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Rigorous estimates on mechanical balance laws in the Boussinesq–Peregrine equations

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This article is dedicated to the memory of Professor Vassilios A. Dougalis.

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Abstract

It is shown that the Boussinesq–Peregrine system, which describes long waves of small amplitude at the surface of an inviscid fluid with variable depth, admits a number of approximate conservation equations. Notably, this paper provides accurate estimations for the approximate conservation of the mechanical balance laws associated with mass, momentum, and energy. These precise estimates offer valuable insights into the behavior and dynamics of the system, shedding light on the conservation principles governing the wave motion.

KEYWORDS

Boussinesq–Peregrine equations, conservation laws, mechanical balance laws

1 | INTRODUCTION

In the present work, approximate conservation equations associated with model systems describing the evolution of surface water waves are under consideration. The water-wave problem can be studied from a number of angles. In the case of a flat bed, the governing equations constitute a Hamiltonian system[7,30] with a finite number of symmetries and corresponding conserved integrals.[3] Recent years have seen intense activity in particular pertaining to the existence and uniqueness of solutions in various function classes.[5,9,21,23,25,29] In practical applications in coastal engineering, the bathymetry is typically not flat (see, e.g., Refs. 4, 12, 16, 23 and many others), and the resulting conservation laws are generally approximate.[18] In this context, the present research aims to establish a robust mathematical framework to comprehend the approximate conservation of fundamental mechanical balance laws which hold in the same order as the governing equations. Differential balance laws in continuum mechanics play a pivotal role, as they give a mathematical expression of the fundamental principles of mass, momentum, and energy conservation. For example, mechanical balance laws can be used to study the problem of wave shoaling, when waves enter a shallower area[22,27] as well as de-shoaling (see, e.g., Refs. 10, 31). Moreover, approximate mechanical balance laws also naturally connect to measures of angular momentum in the fluid flow.[11] These balance laws provide not only physical insights but also serve as a valuable tool for mathematical analysis. By investigating the approximate conservation laws in the context of nontrivial bathymetry, this research endeavors to enhance our understanding of the underlying dynamics and behavior of surface water waves.

We begin by recalling the formulation of the water-wave problem. After briefly introducing two equivalent formulations of the water-wave equations, we present the general dimensionless equations of this problem in terms of the long wave parameter ε, which measures the relative size of the water depth against a typical wavelength.

In Section 2, we recall the asymptotic derivation of the Boussinesq–Peregrine system as an ε^2-approximation of the water wave problem and proceed to define the pressure and conservation equations in the presence of bathymetry. We define the mechanical balance laws in Section 3 before embarking on the mathematical proof of the approximate balance laws in Section 4, where it is shown that the mass balance law succinctly denoted by $\mathcal{M}$ (see Equation (49)) is zero to the same order as the governing evolution system, that is, that $\mathcal{M}$ is of the order of ε^2. Similar estimates are obtained for the momentum balance laws denoted by $\mathcal{I}$ (see Equation (53)) and for the energy balance law $\mathcal{E}$ defined in (57). More precisely, for a constant C depending only on the initial data and for all time $t \in [0, T]$, we have the estimates

$$\max\left\{ \|\mathcal{M}\|_{L^\infty(\mathbb{R})}, \|\mathcal{I}\|_{L^\infty(\mathbb{R})}, \|\mathcal{E}\|_{L^\infty(\mathbb{R})} \right\} \leq C\varepsilon^2 .$$

In addition, we show that $\int_{\mathbb{R}} \mathcal{M}\, dx = 0$, and

$$\max\left\{ \Big\| \int_{\mathbb{R}} \mathcal{I}\, dx \Big\|_{L^\infty(\mathbb{R})}, \Big\| \int_{\mathbb{R}} \mathcal{E}\, dx \Big\|_{L^\infty(\mathbb{R})} \right\} \leq C\varepsilon^2 .$$

We would like to highlight in particular that the estimation parameters remain independent even of nontrivial bathymetry. Another notable observation pertains to the reduced regularity requirements for maintaining precision in integral estimates with respect to initial data. This observation suggests that the integrated mechanical quantities exhibit a more robust behavior and are less sensitive to variations in the initial data. It implies that the collective behavior of these quantities, encompassing mass, momentum, and energy, is primarily influenced by minor fluctuations or irregularities within the system.

