Transferring model structures to operads

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Contents

1 Introduction

This thesis will be an extended (though not complete) exposition of the material in the article "Axiomatic homotopy theory for operads", by Berger and Moerdijk [2]. We will assume the reader is familiar with the theory of monoidal and model categories, but we will not assume any previous knowledge of operads. By operad we mean symmetric operad unless stated otherwise, and we give the definition in section 2. There we will also establish some conventions which will be used thoughout the thesis. Section 3 will be about the construction of the monad associated to an operad, and using it to define algebras over operads. Section 4 will deal with the transfer of model structure to the category of operads. In section 5 we will construct the functor from collections to operads. Section 6 will consider the relationship between symmetric and braided operads.

2 Definitions and conventions

We begin by establishing some conventions. The symbol $\mathcal{E}$ will be used to refer to some closed symmetric monoidal category which is complete and cocomplete. That $\mathcal{E}$ is closed means that there is for each object X of $\mathcal{E}$ a right adjoint to the functor $-\otimes X$, denoted by $(-)^X$. This means, among other things, that colimits commute with tensor products in the following sense; if $\mathcal{D}_1, \ldots, \mathcal{D}_n$ is a collection of small categories, and $F_1 : \mathcal{D}_1 \to \mathcal{E}, \ldots, F_n : \mathcal{D}_n \to \mathcal{E}$ is a collection of functors, then we have an isomorphism

$$\mathrm{colim}_{\mathcal{D}_1} F_1 \otimes \cdots \otimes \mathrm{colim}_{\mathcal{D}_n} F_n \cong \mathrm{colim}_{\mathcal{D}_1 \times \cdots \times \mathcal{D}_n} F_1 \otimes \cdots \otimes F_n,$$

where $F_1 \otimes \cdots \otimes F_n : \mathcal{D}_1 \otimes \cdots \otimes \mathcal{D}_n \to \mathcal{E}$ denotes the functor defined by

$$(X_1, \ldots, X_n) \mapsto F_1(X_1) \otimes \cdots \otimes F_n(X_n),$$

for $(X_1, \ldots, X_n)$ some object of $\mathcal{D}_1 \times \cdots \times \mathcal{D}_n$, and

$$(f_1, \ldots, f_n) \mapsto F_1(f_1) \otimes \cdots \otimes F_n(f_n),$$

for $(f_1, \ldots, f_n) : (X_1, \ldots, X_n) \to (Y_1, \ldots, Y_n)$. This is a fact we will use frequently throughout the thesis, and when we do we may sometimes omit mentioning it. We will also often omit mentioning the domain of a functor when taking its colimit, e.g. we will generally write $\mathrm{colim} F_1$ instead of $\mathrm{colim}_{\mathcal{D}_1} F_1$. Sometimes the monoidal structure of $\mathcal{E}$ will be the cartesian product, in this case we keep writing $\otimes$ for the monoidal product.

Given some group G, which we may consider as a category with a single object denoted by $*$, and some object X of $\mathcal{E}$, a left G-action on X is a functor $G \to \mathcal{E}$ with $* \mapsto X$, and we usually denote the morphism $g \in G$ is sent to by g_*. By a right G-action, we shall mean a similar functor $G^{\mathrm{op}} \to \mathcal{E}$, which sends $g^{\mathrm{op}} \in G^{\mathrm{op}}$ to g^*. When talking about a right action of G on X we will usually not consider the opposite category of G but instead consider it as a contravariant functor sending $g \in G$ to g^*. If X and Y are some objects of $\mathcal{E}$ with a right and a left G-action, respectively, we will mean by $X \otimes_G Y$ the coend of the functor

$$G^{\mathrm{op}} \times G \to \mathcal{E},$$

defined by $* \mapsto X \otimes Y$, $(g, h) \mapsto g^* \otimes h_*$. Since G is a group the elements are invertible, and the coend is the same as the colimit of the functor

$$G^{\mathrm{op}} \to \mathcal{E},$$

the diagonal right action on $X \otimes Y$, defined by $* \mapsto X \otimes Y$ and $g \mapsto g^* \otimes g^*$, where the action in the second factor is defined by $g^* = (g^{-1})_* = (g_*)^{-1}$. This is the same as the colimit of the functor

$$G \to \mathcal{E},$$

where we instead invert the action in the first factor.

We denote the category of objects of $\mathcal{E}$ with a right G-action by $\mathcal{E}^G$, it is the category of functors from G^{op} to $\mathcal{E}$. In particular we can let G run over the symmetric groups, Σ_n for $n \geq 0$, and define the category of collections:

$$\mathrm{Coll}(\mathcal{E}) = \prod_{n \geq 0} \mathcal{E}^{\Sigma_n}.$$

Thus an object of this category, called a collection, is a sequence of objects of $\mathcal{E}$, $\mathcal{K} = (\mathcal{K}(n))_{n \geq 0}$, with a Σ_n action in degree n. A morphism of collections is a sequence of morphisms of $\mathcal{E}$, with the one in degree n being Σ_n-equivariant.

The following definition is taken from [12]. We remark that in the cases one takes a tensor product of zero things, it should be interpreted as being the monoidal unit, I, and that the composition map $P(0) \otimes I \to P(0)$ should be the canonical isomorphism.

Definition 1. An operad in $\mathcal{E}$ is a collection, P, in $\mathcal{E}$, together with a unit map, $e : I \to P(1)$, and for each collection of numbers $n, k_1, \ldots, k_n$ a composition map

$$\gamma : P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) \to P(\Sigma_i k_i),$$

which is associative, unital and equivariant in the following ways.

The following associativity diagram should commute for each collection of numbers $n, k_1, \ldots, k_n$ and $i_s^1, \ldots, i_s^{k_s}$, for each $1 \leq s \leq n$, where we have set $k = k_1 + \cdots + k_n$, $i_s = i_s^1 + \cdots + i_s^{k_s}$:

$$\begin{tikzcd}
P(n) \otimes \left(\bigotimes_{s=1}^{n} P(k_s) \right) \otimes \bigotimes_{s=1}^{n} \bigotimes_{t=1}^{k_s} P(i_s^t) \arrow[r, "\gamma \otimes id"] \arrow[dd, "\text{shuffle}"'] & P(k) \otimes \bigotimes_{s=1}^{n} \bigotimes_{t=1}^{k_s} P(i_s^t) \arrow[d, "\gamma"] \\
 & P(i_1 + \cdots + i_n) \\
P(n) \otimes \bigotimes_{s=1}^{n} \left(P(k_s) \otimes \bigotimes_{t=1}^{k_s} P(i_s^t) \right) \arrow[r, "id \otimes \bigotimes_{s=1}^{n} \gamma"] & P(n) \otimes \bigotimes_{s=1}^{n} P(i_s). \arrow[u, "\gamma"']
\end{tikzcd} \tag{1}$$

The following unit diagrams should commute:

$$\begin{tikzcd}
I \otimes P(n) \arrow[r, "\cong"] \arrow[d, "e \otimes id"'] & P(n) \\
P(1) \otimes P(n) \arrow[ur, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
P(n) \otimes I^{\otimes n} \arrow[r, "\cong"] \arrow[d, "id \otimes e^{\otimes n}"'] & P(n) \\
P(n) \otimes P(1)^{\otimes n} \arrow[ur, "\gamma"'] &
\end{tikzcd}
. \tag{2}$$

Let $\sigma \in \Sigma_n$ and $\tau_i \in \Sigma_{k_i}$, for $1 \leq i \leq n$. Let $\sigma(k_1, \ldots, k_n) \in \Sigma_{k_1 + \cdots + k_n}$ denote the permutation that permutes the blocks of $k_1, k_2, \ldots,$ and k_n numbers as σ permutes the numbers from 1 to n, and let $\tau_1 \oplus \cdots \oplus \tau_n \in \Sigma_{k_1 + \cdots + k_n}$ denote the one which permutes the first k_1 numbers as τ_1 does, the next k_2 ones as τ_2 does, etc. Then the following equivariance diagrams should commute:

$$\begin{tikzcd}
P(n) \otimes \bigotimes_{i=1}^{n} P(k_i) \arrow[r, "\sigma^* \otimes \sigma^{-1}"] \arrow[d, "\gamma"'] & P(n) \otimes \bigotimes_{i=1}^{n} P(k_{\sigma(i)}) \arrow[d, "\gamma"] \\
P(k_1 + \cdots + k_n) \arrow[r, "\sigma\left(k_{\sigma(1)}, \ldots, k_{\sigma(n)}\right)^*"] & P(k_{\sigma(1)} + \cdots + k_{\sigma(n)})
\end{tikzcd} \tag{3}$$

and

$$\begin{tikzcd}
P(n) \otimes \bigotimes_{i=1}^{n} P(k_i) \arrow[r, "id \otimes \bigotimes_{i=1}^{n} \tau_i^*"] \arrow[d, "\gamma"'] & P(n) \otimes \bigotimes_{i=1}^{n} P(k_i) \arrow[d, "\gamma"] \\
P(k_1 + \cdots + k_n) \arrow[r, "(\tau_1 \oplus \cdots \oplus \tau_n)^*"] & P(k_1 + \cdots + k_n).
\end{tikzcd} \tag{4}$$

Definition 2. For two operads P and Q, a morphism or map of operads $f : P \to Q$ is a morphism of the underlying collections, $f = (f_n)_{n\geq 0} : (P(n))_{n\geq 0} \to (Q(n))_{n\geq 0}$, making the diagram

$$\begin{tikzcd} & I \arrow[dl, "e"'] \arrow[dr, "e"] & \\ P(1) \arrow[rr, "f_1"] & & Q(1) \end{tikzcd}$$

commute, as well as all diagrams of the form

$$\begin{tikzcd} P(n) \otimes \bigotimes_{i=1}^{n} P(k_i) \arrow[r, "\gamma"] \arrow[d, "f_n \otimes \bigotimes_{i=1}^{n} f_{k_i}"'] & P(k_1 + \cdots + k_n) \arrow[d, "f_{k_1+\cdots+k_n}"] \\ Q(n) \otimes \bigotimes_{i=1}^{n} Q(k_i) \arrow[r, "\gamma"] & Q(k_1 + \cdots + k_n). \end{tikzcd}$$

The operads in $\mathcal{E}$ form a category, denoted by $\mathrm{Oper}(\mathcal{E})$.

3 The algebras over an operad

The article [2] mentions that any operad in $\mathcal{E}$ gives rise to a monad on $\mathcal{E}$, though does not prove this. We therefore write out the details, and afterwards we use this to give a definiton of an algebra over an operad.

Any collection $P = (P(n))_{n\geq 0}$ induces a functor from $\mathcal{E}$ to itself by sending an object X of $\mathcal{E}$ to

$$P(X) = \coprod_{n\geq 0} P(n) \otimes_{\Sigma_n} X^{\otimes n},$$

and sending any morphism $f : X \to Y$ to the morphism induced by the morphisms $id_{P(n)} \otimes f^{\otimes n} : P(n) \otimes X^{\otimes n} \to P(n) \otimes Y^{\otimes n}$. If additionally P has the structure of an operad, the functor P forms a monad with unit defined by the composition

$$X \xrightarrow{\cong} I \otimes X \xrightarrow{e\otimes id_X} P(1) \otimes X \hookrightarrow P(X)\ ,$$

and multiplication defined as follows.

We have isomorphisms

$$
\begin{aligned}
P(P(X)) &= \coprod_{m\geq 0} P(m) \otimes_{\Sigma_m} \Big(\coprod_{n \geq 0} P(n) \otimes_{\Sigma_n} X^{\otimes n} \Big)^{\otimes m} \\
&\xrightarrow{\cong} \coprod_{m \geq 0} P(m) \otimes_{\Sigma_m} \Bigg(\coprod_{n_1 \geq 0, \dots, n_m \geq 0} \otimes \bigotimes_{i=1}^{m} \Big(P(n_i) \otimes_{\Sigma_{n_i}} X^{\otimes n_i} \Big) \Bigg) \\
&\xrightarrow{\cong} \coprod_{m \geq 0} \Bigg(\coprod_{n_1 \geq 0, \dots, n_m \geq 0} P(m) \otimes \bigotimes_{i=1}^{m} \Big(P(n_i) \otimes_{\Sigma_{n_i}} X^{\otimes n_i} \Big) \Bigg) / \Sigma_m
\end{aligned}
$$

so in order to define the multiplication of the monad, we need only define it on each component $P(m) \otimes \bigotimes_{i=1}^{m} \Big(P(n_i) \otimes_{\Sigma_{n_i}} X^{\otimes n_i} \Big)$, and show that it is consistent with the colimits.

We remark that

$$
P(m) \otimes \bigotimes_{i=1}^{m} \Big(P(n_i) \otimes_{\Sigma_{n_i}} X^{\otimes n_i} \Big) \cong \Bigg(P(m) \otimes \bigotimes_{i=1}^{m} \Big(P(n_i) \otimes X^{\otimes n_i} \Big) \Bigg) / \sim,
$$

where the $\sim$ denotes taking the colimit over $\Sigma_{n_1} \times \cdots \times \Sigma_{n_m}$ of the right action $(\sigma_1, \dots, \sigma_m) \mapsto id \otimes \bigotimes_{i=1}^{m} \sigma_i^* \otimes \sigma_i^{-1}$. Thus, we (again) define the map on each component $P(m) \otimes \bigotimes_{i=1}^{m} \big(P(n_i) \otimes X^{\otimes n_i} \big)$ and show that it is consistent with the colimits.

