# Representing Reliability-Based Peer Pressure

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## Abstract

This thesis proposes and explores three strategies for aggregating individual preferences: the non-contentious, majority, and plurality methods. The proposed methods rely on the lexicographic rule but use a reliability ordering over sets of agents instead of only agents. The preservation of properties (reflexivity, transitivity, totality, antisymmetry, and unanimity) by each method are examined and compared. The thesis contributes to understanding preference aggregation strategies and their relevance in real-life decision-making contexts.

# Contents

# 1 Introduction

Imagine a group of four friends, Alice, Bob, Carol, and Dave, who wants to play a board game. There are three different options, option 1, option 2 and option 3. Alice prefers option 1 over option 2 and option 2 over option 3. Bob prefers options 1 and 3 over option 2, but can't compare option 1 and 3. Carol prefers option 2 and 3 over option 1, and Dave considers the three options equal. The group needs to find a way to aggregate their preferences in a way that the group as a whole is happy with. How can these four preferences be aggregated into one? And what properties of the original preferences are transferred into the new, updated preference ordering?

The friends could try to use an aggregation method to aggregate their preferences into one. There are several methods to choose from, it is choosing the right one that is difficult. Most preference aggregation methods have a property called anonymity, meaning they see all the people as equally important. When choosing a board game to play with friends, anonymity might only sometimes be desirable as the preferences of experienced players might be more helpful and therefore of more weight, ensuring a more enjoyable and engaging gaming experience for the group rather than relying solely on anonymous suggestions.

In situations involving peer pressure, there is no explicit search for the group's preference like in voting systems. Instead, the focus is on how an individual is affected by the opinions or preferences of others, in particular when they consider some people more important than others. In such cases, individuals may prioritize the preferences of certain friends over others. In these and similar scenarios, there is a desire for an aggregation method that is not anonymous, such as the lexicographic rule. The lexicographic rule uses a priority-based reliability ordering to rank each person.

In their work on collective decision-making, *Velázquez-Quesada* (2017) employed the lexicographic rule to model a form of deliberation where each agent updates their preferences based on the preferences of others and their own reliability ordering. In contrast, this thesis adopts a simplified approach by using a single reliability ordering, aggregating agents' preferences similar to a voting method without requiring individual preference updates by each agent.

This thesis introduces a novel approach by extending the lexicographic rule to encompass preferences with a reliability ordering over sets of agents (instead of a reliability ordering over agents), allowing for situations where the number of people involved becomes significant. This generalization enables the application of the lexicographic rule even when dealing with scenarios where the preferences of multiple agents need to be considered. Three different strategies will be presented: the non-contentious, majority, and plurality methods. These strategies share the concept of certain sets of agents being more reliable than others, and their respective properties will be explored and compared to assess their individual strengths and weaknesses.

This thesis will start by looking at relevant theory and related areas of preference aggregation in Chapter 2. This includes theory on friendship dynamics since this thesis revolves around peer pressure and the lexicographic rule, which separates the preference aggregation methods in this thesis from, e.g., voting systems aggregation procedures. The thesis then, in Chapter 3, presents the basic definitions needed for understanding and looking at the properties of the three strategies, which will be presented in the same Chapter. In Chapter 4, the strategies will be examined more extensively, and some properties of each strategy will be presented. There will also be a small discussion section after each property. The thesis concludes with chapter 5, including the conclusion of the thesis and possible future work.

## 2 Background and Related Areas

Peer pressure and friendship dynamics are well-established phenomena extensively studied in the social sciences. Peer pressure is pressure by peer group members to take a specific action, adapt specific values, or otherwise conform to be accepted (*Shamsa Aziz et al.*, 2011). This thesis portrays peer pressure by a reliability ordering over sets of agents with different levels of influence. Friendship dynamics, characterized by the close relationships and interactions between individuals, shape preferences and decision-making processes. The strength of a friendship someone has with their peers also plays a part (*Shah and Jehn*, 1993). These dynamics can significantly influence an individual's decision-making process, leading them to conform to group norms or adopt the beliefs and preferences of their peers. In the context of collective decision-making and preference aggregation, this can result in a process where individual preferences are merged or fused with those of others, ultimately affecting the outcome of the decision-making process.

One approach to decision-making and preference aggregation is voting systems, which consider individual preferences and combine them to determine a group decision. However, these systems have limitations, particularly in situations where individuals have different levels of reliability in the decision-making process. In these cases, the aggregation of preferences may not accurately reflect the nuances or diversity of preferences within the group.

A method known as the lexicographic rule can be employed to address this issue. It involves considering the preferences of individuals based on their reliability or knowledge in decision-making, thereby prioritizing their inputs. This approach allows for a more accurate representation of group preferences, especially in situations involving peer pressure or friendship dynamics (*Fishburn*, 1974).

This thesis focuses on representing reliability-based peer pressure and how it can utilize the lexicographic rule. While the lexicographic rule relies on a reliability ordering over agents, this thesis aims to repurpose it to rely on a reliability ordering over sets of agents. Using a reliability ordering over sets of agents instead of just over agents could offer a more nuanced and realistic approach to understanding decision-making processes in group settings, incorporating social influence dynamics, and capturing the complexity of preference aggregation. It may also facilitate more dynamic and realistic modeling of collective decision-making processes.

### 2.1 Friendship Dynamics

While the concept of friendship appears to be universally understood, there is no clear, agreed-upon definition. The term "friend" is not only a categorical label, such as cousin or colleague, which indicates special social positions, but rather weighted on quality. Since people have their own measurement for where the line between acquaintance and friend lies, it might be impossible to create a single definition for friendship (*Allan*, 1989).

The extent to which a person is susceptible to being influenced by their friends depends on how strong the relationship is. People are more susceptible to being influenced by their close friends than by more casual acquaintances because close friends are perceived as being more trustworthy (*Hallinan and Williams*, 1990).

Peer pressure is a social phenomenon in which individuals are influenced by their peers' beliefs, attitudes, or behaviors. It occurs when individuals feel the urge to conform to group norms in order to gain acceptance or avoid rejection. This pressure can be exerted both explicitly, through direct persuasion or coercion, and implicitly, through unspoken expectations and cues. Peer pressure can have positive or negative consequences, depending on the situation and the behaviors involved (*Shamsa Aziz et al.*, 2011).

## 2.2 Voting Systems

Voting systems, an integral component of democratic decision-making, provide the means to aggregate individual preferences and translate them into collective outcomes. These systems, which vary in complexity, operational mechanisms, and information requirements, aim to balance representation, fairness, and simplicity to ensure the legitimacy of the decision-making process. Several prominent voting systems include the plurality, majority, Borda count, and Condorcet methods. Each of these methods has a property called anonymity, meaning they consider every voter as important as any other. The lexicographic rule, and thus also this thesis, does not possess that property and is based on the idea that, due to potentially different reasons, the preferences/opinions of some people are more important than those of others (*Burgman et al.*, 2014).

As the name suggests, the majority method decides that a candidate is the winner if and only if the candidate gets more than half of the available votes. Because of this strong requirement, the majority method might not produce a winner, especially when more than two alternatives are available.

The plurality method, or first-past-the-post system, is a simple and widely used method that declares the candidate with the most votes as the winner, often leading to outcomes with less than majority support. For the plurality and majority methods, the information required from voters is minimal; they only need to indicate their preferred candidate or alternative.

The Borda count assigns points to candidates based on their rankings in voters preference lists, resulting in the candidate with the highest cumulative score being declared the winner. Condorcet methods, on the other hand, seek to identify a candidate who would prevail in pairwise comparisons against every other candidate. Both Borda count and Condorcet methods require more detailed information from voters, as they necessitate the full preference ordering of all alternatives (*Hemaspaandra et al.*, 1997).

Regarding the output of these methods, majority and plurality methods yield only the winner, while Borda count and Condorcet methods produce a full preference ordering of candidates. This distinction is important when the final decision requires more than just identifying a single winner, as in the case of the method discussed in the thesis, which also returns a full preference ordering.

Each of these voting systems presents its own set of advantages and shortcomings, necessitating careful consideration and contextual analysis when selecting the most suitable method for a given situation. The information required from voters and the type of output generated by each method are crucial factors to consider when evaluating the appropriateness of a specific voting system (*Lujak and Slavkovik*, 2012).

## 2.3 Preference Aggregation

Preference aggregation can be viewed as a broader concept that encompasses voting systems. While both preference aggregation and voting systems involve the process of combining individual preferences to produce a collective outcome, preference aggregation refers to a wider range of methods, including those used in situations where a collective ranking or preference order is desired rather than simply determining a single winner. Preference aggregation by voting is a fundamental problem in social choice theory, which aims to aggregate individual preferences into a collective decision (*Baumeister and Rothe*, 2016).

Research on preference aggregation follows different directions. The literature contains proposals introducing various aggregation methods ([e.g., Plurality (*Courtney*, 1999), Instant-runoff (*Cary*, 2011), Borda count (*Emerson*, 2013)]) and then, similarly to this thesis, discuss their advantages and limitations. Other approaches are more abstract, examining whether there are aggregation methods that satisfy specific conditions. This research line has produced some of the most significant results in the preference aggregation literature, with Arrow's impossibility theorem being the most well-known. Arrow's impossibility theorem states that no voting system can be both equitable and decisive since equity seeks to ensure fairness and equality of treatment. In contrast, decisiveness involves making conclusive decisions that may not always prioritize perfect equality (*Arrow*, 2012).

## 2.4 Belief Merging

The general problem of aggregating preferences shows up in other fields, sometimes under disguises. One interesting instance is the problem of belief merging within AI, which involves combining the beliefs of multiple agents into one single belief. This problem arises in many contexts, such as decision-making, information fusion, and knowledge representation, and has numerous applications in fields such as robotics, natural language processing, and machine learning. Belief merging is a challenging problem because the beliefs of different agents may be inconsistent, incomplete, or uncertain. The merging process needs to consider these factors while preserving the consistency and accuracy of the resulting belief (*Pigozzi*, 2006).

In recent years, there has been growing interest in the problem of belief merging, and several approaches have been proposed to address it. These approaches can be broadly classified into three categories: axiomatic approaches, algorithmic approaches, and logical approaches. Similarly to preference aggregation, axiomatic approaches seek to identify the desirable properties a merging function should satisfy, such as unanimity, independence, or consistency, and derive a merging function that satisfies these properties. Algorithmic approaches, on the other hand, provide explicit algorithms for merging beliefs based on some criteria, such as minimizing the degree of inconsistency or maximizing the degree of agreement. Logical approaches use formal logic to represent and manipulate beliefs and merge them by applying logical operations such as conjunction, disjunction, or negation. The choice of approach depends on the specific requirements of the application and the characteristics of the beliefs to be merged (*Eckert and Pigozzi*, 2005).

Consistency is crucial in belief merging because it ensures that the merged belief is both logically sound and coherent with the available evidence. Without consistency, conflicting information may be combined, leading to illogical or inaccurate conclusions. Achieving consistency in belief merging can be particularly challenging when the merged beliefs are uncertain or incomplete. In these cases, it may be necessary to weigh the available evidence carefully and use probabilistic reasoning or fuzzy logic methods to arrive at a consistent and accurate merged belief.

## 2.5 Belief Fusion

Belief fusion is an advanced technique that builds upon the foundation of belief merging to incorporate the credibility of the sources behind the combined beliefs. While belief merging focuses on synthesizing information from different agents to create a unified belief state, it does not account for the source's credibility or reliability. Belief fusion extends this process by considering the trustworthiness of each source, adding a new layer of nuance and accuracy to the merged beliefs (*Lian*, 2000).