1.1 | The classical equations

The motion of an ideal fluid with a free surface is described by the Euler equations with free-surface boundary conditions. The unknowns are the velocity $\mathbf{V}(t, x, y, z)$ and the pressure $P(t, x, y, z)$, given at a point (x, y, z) in the fluid domain and at a time t. In the present work, it

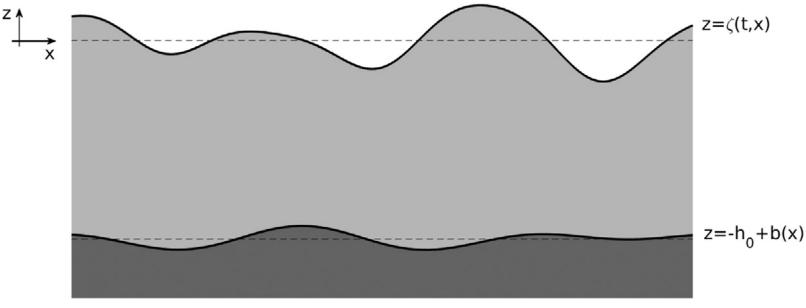

FIGURE 1 Geometric setup of the problem. The function $\zeta(x,t)$ denotes the excursion of the free surface from the rest position. The function $b(x)$ represents variations in the fluid bed, and h_0 designates the mean depth with the fluid at rest.

is assumed that the waves are propagating in the x direction and that the motion is uniform in the direction transverse to the wave propagation direction, that is, in the y direction. The system can then be reduced to two spatial variables (x,z), describing a position in the fluid domain Ω_t. This domain is delimited below by the uneven bottom $-h_0 + b(x)$ and above by a time-dependent free surface $\zeta(t,x)$ and is defined formally as $\Omega_t = \mathbb{R}_+ \times \mathbb{R} \times (-h_0 + b(x), \zeta(t,x))$. The geometric setup of the problem is shown in Figure 1. The Euler momentum equations are coupled with the irrotationality and incompressibility conditions, and the complete system is written as

$$\begin{cases} \partial_t \mathbf{V} + (\mathbf{V}\cdot\nabla_{x,z})\mathbf{V} = -g\mathbf{e}_z - \dfrac{1}{\rho}\nabla_{x,z}P & \text{in} \quad (x,z)\in\Omega_t \ , \\ \nabla_{x,z}\cdot\mathbf{V} = 0 & \text{in} \quad (x,z)\in\Omega_t \ , \\ \nabla_{x,z}\times\mathbf{V} = 0 & \text{in} \quad (x,z)\in\Omega_t \ . \end{cases} \tag{1}$$

The fluid is assumed to be of constant density ρ, and the constant $g > 0$ denotes the gravitational acceleration which is acting in the negative z-direction. The unit vector in the vertical direction is denoted by $\mathbf{e}_z = (0,1)^\top$, and the gradient with respect to horizontal and vertical variables (x,z) is denoted by $\nabla_{x,z} = (\partial/\partial_x, \partial/\partial_z)^\top$. These equations are complemented with the following boundary conditions:

$$\begin{cases} \zeta_t - \sqrt{1+|\partial_x\zeta|^2}\,\mathbf{V}\cdot\mathbf{N}_{surf} = 0 & \text{at} \quad z = \zeta(t,x), \\ P(t,x,z) - P_{atm} = 0 & \text{at} \quad z = \zeta(t,x), \\ -V\cdot\mathbf{N}_{bott} = 0 & \text{at} \quad z = -h_0 + b(x), \\ |\zeta(t,x)| + |V(t,x,z)| \underset{|(x,z)|\to+\infty}{\longrightarrow} 0 & \text{in} \quad (x,z)\in\Omega_t \ . \end{cases} \tag{2}$$

The first two relations at the free surface represent the kinematic and dynamic conditions, respectively. P_{atm} is the atmospheric pressure, which can be assumed to be constant in the present situation. We denote by $\mathbf{N}_{surf} = (1+\zeta_x^2)^{-1/2}(-\zeta_x, 1)^\top$ and $\mathbf{N}_{bott} = (1+b_x^2)^{-1/2}(b_x, -1)^\top$ the outward unit normal to the upper and lower boundaries. The third equation represents a kinematic condition at the bottom, and the fourth condition guarantees that the fluid will be at rest at infinity.

