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# Convexity, convolution and competitive equilibrium

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**ABSTRACT**
This paper considers a chief interface between mathematical programming and economics, namely: money-based trade of perfectly divisible and transferable goods. Three important and related features are singled out here: first, *convexity* enters via acceptable payments, second, *convolution* of monetary criteria secures Pareto efficiency, and third, *competitive equilibrium* obtains when agents' subdifferentials intersect.

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## 1. Introduction

To start simple, first consider just *one* generic economic agent. Seen as prototype, he holds *money reserve* – or liquid *bank roll* - $r \in \mathbb{R}$ alongside a bundle $y$ of perfectly divisible and transferable goods. Construe and record that bundle as a vector in a real Euclidean space $\mathbb{Y}$. The pair $(r, y) =: w$ denotes the agent's *wealth* or his *endowment*. Write $\mathbb{R} \times \mathbb{Y} =: \mathbb{W}$ for the *endowment space*.

While holding wealth $w \in \mathbb{W}$, if the agent contemplates to transact – that is, to sell or buy – a real-good bundle $\Delta y \in \mathbb{Y}$ for money, he uses a monetary *criterion*

$$\Delta y \in \mathbb{Y} \mapsto c(\Delta y\,|w) \in \mathbb{R} \cup \{+\infty\} \tag{1}$$

- seen as *cost* – to calculate or express own economic interest. [1]

Enters next a fixed finite ensemble $I$ of such economic agents, not necessarily many, but certainly more than one. Its members are consumers, producers or traders of diverse sorts. Together they form an *economy* in which member $i \in I$ has 'cost' criterion $c_i$ (1) and wealth $w_i \in \mathbb{W}$.

Typically, somebody owns, to his taste, too little of at least one good but comparatively too much of another. Therefore, using money as means of payment, *agents trade*. That activity may fit into the idealized form of an *infimal convolution*

$$c_I(\Delta y\,|\mathbf{w}) := \inf \left\{ \sum_{i \in I} c_i(\Delta y_i\,|w_i) \,\middle|\, \sum_{i \in I} \Delta y_i = \Delta y \right\}, \tag{2}$$

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featuring $\Delta y = 0$ and $\mathbf{w} := (w_i)$. That is, given a wealth profile $\mathbf{w} = (w_i)$ for which overall cost $c_I(0|\mathbf{w})$ is finite, some real-good *redistribution* $(\Delta y_i)$, $\sum_{i \in I} \Delta y_i = 0$, should, if any, minimize that cost. Thereby, a best choice would be *Pareto efficient*.

This optic on market interactions motivates the present paper to inquire: *Might the agents themselves solve problem* (2)? [2] *Can Pareto efficiency be characterized by prices?* For that, *what role does convexity play?* Because convolution (2) reflects trade, *how might market equilibrium be described?* And, while the economy still stays out of equilibrium, *which mechanisms drive trade?*

Addressing these questions, the paper is planned as follows. Following [1], Section 2 elaborates on the monetary nature of the individual agent's criterion (1). Section 3 uses convolution (2) of such criteria to define price-taking balance of markets. What comes up there is a novel and remarkably simple description of *competitive equilibrium* (Thm. 3.1). Section 4 briefly considers out-of-equilibrium trade.

The paper bridges between selected parts of mathematical optimization and economics. Central in those fields are issues as to *convexity*, *convolution* and *differential calculus* of the agents' criteria. Reflecting on those issues – as they relate to convolution (2) – this paper seeks to:

- emphasize where *convexity* enters most constructively,
- justify *extremal convolution* by way of *monetary criteria*, [3]
- reinforce the importance of *differential calculus* – albeit generalized here,
- describe competitive equilibrium by the annulment of pure profits, and finally, to
- indicate that agent-based, decentralized deals may bring about such equilibrium.

The paper's motivation is composite. It's *practical* because differences between agents' economic margins drive trade; it's *theoretical* because trade continues until agents' *subdifferentials intersect*. Put differently: transactions proceed as long as some *bid-ask spreads* prevail, each reflecting a *gap between subdifferentials*. Novelties come by:

- characterizing competitive equilibrium as a steady state which annuls profits,
- the role that disjoint subdifferentials play as chief drivers of trade, and by
- a unifying view on agents themselves getting to equilibrium.

The paper addresses mathematically inclined optimizers and economists – especially those concerned with agent-driven dynamics, distributed optimization or price emergence [2]. Yet, as invitation to consider interfaces between diverse fields, it presumes almost no special knowledge.

## 2. The agent's preferences and criterion

This section steps back to derive the generic agent's cost criterion (1) from his underlying *preferences*. These are captured by a binary order $\succsim$ over the endowment space $\mathbb{W}$. Write $w \in dom \succsim$ iff the upper level, *preferred set*

$$\{\hat{w} \in \mathbb{W} \,|\, \hat{w} \succsim w\} \tag{3}$$

contains $w$. Hence, $\succsim$ is *reflexive* on $dom \succsim$. By hypothesis, it's also *transitive* there.

Of chief interest are eventual *changes* in the agent's holding of transferable goods – changes accepted and rationalized by pecuniary payments in the opposite direction.