For example, imagine two agents presenting conflicting information about a topic. In a standard belief merging scenario, their beliefs might be combined without considering the credibility of each source, potentially leading to an unreliable or neutral conclusion. On the other hand, belief fusion would weigh the credibility of each source before merging the beliefs, resulting in a more reliable and accurate representation of the truth.

In multi-agent systems, belief fusion plays a pivotal role in synthesizing disparate belief states originating from individual agents while considering the sources credibility. By integrating this additional layer of source evaluation, belief fusion outperforms traditional belief merging techniques in terms of producing more accurate and trustworthy unified belief states. This enhanced capability allows for more effective collaboration, decision-making, and collective intelligence in complex multi-agent environments.

While belief merging and belief fusion share similarities in their goals of combining beliefs from multiple sources, belief fusion distinguishes itself by factoring in the credibility of the sources, leading to a more accurate and reliable merged belief state. This distinction makes belief fusion a superior technique for promoting cooperation, coordination, and collective intelligence in multi-agent systems (*Maynard-Reid and Shoham*, 1998). Belief fusion is thus closer to the scenario studied in this thesis, which also requires some form of priority over the 'agents' whose preferences will be aggregated.

## 2.6 Lexicographic Rule

This section will discuss the lexicographic rule since a version of it is used in all three of the proposals in this thesis. This thesis will use a version of the lexicographic rule, modified to fit the purpose of the thesis. The lexicographic rule is a decision-making heuristic employed in the context of multi-attribute decision problems, where individuals are tasked with choosing between alternatives based on multiple criteria. This rule is predicated on the assumption that decision-makers have an ordered set of criteria for evaluating and comparing alternatives. The lexicographic rule posits that decision-makers prioritize the most important criterion, comparing alternatives solely on this basis, and only resort to considering subsequent criteria in the case of a tie (*Todd and Dieckmann*, 2004).

In applying the lexicographic rule, decision-makers first identify the criterion of utmost importance and compare the alternatives based on this criterion alone. Should two or more alternatives perform equally on this primary criterion, the decision-maker evaluates the tied alternatives according to the next most important criterion. This process is repeated, moving through the criteria in descending order of importance until a decision is made. By relying on this method, decision-makers simplify the decision-making process, effectively reducing the cognitive load involved in evaluating numerous criteria simultaneously (*Fishburn*, 1974).

The lexicographic rule can be formally defined as follows: Given a set of $n$ criteria $C = \{c_1, c_2, \ldots, c_n\}$, with $c_1$ being the most important criterion and $c_n$ being the least important criterion, and a set of $m$ alternatives $A = \{a_1, a_2, \ldots, a_m\}$, alternative $a_i$ is at least as good as alternative $a_j$ if and only if either the least important criterion considers $a_i$ at least as good as $a_j$ and all other criteria consider them equal, or there is a criterion that strictly favors $a_i$ over $a_j$ and all strictly more important criteria consider them equal. Using $\preccurlyeq_{c_i}$ for the order imposed by the criterion $c_i$, $\prec_{c_i}$ for its strict version and $\simeq_{c_i}$ for the 'tying' relation according to the same criterion (as well as propositional connectives), this idea can be formalized as

$$a_j \leqslant a_i \quad \text{iff} \quad \left( a_j \preccurlyeq_{c_n} a_i \wedge \bigwedge_{k=1}^{n-1} a_j \simeq_{c_k} a_i \right) \vee \bigvee_{\ell=1}^{n-1} \left( a_j \prec_{c_\ell} a_i \wedge \bigwedge_{k=1}^{\ell-1} a_j \simeq_{c_k} a_i \right)$$

One can follow a similar lexicographic strategy for aggregating the preferences of a set of agents. Assume, thus, that there is a relation indicating the priority/reliability/importance of the agents whose preferences will be combined. Then, when comparing two alternatives, ask first for the opinion of the most important agent. If she can make a decision, the comparison is finished. Otherwise, move on and ask the second most important agent, and so on.

In group settings, decisions are frequently made collectively rather than individually. Individual decision-making differs fundamentally from group decision-making. When a group confronts a common problem, they tend to reach a consensus as a cohesive unit (*Bosman et al.*, 2006). When Robert considers purchasing a pair of shoes and solicits advice from his peers, the final decision remains solely with him.

Within larger groups of people, there tend to be smaller groupings (i.e., in a family, the parents, brothers, and sisters can be smaller groupings.). If Robert has two parents, two sisters, and a brother, he might think that his parents are the most reliable, his sisters are the least reliable, and he and his brother are in the middle. His hierarchy of reliability doesn't necessarily rely on individuals or the group but rather on subgroups within the family. These kinds of subgroups can also be found in groups of friends, only not as easily definable.

As stated previously, this thesis proposes a novel variant of the lexicographic rule that utilizes an ordering over sets of agents rather than individual agents and focuses specifically on reliability-based peer pressure. This extended approach aims to represent the collective influence of subsets of agents, allowing us to give more weight to the majority and facilitating more dynamic and realistic modeling of decision-making processes than the traditional lexicographic rule. In a family discussion, for instance, it may be appropriate to compare the collective knowledge and beliefs of the subgroups of the family (e.g., parents, siblings) rather than focusing on the individual team members. By employing an ordering based on reliability over these subgroups, we can better account for the dynamics of trust and expertise that drive group decision-making processes, resulting in more accurate and insightful models of complex social interactions.

## 2.7 Lexicographic Template Rule

This thesis needed a generalized version of the lexicographic rule (the one seen in the section before) to represent the wanted situations fully. This section will explain the process of creating the template lexicographic rule used in this thesis and the thoughts behind it. The template is based on considerations suggested by friendship and friend-group behavior. This thesis also considers similar fields and how they solve similar problems. The field this thesis has taken the most inspiration from is preference aggregation, specifically how voting systems define the combination of preference aggregation. This thesis represents three ways of aggregating preferences in groups of friends. However, the most important part of this thesis is finding the template rule in which variables can easily be replaced to represent different types of strategies. The thesis also takes inspiration from the *Velázquez-Quesada* (2017) paper. This thesis utilizes the variables $\alpha$, $\beta$, and $\gamma$ to represent certain values or concepts. Although the variables themselves are constant, the values or concepts they represent can be changed as needed. This part of the thesis will explain in detail how the template uses the variables in general and how the different strategies take advantage of this.

Here is a more general formulation of the lexicographic rule. Let $C = \{c_1, \ldots, c_n\}$ be the set of criteria (which, in this case, is the set of subset of agents $\wp(A)$), with $c_1$ being the most important criterion and $c_n$ being the least important. Using the propositional connectives $\neg$, $\wedge$ and $\vee$, the formulation is

$$u \leqslant v \quad \text{iff} \quad \left(\beta_{c_n}(u,v) \wedge \bigwedge_{k=1}^{n-1} \neg\alpha_{c_k}(u,v)\right) \vee \bigvee_{\ell=1}^{n-1} \left(\alpha_{\ell}(u,v) \wedge \bigwedge_{k=1}^{\ell-1} \neg\alpha_{c_k}(u,v)\right)$$

with $\alpha_c(u,v)$ the condition a criterion $c$ different from the least important needs to satisfy for $u$ and $v$ for it to be decisive (i.e., for it to guarantee an arrow from $u$ to $v$ in the resulting ordering), and $\beta_c(u,v)$ the condition the least important criterion needs to satisfy for $u$ and $v$ for it to be the decisive one. In words, the formulation says that $u$ will be at most as preferred as $v$ if and only if the least important criterion satisfies a beta condition for $u$ and $v$ and all others fail to satisfy the alpha condition, or there is a criterion that is not the least important, and that satisfies alpha, with all more important criterions failing to satisfy alpha.

The general formulation created above seemed to be working perfectly, but it was missing a condition when checking for properties and general testing. In the formulation above, the formula is missing a more strict condition for being allowed to ask a less reliable set of agents, instead of just $\neg\alpha$. This made relations end up being drawn at times they shouldn't. This led to introducing the $\gamma$-condition, a condition that a set of agents should satisfy to allow less important sets of agents to give their opinions. The updated template rule formulation is

$$u \leqslant v \quad \text{iff} \quad \left(\beta_{c_n}(u,v) \wedge \bigwedge_{k=1}^{n-1} \gamma_{c_k}(u,v)\right) \vee \bigvee_{\ell=1}^{n-1} \left(\alpha_\ell(u,v) \wedge \bigwedge_{k=1}^{\ell-1} \gamma_{c_k}(u,v)\right)$$

with $\gamma_c(u,v)$ the condition a criterion $c$ needs to satisfy for $u$ and $v$ to allow checking less important criteria.

The variables in this template rule are placed with that they represent different things/have different roles in mind. The $\alpha$-condition is what is asked of every set of agents in the reliability ordering, except the bottom $c_n$, to guarantee that there will be an arrow from $u$ to $v$. The $\alpha$-condition can be seen as the main requirement for drawing relations.

The $\beta$-condition is what is asked of the least reliable set of agents, $c_n$. This is similar to the $\alpha$ condition in that it is asked of a set of agents to determine if a relation should be drawn. However, the $\beta$-condition is designed to be a less strict version of the alpha that is only asked to the least reliable set of agents. The idea behind the $\beta$ is that if a relation isn't drawn for any of the sets of agents above, then the least reliable set of agents should be asked a milder condition to avoid incomparability when that is not necessary. The $\beta$ can be seen as the secondary requirement for drawing relations.

The $\gamma$-condition is used as a permission requirement. It was created to ensure that the most reliable set of agents is consulted first and so that the lexicographic rule is followed. As it will be seen, the $\gamma$-condition is identical for all three proposed strategies, and it contains both $\neg\alpha(u,v)$ and $\neg\alpha(v,u)$. By including the negation of alpha from both directions of the relation being checked, one can ensure that no higher-priority set is being overlooked. The $\gamma$-condition can be seen as the permission requirement, allowing you to consult the next set of agents.

# 3 Strategies and Definitions

This section presents three original strategies for preference aggregation: the non-contentious, plurality, and majority methods. These methods are developed and evaluated, and their prospective benefits and limitations are discussed. The objective is to contribute to the ongoing dialogue in the field of preference aggregation and to suggest new research directions.

## 3.1 Basic Definitions

The preference and reliability (PR) frame is a formal structure that captures an agent's preferences over a set of worlds as well as a reliability ordering over sets of agents. The frame is represented as a tuple consisting of worlds and each agent's preferences over those worlds. Each preference relation is a preorder (a reflexive and transitive relation) that indicates the relative desirability of one world compared to another from the agent's perspective. The reliability relation is a linear order(a reflexive, transitive, antisymmetric and total relation) that indicates the relative trustworthiness of one non-empty set of agents as compared to another.

The two definitions have taken inspiration from *Velázquez-Quesada* (2017). They are rewritten and modified to fit the goal of this thesis.

**Definition 1 (PR frame)**
A preference and reliability (PR) frame F is a tuple $\langle W, \{\leqslant_i\}_{i \in A}, \preccurlyeq \rangle$ where

- W is a finite nonempty set of worlds.
- $\leqslant_i \subseteq (W \times W)$ is a preorder (a reflexive and transitive relation), the agent $i$ preference relationship over the worlds in $W$ $(u \leqslant_i v$ is read as 'for agent i, world $v$ is at least as preferable as world $u$').
- $\preccurlyeq \subseteq (\wp(A) \setminus \{\varnothing\} \times \wp(A) \setminus \{\varnothing\})$ is a linear total order without the empty set, representing the reliability relation over sets of agents in A $(B \preccurlyeq C$ is read as 'the group of agents $C$ is at least as reliable as the group of agents $C$').

