



Ulrich bundles on a general blow-up of the plane

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Abstract

We prove that on X_n , the plane blown-up at n very general points, there are Ulrich line bundles with respect to a line bundle corresponding to curves of degree m passing simply through the n blown-up points, with $m \leq 2\sqrt{n}$ and such that the line bundle in question is very ample on X_n . We prove that the number of these Ulrich line bundles tends to infinity with n . We also prove the existence of slope-stable rank- r Ulrich vector bundles on X_n , for $n \geq 2$ and any $r \geq 1$ and we compute the dimensions of their moduli spaces. These computations imply that X_n is Ulrich wild.

Keywords Ulrich vector bundles · Stability · Moduli spaces

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1 Introduction

Let X be a smooth irreducible projective variety and let H be a very ample divisor on X . A vector bundle \mathcal{E} on X is said to be an *Ulrich vector bundle* with respect to the polarization H if $h^i(\mathcal{E}(-pH)) = 0$ for all $i \geq 0$ and all $1 \leq p \leq \dim(X)$.

Ulrich bundles first appeared in commutative algebra in the 1980s, where they have been considered because they enjoy some extremal cohomological properties. After that the attention of the algebraic geometers on these bundles has been carried by the beautiful paper [15], where, among other things, the authors compute the Chow form of a projective variety X using Ulrich bundles on X , if they exist.

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In recent years there has been a good amount of work on Ulrich bundles (for surveys see, for instance, [4, 11, 13]), mainly investigating the following problems: given any polarization H on a variety X , does there exist an Ulrich vector bundle with respect to H ? Or, even more generally, given a variety X , does there exist a very ample line bundle H on X and an Ulrich vector bundle on X with respect to H ? What is the smallest possible rank for an Ulrich bundle on a given variety X ? If Ulrich bundles exist, are they stable, and what are their moduli?

Although a lot is known about these problems for some specific classes of varieties (curves, Segre, Veronese, Grassmann varieties, rational scrolls, complete intersections, some classes of surfaces like Del Pezzo, abelian, K3 surfaces, some surfaces of general type, etc.) the above questions are still open in their full generality even for surfaces. In particular, the question for which pairs (X, H) there are Ulrich *line bundles* on X with respect to H is open, though this occurrence seems to be rather rare. Moreover, in the few known cases in which they have been proved to exist, except the case of curves, they are finitely many and of a low number.

In the present paper we investigate the existence of Ulrich bundles on the blow-up X_n of the complex projective plane at n very general points with respect to line bundles $\xi_{n,m}$ corresponding to curves of degree m passing simply through the n blown-up points, with $m \leq 2\sqrt{n}$ and such that the line bundle in question is very ample. Surprisingly enough, our results show that such a surface carries several Ulrich line bundles, and actually their number increases and tends to infinity with n , see Theorem 3.1 and Corollary 3.2. In Theorem 4.2 we classify all these line bundles for $7 \leq n \leq 10$ and $m = 4$ (the cases $n \leq 6$ and $m = 3$ corresponding to the Del Pezzo case being well known already, see [12, 19]). Then, making iterated extensions and deforming, we prove the existence of slope-stable rank- r Ulrich vector bundles on X_n for $n \geq 2$ and any $r \geq 1$ (see Theorem 5.8), we compute the dimension of the moduli spaces of the bundles in question and prove that they are reduced. These computations show that X_n , with $n \geq 2$, is *Ulrich wild*.

Recall that a variety X is said to be Ulrich wild if it possesses families of dimension p of pairwise non-isomorphic, indecomposable, Ulrich bundles for arbitrarily large p (for the introduction of the notion of Ulrich, or more generally, of Cohen–Macaulay *wildness*, see, for instance, [14, 16]). Note that in the literature there are only very few cases of varieties known to carry stable Ulrich bundles of infinitely many ranks, and even fewer of *any* rank (namely, curves and Del Pezzo surfaces). Note that it is known that not all varieties can support Ulrich vector bundles of every rank (see e.g. [10, Cor. 2.], [1, Cor. 3.2], [4, §4] and [8, Cor. 3.7 and 3.9]). In [2, Cor. 3.8] the existence of rank 2 Ulrich vector bundles is proved on the blown-up planes with respect to an ample line bundle of the form $\xi_{n,m}$, without investigating stability or moduli. The description of Ulrich vector bundles on some surfaces of type X_n which are not Del Pezzo surfaces and the related Ulrich wildness can be found in [18]. The fact that X_n supports slope stable rank-two Ulrich bundles and is Ulrich wild for each $n \geq 2$ is a particular case of the results in [6].

It is an interesting problem to study the presence of Ulrich (line) bundles on blow-ups at very general points of other surfaces than the projective plane. The ideas in this paper might be useful to attack this problem.

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2 Preliminaries

2.1 In this paper we will denote by $\pi : X_n \rightarrow \mathbb{P}^2$ the blow-up of the complex projective plane at $n \geq 1$ very general points p_1, \dots, p_n . (The n points are supposed to be very general because one has to take off the a priori countably many closed subsets of n -tuples of points for which the very ampleness of the line bundles $\xi_{m,n}$ considered below or the property of imposing independent conditions to suitable linear systems are not satisfied.) We will denote by E_1, \dots, E_n the exceptional divisors contracted by π to p_1, \dots, p_n , respectively, and by L the pull-back via π of a general line of \mathbb{P}^2 . For integers d, m_1, \dots, m_n , consider the linear system

$$\left| dL - \sum_{i=1}^n m_i E_i \right| \quad (1)$$

on X_n . If d, m_1, \dots, m_n are non-negative, this is the strict transform on X_n of the linear system of curves of degree d in \mathbb{P}^2 with multiplicities at least m_1, \dots, m_n at p_1, \dots, p_n , respectively. If one of the integers m_1, \dots, m_n is negative; for example, m_1 is negative, then this means that $m_1 E_1$ is in the fixed part of the linear system (1), if this system is non-empty. We will denote the linear system (1), or the corresponding line bundle $\mathcal{O}_{X_n}(dL - \sum_{i=1}^n m_i E_i)$, by

$$(d; m_1, \dots, m_n)$$

and we will use exponential notation for repeated multiplicities.

In the sequel we will consider various linear systems of the form $(d; m_1, \dots, m_n)$ as above (cf. e.g. Lemma 2.2, Proposition 2.3, Theorems 3.1, 4.2 and Sect. 5). We will consider the linear systems

$$\xi_{n,m} := (m; 1^n),$$

i.e. in the notation $(d; m_1, \dots, m_n)$, we have $d = m$ and $m_i = 1$ for all $i = 1, \dots, n$.

The line bundle $\xi_{n,m}$ is ample as soon as $n \geq 3$ and $\xi_{n,m}^2 = m^2 - n > 0$ (see [17, Cor.p. 154]) and it is very ample if $m \geq 5$ and

$$m \geq 2\sqrt{n+4} - 3$$

(see [20, Thm. 1]). This result is not optimal and a natural conjecture is that $\xi_{n,m}$ is very ample as soon as

$$\frac{m(m+3)}{2} - n \geq 5$$

(see again [20, Conj. p. 2521]). Moreover, it is classically known that $\xi_{10,4}$ is very ample, the image of X_{10} via the map determined by $\xi_{10,4}$ is the famous smooth *Bordiga surface* of degree 6 in \mathbb{P}^4 . Accordingly $\xi_{n,4}$ is very ample if $n \leq 10$. In any event, if m_n is the minimum m such that $\xi_{n,m}$ is very ample, we have

$$m_n \leq \left\lceil 2\sqrt{n+4} - 3 \right\rceil \quad (2)$$

if $n \geq 3$, whereas $m_1 = 2$ and $m_2 = 3$.

We will need the following:

Lemma 2.1 *If $n \geq 3$, there is an $m < 2\sqrt{n}$ such that $\xi_{n,m}$ is very ample.*

Proof To prove the assertion one has to prove that $2\sqrt{n} > m_n$. By (2), this is implied by $2\sqrt{n} > \lceil 2\sqrt{n+4} - 3 \rceil$, which holds if $2\sqrt{n} \geq 2\sqrt{n+4} - 2$. This is true as soon as $n \geq 3$. \square

2.2 Given a linear system $\mathcal{L} = (d; m_1, \dots, m_n)$, its virtual dimension is

$$\dim_v(\mathcal{L}) = \frac{d(d+3)}{2} - \sum_{i=1}^n \frac{m_i(m_i+1)}{2}.$$

We have $\dim(\mathcal{L}) \geq \dim_v(\mathcal{L})$, and if equality holds we will say that the system \mathcal{L} is regular. One computes

$$\dim_v(\mathcal{L}) = \chi(\mathcal{L}) - 1$$

hence, by the Riemann–Roch theorem, one has

$$\dim(\mathcal{L}) = \dim_v(\mathcal{L}) + h^1(\mathcal{L}) - h^2(\mathcal{L}).$$

Since $h^2(\mathcal{L}) = h^0(K_n \otimes \mathcal{L}^\vee)$, where $K_n = (-3; -1^n)$ denotes the canonical bundle of X_n , and

$$K_n \otimes \mathcal{L}^\vee = (-3 - d; -m_1 - 1, \dots, -m_n - 1)$$

one has $h^2(\mathcal{L}) = 0$ as soon as $d > -3$. In this case one has

$$\dim(\mathcal{L}) = \dim_v(\mathcal{L}) + h^1(\mathcal{L}) \tag{3}$$

and \mathcal{L} is regular if and only if $h^1(\mathcal{L}) = 0$.

