynomial in the size of G.*

Proof. Let $G = (V, E)$ be a chordal graph with a dominating diametral path. We run a BFS search on one of the endpoints of the diametral path which we will call p_0. We let k denote the number of layers of the BFS tree and let p_i be the vertex of the diametral path in L_i where $0 \leq i \leq k$. To construct a rainbow colouring $c : V \to [k-1]$ for G we assign the colouring for $v \in L_i$:

$$c(v) = \begin{cases} k-1 & \text{if } i = 0 \\ 1 & \text{if } i = k \\ 1 & \text{if } \ i = 2 \text{ and for some } v_2 \in (N(v) \cap L_2) : \text{dist}(v_2, p_k) = k \\ i & \text{otherwise.} \end{cases}$$

The combination of being chordal and having a diametral path gives the graph a lot properties which will come in use for this proof. In particular, once a BFS is performed on a chordal graph with a dominating diametral path, we will see that some special edges must exist within a layer, otherwise we would have long induced cycles. We will formulate some of these properties before delving into the details of the proof as they will be of great use.

Claim 3.13. *If we have a path $P_3 = abc$ such that $a, c \in L_i$ and $b \in L_{i+1}$ then the edge (a, c) must be in $E(G)$.*

Proof. There is a direct path to both a and c from p_0. These paths in combination with the P_3 form a larger cycle. We denote $G[L_0 \cup ... \cup L_{i-1} \cup \{a, c\}]$ with G'. The shortest path between a to c within G' we denote P. If $|E(P)| = 1$ then P is just (a, c), therefore let us assume $|E(P)| > 1$. For $|E(P)| = 3$ we denote P as the path $axyc$ where $x, y \in G'$. Neither (a, y) nor (x, c) are in $E(G)$ as this would contradict our assumption about P being the shortest path. We are left with the $C_5 = abcyx$ and by Observation 3.1 we know neither $(x, b), (y, b)$ are edges in $E(G)$, thus the only possible chord (a, c) still leaves us with the induced C_4: $acyx$. This cycle will be greater the longer P is, therefore $|E(P)| \leq 2$. If $|E(P)| = 2$ the internal node of P cannot be connected to b and thus (a, c) must be an edge in G also in this case. ⌟

A simple observation which becomes apparent from the previous claim is the following.

Observation 3.14. *A vertex $x \in L_i$ in G cannot only be dominated to the right.*

Proof. If x were only dominated by p_{i+1} we end up with the $P_3 = p_i p_{i+1} x$ which according to Claim 3.13 means x must also be dominated in layer. ⌟

This observation is important as it reduces the number of cases we have to examine for the proof. The next claim is quite similar to the previous claim and the proof will be familiar as well.

Claim 3.15. *If we have some $P_4 = abcd$ where $a, d \in L_i$ and $b, c \in L_{i+1}$, then the edge (a, d) must be in $E(G)$.*

Proof. There is a direct path to both a and d from p_0. These paths in conjunction with P_4 will form a larger cycle possibly meeting at p_0. We denote $G[L_0 \cup ... \cup L_{i-1} \cup \{a, d\}]$ with G'. The shortest path between a and d within G' we denote P. This path implies a cycle of length at least 4. Since all internal vertices of P are in layers $L_j, j < i$, none can form chords with b and c. We therefore, for any size of P, observe that either (a, c) or (b, d) must be in $E(G)$. Using this we see that both (a, c) and (b, d) form P_3's: abd and acd respectively, and by Claim 3.13 we therefore know that for either case of P_3 (a, d) must an edge in $E(G)$. ⌟

Claim 3.16. *If a vertex $x \in L_i$, where $x \neq p_i$, has a neighbour $y \in L_{i+1}$ then x must be dominated in layer.*

Proof. Firstly, let us look at the case when $y = p_{i+1}$. This would imply the following $P_3 = p_i p_{i+1} x$, which according to Claim 3.13 means (p_i, x) must be in $E(G)$. When $y \neq p_{i+1}$ we investigate the three possible cases of y. If y is dominated to the left we have the following $P_3 = p_i y x$ and using Claim 3.13 we know x is dominated in layer. If y is dominated only in layer we have the following $P_4 = p_i p_{i+1} y x$ and using Claim 3.15

we know (x, p_i) is in $E(G)$. By Observation 3.14 y can never only be dominated to the right. We can thus conclude that x must be dominated in layer if it has a neighbour in L_{i+1}. ⌟

Another simple observation we will make in terms of our colouring for G is the following.