The definition below provides three important relations for comparing preferences between different elements in a set $W$. The first relation is "$<_i$", which indicates that one element is less preferred than another. This relation is defined as an element $u$ being less preferred than element $v$ if and only if $u$ is less or equal over $v$ ($u \leqslant_i v$), but $u$ is not preferred to $v$ ($v \nleqslant_i u$). The second relation is "$\sim_i$", which indicates that two elements are comparable. This relation is defined as $u$ and $v$ being comparable if and only if either $u$ is preferred to $v$ ($u \leqslant_i v$) or $v$ is preferred to $u$ ($v \leqslant_i u$). The third relation is "$\simeq_i$", which indicates that two elements are equally preferred. This relation is defined as $u$ and $v$ being equally preferred if and only if $u$ is less or equal to $v$ ($u \leqslant_i v$) and $v$ is less or equal to $u$ ($v \leqslant_i u$). These relations provide a useful framework for understanding the relationships between different preferences and their relative importance.

**Definition 2**
Let $\leqslant_i \subseteq (W \times W)$ be a preference relation (preorder) over $W$.

- The relation $<_i \subseteq (W \text{ x } W)$, with $u <_i v$ read as 'agent i thinks $u$ is less preferred than $v$' ('agent i thinks $v$ is more preferred than $u$' or simply 'agent i thinks $v$ is preferred over $u$'), is defined as $u <_i v$ iff$_{\text{def}}$ $u \leqslant_i v$ and $v \nleqslant_i u$.
- The relation $\sim_i \subseteq (W \times W)$, with $u \sim_i v$ read as 'agent i thinks $u$ and $v$ are comparable', and is defined as $u \sim_i v$ iff$_{\text{def}}$ $u \leqslant_i v$ or $v \leqslant_i u$.
- The relation $\simeq_i \subseteq (W \times W)$, with $u \simeq_i v$ read as 'agent i thinks $u$ and $v$ are equally preferred', and is defined as $u \simeq_i v$ iff$_{\text{def}}$ $u \leqslant_i v$ and $v \leqslant_i u$.

Definition 2 provides different relations (strictly better, comparable and equally preferred) derived from the preference relations $\leqslant_i$. We could do the same for the reliability relation $\preccurlyeq$, but this is not really needed: since $\preccurlyeq$ is a linear order, all sets of agents are comparable, and $B$ is equally reliable as $C$ if and only if $B$ and $C$ are equal.

Since the reliability relation is a linear order on $\wp(A)$ and the set $A$ is finite, we can think of $\preccurlyeq$ as an array $c$, with the set of agents in position 1 ($c_1$) being the most reliable, and the set of agents in the last position (call it $c_n$) being the least reliable. This notation will be used in this thesis.

## 3.2 The Strategies

In the context of preference aggregation, novel strategies may offer unique benefits and insights into decision-making processes. Instead of examining individual agents, the novel concept of this thesis is that the reliability relation will consist of groups of agents. This section introduces three original methods for aggregating preferences: the non-contentious method, the majority method, and the plurality method. These strategies are developed and evaluated in the context of my own research, and their potential advantages and limitations are discussed. By presenting these methods, the aim is to contribute to the ongoing dialogue in the field of preference aggregation and provide new avenues for future research.

### 3.2.1 Non-Contentious

Recall the pattern of the lexicographic rule identified in the previous chapter:

$$u \leqslant v \quad \text{iff} \quad \left(\beta_{c_n}(u,v) \wedge \bigwedge_{k=1}^{n-1} \gamma_{c_k}(u,v)\right) \vee \bigvee_{\ell=1}^{n-1} \left(\alpha_{\ell}(u,v) \wedge \bigwedge_{k=1}^{\ell-1} \gamma_{c_k}(u,v)\right)$$

The first proposal follows the idea of letting a set of agents $c$ different from the least important to be decisive (for drawing an arrow from $u$ to $v$) when the set of agents in $c$ that believe $v$ to be strictly better than $u$ (denoted as $\text{ags}_c(u < v)$) is non-empty and the union of the set of agents in $c$ who thinks the opposite, $u$ to be strictly better than $v$ (denoted as $\text{ags}_c(v < u)$) and the set of agents thinking the alternatives are incomparable (denoted as $\text{ags}_c(u \nleq v)$) to be empty. This means that the condition $\boldsymbol{\alpha}$ is defined as

$$\alpha_c(u,v) := \left(\text{ags}_c(u < v) \neq \varnothing\right) \wedge \left(\text{ags}_c(v < u) \cup \text{ags}_c(v \nleq u) = \varnothing\right).$$

Note that this essentially says that everybody in $c$ considers $v$ at least as good as $u$ (second conjunct) and that at least one agent in $c$ considers $v$ to be strictly better than $u$ (first conjunct).

The least important set of agents will be decisive when it satisfies the following (slightly easier to pass) condition: the union of the set of agents in $c$ who thinks $v$ is strictly better than $u$ and the set of agents in $c$ who think the alternatives are equally preferred (the set $\text{ags}_c(u < v) \cup \text{ags}_c(u \simeq v)$) is non-empty, and the set of agents in $c$ who thinks $u$ is strictly better than $v$ is empty. This means that the condition $\boldsymbol{\beta}$ is defined as

$$\beta_c(u,v) := \left(\text{ags}_c(u < v) \cup \text{ags}_c(v \simeq u) \neq \varnothing\right) \wedge \left(\text{ags}_c(v < u) = \varnothing\right).$$

This asks for at least one agent in $c$ to considers $v$ to be at least as good as $u$ (first conjunct), and for no agent in $c$ to consider $u$ strictly better than $v$ (second conjunct).

Finally, the condition a set of agents $c$ needs to satisfy to ‘allow’ less important sets of agents to be consulted is the following: $\neg\boldsymbol{\alpha}(u,v)$ to ensure that the set is not decisive from $u$ to $v$, and $\neg\boldsymbol{\alpha}(u,v)$, so the set is not decisive in the opposite direction.

$$\gamma_c(u,v) := \neg\boldsymbol{\alpha}_c(u,v) \wedge \neg\boldsymbol{\alpha}_c(v,u)$$

Thus, a set of agents $c$ allows less important sets of agents to be consulted when it is not decisive in either direction.

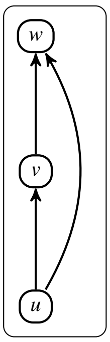
As a strategy for aggregating preferences, the non-contentious method was motivated to facilitate a smooth decision-making process and minimize agent conflicts. The approach aimed to reach a consensus among all agents. By focusing on combining preferences through agents’ agreement, the non-contentious method aimed to ensure that the outcome of the decision-making process would be satisfactory to all parties. The intuition behind this method is that it would be difficult for individuals to object if a mutual agreement can be reached, resulting in a preferred outcome for everyone involved.

The non-contentious method presents a novel approach to aggregating agent preferences by incorporating a reliability relation ordering over sets of agents. This method capitalizes on the tendency of individuals to engage in actions with which they concur while refraining from activities subject to strong disagreement. When the most reliable set of agents reaches a consensus on a particular course of action, the non-contentious method yields a clear and decisive outcome. Conversely, the presence of dissent among agents serves as a deterrent, preventing the implementation of the contentious decision. By emphasizing the importance of agreement and accounting for the influence of dissent, the non-contentious method offers a sophisticated and pragmatic framework for collective decision-making processes, fostering collaboration and reducing the likelihood of disputes among agents.

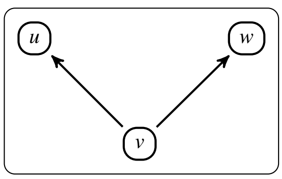
This method is a weaker version of the consensus method (*Bressen*, 2007), which needs every group member to agree on a decision. The non-contentious method is similar to the consensus method in that if someone disagrees, then no agreement is reached. The non-contentious method differs in that agents can be indifferent about a decision, which will still lead to an agreement. The consensus method also does not consider a reliability ordering, although it could easily be implemented using one, which this non-contentious method does. Since the non-contentious method uses the lexicographic rule to go through a reliability ordering over sets of agents, an agreement can be reached if the most reliable set of agents all agrees on a decision.

#### Example

Let's explore an example showcasing how the non-contentious method aggregates preferences. The four figures below represents the preferences of four different agents. Recall the requirements for preference orderings, they are reflexive and transitive. The reflexive arrow will not be drawn, since every world is reflexive. Agent $a$ has a clear preference, the agent prefers world $w$ over world $v$ and world $v$ over world $u$. Agent $b$ is not as sure, the agent believes world $v$ to be the least preferred, but cant compare the other two worlds. Agent $c$ prefers world $w$ the least, and the worlds $u$ and $v$ equally better that $w$. Agent $d$ thinks of all worlds as equally preferred.




*a*

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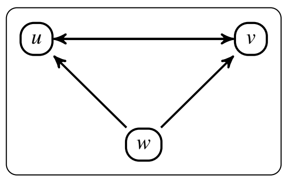


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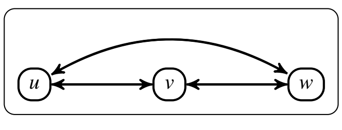
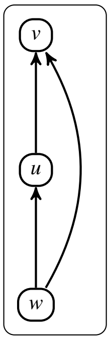



*d*

Now that we have established the preference orderings, we need a reliability ordering to illustrate which set of agents are the most reliable. Recall the requirements for reliability orderings, it is a linear order, meaning it is reflexive, transitive, antisymmetric and total. The reliability relation over the sets of agents can be seen below. The reliability ordering has the set including all four agents at the top, then comes the singleton including only agent $c$, then comes the set including agents $a$, $c$ and $d$, and at the bottom we find the set including only agent $b$. The sets of agents below $\{b\}$ are not needed because they are not consulted in this example.

$$\cdots \longrightarrow \{b\} \longrightarrow \{a,c,d\} \longrightarrow \{c\} \longrightarrow \{a,b,c,d\}$$

If we use the non-contentious method to aggregate these preferences using the reliability ordering shown above, the result will be the new updated preference ordering shown below.




*Non-contentious*

As seen above, if we aggregate the four preferences into one, using the reliability ordering provided, this would be the resulting outcome. The new updated preference ordering prefers $v$ the most, then $u$ is the second most preferred, and world $w$ becomes the least preferred. There is an arrow drawn from $u$ to $v$ because, in the most reliable set, the $\alpha$-condition fails, but since the gamma holds, we ask the $\alpha$-condition to the next set, which is a singleton, and in this set both the $\alpha$- and $\gamma$-conditions hold, drawing the arrow. The reason there is not an arrow from $v$ to $u$ is firstly because the $\alpha$-condition fails for the most reliable set, but most importantly because the $\gamma$-condition will fail in all other sets because of the $\alpha$-condition holding the other way in $C_2$. There is also no arrow from $v$ to $w$ because of the same thing. The $\alpha$-condition holding in $C_2$ from $w$ to $v$, makes the $\gamma$-condition fail in all sets except $C_1$ where the $\alpha$-condition fails. For formal properties, see Chapter 4.

### 3.2.2 Majority Decides

Again, recall the pattern of the lexicographic rule identified in the previous chapter:

$$u \leqslant v \quad \text{iff} \quad \left(\boldsymbol{\beta}_{c_n}(u,v) \wedge \bigwedge_{k=1}^{n-1} \boldsymbol{\gamma}_{c_k}(u,v)\right) \vee \bigvee_{\ell=1}^{n-1} \left(\boldsymbol{\alpha}_{\ell}(u,v) \wedge \bigwedge_{k=1}^{\ell-1} \boldsymbol{\gamma}_{c_k}(u,v)\right)$$

This second proposal follows the idea of letting a set of agents different from the least important to be decisive (for drawing an arrow from $u$ to $v$) when the set of agents in $c$ that believe $v$ to be strictly better than $u$ (denoted as $\mathrm{ags}_c(u < v)$) is strictly larger than the union of the set of agents in $c$ who thinks the opposite, $u$ to be strictly better than $v$ (denoted as $\mathrm{ags}_c(v < u)$) and the set of agents thinking the alternatives are incomparable (denoted as $\mathrm{ags}_c(u \nless v)$). This means that the condition $\boldsymbol{\alpha}$ is defined as

$$\boldsymbol{\alpha}_c(u,v) := |\mathrm{ags}_c(u < v)| > |\mathrm{ags}_c(v < u) \cup \mathrm{ags}_c(v \nless u) \cup \mathrm{ags}_c(v \simeq u)|$$

Note that this essentially says that there are more agents in $c$ that consider $v$ to be strictly better than $u$ than there are agents in $c$ who consider anything else.