For the rest of this section, take N to mean $n_1 + \cdots + n_m$, and take $\coprod\limits_{n_1, \dots, n_m}$ to mean $\coprod\limits_{n_1 \geq 0, \dots, n_m \geq 0}$. We define the map to be the composition

$$
\begin{aligned}
P(m) \otimes \bigotimes_{i=1}^{m} \Big(P(n_i) \otimes X^{\otimes n_i} \Big) &\xrightarrow{\cong} \Big(P(m) \otimes \bigotimes_{i=1}^{m} P(n_i) \Big) \otimes \bigotimes_{i=1}^{m} X^{\otimes n_i} \\
&\xrightarrow{\gamma \otimes \cong} P(N) \otimes X^{\otimes N}.
\end{aligned}
$$

Given a group element $(\sigma_1, \dots, \sigma_m) \in \Sigma_{n_1} \times \cdots \times \Sigma_{n_m}$, in the following diagram

$$
\begin{tikzcd}
P(m) \otimes \bigotimes\limits_{i=1}^{m} \Big(P(n_i) \otimes X^{\otimes n_i} \Big) \arrow[r, "\cong"] \arrow[dd, "id \otimes \bigotimes\limits_{i=1}^{m} \sigma_i^* \otimes \sigma_i^{-1}"'] & \Big(P(m) \otimes \bigotimes\limits_{i=1}^{m} P(n_i) \Big) \otimes \bigotimes\limits_{i=1}^{m} X^{\otimes n_i} \arrow[dd, "\Big(id \otimes \bigotimes\limits_{i=1}^{m} \sigma_i^* \Big) \otimes \bigotimes\limits_{i=1}^{m} \sigma_i^{-1}"'] \arrow[dr, "\gamma \otimes \cong"] & \\
 & & P(N) \otimes_{\Sigma_N} X^{\otimes N} \\
P(m) \otimes \bigotimes\limits_{i=1}^{m} \Big(P(n_i) \otimes X^{\otimes n_i} \Big) \arrow[r, "\cong"] & \Big(P(m) \otimes \bigotimes\limits_{i=1}^{m} P(n_i) \Big) \otimes \bigotimes\limits_{i=1}^{m} X^{\otimes n_i} \arrow[ur, "\gamma \otimes \cong"'] &
\end{tikzcd}
$$

the left square commutes by Mac Lane's coherence theorem [9], and the right triangle is seen to commute by expanding it to the following diagram

$$\begin{tikzcd}
\left(P(n) \otimes \bigotimes_{i=1}^{m} P(n_i)\right) \otimes \bigotimes_{i=1}^{m} X^{\otimes n_i} \arrow[r, "\gamma\otimes\cong"] \arrow[d, "\left(id\otimes\bigotimes_{i=1}^{m}\sigma_i^*\right)\otimes\bigotimes_{i=1}^{m}\sigma_i^{-1}"'] & P(N) \otimes X^{\otimes N} \arrow[dr, "proj."] \arrow[d, "\left(\bigoplus_{i=1}^{m}\sigma_i\right)^*\otimes\bigotimes_{i=1}^{m}\sigma_i^{-1}"'] & \\
& & P(N) \otimes_{\Sigma_N} X^{\otimes N} \\
\left(P(n) \otimes \bigotimes_{i=1}^{m} P(n_i)\right) \otimes \bigotimes_{i=1}^{m} X^{\otimes n_i} \arrow[r, "\gamma\otimes\cong"] & P(N) \otimes X^{\otimes N} \arrow[ur, "proj."'] &
\end{tikzcd}$$

where the left square commutes due to the axiom (4) for an operad and the canonicity of shuffling isomorphisms, and the triangle commutes due to the definition of the colimit.

Now to move out through parentheses and show that the definition works well with the group action of Σ_m. Suppose we have some element $\sigma \in \Sigma_m$. Then one has a commutative diagram

$$\begin{tikzcd}
P(m) \otimes \left(\coprod_{n\geq 0} P(n) \otimes_{\Sigma_n} X^{\otimes n}\right)^{\otimes m} \arrow[r, "\cong"] & \coprod_{n_1,\ldots,n_m} P(m) \otimes \bigotimes_{i=1}^{m} \left(P(n_i) \otimes_{\Sigma_{n_i}} X^{\otimes n_i}\right) \\
P(m) \otimes \left(\coprod_{n\geq 0} P(n) \otimes_{\Sigma_n} X^{\otimes n}\right)^{\otimes m} \arrow[r, "\cong"] \arrow[u, "\sigma^*\otimes id"] \arrow[d, "id\otimes\sigma"'] & \coprod_{n_1,\ldots,n_m} P(m) \otimes \bigotimes_{i=1}^{m} \left(P(n_i) \otimes_{\Sigma_{n_i}} X^{\otimes n_i}\right) \arrow[u, "\coprod_{n_1,\ldots,n_m} \sigma^*\otimes\bigotimes_{i=1}^{m} id"'] \arrow[d] \\
P(m) \otimes \left(\coprod_{n\geq 0} P(n) \otimes_{\Sigma_n} X^{\otimes n}\right)^{\otimes m} \arrow[r, "\cong"] & \coprod_{n_1,\ldots,n_m} P(m) \otimes \bigotimes_{i=1}^{m} \left(P(n_i) \otimes_{\Sigma_{n_i}} X^{\otimes n_i}\right),
\end{tikzcd}$$

where the bottom right vertical map is the one which sends the factor indexed by $(n_1, \ldots, , n_m)$ to the factor indexed by $(n_{\sigma^{-1}(1)}, \ldots, n_{\sigma^{-1}(m)})$, and with each component being $id \otimes \sigma$. Alternatively, it is the map which is induced by $in_{(n_{\sigma^{-1}(1)},\ldots,n_{\sigma^{-1}(m)})} \circ (id \otimes \sigma)$ on the factor indexed by $(n_1, \ldots, n_m)$. Thus, to see that we have a well defined map we need only show that

$$
\begin{tikzcd}
\coprod_{n_1,\dots,n_m} P(m) \otimes \bigotimes_{i=1}^{m} \left(P(n_i) \otimes_{\Sigma_{n_i}} X^{\otimes n_i}\right) \arrow[dr, "\coprod \gamma \circ \cong"] & \\
\coprod_{n_1,\dots,n_m} P(m) \otimes \bigotimes_{i=1}^{m} \left(P(n_i) \otimes_{\Sigma_{n_i}} X^{\otimes n_i}\right) \arrow[u, "\coprod_{n_1,\dots,n_m} \sigma^* \otimes \bigotimes_{i=1}^{m} id"] \arrow[d] & \coprod_{n_1,\dots,n_m} P(N) \otimes_{\Sigma_N} X^{\otimes N} \\
\coprod_{n_1,\dots,n_m} P(m) \otimes \bigotimes_{i=1}^{m} \left(P(n_i) \otimes_{\Sigma_{n_i}} X^{\otimes n_i}\right) \arrow[ur, "\coprod \gamma \circ \cong"'] &
\end{tikzcd}
$$

commutes, where the unnamed arrow is as before. This is easily seen from the fact that

$$
\begin{tikzcd}
P(m) \otimes \bigotimes_{i=1}^{m} \left(P(n_i) \otimes_{\Sigma_{n_i}} X^{\otimes n_i}\right) \arrow[r, "\sigma^* \otimes id"] \arrow[d, "id \otimes \sigma"] & P(m) \otimes \bigotimes_{i=1}^{m} \left(P(n_i) \otimes_{\Sigma_{n_i}} X^{\otimes n_i}\right) \arrow[d, "\gamma \circ \cong"] \\
P(m) \otimes \bigotimes_{i=1}^{m} \left(P(n_{\sigma^{-1}(i)}) \otimes_{\Sigma_{n_{\sigma^{-1}(i)}}} X^{\otimes n_{\sigma^{-1}(i)}}\right) \arrow[r, "\gamma \circ \cong"] & P(N) \otimes_{\Sigma_N} X^{\otimes N}
\end{tikzcd}
$$

commutes, which it does by the equivariance axiom (3) and the properties of a coend.

For the associativity and unit axioms of the monad, it is easy to see that they are satisfied before quotienting out by the group action, so they hold after doing so.

We introduce now the notion of an algebra over an operad.

Definition 3. Let P be some operad in $\mathcal{E}$. An algebra over P is an algebra over the associated monad. That is, it is an object A of $\mathcal{E}$, together with a morphism $a : P(A) \to A$, making the diagrams

$$
\begin{tikzcd}
P(P(A)) \arrow[r, "P(a)"] \arrow[d, "\mu_A"'] & P(A) \arrow[d, "a"] \\
P(A) \arrow[r, "a"] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "id_A"'] & P(A) \arrow[d, "a"] \\
& A
\end{tikzcd}
$$

commute. These diagrams are called the associativity and unit diagrams for the algebra, respectively (algebras over monads are described in [9, VI.2]).

Note that this definition is different from the usual one (see e.g. [12]), where one defines an algebra over an operad to be a collection of maps $P(n) \otimes A^{\otimes n} \to A$ satisfying some conditions. The two definitions are easily seen to be equivalent.

Example 1. Let P be the commutative operad, $P = (I)_{n \geq 0}$, with unit and composition maps the canonical isomorphisms. An algebra over P is the same as a commutative monoid in $\mathcal{E}$. The map $a : P(A) \to A$ is the same as a sequence of maps $A^n/\Sigma_n \to A$, $n \geq 0$, that is, a sequence of commutative maps $A^n \to A$. The associative diagram ensures that all ways of getting from A^n to A gives the same map, i.e. associativity, while unit diagram implies the unit axioms for the monoid.

The algebras are a large part of the motivation for studying operads, and the possibility of transferring model structures from operads to their algebras are a large part of the motivation for transferring model strucures to operads. We will not do so in this thesis, however.

4 The model structure on operads

In this section we give suficient conditions for there to be a model structure on the category of operads in $\mathcal{E}$. In [2], they transfer model structures from $\mathcal{E}$ to the category of reduced operads, as well as to the category of operads when $\mathcal{E}$ is cartesian. We will, however, only do the latter.

4.1 The transfer principle and path-object argument

We state here the transfer principle, in the form it is stated in [2].

Theorem 1. *Let $\mathcal{E}$ be a cofibrantly generated model category, and let $\mathcal{C}$ be a category which is complete and cocomplete. Let $F : \mathcal{E} \rightleftarrows \mathcal{C} : U$ be an adjunction between the two, with left adjoint F and right adjoint U. Call a morphism f in $\mathcal{C}$ a fibration or a weak equivalence if $U(f)$ is one. Then this defines a cofibrantly generated model structure on $\mathcal{C}$ if the following two conditions are satisfied:*

(i) the functor F preserves small objects;

(ii) any sequential colimit of pushouts of images under F of the generating trivial cofibrations of $\mathcal{E}$ yields a weak equivalence in $\mathcal{C}$.

Proof. It follows quickly from either Theorem 2.1.19 in [7], or from Theorem 11.3.1 in [6]. In the latter case, the details are written out right afterwards in Theorem 11.3.2. □

We recall some notions of model categories that will be of particular importance to us:

Definition 4. For some object X of a model category $\mathcal{E}$, a fibrant replacement of X is a fibrant object $\tilde{X}$ of $\mathcal{E}$ together with a weak equivalence $X \xrightarrow{\sim} \tilde{X}$. A fibrant replacement functor is a functor $(-)\tilde{} : \mathcal{E} \to \mathcal{E}$, sending each object onto a fibrant replacement, with the weak equivalences forming a natural transformation $id_{\mathcal{E}} \stackrel{\cdot}{\longrightarrow} (-)\tilde{}$.

Sometimes, if we have some other (non-model) category $\mathcal{C}$ and a forgetful functor $\mathcal{U} : \mathcal{C} \to \mathcal{E}$, we shall call something in $\mathcal{C}$ a fibrant replacement or fibrant replacement functor if it is one when forgetting down to $\mathcal{E}$.

An example of fibrant replacement functor is the identity functor and identity natural transformation, if all objexts of the model category are fibrant. This is the case for *Top* when given the model structure where the weak equivalences are weak homotopy equivalences, and the fibrations are Serre fibrations [4]. It is also the case if *Top* is given the model structure where the weak equivalences are (ordinary) homotopy equivalences, and where the fibrations are Hurewicz fibrations [14]. For a less trivial example, there is the category of simplicial sets with weak equivalences the maps that are weak homotopy equivalences when taking the geometric realization, and cofibrations the injective maps. In this case many objects are non-fibrant, e.g. all the finite non-discrete simplicial sets. In this example, a fibrant replacement functor is passing to geometric realization and then taking the singular complex, with weak equivalences the obvious inclusions.

Definition 5. For some object X of a model category $\mathcal{E}$, a path-object for X is an object $\mathrm{Path}(X)$ of $\mathcal{E}$, a weak equivalence $X \xrightarrow{\sim} \mathrm{Path}(X)$, and a fibration $\mathrm{Path}(X) \twoheadrightarrow X \times X$, such that the composition of the two morphisms is the diagonal.

As with the case of fibrant replacements, we shall sometimes refer to some object and morphisms in a (non-model) category as a path-object if it is when we forget down to $\mathcal{E}$.

There is a notion of functorial path-object, where $\mathrm{Path} : \mathcal{E} \to \mathcal{E}$ is a functor and the morphisms form natural transformations $id_{\mathcal{E}} \overset{\cdot}{\longrightarrow} \mathrm{Path}$ and $\mathrm{Path} \overset{\cdot}{\longrightarrow} \Delta$ (Δ being the diagonal functor), but it does not seem necessary in order for the path-object argument to work.

We will need the following lemma for our proof of the path-object argument.

Lemma 1. *For any commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F, \end{tikzcd}$$

where the horizontal compositions are isomorphisms, if the middle vertical morphism is an isomorphism, then so are the left and right vertical morphisms.

Proof. Denote the morphism from A to B by AB, the one from B to C by BC, etc. Denote $(BC \circ AB)^{-1} \circ BC \circ BE^{-1} \circ DE$ by X and $BC \circ BE^{-1} \circ DE \circ (EF \circ DE)^{-1}$ by Y. Then X and Y are the inverses of AD and CF, respectively. Proving this is pure algebra, for instance, to show that X is a left inverse to

AD, write

$$\begin{aligned} X \circ AD &= (BC \circ AB)^{-1} \circ BC \circ BE^{-1} \circ DE \circ AD \\ &= (BC \circ AB)^{-1} \circ BC \circ BE^{-1} \circ BE \circ AB \\ &= (BC \circ AB)^{-1} \circ BC \circ AB \\ &= id_A. \end{aligned}$$

To show that X is a right inverse one must remark that $AD \circ (BC \circ AB)^{-1} = (EF \circ DE)^{-1} \circ CF$, but other than that the calculations are equally trivial, and we omit them. The proof that Y is the inverse of CF is similar, and we omit it. □

In [2] the authors cite several different sources for Quillen's path-object argument, but none of the versions cited seem to precisely fit what they need. We therefore state a version of the argument which does. Specifically, our version is based on [13, A.3], generalizing it to any model category.