Lemma 2.2 *Let \mathcal{L} be a linear system on X_n of the form*

$$(d; 1^h, 0^k, (-1)^{n-h-k})$$

where $d \geq 0$. If $\dim_v(\mathcal{L}) \geq -1$, then \mathcal{L} is regular.

Proof Set $E = E_{h+k+1} + \dots + E_n$. For each curve $E_i \simeq \mathbb{P}^1$, for $h+k+1 \leq i \leq n$ one has $\mathcal{L} \cdot E_i = -1$, hence E is in the fixed part of \mathcal{L} , if this is non-empty. So we have $\dim(\mathcal{L}) = \dim(\mathcal{L}(-E))$. The linear system $\mathcal{L}(-E)$ is $(d; 1^h, 0^k)$. One has $\dim_v(\mathcal{L}) = \dim_v(\mathcal{L}(-E))$. By $\dim_v(\mathcal{L}(-E)) \geq -1$ and by the very generality of the imposed simple base points p_1, \dots, p_h , it follows that $\mathcal{L}(-E)$ is regular. Hence, we have

$$\dim(\mathcal{L}) = \dim(\mathcal{L}(-E)) = \dim_v(\mathcal{L}(-E)) = \dim_v(\mathcal{L})$$

as wanted. \square

2.3 We will use the following result:

Proposition 2.3 *Consider a linear system of the form*

$$\mathcal{L}_{d,n,\delta,k} = (d; 2^\delta, 1^k, 0^{n-\delta-k})$$

(and permuted multiplicities), with $k \geq 1$. If

$$\dim_v(\mathcal{L}_{d,n,\delta,k}) = \frac{d(d+3)}{2} - 3\delta - k \geq 0$$

then $\mathcal{L}_{d,n,\delta,k}$ is regular, i.e.

$$h^1(X_n, \mathcal{L}_{d,n,\delta,k}) = 0$$

and the general curve in $\mathcal{L}_{d,n,\delta,k}$ on X_n is smooth and irreducible.

Proof This is a straightforward application of [9, Thm. 2.3] (see also [3, Cor. 4.6] and [7, Thms. 1.1, 1.3 and 1.4]). \square

2.4 Let S be a smooth irreducible projective surface and let H be a very ample divisor on S . It is well known and easily seen that \mathcal{O}_S is Ulrich with respect to H if and only if $S = \mathbb{P}^2$ and H is a line.

We will need the following proposition (cf. also [6, Prop. (2.1)]):

Proposition 2.4 *Let S be a smooth irreducible projective surface and let H be a very ample divisor on S .*

A line bundle $\mathcal{L} \not\cong \mathcal{O}_S$ is Ulrich if and only if it is of the form $\mathcal{L} = \mathcal{O}_S(C)$, where C is an effective divisor on S satisfying

- (i) $C \cdot H = \frac{1}{2}H \cdot (3H + K_S)$;
- (ii) $\frac{1}{2}(C^2 - C \cdot K_S) + \chi(\mathcal{O}_S) - H^2 = 0$;
- (iii) $h^1(\mathcal{O}_C(K_S + H)) = 0$;
- (iv) *the restriction map $r : H^0(\mathcal{O}_S(K_S + 2H)) \rightarrow H^0(\mathcal{O}_C(K_S + 2H))$ is injective or surjective.*

Moreover, the divisor C can be taken to be a smooth curve.

We will say that \mathcal{L} as above is *determined* by C .

Proof It is well known that an Ulrich bundle is globally generated, whence it is of the form $\mathcal{L} = \mathcal{O}_S(C)$ for C an effective nonzero divisor, which can even be taken to be a smooth curve. Hence, \mathcal{L} is Ulrich if and only if $h^i(\mathcal{O}_S(C - jH)) = 0$ for $i = 0, 1, 2$, $j = 1, 2$. From the short exact sequence

$$0 \longrightarrow \mathcal{O}_S(-jH) \longrightarrow \mathcal{O}_S(C - jH) \longrightarrow \omega_C(-K_S - jH) \longrightarrow 0 \quad (4)$$

we see that \mathcal{L} is Ulrich if and only if

$$h^0(\omega_C(-K_S - jH)) = 0, \quad j = 1, 2, \quad (5)$$

$$\begin{aligned} &\text{the coboundary maps } H^1(\omega_C(-K_S - jH)) \rightarrow H^2(\mathcal{O}_S(-jH)) \\ &\text{are isomorphisms, } \quad j = 1, 2. \end{aligned} \quad (6)$$

Clearly the vanishing for $j = 2$ in (5) is implied by the one for $j = 1$. Thus, by Serre duality (5)–(6) are equivalent to

$$h^1(\mathcal{O}_C(K_S + H)) = 0, \quad (7)$$

$$\begin{aligned} &\text{the restriction maps } r_j : H^0(\mathcal{O}_S(K_S + jH)) \rightarrow H^0(\mathcal{O}_C(K_S + jH)) \\ &\text{are isomorphisms, } \quad j = 1, 2. \end{aligned} \quad (8)$$

By (7) and the fact that $h^1(\mathcal{O}_S(K_S + jH)) = h^2(\mathcal{O}_S(K_S + jH)) = 0$, the domain and target of r_j have dimensions $\chi(\mathcal{O}_S(K_S + jH))$ and $\chi(\mathcal{O}_C(K_S + jH))$, respectively. Moreover, it is easy to see that r_1 is injective as soon as r_2 is. Hence, given (7), condition (8) is equivalent to

$$\chi(\mathcal{O}_S(K_S + jH)) = \chi(\mathcal{O}_C(K_S + jH)), \quad j = 1, 2, \quad (9)$$

$$\text{the restriction map } r_2 : H^0(\mathcal{O}_S(K_S + 2H)) \rightarrow H^0(\mathcal{O}_C(K_S + 2H)) \text{ is injective.} \quad (10)$$

Thus, $\mathcal{O}_S(C)$ is Ulrich if and only if (7), (9) and (10) are satisfied. Condition (7) is condition (iii) in the statement of the proposition, whereas (9) is equivalent to (i)–(ii) by Riemann–Roch. Finally, (10) is equivalent to (iv), as the domain and target have the same dimensions, again by (9). \square

3 Ulrich line bundles on X_n

In this section we will prove the existence of Ulrich line bundles on X_n . We will assume $m \geq 4$, since the Del Pezzo case ($n \leq 6$ and $m = 3$ in the notation below) has already been worked out in [12, Prop. 2.19] and [19, Thm. 1.1]. This is our result:

Theorem 3.1 *Let $n \geq 3$ be an integer and let $m \geq 4$ be an integer such that $\xi_{n,m}$ is very ample on X_n and $m \leq 2\sqrt{n}$. Let d be a positive integer such that*

$$\frac{2m - 3 - \sqrt{8n + 1}}{2} \leq d \leq \frac{2m - 3 + \sqrt{8n + 1}}{2} \tag{11}$$

and

$$\frac{3(m - 1) - \sqrt{4n - m^2 + 1}}{2} < d < \frac{3(m - 1) + \sqrt{4n - m^2 + 1}}{2}. \tag{12}$$

Set

$$\delta = \frac{m^2}{2} - \frac{m}{2}(2d + 3) + \frac{d^2 + 3d + 2}{2} = \frac{(d - m)(d - m + 3)}{2} + 1$$

and

$$k = n + 3m(d + 1) - \frac{m(5m - 3)}{2} - (d^2 + 3d + 2).$$

Then δ and k are integers such that

$$0 \leq \delta \leq n, \tag{13}$$

$$1 \leq k \leq n, \tag{14}$$

and

$$\delta + k \leq n. \tag{15}$$

Moreover,

$$\mathcal{L}_{d,n,k,\delta} = (d; 2^\delta, 1^k, 0^{n-\delta-k})$$

(and permuted multiplicities) is an Ulrich line bundle on X_n with respect to $\xi_{n,m}$.

Proof Take for granted (13), (14) and (15) for the time being. A direct computation of $\dim_v(\mathcal{L}_{d,n,k,\delta})$ as in Proposition 2.3 shows that

$$\dim_v(\mathcal{L}_{d,n,k,\delta}) = m^2 - n - 1 \geq 0$$

because

$$m^2 - n = \xi_{n,m}^2 > 0.$$

By Proposition 2.3, the system $\mathcal{L}_{d,n,k,\delta}$ is non-empty and the general curve C in $\mathcal{L}_{d,n,k,\delta}$ is smooth since $k \geq 1$ by (14). Now we apply Proposition 2.4 to $S = X_n$, $H = \xi_{n,m}$ and

C as above. To prove the theorem we have to check conditions (i)–(iv) in the statement of Proposition 2.4.

As for (i), with an easy computation we have

$$C \cdot \xi_{n,m} = dm - 2\delta - k = \frac{3m(m-1)}{2} - n = \frac{1}{2}\xi_{n,m} \cdot (3\xi_{n,m} + K_n)$$

as wanted.

As for (ii), note that $\chi(\mathcal{O}_{X_n}) = 1$, so

$$\frac{1}{2}(C^2 - C \cdot K_n) + \chi(\mathcal{O}_S) - \xi_{n,m}^2 = \frac{1}{2}(d^2 + 3d - 6\delta - 2k) - m^2 + n + 1$$

which is easily computed to be 0, as desired.