Specifically, if the agent holds $(r, y) = w \in dom \succsim$, and contemplates an improved position $(\hat{r}, \hat{y}) = \hat{w} \succsim w$ (3) – by receiving *revenue* $\Delta r := \hat{r} - r$ for *supply* $\Delta y := -(\hat{y} - y)$ – he *asks* no less money for the latter than

$$c(\Delta y\,|w) := \inf\left\{\Delta r \in \mathbb{R} \,\middle|\, \hat{w} = (\Delta r, -\Delta y) + w \succsim w\right\} \in \mathbb{R} \cup \{+\infty\}. \tag{4}$$

[The customary convention $\inf \varnothing = +\infty$ applies.] By contrast, if he holds wealth $w \in dom \succsim$, but rather is a *customer* who *demands* $\Delta y \in \mathbb{Y}$, he would *bid* no more money for that bundle than

$$b(\Delta y\,|w) := \sup\left\{\Delta r \in \mathbb{R} \,\middle|\, \hat{w} = (-\Delta r, \Delta y) + w \succsim w\right\} \in \mathbb{R} \cup \{-\infty\}. \tag{5}$$

Since *expense is negative revenue*, and *demand is negative supply* – and formally, because $b(\Delta y|w) = -c(-\Delta y|w)$ – henceforth let $c$ (1) be a unifying criterion, derived by (4) and construed as cost. Henceforth assuming $c > -\infty$, it follows forthwith:

**Proposition 2.1 (on monetary criteria):** *For each $w \in dom \succsim$ it holds that $c(0|w) \leq 0$ - to the effect that $c(\cdot|w)$ is proper, meaning finite somewhere.*

*Provided the preferred set (3) be closed ( convex), the cost criterion $\Delta y \in \mathbb{Y} \mapsto c(\Delta y|w)$ (1) also becomes closed [4] (resp. convex).*

*As function, $c(\cdot|w)$ extends additively in the money variable from $\mathbb{Y}$ to $\mathbb{W}$ by*

$$\begin{aligned} \Delta\hat{w} := (\Delta\hat{r}, \Delta\hat{y}) \Rightarrow c(\Delta\hat{w}\,|w) &:= \inf\left\{\Delta r \,\middle|\, (\Delta r, 0) - \Delta\hat{w} + w \succsim w\right\} \\ &= \Delta\hat{r} + c(\Delta\hat{y}\,|w). \qquad \square \end{aligned} \tag{6}$$

(6) tells that cost $c(\cdot|w)$ is linear in money on its domain. So, *benefit* $b$ (5), seen as *utility* $u(\cdot) = -c(-\cdot)$, becomes *quasilinear* there – a property long and widely presumed in analysis of benefits versus costs [1].

Let $y^* \in \mathbb{Y}^*$ be shorthand for a linear price regime $y \in \mathbb{Y} \mapsto y^*y := y^*(y) \in \mathbb{R}$. If the agent faces a fixed price $y^* \in \mathbb{Y}^*$, he may aim at non-negative, *price-taking* *profit*

$$c^*(y^*|w) := \sup\left\{y^*\Delta y - c(\Delta y|w)\,\middle|\,\Delta y \in \mathbb{Y}\right\} \in \mathbb{R}_+ \cup \{+\infty\}. \quad (7)$$

*Fenchel conjugate* (7) speaks in plain economic terms. Further, its attainment links directly to calculus and optimality conditions. To clarify this, call $y^* \in \mathbb{Y}^*$ a *subgradient of* $c(\cdot|w)$ *at* $\Delta y$, written $y^* \in \partial c(\Delta y|w)$, iff

$$\Delta y \in \arg\max\left\{y^* - c(\cdot\,|w)\right\} \text{ with finite maximal value.}$$

Equivalently,

$$y^* \in \partial c(\Delta y|w) \Longleftrightarrow y^*\Delta y = c^*(y^*|w) + c(\Delta y|w) \in \mathbb{R}. \quad (8)$$

That is, price-taking *revenue* $y^*\Delta y$ should equal *profit* $c^*(y^*|w)$ atop full cover of *cost* $c(\Delta y|w)$.

**Proposition 2.2 (on profit and expenditure):** *In terms of any price regime* $(r^*, y^*) = w^* \in \mathbb{W}^* := \mathbb{R}^* \times \mathbb{Y}^*$ *on* $\mathbb{W}$, *infimal* ***expenditure***

$$\mathcal{E}(w^*|w) := \inf\left\{w^*\hat{w}\,\middle|\,\hat{w} \succsim w\right\}$$

*and supremal* ***profit***

$$c^*(w^*|w) := \sup\left\{w^*\hat{w} - c(\hat{w}|w)\,\middle|\,\hat{w} \in \mathbb{W}\right\},$$

*satisfy*

$$c^*(w^*|w) = w^*w - \mathcal{E}(w^*|w) \textit{ if } r^* = 1, +\infty \textit{ otherwise.} \quad (9)$$

*Thus, if money commands fixed unit price* $r^* = 1$ *– that is, when* $w^* = (1, y^*)$*:*

$$c^*(w^*|w) = c^*(y^*|w). \quad (10)$$

***Proof:*** Recall (6) to see that $(r^*, y^*) = w^*$ yields (9) by

$$\begin{aligned} c^*(w^*|w) &= \sup\left\{w^*\Delta w - \Delta r\,\middle|\,\hat{w} := (\Delta r, 0) - \Delta w + w \succsim w, \Delta w \in \mathbb{W}, \Delta r \in \mathbb{R}\right\} \\ &= \sup\left\{w^*(w - \hat{w}) + (r^* - 1)\Delta r\,\middle|\,\hat{w} \succsim w, \Delta r \in \mathbb{R}\right\} \\ &= \sup\left\{w^*(w - \hat{w})\,\middle|\,\hat{w} \succsim w\right\} \text{ if } r^* = 1, \text{and} + \infty \text{ otherwise} \\ &= w^*w - \mathcal{E}(w^*|w) \text{ if } r^* = 1, \text{ and} + \infty \text{ otherwise.} \end{aligned}$$

Now (10) follows from (6) and (9 ). ■

**Remarks:** Propositions 2.1 and 2.2 indicate *two* lines of subsequent arguments. *First*, for analysis, it would be convenient to have the agent's cost criterion $c$ (4) *subdifferentiable*, meaning $\partial c(\Delta y|w) \neq \varnothing$ (8) for each feasible pair $(\Delta y, w) \in \mathbb{Y} \times \mathbb{W}$ of interest. *Second*, for intuition, one might expect that his pure profit $c^*(y^*|w)$ (7) will dwindle by way of repeated trades.