Lemma 2. *Let $\mathcal{E}$ be some model category. Let $\mathcal{C}$ be a category which is complete and cocomplete. Let $F : \mathcal{E} \rightleftarrows \mathcal{C} : U$ be an adjunction. Call a morphism of $\mathcal{C}$ a weak equivalence or fibration if it is sent to one by U. If $\mathcal{C}$ satisfies the following conditions*

- *$\mathcal{C}$ has a fibrant replacement functor*
- *$\mathcal{C}$ has path-objects for fibrant objects*

then any map in $\mathcal{C}$ having the left lifting property (llp) with respect to fibrations is a weak equivalence.

Note in particular that if $\mathcal{E}$ is a cofibrantly generated model category, then satisfying the conditions of the lemma ensures that condition (ii) of transfer is satisfied, since any sequential colimit of pushouts of images under F of the generating trivial cofibrations has the llp with respect to fibrations.

Proof. Let $X \xrightarrow{\gamma_X} \tilde{X}$ denote the fibrant replacement of X, and let $X \xrightarrow{\alpha_X} \text{Path}(X) \xrightarrow{\beta_X} X \times X$ denote the path-object. Suppose $f : X \to Y$ is some morphism in $\mathcal{C}$ which has the llp with respect to fibrations. We can then find a lift in the following diagram:

$$\begin{tikzcd} X \arrow[r, "\gamma_X"] \arrow[d, "f"'] & \tilde{X} \arrow[d, two heads] \\ Y \arrow[r] \arrow[ur, dashed, "r"'] & * \end{tikzcd}$$

We then find a lift in the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[r, "\gamma_Y"] & \tilde{Y} \arrow[r, "\alpha_{\tilde{Y}}"] & \mathrm{Path}(\tilde{Y}) \arrow[d, "\beta_{\tilde{Y}}"] \\
Y \arrow[rrr, "(\gamma_Y, \tilde{f}\circ r)"'] \arrow[rrru, dashed] & & & \tilde{Y} \times \tilde{Y}
\end{tikzcd}$$

The solid diagram commutes since it commutes in each factor of $\tilde{Y} \times \tilde{Y}$.

Applying U to the lower triangle yields a right homotopy from $U(\gamma_Y)$ to $U(\tilde{f} \circ r)$. This implies that they get sent to the same map in the homotopy category [4, 5.10]. Since $U(\gamma_Y)$ is sent onto an isomorpism [4, 5.8], so is $U(\tilde{f} \circ r)$. Apply $\mathrm{Ho} \circ U$ on the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "id"] \arrow[d, "f"] & X \arrow[r, "\gamma_X"] \arrow[d, "\gamma_X"] & \tilde{X} \arrow[d, "\tilde{f}"] \\
Y \arrow[r, "r"] & \tilde{X} \arrow[r, "\tilde{f}"] & \tilde{Y}
\end{tikzcd}$$

and notice that then the two horizontal compositions are isomorphisms (Ho denotes the homotopy category functor). By applying Lemma 1 we see that $\mathrm{Ho}(U(f))$ is an isomorphism. This means that $U(f)$ is a weak equivalence [4, 5.8], which is what we wanted to prove. $\square$

4.2 The model structure on operads

We say that a functor, $F : (\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, \tau_{\mathcal{C}}) \to (\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, \tau_{\mathcal{D}})$ between symmetric monoidal categories is symmetric monoidal if it comes equipped with a morphism $I_{\mathcal{D}} \to F(I_{\mathcal{C}})$, and a collection of morphisms $F(X) \otimes_{\mathcal{D}} F(Y) \to F(X \otimes_{\mathcal{C}} Y)$, natural in X and Y, making some diagrams commute [2, 2.4]. Informally, the axioms say that it doesn't matter in which order one "takes out" F from tensor products, that removing the monoidal unit before applying the functor is the same as doing it after, and that applying a twist and "moving out" F is the same as fist moving out and then twisting. For details, see [9, XI.2]. One thing to note is that some sources call what we have defined here a "lax" monoidal functor, and when not specifying that their monoidal functor is lax mean that the equipped morphisms are all isomorphisms. We call such monoidal functors strong. An important property of symmetric monoidal functors is that they preserve operads: if $P = (P(n))_{n \geq 0}$ is an operad in $\mathcal{C}$, then $F(P) = (F(P(n)))_{n \geq 0}$ is an operad in $\mathcal{D}$ with unit

$$I_{\mathcal{D}} \to F(I_{\mathcal{C}}) \xrightarrow{F(e)} F(P(1)),$$

and with operad composition

$$\begin{aligned}
&F(P(n)) \otimes_{\mathcal{D}} F(P(k_1)) \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} F(P(k_n)) \\
&\qquad \longrightarrow F(P(n) \otimes_{\mathcal{C}} P(k_1) \otimes_{\mathcal{C}} \cdots \otimes_{\mathcal{C}} P(k_n)) \\
&\qquad \xrightarrow{F(\gamma)} F(P(k_1 + \cdots + k_n)).
\end{aligned}$$

Checking that the associativity, unit, and equivariance diagrams commute is simple, and we omit the proof. In particular, if $\mathcal{C}$ is some cartesian closed category, then for any object A of $\mathcal{C}$ the functor $(-)^A : (\mathcal{C}, \otimes, I, \tau) \to (\mathcal{C}, \otimes, I, \tau)$ is strong symmetric monoidal since right adjoints preserve limits, and so for any operad P in $\mathcal{C}$ there is an operad $P^A = (P(n)^A)_{n \geq 0}$. From now on when dealing with symmetric monoidal functors, we shall state that they are so, and write them simply as $F : \mathcal{C} \to \mathcal{D}$, omitting giving special names to the tensor products in the monoidal categories.

For symmetric monoidal functors there is a notion of symmetric monoidal natural transformations; for two symmetric monoidal functors $F, G : \mathcal{C} \to \mathcal{D}$, a symmetric monoidal natural transformation from F to G is an ordinary natural transformation $\alpha : F \stackrel{\cdot}{\longrightarrow} G$, which makes all diagrams

$$\begin{tikzcd} & I \arrow[dl] \arrow[dr] & \\ F(I) \arrow[rr, "\alpha_I"] & & G(I) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F(X) \otimes F(Y) \arrow[r, "\alpha_X \otimes \alpha_Y"] \arrow[d] & G(X) \otimes G(Y) \arrow[d] \\ F(X \otimes Y) \arrow[r, "\alpha_{X \otimes Y}"] & G(X \otimes Y) \end{tikzcd}$$

commute [10, Definition 20.3]. A symmetric monoidal fibrant replacement functor is a fibrant replacement functor which is also symmetric monoidal, and where the natural transformation is symmetric monoidal.

A symmetric monoidal model category is a category which is closed symmetric monoidal and a model category, and satisfies some axioms that relate the two structures (see [7, Definition 4.2.6]). If the monoidal structure is cartesian, we call it a cartesian closed model category.

We introduce one final notion of monoidal model categories before proving what is Theorem 3.2 in [2]. An interval is a pair of morphisms

$$I \amalg I \rightarrowtail J \xrightarrow{\sim} I,$$

the first one being a cofibration and the second one a weak equivalence, such that the composition is the folding map $I \amalg I \xrightarrow{id+id} I$. Every monoidal model category has an interval, which can be seen by using the appropriate factorization axiom on the folding map.

Theorem 2. *Let $\mathcal{E}$ be a cartesian closed model category such that*

- *$\mathcal{E}$ is cofibrantly generated and the terminal object of $\mathcal{E}$ is cofibrant;*
- *$\mathcal{E}$ has a symmetric monoidal fibrant replacement functor.*

Then, there is a cofibrantly generated model structure on the category of operads, in which a map $P \to Q$ is a weak equivalence (resp. fibration) iff for each n, the map $P(n) \to Q(n)$ is a weak equivalence (resp. fibration) in $\mathcal{E}$.

Proof. The proof follows [2] closely. We shall construct the model structure on operads by transfer (Theorem 1) using the path-object argument (Lemma

2). One can show that the forgetful functor is monadic and preserves filtered colimits (we will not show this), and this together with the category of collections in $\mathcal{E}$ being complete and cocomplete ensures that the category of operads in $\mathcal{E}$ is complete and cocomplete [3, Proposition 4.3.6]. That the forgetful functor preserves filtered colimits also ensures that condition (i) of the transfer principle is satisfied.

Let P be an operad and let $\tilde{P}$ be the collection defined by $\tilde{P}(n) = P(n)\tilde{}$ for $n \geq 0$, where $X \mapsto X\tilde{}$ is the symmetric monoidal fibrant replacement functor in $\mathcal{E}$. Since the fibrant replacement functor is symmetric monoidal, the operad structure on P induces an operad structure on $\tilde{P}$, so that $\tilde{P}$ is a fibrant replacement for P in the category of operads. Thus, the first condition in Lemma 2 holds.

For the construction of a path-object for fibrant operads, we use the fact that in a cartesian closed category exponentiation is product preserving and hence strong symmetric monoidal. This implies that for any interval $I \amalg I \rightarrowtail J \xrightarrow{\sim} I$, mapping into a fibrant operad P yields a sequence of morphisms of operads: $P^I \xrightarrow{\sim} P^J \twoheadrightarrow P^{I \amalg I}$, where the first morphism is a weak equivalence since I is cofibrant [2, Lemma 2.3], and the second morphism is a fibration since P is fibrant [7, Lemma 4.2.2]. The fact that $P \cong P^I$ can be seen by chasing around elements in hom-sets related to the adjunction $- \otimes I \rightleftarrows (-)^I$, and the fact that $P^{I \amalg I} \cong P \times P$ follows from $\mathcal{E}$ being cartesian closed. Thus we have a path-object for P. $\square$

5 Constructing the free functor

There is an obvious forgetful functor $\mathrm{Oper}(\mathcal{E}) \to \mathrm{Coll}(\mathcal{E})$, and it turns out that there is also a free one $\mathrm{Coll}(\mathcal{E}) \to \mathrm{Oper}(\mathcal{E})$. The purpose of this section is to construct it, and to show that it together with the forgetful one form an adjunction. To do this we first introduce a kind of trees as well as graftings of them, which will be used in the construction. We will then do the construction, before finally showing that we have the adjunction.

5.1 Trees

The trees we are going to use are the combinatorial trees defined in [8]. These trees differ from what is usually meant by tree in that they are planar and rooted, and that the edges are allowed to not have vertices at their ends. We will only give a short description of the trees themselves, for details, see [8]. We will however give in full detail some definitions which are not to be found in Leinsters book, such as that of (non-planar) isomorphisms of trees.

Definition 6. A (finite, planar, rooted) tree T consists of

- a finite set $v(T)$ (the vertices)
- a finite set $e(T)$ (the edges), a subset $\mathrm{in}(T) \subseteq e(T)$ (the input edges, or leaves), and an element $o \in e(T)$ (the output edge)

- a function $s : e(T) \setminus \mathrm{in}(T) \to v(T)$ (source) and a function $t : e(T) \setminus \{o\} \to v(T)$ (target)
- for each $v \in v(T)$, a total order on $\mathrm{in}(v) := t^{-1}(v)$,

such that there is a unique path from each vertex to the root edge.

The ordering of each set $\mathrm{in}(v)$ gives the order from left to right we would put the edges in if we were to embed the tree in the plane, explaining why we call them planar. We note in particular the case of having just one edge and no vertices, in this case we will call the tree trivial. We pick a representative of these and denote it by $|$. We define $|T|$ to be $|\mathrm{in}(T)|$ and for $v \in v(T)$ we define $|v|$ to be $|\mathrm{in}(v)|$. For non-trivial trees, the source of the output edge may be of special interest; we call it the root vertex or simply the root, and denote it by $r(T) = s(o)$. If a tree has only one vertex we call it a corolla, and denote an n-leaved corolla, $n \geq 0$, by c_n. We now define isomorphisms of trees:

Definition 7. For two trees T, T', an isomorphism $\phi : T \to T'$ is a pair of bijections (denoted by the same name) $\phi : v(T) \to v(T')$, $\phi : e(T) \to e(T')$, satisfying the expressions $s(\phi(e)) = \phi(s(e))$ and $t(\phi(e)) = \phi(t(e))$, e an edge of T, whenever they are defined. It follows from this that leaves are sent to leaves and that the root edge is sent to the root edge.

Note in particular that the isomorphisms defined here are non-planar; the definition makes no use of the planar structure. We shall sometimes consider the induced maps $\mathrm{in}(\phi) : \mathrm{in}(T) \to \mathrm{in}(T')$ and $\phi_v : \mathrm{in}(v) \to \mathrm{in}(\phi(v))$ as elements of symmetric groups, through their planar structures. More explicitly, if $p : \underline{n} \to \mathrm{in}(T)$ and $p' : \underline{n} \to \mathrm{in}(T')$ are the planar orderings of the input edges of T and T' respectively, we may sometimes associate $\mathrm{in}(\phi)$ with

$$\underline{n} \xrightarrow{p} \mathrm{in}(T) \xrightarrow{\mathrm{in}(\phi)} \mathrm{in}(T') \xrightarrow{(p')^{-1}} \underline{n},$$

and similarly with ϕ_v. Trees and isomorphisms form a category (in fact a groupoid) which we denote by $\mathbb{T}$, which splits up further into the categories of trees with n leaves, $\mathbb{T}(n)$, for $n \geq 0$.

There is a functor $\lambda : \mathbb{T} \to \mathbf{Set}$, sending a tree T to the set of all labelings of its leaves,

$$\lambda(T) = \{\tau : \{1, \ldots, |T|\} \to \mathrm{in}(T) \mid \tau \text{ bijection}\}.$$

Each isomorphism, $\phi : T \to T'$, is sent to a function defined by post-composition with the bijections $\mathrm{in}(\phi) : \mathrm{in}(T) \to \mathrm{in}(T')$, like so:

$$\lambda(\phi)\left(\{1, \ldots, |T|\} \xrightarrow{\tau} \mathrm{in}(T)\right) = \{1, \ldots, |T|\} \xrightarrow{\tau} \mathrm{in}(T) \xrightarrow{\mathrm{in}(\phi)} \mathrm{in}(T').$$

Restricting λ to $\mathbb{T}(n)$, we get for each $\sigma \in \Sigma_n$ a natural transformation $\sigma^* : \lambda \overset{\cdot}{\longrightarrow} \lambda$, defined by pre-composition/relabeling leaves:

$$\sigma^*_T\left(\{1, \ldots, n\} \xrightarrow{\tau} in(T)\right) = \{1, \ldots, n\} \xrightarrow{\sigma} \{1, \ldots, n\} \xrightarrow{\tau} \mathrm{in}(T).$$

To see that σ^* is natural, we would like, for each $\phi : T \to T'$, the following diagram to commute:

$$\begin{tikzcd}
\lambda(T) \arrow[r, "\sigma^*"] \arrow[d, "\phi_*"] & \lambda(T) \arrow[d, "\phi_*"] \\
\lambda(T') \arrow[r, "\sigma^*"] & \lambda(T')
\end{tikzcd},$$

which it does, which can be seen by casing an element around:

$$\begin{tikzcd}
\tau \arrow[r, mapsto] \arrow[d, mapsto] & \tau \circ \sigma \arrow[d, mapsto] \\
\mathrm{in}(\phi) \circ \tau \arrow[r, mapsto] & \mathrm{in}(\phi) \circ \tau \circ \sigma
\end{tikzcd}.$$

We note also that these natural transformations are functorial on Σ_n^{op}, and thus they constitute a right action on λ restricted to n.