As for (iii), suppose, by contradiction, that $h^1(\mathcal{O}_C(K_n + \xi_{n,m})) > 0$. So the divisors cut out by the system $K_n + \xi_{n,m}$ on C are special, i.e. they are contained in divisors of the canonical series, cut out on C by the system $K_n + \mathcal{L}_{d,n,k,\delta}$. Note that

$$K_n + \xi_{n,m} = (m-3; 0^n) \quad \text{and} \quad K_n + \mathcal{L}_{d,n,k,\delta} = (d-3; 1^\delta, 0^k, (-1)^{n-\delta-k}).$$

Let D be a general curve in the system $K_n + \xi_{n,m}$ (i.e. a general curve of degree $m-3$) that cuts on C a divisor consisting of $d(m-3)$ distinct points. This divisor has to be contained in a curve D' of the system $K_n + \mathcal{L}_{d,n,k,\delta}$. Since D has degree $m-3$ and D' has degree $d-3$ and $d(m-3) > (d-3)(m-3)$ (because $m \geq 4$), by Bézout's theorem D is contained in D' . This implies that the system $K_n + \mathcal{L}_{d,n,k,\delta} - (K_n + \xi_{n,m}) = \mathcal{L}_{d,n,k,\delta} - \xi_{n,m}$ is effective. We have

$$\mathcal{L}_{d,n,k,\delta} - \xi_{n,m} = (d-m; 1^\delta, 0^k, (-1)^{n-\delta-k})$$

whose virtual dimension is

$$\frac{(d-m)(d-m+3)}{2} - \delta = -1.$$

Then, by Lemma 2.2, $\mathcal{L}_{d,n,k,\delta} - \xi_{n,m}$ is empty, a contradiction.

Finally, as for (iv) we want to prove that the map

$$r : H^0(K_n + 2\xi_{n,m}) \rightarrow H^0((K_n + 2\xi_{n,m})|_C)$$

is injective. We have

$$\ker(r) = H^0(K_n + 2\xi_{n,m} - \mathcal{L}_{d,n,k,\delta})$$

and

$$K_n + 2\xi_{n,m} - \mathcal{L}_{d,n,k,\delta} = (2m-3-d; (-1)^\delta, 0^k, 1^{n-\delta-k}).$$

If $2m-3 < d$ it is clear that r is injective because of course $h^0(K_n + 2\xi_{n,m} - \mathcal{L}_{d,n,k,\delta}) = 0$. Assume that $2m-3 \geq d$. Note that the virtual dimension of $K_n + 2\xi_{n,m} - \mathcal{L}_{d,n,k,\delta}$ is

$$\frac{(2m-3-d)(2m-d)}{2} - n + k + \delta = -1.$$

Then by Lemma 2.2 we again have that $h^0(K_n + 2\xi_{n,m} - \mathcal{L}_{d,n,k,\delta}) = 0$ and r is injective, as wanted.

To finish the proof, we are left to prove (13), (14) and (15). This is a mere computation. For instance, $\delta \geq 0$ is equivalent to

$$d^2 - (2m-3)d + m^2 - 3m + 2 \geq 0.$$

The discriminant of this quadratic in d is

$$(2m - 3)^2 - 4(m^2 - 3m + 2) = 1,$$

hence we need

$$\text{either } d \geq \frac{2m - 3 + 1}{2} = m - 1, \quad \text{or } d \leq \frac{2m - 3 - 1}{2} = m - 2$$

which is always true.

The inequality $\delta \leq n$ is equivalent to

$$d^2 - (2m - 3)d + m^2 - 3m + 2 - 2n \leq 0. \quad (16)$$

The discriminant in d is computed to be $8n + 1$. Since (11) holds, (16) also holds.

The inequality $k > 0$ is equivalent to

$$2d^2 - 6d(m - 1) + 5m^2 - 9m + 4 - 2n < 0. \quad (17)$$

The discriminant in d is computed to be $4n - m^2 + 1$ which is non-negative by assumption. Since (12) holds, then (17) holds as well.

The inequality $k \leq n$ is equivalent to

$$2d^2 - 6d(m - 1) + 5m^2 - 9m + 4 \geq 0.$$

The discriminant is computed to be $4(1 - m^2) < 0$ (because $m \geq 4$), hence $k \leq n$ holds.

Finally $\delta + k \leq n$ is equivalent to

$$d^2 - (4m - 3)d + 4m^2 - 6m + 2 \geq 0$$

whose discriminant is 1, so that $\delta + k \leq n$ if

$$\text{either } d \geq \frac{4m - 3 + 1}{2} = 2m - 1, \quad \text{or } d \leq \frac{4m - 3 - 1}{2} = 2m - 2$$

which is always true. \square

As an immediate consequence we have:

Corollary 3.2 *There is a sequence $\{h_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow +\infty} h_n = +\infty$ and such that for for all $n \geq 3$ there are h_n pairwise non-isomorphic Ulrich line bundles on X_n with respect to $\xi_{n,m}$.*

Proof The h_n line bundles in question are the ones obtained by permutation of the multiplicities from the line bundles $\mathcal{L}_{d,n,k,\delta}$ as in Theorem 3.1. \square

As a warning, we explicitly stress that Corollary 3.2 above does not imply that each X_n supports at most a finite number h_n of Ulrich line bundles which grows as n tends to infinity.

Remark 3.3 The hypotheses of Theorem 3.1 could be a bit relaxed. The strict inequalities in (12) are required in order that $k \geq 1$, and this is used to imply that the general curve C in $\mathcal{L}_{d,n,\delta,k}$ is smooth. However $k \geq 1$ is not necessary for the general curve C in $\mathcal{L}_{d,n,\delta,k}$ to be smooth. For example, for $n = 2$ and $m = 3$, the general curve in $\mathcal{L}_{3,2,1,0} = (3; 2, 0)$ is smooth and this line bundle is Ulrich (see also [12, Prop. 2.19] and [19, Thm. 1.1]). We will use this later.

4 Classification of Ulrich line bundles on X_n with $7 \leq n \leq 10$

Theorem 3.1 proves the existence of several Ulrich line bundles on X_n but does not give a full classification of such line bundles. However, in principle, it is possible to pursue such a classification. We want to show how this works for Ulrich line bundles with respect to $\xi_{n,4}$, in the case in which $m = 4$ is the minimum such that $\xi_{n,m}$ is very ample. This corresponds to $7 \leq n \leq 10$. As we said already, the Del Pezzo case $m = 3$ is well known, and hence we do not dwell on it here.

As we know from Proposition 2.4, an Ulrich line bundle on X_n is of the form $\mathcal{O}_{X_n}(C)$ where C is a smooth curve on X_n verifying properties (i)–(iv). Let us restate properties (i)–(iv) in our setting, with $7 \leq n \leq 10$, in which $m = 4$. We have $H = \xi_{n,4}$ and C belongs to a system $(d; m_1, \dots, m_n)$, and it is easy to check that properties (i)–(iv) read now as follows:

- (i) $\xi_{n,4} \cdot C = 18 - n$;
- (ii) $p_a(C) = d - 2$;
- (iii) $h^1(\mathcal{O}_C(L)) = 0$ (recall that L is the pull-back to X_n of a general line of \mathbb{P}^2);
- (iv) the restriction map $r : H^0(\mathcal{O}_S(K_n + 2\xi_{n,4})) \rightarrow H^0(\mathcal{O}_C(K_n + 2\xi_{n,4}))$ is either injective or surjective, where $K_n + 2\xi_{n,4} = (5; 1^n)$.

We need a preliminary lemma:

Lemma 4.1 *In the above setting, the curve C is irreducible.*

Proof First, we notice that the case $d = 1$ cannot happen. In this case in fact, since $\mathcal{O}_{X_n}(C)$ is globally generated, C would be clearly irreducible while from (ii) we would have $p_a(C) = -1$ a contradiction. So we have $d \geq 2$ and therefore $p_a(C) = d - 2 \geq 0$.

If C were reducible, since $\mathcal{O}_{X_n}(C)$ is globally generated, this means that $|C|$ would be composed with a base point free pencil $|F|$, i.e. $C \sim aF$ with $a \geq 2$. Then $p_a(C) = ap_a(F) - a + 1 \geq 0$ hence $p_a(F) \geq 1$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{X_n}((a-1)F) \longrightarrow \mathcal{O}_{X_n}(aF) = \mathcal{O}_{X_n}(C) \longrightarrow \mathcal{O}_F(C) = \mathcal{O}_F \longrightarrow 0$$

and the cohomology sequence

$$H^1(\mathcal{O}_{X_n}(C)) \longrightarrow H^1(\mathcal{O}_F) \longrightarrow H^2(\mathcal{O}_{X_n}((a-1)F)).$$

We would have $h^1(\mathcal{O}_F) = p_a(F) \geq 1$ and $h^2(\mathcal{O}_{X_n}((a-1)F)) = 0$, hence $h^1(\mathcal{O}_{X_n}(C)) > 0$. This gives a contradiction since $h^1(\mathcal{O}_{X_n}(C)) = 0$ because Ulrich bundles have vanishing intermediate cohomology (see, for instance, [4, (3.1)]). \square

Theorem 4.2 *The Ulrich line bundles on X_n with $7 \leq n \leq 10$ with respect to $\xi_{n,4}$ are:*

- (a) $(6; 2^6, 1^{n-6})$;
- (b) $(5; 2^3, 1^{n-4}, 0)$;
- (c) $(4; 2, 1^{n-4}, 0^3)$;
- (d) $(3; 1^{n-6}, 0^6)$;
- (e) $(2; 0^{10})$ only for $n = 10$;
- (f) $(7; 2^{10})$ only for $n = 10$,

with permutations of the multiplicities.