It seems, however, more realistic to hope that these features be satisfied *in the large* - that is, by the convoluted items $c_I$,$c_I^*$ (2), (13), but not necessarily *in the small*, at the level of each pair $c_i, c_i^*$ (7 ). Also, because convolution tends to regularize data, the said features appear easier to justify in the aggregate.

Anyway, (2) leads directly to a novel definition of *price-taking behavior and steady states in markets* – as considered next.

## 3. Competitive equilibrium

This section aims at a simple and speaking concept of competitive equilibrium.

As data, presume that agent $i \in I$ has a reflexive and transitive preference order $\succsim_i$ over a non-empty subset *dom* $\succsim_i$ of the endowment space $\mathbb{W}$. Given any $w_i \in dom \succsim_i$, he derives his cost criterion $\Delta y_i \in \mathbb{Y} \mapsto c_i(\Delta y_i|w_i) \in \mathbb{R}\cup\{+\infty\}$ as explained in Section 2. For interpretation, it's convenient to regard him here as a producer.

At the outset, the economy features a wealth profile $i \in I \mapsto w_i^0 \in dom \succsim_i$. Let

$$\mathbf{W} := \left\{ \mathbf{w} = (w_i) \in \mathbb{W}^I \,\middle|\, w_i \in dom \succsim_i \ \& \sum_{i\in I} w_i = \sum_{i\in I} w_i^0 \right\} \tag{11}$$

be the set of feasible profiles. With $\mathbf{w} = (w_i) \in \mathbf{W}$ fixed, the *inf-convolution*

$$c_I(0\,|\mathbf{w}) := \inf \left\{ \sum_{i\in I} c_i(\Delta y_i\,|w_i) \,\middle|\, \sum_{i\in I} \Delta y_i = 0 \right\}$$

models best change in overall cost, obtained by *reallocation* $(\Delta y_i)$, $\sum_{i\in I} \Delta y_i = 0$, of goods. In particular, by (4), because $c_i(0|w_i) \le 0$, it follows that $c_I(0|\mathbf{w}) \le 0$.

Thus, the special instance $c_I(0|\mathbf{w}) = 0$ stands out. Then, *potential reduction of aggregate cost is already minimal and nil*, with each $c_i(\Delta y_i|w_i) = 0$ realized by a best choice $\Delta y_i = 0$. In short, *no* improvement is possible, be it in the large or the small; Pareto efficiency prevails already.

If moreover, a common price $y^* \in \mathbb{Y}^*$ yields $c_I^*(y^*|\mathbf{w}) = 0$, *no* added surplus can be had: *aggregate profit is also minimal and nil*. Together these simple observations motivate the following:

**Definition 3.1 (competitive equilibrium):** *A price-cum-allocation* $(y^*, \mathbf{w}) \in \mathbb{Y}^* \times \mathbf{W}$ (11) *constitutes a* ***competitive equilibrium*** *iff* $c_I^*(y^*|\mathbf{w}) = 0$.

**Theorem 3.1 (on competitive equilibrium):** *For any competitive equilibrium* $(y^*, \mathbf{w})$ *it holds:*

(1). *Overall surplus* $c_I^*(\cdot|\mathbf{w})$ *is globally minimal and null at the equilibrium price* $y^*$. *Consequently,* $0 \in \partial c_I^*(y^*|\mathbf{w})$.

(2). *No agent can collect additional profit:* $c_i^*(y^*|w_i) = 0$ *and* $0 \in \partial c_i^*(y^*|w_i)$ $\forall i \in I$.

(3). *No more trade is undertaken:* $c_i(\Delta y_i|w_i) = 0$ *with a best choice* $\Delta y_i = 0$ $\forall i \in I$.

(4). *If* $c_I(\cdot|\mathbf{w})$ *(2) coincides with its closed convex envelope at* $\Delta y = 0$, *equilibrium pricing is common, meaning*

$$y^* \in \partial c_I(0\,|\mathbf{w}) \subseteq \cap_{i\in I}\partial c_i(0\,|w_i). \tag{12}$$

*Conversely, if a price be common in that* $y^* \in \cap_{i\in I}\partial c_i(0|w_i)$, *then* $\partial c_I(0|\mathbf{w}) \supseteq \cap_{i\in I}\partial c_i(0|w_i)$, *and* $(y^*, \mathbf{w})$ *is a competitive equilibrium.*

***Proof:*** Since $w_i \in dom \succsim_i$ implies $c_i(0|w_i) \leq 0$, it follows from (7) and $c_i^*(y^*|w_i) \geq y^*0 - c_i(0|w_i) \geq 0$ that $c_i^*(y^*|w_i) \geq 0$ for each $y^* \in \mathbb{Y}^*$. Therefore, given any wealth profile $(w_i) = \mathbf{w} \in \mathbf{W}$ (11), because

$$c_I^*(\cdot\,|\mathbf{w}) = \sum_{i\in I} c_i^*(\cdot\,|w_i) \geq 0, \tag{13}$$

$c_I^*(y^*|\mathbf{w}) = 0$ is indeed minimal in equilibrium – to the effect that $0 \in \partial c_I^*(y^*|\mathbf{w})$, with each $c_i^*(y^*|w_i) = 0$, and thereby $0 \in \partial c_i^*(y^*|w_i)$. This takes care of assertions 1&2). For 3) let $c_I^{**}(\cdot|\mathbf{w})$ be the Fenchel conjugate of $c_I^*(\cdot|\mathbf{w})$. Then $c_I^{**}(0|\mathbf{w}) \leq c_I(0|\mathbf{w}) \leq 0$ and

$$y^*0 = c_I^*(y^*\,|\mathbf{w}) + c_I^{**}(0\,|\mathbf{w}) = 0 \Longrightarrow$$
$$c_I^{**}(0\,|\mathbf{w}) = 0 \Longrightarrow c_I(0\,|\mathbf{w}) = 0 \Longrightarrow \text{each } c_i(0\,|w_i) = 0.$$