The least important set of agents will be decisive when it satisfies the following (slightly easier to pass) condition: the union of the set of agents in $c$ who thinks $v$ is strictly better than $u$ and the set of agents in $c$ who think the alternatives are equally preferred (the set $\mathrm{ags}_c(u < v) \cup \mathrm{ags}_c(u \simeq v)$) is non-empty, and to be strictly larger than the union of the set of agents in $c$ who thinks $u$ is strictly better than $v$ and the set of agents in $c$ who thinks $u$ and $v$ to be incomparable. This means that the condition $\boldsymbol{\beta}$ is defined as

$$\boldsymbol{\beta}_c(u,v) := |\mathrm{ags}_c(u < v) \cup \mathrm{ags}_c(u \simeq v)| \ > \ |\mathrm{ags}_c(v < u) \cup \mathrm{ags}_c(v \nsim u)|.$$

This asks for the set of agents in $c$ who considers $v$ to be at least as good as $u$ to be strictly better than the union of the set of agents in $c$ who considers $u$ to be strictly better than $v$ and the set of agents who considers $u$ and $v$ to be incomparable.

Finally, the condition a set of agents $c$ needs to satisfy to 'allow' less important sets of agents to be checked is the following: $\neg\boldsymbol{\alpha}(u,v)$ to ensure that the set is not decisive from $u$ to $v$, and $\neg\boldsymbol{\alpha}(u,v)$, so the set is not decisive in the opposite direction.

$$\boldsymbol{\gamma}_c(u,v) := \neg\boldsymbol{\alpha}_c(u,v) \wedge \neg\boldsymbol{\alpha}_c(v,u)$$

Thus, a set of agents $c$ allows less important sets of agents to be consulted when it is not decisive in either direction.

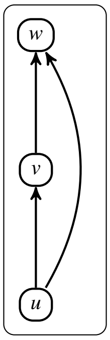
The majority aggregation method serves as a compelling approach for preference aggregation due to its intuitive appeal and ability to capture the majority's will. By prioritizing the preference that receives the majority of the votes, the majority rule ensures that the decision reflects the collective choice of the majority of the group. This approach promotes inclusivity and simplifies the decision-making process by providing a clear and decisive outcome. Additionally, the majority rule can minimize conflicts and encourage consensus-building by encouraging individuals to align their preferences with the general opinion, enabling a sense of unity and cooperation within the group.

Majority voting is a common method for decision-making in a wide range of contexts, including politics, business, and social settings. In voting systems, majority rule means that the option with the most votes is chosen as the final decision. The concept of majority rule is based on the idea that decisions made by a group are more likely to be accurate and fair if they reflect the preferences of the majority (*Dahl*, 1956).

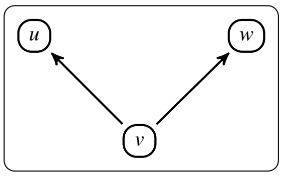
The majority rule can also be applied to decision-making among friends or social groups. In these settings, the majority rule means that the option preferred by the majority of group members is chosen as the final decision. This decision-making method can be useful when group members have different preferences or opinions and must come to a consensus.

#### Example

Let's explore an example showcasing how the majority method aggregates preferences. The four figures below represents the preferences of four different agents. Recall the requirements for preference orderings, they are reflexive and transitive. The reflexive arrow will not be drawn, since every world is reflexive. Consider agents with the same preferences as the previous example.




*a*

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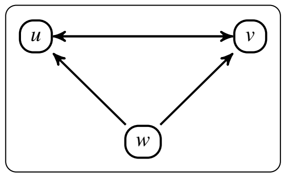


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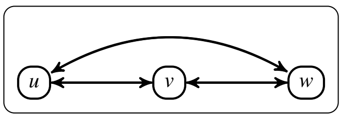
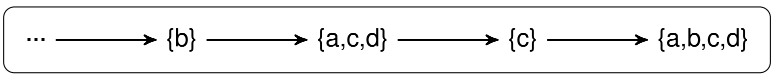
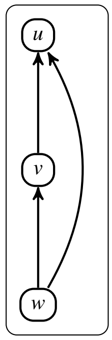



*d*

Now that we have established the preference orderings, we need a reliability ordering to illustrate which set of agents are the most reliable. Recall the requirements for reliability orderings, it is a linear order, meaning it is reflexive, transitive, antisymmetric and total. The reliability relation over the sets of agents can be seen below. The reliability ordering is identical to the one in the previous example.



If we use the majority method to aggregate these preferences using the reliability ordering shown above, the result will be the new updated preference ordering shown below.




*Majority*

As seen above, if we aggregate the four preferences into one, using the reliability ordering provided, this would be the resulting outcome. The new updated preference ordering prefers $u$ the most, then $v$ is the second most preferred, and world $w$ becomes the least preferred. The reason there is an arrow from $v$ to $u$ and not the other way, from $u$ to $v$, is that the $\alpha$-condition fails both from $v$ to $u$ and from $u$ to $v$ in sets $C_1$,$C_2$ and $C_3$. However, in set $C_4$, the $\alpha$-condition holds since agent $b$ has a strict arrow from $v$ to $u$, making it a majority, drawing the arrow. This also makes the $\gamma$-condition from $u$ to $v$ fail, assuring there won't be an arrow from $u$ to $v$. The reason there is no arrow from $v$ to $w$ is that in $C_1$ the $\alpha$-condition fails both ways, but in $C_2$ the $\alpha$-condition holds together with the $\gamma$-condition from $w$ to $v$. This means that the *gamma*-condition will fail in set $C_2$ from $v$ to $w$ and all sets below, making the arre from $v$ to $w$ not be drawn. For formal properties, see Chapter 4.

### 3.2.3 Plurality

Again, recall the pattern of the lexicographic rule identified in the previous chapter:

$$u \leqslant v \quad \text{iff} \quad \left( \boldsymbol{\beta}_{c_n}(u,v) \wedge \bigwedge_{k=1}^{n-1} \boldsymbol{\gamma}_{c_k}(u,v) \right) \vee \bigvee_{\ell=1}^{n-1} \left( \boldsymbol{\alpha}_{\ell}(u,v) \wedge \bigwedge_{k=1}^{\ell-1} \boldsymbol{\gamma}_{c_k}(u,v) \right)$$

This third proposal follows the idea of letting a set of agents different from the least important to be decisive (for drawing an arrow from $u$ to $v$) when the set of agents in $c$ that believe $v$ to be strictly better than $u$ (denoted as $\text{ags}_c(u < v)$) is strictly larger than the set of agents in $c$ who thinks the opposite, $u$ to be strictly better than $v$ (denoted as $\text{ags}_c(v < u)$) and the set of agents thinking the alternatives are incomparable (denoted as $\text{ags}_c(u \not\leqslant v)$) individually. This means that the condition $\boldsymbol{\alpha}$ is defined as

$$\boldsymbol{\alpha}_c(u,v) := \big(|\text{ags}_c(u < v)| > |\text{ags}_c(v < u)|\big) \wedge \big(|\text{ags}_c(u < v)| > |\text{ags}_c(v \simeq u)|\big)$$
$$\wedge \big(|\text{ags}_c(u < v)| > |\text{ags}_c(v \nsim u)|\big)$$

Note that this says, essentially, that there are more agents in $c$ that consider $v$ strictly better than $u$ than there are agents in $c$ in the other three sets individually$(>, \simeq, \nsim)$.

The least important set of agents will be decisive when it satisfies the following (slightly easier to pass) condition: the union of the set of agents in $c$ who thinks $v$ is strictly better than $u$ and the set of agents in $c$ who think the alternatives are equally preferred (the set $\text{ags}_c(u < v) \cup \text{ags}_c(u \simeq v)$) is non-empty, and to be strictly larger than the union of the set of agents in $c$ who thinks $u$ is strictly better than $v$ and the set of agents in $c$ who thinks $u$ and $v$ to be incomparable. This means that the condition $\beta$ is defined as

$$\beta_c(u,v) := |\text{ags}_c(u < v) \cup \text{ags}_c(u \simeq v)| \; > \; |\text{ags}_c(v < u) \cup \text{ags}_c(v \not\sim u)|.$$

This asks for the set of agents in $c$ who considers $v$ to be at least as good as $u$ to be strictly better than the union of the set of agents in $c$ who considers $u$ to be strictly better than $v$ and the set of agents who considers $u$ and $v$ to be incomparable.

Finally, the condition a set of agents $c$ needs to satisfy to 'allow' less important sets of agents to be checked is the following: $\neg\alpha(u,v)$ to ensure that the set is not decisive from $u$ to $v$, and $\neg\alpha(u,v)$, so the set is not decisive in the opposite direction.

$$\gamma_c(u,v) := \neg\alpha_c(u,v) \wedge \neg\alpha_c(v,u)$$

Thus, a set of agents $c$ allows less important sets of agents to be consulted when it is not decisive in either direction. In the context of decision-making among a group, the plurality strategy has been shown to have several advantages. This approach involves each group member expressing their preference for a particular outcome, and the outcome with the most support being selected, has several desirable properties. Firstly, it preserves all possible states in the event of unanimity among group members.

The plurality method provides a flexible and instinctive approach to preference aggregation. The plurality aggregation method is motivated by the desire to capture the most preferred choice between diverse preferences. The plurality rule acknowledges the existence of multiple valid perspectives by selecting the preference with the highest number of votes, regardless of whether it constitutes a majority. Plurality is well-suited for simple, everyday decisions such as choosing a movie or a board game. However, it may not be as reliable for more significant decisions, such as elections, requiring a clear winner.

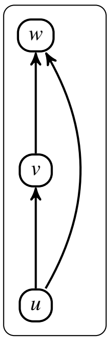
While majority voting is a common method for decision-making in various contexts, it may not always accurately represent the preferences of a group, particularly in informal settings such as among friends. In contrast, using plurality in voting systems may better represent the dynamics of friend groups.

Plurality voting is a decision-making method that selects the option with the most votes, even if it does not receive a majority of the votes (i.e., more than 50%). In friend group decision-making, plurality voting allows for a more informal and relaxed decision-making process without requiring a strict majority to be reached.

The use of plurality voting in friend group decision-making can better represent these groups' informal and relaxed dynamics. While it may not accurately represent the preferences of all group members, it allows for a more flexible and efficient decision-making process without requiring a strict majority to be reached. Further research is needed to explore the effectiveness of plurality voting in different social contexts and to identify the factors that influence group decision-making processes.

#### Example

Let's explore an example showcasing how the plurality method aggregates preferences. The four figures below represents the preferences of four different agents. Recall the requirements for preference orderings, they are reflexive and transitive. The reflexive arrow will not be drawn, since every world is reflexive. The preferences are identical to those in the two previous examples.