Proof Let $\mathcal{O}_{X_n}(C)$ be an Ulrich line bundle on X_n , with C smooth and irreducible, by Lemma 4.1. We suppose $\mathcal{O}_{X_n}(C)$ is of the form $(d; m_1, \dots, m_n)$ on X_n .

Consider the embedding $\phi_{\xi_{n,4}} : X_n \rightarrow S \subset \mathbb{P}^{14-n}$, where S is the image of X_n , and denote still by C the image of C on S . If C is degenerate, then clearly $d \leq 4$. Otherwise C is a non-degenerate smooth irreducible curve of degree $18 - n$ in \mathbb{P}^{14-n} (see property (i) above). By Castelnuovo’s bound, we have $p_a(C) \leq 4$ if $7 \leq n \leq 9$ and $g(C) \leq 5$ if $n = 10$. In addition, if $n = 10$ and $g(C) = 5$, then $C \subset \mathbb{P}^4$ is a canonical curve, i.e. $\xi_{10,4}$ cuts out on C the canonical series, hence we are in case (f). By property (ii), we have $d \leq 6$ if $7 \leq n \leq 9$ and $d \leq 7$ if $n = 10$. On the other hand we have $d \geq 2$ (see the proof of Lemma 4.1).

Suppose $d = 6$, hence $p_a(C) = 4$. Then we have

$$10 - \sum_{i=1}^n \frac{m_i(m_i - 1)}{2} = 4$$

$$24 - \sum_{i=1}^n m_i = 18 - n,$$

i.e.

$$6 - \sum_{i=1}^n \frac{m_i(m_i - 1)}{2} = 0$$

$$6 - \sum_{i=1}^n (m_i - 1) = 0$$

whence

$$\sum_{i=1}^n \frac{m_i(m_i - 1)}{2} = \sum_{i=1}^n (m_i - 1),$$

i.e.

$$\sum_{i=1}^n \frac{(m_i - 1)(m_i - 2)}{2} = 0$$

which yields $m_i \leq 2$ for $1 \leq i \leq n$. This implies that we are in case (a). A similar computation shows that in case $d = 5$ the only possible cases are the one in (b) and $(5; 3, 1^{n-1})$. This line bundle however is not Ulrich because property (iv) is not verified, since the kernel of the map r is $H^0(\mathcal{L})$ with $\mathcal{L} = K_n + 2\xi_{n,m} - C = (0; (-2), 0^{n-1})$, that is nonzero. If $2 \leq d \leq 4$ it is easily seen that the remaining possible cases are the ones in (c)–(e).

Finally one has to check that in all cases (a)–(f) we do have Ulrich line bundles. To see this, by Proposition 2.4 we have only to prove that properties (iii) and (iv) above are verified. Property (iii) is verified in cases (b)–(e) because $\deg(\mathcal{O}_C(L)) > 2p_a(C) - 2$. In case (a) it is verified because otherwise the system $(2; 1^6, 0^{n-6})$ would be non-empty, contradicting the very generality of the blown-up points. Indeed, if C is a general curve of the system $(6; 2^6, 1^{n-6})$, then $h^1(\mathcal{O}_C(L)) > 0$ implies, by Serre duality, that $h^0(\mathcal{O}_C(K_C - L)) > 0$ and, by adjunction, the linear series $|K_C - L|$ is cut out on C by the linear system $(2; 1^6, 0^{n-6})$. In case (f) it is verified because otherwise the system $(3; 1^{10})$ would be non-empty, a contradiction again. As for property (iv), note that the map r is clearly injective in cases (a) and (f), because $\ker(r)$ consists in these cases of sections of line bundles of negative degrees. As for the other cases one looks at the kernel of the map r , which is the H^0 of the following line bundles:

- $(0; (-1)^3, 0^{n-4}, 1)$ in case (b);
- $(1; -1, 0^{n-4}, 1^3)$ in case (c);
- $(2; 0^{n-6}, 1^6)$ in case (d);
- $(3; 1^{10})$ in case (e),

and in all these cases this H^0 is zero by the very generality of the blown-up points, so r is injective. □

5 Higher rank Ulrich vector bundles on X_n

In this section we will construct higher rank slope-stable Ulrich vector bundles on X_n and we will compute the dimensions of the moduli spaces of the constructed bundles.

In the whole section $n \geq 2$ will be an integer and m will be an integer such that $\xi_{n,m}$ is very ample with $m = 3$ if $n = 2$ and $m < 2\sqrt{n}$ if $n \geq 3$ (cf. Lemma 2.1).

We start by defining

$$L_1 = \begin{cases} \left(\frac{3(m-1)}{2}; 2^{\frac{m^2-1}{8}}, 1^{n-\frac{m^2-1}{4}}, 0^{\frac{m^2-1}{8}} \right), & \text{if } m \text{ is odd,} \\ \left(\frac{3}{2}m - 1; 2^{\frac{m(m+2)}{8}}, 1^{n-\frac{m^2}{4}}, 0^{\frac{m(m-2)}{8}} \right), & \text{if } m \text{ is even,} \end{cases}$$

$$L_0 = \begin{cases} \left(\frac{3(m-1)}{2}; 0^{\frac{m^2-1}{8}}, 1^{n-\frac{m^2-1}{4}}, 2^{\frac{m^2-1}{8}} \right), & \text{if } m \text{ is odd,} \\ \left(\frac{3}{2}m - 1; 0^{\frac{m(m-2)}{8}}, 1^{n-\frac{m^2}{4}}, 2^{\frac{m(m+2)}{8}} \right), & \text{if } m \text{ is even.} \end{cases}$$

Lemma 5.1 *The line bundles L_0 and L_1 are Ulrich with respect to $\xi_{n,m}$ and satisfy*

$$h^i(L_0 - L_1) = h^i(L_1 - L_0) = 0, \text{ for } i = 0, 2, \tag{18}$$

and

$$h^1(L_0 - L_1) = h^1(L_1 - L_0) = \begin{cases} \frac{m^2-3}{2}, & \text{if } m \text{ is odd} \\ \frac{m^2-m-2}{2}, & \text{if } m \text{ is even.} \end{cases} \tag{19}$$

Note that L_0 and L_1 are Ulrich dual in the sense of [6, Prop. 2.1(1)-(2)].

Proof of Lemma 5.1 Let δ be as in Theorem 3.1. Note that

$$n - \delta - k = \delta + \frac{3m(m-1)}{2} - md. \tag{20}$$

We first treat the case where m is odd.

Take $d = \frac{3(m-1)}{2}$ and assume first $n \geq 3$. Such a value of d is clearly compatible with (12). It is also compatible with (11). Indeed

$$3(m-1) - (2m-3) + \sqrt{8n+1} = m + \sqrt{8n+1} > 0$$

and

$$2m-3 + \sqrt{8n+1} - 3(m-1) = \sqrt{8n+1} - m > 0$$

because $m < 2\sqrt{n}$. For such a value of d , we get $\delta = n - \delta - k$ from (20). More precisely, an easy computation shows that for such a d one has

$$\delta = \frac{m^2-1}{8}, \text{ hence } k = n - \frac{m^2-1}{4}.$$

Hence, we get Ulrich line bundles on X_n of the form

$$\mathcal{L}_{\frac{3(m-1)}{2}, n, \frac{m^2-1}{8}, n, \frac{m^2-1}{4}} = \left(\frac{3(m-1)}{2}, 2^{\frac{m^2-1}{8}}, 1^{n-\frac{m^2-1}{4}}, 0^{\frac{m^2-1}{8}} \right)$$

and permutations of the multiplicities. In particular, we see that L_0 and L_1 defined above are Ulrich.

Up to permutations of multiplicities, one has

$$L_1 - L_0 = L_0 - L_1 = \left(0; 2^{\frac{m^2-1}{8}}, 0^{n-\frac{m^2-1}{4}}, (-2)^{\frac{m^2-1}{8}} \right).$$

Hence, we see that $h^0(L_1 - L_0) = h^0(L_0 - L_1) = 0$ and, by Serre duality, $h^2(L_1 - L_0) = h^0(K_n + L_0 - L_1) = 0$. By Riemann–Roch and an easy computation,

$$h^1(L_1 - L_0) = -\chi(L_1 - L_0) = \frac{m^2 - 3}{2}.$$

The cohomology of $L_0 - L_1$ is computed in the same way.

In the case $n = 2, m = 3$, the line bundles L_0 and L_1 are Ulrich (see Remark 3.3) and (18) and (19) are verified as before.

We next treat the case where m is even.