For 4) invoke an auxiliary result – one which presumes no convexity: ■

**Lemma 3.1 (on subdifferentials of inf-convolutions):** *Given a real vector space* $\mathbb{Y}$ *and finite family of proper functions* $f_i : \mathbb{Y} \mapsto \mathbb{R}\cup\{+\infty\}$, $i \in I$, *for any profile* $(y_i)$ *that solves*

$$f_I(y_I) := \inf\left\{\sum_{i\in I} f_i(y_i)\,\middle|\,\sum_{i\in I} y_i = y_I\right\}, \tag{14}$$

*it follows that* $\partial f_I(y_I) = \cap_{i\in I}\partial f_i(y_i)$.

***Proof:*** from [3] is included for convenience. If $(y_i)$ solves (14) and $\sum_{i\in I}\hat{y}_i =: \hat{y}_I \in \mathbb{Y}$, then $y^* \in \partial f_I(y_I)$ implies

$$\sum_{i\in I} f_i(\hat{y}_i) \geq f_I(\hat{y}_I) \geq f_I(y_I) + y^*(\hat{y}_I - y_I) = \sum_{i\in I}\left[f_i(y_i) + y^*(\hat{y}_i - y_i)\right]. \tag{15}$$

In this string, posit $\hat{y}_j = y_j$ for each $j \in I\setminus i$ to get

$$f_i(\hat{y}_i) \geq f_i(y_i) + y^*(\hat{y}_i - y_i). \tag{16}$$

Since $i \in I$ and $\hat{y}_i \in \mathbb{Y}$ were arbitrary, it follows that $y^* \in \partial f_i(y_i)$ for all $i \in I$, hence $\partial f_I(y_I) \subseteq \cap_{i\in I}\partial f_i(y_i)$.

For the turned-around inclusion, given any $y^* \in \cap_{i\in I}\partial f_i(y_i)$ with $\sum_{i\in I} y_i = y_I$, summation of (16) across $I$, subject to $\sum_{i\in I}\hat{y}_i = y_I$, proves the optimality of allocation $(y_i)$. Further, the same summation of (16), but now with $\sum_{i\in I}\hat{y}_i = \hat{y}_I$, gives the two inequalities in (15) – and thereby $y^* \in \partial f_I(y_I)$, hence $\cap_{i\in I}\partial f_i(y_i) \subseteq \partial f_I(y_I)$. ■

Returning to claim 4), first let $\check{c}_I(\cdot|\mathbf{w})$ denote the closed convex envelope of $c_I(\cdot|\mathbf{w})$. The conjugate of the said envelope equals that of $c_I(\cdot|\mathbf{w})$. Consequently, $0 \in \partial c_I^*(y^*|\mathbf{w}) \Longrightarrow y^* \in \partial \check{c}_I(0|\mathbf{w}) \subseteq \partial c_I(0|\mathbf{w})$. As upshot, $\partial c_I(0|\mathbf{w})$ *is non-empty*. Now, use Lemma 3.1 with

$$f_i = c_i(\cdot\,|w_i), y_i = 0, \quad \text{and} \quad f_I(0) = c_I(0\,|\mathbf{w}) = \sum_{i\in I} c_i(0\,|w_i) = \sum_{i\in I} f_i(0)$$

to see that existence of any $y^* \in \partial c_I(0|\mathbf{w})$ implies $y^* \in \partial c_I(0|\mathbf{w}) = \cap_{i\in I}\partial c_i(0|w_i)$. This proves the theorem.

**Corollary 3.1 (on improvement and equilibrium):** *Given any equilibrium allocation* $\mathbf{w}$*, aggregate cost can not be reduced:* $c_I(0|\mathbf{w}) = 0$*. Conversely, given* $c_I(0|\mathbf{w}) = 0$*, then* $(y^*, \mathbf{w})$ *is a competitive equilibrium for any* $y^* \in \partial c_I(0|\mathbf{w}) = \cap_{i\in I}\partial c_i(0|w_i)$.

***Proof:*** The first assertion was already proven. For the second, $c_I(0|\mathbf{w}) = 0$ implies that $c_I(0|\mathbf{w}) = \sum_{i\in I} c_i(0|w_i) = 0$. Thus by Lemma 3.1, $\partial c_I(0|\mathbf{w}) = \cap_{i\in I}\partial c_i(0|w_i)$. Further, for any $y^* \in \partial c_I(0|\mathbf{w})$ it holds $0 = y^*0 = c_I^*(y^*|\mathbf{w}) + c_I(0|\mathbf{w})$. Consequently, $c_I(0|\mathbf{w}) = 0$ implies $c_I^*(y^*|\mathbf{w}) = 0$. ■

**Remarks (*On closure and convexity*):** Closure (alias lower semicontinuity) of $c_I(\cdot|\mathbf{w})$ at 0 obtains when $\partial c_I(0|\mathbf{w})$ is non-empty. Convolution $c_I(\cdot|\mathbf{w})$ (2) would be convex if each term $c_i(\cdot|w_i)$ were so. But convexity entered here just for $c_I(\cdot|\mathbf{w})$ and just at 0.