1.2 Equivalent formulations

Since the unknown fluid velocity $\mathbf{V}(t,x,z)$ is defined on a time-dependent domain Ω_t bounded by the bottom and the free surface itself which is part of the solution, the problem poses a number of theoretical and numerical challenges. The key assumption allowing for simplification of the problem is the irrotationality of the flow. This property ensures the existence of a velocity potential $\varphi(t,x,z)$ with the property that $\nabla_{x,z}\varphi = \mathbf{V}$. The incompressibility of the fluid then guarantees that $\varphi(t,x,z)$ is harmonic in the fluid domain:

$$\Delta_{x,z}\varphi = \varphi_{xx} + \varphi_{zz} = 0 \qquad \text{in}\Omega_t\ . \tag{3}$$

The kinematic boundary conditions at the free surface and at the bottom are formulated in terms of the potential velocity as follows:

$$\begin{cases} \zeta_t + \varphi_x\zeta_x - \varphi_z = 0 & \text{at} \quad z = \zeta(t,x)\,, \\ \varphi_z - b_x\varphi_x = 0 & \text{at} \quad z = -h_0 + b(x)\,. \end{cases} \tag{4}$$

Moreover, replacing $\mathbf{V}$ by $\nabla_{x,z}\varphi$ in the Euler equation (1), then integrating in (x,z) and using the fact that the fluid is at rest at infinity, the Bernoulli equation reads

$$\partial_t\varphi + \frac{1}{2}|\nabla_{x,z}\varphi|^2 = -gz - \frac{1}{\rho}(P - P_{atm}) \qquad \text{in}(x,z) \in \Omega_t\ . \tag{5}$$

Equations (3)–(5) and boundary conditions (4) are called the free-surface Bernoulli formulation of the water-wave problem.

Although the system is now given in a simpler form, the complication of the unknown domain is still present. In order to avoid this issue, the Dirichlet–Neumann operator will be introduced. First, we define the trace of the velocity potential at the free surface as

$$\psi(t,x) = \varphi(t,x,\zeta(t,x)) = \varphi_{|z=\zeta}\ .$$

The Dirichlet–Neumann operator is then defined as a mapping

$$\mathcal{G}[\zeta,b] : \psi \mapsto \sqrt{1+\zeta_x^2}\,\partial_{\mathbf{N}_{surf}}\varphi_{|z=\zeta} = -\zeta_x(\varphi_x)_{|z=\zeta} + (\varphi_z)_{|z=\zeta}\ .$$

The function φ is a function which is harmonic in the fluid domain and satisfies the bottom and given free-surface conditions (see Ref. 23 for a careful analysis):

$$\begin{cases} \varphi_{xx} + \varphi_{zz} = 0, & \text{in}\Omega_t\ , \\ \partial_{\mathbf{N}_{bott}}\varphi_{|z=-h_0+b} = 0, & \text{at}z = -h_0 + b\ , \\ \varphi_{|z=\zeta} = \psi(t,x), & \text{at}z = \zeta\ . \end{cases}$$

Finally, as shown in Refs. 7, 8, and 30, the problem can be written as a coupled system of evolution equations in the unknowns (ζ,ψ). This system is known as the Zakharov–Craig–Sulem formulation of the water-wave equations, written as

$$\begin{cases} \partial_t \zeta - \mathcal{G}[\zeta, b]\psi = 0 \,, \\ \partial_t \psi + \zeta + \frac{1}{2}\psi_x^2 - \frac{1}{2}(1+\zeta_x^2)^{-1}\left(\mathcal{G}[\zeta, b]\psi + \zeta_x \psi_x\right)^2 = 0 \,. \end{cases} \tag{6}$$

The first equation of the model above is obtained by using the Dirichlet–Neumann operator in the kinematic condition at the free surface (4), and the second evolution equation on ψ is a consequence of the chain rule where the three relations below are used:

$$\begin{cases} (\varphi_t)_{|z=\zeta} = \psi_t - \zeta_t (\varphi_z)_{|z=\zeta} \,, \\ (\varphi_x)_{|z=\zeta} = \psi_x - \zeta_x (\varphi_z)_{|z=\zeta} \,, \\ (\varphi_z)_{|z=\zeta} = (1+\zeta_x^2)^{-1}(\mathcal{G}[\zeta, b]\psi + \zeta_x \psi_x) \,. \end{cases}$$

Combining the third relation with the first two and after replacing $\partial_t \zeta$ with $\mathcal{G}[\zeta, b]\psi$, the desired system (6) appears. This formulation of the problem has the advantage that the functions are evaluated at the free surface, and the vertical direction is removed.