Take $d = \frac{3}{2}m - 1$. As in the odd case, such a value of d is easily seen to be compatible with (11) and with (12) (for verifying (12) one needs $m < 2\sqrt{n}$). For this value of d one easily computes

$$\delta = \frac{m(m+2)}{8}, \quad k = n - \frac{m^2}{4} \quad \text{and} \quad n - \delta - k = \frac{m(m-2)}{8}.$$

Hence, we get Ulrich line bundles of the form

$$\mathcal{L}_{\frac{3}{2}m-1, n, \frac{m(m+2)}{8}, n, \frac{m^2}{4}} = \left(\frac{3}{2}m - 1, 2^{\frac{m(m+2)}{8}}, 1^{n-\frac{m^2}{4}}, 0^{\frac{m(m-2)}{8}} \right)$$

and permutations of the multiplicities. In particular, L_0 and L_1 defined above are Ulrich.

One has

$$L_1 - L_0 = \left(0; 2^{\frac{m(m-2)}{8}}, 1^{\frac{m}{2}}, 0^{n-\frac{m(m+2)}{4}}, (-1)^{\frac{m}{2}}, (-2)^{\frac{m(m-2)}{8}} \right).$$

Again we see that $h^0(L_1 - L_0) = h^2(L_1 - L_0) = 0$. The computation of $h^1(L_1 - L_0)$ then again follows by Riemann–Roch. The cohomology of $L_0 - L_1$ is computed in the same way.

□

For simplicity we will set

$$h := h^1(L_0 - L_1) = h^1(L_1 - L_0) = \begin{cases} \frac{m^2-3}{2} & \text{if } m \text{ is odd} \\ \frac{m^2-m-2}{2}, & \text{if } m \text{ is even.} \end{cases}$$

As $n \geq 2$ then $m \geq 3$, whence $h \geq 3$.

To construct higher rank Ulrich bundles on X_n we proceed as follows.

Since $\text{Ext}^1(L_0, L_1) \simeq H^1(L_1 - L_0) \simeq \mathbb{C}^h$, we have a non-split extension

$$0 \longrightarrow \mathcal{E}_1 := L_1 \longrightarrow \mathcal{E}_2 \longrightarrow L_0 \longrightarrow 0, \tag{21}$$

where \mathcal{E}_2 is a rank-two vector bundle, necessarily Ulrich, as L_0 and L_1 are. We proceed taking extensions

$$\begin{aligned} 0 &\longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_3 \longrightarrow L_1 \longrightarrow 0, \\ 0 &\longrightarrow \mathcal{E}_3 \longrightarrow \mathcal{E}_4 \longrightarrow L_0 \longrightarrow 0, \end{aligned}$$

and so on; that is, defining

$$\epsilon_r = \begin{cases} 0, & \text{if } r \text{ is even,} \\ 1, & \text{if } r \text{ is odd,} \end{cases} \tag{22}$$

we take successive extensions $[\mathcal{E}_{r+1}] \in \text{Ext}^1(L_{\epsilon_{r+1}}, \mathcal{E}_r)$ for all $r \geq 1$:

$$0 \longrightarrow \mathcal{E}_r \longrightarrow \mathcal{E}_{r+1} \longrightarrow L_{\epsilon_{r+1}} \longrightarrow 0. \tag{23}$$

A priori we do not know that we can always take *non-split* such extensions; this we will prove in a moment. In any case, all \mathcal{E}_r are Ulrich vector bundles of rank r , as extensions of Ulrich bundles are again Ulrich.

Lemma 5.2 *Let L be L_0 or L_1 . Then, for all $r \geq 1$ we have*

- (i) $h^2(\mathcal{E}_r \otimes L^*) = 0$,
- (ii) $h^2(\mathcal{E}_r^* \otimes L) = 0$,
- (iii) $h^1(\mathcal{E}_r \otimes L_{\epsilon_{r+1}}^*) \geq h \geq 3$.

Proof We prove (i)–(ii) by induction on r . Assertion (i) follows from Lemma 5.1 because $\mathcal{E}_1 = L_1$. Assuming it holds for r , we have, by tensoring (23) by L^* , that

$$h^2(\mathcal{E}_{r+1} \otimes L^*) \leq h^2(\mathcal{E}_r \otimes L^*) + h^2(L_{\epsilon_{r+1}} \otimes L^*) = 0,$$

by the induction hypothesis and Lemma 5.1, since $L_{\epsilon_{r+1}} \otimes L^*$ equals one of \mathcal{O}_X , $L_0 - L_1$ or $L_1 - L_0$.

A similar reasoning, tensoring the dual of (23) by L , proves (ii).

To prove (iii), first note that it holds for $r = 1$, as $h^1(\mathcal{E}_1 \otimes L_{\epsilon_2}^*) = h^1(L_1 - L_0) = h$ by Lemma 5.1. Then, for any $r \geq 1$, tensor (23) by $L_{\epsilon_{r+2}}^*$. Using that $h^2(\mathcal{E}_r \otimes L_{\epsilon_{r+2}}^*) = 0$ by (i), we see that $h^1(\mathcal{E}_{r+1} \otimes L_{\epsilon_{r+2}}^*) \geq h^1(L_{\epsilon_{r+1}} \otimes L_{\epsilon_{r+2}}^*) = h \geq 3$, by Lemma 5.1. \square

By (iii) of the last lemma, we have that $\dim(\text{Ext}^1(L_{\epsilon_{r+1}}, \mathcal{E}_r)) = h^1(\mathcal{E}_r \otimes L_{\epsilon_{r+1}}^*) > 0$ for all $r \geq 1$, which means that we can always pick *non-split* extensions of the form (23). We will henceforth do so.

Lemma 5.3 *For all $r \geq 1$ we have*

- (i) $h^1(\mathcal{E}_{r+1} \otimes L_{\epsilon_{r+1}}^*) = h^1(\mathcal{E}_r \otimes L_{\epsilon_{r+1}}^*) - 1$,
- (ii) $h^1(\mathcal{E}_r \otimes L_{\epsilon_{r+1}}^*) = \lfloor \frac{r+1}{2} \rfloor (h - 1) + 1$,
- (iii) $h^2(\mathcal{E}_r \otimes \mathcal{E}_r^*) = 0$,
- (iv) $\chi(\mathcal{E}_r \otimes L_{\epsilon_{r+1}}^*) = -\lfloor \frac{r+1}{2} \rfloor (h - 1) - \epsilon_r$,
- (v) $\chi(L_{\epsilon_r} \otimes \mathcal{E}_r^*) = -\lfloor \frac{r+1}{2} \rfloor (h - 1) + \epsilon_r h$,
- (vi) $\chi(\mathcal{E}_r \otimes \mathcal{E}_r^*) = -\frac{1}{2}(r^2 - \epsilon_r)(h - 1) + \epsilon_r$,
- (vii) *the slope of \mathcal{E}_r is $\mu(\mathcal{E}_r) = L_0 \cdot \xi_{m,n} = L_1 \cdot \xi_{m,n}$.*

Proof (i) Since $\text{Ext}^1(L_{\epsilon_{r+1}}, \mathcal{E}_r) \simeq H^1(\mathcal{E}_r \otimes L_{\epsilon_{r+1}}^*)$ and the sequence (23) is constructed by taking a nonzero element therein, the coboundary map $H^0(\mathcal{O}_X) \rightarrow H^1(\mathcal{E}_r \otimes L_{\epsilon_{r+1}}^*)$ of (23) tensored by $L_{\epsilon_{r+1}}^*$, i.e.

$$0 \longrightarrow \mathcal{E}_r \otimes L_{\epsilon_{r+1}}^* \longrightarrow \mathcal{E}_{r+1} \otimes L_{\epsilon_{r+1}}^* \longrightarrow \mathcal{O}_{X_n} \longrightarrow 0, \tag{24}$$

is nonzero. Thus, (i) follows from the cohomology of (24).

(ii) We use induction on r . For $r = 1$, the right-hand side of the formula yields h , whereas the left-hand side equals $h^1(\mathcal{E}_1 \otimes L_0^*) = h^1(L_1 - L_0)$; thus the formula is correct by Lemma 5.1.

Assume now that the formula holds for r . Tensoring (23) by $L_{\epsilon_{r+2}}^*$, we obtain

$$0 \longrightarrow \mathcal{E}_r \otimes L_{\epsilon_{r+2}}^* \longrightarrow \mathcal{E}_{r+1} \otimes L_{\epsilon_{r+2}}^* \longrightarrow L_{\epsilon_{r+1}} \otimes L_{\epsilon_{r+2}}^* \longrightarrow 0. \tag{25}$$

We have $h^0(L_{\epsilon_{r+1}} \otimes L_{\epsilon_{r+2}}^*) = 0$ and $h^1(L_{\epsilon_{r+1}} \otimes L_{\epsilon_{r+2}}^*) = h$ by Lemma 5.1, and $h^2(\mathcal{E}_r \otimes L_{\epsilon_{r+2}}^*) = 0$ by Lemma 5.2. Thus,

$$h^1(\mathcal{E}_{r+1} \otimes L_{\epsilon_{r+2}}^*) = h + h^1(\mathcal{E}_r \otimes L_{\epsilon_{r+2}}^*) = h + h^1(\mathcal{E}_r \otimes L_{\epsilon_r}^*)$$

Using (i) and the induction hypothesis, this equals

$$\begin{aligned} h + (h^1(\mathcal{E}_{r-1} \otimes L_{\epsilon_r}^*) - 1) &= h + \left(\left\lfloor \frac{(r-1)+1}{2} \right\rfloor (h-1) + 1 \right) - 1 \\ &= \left\lfloor \frac{(r+1)+1}{2} \right\rfloor (h-1) + 1, \end{aligned}$$

showing that the formula holds for $r + 1$.