(*On Debreu versus Walras*). Definition 3.1 reports the wealth profile *ex post*, *in equilibrium*, as did Debreu [4]. Accordingly, Theorem 3.1 obviates trade or dispenses with it. All transactions have already been undertaken – *out of equilibrium*. By contrast, Walras fixed the wealth profile *ex ante*, *out of equilibrium*, prior to trade, by liquidating the initial endowments at equilibrium prices. This done, he allowed trade, but only *in equilibrium* at corresponding prices.

Thus, regarding competitive markets, two sorts of steady states have been conceptualized as polar extremes. It appears fitting therefore to ask: If any, how might a competitive equilibrium emerge? That question have generated a large literature with no simple answers [3, 5–10]. The next section concludes by considering these matters.

## 4. Getting to equilibrium

Adam Smith (1776) alluded to an 'invisible hand', and Leon Walras (1874) suggested 'tâtonnement' in prices. These metaphors remain largely fictional because neither offers any clear or constructive guidance. Moreover, they make no mention of market mechanisms or money.

Yet (12) tells that *disequilibrium prevails iff subdifferentials do not insect*; that is, iff $\cap_{i\in I}\partial c_i(0|w_i) = \varnothing$. For this event, it suffices that *just two* agents disagree – in fact, even when they value *just one* good. The next proposition, which spells this out, can be skipped, but it motivates Theorem 4.1.

**Proposition 4.1 (on strict improvements and direct deals):** *Omitting mention of wealth, let the criteria $c_i = c_i(\cdot|w_i)$ all be convex, finite near 0, with $c_i(0) \leq 0$, and suppose $\cap_{i\in I}\partial c_i(0) = \varnothing$. Then $c_I(0) < 0$, and still $\cap_i \partial c_{i\in I}(\Delta y_i) = \varnothing$ for sufficiently small $\Delta y_i$.*

*In that case, with $I = \{i, j\}$ and $\mathbb{Y} = \mathbb{R}$, a unit price $y^*$ - between $\partial c_i(\Delta y)$ and $\partial c_j(-\Delta y)$ - applied to some suitably small quantity $\Delta y$, gives more revenue to the seller and less expense to the buyer, hence strict improvement for either party.*

***Proof:*** Invoke Lemma 3.1 to see that infimal cost $c_I(0)$ (2) can *not* be attained by choosing all $\Delta y_i = 0$. Small perturbations $\Delta y_i$ maintain disjoint subdifferentials because these are outer semicontinuous [11]. Finally – for a strictly improving, single-good, bilateral and direct deal – let agent:

- $i$ be a seller (4) who *asks* unit price $y_i^* = \max \partial c_i(\Delta y_i)$, or more, for quantity $\Delta y_i > 0$, and let
- $j$ be a buyer (5) who uses *supdifferential* $\hat{\partial}(\cdot) := -\partial(-\cdot)$ and *bids* unit price $y_j^* = \min \hat{\partial} b_j(\Delta y_j)$, or less, for quantity $\Delta y_j > 0$. Then, given *spread* $y_j^* - y_i^* > 0$, any unit price $y^* \in (y_i^*, y_j^*)$ for the quantity $\Delta y := \min\{\Delta y_i, \Delta y_j\}$, gives

$$\text{seller } i \text{ revenue } r_i := y^*\Delta y > c_i(\Delta y) \text{ and buer } j \text{ expense } r_j := y^*\Delta y < b_j(\Delta y). \tag{17}$$

Indeed, from $y_i^* < y^*$, $c_i(0) \leq 0$ and $\partial c_i \leq y_i^*$ on $[0, \Delta y]$ it follows that

$$r_i - c_i(\Delta y) > y_i^*\Delta y - c_i(\Delta y) \geq y_i^*\Delta y - [c_i(\Delta y) - c_i(0)]$$
$$= \int_0^{\Delta y} [y_i^* - \partial c_i] \geq 0.$$

The second inequality in (17) is proven likewise, using the concavity of $-c_j(\cdot) = b_j(-\cdot)$. ■

An auxiliary result adds to Proposition 4.1 and prepares for Theorem 4.1.

**Lemma 4.1 (bid-ask spreads):** *Suppose the member $i \in I$ uses a price set $Y_i^* \subset \mathbb{Y}^*$. Let $\Delta Y_i \subset \mathbb{Y}$ be a bounded symmetric neighborhood of 0. Then, if all these sets are non-empty closed and convex, ensemble I features a non-negative **bid-ask spread***

$$S_I := \inf_{(y_i^*)} \sup_{(\Delta y_i)} \left\{ \sum_{i \in I} y_i^*(\Delta y_i) \middle| y_i^* \in Y_i^*, \Delta y_i \in \Delta Y_i \& \sum_{i \in I} \Delta y_i = 0 \right\}. \qquad (18)$$

*That spread is nil if there is a common price $y^* \in \cap_{i \in I} Y_i^*$. Conversely, let at least one set $Y_i^*$ be compact. Then, disagreement on prices, meaning $\cap_{i \in I} Y_i^* = \varnothing$, implies that $S_I > 0$. Thus, granted at least one compact price set $Y_i^*$,*

$$\cap_{i \in I} Y_i^* = \varnothing \iff S_I > 0.$$

***Proof:*** For $S_I \geq 0$, just take each $\Delta y_i = 0$. Given any $y^* \in \cap_{i \in I} Y_i^*$, clearly, $\sum_{i \in I} \Delta y_i = 0$ implies $\sum_{i \in I} y^*(\Delta y_i) = 0$.