Applying the Groethendieck construction (see e.g. [1]) to $\lambda : \mathbb{T} \to \mathbf{Set}$ gives us a category $\mathbb{T}[\lambda]$ and a functor $\pi : \mathbb{T}[\lambda] \to \mathbb{T}$. The Groethendieck construction, $\mathbb{T}[\lambda]$, is the category of labeled trees, and the functor π is defined by forgetting the labels. Explicitly, a labeled tree is a tree, T, together with a bijection $\tau : \underline{n} \to \mathrm{in}(T)$, where $n = |T|$. An isomorphism of labeled trees is an isomorphism of trees which also preserves the labels, i.e. such that

$$\begin{tikzcd}
\mathrm{in}(T) \arrow[rr, "in(\phi)"] & & \mathrm{in}(T') \\
& \underline{n} \arrow[ul, "\tau"] \arrow[ur, "\tau'"'] &
\end{tikzcd}$$

commutes. That labeled trees together with isomorphisms of labeled trees form a category $\mathbb{T}[\lambda]$ is immediate. Restriction to labeled trees with n leaves gives us the category $\mathbb{T}[\lambda](n)$, and we may sometimes denote π restricted to this subcategory as π_n. We will sometimes regard a labeling as an element of a symmetric group; if $p : \underline{n} \to \mathrm{in}(T)$ is the labeling arising from the planar structure, we associate a labeling $\tau : \underline{n} \to \mathrm{in}(T)$ with

$$\underline{n} \xrightarrow{\tau} \mathrm{in}(T) \xrightarrow{p^{-1}} \underline{n}.$$

In particular, the labeling arising from the planar structure will sometimes be denoted as id.

Trees, labeled or not, may be grafted onto each other. This procedure can be imagined as taking some tree with, say, n leaves, and n other trees and glueing the roots of those n trees onto the leaves of the n-leaved one. Explicitly, let the numbers $n, k_1, \ldots, k_n$ be given, then there is a functor

$$-(-, \ldots, -) : \mathbb{T}(n) \times \prod_{i=1}^{n} \mathbb{T}(k_i) \to \mathbb{T}(k_1 + \cdots + k_n),$$

defined by sending $(S, T_1, \ldots, T_n)$ to the tree $S(T_1, \ldots, T_n)$, which has vertices and edges respectively

$$v(S(T_1, \ldots, T_n)) = v(S) \amalg \coprod_{i=1}^{n} v(T_i)$$

and

$$e(S(T_1, \ldots, T_n)) = e(S) \amalg \coprod_{i=1}^{n} e(T_i) \setminus \{o_i\},$$

the o_i denoting the output edge of T_i. We let the output edge be the output edge of S and the input edges the input edges of the T_i. The source and target maps are defined by

$$s(e) = \begin{cases} r(T_i) & \text{if } e \text{ is the } i\text{-th leaf of } S \\ s(e) & \text{otherwise} \end{cases}$$

and

$$t(e) = t(e),$$

where the s and t on the right hand side of the equal mark is the appropriate source and target maps for one of $S, T_1, \ldots, T_n$. On isomorphisms, $-(-, \ldots, -)$ is defined in the obvious way.

Grafting of labeled trees is defined in the following way: for each collection of numbers $n, k_1, \ldots, k_n$ there is a functor

$$-(-, \ldots, -) : \mathbb{T}[\lambda](n) \times \prod_{i=1}^{n} \mathbb{T}[\lambda](k_i) \to \mathbb{T}[\lambda](k_1 + \cdots + k_n),$$

defined by sending $((S, \sigma), (T_1, \tau_1), \ldots, (T_n, \tau_n))$ to the labeled tree with underlying tree $S(T_{\sigma^{-1}(1)}, \ldots, T_{\sigma^{-1}(n)})$ and with label the composition

$$\underline{k} \xrightarrow{\cong} \underline{k_1} \amalg \cdots \amalg \underline{k_n} \xrightarrow{\tau_1 \amalg \cdots \amalg \tau_n} \mathrm{in}(T_1) \amalg \cdots \amalg \mathrm{in}(T_n),$$

where we've set $k = k_1 + \cdots + k_n$. The definition of what the functor does on isomorphisms is again the obvious one. Note that we have used the same symbol for grafting of labeled and unlabeled trees; when using it later on it's meaning must be understood from context.

When looking at the domain and codomain of the grafting functor for labeled trees, one might think that it makes labeled trees into an operad. In thinking so one would be almost right, for it makes labeled trees into an operad (in groupoids) up to canonical natural isomorphism. The right action of the symmetric groups is defined through relabeling of the leaves, some $\sigma \in \Sigma_n$ sending (T, τ) to $(T, \tau \circ \sigma)$. We remark that [2] incorrectly states that it is an actual (strict) operad, but that this oversight is unimportant, as any problems disappear when taking colimits. Grafting of unlabeled trees, on the other hand, does not form an operad, as there is no right action of the symmetric groups. It is, however, a non-Σ operad [8].

5.2 Constructing the functor

To define the free functor we first construct some functors on the category of trees, from which it will be built. We define for each collection $\mathcal{K}$ a functor $\underline{\mathcal{K}} : \mathbb{T}^{\mathrm{op}} \to \mathcal{E}$. We do so inductively on the number of vertices in the trees, first for the objects, and then for the morphisms. If the number of vertices in a tree T is 0, then $T = |$ and we define $\mathcal{K}(|)$ to be I, the unit in the monoidal category. If the number of vertices of T is strictly larger than 0, then T decomposes uniquely as a grafting of trees with a corolla, $T = c_n(T_1, \ldots, T_n)$, and we define

$$\underline{\mathcal{K}}(T) = \underline{\mathcal{K}}(c_n(T_1, \ldots, T_n)) = \mathcal{K}(n) \otimes \underline{\mathcal{K}}(T_1) \otimes \cdots \otimes \underline{\mathcal{K}}(T_n).$$

We see that $\underline{\mathcal{K}}(T)$ is a tensor product over the vertices v of T of $K(|in(v)|)$, as well as a monoidal unit for each leaf, with the order depending on the planar structure. To see this, consider the following inductive argument:

If the number of vertices of T is 1, then $\underline{\mathcal{K}}(T) = \mathcal{K}(|v|) \otimes I \otimes \cdots \otimes I$, where v is the single vertex in the tree and the number of monoidal units is equal to the valence of v, the number of leaves. Now, if T has more than 1 vertex, then any vertex v of T is either the root, or it is not. If v is the root, then $\mathcal{K}(|v|)$ is the first term of $\underline{\mathcal{K}}(T)$, and any leaf on v is some $|$ grafted onto the root, and so gives a monoidal unit in the tensor product. If v is not the root of T, v is in one of the trees T_i, and by induction $\mathcal{K}(|v|)$ and one monoidal unit for each leaf on v appears in $\underline{\mathcal{K}}(T_i)$ which appears in $\underline{\mathcal{K}}(T)$. It is clear that the planar structure determines the order in which things appear, since if $v \in v(T_i)$ and $v' \in v(T_j)$, $i < j$, then $\mathcal{K}(|v|)$ appears before $\mathcal{K}(|v'|)$, and similarly if we decompose the trees $T_i = c_m((T_i)_1, \ldots, (T_i)_m)$.

To define $\underline{\mathcal{K}}$ on morphisms, suppose first that the number of vertices in T is 0. Then $T = |$, and any isomorphism $\phi : T \to T'$ is unique, and we let $\underline{\mathcal{K}}(\phi)$, which we denote by ϕ^*, be id_I. Next, suppose T has 1 or more vertices. Then $\phi : T \to T'$ decomposes into $\sigma(\phi_1, \ldots, \phi_n)$, where $\sigma : c_n \to c'_n$ and $\phi_i : T_i \to T'_{\sigma(i)}$. The map σ induces a map $\mathrm{in}(v) \to \mathrm{in}(v')$, where v and v' are the roots of T and T', which we can regard as an element of Σ_n through the planar structure. We shall denote this element as σ also, and we define $\phi^* : \underline{\mathcal{K}}(T') \to \underline{\mathcal{K}}(T)$ to be the composition

$$\begin{aligned}\mathcal{K}(n) \otimes \underline{\mathcal{K}}(T'_1) \otimes \cdots \otimes \underline{\mathcal{K}}(T'_n) &\xrightarrow{id \otimes \sigma^{-1}} \mathcal{K}(n) \otimes \underline{\mathcal{K}}(T'_{\sigma(1)}) \otimes \cdots \otimes \underline{\mathcal{K}}(T'_{\sigma(n)}) \\ &\xrightarrow{\sigma^* \otimes \phi_1^* \otimes \cdots \otimes \phi_n^*} \mathcal{K}(n) \otimes \underline{\mathcal{K}}(T_1) \otimes \cdots \otimes \underline{\mathcal{K}}(T_n),\end{aligned}$$

or equivalently

$$\phi^* = \sigma^* \otimes \left((\phi_1^* \otimes \cdots \otimes \phi_n^*) \circ \sigma^{-1} \right).$$

It will sometimes be necessary to work with a different, non-inductive definition of ϕ^*. To give this new definition, we introduce some notation: if T is a tree, we use $\bigotimes_{v \in v(T)}$ to denote a tensor product of $|v(T)|$ things, with order depending on the planar structure of T. For instance, if $T = c_m(T_1, \ldots, T_m)$, the first tensor factor in $\bigotimes_{v \in v(T)} F(v)$ is $F(r(T))$, and if $v \in v(T_i)$ and $w \in v(T_j)$, $i < j$, then $F(v)$ comes before $F(w)$. Now, as before let $\phi : T \to T'$ be an isomorphism of trees, and in addition set $m = |v(T)| = |v(T')|$. By removing the monoidal units in $\underline{\mathcal{K}}(T)$ and $\underline{\mathcal{K}}(T')$ corresponding to non-output leaves one gets

$$\underline{\mathcal{K}}(T) \cong \bigotimes_{v \in v(T)} \mathcal{K}(|v|), \quad \underline{\mathcal{K}}(T') \cong \bigotimes_{v' \in v(T')} \mathcal{K}(|v'|).$$

(The reason for the use of non-output leaves is the case $m = 0$, in this case interpret $\bigotimes_{v \in v(T)} \mathcal{K}(|v|)$ as I.) Using these two isomorphisms, and letting ϕ_v denote the element of $\Sigma_{|v|}$ corresponding to the induced map $\mathrm{in}(v) \to \mathrm{in}(\phi(v))$, we define ϕ^* to be the composition

$$\bigotimes_{v' \in v(T')} \mathcal{K}(|v'|) \xrightarrow{\phi^{-1}} \bigotimes_{v \in v(T)} \mathcal{K}(|\phi(v)|) \xrightarrow{\bigotimes_{v \in v(T)} \phi_v^*} \bigotimes_{v \in v(T)} \mathcal{K}(|v|).$$

This can be imagined as first moving all the tensor factors into the correct place, and then using the group actions on them all at once.

Lemma 3. *The two definitions of ϕ^* are equivalent.*

Proof. The proof is by induction on the number of vertices. If the number of vertices is equal to 0, both definitions give $\phi^* = id$, so it is clear. If the number of vertices is greater than 0, then T and T' decompose as $c_n(T_1, \dots, T_n)$ and $c_n(T'_1, \dots, T'_n)$, respectively, and ϕ decomposes into $\sigma : c_n \to c_n$ and $\phi_i : T_i \to T'_{\sigma(i)}$, $1 \le i \le n$. Consider the following diagram

$$\begin{tikzcd}
\mathcal{K}(n) \otimes \bigotimes_{i=1}^{n} \underline{\mathcal{K}}(T'_i) \arrow[r, "\cong"] \arrow[d, "\sigma^* \otimes \sigma^{-1}"'] & \mathcal{K}(n) \otimes \bigotimes_{i=1}^{n} \bigotimes_{v' \in v(T'_i)} \underline{\mathcal{K}}(|v'|) \arrow[d, "\sigma^* \otimes \sigma^{-1}"] \\
\mathcal{K}(n) \otimes \bigotimes_{i=1}^{n} \underline{\mathcal{K}}(T'_{\sigma(i)}) \arrow[r, "id \otimes \bigotimes_{i=1}^{n} \cong"] \arrow[d, "id \otimes \bigotimes_{i=1}^{n} \phi_i^*"'] & \mathcal{K}(n) \otimes \bigotimes_{i=1}^{n} \bigotimes_{v' \in v(T'_{\sigma(i)})} \underline{\mathcal{K}}(|v'|) \arrow[d, "id \otimes \bigotimes_{i=1}^{n} \phi_i^*"] \\
\mathcal{K}(n) \otimes \bigotimes_{i=1}^{n} \underline{\mathcal{K}}(T_i) \arrow[r, "id \otimes \bigotimes_{i=1}^{n} \cong"] & \mathcal{K}(n) \otimes \bigotimes_{i=1}^{n} \bigotimes_{v \in v(T_i)} \underline{\mathcal{K}}(|v|),
\end{tikzcd}$$

where the phis on the left are defined inductively and the ones on the right directly. The upper half commutes by coherence, and the lower by induction. $\square$

For each map of collections $f : \mathcal{K} \to \mathcal{C}$, there is a natural transformation $\underline{f} : \underline{\mathcal{K}} \to \underline{\mathcal{C}}$. It is also defined inductively on the number of vertices of each tree T. If T has 0 vertices then $T = |$, and we define $\underline{f}_T = id_I$. If T has more than 0 vertices, then T decomposes uniquely as $c_m(T_1, \dots, T_m)$, and we define

$$\underline{f}_T = f_m \otimes \underline{f}_{T_1} \otimes \cdots \otimes \underline{f}_{T_m}.$$

To see that this is natural, let $\phi : T \to T'$ be given, then if the number of vertices is 0 it is trivially natural, while if the number of vertices is greater than 0, T again decomposes as $c_m(T_1, \dots, T_m)$ and the naturality diagram becomes

$$\begin{tikzcd}
\mathcal{K}(m) \otimes \bigotimes_{i=1}^{m} \underline{\mathcal{K}}(T'_i) \arrow[r, "f_m \otimes \bigotimes_{i=1}^{m} \underline{f}_{T'_i}"] \arrow[d, "id \otimes \sigma^{-1}"'] & \mathcal{C}(m) \otimes \bigotimes_{i=1}^{m} \underline{\mathcal{C}}(T'_i) \arrow[d, "id \otimes \sigma^{-1}"] \\
\mathcal{K}(m) \otimes \bigotimes_{i=1}^{m} \underline{\mathcal{K}}(T'_{\sigma(i)}) \arrow[r, "f_m \otimes \bigotimes_{i=1}^{m} \underline{f}_{T'_{\sigma(i)}}"] \arrow[d, "\sigma^* \otimes \phi_1^* \otimes \cdots \otimes \phi_m^*"'] & \mathcal{C}(m) \otimes \bigotimes_{i=1}^{m} \underline{\mathcal{C}}(T'_{\sigma(i)}) \arrow[d, "\sigma^* \otimes \phi_1^* \otimes \cdots \otimes \phi_m^*"] \\
\mathcal{K}(m) \otimes \bigotimes_{i=1}^{m} \underline{\mathcal{K}}(T_i) \arrow[r, "f_m \otimes \bigotimes_{i=1}^{m} \underline{f}_{T_i}"] & \mathcal{C}(m) \otimes \bigotimes_{i=1}^{m} \underline{\mathcal{C}}(T_i)
\end{tikzcd}$$

The top square commutes by coherence, and the bottom square commutes in the first factor since f is a map of collections, and in the other factors by induction.