(iii) We again use induction on r . For $r = 1$ (iii) says that $h^2(L_1 - L_1) = h^2(\mathcal{O}_{X_n}) = 0$, which is true. Assume now that (iii) holds for r . The cohomology of (23) tensored by \mathcal{E}_{r+1}^* and Lemma 5.2(ii) yield

$$h^2(\mathcal{E}_{r+1} \otimes \mathcal{E}_{r+1}^*) \leq h^2(\mathcal{E}_r \otimes \mathcal{E}_{r+1}^*) + h^2(L_{\epsilon_{r+1}} \otimes \mathcal{E}_{r+1}^*) = h^2(\mathcal{E}_r \otimes \mathcal{E}_{r+1}^*). \tag{26}$$

The cohomology of the dual of (23) tensored by \mathcal{E}_r and Lemma 5.2(i) yields

$$h^2(\mathcal{E}_r \otimes \mathcal{E}_{r+1}^*) \leq h^2(\mathcal{E}_r \otimes L_{\epsilon_{r+1}}^*) + h^2(\mathcal{E}_r \otimes \mathcal{E}_r^*) = h^2(\mathcal{E}_r \otimes \mathcal{E}_r^*). \tag{27}$$

Now (26)–(27) and the induction hypothesis yield $h^2(\mathcal{E}_{r+1} \otimes \mathcal{E}_{r+1}^*) = 0$, as desired.

(iv) For $r = 1$ (iv) reads $\chi(L_1 - L_0) = -h$, which is true by Lemma 5.1. For $r = 2$ (iv) reads $\chi(\mathcal{E}_2 \otimes L_1^*) = -h + 1$; using sequence (25) with $r = 1$, one computes $\chi(\mathcal{E}_2 \otimes L_1^*) = \chi(\mathcal{O}_{X_n}) + \chi(L_0 - L_1) = 1 - h$, by Lemma 5.1, so again the formula is correct.

Assume now that the formula holds up to a certain $r \geq 2$. From (25) and Lemma 5.1 we find

$$\chi(\mathcal{E}_{r+1} \otimes L_{\epsilon_{r+2}}^*) = \chi(\mathcal{E}_r \otimes L_{\epsilon_{r+2}}^*) + \chi(L_{\epsilon_{r+1}} \otimes L_{\epsilon_{r+2}}^*) = \chi(\mathcal{E}_r \otimes L_{\epsilon_r}^*) - h. \tag{28}$$

Then (24) (with r replaced by $r - 1$) yields

$$\chi(\mathcal{E}_r \otimes L_{\epsilon_r}^*) = \chi(\mathcal{E}_{r-1} \otimes L_{\epsilon_r}^*) + \chi(\mathcal{O}_{X_n}) = \chi(\mathcal{E}_{r-1} \otimes L_{\epsilon_r}^*) + 1. \tag{29}$$

Inserting into (28) and using the induction hypothesis, we get

$$\begin{aligned} \chi(\mathcal{E}_{r+1} \otimes L_{\epsilon_{r+2}}^*) &= \chi(\mathcal{E}_{r-1} \otimes L_{\epsilon_r}^*) + 1 - h \\ &= -(h - 1) \left\lfloor \frac{(r - 1) + 1}{2} \right\rfloor - \epsilon_{r-1} + 1 - h \\ &= -(h - 1) \left(\left\lfloor \frac{r}{2} \right\rfloor + 1 \right) - \epsilon_{r-1} \\ &= -(h - 1) \left\lfloor \frac{(r + 1) + 1}{2} \right\rfloor - \epsilon_{r+1}, \end{aligned}$$

proving that the formula holds also for $r + 1$.

(v) For $r = 1$ (v) reads $\chi(L_1 - L_1) = 1$, which is true. For $r = 2$ (v) reads $\chi(L_0 \otimes \mathcal{E}_2^*) = -h + 1$; from the dual of sequence (21) tensored by L_0 , one computes $\chi(L_0 \otimes \mathcal{E}_2^*) = \chi(\mathcal{O}_{X_n}) + \chi(L_0 - L_1) = 1 - h$, by Lemma 5.1, so again the formula is correct.

Assume now that the formula holds up to a certain $r \geq 2$. From the dual of sequence (23) tensored by $L_{\epsilon_{r+1}}$ we find

$$\begin{aligned} \chi(L_{\epsilon_{r+1}} \otimes \mathcal{E}_{r+1}^*) &= \chi(L_{\epsilon_{r+1}} \otimes L_{\epsilon_{r+1}}^*) + \chi(L_{\epsilon_{r+1}} \otimes \mathcal{E}_r^*) \\ &= \chi(\mathcal{O}_{X_n}) + \chi(L_{\epsilon_{r+1}} \otimes \mathcal{E}_r^*) = 1 + \chi(L_{\epsilon_{r+1}} \otimes \mathcal{E}_r^*). \end{aligned} \tag{30}$$

The dual of sequence (23) with r replaced by $r - 1$ tensored by $L_{\epsilon_{r+1}}$, together with Lemma 5.1, yields

$$\chi(L_{\epsilon_{r+1}} \otimes \mathcal{E}_r^*) = \chi(L_{\epsilon_{r+1}} \otimes L_{\epsilon_r}^*) + \chi(L_{\epsilon_{r+1}} \otimes \mathcal{E}_{r-1}^*) = -h + \chi(L_{\epsilon_{r-1}} \otimes \mathcal{E}_{r-1}^*). \tag{31}$$

Inserting into (30) and using the induction hypothesis, we get

$$\begin{aligned} \chi(L_{\epsilon_{r+1}} \otimes \mathcal{E}_{r+1}^*) &= 1 - h + \chi(L_{\epsilon_{r-1}} \otimes \mathcal{E}_{r-1}^*) \\ &= 1 - h - (h - 1) \left\lfloor \frac{(r - 1) + 1}{2} \right\rfloor + \epsilon_{r-1}h \\ &= -(h - 1) \left(\left\lfloor \frac{r}{2} \right\rfloor + 1 \right) + \epsilon_{r-1}h \\ &= -(h - 1) \left\lfloor \frac{(r + 1) + 1}{2} \right\rfloor + \epsilon_{r+1}h, \end{aligned}$$

proving that the formula holds also for $r + 1$.

(vi) We check the given formula for $r = 1, 2$.

We have $\chi(\mathcal{E}_1 \otimes \mathcal{E}_1^*) = \chi(L_1 - L_1) = \chi(\mathcal{O}_X) = 1$, which fits with the given formula for $r = 1$.

From (21) tensored by \mathcal{E}_2^* we get

$$\chi(\mathcal{E}_2 \otimes \mathcal{E}_2^*) = \chi(L_1 \otimes \mathcal{E}_2^*) + \chi(L_0 \otimes \mathcal{E}_2^*) \stackrel{(v)}{=} \chi(L_1 \otimes \mathcal{E}_2^*) - (h - 1). \tag{32}$$

From the dual of (21) tensored by L_1 and Lemma 5.1 we get

$$\chi(L_1 \otimes \mathcal{E}_2^*) = \chi(L_1 - L_1) + \chi(L_1 - L_0) = \chi(\mathcal{O}_{X_n}) - h = 1 - h. \tag{33}$$

Combining (32) and (33), we get $\chi(\mathcal{E}_2 \otimes \mathcal{E}_2^*) = -2(h - 1)$, which again fits with the given formula for $r = 2$.

Assume now that the given formula is valid up to a certain $r \geq 2$. From (23) tensored by \mathcal{E}_{r+1}^* and successively the dual of (23) tensored by \mathcal{E}_r we get

$$\begin{aligned} \chi(\mathcal{E}_{r+1} \otimes \mathcal{E}_{r+1}^*) &= \chi(\mathcal{E}_r \otimes \mathcal{E}_{r+1}^*) + \chi(L_{\epsilon_{r+1}} \otimes \mathcal{E}_{r+1}^*) \\ &= \chi(\mathcal{E}_r \otimes \mathcal{E}_r^*) + \chi(\mathcal{E}_r \otimes L_{\epsilon_{r+1}}^*) + \chi(L_{\epsilon_{r+1}} \otimes \mathcal{E}_{r+1}^*). \end{aligned}$$

Using the induction hypothesis, together with (iv) and (v) (with r substituted by $r + 1$), the right-hand side can be written as

$$\left[-\frac{1}{2} (r^2 - \epsilon_r) (h - 1) + \epsilon_r \right] + \left[-(h - 1) \left\lfloor \frac{r + 1}{2} \right\rfloor - \epsilon_r \right] + \left[-(h - 1) \left\lfloor \frac{r + 2}{2} \right\rfloor + \epsilon_{r+1} h \right]$$

An easy computation shows that this equals

$$-\frac{1}{2} ((r + 1)^2 - \epsilon_{r+1}) (h - 1) + \epsilon_{r+1},$$

finishing the inductive step.