For the converse, let $C$ equal the product set $\Pi_{i \in I} Y_i^*$. So defined, $C$ is a closed convex and non-empty subset of $\mathbb{Y}^{*I}$. If needed for compactness, intersect each $Y_i^*$ with one among these which is compact. This done, $C$ becomes compact. Now, $\cap_{i \in I} Y_i^* = \varnothing$ iff $C$ is doesn't intersect the *diagonal* $D := \{(y_i^*) \mid \text{all } y_i^* \in \mathbb{Y}^* \text{ are equal}\}$. Then, $C$ and $D$ are strictly separated by some non-zero $(\Delta y_i) \in \mathbb{Y}^I$, meaning

$$\sup \left\{ \sum_{i \in I} y_i^*(\Delta y_i) \,\middle|\, y_i^* \in Y_i^* \right\} < \inf \left\{ \sum_{i \in I} y^*(\Delta y_i) \,\middle|\, y^* \in \mathbb{Y}^* \right\}.$$

This inequality can not hold unless $\sum_{i \in I} \Delta y_i = 0$ – whence the right hand side equals 0. By suitable scaling (if necessary), it entails no loss to presume that each $\Delta y_i \in \Delta Y_i$. Now conclude because by (18):

$$\begin{aligned} -S_I &= \sup_{(y_i^*)} \inf_{(\Delta \hat{y}_i)} \left\{ \sum_{i \in I} y_i^*(\Delta \hat{y}_i) \middle| y_i^* \in Y_i^*, \Delta \hat{y}_i \in \Delta Y_i \& \sum_{i \in I} \Delta \hat{y}_i = 0 \right\} \\ &\leq \sup_{(y_i^*)} \left\{ \sum_{i \in I} y_i^*(\Delta y_i) \,\middle|\, y_i^* \in Y_i^* \right\} < 0. \end{aligned}$$

■

**Theorem 4.1 (disequilibrium and trade):** *Given a wealth profile $\mathbf{w} = (w_i) \in \mathbf{W}$, suppose agent $i \in I$ uses a closed convex price set $Y_i^*$ which contains $\partial c_i(0 \,|w_i) \neq \varnothing$, and he contemplates a transaction $\Delta y_i$ within a bounded closed convex, symmetric neighborhood $\Delta Y_i$ of 0. Then, provided at least one $Y_i^*$ be bounded, if*

*$\cap_{i\in I} Y_i^* = \varnothing$, there exists a redistribution $(\Delta y_i) \neq 0$, $\sum_{i\in I} \Delta y_i = 0$, such that*

$$\sup\left\{\sum_{i\in I} y_i^*(\Delta y_i)\left|y_i^* \in \partial c_i(0\,|w_i)\right.\right\} \leq \sup\left\{\sum_{i\in I} y_i^*(\Delta y_i) y_i^* \in Y_i^*\right\} < 0. \quad (19)$$

*In particular, if each $\partial c_i(0\,|w_i) = Y_i^*$ is compact, then $\sum_{i\in I} c_i'(0; \Delta y_i|w_i) < 0$, and for a possibly shortened profile $(\Delta y_i) \longleftarrow (s\Delta y_i)$, $s > 0$,*

$$c_I(0\,|\mathbf{w}) \leq \sum_{i\in I} c_i(\Delta y_i\,|w_i) < \sum_{i\in I} c_i(0\,|w_i) \leq 0. \quad (20)$$

*So, modulo suitable, zero-sum money transfers $(\Delta r_i) \neq 0$, $\sum_{i\in I} \Delta r_i = 0$, each $\Delta r_i > c_i(\Delta y_i|w_i)$. Clearly, such trade complies with agents' incentives.*

***Proof:*** Invoke Lemma 4.1 and use $c_i'(0; \Delta y_i|w_i) = \sup\{y_i^*(\Delta y_i)|y_i^* \in \partial c_i(0|w_i)\}$ in (19). Finally, for (20), provided $s > 0$ and $\varepsilon > 0$ both be sufficiently small,

$$\begin{aligned} c_I(0\,|\mathbf{w}) &\leq \sum_{i\in I} c_i(s\Delta y_i\,|w_i) \leq \sum_{i\in I}[c_i(0\,|w_i) + sc_i'(0; \Delta y_i\,|w_i) + s\varepsilon] \\ &< \sum_{i\in I} c_i(0\,|w_i) \leq 0. \end{aligned}$$

■

**Remarks (*On active or restrictive traders*):** In Lemma 4.1 and Theorem 4.1 a strictly smaller ensemble $\mathcal{I} \subset I$, $\#\mathcal{I} \geq 2$, could come onto stage. Moreover, its members might just trade goods recorded in a lower-dimensional commodity space $Y \subset \mathbb{Y}$. Also, instead of demanding that at least one price set $Y_i^*$ be non-empty compact, it suffices that $\cap_{i\in\mathcal{I}} Y_i^*$ be such for some subset $\mathcal{I} \subsetneq I$.

(*On market mechanisms*). Anyway, trade proceeds via various mechanisms – say, via auctions, direct deals or order markets. Most likely, active traders vary, maybe randomly, in their names, numbers or proximity – or in their focus on selected goods.

(*On incentive compatibility*). Reasonably, no party ever accepts a set-back compared to his pre-trade position. That is, each deal should be voluntary:

**Assumption 4.1 (on acceptable deals and updates):** *If agent $i \in I$ enters a deal with endowment $(r_i, y_i) =: w_i \in \text{dom} \succsim_i$, he exits with an 'improved' updated version $(\hat{r}_i, \hat{y}_i) =:$*

$$\hat{w}_i =: w_i^{+1} = (\Delta r_i, -\Delta y_i) + w_i \succsim_i w_i, \quad (21)$$

*featuring a money transfer $\Delta r_i := \hat{r}_i - r_i$ for some bundle $\Delta y_i := \hat{y}_i - y_i$ such that*

$$\Delta r_i \geq \partial c_i(\Delta y_i\,|w_i) \quad \text{and} \quad \sum_{i\in I}(\Delta r_i, \Delta y_i) = (0, 0). \quad (22)$$