We now define the free functor $\mathcal{F} : \mathrm{Coll}(\mathcal{E}) \to \mathrm{Oper}(\mathcal{E})$. Let $\mathcal{K}$ be some collection, and let $f : \mathcal{K} \to \mathcal{C}$ be some map of collecions. Then, define

$$\mathcal{F}\mathcal{K} = \mathrm{colim}_{\mathbb{T}[\lambda]} \underline{\mathcal{K}}\pi$$

and

$$\mathcal{F}f = \mathrm{colim}_{\mathbb{T}[\lambda]} \underline{f}\pi.$$

We remark that [2] uses a different definition using coends, however the author found this definition easier to work with. Since $\mathbb{T}[\lambda]$ splits as a coproduct into smaller subgroupoids $\mathbb{T}[\lambda](n)$ of labeled trees with n leaves, so does our colimit split as a coproduct of $\mathcal{F}\mathcal{K}(n) = \mathrm{colim}_{\mathbb{T}[\lambda](n)} \underline{\mathcal{K}}\pi_n$.

The object $\mathcal{F}\mathcal{K} = \coprod_{n \geq 0} \mathcal{F}\mathcal{K}(n)$ has the structure of a collection, i.e. each $\mathcal{F}\mathcal{K}(n)$ has a right action of Σ_n. For each $\sigma \in \Sigma_n$, this action is induced by the maps

$$\underline{\mathcal{K}}\pi(T, \tau) = \underline{\mathcal{K}}\pi(T, \tau\sigma) \xrightarrow{in_{(T,\tau\sigma)}} \mathrm{colim}_{\mathbb{T}[\lambda](n)} \underline{\mathcal{K}}\pi_n,$$

where (T, τ) is an object of $\mathbb{T}[\lambda](n)$. This is well defined, for for each isomorphism of labeled trees $\phi : (T, \tau) \to (T', \tau')$, the following diagram commutes

$$\begin{tikzcd}
\underline{\mathcal{K}}\pi(T', \tau') \arrow[r, "="] \arrow[d, "\phi^*"'] & \underline{\mathcal{K}}\pi(T', \tau'\sigma) \arrow[d, "\phi^*"] \arrow[dr, "in_{(T',\tau'\sigma)}"] & \\
 & & \mathrm{colim}_{\mathbb{T}[\lambda](n)} \underline{\mathcal{K}}\pi_n \\
\underline{\mathcal{K}}\pi(T, \tau) \arrow[r, "="] & \underline{\mathcal{K}}\pi(T, \tau\sigma) \arrow[ur, "in_{(T,\tau\sigma)}"'] &
\end{tikzcd}$$

once we have shown that $\phi : (T, \tau\sigma) \to (T', \tau'\sigma)$ is an isomorphism of labeled trees. It is, since $\mathrm{in}(\phi) \circ (\tau \circ \sigma) = (\mathrm{in}(\phi) \circ \tau) \circ \sigma = \tau' \circ \sigma$.

Having now shown that $\mathcal{FK}$ is a collection, we would like to show that it is also an operad. We start by defining the unit: we let it be the inclusion map

$$I = \underline{\mathcal{K}}(|) \xrightarrow{in_{(|,id)}} \mathcal{FK}(1).$$

We define the operad composition to be the composition

$$\begin{aligned}
\mathcal{FK}(n) \otimes \bigotimes_{i=1}^{n} \mathcal{FK}(k_i) &= \mathrm{colim}_{\mathbb{T}[\lambda](n)} \underline{\mathcal{K}}\pi_n \otimes \bigotimes_{i=1}^{n} \mathrm{colim}_{\mathbb{T}[\lambda](k_i)} \underline{\mathcal{K}}\pi_{k_i} \\
&\cong \mathrm{colim}_{\mathbb{T}[\lambda](n) \times \prod_{i=1}^{n} \mathbb{T}[\lambda](k_i)} \left(\underline{\mathcal{K}}\pi_n \otimes \bigotimes_{i=1}^{n} \underline{\mathcal{K}}\pi_{k_i} \right) \\
&\cong \mathrm{colim}_{\mathbb{T}[\lambda](n) \times \prod_{i=1}^{n} \mathbb{T}[\lambda](k_i)} \underline{\mathcal{K}}\pi_{\Sigma_i k_i} \circ -(-, \dots, -) \\
&\to \mathrm{colim}_{\mathbb{T}[\lambda](\Sigma_i k_i)} \underline{\mathcal{K}}\pi_{\Sigma_i k_i} \\
&= \mathcal{FK}(\Sigma_i k_i),
\end{aligned}$$

where we on the second line use the fact that tensor products commute with colimits since $\mathcal{E}$ is closed, on the fourth line the map is the one that is induced $\mathrm{colim} F \circ G \to \mathrm{colim} F$ whenever one has two composable functors F and G, and on the third line we use the colimit functor on a natural isomorphism which we will now describe.

Let $((S, \sigma), (T_1, \tau_1), \dots, (T_n, \tau_n))$ be an object of $\mathbb{T}[\lambda](n) \times \prod_{i=1}^{n} \mathbb{T}[\lambda](k_i)$, and let $(T, \tau) = (S, \sigma)((T_1, \tau_1), \dots, (T_n, \tau_n))$. The two objects $\underline{\mathcal{K}}(S) \otimes \bigotimes_{i=1}^{n} \underline{\mathcal{K}}(T_i)$ and $\underline{\mathcal{K}}(T)$ are isomorphic to tensor products over the vertices of T:

$$\begin{aligned}
\underline{\mathcal{K}}(S) \otimes \bigotimes_{i=1}^{n} \underline{\mathcal{K}}(T_i) &\cong \bigotimes_{v \in v(S)} \mathcal{K}(|v|) \otimes \bigotimes_{i=i}^{n} \bigotimes_{v \in v(T_i)} \mathcal{K}(|v|) \\
&\cong \bigotimes_{v \in v(T)} \mathcal{K}(|v|) \\
&\cong \underline{\mathcal{K}}(T).
\end{aligned}$$

We call this composition / symmetry isomorphism $sym_{(S,\sigma), T_1, \dots, T_n}$, or simply sym if it is clear from context. Our desired natural isomorphism is then

$$(sym_{(S,\sigma), T_1, \dots, T_n})_{((S,\sigma), (T_1, \tau_1), \dots, (T_n, \tau_n))},$$

where $((S, \sigma), (T_1, \tau_1), \dots, (T_n, \tau_n))$ ranges over $ob(\mathbb{T}[\lambda](n) \times \prod_{i=1}^{n} \mathbb{T}[\lambda](k_i))$. To see that this transformation is natural let

$$(\psi, \phi_1, \dots, \phi_n) : ((S, \sigma), (T_1, \tau_1), \dots, (T_n, \tau_n)) \to ((S', \sigma'), (T_1'.\tau_1'), \dots, (T_n', \tau_n'))$$

be given, with $|S| = |S'| = n$. Call $\psi(\phi_1, \dots, \phi_n) : T \to T'$ for ϕ. Using Lemma 3 in order to factor $\phi^*, \psi, \phi_1^*, \dots, \phi_n^*$, and using the natural isomorphism $\underline{\mathcal{K}}(-) \cong \bigotimes_{v \in v(-)} \mathcal{K}(|v|)$, the commutativity of the naturality square becomes equivalent to the commutativity of the following diagram.

$$\begin{tikzcd}
\displaystyle\bigotimes_{v'\in v(S')} \mathcal{K}(|v'|) \otimes \bigotimes_{i=1}^{n} \left(\bigotimes_{v'\in v(T_i')} \mathcal{K}(|v'|) \right) \arrow[r, "sym", "\cong"'] \arrow[d, "\psi^{-1}\otimes\bigotimes_{i=1}^{n}\phi_i^{-1}"'] & \displaystyle\bigotimes_{v'\in v(T')} \mathcal{K}(|v'|) \arrow[d, "\phi^{-1}"] \\
\displaystyle\bigotimes_{v\in v(S)} \mathcal{K}(|\psi(v)|) \otimes \bigotimes_{i=1}^{n} \left(\bigotimes_{v\in v(T_i)} \mathcal{K}(|\phi(v)|) \right) \arrow[r, "sym", "\cong"'] \arrow[d, "\bigotimes_{v\in v(S)}\psi_v^*\otimes\bigotimes_{i=1}^{n}\left(\bigotimes_{v\in v(T_i)}\phi_v^*\right)"'] & \displaystyle\bigotimes_{v\in v(T)} \mathcal{K}(|\phi(v)|) \arrow[d, "\bigotimes_{v\in v(T)}\phi_v^*"] \\
\displaystyle\bigotimes_{v\in v(S)} \mathcal{K}(|v|) \otimes \bigotimes_{i=1}^{n} \left(\bigotimes_{v\in v(T_i)} \mathcal{K}(|v|) \right) \arrow[r, "sym", "\cong"'] & \displaystyle\bigotimes_{v\in v(T)} \mathcal{K}(|v|)
\end{tikzcd}$$

Both the upper and lower square commute by Mac Lanes coherence theorem [9].

Having now defined the operad structure, we proceed to show that it satisfies the operad axioms. We begin with the unit axioms, the diagrams (2): let $(T,\tau) \in \mathbb{T}[\lambda](n)$ be given, and consider the following diagram.

$$\begin{tikzcd}
I \otimes \underline{\mathcal{K}}(T) \arrow[r, "id\otimes id"] \arrow[d, "id\otimes in_{(T,\tau)}"] & \underline{\mathcal{K}}(|) \otimes \underline{\mathcal{K}}(T) \arrow[r, "\cong"] \arrow[d, "in_{(|,id)}\otimes in_{(T,\tau)}"] & \underline{\mathcal{K}}(T) \arrow[r, "id"] \arrow[d, "in_{(T,\tau)}"] & \underline{\mathcal{K}}(T) \arrow[d, "in_{(T,\tau)}"] \\
I \otimes \mathcal{F}\mathcal{K}(n) \arrow[r, "e\otimes id"] & \mathcal{F}\mathcal{K}(1) \otimes \mathcal{F}\mathcal{K}(n) \arrow[r, "\cong"] & \operatorname{colim}\underline{\mathcal{K}}\pi_n \circ -(-) \arrow[r] & \mathcal{F}\mathcal{K}(n)
\end{tikzcd}$$

Clearly, the composition along the top is the canonical isomorphism $I\otimes\underline{\mathcal{K}}(T) \to \underline{\mathcal{K}}(T)$, which implies that the bottom composition is the appropriate canonical isomorphism, proving the left unit axiom. Proving the right unit axiom is entirely analogous, and we omit it.