(vii) This can either easily be checked by induction again or follows from the fact that the slope of an Ulrich bundle only depends on the ambient polarized variety, cf. e.g. [6, Prop. 2.1(3)], and must therefore equal the slope of L_0 and L_1 for all ranks. \square

We now define, for each $r \geq 1$, the scheme $\mathfrak{U}(r)$ to be the modular family of the vector bundles \mathcal{E}_r defined above. For $r \geq 2$, the scheme $\mathfrak{U}(r)$ contains a subscheme $\mathfrak{U}(r)^{\text{ext}}$ parametrizing bundles \mathcal{F}_r that are non-split extensions of the form

$$0 \longrightarrow \mathcal{F}_{r-1} \longrightarrow \mathcal{F}_r \longrightarrow L_{\epsilon_r} \longrightarrow 0, \tag{34}$$

with $[\mathcal{F}_{r-1}] \in \mathfrak{U}(r - 1)$.

Lemma 5.4 *Let \mathcal{F}_r be a general member of $\mathfrak{U}(r)$. Then \mathcal{F}_r is Ulrich of rank r with slope $\mu := L_0 \cdot \xi_{m,n} = L_1 \cdot \xi_{m,n}$.*

Moreover,

- (i) $\chi(\mathcal{F}_r \otimes \mathcal{F}_r^*) = -\frac{1}{2} (r^2 - \epsilon_r) (h - 1) + \epsilon_r,$
- (ii) $h^2(\mathcal{F}_r \otimes \mathcal{F}_r^*) = 0,$
- (iii) $h^1(\mathcal{F}_r \otimes L_{\epsilon_{r+1}}^*) \leq \lfloor \frac{r+1}{2} \rfloor (h - 1) + 1.$

Proof Ulrichness is an open property in the family, and the rank and slope are constant, so the general member of $\mathfrak{U}(r)$ is Ulrich of rank r and slope μ as each \mathcal{E}_r constructed above is (cf. Lemma 4.3(vii)) or use the fact that the slope of an Ulrich bundle only depends on the ambient polarized variety, cf. e.g. [6, Prop. 2.1(3)].

Properties (ii) and (iii) follows by specializing \mathcal{F}_r to an \mathcal{E}_r constructed above, and using semicontinuity and Lemma 5.3(iii) and (ii), respectively. Property (i) follows by Lemma 5.3(vi), since the given χ depends only on the Chern classes of the two factors and of X_n , which are constant in the family $\mathfrak{U}(r)$. \square

We wish to prove that the general member of $\mathfrak{U}(r)$ is slope-stable. To this end we will need a couple of auxiliary results.

Lemma 5.5 *Let $r \geq 2$ and assume that $[\mathcal{F}_r] \in \mathfrak{U}(r)^{\text{ext}}$ sits in a non-split sequence like (34) with $[\mathcal{F}_{r-1}] \in \mathfrak{U}(r - 1)$ being slope-stable. Then*

- (i) \mathcal{F}_r is simple (that is, $h^0(\mathcal{F}_r \otimes \mathcal{F}_r^*) = 1$);

(ii) if \mathcal{G} is a destabilizing subsheaf of \mathcal{F}_r , then $\mathcal{G}^* \simeq \mathcal{F}_{r-1}^*$ and $(\mathcal{F}_r/\mathcal{G})^* \simeq L_{\epsilon_r}^*$; if, furthermore, $\mathcal{F}_r/\mathcal{G}$ is torsion-free, then $\mathcal{G} \simeq \mathcal{F}_{r-1}$ and $\mathcal{F}_r/\mathcal{G} \simeq L_{\epsilon_r}$.

Proof We first prove (ii).

Assume that \mathcal{G} is a destabilizing subsheaf of \mathcal{F}_r , that is $0 < \text{rk}(\mathcal{G}) < \text{rk}(\mathcal{F}_r) = r$ and $\mu(\mathcal{G}) \geq \mu := \mu(\mathcal{F}_r)$. Define

$$\mathcal{Q} := \text{im}\{\mathcal{G} \subset \mathcal{F}_r \rightarrow L_{\epsilon_r}\} \text{ and } \mathcal{K} := \ker\{\mathcal{G} \rightarrow \mathcal{Q}\}.$$

Then we may put (34) into a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{Q} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{F}_{r-1} & \longrightarrow & \mathcal{F}_r & \longrightarrow & L_{\epsilon_r} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{K}' & \longrightarrow & \mathcal{F}_r/\mathcal{G} & \longrightarrow & \mathcal{Q}' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

defining \mathcal{K}' and \mathcal{Q}' . We have $\text{rk}(\mathcal{Q}) \leq 1$.

Assume that $\text{rk}(\mathcal{Q}) = 0$. Then $\mathcal{Q} = 0$, whence $\mathcal{K} \simeq \mathcal{G}$ and $\mathcal{Q}' \simeq L_{\epsilon_r}$. Since $\mu(\mathcal{K}) = \mu(\mathcal{G}) \geq \mu = \mu(\mathcal{F}_{r-1})$ and \mathcal{F}_{r-1} is slope-stable, we must have $\text{rk}(\mathcal{K}) = \text{rk}(\mathcal{F}_{r-1}) = r - 1$. It follows that $\text{rk}(\mathcal{K}') = 0$. Since

$$c_1(\mathcal{K}) = c_1(\mathcal{F}_{r-1}) - c_1(\mathcal{K}') = c_1(\mathcal{F}_{r-1}) - D',$$

where D' is an effective divisor supported on the codimension one locus of the support of \mathcal{K}' , we have

$$\mu \leq \mu(\mathcal{K}) = \frac{(c_1(\mathcal{F}_{r-1}) - D') \cdot \xi_{m,n}}{r - 1} = \frac{c_1(\mathcal{F}_{r-1}) \cdot \xi_{m,n}}{r - 1} - \frac{D' \cdot \xi_{m,n}}{r - 1} = \mu - \frac{D' \cdot \xi_{m,n}}{r - 1}.$$

Hence $D' = 0$, which means that \mathcal{K}' is supported in codimension at least two. Thus, $\mathcal{E}xt^i(\mathcal{K}', \mathcal{O}_X) = 0$ for $i \leq 1$, and it follows that $\mathcal{G}^* \simeq \mathcal{K}^* \simeq \mathcal{F}_{r-1}^*$ and $(\mathcal{F}_r/\mathcal{G})^* \simeq \mathcal{Q}'^* \simeq L_{\epsilon_r}^*$, as desired. If, furthermore, $\mathcal{F}_r/\mathcal{G}$ is torsion-free, then we must have $\mathcal{K}' = 0$, whence $\mathcal{G} \simeq \mathcal{F}_{r-1}$ and $\mathcal{F}_r/\mathcal{G} \simeq L_{\epsilon_r}$.

Next we prove that $\text{rk}(\mathcal{Q}) = 1$ cannot happen. Indeed, if $\text{rk}(\mathcal{Q}) = 1$, then $\text{rk}(\mathcal{K}) = \text{rk}(\mathcal{G}) - 1 \leq r - 2 < r - 1 = \text{rk}(\mathcal{F}_{r-1})$ and $\text{rk}(\mathcal{Q}') = 0$; in particular, \mathcal{Q}' is a torsion sheaf. Since

$$c_1(\mathcal{K}) = c_1(\mathcal{G}) - c_1(\mathcal{Q}) = c_1(\mathcal{G}) - c_1(L_{\epsilon_r}) + c_1(\mathcal{Q}') = c_1(\mathcal{G}) - c_1(L_{\epsilon_r}) + D,$$

where D is an effective divisor supported on the codimension one locus of the support of \mathcal{Q}' , we have

$$\begin{aligned} \mu(\mathcal{K}) &= \frac{(c_1(\mathcal{G}) - c_1(L_{\epsilon_r}) + D) \cdot \xi_{m,n}}{\text{rk}(\mathcal{K})} \geq \frac{(c_1(\mathcal{G}) - c_1(L_{\epsilon_r})) \cdot \xi_{m,n}}{\text{rk}(\mathcal{K})} \\ &= \frac{\mu(\mathcal{G}) \text{rk}(\mathcal{G}) - c_1(L_{\epsilon_r}) \cdot \xi_{m,n}}{\text{rk}(\mathcal{K})} = \frac{\mu(\mathcal{G}) \text{rk}(\mathcal{G}) - \mu}{\text{rk}(\mathcal{G}) - 1} \geq \frac{\mu \text{rk}(\mathcal{G}) - \mu}{\text{rk}(\mathcal{G}) - 1} = \mu \end{aligned}$$

This contradicts slope-stability of \mathcal{F}_{r-1} .