*As modelled, trade complies with incentives because $\Delta r_i \geq \partial c_i(\Delta y_i|w_i)$, and the actions are purely redistributive in that $\sum_{i\in I}(\Delta r_i, \Delta y_i) = (0, 0)$. Write $\hat{w}_i \succ_i w_i$ if $\Delta r_i > \partial c_i(\Delta y_i|w_i)$. These features motivate the following:*

**Assumption 4.2 (on trades):** *Agents' transactions fit the algorithmic form*

$$\mathbf{w} \in \mathbf{W} \rightrightarrows A(\mathbf{w}) := \left\{ \mathbf{w}^{+1} = (w_i^{+1}) \in \mathbf{W} \,\middle|\, w_i^{+1} \succsim_i w_i \; \forall\, i \in I \right\}. \quad (23)$$

*Any instance A has the solution set*

$$\mathbf{E} := \left\{ \mathbf{w} \in \mathbf{W} \,\middle|\, (y^*, \mathbf{w}) \text{ is a competitive equilibrium for some } y^* \in \mathbb{Y}^* \right\}.$$

**Proposition 4.2 (on existence of equilibrium):** *Suppose each upper level set* $\{\cdot \succsim_i w_i\}$ *(3) is convex. With* $\mathbf{W}$ *compact, also suppose the correspondence* $A(\cdot)$ *is outer semicontinous. Then – by Kakutani's theorem [11] - there exists an equilibrium.*

**Proposition 4.3 (on strictly improving trades):** *Suppose* $c_I(0|\mathbf{w})$ *is attained and* $\partial c_I(0|\mathbf{w})$ *be non-empty for each* $\mathbf{w} \in \mathbf{W}$ *with at least one* $\partial c_i(0|w_i)$ *compact. Let* $\mathbf{w} \in \mathbf{W}$ *qualify for update by A (23) iff* $S_I(\mathbf{w}) > 0$ *(18) or* $c_I(0|\mathbf{w}) < 0$*. Then some* $\mathbf{w}^{+1} \in A(\mathbf{w})$ *satisfies* $w_i^{+1} \succ_i w_i$ *for each* $i \in I$*. Otherwise, if* $S_I(\mathbf{w}) = 0$ *and* $c_I(0|\mathbf{w}) = 0$*,then* $(y^*, \mathbf{w})$ *is an equilibrium for any* $y^* \in \partial c_I(0|\mathbf{w})$*.*

***Proof:*** From $S_I(\mathbf{w}) > 0$ and Theorem 3.1 it follows that $\cap_{i\in I}\partial c_i(0|w_i) = \emptyset$. Hence by (20) in Theorem 4.1 it holds $c_I(0|\mathbf{w}) < 0$ and the system

$$w_i^{+1} = (\Delta r_i, -\Delta y_i) + w_i, \quad \Delta r_i > c(\Delta y_i\,|w_i), \quad \forall\, i \in I,$$

is solvable. Then $\mathbf{w}^{+1} \in A(\mathbf{w})$ with each $w_i^{+1} \succ_i w_i$.

Otherwise, if $S_I(\mathbf{w}) = 0$, the intersection $\cap_{i\in I}\partial c_i(\Delta y_i|w_i)$ is non-empty for some allocation $(\Delta y_i)$, $\sum_{i\in I} \Delta y_i = 0$. Then, by Lemma 3.1, $\partial c_I(0|\mathbf{w}) = \cap_{i\in I}\partial c_i(\Delta y_i|w_i)$. Now, for any $y^* \in \partial c_I(0|\mathbf{w})$ the conclusion follows from

$$0 = y^*0 = c_I^*(y^*\,|\mathbf{w}) + c_I(0\,|\mathbf{w}) = c_I^*(y^*\,|\mathbf{w}).$$

■

Suppose the market features non-overlapping, sequential *sessions*, each closing by clearance or clock. Also for argument, suppose that when a session closes, the very last transactions (21) are rationalized *ex post* – at closure time – by the parties themselves, and by Lemma 3.1, as follows:

**Assumption 4.3 (on session closure):** *Each market session closes by some last reallocation* $(\Delta y_i)$, $\sum_{i\in I} \Delta y_i = 0$, *supported by a clearing price* $y^* \in \cap_{i\in I}\partial c_i(\Delta y_i|w_i)$ *and revenues* $r_i = y^*\Delta y_i$ *(21). Immediately thereafter, agent i updates his holding to* $\hat{w}_i = w_i^{+1}$ *(21). With that update he enters the subsequent session. If worthwhile, the latter begins with* $\cap_{i\in I}\partial c_i(0|w_i^{+1}) = \emptyset$.

*What stands sharply out is the special case where* $w_i^{+1} = w_i$ *and* $\cap_{i\in I}\partial c_i(0|w_i) \neq \emptyset$.

*Then, another session has no effect: a best option for every agent, given his endowment, is to stay put. Can iterated sessions bring the agents towards such a state? Recall that, given any $y^* \in \mathbb{Y}^*$ and $\mathbf{w} = (w_i) \in \mathbf{W}$, total profit equals*

$$c_I^*(y^* | \mathbf{w}) = \sum_{i \in I} c_i^*(y^* | w_i) \text{ with each } c_i^*(y^* | w_i) \geq 0.$$

*Specialize here to session closure with $y^* \in \cap_{i\in I} \partial c_i(\Delta y_i | w_i)$ and $\sum_{i\in I} \Delta y_i = 0$. So, to focus the above question, I rather ask: will total profit $c_I^*(y^* | \mathbf{w})$ decrease from one session to the subsequent? Indeed, it does, as is confirmed by the following:*

**Proposition 4.4 (monotone decreasing profit):** *Passing from the penultimate price-cum-endowment profile $(y^*, \boldsymbol{w})$, at the closure of one session, to its version $(y^{*+1}, \boldsymbol{w}^{+1})$ when the subsequent session closes, it holds $c_I^*(y^{*+1} | \mathbf{w}^{+1}) \leq c_I^*(y^* | \mathbf{w})$.*