Observation 3.17. *A vertex $x \in L_2$ with a direct path to p_k will always be coloured 2.*

Proof. For a vertex $x \in L_2$ to be coloured 1 it must have some neighbour $v_2 \in L_2$ such that $\text{dist}(v_2, p_k) = k$, but this can never happen for x as being a neighbour of x would imply $\text{dist}(v_2, p_k) \leq k - 1$. ⌟

Claim 3.18. *There will never exist a direct path between some vertex $x \in L_2$ coloured 1 and some vertex $y \in L_i$, $2 < i \leq k$, where y is only dominated in layer.*

Proof. Let us assume for the sake of contradiction that such a direct path $P = x...v_{i-1}y$ exists, while x is coloured 1. We know that y is dominated in layer while also having some neighbour $v_{i-1} \in L_{i-1}$ where $v_{i-1} \neq p_{i-1}$. We can thus see the formation of the following $P_4 = p_{i-1}p_iyv_{i-1}$ which according to Claim 3.15 necessitates that the edge (v_{i-1}, p_{i-1}) is in $E(G)$. This forms a cycle $p_{i-1}p_iyv_{i-1}$, thus one of the edges (p_{i-1}, y) and (v_{i-1}, p_i) must also be in $E(G)$. According to our assumption y is only dominated in layer, therefore only (v_{i-1}, p_i) can exist in $E(G)$, but this implies a direct path $x...v_{i-1}p_i...p_k$ between x and p_k - which according to Observation 3.17 implies x is never coloured 1, a contradiction. ⌟

As mentioned earlier in this thesis, a general problem in terms of our strategy for rainbow vertex colouring is when the path between two points repeats a layer, either in terms of a zig-zag or by using an edge within a layer. Therefore a nice property in terms of chordal graphs with a dominating diametral path is the following:

Claim 3.19. *If for a pair of vertices $x \in L_i$ and $y \in L_j$ where $j > i$ and $\text{dist}(x, y) = j - i + 1$, then there will always exist some path between x and y such that it repeats layers in L_i.*

Proof. For the sake of contradiction, let us assume there is some path between x and y, such that it only repeats in layer L_l, for some $l > i$. We have the following path: $x...v_{l-1}v_lv'_lv_{l+1}...y$ where $v_l \neq v'_l$ and both vertices belong to layer L_l. We know that v'_l must have some neighbour in L_{l-1}. If this neighbour is p_{l-1} we have, by using Claim 3.16 on x, the following path: $xp_i...p_{l-1}v'_l...y$. Thus, let us assume v'_l has some other neighbour v'_{l-1} in L_{l-1}. Notice we now have the $P_4 = v_{l-1}v_lv'_lv'_{l-1}$ which, according to Claim 3.15, necessitates that the edge (v_{l-1}, v'_{l-1}) is in the graph. We can now use the path $x...v_{l-1}v'_{l-1}v'_l...y$ between x and y, shifting the repetition one layer to the left. This pattern will repeat for v'_{l-1}, where it can either be dominated by p_{l-2} or have some other neighbour in L_{l-2}, and thus shifting the path a further step to the left. Continuing until we are in L_i where we with some v'_i end up with the following path: $xv'_i...y$, proving there will always exist some path such that the repetition of layers happens in L_i. ⌟

To argue the correctness of the colouring let us look at a pair of arbitrary vertices $u \in L_i$ and $v \in L_j$. Firstly we consider the case when $i = j$. By Claim 3.6 we know there will be a rainbow path for all cases of G except for when $\text{diam}(G) = 3$. For this instance there are three possibilities for i and j.

1. $i = j = 1$. The path up_0v will always exist and is rainbow.
2. $i = j = 2$. Because of Observation 3.14 we know u and v must be dominated by either p_1, p_2 or both. If they are dominated by the same vertex there is a rainbow path, while if they are dominated by different vertices either path up_1p_2v and up_2p_1v is rainbow.
3. $i = j = 3$. The vertices are either dominated by p_2, p_3 or both. If they are dominated by the same vertex there is a rainbow path, and if they are dominated by different vertices either path up_2p_3v and up_3p_2v is rainbow.

We will now examine all cases of u and v such that $0 \leq i < j \leq k$. Firstly we can observe by Claim 3.5 that for cases where $i \geq 3$ we can always use the dominating path. We will look at the cases where the dominating path cannot be arbitrarily used, i.e. when $i < 3$ more closely.