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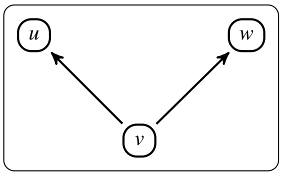


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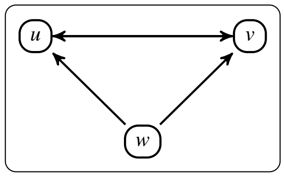


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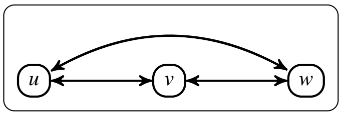
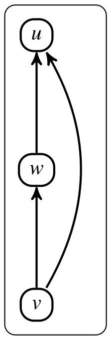



*d*

Now that we have established the preference orderings, we need a reliability ordering to illustrate which set of agents are the most reliable. Recall the requirements for reliability orderings, it is a linear order, meaning it is reflexive, transitive, antisymmetric and total. The reliability relation over the sets of agents can be seen below. The reliability ordering is identical to the ones in the two previous examples.

$$\cdots \longrightarrow \{b\} \longrightarrow \{a,c,d\} \longrightarrow \{c\} \longrightarrow \{a,b,c,d\}$$

If we use the plurality method to aggregate these preferences using the reliability ordering shown above, the result will be the new updated preference ordering shown below.




*Plurality*

As seen above, if we aggregate the four preferences into one, using the reliability ordering provided, this would be the resulting outcome. The new updated preference ordering prefers $u$ the most, then $w$ is the second most preferred, and world $v$ becomes the least preferred. The reason there is no arrow from $u$ to $v$ is that the $\alpha$-condition fails in $C_1$, $C_2$ and $C_3$, while the $\gamma$-condition holds, and in the set $C_4$ where we find agent $b$ alone, there is a plurality from $v$ to $u$, making the $\gamma$-condition fail in all sets below, and $\alpha$-condition fail from $u$ to $v$ in $C_4$, meaning there won't be drawn an arrow from $u$ to $v$ and will be an arrow from $v$ to $u$. The reason there is an arrow from $v$ to $w$ is quite interesting, as it clearly highlights how the plurality method differs from the majority method. In set $C_1$, there is a plurality of strict arrows from $v$ to $w$, making the plurality method draw the arrow, while the majority method needs to look further down the reliability ordering. This will be discussed further in Chapter 4, along with the formal properties.

# 4 Results and Discussion

This section presents the thesis findings on the properties of the three presented preference aggregation methods. The section focuses on the properties of reflexivity, transitivity, totality, and antisymmetry. Additionally, the study explores whether the aggregation methods preserve unanimity, ensuring that the resulting preference order coincides with unanimous agreements among agents. Before reading this section it is important to recall the required properties the thesis asks for. The preference ordering is a preorder, meaning it is reflexive and transitive. The reliability ordering is a linear order over non-empty sets of agents, meaning it is reflexive, transitive, total and antisymmetric. It is also useful to recall that the domain $W$ is nonempty.

## 4.1 Reflexivity

Recall the template presented in Chapter 2.

$$u \leqslant u \quad \text{iff} \quad \underbrace{\left(\beta_{C_n}(u,u) \wedge \bigwedge_{k=1}^{n-1} \gamma_{C_k}(u,u)\right)}_{1} \vee \underbrace{\bigvee_{\ell=1}^{n-1} \left(\alpha_\ell(u,u) \wedge \bigwedge_{k=1}^{\ell-1} \gamma_{C_k}(u,u)\right)}_{2}$$

Let $\{\leqslant_i\}_{i\in A}$ be a finite collection of binary preference orderings over a domain $W$; Recall that the reliability ordering over non-empty sets of agents can be seen as an array $C$ where the most reliable set of agents appears in position 1. The least reliable set of agents appears in the last position $n$. We want to discuss whether the proposed strategies preserve reflexivity, that is, whether the resulting preference ordering is reflexive when the preferences of all agents are reflexive.

Definition of reflexivity: For all worlds $u \in W$, $u \leqslant u$.

#### Non-Contentious

Recall the conditions for the non-contentious method,

$$\alpha_c(u,u) := \left(\operatorname{ags}_c(u<u) \neq \varnothing\right) \wedge \left(\operatorname{ags}_c(u<u) \cup \operatorname{ags}_c(u \not\leqslant u) = \varnothing\right).$$

$$\beta_c(u,u) := \left(\operatorname{ags}_c(u<u) \cup \operatorname{ags}_c(u \simeq u) \neq \varnothing\right) \wedge \left(\operatorname{ags}_c(u<u) = \varnothing\right).$$

$$\gamma_c(u,u) := \neg\alpha_c(u,u) \wedge \neg\alpha_c(u,u)$$

The non-contentious method preserves reflexivity. To prove this, take any $u \in W$. The second conjunct of the formula will fail since no set of agents can satisfy $\alpha$, as no set of agents contains an agent that considers u strictly better than u. We then look at the first part of the disjunction, which always holds. The $\beta$ condition will hold because of the reflexivity of the preference relations, any (non-empty) set of agents satisfies the beta condition since every set contains at least one agent considering $u$ and $u$ as equally preferred and contains no agent considering $u$ strictly better than $u$ or considering $u$ and $u$ incomparable. The $\gamma$ condition will also always hold since no set of agents can satisfy $\alpha$, as mentioned, making it fail both directions. Thus making the plurality method preserve reflexivity.

If the empty set was included and it happened to appear at the bottom of the reliability ordering, reflexivity would not hold. As mentioned before, the $\alpha$ fails because strict arrows cannot appear from one world to an identical world because it would then count as going both ways, making the $\alpha$ fail. The $\beta$ condition is what is asked of the least reliable set of agents, and if that set is empty, it will fail. Thus if the empty set were included as the bottom set, reflexivity would not be preserved by the non-contentious method.

#### Majority

Recall the conditions for the majority method,

$$\alpha_c(u,u) := |\text{ags}_c(u < u)| > |\text{ags}_c(u < u) \cup \text{ags}_c(u \nsim u) \cup \text{ags}_c(u \simeq u)|.$$

$$\beta_c(u,u) := |\text{ags}_c(u < u) \cup \text{ags}_c(u \simeq u)| > |\text{ags}_c(u < u) \cup \text{ags}_c(u \nsim u)|.$$

$$\gamma_c(u,u) := \neg\alpha_c(u,u) \wedge \neg\alpha_c(u,u)$$

The majority method also preserves reflexivity. To prove this, take any $u \in W$. Similarly to the previous proof, the second conjunct of the formula will fail since no set of agents can satisfy $\alpha$, as no set of agents contains an agent that considers u strictly better than u. We then look at the first part of the disjunction, which always holds. The $\beta$ condition will hold because of the reflexivity of the preference relations, any (non-empty) set of agents satisfies the beta condition since every set contains at least one agent considering $u$ and $u$ as equally preferred and contains no agent considering $u$ strictly better than $u$ or considering $u$ and $u$ incomparable. The $\gamma$ condition will also always hold. This is because no set of agents can satisfy $\alpha$, as mentioned, making it fail in both directions. Thus making the majority method preserve reflexivity.

If the empty set was included and it happened to appear at the bottom of the reliability ordering, reflexivity would not hold. As mentioned before, the $\alpha$ fails because strict arrows cannot appear from one world to an identical world because it would then count as going both ways, making the $\alpha$ fail. The $\beta$ condition is what is asked of the least reliable set of agents, and if that set is empty, it will fail. Thus if the empty set were included as the bottom set, reflexivity would not be preserved by the majority method.

#### Plurality

Recall the conditions for the plurality method,

$$\alpha_c(u,u) := \big(|\text{ags}_c(u < u)| > |\text{ags}_c(u < u)|\big) \wedge \big(|\text{ags}_c(u < u)| > |\text{ags}_c(u \simeq u)|\big)$$
$$\wedge \big(|\text{ags}_c(u < u)| > |\text{ags}_c(u \nsim u)|\big)$$

$$\beta_c(u,u) := |\text{ags}_c(u < u) \cup \text{ags}_c(u \simeq u)| > |\text{ags}_c(u < u) \cup \text{ags}_c(u \nsim u)|$$

$$\gamma_c(u,u) := \neg\alpha_c(u,u) \wedge \neg\alpha_c(u,u)$$

The plurality method also preserves reflexivity, meaning all three methods do so. To prove this, take any $u \in W$. Same scenario as the two previous proofs, the second conjunct of the formula will fail since no set of agents can satisfy $\alpha$, as no set of agents contains an agent that considers u strictly better than u. We then, again, look at the first part of the disjunction, which always holds. The $\beta$ condition will hold because of the reflexivity of the preference relations. Any (non-empty) set of agents satisfies the beta condition since every set contains at least one agent considering $u$ and $u$ as equally preferred and contains no agent considering $u$ strictly better than $u$ or considering $u$ and $u$ incomparable. The $\gamma$ condition will also always hold. This is because no set of agents can satisfy $\alpha$, as mentioned, making it fail in both directions. Thus making the plurality method preserve reflexivity.

If the empty set was included and it happened to appear at the bottom of the reliability ordering, reflexivity would not hold. As mentioned before, the $\alpha$ fails because strict arrows cannot appear from one world to an identical world because it would then count as going both ways, making the $\alpha$ fail. The $\beta$ condition is what is asked of the least reliable set of agents, and if that set is empty, it will fail. Thus if the empty set were included as the bottom set, reflexivity would not be preserved by the plurality method.

#### Discussion

The section above examines whether the three different preference aggregation methods preserve the property of reflexivity. Reflexivity is a crucial property in preference orderings, ensuring that every element is at least as preferred as itself.

Reflexivity, defined as for all worlds $u \in W$, $u \leqslant u$, serves as a basis for preference orderings. In the context of preference aggregation methods, assessing whether the proposed aggregation methods preserve the property of reflexivity is essential.

Based on the proofs provided in the previous section, it can be concluded that all three preference aggregation methods, non-contentious, majority, and plurality, preserve the property of reflexivity. Regardless of the specific method used, the resulting preference ordering ensures that each world is at least as preferred as itself. This observation demonstrates the consistency of these methods in upholding this essential characteristic of preference orderings.

The proof shows that if the empty set were to appear at the bottom of the reliability ordering, none of the proposed strategies would preserve reflexivity. This happens in all three strategies because of the same factors. The $\alpha$-conditions all fail since all three require strict arrows, which can't happen from $u$ to $u$. The $\beta$-conditions (the ones asked to the least reliable set) are the ones assuring the reflexive arrow in all three, so if the least reliable set is empty, reflexivity will not be preserved. The empty set is not included in this thesis, but it is discussed to some extent in the future work section.

## 4.2 Transitivity

To see if the property of transitivity preserves for each of the methods, we will assume all agents to be transitive to see if the combined preference ordering preserves transitivity.

Let $\{\leqslant_i\}_{i \in A}$ be a finite collection of binary preference orderings over a domain $W$; Recall that the reliability ordering over non-empty sets of agents can be seen as an array

$C$ where the most reliable set of agents appears in position 1. The least reliable set of agents appears in the last position $n$. We want to discuss whether the proposed strategies preserve transitivity, that is, whether the resulting preference ordering is transitive when the preferences of all agents are transitive.

Definition of transitivity: for all worlds $u$, $v$ and $w \in W$ if $u \leqslant v \wedge v \leqslant w$ then $u \leqslant w$.