The associativity axiom is satisfied essentially because it holds before taking colimits. Consider diagram (1), but with $P = \mathcal{F}\mathcal{K}$. As the first entry is a colimit, we can show that $\gamma \circ (\gamma \otimes id) = \gamma \circ (id \otimes (\otimes_s \gamma)) \circ \mathit{shuffle}$ if we show that $\gamma \circ (\gamma \otimes id) \circ in = \gamma \circ (id \otimes (\otimes_s \gamma)) \circ \mathit{shuffle} \circ in$, where in is the inclusion of some factor. To do this, let the labeled trees (S,σ), $(T_1,\tau_1),\dots,(T_n,\tau_n)$, and $(R_s^1,\rho_s^1),\dots,(R_s^{k_s},\rho_s^{k_s})$, $1 \le s \le n$, be given. Let (A,α) be the composition of them all, i.e. (let $(T,\tau) = (S,\sigma)((T_1,\tau_1),\dots,(T_n,\tau_n))$ and $(B_s,\beta_s) = (T_s,\tau_s)((R_s^1,\rho_s^1),\dots,(R_s^{k_s},\rho_s^{k_s}))$)

$$\begin{aligned}
(A,\alpha) &= (T,\tau)((R_1^1,\rho_1^1),\dots,(R_n^{k_n},\rho_n^{k_n})) \\
&\cong (S,\sigma)((B_1,\beta_1),\dots,(B_n,\beta_n)).
\end{aligned}$$

Then one can use the coherence theorem to show that $\gamma \circ (\gamma \otimes id) \circ in$ and $\gamma \circ (id \otimes (\otimes_s \gamma)) \circ \mathit{shuffle} \circ in$ are both equal to the composition

$$\underline{\mathcal{K}}(S) \otimes \left(\bigotimes_{s=1}^{n} \underline{\mathcal{K}}(T_s) \right) \otimes \bigotimes_{s=1}^{n} \bigotimes_{t=1}^{k_s} \underline{\mathcal{K}}(R_s^t) \xrightarrow{\cong} \underline{\mathcal{K}}(A) \xrightarrow{in_{(A,\alpha)}} \mathcal{F}\mathcal{K}(i_1 + \cdots + i_n).$$

Finally, we show that equivariance holds. Consider diagram (4), again with $P = \mathcal{FK}$. Again, the top right entry in the diagram is a colimit, so we see what the maps are on each component. Let $(S, s), (T_1, t_1), \ldots, (T_n, t_n)$ be labeled trees with $|S| = n, |T_1| = k_1, \ldots, |T_n| = k_n$. One can then easily show that both

$$\gamma \circ (id \otimes \tau_1 \otimes \cdots \otimes \tau_n) \circ in_{(S,s),(T_1,t_1),\ldots,(T_n,t_n)}$$

and

$$(\tau_1 \oplus \cdots \oplus \tau_n) \circ \gamma \circ in_{(S,s),(T_1,t_1),\ldots,(T_n,t_n)}$$

are equal to the composition

$$\underline{\mathcal{K}}(S) \otimes \bigotimes_{i=1}^{n} \underline{\mathcal{K}}(T_i) \xrightarrow{\cong} \underline{\mathcal{K}}(T) \xrightarrow{in_{(T, t \circ (\tau_1 \oplus \cdots \oplus \tau_n))}} \mathcal{FK}(k_1 + \cdots + k_n),$$

by using the coherence theorem. Checking that the first equivariance diagram (3) commutes is similar to checking the second one, although it is made a bit more complicated by the shuffling around of tensor factors. In the process of showing that both $\gamma \circ (\sigma^* \otimes \sigma^{-1}) \circ in$ and $\sigma(k_{\sigma(1)}, \ldots, k_{\sigma(n)}) \circ \gamma \circ in$ are equal to the composition

$$\underline{\mathcal{K}}(S) \otimes \bigotimes_{i=1}^{n} \underline{\mathcal{K}}(T_i) \xrightarrow{\cong} \underline{\mathcal{K}}(T) \xrightarrow{in_{\left(T, t \circ \sigma\left(k_{\sigma(1)}, \ldots, k_{\sigma(n)}\right)\right)}} \mathcal{FK}(k_1 + \cdots + k_n),$$

one is required to show that

$$\left(T, t \circ \sigma\left(k_{\sigma(1)}, \ldots, k_{\sigma(n)}\right)\right) = (S, s \circ \sigma)\left(\left(T_{\sigma(1)}, t_{\sigma(1)}\right), \ldots, \left(T_{\sigma(n)}, t_{\sigma(n)}\right)\right).$$

To see that this equality holds, consider the first equivariance diagram (3), but with $P = \mathbb{T}[\lambda]$. Although $\mathbb{T}[\lambda]$ is only an operad up to natural isomorphism, the indicated equivariance diagram strictly commutes, which yields the equality.

5.3 Proving adjointedness

Before proving that the functor $\mathcal{F}$ is left adjoint to the forgetful functor, we first recall some facts about adjoint functors. The adjointness of the two functors $\mathcal{F}$ and $\mathcal{U}$ is equivalent to the existence of two natural transformations

$$\eta : id_{\mathrm{Coll}(\mathcal{E})} \stackrel{\cdot}{\longrightarrow} \mathcal{UF}, \quad \varepsilon : \mathcal{FU} \stackrel{\cdot}{\longrightarrow} id_{\mathrm{Oper}(\mathcal{E})},$$

called the unit and counit, respectively, of the adjunction, making the following diagrams commute [9, IV.1]:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mathcal{F}\eta"] \arrow[dr, "="'] & \mathcal{FUF} \arrow[d, "\varepsilon\mathcal{F}"] \\ & \mathcal{F} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{UFU} \arrow[d, "\mathcal{U}\varepsilon"'] & \mathcal{U} \arrow[l, "\eta\mathcal{U}"'] \arrow[dl, "="] \\ \mathcal{U} & \end{tikzcd} \tag{5}$$

These diagrams are referred to as the triangle identities, and written algebraically becomes

$$\varepsilon\mathcal{F} \circ \mathcal{F}\eta = id_{\mathcal{F}}, \quad \mathcal{U}\varepsilon \circ \eta\mathcal{U} = id_{\mathcal{U}}.$$

We shall now construct two such natural transformations.

The unit η is simple to define. Let $\mathcal{K}$ be some collection. Define $\eta_{\mathcal{K}} : \mathcal{K} \to \mathcal{U}\mathcal{F}(\mathcal{K})$ by defining it to be for each $n \geq 0$ the composition

$$\mathcal{K}(n) \xrightarrow{\cong} \underline{\mathcal{K}}\pi(c_n, id) \xrightarrow{in_{(c_n,id)}} \text{colim}\underline{\mathcal{K}}\pi_n = \mathcal{U}\mathcal{F}(\mathcal{K})(n).$$

These form a map of collections, i.e. they are Σ_n-invariant, since given $\sigma \in \Sigma_n$ the diagram

$$\begin{tikzcd}
\mathcal{K}(n) \arrow[r, "\cong"] \arrow[d, "\sigma^*"] & \underline{\mathcal{K}}\pi(c_n, id) \arrow[r, "in_{(c_n,id)}"] \arrow[d, "\sigma^* \circ id"] & \text{colim}\underline{\mathcal{K}}\pi_n \arrow[d, "\sigma^*"] \\
\mathcal{K}(n) \arrow[r, "\cong"] & \underline{\mathcal{K}}\pi(c_n, id) \arrow[r, "in_{(c_n,id)}"] & \text{colim}\underline{\mathcal{K}}\pi_n
\end{tikzcd}$$

commutes. Note here that the left vertical and right vertical maps are the group action of $\sigma \in \Sigma_n$ on $\mathcal{K}(n)$ and colim$\underline{\mathcal{K}}\pi$, respectively, while in the middle vertical map we consider σ a map $c_n \to c_n$ and apply $\underline{\mathcal{K}}\pi$ on it. The left square in the diagram obviously commutes, while the right square can be expanded to

$$\begin{tikzcd}
\underline{\mathcal{K}}\pi(c_n, id) \arrow[r, "in_{(c_n,id)}"] \arrow[d, "id"'] & \text{colim}\underline{\mathcal{K}}\pi_n \arrow[d, "\sigma^*"] \\
\underline{\mathcal{K}}\pi(c_n, \sigma) \arrow[r, "in_{(c_n,\sigma)}"] \arrow[d, "\sigma^*"'] & \text{colim}\underline{\mathcal{K}}\pi_n \\
\underline{\mathcal{K}}\pi(c_n, id). \arrow[ur, "in_{(c_n,id)}"'] &
\end{tikzcd}$$

Next, we show that η is natural. Let $f = (f_n)_{n \geq 0} : \mathcal{K} \to \mathcal{C}$ be some map of collections. Then to check naturalilty we check it in each degree:

$$\begin{tikzcd}
\mathcal{K}(n) \arrow[r, "\cong"] \arrow[d, "f_n"] & \underline{\mathcal{K}}\pi(c_n, id) \arrow[r, "in_{(c_n,id)}"] \arrow[d, "f_n \otimes id_{I^{\otimes n}}"] & \text{colim}\underline{\mathcal{K}}\pi \arrow[d, "\mathcal{U}\mathcal{F}(f)(n)"] \\
\mathcal{K}(n) \arrow[r, "\cong"] & \underline{\mathcal{K}}\pi(c_n, id) \arrow[r, "in_{(c_n,id)}"] & \text{colim}\underline{\mathcal{K}}\pi
\end{tikzcd}$$

The right square commutes by coherence, and the left one by definition.

The counit is a bit more tricky to define as it is requires use of the operad structure. We start by defining for each operad P and each tree T a map $\varepsilon'_{P,T} : \underline{\mathcal{U}P}(T) \to P(|T|)$, which we do by induction on the number of vertices in T. For the base step, suppose $|v(T)| = 0$, i.e. $T = |$. In this case $\underline{\mathcal{U}P}(T) = I$, and we define $\varepsilon'_{P,T}$ to be the unit of P, namely $e : I \to P(1)$. Suppose now that $|v(T)| > 0$, and that the map has been defined for all trees with fewer vertices. Then T decomposes as $T = c_m(T_1, \dots, T_m)$. By induction, the map is defined for each T_i, and so we define $\varepsilon'_{P,T}$ to be the composition

$$\begin{aligned}\underline{\mathcal{U}P}(T) &= \underline{\mathcal{U}P}(c_m(T_1, \dots, T_m)) \\ &= P(m) \otimes \underline{\mathcal{U}P}(T_1) \otimes \cdots \otimes \underline{\mathcal{U}P}(T_m) \\ &\xrightarrow{id \otimes \varepsilon'_{P,T_1} \otimes \cdots \otimes \varepsilon'_{P,T_m}} P(m) \otimes P(|T_1|) \otimes \cdots \otimes P(|T_m|) \\ &\xrightarrow{\gamma} P(|T|).\end{aligned}$$

We will show later that the maps $\varepsilon'_{P,T}$ are natural in T (Lemma 4).

We now define the counit: define each component ε_P of ε to be in each degree induced, for each (T, τ) in $\mathbb{T}[\lambda](n)$, by the maps

$$\underline{\mathcal{U}P}(T) \xrightarrow{\varepsilon'_{P,T}} P(|T|) \xrightarrow{\tau^*} P(n).$$

This induces a map out of $\mathcal{F}\mathcal{U}(P)(n) = \mathrm{colim}_{\mathbb{T}[\lambda](n)} \underline{\mathcal{U}P}\pi_n$, because the following diagram commutes for each $\phi : (T, \tau) \to (T', \tau')$.

$$\begin{tikzcd} \underline{\mathcal{U}P}(T') \arrow[r, "\varepsilon'_{P,T'}"] \arrow[d, "\phi^*"] & P(|T'|) \arrow[d, "\phi^*"] \arrow[dr, "(\tau')^*"] & \\ & & P(n) \\ \underline{\mathcal{U}P}(T) \arrow[r, "\varepsilon'_{P,T}"] & P(|T|) \arrow[ur, "\tau^*"'] & \end{tikzcd}$$

The square commutes by Lemma 4, and the triangle commutes because $\tau^*\phi^* = \phi\tau^* = (\tau')^*$.

Now that we have shown that ε_P is well defined, we need to show that it is a map of operads. That is, for $n, k_1, \dots, k_n \in \mathbb{N}$, we need to show that

$$\begin{tikzcd} \mathcal{F}\mathcal{U}(P)(n) \otimes \bigotimes_{i=1}^{n} \mathcal{F}\mathcal{U}(P)(k_i) \arrow[r, "\gamma"] \arrow[d, "\varepsilon_P(n) \otimes \bigotimes_{i=1}^{n} \varepsilon_P(k_i)"'] & \mathcal{F}\mathcal{U}(P)\left(\sum_{i=1}^{n} k_i\right) \arrow[d, "\varepsilon_P\left(\sum_{i=1}^{n} k_i\right)"] \\ P(n) \otimes \bigotimes_{i=1}^{n} P(k_i) \arrow[r, "\gamma"] & P\left(\sum_{i=1}^{n} k_i\right) \end{tikzcd}$$

commutes. To show this we first show that, for all $(S, \sigma), (T_1, \tau_1), \dots, (T_n, \tau_n)$,

$|S| = n$, $(T, \tau) = (S, \sigma)((T_1, \tau_1), \ldots, (T_n, \tau_n))$, the outer diagram commutes:

$$\begin{tikzcd}
\underline{\mathcal{U}P}(\pi(S,\sigma)) \otimes \bigotimes_{i=1}^{n} \underline{\mathcal{U}P}(\pi(T_i,\tau_i)) \arrow[rr, "sym_\sigma"] \arrow[d, "\varepsilon'_S \otimes \bigotimes_{i=1}^{n} \varepsilon'_{T_i}"'] & & \underline{\mathcal{U}P}(\pi(T,\tau)) \arrow[d, "\varepsilon'_T"] \\
P(|S|) \otimes \bigotimes_{i=1}^{n} P(|T_i|) \arrow[r, "id\otimes\sigma"] \arrow[d, "\sigma^* \otimes \bigotimes_{i=1}^{n} \tau_i^*"'] & P(|S|) \otimes \bigotimes_{i=i}^{n} P(|T_{\sigma^{-1}(i)}|) \arrow[r, "\gamma"] & P(|T|) \arrow[d, "\tau^*"] \\
P(n) \otimes \bigotimes_{i=1}^{n} P(k_i) \arrow[rr, "\gamma"] & & P\left(\sum_{i=1}^{n} k_i\right),
\end{tikzcd} \tag{6}$$

with the maps out of the top two entries being natural with respect to maps of labeled trees. We have here removed P from each ε' for simplicity, and we have added a subscript to *sym* to clarify that it is dependent on σ. The fact that the lower square commutes is the equivariance diagram (3), so we need only to show that the top half commutes. Furthermore, we need only show it commutes in the special case $\sigma = id$, which can be seen by observing the diagram

$$\begin{tikzcd}
\underline{\mathcal{U}P}(\pi(S,\sigma)) \otimes \bigotimes_{i=1}^{n} \underline{\mathcal{U}P}(\pi(T_i,\tau_i)) \arrow[d, "id\otimes\sigma"] \arrow[dr, "sym_\sigma"] & \\
\underline{\mathcal{U}P}(\pi(S,id)) \otimes \bigotimes_{i=1}^{n} \underline{\mathcal{U}P}(\pi(T_{\sigma^{-1}(i)},\tau_{\sigma^{-1}(i)})) \arrow[r, "sym_{id}"] \arrow[d, "\varepsilon'_S \otimes \bigotimes_{i=1}^{n} \tau^*_{\sigma^{-1}(i)} \circ \varepsilon'_{T_{\sigma^{-1}(i)}}"'] & \underline{\mathcal{U}P}(\pi(T,\tau)) \arrow[d, "\left(\tau_{\sigma^{-1}(1)} \oplus \cdots \oplus \tau_{\sigma^{-1}(n)}\right)^* \circ \varepsilon'_T"'] \\
P(n) \otimes \bigotimes_{i=1}^{n} P(k_{\sigma^{-1}(i)}) \arrow[r, "\gamma"] \arrow[d, "\sigma^* \otimes \sigma^{-1}"'] & P(k_{\sigma^{-1}(1)} + \cdots + k_{\sigma^{-1}(n)}) \arrow[d, "\sigma(k_1,\ldots,k_n)^*"] \\
P(n) \otimes \bigotimes_{i=1}^{n} P(k_i) \arrow[r, "\gamma"] & P(k_1 + \cdots + k_n).
\end{tikzcd}$$

Note that the left and right vertical compositions are $\sigma^* \circ \varepsilon'_S \otimes \bigotimes_{i=1}^{n} \tau_i^* \circ \varepsilon'_{T_i}$ and $\tau^* \circ \varepsilon'_T$, respectively. We show that the top half of (6) commutes in the case $\sigma = id$ by induction on the number of vertices of T.