Case 1. $i = 0$.
The only case we cannot use the dominating path is if $j = k$ and v is only dominated in layer. We know by Observation 3.4 that p_0 must have a direct path to every vertex including v. Because of Claim 3.18 we know this path will never intersect some vertex coloured 1 in L_2 and thus implies the path is rainbow.

Case 2. $i = 1$.
If u is dominated to the right the diametral path will be rainbow for every case of v. This leaves us two cases of u.

1. u is dominated in layer. The only case of v we have to look at is when $j = k$ while v is only dominated in layer. There are two subcases we have to examine for this case.

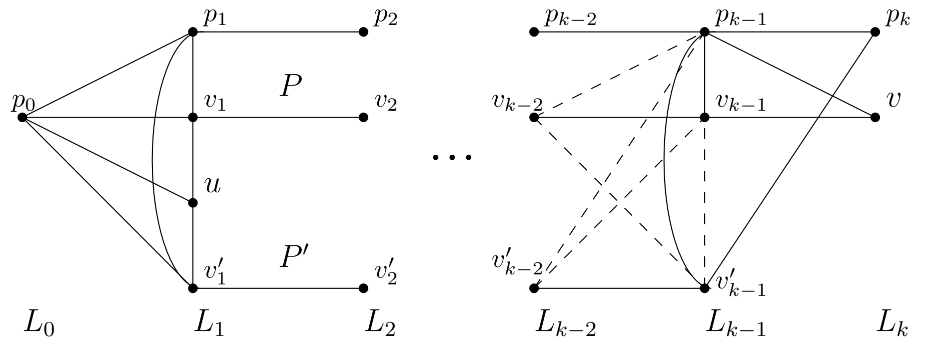

Figure 3.4: If $(v_{k-2}, v'_{k-1}) \in E(G)$ then v_2 is never coloured 1, while if $(v'_{k-2}, v_{k-1}) \in E(G)$ then u has a rainbow coloured path to v: $uv'_1v'_2...v'_{k-2}v_{k-1}v$. Similarly if $(v_{k-2}, p_{k-1}) \in E(G)$ then v_2 is never coloured 1, while if $(v'_{k-2}, p_{k-1}) \in E(G)$ then u has a rainbow path to v: $uv'_1v_2...v'_{k-2}p_{k-1}v$. Thus (v_{k-1}, v'_{k-1}) must be an edge in the graph.

(a) $\text{dist}(u, v) = k - 1$. Since the path between u and v is direct and v is only dominated in layer we know by Claim 3.18 that no internal vertex is coloured 1 and thus the path is rainbow.

(b) $\text{dist}(u, v) = k$. Using Claim 3.19 we know there will be some path repeating in L_1 and from there go directly to v. Since the path from L_1 to v is direct we can again use Claim 3.18 and thus we know that all internal vertices are distinctly coloured.

2. u is only dominated by p_0. For this case in both instances when $j = k - 1$ and $j = k$ we cannot arbitrarily use the diametral path.

(a) $j = k-1$. If v is dominated to the left we can use the diametral path, thus let us assume v is only dominated in layer. From p_0 there must exist some direct path to v. We can thus go from u to p_0 and then traverse the direct path to v. By Claim 3.18, since v is dominated in layer, the vertex in L_2 of the direct path will never be coloured 1.

(b) $j = k$. First we observe that the combination of u not being dominated in layer and Claim 3.16 means that $\text{dist}(u, v) = k$. By Claim 3.19 we therefore know there must exist some path $P = uv_1v_2...v_{k-1}v$, where $v_1 \neq p_1$. If v is only dominated in layer, Claim 3.18 states that v_2 must be coloured 2 and thus P is rainbow. Therefore let us assume v is dominated to the left. As argued before, the fact that u is not dominated in layer and Claim 3.16, imply that by Claim 3.19 we can identify that the following path $P' = uv'_1v'_2...v'_{k-1}p_k$ also must be in G. By Observation 3.17 v'_2 is not coloured 1 as this vertex has a direct path to p_k. If for some $i < k$, $v'_i = p_i$ then we have the following rainbow path: $uv'_1...v'_{i-1}p_i...p_{k-1}v$, thus let us assume for all $i < k$, $v'_i \neq p_i$.