1.3 | Nondimensionalization

The identification of small physical parameters is essential for the derivation of simplified asymptotic models from the full system of equations (6). The first step is to nondimensionalize these equations using the physical characteristics of the flow. From a physical point of view, wave motion at the free surface can be characterized by the following scales:

$$a_{surf} \text{ : typical wave amplitude,} \qquad \lambda \text{ : typical wavelength ,}$$

$$a_{bott} \text{ : typical topographic variation,} \qquad h_0 \text{ : mean water depth .}$$

Based on these scales, let us introduce the nondimensionalized variables in the form

$$\frac{x}{x'} = \lambda, \qquad \frac{z}{z'} = h_0, \qquad \frac{\zeta}{\zeta'} = a_{surf}, \qquad \frac{b}{b'} = a_{bott}, \qquad \frac{\varphi}{\varphi'} = \frac{a_{surf}\sqrt{gh_0}}{h_0}, \qquad \frac{t}{t'} = \frac{\lambda}{\sqrt{gh_0}},$$
$$\frac{P}{P'} = \rho \,.$$

When these nondimensional variables are introduced into the water-wave system, three dimensionless parameters appear in the equations:

$$\varepsilon = \frac{a_{surf}}{h_0}, \qquad \mu = \frac{h_0^2}{\lambda^2}, \qquad \beta = \frac{a_{bott}}{h_0}, \tag{7}$$

where ε, μ, and β, are usually denoted as the nonlinearity, shallowness, and topography parameters, respectively. In coastal oceanography, typical values of these dimensionless parameters are usually such that ε, μ, and $\beta \in (0,1]$.

With these scalings the nondimensional Bernoulli formulation becomes (we eliminate the primes for the sake of clarity)

$$\begin{cases} \mu\partial_x^2\varphi + \partial_z^2\varphi = 0 & \text{in} \quad -1+\beta b(x) < z < \varepsilon\zeta(t,x)\,, \\ \partial_t\varphi + \dfrac{1}{2}\left(\varepsilon\varphi_x^2 + \dfrac{\varepsilon}{\mu}\varphi_z^2\right) = -\dfrac{1}{\varepsilon}z - \dfrac{1}{ag}(P - P_{atm}) & \text{in} \quad -1+\beta b(x) < z < \varepsilon\zeta(t,x)\,, \\ \partial_z\varphi - \mu\beta b_x\varphi_x = 0 & \text{at} \quad z = -1+\beta b(x)\,, \\ \partial_t\zeta + \mu^{-1}(\mu\varepsilon\zeta_x\varphi_x - \varphi_z) = 0 & \text{at} \quad z = \varepsilon\zeta(t,x)\,. \end{cases} \tag{8}$$

Notice that the Bernoulli equation at the free surface reads

$$\partial_t\varphi_{|_{z=\varepsilon\zeta}} + \frac{1}{2}(\varepsilon\varphi_x^2 + \frac{\varepsilon}{\mu}\varphi_z^2)_{|_{z=\varepsilon\zeta}} = -\zeta\,, \tag{9}$$

and the dimensionless version of Zakharov–Craig–Sulem (6) formulation of the water-wave equations reads

$$\begin{cases} \partial_t\zeta - \dfrac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi = 0\,, \\ \partial_t\psi + \zeta + \dfrac{1}{2}\varepsilon\psi_x^2 - \dfrac{1}{2}\varepsilon\mu\left(1+\varepsilon^2\mu\zeta_x^2\right)^{-1}\left(\dfrac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi + \varepsilon\zeta_x\psi_x\right)^2 = 0\,. \end{cases} \tag{10}$$