***Proof:*** from [3]. By (21) $w_i^+ \succsim_i w_i$ for each $i \in I$. So, granted transitive preference orders, expenditures 'increase': $\mathcal{E}_i(\cdot | w_i^+) \geq \mathcal{E}_i(\cdot | w_i)$ for all $i \in I$. The implication $y^* \in \partial c_I(0 | \mathbf{w}) \Longrightarrow 0 \in \partial c_I^*(y^* | \mathbf{w})$ tells that $c^*(\cdot | \mathbf{w})$ is minimal at $y^*$. Collecting these facts, and letting $y^0 := \sum_{i \in I} y_i^0$ with initial endowments $(r_i^0, y_i^0) = w_i^0$, it follows that from Lemma that:

$$c_I^*(y^{*+} | \mathbf{w}^+) = \inf_{\hat{y}^*} \left\{ \hat{y}^* y^0 - \sum_{i \in I} \mathcal{E}_i(\hat{y}^* | w_i^+) \right\}$$

$$\leq \left\{ y^* y^0 - \sum_{i \in I} \mathcal{E}_i(y^* | w_i) \right\} = c_I^*(y^* | \mathbf{w}).$$

■

**Theorem 4.2 (convergence to competitive equilibrium):** *For any $\boldsymbol{w} \in \mathbf{W}$ and $y^* \in \partial c_I(0 | \mathbf{w})$ with $c_I^*(y^* | \mathbf{w}) > 0$, suppose that $\mathcal{E}_i(\cdot | w_i^+) \geq \mathcal{E}_i(\cdot | w_i)$ $\forall i \in I$ implies*

$$\inf_{y^*} c_I^*(y^* | \mathbf{w}^+) < c_I^*(y^* | \mathbf{w}). \tag{24}$$

*Also suppose $c_I^*(y^* | \mathbf{w})$ is jointly closed, meaning lower semicontinuous in $(y^*, \boldsymbol{w})$. Then, by iterated sessions, total profit $c_I^*(y^* | \mathbf{w})$ converges to 0. That is, each cluster point $\boldsymbol{w}$ of the generated sequence $(\boldsymbol{w}^k)$ qualifies as competitive equilibrium $(y^*, \boldsymbol{w})$ for any $y^* \in \partial c_I(0 | \mathbf{w})$.*

***Proof:*** Consider the sequence $(\boldsymbol{w}^k)$ emanating from $\boldsymbol{w}^0 = (w_i^0) \in \mathbf{W}$, where $\boldsymbol{w}^k = (w_i^k)$ is the penultimate endowment profile just prior to closure of session $k$. During that session each agent $i$ may have secured finitely many 'improving' updates (21). These can be seen as interim *spacer steps*; see [12] Theorem 7.3.4. By Proposition 5.1, the sequence $c_I^*(y^{k*} | \mathbf{w}^k)$ decreases monotonically. Being bounded below by 0, total surplus converges to some limit $L \geq 0$.

Consider any cluster point $\boldsymbol{w}$ of $(\boldsymbol{w}^k)$. It suffices by Proposition 4.2 to show that $c_I^*(y^*|\mathbf{w}) = 0$ for each $y^* \in \partial c_I(0|\mathbf{w})$. But otherwise, (24) would yield the contradiction $\lim c_I^*(y^{*k+1}|\mathbf{w}^{k+1}) < L = \lim c_I^*(y^{k*}|\mathbf{w}^k)$. ■

## Notes

1. Throughout, the 'difference operator' Δ helps to emphasize change and dynamics.
2. At best, no auctioneer, invisible hand or system operator would be needed.
3. Two 'mechanisms' meet here – one rather modern, the other ancient – namely: the mathematics of *inf-convolution* [11] versus the economics of *money* [10].
4. That is, *lower semicontinuous.*

## Disclosure statement

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## References

[1] Luenberger DG. Microeconomic theory. New York: McGraw-Hill; 1995.
[2] Flåm SD. Emergence of price-taking behavior. Econ Theory. 2020;70:847–870. doi: 10.1007/s00199-019-01232-5
[3] Flåm SD. Towards competitive equilibrium by double auctions. Pure Appl Funct Anal. 2021;6(6):1211–1225.
[4] Debreu G. Valuation equilibrium and pareto optimum. Proc Nat Acad Sci. 1954;40: 588–592. doi: 10.1073/pnas.40.7.588
[5] Feldman AM. Bilateral trading processes, pairwise optimality, and pareto optimality. Rev Econ Stud. 1972;40(4):463–473. doi: 10.2307/2296581
[6] Flåm SD. Market equilibria and money. Fixed Point Theory Algo Sci Eng. 2021;2021: Article Id 20. doi: 10.1186/s13663-021-00705-4
[7] Gintis H. Individuality and entanglement. Princeton, NJ: Princeton Univ Press; 2017.
[8] Keisler HJ. Getting to competitive equilibrium. Econometrica. 1996;64(1):29–49. doi: 10.2307/2171923
[9] Negishi T. Welfare economics and existence of equilibrium for a competitive economy. Metroeconomica. 1960;12:92–97. doi: 10.1111/meca.1960.12.issue-2-3
[10] Shapley L, Shubik M. Trade using one commodity as means of payment. J Pol Econ. 1977;85:937–968. doi: 10.1086/260616
[11] Borwein JM, Lewis AS. Convex analysis and nonlinear optimization. Berlin: Springer; 2000.
[12] Bazaraa MS, Sherali HD, Shetty CM. Nonlinear programming: theory and algorithms. 3rd ed., New York: Wiley-Interscience; 2016.