Using Claim 3.16 we know both v'_{k-1} and v_{k-1} must be dominated in layer. With P and P' we can as a result see the formation of a large cycle starting at u and meeting at p_{k-1}. In particular in L_{k-1} we have a P_3 on the vetices $v'_{k-1}p_{k-1}v_{k-1}$. We let P'' denote the shortest path between v_{k-1} and v'_{k-1} in $G' = G[\{u\} \cup \{v_1, v'_1\} \cup ... \cup \{v_{k-2}, v'_{k-2}\} \cup \{v_{k-1}, v'_{k-1}\}]$. If $|E(P'')| = 2$ then either v_{k-2} or v'_{k-2} is an internal node of P''. If v_{k-2} is the internal node this means the edge (v_{k-2}, v'_{k-1}) is in $E(G)$, which means there is a direct path from v_2 to p_k and therefore v_2 would be coloured 2, making P a rainbow path. We therefore check for when v'_{k-2} is the internal node. This means the edge (v'_{k-2}, v_{k-1}) is in $E(G)$, but this results in the rainbow path: $uv'_2...v'_{k-2}v_{k-1}v$. Since $|E(P'')| = 2$ does not work we check for when $|E(P'')| = 3$ in G'. This means that the edge (v_{k-2}, v'_{k-2}) is in $E(G)$. We already established earlier that neither edge (v'_{k-2}, v_{k-1}) or (v_{k-2}, v'_{k-1}) can be in $E(G)$. If (v_{k-2}, p_{k-1}) is in the graph then, again, v_2 will have a direct path to p_k and if (v'_{k-2}, p_{k-1}) there is a rainbow path $uv'_1...v'_{k-2}p_{k-1}v$ between u and v. As we currently have a 5-cycle $v_{k-2}v_{k-1}p_{k-1}v'_{k-1}v_{k-2}$ and no other edges can be added to G' in these layers we see that for $|E(P'')| > 3$ this cycle will only grow larger. Therefore $|E(P'')| = 1$ and by applying Claim 3.15 on the $P_4 = v_{k-2}v_{k-1}v'_{k-1}v'_{k-2}$ we conclude (v_{k-2}, v'_{k-2}) is an edge in G. However as argued above, (v_{k-2}, v'_{k-1}) and (v'_{k-2}, v_{k-1}) are not in G. We then have an induced cycle of length four in G, a contradiction since G is chordal. All the potential edges in the cycle is shown in Figure 3.4.

Case 3. $i = 2$.
If u is dominated to the right or in layer the dominating path will always be rainbow, therefore let us assume u is only dominated to the left. For $j \leq k-1$ the dominating path is rainbow. The only case the dominating path is not rainbow is if $j = k$ while v is only dominated in layer. Here we must use some alternative path.

1. dist$(u, v) = k-2$. As u is only dominated to the left Claim 3.16 states u cannot have a neighbour in L_3 and therefore u can never have a direct path to v. Thus this case will never happen.

2. dist$(u, v) = k-1$. Because of Claim 3.19 the path will repeat layers in L_2 and so it does not matter if the first internal node is coloured 1 or 2 as the rest of the internal path will be coloured $3...k-1$, which is rainbow either way.

3. dist$(u, v) = k$. For most cases until now there has been no need to discuss when the path zig-zags as the distance between the vertices has not allowed for this to

happen. In this case however this is something we have to be cautious about. Fortunately, it turns out the path can only zig-zag in L_2. This can be seen using the contrapositive of Claim 3.16 which states u cannot have a neighbour in L_3 as this would mean it being dominated in layer. Therefore any path between u and v, of length k, must either go to L_1 and from there directly to v, or repeat layers in L_2 and from there go to v, repeating a layer again somewhere along this path. In the first case, the zig-zag path will be of the following structure: $uv_1v_2...v_{k-1}v$. Using Claim 3.18 we know v_2 will not be coloured 1 and thus the path is rainbow.

Since we, in the previous paragraph, established a zig-zag in L_2 is rainbow we now consider the case when no such zig-zagging path exists between u and v. This means there must be some neighbour v_2 of u in L_2 such that $\text{dist}(v_2, v) = k - 1$. Using Claim 3.19 we know there is some path $v_2v_2'...v$. We thus have the following path to v: $uv_2v_2'...v$. The colouring of the path is dependent on these two cases.

(a) $\text{dist}(u, p_k) = k$. In this case all neighbours of u in L_2 are coloured 1, thus v_2 would be coloured 1. By Claim 3.18 v_2' is coloured 2 and thus the rest of the path would be coloured $2, ..., k - 1$, which means the path is rainbow.