If $|v(T)| = 0$, then $S = T_1 = T = |$ and diagram (6) reduces to

$$\begin{tikzcd}
I \otimes I \arrow[r, "\lambda=\rho"] \arrow[d, "e\otimes e"'] & I \arrow[d, "e"] \\
P(1) \otimes P(1) \arrow[r, "\gamma"] & P(1),
\end{tikzcd}$$

which commutes by the unit laws of an operad, and which satisfies the (trivial) naturality requirement.

Now suppose that $|v(T)| > 0$, and that we have shown that the required properties hold for all trees with fewer vertices. Then we can decompose T and S as $T = c_m(R_1, \ldots, R_m)$ (think R for "cut at the Root") and $S = c_m(S_1, \ldots, S_m)$. Let $h_j = |S_j|$. We give the trees $T_1, \ldots, T_n$ some additional names: let $T_i^j = T_{k_1} + \cdots + k_{j-1} + i$, where $1 \leq i \leq h_j$. We then have the relation $R_j = S_j(T_1^j, \ldots, T_{h_j}^j)$. We expand the upper half of diagram (6) to

$$\begin{tikzcd}
\underline{\mathcal{U}P}(\pi(S,id)) \otimes \bigotimes_{i=1}^{n} \underline{\mathcal{U}P}(\pi(T_i,\tau_i)) \arrow[r, "sym"] \arrow[d, "="'] & \underline{\mathcal{U}P}(\pi(T,\tau)) \arrow[d, "="] \\
\underline{\mathcal{U}P}(S) \otimes \bigotimes_{i=1}^{n} \underline{\mathcal{U}P}(T_i) \arrow[r, "sym"] \arrow[d, "\left(id \otimes \bigotimes_{j=1}^{m} \varepsilon'_{S_j}\right) \otimes \bigotimes_{i=1}^{n} \varepsilon'_{T_i}"'] & \underline{\mathcal{U}P}(T) \arrow[d, "id \otimes \bigotimes_{j=1}^{m} \varepsilon'_{R_i}"] \\
\left(P(m) \otimes \bigotimes_{j=1}^{m} P(|S_j|)\right) \otimes \bigotimes_{i=1}^{n} P(|T_i|) \arrow[r, "\left(id \otimes \bigotimes_{j=1}^{m} \gamma\right) \circ \cong"] \arrow[d, "\gamma \otimes \bigotimes_{i=1}^{n} id"'] & P(m) \otimes \bigotimes_{j=1}^{m} P(|R_j|) \arrow[d, "\gamma"] \\
P(|S|) \otimes \bigotimes_{i=1}^{n} P(|T_i|) \arrow[r, "\gamma"] & P(|T|)
\end{tikzcd} \tag{7}$$

The top square obviously commutes, and the bottom square commutes as it is the associativity axiom for an operad. To see that the middle square commutes, transpose it and expand it to

$$\begin{tikzcd}
P(m) \otimes \bigotimes_{j=1}^{m} \underline{\mathcal{U}P}(S_j) \otimes \bigotimes_{i=1}^{n} \underline{\mathcal{U}P}(T_i) \arrow[r, "id \otimes \bigotimes_{j=1}^{m} \varepsilon'_{S_j} \otimes \bigotimes_{i=1}^{n} \varepsilon'_{T_i}"] \arrow[d, "\cong"'] & P(m) \otimes \bigotimes_{j=1}^{m} P(S_j) \otimes \bigotimes_{i=1}^{n} P(T_i) \arrow[d, "\cong"] \\
P(m) \otimes \bigotimes_{j=1}^{m} \left(\underline{\mathcal{U}P}(S_j) \otimes \bigotimes_{i=1}^{h_j} \underline{\mathcal{U}P}(T_i^j)\right) \arrow[r, "id \otimes \bigotimes_{j=1}^{m} \left(\varepsilon'_{S_j} \otimes \bigotimes_{i=1}^{h_j} \varepsilon'_{T_i^j}\right)"] \arrow[d, "id \otimes \bigotimes_{j=1}^{m} sym_{id}"'] & P(m) \otimes \bigotimes_{j=1}^{m} \left(P(S_j) \otimes \bigotimes_{i=1}^{h_j} P(T_i^j)\right) \arrow[d, "id \otimes \bigotimes_{j=1}^{m} \gamma"] \\
P(m) \otimes \bigotimes_{j=1}^{m} \underline{\mathcal{U}P}(R_j) \arrow[r, "id \otimes \bigotimes_{i=1}^{m} \varepsilon'_{R_i}"] & P(m) \otimes \bigotimes_{j=1}^{m} P(|R_j|).
\end{tikzcd}$$

The top square commutes by coherence, and the lower one commutes by induction since each R_i has fewer vertices than T. The fact that the diagram satisfies the naturality requirement follows quickly from the fact that it hold for each R_i.

Finally, the transformation just defined is natural since any map of operads $f = (f_n)_{n\geq 0} : P \to Q$ commutes with the operad compositions γ, and each component of ε is more or less a large composition of γs.

Theorem 3. *The functor $\mathcal{F} : \mathrm{Coll}(\mathcal{E}) \to \mathrm{Oper}(\mathcal{E})$ is left adjoint to the forgetful functor $\mathcal{U} : \mathrm{Oper}(\mathcal{E}) \to \mathrm{Coll}(\mathcal{E})$.*

Proof. The proof is based on the proof for Theorem 1.91 in [11]. All that remains is to show that the natural transformations just defined satisfy the triangle identities.

Now that we have defined the unit $\eta : id_{Coll(\mathcal{E})} \overset{\cdot}{\longrightarrow} \mathcal{UF}$ and the counit $\varepsilon : \mathcal{FU} \overset{\cdot}{\longrightarrow} id_{Oper(\mathcal{E})}$ we show that the triangle identities are satisfied.

First we show that the second diagram in (5) commutes. We pick an operad P and $n \geq 0$. The following diagram clearly commutes, which proves it.

$$\begin{tikzcd}
P(n) \arrow[r, "\eta\mathcal{U}_P(n)"] \arrow[dr, "\cong"'] & \mathcal{FU}P(n) \arrow[r, "\mathcal{U}\varepsilon_P(n)"] & P(n) \\
& \underline{\mathcal{U}P}(c_n) \arrow[u, "in_{(c_n,id)}"'] \arrow[ur, bend right, "\cong"'] &
\end{tikzcd}$$

To show that first diagram in (5) commutes, we show that it commutes for each collection, $\mathcal{K}$, and for each degree $n \geq 0$. Let (T, τ) be some n-leafed labeled tree. Consider the following diagram:

$$\begin{tikzcd}
\underline{\mathcal{K}}\pi_n(T,\tau) \arrow[r, "\underline{\eta_{\mathcal{K}}}\pi_{(T,\tau)}"] \arrow[d, "in_{(T,\tau)}"] & \underline{\mathcal{UFK}}\pi_n(T,\tau) \arrow[r, "\varepsilon'_{\mathcal{FK},T}"] \arrow[d, "in_{(T,\tau)}"] & \mathcal{FK}(n) \arrow[d, "\tau^*"] \\
\mathcal{FK}(n) \arrow[r, "\mathcal{F}(\eta_{\mathcal{K}})(n)"] & \mathcal{FUFK}(n) \arrow[r, "\varepsilon_{\mathcal{FK}}(n)"] & \mathcal{FK}(n)
\end{tikzcd} \tag{8}$$

Both squares in this diagram commute by definition. We would like to show that the composition of the bottom two horizontal maps is the identity. To do this we include into the colimit $\mathcal{FK}(n)$ a factor $\underline{\mathcal{K}}\pi_n(T,\tau)$ and show that the composition is $in_{(T,\tau)}$. We do this with an inductive argument on the number of vertices in the tree. We first do the argument assuming all labelings to be the trivial/planar labelings, and afterwards show that it holds for all labelings.

For the base case of T having 0 vertices, $T = |$, and diagram (8) becomes

$$\begin{tikzcd}
I \arrow[r, "id_I"] \arrow[d, "in_|"] & I \arrow[r, "in_|"] \arrow[d, "in_|"] & \mathcal{FK}(1) \arrow[d, "id_{\mathcal{FK}(1)}"] \\
\mathcal{FK}(1) \arrow[r, "\mathcal{F}(\eta_{\mathcal{K}})(1)"] & \mathcal{FUFK}(1) \arrow[r, "\varepsilon_{\mathcal{FK}}(1)"] & \mathcal{FK}(1),
\end{tikzcd}$$

and it is clear. Suppose now that T has more than 0 vertices. Then (T, id) decomposes uniquely as $(c_n, id)((T_1, id), \ldots, (T_m, id))$, and by definition the composition along the top and right sides of diagram (8) becomes

$$\underline{\mathcal{K}}(T) \xrightarrow{\eta_{\mathcal{K}}(m)\otimes\bigotimes_{i=1}^{m} \underline{\eta_{\mathcal{K}}}_{T_i}} \mathcal{F}\mathcal{K}(m)\otimes\bigotimes_{i=1}^{m} \underline{\mathcal{U}\mathcal{F}\mathcal{K}}(T_i) \xrightarrow{id\otimes\bigotimes_{i=1}^{m} \varepsilon'_{\mathcal{K},T_i}} \mathcal{F}\mathcal{K}(m)\otimes\bigotimes_{i=1}^{m} \mathcal{F}\mathcal{K}(|T_i|) \xrightarrow{\gamma} \mathcal{F}\mathcal{K}(n).$$

Since the compositions $\varepsilon'_{\mathcal{K},T_i} \circ \underline{\eta_{\mathcal{K}}}_{T_i}$ are $in_{(T_i,id)}$ by induction hypothesis, we need only check that $\gamma \circ \eta_{\mathcal{K}}(m) \otimes \bigotimes_{i=1}^{m} in_{T_i,id} = in_{(T,id)}$. To see this, consider the following commutative diagram.

$$\begin{tikzcd}
\mathcal{K}(m)\otimes\bigotimes_{i=1}^{m}\underline{\mathcal{K}}(T_i) \arrow[r, "\eta_{\mathcal{K}}\otimes\bigotimes_{i=1}^{m} in"] \arrow[d, "\cong\otimes\bigotimes_{i=1}^{m} id"'] & \mathcal{F}\mathcal{K}(m)\otimes\bigotimes_{i=1}^{m}\mathcal{F}\mathcal{K}(|T_i|) \arrow[r, "\gamma"] \arrow[d, "\cong"] & \mathcal{F}\mathcal{K}(n) \\
\underline{\mathcal{K}}(c_m)\otimes\bigotimes_{i=1}^{m}\underline{\mathcal{K}}(T_i) \arrow[r, "in"] \arrow[ur, "in\otimes\bigotimes_{i=1}^{m} in"] \arrow[drr, "\cong"'] & \mathrm{colim}\underline{\mathcal{K}}\pi_m\otimes\bigotimes_{i=1}^{m}\underline{\mathcal{K}}\pi_{|T_i|} \arrow[r, "\cong"] & \mathrm{colim}\underline{\mathcal{K}}\pi_n\circ -(-,\ldots,-) \arrow[u] \\
 & & \underline{\mathcal{K}}\pi_n(c_m(T_1,\ldots,T_m)) \arrow[u, "in_{(c_m,T_1,\ldots,T_m)}"]
\end{tikzcd}$$

The top right vertical morphism is the one described after defining the operad composition on page 21. Starting in the upper left corner and going down and then down right gives us the identity, by coherence of symmetric monoidal categories. Going from the lower right to the top right is the inclusion, and so we have shown that the composition along the top is the inclusion.

Now to show that it holds for all labeled trees, and not just the trivially labeled ones. Assume now that the label of (T, τ) need not be trivial. Consider then the diagram

$$\begin{tikzcd}
\underline{\mathcal{K}}\pi_n(T,id) \arrow[r, "in_{(T,id)}"] \arrow[d, "id"] & \mathcal{F}\mathcal{K}(n) \arrow[r, "\varepsilon_{\mathcal{F}\mathcal{K}}\circ\mathcal{F}(\eta_{\mathcal{K}})(n)"] \arrow[d, "\tau^*"] & \mathcal{F}\mathcal{K}(n) \arrow[d, "\tau^*"] \\
\underline{\mathcal{K}}\pi_n(T,\tau) \arrow[r, "in_{(T,\tau)}"] & \mathcal{F}\mathcal{K}(n) \arrow[r, "\varepsilon_{\mathcal{F}\mathcal{K}}\circ\mathcal{F}(\eta_{\mathcal{K}})(n)"] & \mathcal{F}\mathcal{K}(n).
\end{tikzcd}$$

The left square commutes by definition and the right square commutes since $\varepsilon_{\mathcal{F}\mathcal{K}} \circ \mathcal{F}(\eta_{\mathcal{K}})$ is a map of operads. The composition of the top two horizontal morphisms is $in_{(T,id)}$, and since $\tau^* \circ in_{(T,id)} = in_{(T,\tau)}$, going from the top left to the bottom right in the diagram is equal to $in_{(T,\tau)}$, and this is equal to the composition of the bottom two horizontal maps. $\square$

Lemma 4. *The maps $\varepsilon'_{P,T} : \underline{\mathcal{U}P}(T) \to P(|T|)$ are natural in T, i.e. for each isomorphism of trees $\phi : T \to T'$, the diagram*

$$\begin{tikzcd}
\underline{\mathcal{U}P}(T') \arrow[r, "\phi^*"] \arrow[d, "\varepsilon'_{P,T'}"] & \underline{\mathcal{U}P}(T) \arrow[d, "\varepsilon'_{P,T}"] \\
P(|T'|) \arrow[r, "\phi^*"] & P(|T|)
\end{tikzcd} \tag{9}$$

commutes.