In this formula, the trace ϕ of the velocity potential at the free surface is given as before by $\psi(t,x) = \varphi_{|_{z=\varepsilon\zeta}}$, and the Dirichlet–Neumann operator $\mathcal{G}_\mu[\varepsilon\zeta,\beta b]$ is defined as before with the help of a harmonic function in the fluid domain. More precisely, if φ is a function satisfying the first, second, and third equations in (8), we have the system

$$\begin{cases} \mu\varphi_{xx} + \varphi_{zz} = 0 & \text{in} \quad -1+\beta b(x) < z < \varepsilon\zeta(t,x)\,, \\ \partial_{\mathbf{n}}\varphi_{|_{z=-1+\beta b}} = 0, & \text{at} \quad z = -1+\beta b(x)\,, \\ \varphi_{|_{z=\varepsilon\zeta}} = \psi(t,x)\,. & \text{at} \quad z = \varepsilon\zeta(t,x)\,, \end{cases} \tag{11}$$

The second equation in (8) enters as a boundary condition in the second equation of (10) in terms of the Dirichlet–Neumann operator defined as

$$\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi = \sqrt{1+\mu\varepsilon^2\zeta_x^2}\left(\partial_{\mathbf{N}_{surf}}\varphi\right)_{|_{z=\varepsilon\zeta}} = -\varepsilon\mu\zeta_x(\varphi_x)_{|_{z=\varepsilon\zeta}} + (\varphi_z)_{|_{z=\varepsilon\zeta}}\,. \tag{12}$$

2 | ASYMPTOTIC ANALYSIS IN THE BOUSSINESQ–PEREGRINE REGIME

We are interested in waves of small amplitude in shallow water, and we thus assume $\varepsilon \sim \mu \ll 1$. On the other hand, we allow for large variations in topography and make no assumption on the size of β. It is thus possible to set $\beta \sim 1$ throughout this article, and indeed the original Boussinesq–Peregrine system was derived by Boussinesq[6] and Peregrine[28] under this assumption.

2.1 | Derivation of the Boussinesq–Peregrine equations

We give a brief review of the asymptotic derivation of the model system following the method introduced in Ref. 24. Focusing on the first equation, we define v as the average horizontal velocity in the fluid column. Using the kinematic boundary conditions at the free surface and at the bottom, it can be shown that

$$hv = \int_{-1+\beta b}^{\varepsilon\zeta} \varphi_x \, dz \,, \qquad \text{where} \qquad h = h_b + \varepsilon\zeta \quad \text{and} \quad h_b = 1 - \beta b \,. \tag{13}$$

Hence, the first equation of (10) coincides with the mass conservation equation $\zeta_t + \partial_x(hv) = 0$.

From the second equation of (10), we seek an evolution equation for v in terms of (ζ, v). In other words, we expand the velocity potential φ to second-order in ε using the assumption $\varepsilon \sim \mu \ll 1$. Indeed, we stipulate an asymptotic expansion of the potential in the form

$$\varphi = \varphi_{app} + O(\varepsilon^2) = \varphi_0 + \varepsilon\varphi_1 + O(\varepsilon^2) \,, \tag{14}$$

where φ_0 and φ_1 are solutions to the following differential equation:

$$\partial_z^2\varphi_0 + \varepsilon\partial_z^2(\varphi_1) = -\varepsilon\partial_x^2(\varphi_0) + O(\varepsilon^2) \,.$$

In fact, when $\varepsilon \sim \mu \ll 1$, the boundary conditions above (11) ensure that

$$\text{for } \; i = 0, 1, \qquad \partial_z \varphi_i{}_{|_{z=-1+\beta b}} = \beta b_x \partial_x \varphi_{i-1}{}_{|_{z=-1+\beta b}} \,, \qquad \text{and} \qquad \varphi_i{}_{|_{z=\varepsilon\zeta}} = \delta_{0,i}\psi \,, \tag{15}$$

with $\varphi_{-1} = 0$. Therefore, by solving the above differential equations in the form $\partial_z^2\varphi_0 = 0$, $\partial_z^2\varphi_1 = -\partial_x^2\varphi_0$, the approximate solutions to the boundary-value problem (11) after dropping the $O(\varepsilon^2)$ terms are the following polynomials in z:

$$\varphi_0(t,x) = \psi, \qquad \qquad \varphi_1(t,x,z) = -\frac{1}{2}(z^2 + 2z - 2\beta b z)\psi_{xx} + \beta b_x z \psi_x \,. \tag{16}$$