(b) $\text{dist}(u, p_k) < k$. This case implies there is some neighbour v_2 of u with a direct path to p_k. Since v is dominated in layer we have the following path: $uv_2...p_kv$ which is also rainbow. Indeed, p_k is coloured 1 and by Claim 3.18 v_2 is coloured 2.

This proves c is indeed a rainbow colouring for G with $\text{diam}(G) - 1$ colours. Finding the root vertex of the dominating diametral path is a polynomial time operation while the BFS takes linear time. Checking if nodes in L_2 must be coloured 1 is done by checking for each vertex in L_2 their distance to p_k. This is a $\mathcal{O}(n^2)$ time operation, thus when summarizing all these times together we end up with a polynomial time algorithm. □

A subclass of the chordal graphs with a dominating diametral path is that of interval graphs for which we, in the following chapter, will achieve an even stronger result.

3.4 Interval graphs

In the previous section we presented a proof which served as a direct improvement on an earlier result concerning interval graphs as it dealt with a graph class in which interval graphs is a subclass. For this section we will expand on the theme of interval graphs as we prove that also SRVC can be efficiently solved on interval graphs. Only an algorithm computing an optimal rainbow verex colouring was known [8]. For this result we first have to go through some of the notation related to interval models I. An interval model I consists of n intervals on a real line, each interval corresponding to some vertex of the graph its representing. Two adjacent vertices in G must have a non-empty intersection in I. We denote by I_v the interval corresponding to v in G. The left endpoint of some interval I_v is denoted $\ell(I_v)$, while the right one is denoted $r(I_v)$.

Theorem 3.20. *If G is an interval graph, then* $\mathbf{srvc}(G) = \mathrm{diam}(G) - 1$ *and the corresponding strong rainbow vertex colouring can be found in time that is linear in the size of G.*

Proof. Let $G = (V, E)$ be an interval graph, I be an interval model for G and p_0 be the vertex corresponding to the interval in I such that $r(p_0) \leq r(v)$ for all $v \in I$. We run a BFS with p_0 as the root and let k be the number of layers in the resulting BFS tree, excluding p_0. We will first establish some important properties of our graph which will prove useful as we continue.

Claim 3.21. *For each layer L_i of G, where $0 \leq i < k$, there exists a vertex $x \in L_i$ such that $L_{i+1} \subset N(x)$.*

Proof. Let x be the vertex in L_i with the rightmost right endpoint, i.e. $r(I_x) \geq r(I_v)$ for all $v \in L_i$. We will argue that $L_{i+1} \subset N(x)$. For the sake of contradiction let us assume there is some vertex $u \in L_{i+1}$ which is not a neighbour of x. Since u is in L_{i+1} its interval must intersect some interval in L_i. Therefore we assume I_u intersects some I_y such that $y \in L_i$ while $y \neq x$. Since $I_x \cap I_u = \emptyset$ this means that I_u must either be completely left or completely right of I_x. If we assume its completely left, then since I_x must intersect some interval in L_{i-1} this would imply I_u intersecting some interval in L_l, where $l \leq i - 1$. Therefore I_u is completely right of I_x. Since I_u intersects I_y this would imply that $r(I_y) > r(I_x)$, contradicting our assumption about I_x having the rightmost right endpoint in L_i. ⌟

We will from this point on denote let a_i denote the vertex in layer L_i which is connected to all vertices in L_{i+1}.

Claim 3.22. *L_1 is a clique.*

Proof. All intervals corresponding to the vertices in L_1 intersect the interval corresponding to p_0. This implies that the left endpoints of all intervals in L_1 starts before $r(I_{p_0})$. This combined with our knowing that p_0 is the interval finishing first in G, meaning all intervals in L_1 has its right endpoint further to the right than I_{p_0}, we can see that all intervals in L_1 must intersect in $r(I_{p_0})$ thus proving that L_1 is a clique. ⌟

To construct a strong rainbow vertex colouring $c : V \to [k-1]$ for G we assign the colouring for $v \in L_i$:

$$c(v) = \begin{cases} i & \text{if } 1 \leq i \leq k-1 \\ \text{arbitrary colour from } [k-1] & \text{otherwise.} \end{cases}$$

To see that this colouring is correct we distinguish between all three possible cases of shortest paths between to vertices $u \in L_i$ and $v \in L_j$. We begin by noting that because of Claim 3.21 $\text{dist}(u, v) \leq j - i + 2$ since there will always exist the path: $ua_{i-1}...a_{j-1}v$. We must also point out that in the case where $i = j$, because of Claim 3.21, there is a vertex a_{i-1} in L_{i-1} and therefore $\text{dist}(u, v) \leq 2$. For the remainder of the proof we will look at cases when $0 \leq i < j \leq k$.