#### Non-Contentious

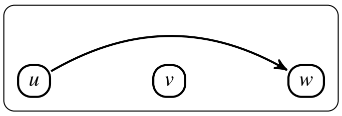
Now recall the conditions for the non-contentious method,

$$\alpha_c(u,v) := \left(\mathrm{ags}_c(u < v) \neq \varnothing\right) \wedge \left(\mathrm{ags}_c(v < u) \cup \mathrm{ags}_c(u \nsim v) = \varnothing\right).$$

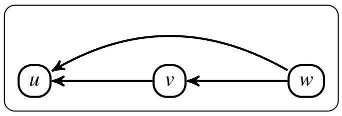
$$\beta_c(u,v) := \left(\mathrm{ags}_c(u < v) \cup \mathrm{ags}_c(u \simeq v) \neq \varnothing\right) \wedge \left(\mathrm{ags}_c(v < u) = \varnothing\right).$$

$$\gamma_c(u,v) := \neg\alpha_c(u,v) \wedge \neg\alpha_c(v,u)$$

Disproof by counterexample: Imagine you have at least two agents; $a$ and $b$, and three worlds; $u$, $v$ and $w$. The transitive preference orderings look like this,




*a*



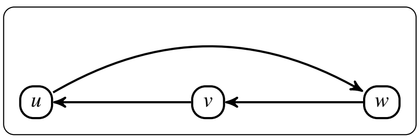

*b*

and the reliability ordering looks like this,

$$\cdots \longrightarrow \{b\} \longrightarrow \{a\}$$

The only sets being consulted are the two most reliable. This is why the rest of the reliability ordering is not displayed. If the two most reliable sets are as shown above, the rest of the reliability ordering could be in any order, and the outcome would be the same.

If we aggregate these preferences using the non-contentious method, we would get this non-transitive preference ordering, proving that transitivity does not hold in the non-contentious method.




There is an arrow from $w$ to $v$ because the $\alpha$ and the $\gamma$ hold in the second most reliable set. An arrow isn't drawn in the most reliable set since the $\alpha$ does not hold, and since the $\alpha$ does not hold either way, this means that $\gamma$ holds for the second most reliable set. Using the same logic, an arrow is also drawn from $v$ to $u$. In the second most reliable set, $\alpha$ and $\gamma$ hold because of the same principle. Now we have an arrow from $w$ to $v$ and an arrow from $v$ to $u$. Based on the transitivity definition, we need an arrow from $w$ to $u$. We don't draw an arrow from $w$ to $u$ because, in the most reliable set, the $\alpha$ holds, making us draw an arrow from $u$ to $w$. This means that the $\gamma$ will fail for every less reliable set than the most reliable one, meaning we don't get an arrow from $w$ to $u$. Thus making transitivity not always be preserved by the non-contentious method.

#### Majority

Recall the conditions for the majority method,

$$\boldsymbol{\alpha}_c(u,v) := |\mathrm{ags}_c(u < v)| \; > \; |\mathrm{ags}_c(v < u) \cup \mathrm{ags}_c(u \nsim v) \cup \mathrm{ags}_c(u \simeq v)|$$

$$\boldsymbol{\beta}_c(u,v) := |\mathrm{ags}_c(u < v) \cup \mathrm{ags}_c(u \simeq v)| \; > \; |\mathrm{ags}_c(v < u) \cup \mathrm{ags}_c(u \nsim v)|$$

$$\boldsymbol{\gamma}_c(u,v) := \neg\boldsymbol{\alpha}_c(u,v) \wedge \neg\boldsymbol{\alpha}_c(v,u)$$

Disproof by counterexample: Imagine you have at least two agents; $a$ and $b$, and three worlds; $u$, $v$ and $w$. The transitive preference orderings look like this,

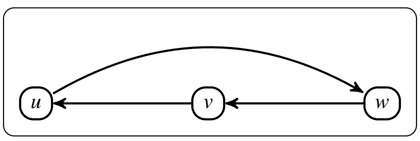

*a* *b*

and the reliability ordering looks like this,

$$\cdots \longrightarrow \{b\} \longrightarrow \{a\}$$

The only sets being consulted are the two most reliable. This is why the rest of the reliability ordering is not displayed. If the two most reliable sets are as shown above, the rest of the reliability ordering could be in any order, and the outcome would be the same.

If we aggregate these preferences using the majority method, we would get this non-transitive preference ordering, proving that transitivity does not be preserved by the majority method.



The new preference ordering is identical to the one we got in the non-contentious proof. The method of getting there is also similar. There is an arrow from $w$ to $v$ because the $\alpha$ and the $\gamma$ hold in the second most reliable set. An arrow isn't drawn in the most reliable set since the $\alpha$ does not hold, and since the $\alpha$ does not hold either way, $\gamma$ holds for the second most reliable set. By using the same logic, an arrow is also drawn from $v$ to $u$. In the second most reliable set, $\alpha$ and $\gamma$ hold because of the same principle. Now we have an arrow from $w$ to $v$ and an arrow from $v$ to $u$. Based on the transitivity definition, we need an arrow from $w$ to $u$. We don't draw an arrow from $w$ to $u$ because, in the most reliable set, the $\alpha$ holds, making us draw an arrow from $u$ to $w$. This means that the $\gamma$ will fail for every less reliable set than the most reliable one, meaning we don't get an arrow from $w$ to $u$. Thus meaning, transitivity is not always preserved by the majority method.

#### Plurality

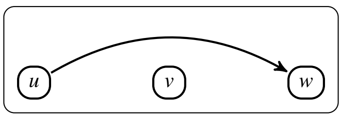
Recall the conditions for the plurality method,

$$\boldsymbol{\alpha}_c(u,v) := \big(|\text{ags}_c(u<v)| > |\text{ags}_c(v<u)|\big) \wedge \big(|\text{ags}_c(u<v)| > |\text{ags}_c(u \simeq v)|\big) \\ \wedge \big(|\text{ags}_c(u<v)| > |\text{ags}_c(u \not\sim v)|\big)$$

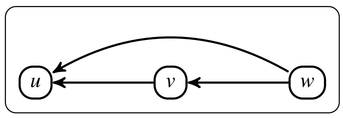
$$\boldsymbol{\beta}_c(u,v) := |\text{ags}_c(u<v) \cup \text{ags}_c(u \simeq v)| > |\text{ags}_c(v<u) \cup \text{ags}_c(u \not\sim v)|$$

$$\boldsymbol{\gamma}_c(u,v) := \neg\boldsymbol{\alpha}_c(u,v) \wedge \neg\boldsymbol{\alpha}_c(v,u)$$

Disproof by counterexample: Imagine you have at least two agents; $a$ and $b$, and three worlds; $u$, $v$ and $w$. The transitive preference orderings look like this,




$a$



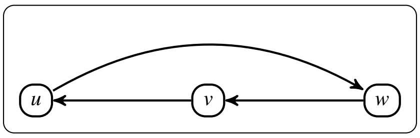

$b$

and the reliability ordering looks like this,

$$\cdots \longrightarrow \{b\} \longrightarrow \{a\}$$

The only sets being consulted are the two most reliable. This is why the rest of the reliability ordering is not displayed. If the two most reliable sets are as shown above, the rest of the reliability ordering could be in any order, and the outcome would be the same.

If we aggregate these preferences using the plurality method, we would get this non-transitive preference ordering, proving that transitivity is not always preserved by the plurality method.




The new preference ordering is identical to the one we got in both previous proofs, and the method of getting there is also similar. There are small technical differences as to why the alphas, betas and gammas fails/hold, but the pattern is identical. Again, there is an arrow from $w$ to $v$ because the $\alpha$ and the $\gamma$ hold in the second most reliable set. An arrow isn't drawn in the most reliable set since the $\alpha$ does not hold, and since the $\alpha$ does not hold either way, $\gamma$ holds for the second most reliable set. By the use of the same logic, an arrow is also drawn from $v$ to $u$. In the second most reliable set, $\alpha$ and $\gamma$ hold because of the same principle. Now we have an arrow from $w$ to $v$ and an arrow from $v$ to $u$. Based on the transitivity definition, we need an arrow from $w$ to $u$. We don't draw an arrow from $w$ to $u$ because, in the most reliable set, the $\alpha$ holds, making us draw an arrow from $u$ to $w$. This means that the $\gamma$ will fail for every less reliable set than the most reliable one, meaning we don't get an arrow from $w$ to $u$. Thus meaning, transitivity is not always preserved by the plurality method.

#### Discussion

The section above examines whether the property of transitivity is preserved in the three different preference aggregation methods. Transitivity is a crucial property in preference orderings, ensuring that the preferences are consistent and allow for meaningful comparisons between different alternatives.

A counterexample is presented in all three of the proposed methods, demonstrating the absence of transitivity. In all cases, the same agents and worlds are presented, and the derived preference orderings does not preserve transitivity by failing to draw an arrow from $w$ to $u$, despite arrows existing between $w$ and $v$ and also between $v$ and $u$.

Transitivity does not hold in the three proposed methods, as the counterexamples demonstrate. This might seem as if the methods are flawed since transitivity is a fundamental property asked of preference orderings, but if we are to believe the Condorcet paradox, this might not be a bad thing. The Condorcet paradox says that because of pairwise comparisons, option 1 can be preferred to option 2, option 2 to option 3, and option 3 to option 1. The paradox shows that even if every agent's preferences are transitive, the aggregated preferences of the group are not necessarily transitive because of pairwise comparisons(*Gehrlein*, 1983).

## 4.3 Totality

Recall the template.

$$u \leqslant v \quad \text{iff} \quad \underbrace{\left(\boldsymbol{\beta}_{c_n}(u,v) \wedge \bigwedge_{k=1}^{n-1} \boldsymbol{\gamma}_{c_k}(u,v)\right)}_{1} \vee \underbrace{\bigvee_{\ell=1}^{n-1} \left(\boldsymbol{\alpha}_{\ell}(u,v) \wedge \bigwedge_{k=1}^{\ell-1} \boldsymbol{\gamma}_{c_k}(u,v)\right)}_{2}$$

Totality states that any two objects are comparable: for any worlds $u$ and $v$, we have either $u \leqslant v$ or $v \leqslant u$. Thus, the preservation of totality (what we want to find out) is the same as the preservation of comparability.

Let $\{\leqslant_i\}_{i\in A}$ be a finite collection of binary preference orderings over a domain $W$; Recall that the reliability ordering over non-empty sets of agents can be seen as an array $C$ where the most reliable set of agents appears in position 1. The least reliable set of agents appears in the last position $n$. We want to discuss whether the proposed strategies preserve totality, that is, whether the resulting preference ordering is total when the preferences of all agents are total.

Definition of totality: For all worlds $u,v \in W$, $u \leqslant v \vee v \leqslant u$.

Due to the similarities in the definitions, there will only be one proof to show that the three methods preserve totality. This can be done because the $\gamma$-condition is the same in all cases and because the beta is similar. Nowhere in the proof will something strategy specific be mentioned. The proof works as proof for all three proposed strategies, even though it is only one proof.

To prove that totality holds in all three methods, we will assume that $u \leqslant v$ fails to show that $v \leqslant u$ will hold.

If $u \leqslant v$ fails, this means that the most important set of agents $c_1$ does not satisfy the alpha condition for $(u,v)$, that is, $\alpha_{c_1}(u,v)$ fails. Now there are two options.

(1) Either $c_1$ does not satisfy the $\gamma$-condition (that is $\gamma_{c_1}(u,v)$ fails), then $\alpha_{c_1}(v,u)$ holds (because $\gamma$-conditino is the negation of alpha in one direction and the negation of alpha in the other direction), enough for the strategy to set $v \leqslant u$.

(2) Or $c_1$ satisfies the $\gamma$-condition, then consider the second set of agents $c_2$. It cannot satisfy the $\gamma$-condition for $(u,v)$ (otherwise, the strategy would set $u \leqslant v$), that is, $\alpha_{c_2}(u,v)$ fails. We are again left with two options.

(1) Either $c_2$ does not satisfy the $\gamma$-condition (that is $\gamma_{c_2}(u,v)$ fails), then $\alpha_{c_2}(v,u)$ holds, enough for the strategy to set $v \leqslant u$.