Substituting the above solutions in (13) and dropping $O(\varepsilon^2)$ components, it appears that

$$\begin{aligned} hv &= h\psi_x + \frac{1}{3}\varepsilon h_b^3 \psi_{xxx} - \varepsilon\beta h_b^2 b_x \psi_{xx} - \frac{1}{2}\varepsilon\beta h_b^2 b_{xx}\psi_x + O(\varepsilon^2) \\ &= h\psi_x + \frac{1}{3}\varepsilon\partial_x\big(h_b^3\psi_{xx}\big) - \frac{1}{2}\varepsilon\beta\big(\partial_x(h_b^2 b_x\psi_x) - h_b^2 b_x \psi_{xx}\big) - \varepsilon\beta^2 h_b b_x^2 \psi_x + O(\varepsilon^2) \,. \end{aligned} \tag{17}$$

At this stage, we are looking for an approximate expansion with respect to ε of ψ_x in terms of ζ and v. Since

$$h^{-1} = h_b^{-1}\frac{1}{1 + \varepsilon h_b^{-1}\zeta} = h_b^{-1}(1 + O(\varepsilon)) \,, \tag{18}$$

we can express (17) as

$$\psi_x = v - \frac{1}{3h_b}\varepsilon\partial_x\big(h_b^3\psi_{xx}\big) + \frac{1}{2h_b}\varepsilon\beta\big(\partial_x(h_b^2 b_x\psi_x) + h_b^2 b_x\psi_{xx}\big) + \varepsilon\beta^2 b_x^2\psi_x + O(\varepsilon^2) \,. \tag{19}$$

Note that this equation implies the simpler relation

$$\psi_x = v + O(\varepsilon)\,. \tag{20}$$

Now, combining (20) with (19) and (17) yields the desired ε^2-approximation of ψ_x and $h\psi_x$ in terms of ζ and v:

$$\psi_x = v - \frac{1}{3h_b}\varepsilon\partial_x\left(h_b^3 v_x\right) + \frac{1}{2h_b}\varepsilon\beta\left(\partial_x(h_b^2 b_x v) - h_b^2 b_x v_x\right) + \varepsilon\beta^2 b_x^2 v + O(\varepsilon^2) \tag{21}$$

$$h\psi_x = hv - \frac{1}{3}\varepsilon\partial_x\left(h_b^3 v_x\right) + \frac{1}{2}\varepsilon\beta\left(\partial_x(h_b^2 b_x v) - h_b^2 b_x v_x\right) + \varepsilon\beta^2 h_b b_x^2 v + O(\varepsilon^2)\,. \tag{22}$$

At this stage, note that the bottom profile is independent of t, so the key point is to take the spatial derivative of the second equation of (10), then multiply it by h and replace the expansion (22). Indeed, one may derive the following approximate equation:

$$\partial_t(h\psi_x) + h\zeta_x + \varepsilon v v_x = O(\varepsilon^2)\,. \tag{23}$$

Again using the mass equation such that $h_t = -\varepsilon v_x + O(\varepsilon^2)$, Equation (23) can be written in the form

$$\left(1 - \frac{1}{3h_b}\varepsilon\partial_x\left(h_b^3\partial_x\cdot\right) + \frac{1}{2h_b}\varepsilon\beta\left(\partial_x(h_b^2 b_x\cdot) - h_b^2 b_x\partial_x\cdot\right) + \varepsilon\beta^2 b_x^2\partial_x\cdot\right)v_t + \zeta_x + \varepsilon v v_x = O(\varepsilon^2)\,. \tag{24}$$