Case 1. $\text{dist}(u, v) = j - i$.
There exists a direct path between u and v.

Case 2. $\text{dist}(u, v) = j - i + 2$.
Using Claim 3.21 we know there will always exist some vertex a_{i-1} which is connected to u. We thus traverse the path $ua_{i-1}a_i...v$, which will be rainbow rainbow in all cases unless $a_{i-1} = p_0$. Fortunately, when $u \in L_1$ there will always exist a shorter path between u and v. This can be observed using Claim 3.22 which states that L_1 is a clique, thereby meaning u and a_1 are neighbours and revealing the following rainbow path: $ua_1...v$.

Case 3. $\text{dist}(u, v) = j - i + 1$.
In this case a shortest path will repeat layers once. If there exists a shortest path repeating layers in L_i then the colouring is correct, so let us assume this is not the case. In particular, we know that in this case $ua_i \notin E(G)$. With these precautions for there to exist a path such that $\text{dist}(u, v) = j - i + 1$, I_u has to intersect some interval $I_x \in L_{i+1}$. We know that I_u does not intersect I_{a_i} and this means that I_u has to end before I_{a_i} starts since I_{a_i} is the interval that finishes last in L_i. Since I_u finishes before I_{a_i} the entire interval of u is contained within $I_{a_{i-1}}$ as I_{a_i} also intersects with $I_{a_{i-1}}$, and u cannot start before $I_{a_{i-1}}$ or else it would belong to L_{i-1}. This means that I_x also intersects with $I_{a_{i-1}}$ which implies that $x \in L_i$, contradicting our assumption about x being in L_{i+1}. We can thus conclude that if $ua_i \notin E(G)$ then there are no edges from u to a vertex in L_{i+1}, meaning any path between u and v of $\text{dist}(u, v) = j - i + 1$ must repeat layers in L_i.

Finding the root vertex using the interval model is constant time, while the BFS is linear therefore adding these two together results in a linear time algorithm. We can thus conclude that in the case where G is an interval graph $\mathbf{srvc}(G) = \text{diam}(G) - 1$. □

3.5 Claw-free diametral path graphs

The final section of this chapter concerns claw-free graphs that have a dominating diametral path. This proof is an improvement on an earlier result concerning unit interval graphs, although for that graph class there already exists a proof for the stronger variant of rainbow vertex colouring [8]. This result however serves more as another argument for the correctness of the Conjecture 3.1 which introduced this entire chapter. As a reminder, what the conjecture argued was that for some diametral path graph G the rainbow connection number would be equal to $\text{diam}(G) - 1$. The results we have presented in this thesis have all been on some variant of graphs which contain a dominating diametral path. Recall that what separates a diametral path graph and a graph containing a dominating diametral path is that for diametral path graphs all connected induced subgraphs have a dominating diametral path. This means that diametral path graphs likely are slightly denser than their counterpart. Not allowing induced claws is another way of making some graph a little more dense. Thus we see this result on claw-free graphs as an argument in favor of the conjecture. For this final proof we show that for graphs with a dominating diametral path while being claw-free, a colouring using $\text{diam}(G) - 1$ colours is achievable in polynomial time.

Theorem 3.23. *If G is a claw-free graph with a dominating diametral path, then $\mathbf{rvc}(G) = \text{diam}(G) - 1$ and the corresponding rainbow vertex colouring can be found in time that is polyonmial in the size of G.*

Proof. Let $G = (V, E)$ be a claw-free diametral path graph. We run a BFS on one of the ends of the diametral path. We denote this vertex p_0 and we let k denote the number of layers of the BFS-tree excluding p_0. The vertex of the diametral path in each layer we denote p_i, $1 \leq i \leq k$. To construct a colouring $c : V \to [k-1]$ on G for $v \in L_i$ we assign the colouring:

$$c(v) = \begin{cases} k-1 & \text{if } i = 0 \\ 1 & \text{if } i = k \\ 1 & \text{if } N(v) \cap L_{i+1} = \emptyset \\ i & \text{otherwise.} \end{cases}$$

Before arguing the correctness of our colouring we will give some observations and claims on our graph to simplify our proof.