To prove (i), assume that \mathcal{F}_r is non-simple, that is, it admits a nontrivial endomorphism. Then there exists a nonzero endomorphism $\varphi : \mathcal{F}_r \rightarrow \mathcal{F}_r$ dropping rank everywhere. Indeed, take any endomorphism α that is not a constant times the identity, pick an eigenvalue λ of $\alpha(x)$ for some $x \in X_n$ and set $\varphi = \alpha - \lambda \text{id}$; then $\det(\varphi) \in H^0(\det(\mathcal{F}_r^*) \otimes \det(\mathcal{F}_r)) = H^0(\mathcal{O}_{X_n}) \simeq \mathbb{C}$ vanishes at x , whence it is identically zero. Both $\ker(\varphi)$ and $\text{im}(\varphi)$, being subsheaves of \mathcal{F}_r , are torsion-free, and the equality

$$\text{rk}(\mathcal{F}_r)\mu(\mathcal{F}_r) = \text{rk}(\ker(\varphi))\mu(\ker(\varphi)) + \text{rk}(\text{im}(\varphi))\mu(\text{im}(\varphi))$$

easily yields that one of them is destabilizing. By part (ii), it follows that either $\ker(\varphi) \simeq \mathcal{F}_{r-1}$ (and $\text{im}(\varphi) \simeq L_{\epsilon_r}$) or $\text{im}(\varphi)^* \simeq \mathcal{F}_{r-1}^*$. In the first case, if $\varphi^2 = 0$, then $L_{\epsilon_r} \subset \mathcal{F}_{r+1}$, contradicting the slope-stability of \mathcal{F}_{r+1} , and if $\varphi^2 \neq 0$, then φ maps L_{ϵ_r} isomorphically to L_{ϵ_r} , implying that (34) splits, a contradiction. In the second case, if $\varphi^2 = 0$, then $r = 2$, $\mathcal{F}_{r-1} = L_1 = \text{im}(\varphi) \subset \ker(\varphi) = L_0$, yielding the contradiction $h^0(L_0 - L_1) > 0$, whereas if $\varphi^2 \neq 0$, then, since \mathcal{F}_{r+1} is slope-stable, φ maps \mathcal{F}_{r+1} isomorphically to \mathcal{F}_{r+1} , implying again that (34) splits, a contradiction. \square

Lemma 5.6 *Let $r \geq 2$ and assume that the general member of $\mathfrak{U}(r - 1)$ is slope-stable. Then $\mathfrak{U}(r)$ is generically smooth of dimension $\frac{1}{2}(r^2 - \epsilon_r)(h - 1) + \epsilon_{r+1}$ and properly contains $\mathfrak{U}(r)^{\text{ext}}$.*

Proof The general member \mathcal{F}_r of $\mathfrak{U}(r)$ satisfies $h^0(\mathcal{F}_r \otimes \mathcal{F}_r^*) = 1$ by Lemma 5.5(i) and $h^2(\mathcal{F}_r \otimes \mathcal{F}_r^*) = 0$ by Lemma 5.4(ii). Hence, one has (see e.g. [5, Prop. 2.10]) that $\mathfrak{U}(r)$ is generically smooth of dimension

$$\begin{aligned} h^1(\mathcal{F}_r \otimes \mathcal{F}_r^*) &= -\chi(\mathcal{F}_r \otimes \mathcal{F}_r^*) + h^0(\mathcal{F}_r \otimes \mathcal{F}_r^*) + h^2(\mathcal{F}_r \otimes \mathcal{F}_r^*) = -\chi(\mathcal{F}_r \otimes \mathcal{F}_r^*) + 1 \\ &= \frac{1}{2}(r^2 - \epsilon_r)(h - 1) - \epsilon_r + 1 = \frac{1}{2}(r^2 - \epsilon_r)(h - 1) + \epsilon_{r+1}, \end{aligned}$$

where we have used Lemma 5.4(i), as claimed.

Similarly, being slope-stable, also the general member \mathcal{F}_{r-1} of $\mathfrak{U}(r - 1)$ satisfies $h^0(\mathcal{F}_{r-1} \otimes \mathcal{F}_{r-1}^*) = 1$, so the same reasoning shows that

$$\dim(\mathfrak{U}(r - 1)) = \frac{1}{2}((r - 1)^2 - \epsilon_{r-1})(h - 1) + \epsilon_r. \tag{35}$$

Moreover,

$$\dim(\text{Ext}^1(L_{\epsilon_r}, \mathcal{F}_{r-1})) = h^1(\mathcal{F}_{r-1} \otimes L_{\epsilon_r}^*) \leq \left\lfloor \frac{r}{2} \right\rfloor (h - 1) + 1. \tag{36}$$

by Lemma 5.4(iii). Hence,

$$\begin{aligned} \dim(\mathfrak{U}(r)^{\text{ext}}) &\leq \dim(\mathfrak{U}(r-1)) + \dim \mathbb{P}(\text{Ext}^1(L_{\epsilon_r}, \mathcal{F}_{r-1})) \\ &\leq \frac{1}{2}((r-1)^2 - \epsilon_{r-1})(h-1) + \epsilon_r + \left\lfloor \frac{r}{2} \right\rfloor (h-1) \\ &= \frac{1}{2}(r^2 - \epsilon_r)(h-1) + \epsilon_{r+1} - \left(\left\lfloor \frac{r+1}{2} \right\rfloor - \epsilon_r \right) (h-1) + \epsilon_r - \epsilon_{r+1} \\ &= \dim(\mathfrak{U}(r)) - \left(\left\lfloor \frac{r+1}{2} \right\rfloor - \epsilon_r \right) (h-1) + \epsilon_r - \epsilon_{r+1}, \end{aligned}$$

and one easily sees that this is strictly less than $\dim(\mathfrak{U}(r))$, since $r \geq 2$ and $h \geq 3$. Thus, $\mathfrak{U}(r)^{\text{ext}}$ is properly contained in $\mathfrak{U}(r)$, as claimed. \square

We can now prove slope-stability of the general member of $\mathfrak{U}(r)$.

Proposition 5.7 *Let $r \geq 1$. The general member of $\mathfrak{U}(r)$ is slope-stable.*

Proof We use induction on r , the result being trivially true for $r = 1$. Assume $r \geq 2$ and that the general member of $\mathfrak{U}(r)$ is not slope-stable, whereas the general member of $\mathfrak{U}(r-1)$ is. Consider a one-parameter flat family $\mathcal{F} \rightarrow \mathbb{D}$ of bundles over the disc \mathbb{D} such that the generic member $\mathcal{F}^{(t)}$ of this family is also a generic member of $\mathfrak{U}(r)$ and the fiber $\mathcal{F}^{(0)}$ of $\mathcal{F} \rightarrow \mathbb{D}$ over $0 \in \mathbb{D}$ lies in $\mathfrak{U}(r)^{\text{ext}}$. Since we are assuming that the general member of $\mathfrak{U}(r)$ is not slope-stable, we have that $\mathcal{F}^{(t)}$ is not stable so that there is a destabilizing sequence

$$0 \longrightarrow \mathcal{G}^{(t)} \longrightarrow \mathcal{F}^{(t)} \longrightarrow \mathcal{Q}^{(t)} \longrightarrow 0 \quad (37)$$

which we can take to be saturated, that is, such that $\mathcal{Q}^{(t)}$ is torsion free, whence so that $\mathcal{G}^{(t)}$ and $\mathcal{Q}^{(t)}$ are (Ulrich) vector bundles (see [5, Thm. 2.9] or [4, (3.2)]). We may also assume that, up to base change, the sequence (37) is defined over the field of meromorphic functions over \mathbb{D} , so that there are two flat families $\mathcal{G} \rightarrow \mathbb{D}$ and $\mathcal{Q} \rightarrow \mathbb{D}$ of bundles over the disc \mathbb{D} such that the generic members of $\mathcal{G} \rightarrow \mathbb{D}$ and $\mathcal{Q} \rightarrow \mathbb{D}$ are $\mathcal{G}^{(t)}$ and $\mathcal{Q}^{(t)}$, respectively.

The limit of $\mathbb{P}(\mathcal{Q}^{(t)}) \subset \mathbb{P}(\mathcal{F}^{(t)})$ defines a subvariety of $\mathbb{P}(\mathcal{F}^{(0)})$ of the same dimension as $\mathbb{P}(\mathcal{Q}^{(t)})$, whence a coherent sheaf $\mathcal{Q}^{(0)}$ of rank $\text{rk}(\mathcal{Q}^{(t)})$ with a surjection $\mathcal{F}^{(0)} \rightarrow \mathcal{Q}^{(0)}$. Denoting by $\mathcal{G}^{(0)}$ its kernel, we have $\text{rk}(\mathcal{G}^{(0)}) = \text{rk}(\mathcal{G}^{(t)})$ and $c_1(\mathcal{G}^{(0)}) = c_1(\mathcal{G}^{(t)})$. Hence, (37) specializes to a destabilizing sequence for $t = 0$. Lemma 5.5 yields that $\mathcal{G}^{(0)*}$ (respectively, $\mathcal{Q}^{(0)*}$) is the dual of a member of $\mathfrak{U}(r-1)$ (resp., the dual of L_{ϵ_r}). It follows that $\mathcal{G}^{(t)*}$ (resp., $\mathcal{Q}^{(t)*}$) is a deformation of the dual of a member of $\mathfrak{U}(r-1)$ (resp., a deformation of $L_{\epsilon_r}^*$), whence that $\mathcal{G}^{(t)}$ is a deformation of a member of $\mathfrak{U}(r-1)$, as both are locally free, and $\mathcal{Q}^{(t)} \simeq L_{\epsilon_r}$, for the same reason.

In other words, the general member of $\mathfrak{U}(r)$ is an extension of L_{ϵ_r} by a member of $\mathfrak{U}(r-1)$. Hence, $\mathfrak{U}(r) = \mathfrak{U}(r)^{\text{ext}}$, contradicting Lemma 5.6. \square

By combining Proposition 5.7 and Lemma 5.4 we deduce:

Theorem 5.8 *For any $r \geq 1$, the blown-up plane X_n , with $n \geq 2$, carries slope-stable rank- r Ulrich bundles, and their moduli space contains a reduced and irreducible component of dimension $\frac{1}{2}(r^2 - \epsilon_r)(h-1) + \epsilon_{r+1}$.*

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