Proof. We prove this by induction on the number of vertices. If the number of vertices in T is 0, then diagram (9) becomes

$$\begin{tikzcd}
I \arrow[r, "id"] \arrow[d, "e"] & I \arrow[d, "e"] \\
P(1) \arrow[r, "id"] & P(1),
\end{tikzcd}$$

which clearly commutes. If the number of vertices in T is greater than 0, then $T = c_m(T_1, \ldots, T_m)$ and $T' = c_m(T'_1, \ldots, T'_m)$ and we can expand the diagram to the following:

$$\begin{tikzcd}
\underline{\mathcal{U}P}(T') \arrow[r, "id\otimes\bigotimes_{i=1}^m \underline{\mathcal{U}P}(\phi_{\sigma^{-1}(i)})"] \arrow[d, "id\otimes\bigotimes_{i=1}^m \varepsilon'_{P,T'_i}"] & P(m) \otimes \bigotimes_{i=1}^m \underline{\mathcal{U}P}(T_{\sigma^{-1}(i)}) \arrow[r, "\sigma^*\otimes\sigma^{-1}"] \arrow[d, "id\otimes\bigotimes_{i=1}^m \varepsilon'_{P,T_{\sigma^{-1}(i)}}"] & \underline{\mathcal{U}P}(T) \arrow[d, "id\otimes\bigotimes_{i=1}^m \varepsilon'_{P,T_i}"] \\
P(m) \otimes \bigotimes_{i=1}^m P(|T'_i|) \arrow[r, "id\otimes\bigotimes_{i=1}^m \phi^*_{\sigma^{-1}(i)}"] \arrow[d, "\gamma"] & P(m) \otimes \bigotimes_{i=1}^m P(|T_{\sigma^{-1}(i)}|) \arrow[r, "\sigma^*\otimes\sigma^{-1}"] \arrow[d, "\gamma"] & P(m) \otimes \bigotimes_{i=1}^m P(|T_i|) \arrow[d, "\gamma"] \\
P(n) \arrow[r, "\bigoplus_{i=1}^m \phi^*_{\sigma^{-1}(i)}"] & P(n) \arrow[r, "\sigma(|T_1|,\ldots,|T_m|)^*"] & P(n)
\end{tikzcd}$$

The top left square commutes by induction, the top right commutes by the coherence of a symmetric monoidal category, and the bottom two squares commute by the equivariance axioms for an operad. Furthermore, the composition of the horizontal maps at the bottom of the diagram is exactly ϕ^*.

□

6 Braided operads

For $n \geq 0$, let B_n denote the braid group on n strings. Let $q_n : B_n \to \Sigma_n$ denote the natural projection, or just q if there is no possibility of confusion.

By considering objects of $\mathcal{E}$ with right actions of the braid groups, we can define the category of braided collections, $\mathrm{Coll}^B(\mathcal{E})$, in the same way we defined the category of symmetric collections. Furthermore, one can define braided operads:

Definition 8. [5, Definition 3.1] A braided operad in a symmetric monoidal category $\mathcal{E}$ is a braided collection, $P = (P(n))_{n\geq 0}$, together with a unit map, $e : I \to P(1)$, and for each collection of nonnegative numbers $n, k_1, \dots, k_n$, a composition map

$$\gamma : P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) \to P(k_1 + \cdots + k_n),$$

making certain associativity, unit, and equivariance diagrams commute. The associativity and unit diagrams are identical to those of a symmetric operad, while the equivariance requirement is as follows. For each collection of braids $\beta \in B_n$, $\alpha_i \in B_{k_i}$ for $1 \leq i \leq n$, let $\beta\,(k_1, \dots, k_n)$ denote the braid in $B_{k_1+\cdots+k_n}$ which braids the blocks of k_1, k_2, ... , and k_n strings as β braids n strings, and let $\alpha_1 \oplus \cdots \oplus \alpha_n$ denote the braid in $B_{k_1+\cdots+k_n}$ which braids the first k_1 strings as α_1 does it, the next k_2 in the way α_2 does it, etc. Then the following two diagrams commute:

$$\begin{tikzcd}
P(n) \otimes \bigotimes\limits_{i=1}^{n} P(k_i) \arrow[r, "\beta^* \otimes q(\beta)^{-1}"] \arrow[d, "\gamma"'] & P(n) \otimes \bigotimes\limits_{i=1}^{n} P(k_{q(\beta)(i)}) \arrow[d, "\gamma"] \\
P(k_1 + \cdots + k_n) \arrow[r, "\beta\left(k_{q(\beta)(1)}, \dots, k_{q(\beta)(n)}\right)^*"] & P(k_{q(\beta)(1)} + \cdots + k_{q(\beta)(n)})
\end{tikzcd}$$

and

$$\begin{tikzcd}
P(n) \otimes \bigotimes\limits_{i=1}^{n} P(k_i) \arrow[r, "id \otimes \bigotimes_{i=1}^{n} \alpha_i"] \arrow[d, "\gamma"'] & P(n) \otimes \bigotimes\limits_{i=1}^{n} P(k_i) \arrow[d, "\gamma"] \\
P(k_1 + \cdots + k_n) \arrow[r, "(\alpha_1 \oplus \cdots \oplus \alpha_n)^*"] & P(k_1 + \cdots + k_n).
\end{tikzcd}$$

Morphisms of braided operads are defined analagously to morphisms of symmetric operads, and so braided operads in $\mathcal{E}$ make up a category, $\mathrm{Oper}^B(\mathcal{E})$. Braided operads occurs in places where it is not only relevant that something moved to a place, but also the path it took to get there. For an example of a braided operad, see section 3 of [5]. We will not investigate whether or not it is possible to transfer the model structure of some monoidal model category to its category of braided operads, but if one were to do so, and attempt it in the manner one does it for symmetric operads, a crucial step would be the construction of a free functor $\mathrm{Coll}^B(\mathcal{E}) \to \mathrm{Oper}^B(\mathcal{E})$. One might hope that one could simply take the construction we have performed in section 5, and replace all permutations with braids, so to speak.

In the process of investigating adjunctions $\mathrm{Coll}(\mathcal{E}) \rightleftarrows \mathrm{Oper}(\mathcal{E})$ and $\mathrm{Coll}^B(\mathcal{E}) \rightleftarrows \mathrm{Oper}^B(\mathcal{E})$, it may be of interest to ask whether or not there exist adjunctions $\mathrm{Coll}^B(\mathcal{E}) \rightleftarrows \mathrm{Coll}(\mathcal{E})$ and $\mathrm{Oper}^B(\mathcal{E}) \rightleftarrows \mathrm{Oper}(\mathcal{E})$, induced by the quotient maps $q_n : B_n \to \Sigma_n$. There is, and the forgetful functors are easy to give: any symmetric collection or operad can be considered to be a braided one by composing the symmetric group actions with the quotient maps, letting the action of $\beta \in B_n$ be that of $q_n(\beta)$. Functoriality is immediate since nothing is changed in the morphisms themselves, and the commutativity of any axiom-diagram is immediately inherited from the corresponding symmetric one. It is also quite easy to define the free functors; they are, roughly, defined by modding out by the kernel of q_n in each degree. It is, however, necessary to do some work in showing that this is well defined, and that the symmetric axiom-diagrams commute.

We begin with the case of the collections. This follow from a more general result, which we state here.

Lemma 5. *Let G be some group, H some normal subgroup, and $\mathcal{C}$ some category with colimits of the shape $H^{\mathrm{op}} \to \mathcal{C}$. Then there is a functor $\mathcal{F} : \mathcal{C}^G \to \mathcal{C}^{G/H}$ which is left adjoint to the forgetful functor $\mathcal{U} : \mathcal{C}^{G/H} \to \mathcal{C}^G$.*

Proof. We start by showing that $\mathcal{F}$ is well defined on objects. Let c be some object of $\mathcal{C}^G$. Let $[g]$ be some element of G/H, and define the action of it, $[g]^*$, to be the map induced by

$$c \xrightarrow{g^*} c \xrightarrow{proj.} c/H.$$

To show that this is well defined, we must show that for each $h \in H$, the following diagram commutes:

$$\begin{tikzcd} c \arrow[d, "h^*"'] \arrow[dr, "proj.\circ g^*"] & \\ c \arrow[r, "proj.\circ g^*"'] & c/H \end{tikzcd} .$$

Since H is normal, we can pick $h' \in H$ such that $hg = gh'$. Expand the above diagram to the following one

$$\begin{tikzcd} c \arrow[r, "g^*"] \arrow[d, "h^*"'] & c \arrow[d, "h'^*"'] \arrow[dr, "proj."] & \\ c \arrow[r, "g^*"] & c \arrow[r, "proj."'] & c/H \end{tikzcd} ,$$

which clearly commutes. We now show that the action is independent of our choice of representative: let $g' \in [g]$ be some other representative, then there exists some $h \in H$ such that $g' = gh$, and so we get the commutative diagram

$$\begin{tikzcd} c \arrow[r, "g'^*"] \arrow[d, "g^*"'] & c \arrow[d, "proj."] \\ c \arrow[ur, "h^*"] \arrow[r, "proj."'] & c/H, \end{tikzcd}$$

which proves it. For some $[g], [g'] \in G/H$, the fact that $[g]^*[g']^* = ([g'][g])^*$ is easily shown to be the case, proving that $\mathcal{F}$ is well defined on objects. In a similar fashion, one sees that $\mathcal{F}$ is well defined on morphisms.

To show that there is an adjunction, let X be some object of $\mathcal{E}^G$ and Y some object of $\mathcal{E}^{G/H}$, and define the isomorphism $\mathcal{E}^{G/H}(\mathcal{F}(X), Y) \to \mathcal{E}^G(X, \mathcal{U}(Y))$ by sending $f/H : X/H \to Y$ to $f : X \to Y$, and define the isomorphism in the other direction to send f to f/H. These are obviously inverse, so we need only show that they are well defined and natural. That the first is well defined is clear, as it is simply pre-composition with the projection $X \to X/H$. That the second is well defined is clear since the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h^*"'] & Y \arrow[d, "h^*=q(h)^*=id_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes, where $h \in H$. That the isomorphism $\mathcal{E}^{G/H}(\mathcal{F}(X), Y) \to \mathcal{E}^G(X, \mathcal{U}(Y))$ is natural can be seen by chasing elements around, and this also proves the naturality of the inverse isomorphism. □

Proposition 1. *There is an adjunction* $\mathcal{F} : \mathrm{Coll}^B(\mathcal{E}) \rightleftarrows \mathrm{Coll}(\mathcal{E}) : \mathcal{U}$*, where* $\mathcal{F}$ *is the left adjoint, defined by quotienting out by* $\ker(q_n)$ *in degree* n*, and* $\mathcal{U}$ *is the the right adjoint, defined to be the obvious forgetful functor.*

Proof. For each $n \geq 0$, use Lemma 5 with $\mathcal{C} = \mathcal{E}$, $G = B_n$, and $H = \ker(q_n)$ to get an adjunction $\mathcal{E}^{B_n} \rightleftarrows \mathcal{E}^{\Sigma_n}$. An adjunction in each degree yields an adjunction on the product, so we have an adjunction $\mathrm{Coll}^B(\mathcal{E}) \rightleftarrows \mathrm{Coll}(\mathcal{E})$. □

Proposition 2. *There is an adjunction* $\mathcal{F} : \mathrm{Oper}^B(\mathcal{E}) \rightleftarrows \mathrm{Oper}(\mathcal{E}) : \mathcal{U}$*, where* $\mathcal{F}$ *is the left adjoint, defined by quotienting out by* $\ker(q_n)$ *in degree* n*, and* $\mathcal{U}$ *is the right adjoint, defined to be the obvious forgetful functor.*

Proof. That the forgetful functor is well defined is clear. The free functor clearly maps to collections and maps of collections, and so we need to define the operad structure and show that all necessary axioms are satisfied, before showing that the two functors are adjoint. Let $P = (P(n))_{n \geq 0}$ be some braided operad. Then, define the operad unit of $\mathcal{F}(P)$ to be $I \xrightarrow{e} P(1) = P(1)/\Sigma_1$, and define the operad composition as follows. Given $n, k_1, \dots, k_n$, define an action of $\ker(q_n) \times \ker(q_{k_1}) \times \cdots \times \ker(q_{k_n})$ on $P(k_1 + \cdots + k_n)$ by sending $(s, s_1, \dots, s_n)$ to $(s(k_1, \dots, k_n) \circ s_1 \oplus \cdots \oplus s_n)^*$. Then the (braid) operad composition in P induces a map

$$P(n)/\ker(q_n) \otimes \bigotimes_{i=1}^{n} P(k_i)/\ker(q_{k_i}) \to P(k_1 + \cdots + k_n)/\ker(q_n) \times \prod_{i=1}^{n} \ker(q_{k_i}),$$

since the necessary diagram

$$\begin{tikzcd}
P(n) \otimes \bigotimes_{i=1}^{n} P(k_i) \arrow[r, "\gamma"] \arrow[d, "s^* \otimes \bigotimes_{i=1}^{n} s_i^*"'] & P(k_1 + \cdots + k_n) \arrow[d, "(s(k_1,\ldots,k_n) \circ s_1 \oplus \cdots \oplus s_n)^*"] \\
P(n) \otimes \bigotimes_{i=1}^{n} P(k_i) \arrow[r, "\gamma"] & P(k_1 + \cdots + k_n)
\end{tikzcd}$$

can be reduced to the two equivariance diagrams (note that we don't get any shuffling around since s is in the kernel). The composition is defined to be this map, followed by the map

$$P(k_1 + \cdots + k_n)/\ker(q_n) \times \prod_{i=1}^{n} \ker(q_{k_i}) \to P(k_1 + \cdots + k_n)/\ker(q_{k_1+\cdots+k_n}),$$

which is the projection map, where we consider $\ker(q_n) \times \ker(q_{k_1}) \times \cdots \times \ker(q_{k_n})$ to be a subgroup of $\ker(q_{k_1+\cdots+k_n})$ by the assignment $(s, s_1, \ldots, s_n) \mapsto s(k_1, \ldots, k_n) \circ s_1 \oplus \cdots \oplus s_n$.

The unit condition is obviously satisfied, while the associativity condition can be checked by checking it on each component of the colimit. That the isomorphisms of hom sets are well defined and natural is easy to check. □

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