Observation 3.24. *A direct path between vertices $x \in L_i$ and $y \in L_j$, where $j > i$, cannot intersect some vertex $z \in L_l$ coloured 1, $i \leq l < j$.*

Proof. Since z has no neighbour in L_{l+1} there is no direct path from z to v and thus the direct path between x and y cannot intersect z on the way, as this would imply the path either repeating or zig-zagging in L_l. ⌟

Claim 3.25. *If a vertex $x \in L_i$, with $i > 1$, is dominated to the left, then x is also dominated in layer.*

Proof. For the sake of contradiction let us assume $x \in L_i$, where $i > 1$, is only dominated to the left. This implies a structure $\{p_{i-1}xp_ip_{i-2}\}$ inducing a $K_{1,3}$ in G. Out of the three possible edges $(p_{i-2}, p_i), (p_{i-2}, x), (p_i, x)$ we know by Observation 3.1 that the only edge which can exist to prevent there being an induced claw in G is (p_i, x) - meaning x must also be dominated in layer. ⌟

To argue the correctness of the colouring we will look at a pair of arbitrary vertices $u \in L_i$ and $v \in L_j$. We start by looking at the case where $i = j$. By Claim 3.6 we know that for all cases except for when $\text{diam}(G) = 3$ there will be a rainbow path between u and v. There are three possible cases for i and j:

1. $i = j = 1$. Both u and v are connected to p_0, leading to the rainbow path up_0v.
2. $i = j = 2$. Claim 3.25 states that neither u nor v can only be dominated to the left, which means they are either dominated by p_2, p_3 or both. If the vertices are dominated by the same vertex there is a rainbow path, and for the case they are dominated by different vertices either path up_2p_3v and up_3p_2v is rainbow.
3. $i = j = 3$. Since u and v are in layer L_3 they cannot be dominated to the right. Claim 3.25 tells us u and v cannot only be dominated left so we therefore know that both u and v must be dominated in layer. We, as a result, have the rainbow path up_3v.

For the rest of the proof we will assume $0 \leq i < j \leq k$. For cases when $i \geq 3$ we know by Claim 3.5 that there exists a rainbow path between u and v. When $i = 0$ we know by Observation 3.4 that there must always be a direct path to v, and this path will never intersect some vertex coloured 1 in L_2 as by Observation 3.24 this would imply the path not being direct. When $i = 2$ we know from Claim 3.25 that u must be dominated in layer and thus for all cases of j the diametral path will have a rainbow path to v. We are thus left with only one case to examine, which is when $i = 1$ while u is not dominated to the right.

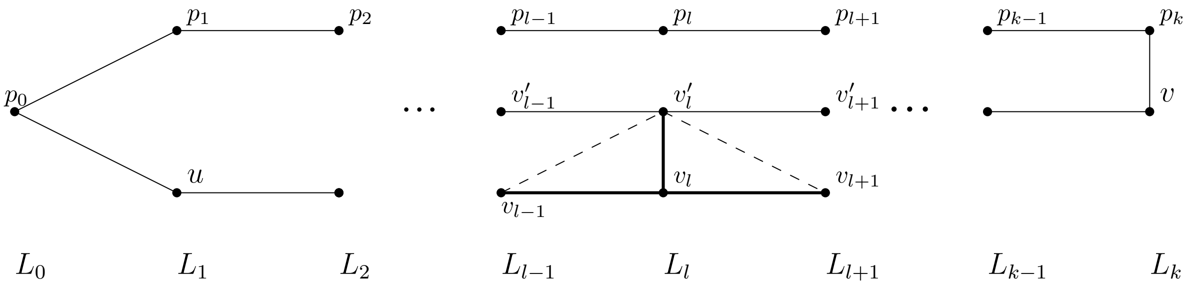

Figure 3.5: Scenario where the path repeats in layer L_l. If v_l has no neighbours in L_{l+1} it is coloured 1, otherwise one of two possible edges (dashed lines) must be in the graph.

1. $j = k - 1$. Consider the path using the edge (u, p_0) and then a direct path between p_0 and v. The direct path between p_0 and v will by Observation 3.24 never intersect a vertex coloured 1 in L_2 and thus the path is rainbow.