Defining the second-order elliptic linear operator

$$\begin{aligned} h_b\mathcal{T}[h_b,\beta b]w &= -\frac{1}{3}\partial_x(h_b^3\partial_x w) + \frac{\beta}{2}\left[\partial_x(h_b^2 b_x w) - h_b^2 b_x\partial_x w\right] + \beta^2 h_b b_x^2 w \\ &= -\frac{1}{3}h_b^3\partial_x^2 w + \beta h_b^2 b_x\partial_x w + \frac{1}{2}\beta h_b^2 b_{xx} w\,, \end{aligned} \tag{25}$$

the Boussinesq–Peregrine equations can be written in the tidy form

$$\begin{cases} \partial_t\zeta + \partial_x(h_b + \varepsilon\zeta v) = 0 \\ (1 + \varepsilon\mathcal{T}[h_b,\beta b])\partial_t v + \zeta_x + \varepsilon v v_x = 0\,. \end{cases} \tag{26}$$

2.2 | Asymptotic expansion of the total pressure

In what follows, it will be important to find an appropriate expression for the dynamic fluid pressure in terms of the surface elevation at order ε^2. For this purpose, we use the nondimensional Bernoulli equation

$$\overline{P}(t,x,z) = \frac{1}{ag}(P - P_{atm}) + \frac{1}{\varepsilon}z = -\partial_t\varphi - \frac{1}{2}\left(\varepsilon\varphi_x^2 + \frac{\varepsilon}{\mu}\varphi_z^2\right)\,. \tag{27}$$

In fact, having an appropriate expression for the pressure is essential for the formulation of the momentum and energy balance (see Sections 3.2 and 3.3). Substituting (14) with (16) in (27) and dropping all terms of order ε^2, we find

$$\overline{P}(t,x,z) = -\psi_t - \frac{\varepsilon}{2}\psi_x^2 + \frac{\varepsilon}{2}(z^2 + 2z - 2\beta bz)\psi_{txx} - \varepsilon\beta b_x z\psi_{tx} + O(\varepsilon^2)\,.$$

From the system of Equations (10), we trade the time derivative on ψ by spatial derivatives such that $\psi_t = -\zeta - \frac{\varepsilon}{2}\psi_x^2 + O(\varepsilon^2)$, and we finally obtain

$$\overline{P}(t,x,z) = \zeta - \frac{\varepsilon}{2}(z^2 + 2z - 2\beta bz)\zeta_{xx} + \varepsilon\beta b_x z\zeta_x + O(\varepsilon^2)\,. \tag{28}$$

It is clear that this expression only depends on the variable $\zeta(t,x)$. Note that the first term on the right-hand side corresponds to the hydrostatic pressure, and the ε-components correspond to the nonhydrostatic pressure. Note that this formulation would change in the presence of capillarity (see for example Ref. 26), but it is assumed here that surface tension is negligible.

2.3 | Exact and almost conservation forms

Clearly, as the bathymetry given by the function $b(x)$ is time independent, the first evolution equation in the system (26) is conservative:

$$\underbrace{\partial_t(h) + \varepsilon\partial_x(hv)}_{C} = 0\,. \tag{29}$$

Moreover, in view of (29) and the second equation of (26) multiplied by h_b, we arrive at the following almost conserved form, given by

$$\underbrace{\partial_t\left(h_b v - \frac{1}{3}\varepsilon h_b^3 v_{xx} + \varepsilon\beta h_b^2 b_x v_x + \frac{1}{2}\varepsilon\beta h_b^2 b_{xx} v\right) + \partial_x\left(h_b\zeta + \frac{1}{2}\varepsilon h_b v^2\right) + \beta b_x\left(\zeta + \frac{1}{2}\varepsilon v^2\right)}_{C_*} = 0\,. \tag{30}$$

Furthermore, multiplying the first equation of (26) by ζ yields

$$h_b v\zeta_x = \frac{1}{2}\partial_t(\zeta^2) + \partial_x(h\zeta v) - \frac{1}{2}\varepsilon\partial_x(\zeta^2 v) + \frac{1}{2}\varepsilon\zeta^2 v_x\,.$$

Using the above equation with the second equation of (26) multiplied by $h_b v$, we arrive at the following almost conservation form, given by

$$\underbrace{\tfrac{1}{2}\partial_t\left(h_b v^2 + \zeta^2 + \tfrac{1}{3}\varepsilon h_b^3 v_x^2 - \varepsilon\beta h_b^2 b_x v v_x + \varepsilon\beta^2 h_b b_x^2 v^2\right) + \partial_x\left(\tfrac{1}{3}\varepsilon h_b v^3 + h\zeta v - \tfrac{1}{2}\varepsilon\zeta^2 v - \tfrac{1}{3}\varepsilon h_b^3 v v_{xt} + \tfrac{1}{2}\varepsilon\beta h_b^2 b_x v v_t\right) + \tfrac{1}{2}\varepsilon\zeta^2 v_x + \tfrac{1}{3}\varepsilon\beta b_x v^3}_{C_{**}} = 0\,. \tag{31}$$