(2) Or $c_2$ satisfies the $\gamma$-condition, and we then get to consider the third set of agents $c_3$. If we get to continue asking less and less reliable sets of agents, we eventually reach the least reliable set of agents $c_n$. Since $u \leqslant v$ fails, this set does not satisfy the beta condition for $(u,v)$, that is, $\beta_{c_n}(u,v)$ fails. Because beta is a conjunction, there are two cases.

(1) The set either contains an agent who considers $u$ strictly better than $v$. This agent appears as a singleton somewhere in the ordering and as a singleton will satisfy the alpha/beta condition for the pair $(v,u)$. Since we reach this least reliable set, every other set satisfies the $\gamma$-condition, so this last set will be asked, and thus it will draw an arrow from $v$ to $u$.

(2) Or the set contains no agents with an arrow from $u$ to $v$. But, since all preference relations are assumed to be total, and since the set is non-empty, there should be at least one agent with an arrow from $v$ to $u$, so this agent considers $v < u$. Then, as in the previous case, this agent appears as a singleton somewhere in the ordering and, as a singleton, will satisfy the alpha/beta condition for the pair $(v,u)$. Since we reach this least reliable set, every other set satisfies the $\gamma$-condition, so this last set will be asked, and thus it will draw an arrow from $v$ to $u$, making totality be preserved in all three methods.

#### Discussion

The property of totality is an essential element in preference orderings. Totality ensures that any two objects or worlds can be compared. The definition states that for any given pair of worlds, one of the following must hold: either the first world is as least as good as the second (denoted as $u \leqslant v$), or the second world is at least as good as the first (denoted as $v \leqslant u$).

Based on the presented proofs, all three preference aggregation methods preserve the property of totality. The fact that all three methods preserve totality is good because this means that we are not left with incomparable worlds when it is not necessary. Incomparable worlds could sometimes be desirable, but not when the preference ordering is total. One might want an aggregated preference ordering to be incomparable when the agent's preference ordering is incomparable, but not in other cases.

Something to highlight from the totality proof is that the assumption of all individual preferences being total is only needed in the last part of the proof, in the least reliable set when checking the $\beta$-condition.

## 4.4 Antisymmetry

Let $\{\leqslant_i\}_{i\in A}$ be a finite collection of binary preference orderings over a domain $W$; Recall that the reliability ordering over non-empty sets of agents can be seen as an array $C$ where the most reliable set of agents appears in position 1. The least reliable set of agents appears in the last position $n$. We want to discuss whether the proposed strategies preserve antisymmetry, that is, whether the resulting preference ordering is antisymmetric when the preferences of all agents are antisymmetric.

Definition of antisymmetry: For all worlds $u, v \in W$, if $u \leqslant v \wedge v \leqslant u$ then $u = v$.

#### Non-Contentious

The only way for two arrows to appear between any two different $u, v \in W$ in the non-contentious method is for in the least reliable set, there has to be at least one agent who prefers $u$ and $v$ equally and for no agents to have a strict preference. The $\gamma$ condition also has to hold, to make sure that there isn't a more reliable set drawing an arrow. This scenario is impossible when every $\leqslant_i$ is antisymmetric. When an arrow is drawn one way, the $\gamma$ condition ensures that another arrow is not drawn in a less reliable set of agents. -it is also not possible for the alpha condition to hold in both directions since for it to hold from $u$ to $v$, you need at least one agent believing $u < v$ and no agent to believe

$v < u$, and for it to hold from $v$ to $u$ you need at least one agent believing $v < u$. This is not possible, thus making antisymmetry preserved by the non-contentious method.

#### Majority

The only way for two arrows to appear between any two different $u, v \in W$ in the majority method is to, in the least reliable set, both have a majority for arrows from $u$ to $v$ over strict arrows from $v$ to $u$ and for arrows from $v$ to $u$ over strict arrows from $u$ to $v$. The $\gamma$ condition also has to hold, to make sure that there isn't a more reliable set drawing an arrow. This scenario is impossible when every $\leqslant_i$ is antisymmetric. When an arrow is drawn one way, the $\gamma$ condition ensures that another arrow is not drawn in a less reliable set of agents. It is also impossible for the alpha condition to hold in both directions in the same set since there can't be a majority for both $u < v$ and $v < u$ simultaneously. This is not possible, thus making antisymmetry preserved by the majority method.

#### Plurality

Identical to the previous proof, the only way for two arrows to appear between any two different $u, v \in W$ in the plurality method is to, in the least reliable set, both have a majority for arrows from $u$ to $v$ over strict arrows from $v$ to $u$ and for arrows from $v$ to $u$ over strict arrows from $u$ to $v$. The $\gamma$ condition also has to hold to ensure there isn't a more reliable set drawing an arrow. This scenario is impossible when every $\leqslant_i$ is antisymmetric. When an arrow is drawn one way, the $\gamma$ condition ensures that another arrow is not drawn in a less reliable set of agents. It is also impossible for the alpha condition to hold in both directions in the same set since there cant be a plurality for both $u < v$ and $v < u$ simultaneously. This is not possible, thus making antisymmetry preserved by the plurality method.

#### Discussion

The proofs provided in the section above shows that the antisymmetry property is preserved in all three methods. This means that if every individual preference ordering is antisymmetric, the aggregated preference ordering will also be antisymmetric. This is desirable because it ensures that if all agents rank two alternatives equally, they are considered identical in the social preference order, meaning no two different alternatives will be ranked equally. This is helpful in situations where the individual preference ordering is also antisymmetric. Preserving the property of antisymmetry in preference aggregation methods promotes consistency and coherence in the decision-making process.

## 4.5 Unanimity

Let $\{\leqslant_i\}_{i \in A}$ be a finite collection of binary preference orderings over a domain $W$; Recall that the reliability ordering over non-empty sets of agents can be seen as an array $C$ where the most reliable set of agents appears in position 1. The least reliable set of agents appears in the last position $n$. The property that we want to discuss is whether the aggregation method respects unanimity. If all agents agree on the relative position of two objects, then the resulting preference will take this as its output. Now between any two objects, there are three possibilities: one is strictly more preferred than the other, or they are equally preferred or incomparable. Those are the three cases we want to check.

## Non-Contentious

Recall the conditions for the non-contentious method,

$$\alpha_c(u,v) := \left(\text{ags}_c(u<v) \neq \varnothing\right) \wedge \left(\text{ags}_c(v<u) \cup \text{ags}_c(u \not\sim v) = \varnothing\right).$$

$$\beta_c(u,v) := \left(\text{ags}_c(u<v) \cup \text{ags}_c(u \simeq v) \neq \varnothing\right) \wedge \left(\text{ags}_c(v<u) = \varnothing\right).$$

$$\gamma_c(u,v) := \neg\alpha_c(u,v) \wedge \neg\alpha_c(v,u)$$

#### Strict Preference

We assume all agents to believe $u<v$ and that at least one agent exists to prove that $<$ holds.

Since all agents believe $u<v$ and there is no empty set, the most reliable set of agents will contain at least one agent who believe $u<v$ making the set $(\text{ags}_c(u<v))$ not empty and the sets $(\text{ags}_c(v<u) \cup \text{ags}_c(u \not\sim v))$ empty, thus making the $\alpha$ condition hold. The $\gamma$ condition will also hold since this is the most reliable set, making it impossible for $\alpha$ to have happened in a more reliable set. The $\gamma$ condition will also make sure that there will be no arrow from $v$ to $u$ since the set $(\text{ags}_c(v<u))$ will always be empty making $\neg\alpha(v,u)$ not hold, thus making unanimity on $<$ to be preserved by the non-contentious method.

#### Equal Preference

We assume all agents to believe $u \simeq v$ and that there exists at least one agent to prove that $\simeq$ holds.

Since every agent believes $u$ and $v$ to be equal, the alpha condition is going to fail no matter what because of the set $(\text{ags}_c(u<v))$ being empty. This means that we have to look at the least reliable set of agents. Since the $\beta$ includes the set $\text{ags}_c(u \simeq v)$, which will include every agent, an arrow will be drawn from $u$ to $v$. We then have to check if there is an arrow from $v$ to $u$. Also this way, the alpha condition is going to fail no matter what because of the set $(\text{ags}_c(v<u))$ being empty. This means that we have to look at the least reliable set of agents. Since the $\beta$ includes the set $\text{ags}_c(u \simeq v)$, which will include every agent, an arrow will be drawn from $v$ to $u$ as well, thus making unanimity on $\simeq$ to be preserved by the non-contentious method.

#### Incomparability

We assume all agents to believe $u \not\sim v$ and that there exists at least one agent to prove that $\not\sim$ holds.

Since every agent believe $u$ and $v$ to be incomparable, the alpha condition is going to fail no matter what because of the set $(\text{ags}_c(u<v))$ being empty. This means that

we have to look at the least reliable set of agents. The $\beta$ condition ask for the sets $(\text{ags}_c(u < v) \cup \text{ags}_c(u \simeq v))$ to not be empty. If every agent believes $u \not\sim v$ then they will be empty no matter how many agents are in the set. This means no arrow will be drawn from $u$ to $v$ and the same goes the other direction, from $v$ to $u$, thus making unanimity on $\not\sim$ to be preserved by the non-contentious method.

## Majority

Recall the conditions for the majority method,

$$\alpha_c(u,v) := |\text{ags}_c(u < v)| > |\text{ags}_c(v < u) \cup \text{ags}_c(u \not\leq v) \cup \text{ags}_c(u \simeq v)|$$

$$\beta_c(u,v) := |\text{ags}_c(u < v) \cup \text{ags}_c(u \simeq v)| > |\text{ags}_c(v < u) \cup \text{ags}_c(u \not\sim v)|$$

$$\gamma_c(u,v) := \neg\alpha_c(u,v) \wedge \neg\alpha_c(v,u)$$

#### Strict Preference

We assume all agents to believe $u < v$ and that there exists at least one agent to prove that $<$ holds.

Since all agents believe $u < v$ and there is no empty set, the most reliable set of agents will contain at least one agent who believe $u < v$ making the set $(\text{ags}_c(u < v))$ be a majority no matter how many agents are in the set, thus making the $\alpha$ condition hold. The $\gamma$ condition will also hold since this is the most reliable set, making it impossible for $\alpha$ to have happened, either way, in a more reliable set. The $\gamma$ condition will also make sure that there will be no arrow from $v$ to $u$ since the set $(\text{ags}_c(u < v))$ will always be in majority making $\neg\alpha(u,v)$ not hold, thus making unanimity on $<$ to be preserved by the majority method.

#### Equal Preference

We assume all agents to believe $u \simeq v$ and that there exists at least one agent to prove that $\simeq$ holds.

Since every agent believes $u$ and $v$ to be equal, the alpha condition is going to fail no matter what because of the set $(\text{ags}_c(u < v))$ being empty, making it not a majority no matter what. This means that we have to look at the least reliable set of agents. Since the $\beta$ includes the set $\text{ags}_c(u \simeq v)$, which will include every agent, an arrow will be drawn from $u$ to $v$.

We then have to check if there is an arrow from $v$ to $u$. Also this way, the alpha condition is going to fail no matter what because of the set $(\text{ags}_c(v < u))$ being empty, making it impossible for it to be in a majority. This means that we have to look at the least reliable set of agents again. Since the $\beta$ includes the set $\text{ags}_c(u \simeq v)$, which will include every agent, an arrow will be drawn from $v$ to $u$ as well, thus making unanimity on $\simeq$ to be preserved by the majority method.

#### Incomparability

We assume all agents to believe $u$ and that there exists at least one agent to prove that $\nsim$ holds.