2. $j = k$. If $\text{dist}(u, v) = k - 1$ then the path goes directly to v and by Observation 3.24 this path will not intersect some vertex coloured 1, thus it is rainbow. If $\text{dist}(u, v) = k$ the path must repeat layers at some point. If the repetition of layers happens in L_i or L_j the path will be rainbow. Therefore let us assume this repetition happens in some layer L_l where $i < l < j$. We have the following path: $P = u...v_{l-1}v_lv'_lv'_{l+1}...v$. If v_l has no neighbour in L_{l+1} it will be coloured 1, making P rainbow, thus let us assume P has some neighbour $v_{l+1} \in L_{l+1}$. If $(v_{l-1}, v'_l) \notin E(G)$ and $(v'_l, v_{l+1}) \notin E(G)$ then $\{v_lv_{l-1}v'_lv_{l+1}\}$ induces a claw in G. As a result, either (v_{l-1}, v'_l) or (v'_l, v_{l+1}) must be an edge in $E(G)$. The former case implies a direct path $u...v_{l-1}v'_lv'_{l+1}...v$ between u and v, thus let us assume only $(v'_l, v_{l+1}) \in E(G)$. Since v'_l must have some neighbour v'_{l-1} in L_{i-1} another claw becomes apparent: $\{v'_lv'_{l-1}v'_{l+1}v_{l+1}\}$. In a similar argument to the proof of Claim 3.25 we have three possible edges to prevent an induced claw, but only one, as seen by Observation 3.1 is allowed within the BFS-structure — thus (v_{l+1}, v'_{l+1}) must be an edge within $E(G)$. The inclusion of this edge means the repetition of layers can be shifted one step further to the right, to L_{i+1}, with the following path: $u...v_{l-1}v_lv_{l+1}v'_{l+1}...v$. For this new path either v_{l+1} has a neighbour in L_{l+2}, which means the repetition can be shifted a further step to the right, the argument being identical to the one presented just now, or v_{l+1} has no neighbour in L_{i+2} meaning its coloured 1, making this new path rainbow. If the path repeats in L_k, in the scenario where the repetition has shifted all the way to the right, the path will also be rainbow. This scenario where the path between u and v repeats in some layer L_l is shown in Figure 3.5.

This proves that c is a rainbow colouring for G using $\text{diam}(G) - 1$ colours. It is simple to see that our colouring algorithm only takes polynomial time. The BFS-search is linear, while finding vertices which are not dominated by the following layer is also linear. This added with the time it takes to find the dominating diametral path results in a polynomial time algorithm. □

Chapter 4

Conclusion

In this thesis we have given efficient algorithms for optimally colouring multiple subclasses of graphs with dominating diametral paths. We have shown that for a graph G with a dominating diametral path, and one of the following properties: chordal, claw-free or bipartite, we can find a colouring in polynomial time using $\mathrm{diam}(G) - 1$ colours. But the status of the of the conjecture of Heggernes et al. [8] still remains an open problem.

Conjecture 4.1. *Let G be a diametral path graph. Then* $\mathbf{rvc}(G) = \mathrm{diam}(G) - 1$.

We believe the conjecture is true because of the property diametral path graphs have where *every induced subgraph* contains a dominating diametral path. This seems to make for a denser graph class and as we have seen for the claw-free and chordal graphs the dominating path in conjunction with some property making for a more dense graph has been beneficial in terms of finding a rainbow vertex colouring using $\mathrm{diam}(G) - 1$ colours. However the conjecture is not true for a slightly bigger graph class. We showed an example of a graph with a dominating diametral path where a colouring using $\mathrm{diam}(G) - 1$ was not possible. In fact, we think computing RVC in this graph class is an NP-complete problem, leading us to make the following conjecture.

Conjecture 4.2. Rainbow vertex colouring *is* NP*-complete for graphs containing a dominating diametral path.*

Since we also proved that for a graph G with a dominating diametral path $\mathbf{rvc}(G) \leq \mathrm{diam}(G)$, what we are essentially arguing in this conjecture is that deciding whether G can be coloured using $\mathrm{diam}(G)$ or $\mathrm{diam}(G) - 1$ colours is an NP-complete problem.

Finally, we showed that for an interval graph G, $\mathbf{srvc}(G) = \mathrm{diam}(G) - 1$. It would be interesting to investigate whether the same holds for chordal diametral path graphs which generalises interval graphs, although the question is interesting for the other graph classes we have examined in this thesis as well.

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