Since every agent believe $u$ and $v$ to be incomparable, the alpha condition is going to fail no matter what because of the set $(\text{ags}_c(u < v))$ being empty, making it impossible for it to be in a majority. This means that we have to look at the least reliable set of agents. The $\beta$ condition ask for the sets $(\text{ags}_c(u < v) \cup \text{ags}_c(u \simeq v))$ larger than the sets $(\text{ags}_c(u > v) \cup \text{ags}_c(u \nsim v))$. If every agent believes $u \nsim v$ then the sets $(\text{ags}_c(u < v) \cup \text{ags}_c(u \simeq v))$ wont be larger. This means no arrow will be drawn from $u$ to $v$ and the same goes the other direction, from $v$ to $u$, thus making unanimity on $\nsim$ to be preserved by the majority method.

### Plurality

Recall the conditions for the plurality method,

$$\alpha_c(u,v) := \big(|\text{ags}_c(u<v)| > |\text{ags}_c(v<u)|\big) \wedge \big(|\text{ags}_c(u<v)| > |\text{ags}_c(u\simeq v)|\big) \\ \wedge \big(|\text{ags}_c(u<v)| > |\text{ags}_c(u \nsim v)|\big)$$

$$\beta_c(u,v) := |\text{ags}_c(u<v) \cup \text{ags}_c(u \simeq v)| > |\text{ags}_c(v<u) \cup \text{ags}_c(u \nsim v)|$$

$$\gamma_c(u,v) := \neg\alpha_c(u,v) \wedge \neg\alpha_c(v,u)$$

#### Strict Preference

We assume all agents to believe $u < v$ and that there exists at least one agent to prove that $<$ holds.

Since all agents believe $u < v$ and there is no empty set, the most reliable set of agents will contain at least one agent who believe $u < v$ making the set $(\text{ags}_c(u < v))$ be a plurality no matter how many agents are in the set, thus making the $\alpha$ condition hold. The $\gamma$ condition will also hold since this is the most reliable set, making it impossible for $\alpha$ to have happened, either way, in a more reliable set. The $\gamma$ condition will also make sure that there will be no arrow from $v$ to $u$ since the set $(\text{ags}_c(u < v))$ will always be in plurality making $\neg\alpha(u, v)$ not hold, thus making unanimity on $<$ to be preserved by the majority method.

#### Equal Preference

We assume all agents to believe $u \simeq v$ and that there exists at least one agent to prove that $\simeq$ holds.

Since every agent believes $u$ and $v$ to be equal, the alpha condition is going to fail no matter what because of the set $(\text{ags}_c(u < v))$ being empty, making it not be in a plurality no matter what. This means that we have to look at the least reliable set of agents.

Since the $\beta$ includes the set $\text{ags}_c(u \simeq v)$, which will include every agent, an arrow will be drawn from $u$ to $v$.

We then have to check if there is an arrow from $v$ to $u$. Also this way, the alpha condition is going to fail no matter what because of the set $(\text{ags}_c(v < u))$ being empty, making it impossible for it to be in a plurality. This means that we have to look at the least reliable set of agents again. Since the $\beta$ includes the set $\text{ags}_c(u \simeq v)$, which will include every agent, an arrow will be drawn from $v$ to $u$ as well, thus making unanimity on $\simeq$ to be preserved by the plurality method.

#### Incomparability

We assume all agents to believe $u \nsim v$ and that at least one agent exists to prove that $\nsim$ holds.

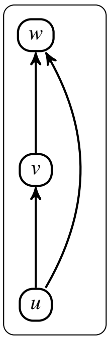
Since every agent believe $u$ and $v$ to be incomparable, the alpha condition is going to fail no matter what because of the set $(\text{ags}_c(u < v))$ being empty, making it impossible for it to be in a plurality. This means that we have to look at the least reliable set of agents. The $\beta$ condition ask for the sets $(\text{ags}_c(u < v) \cup \text{ags}_c(u \simeq v))$ larger than the sets $(\text{ags}_c(u > v) \cup \text{ags}_c(u \nsim v))$. If every agent believes $u \nsim v$ then the sets $(\text{ags}_c(u < v) \cup \text{ags}_c(u \simeq v))$ won't be larger. This means no arrow will be drawn from $u$ to $v$ and the same goes the other direction, from $v$ to $u$, thus making unanimity on $\nsim$ to be preserved by the plurality method.

#### Discussion

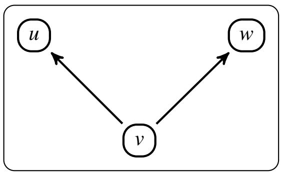
Preserving unanimity as a preference aggregation method is valuable because it ensures fairness by respecting the consensus of all agents. All three proposed aggregation methods preserve unanimity regarding strict preference, equal preference, and incomparability. The three methods accurately reflect the collective intention by respecting unanimity.

## 4.6 Comparing the Examples

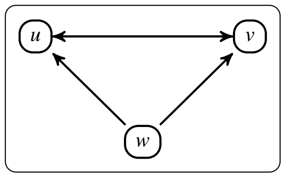
Let's explore the example used throughout this thesis, showcasing how the different methods aggregate preferences. The four figures below represent the preferences of four different agents. Recall the requirements for preference orderings; they are reflexive and transitive. The reflexive arrow will not be drawn, since every world is reflexive. Agent $a$ has a clear preference, the agent prefers world $w$ over world $v$ and world $v$ over world $u$. Agent $b$ is not as sure, the agent believes world $v$ to be the least preferred, but can't compare the other two worlds. Agent $c$ prefers world $w$ the least, and the worlds $u$ and $v$ equally better that $w$. Agent $d$ thinks of all worlds as equally preferred.



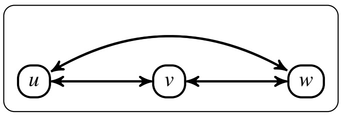

*a*




*b*




*c*

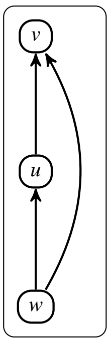



*d*

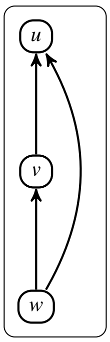
Now that we have established the preference orderings, we need a reliability ordering to illustrate which set of agents are the most reliable. Recall the requirements for reliability orderings, it is a linear order, meaning it is reflexive, transitive, antisymmetric and total. The reliability relation over the sets of agents can be seen below. The reliability ordering has the set including all four agents at the top, then comes the singleton including only agent $c$, then comes the set including agents $a$, $c$ and $d$, and at the bottom we find the set including only agent $b$. The sets of agents between $\{b\}$ and $\{a,c,d\}$ are not needed, because they are not consulted in this example.

{b} ⟶ ··· ⟶ {a,d,c} ⟶ {c} ⟶ {a,b,c,d}

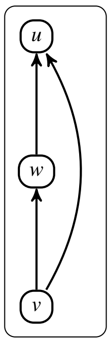
If we use the methods presented in this thesis to aggregate these preferences using the reliability ordering shown above, the result will be the new updated preference ordering shown below.




*Non-contentious*




*Majority*




*Plurality*

All three of the new updated preference orderings have different preferences, showcasing the differences. Even if they all preserve the same properties, they differ in some situations. As seen above, if we aggregate the four preferences into one, using the reliability ordering provided, these would result from the three different aggregation methods. In this example, all three aggregation methods preserve transitivity, even though none preserve transitivity in all situations. This example also shows that the choice of aggregation method matters, as the results can differ depending on the chosen method.

# 5 Conclusion and Future Work

## 5.1 Conclusion

In this thesis, three different strategies for aggregating preferences have been presented. First, the non-contentious method was presented, with its unique way of aggregating preferences. The intuition behind the non-contentious method was simple, groups of people are happier doing things everyone in a said group wants and avoiding doing things not everyone wants. The name "non-contentious" is a term meaning not likely to cause disagreement or an argument. The non-contentious method is the most unique of the three strategies, mostly because the two other strategies are quite similar but also because it is the only strategy of the three where you don't find a similar method in the related fields. The closest preference aggregation method to the non-contentious method is probably the unanimity rule (*Romme*, 2004). The "unanimity rule requires complete consensus in the group for a decision to be made. This is close to what the non-contentious method does, but not quite since the non-contentious method doesn't need complete consensus; it only needs not to have disagreements. An example of how they are different could be three people in a group; two people think one option is better than another, and the last person thinks they are equal. In the unanimity rule, a decision cant be made since there isn't a consensus, but in the non-contentious method, a decision will be made that option is better than the other.

The second preference aggregation method presented is the majority method. The intuition behind this is simple: the easiest approach for getting a group of people to agree to something is to choose what most people want to do. This method is found in many related fields([politics(*Saunders*, 2010), law(*Levmore*, 1989), social sciences(*LAM*, 2000)]). The difference in this thesis' method is that there is a reliability ordering over sets of agents, meaning not all peoples votes are equal.

The last preference aggregation method presented in the thesis is the plurality method. The intuition behind it was that in groups of people with even numbers, the majority method could often end up with exactly 50% of the votes, leading to no decisive outcome. The plurality method was proposed as a slightly more expressive version of the majority method in the sense that in cases where the majority method has exactly 50% of one option, the plurality would be decisive in the cases where the other 50% did not all have the exact same preference. An example of this could be a group of four people where two people have a strict preference for option 1 over option 2, one person has the options as equal, and the last person cant compare them. In this scenario, option 1 would not have a majority even though there are three arrows from option 2 to option 1, but there would be a plurality. The plurality method is also found in several related fields([elections(*Blais and Carty*, 1988), opinion polls(*Katz and Cantril*, 1937)]).

Regarding the properties of the three proposals, all three preserve reflexivity, totality, and antisymmetry and respect unanimity. None of the proposals preserve transitivity in all situations, but as discussed in the previous section, this might not be terrible. At least in the reflexivity proof, it was also discussed that if the empty set was included, this might affect what properties are preserved, this will be discussed further in the next section.

In conclusion, all three aggregation methods share the same properties and work for aggregating preferences. All three aggregation methods behave differently in certain situations and identically in others. It is difficult to say if one of the aggregation procedures is better than another, but all three seem to work well in most situations.

## 5.2 Future Work

One of the most evident works for the future would be to add formal language for describing the various operations happening in this thesis. With a formal language one could go into depth to analyze and describe in greater detail what happens in different scenarios.

There are also a lot of small adjustments that could be made to what is presented in this thesis. What if we don't require the reliability ordering to be a linear order? How does this change the properties and aggregated preferences? Also, if we don't require the preference ordering to be a total order, how does this affect properties? It could also be interesting to have a reliability ordering where, since it is an ordering over sets of agents, what were to happen both if the empty set is taken into consideration and also what would happen if you exclude the singleton sets and only look at set including at least two agents.

The reason this thesis chose to ignore the empty set was not just to help these proposals preserve some properties. It was done so that it might be included in the future. If one includes the empty set of agents, one has to think about what this means, to trust an empty set. If a set of agents are less trusted than an empty set, are they then distrusted? Many possibilities unfold when including the empty set. This thesis, however, opted against including it at this point.

Another interesting approach for future work could be looking more into the psychological part of preference aggregation and trying to optimize the aggregation method rooted more in reality. This thesis is very abstract, talking about "agents"; one could look more into more specific situations where people act differently. There could be differences in how people act based on their age, gender, political standpoint, or country of origin.

Even the gender distribution of a friend group can alter its dynamics. A group of men may respond to and solve problems differently than a group of women. It has long been acknowledged that same-sex alliances are essential for psychological health and social support. However, researchers have given them less attention than romantic relationships and cross-sex friendships. Same-sex friendships have unique dynamics, and they can vary dependent on the gender composition of the group. Same-gender friendships can provide a space for individuals to explore and challenge gender norms but can also reinforce them. For instance, males in same-sex groups may engage in more physical and competitive activities, whereas girls in same-sex groups may prioritize emotional intimacy and social support (*Rose*, 1985).

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