We will see later in Section 4 that the aforementioned identities are crucial in achieving ε^2-estimates. It turns out that the mass, momentum, and the energy balance laws are perfectly fit by C, C_*, and C_{**}, respectively.

2.4 | Well-posedness results

Appropriate well-posedness results are required for the L^∞-estimates of the mechanical balance laws in Section 4. From Chapter 9 of Ref. 23, or similar to work done in Refs. 9, 13 and 19, 20 to similar models, it is possible to deduce the following local existence result for (26) on time scales T of order $\frac{1}{\max(\varepsilon,\beta)}$. In addition, the author in Ref. 25 modifies the structure of system (26) in the hope of eliminating the bottom topography parameter's dependence on time existence. In this case, the time interval scales with ε^{-1}.

We use the following standard notation. For any real constant s, $H^s = H^s(\mathbb{R})$ denotes the Sobolev space of all tempered distributions f with the norm $|f|_{H^s} = |\Lambda^s f|_2 < \infty$, where Λ is the pseudo-differential operator $\Lambda^s = (1-\partial_x^2)^{s/2}$. The symbol $\varepsilon \vee \beta$ denotes the maximum of the values of ε and β.

Theorem 1 (Local well-posedness[25]). *Fix any $s > 3/2$ and $b \in H^{s+1}(\mathbb{R})$ such that the below assumption is satisfied for any $x \in \mathbb{R}$:*

$$\textit{there exist } h_{min} > 0 \textit{ such that } h_b(x) = 1 - \beta b(x) \geq h_{min} \ . \tag{32}$$

Denote by $X^s = H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$, the energy space of solutions endowed with the norm

$$|(p,q)|^2_{X^s} \ =: \ |p|^2_{H^s} + |q|^2_{H^s} + \varepsilon |q_x|^2_{H^s} \ . \tag{33}$$

Let $U_0 = (\zeta_0, v_0)^\top \in X^s$ be initial data such that (32) is satisfied for any $x \in \mathbb{R}$. Then there exists a maximal $T = T(|U_0|_{X^s}) > 0$, uniformly bounded from below with respect to $\varepsilon, \mu, \beta \in (0,1)^3$, such that the Boussinesq–Peregrine equations (26) admit a unique solution $U = (\zeta, v)^\top \in C([0,T), X^s)$. Additionally, we have the solution size estimate

$$|U|_{X^s} \leq C\big(h^{-1}_{min}, |b_x|_{H^s}, |U_0|_{X^s}\big), \qquad for \quad 0 \leq t \leq T \ . \tag{34}$$

This estimate provides an upper bound for the solution U in the Sobolev norm X^s during the time interval $0 \leq t \leq T$.

Remark 1. It is worth noting that the solution of (26) converges to the solution of the water-wave problem with precision ε^2.

3 | MECHANICAL BALANCE LAWS

This section is devoted to the derivation of the mass, momentum and energy densities, and fluxes for the Boussinesq–Peregrine equations (26). The local balance laws consist of terms of the same asymptotic order ε^2 as in the Boussinesq–Peregrine equations. Mechanical balance laws for simpler equations have been found in several previous works,[1,2,15,18] but a proof of convergence has been given in some specific cases.[14,17] In the following, we assume that the fluid density is $\rho = 1$ and recall that the dimensionless fluid domain for each time variable $\Omega_t = \{(x,z) \in \mathbb{R} \times \mathbb{R}, \text{ such that} -1+\beta b < z < \varepsilon\zeta\}$ is bounded from above by the dimensionless free surface wave $\varepsilon\zeta(t,x)$ and from below by $-1+\beta b(x)$ the dimensionless bottom parameterization. We introduce a subdomain $\Omega_{t,\lambda}$, which is also limited in the horizontal direction. This domain is defined as $\Omega_{t,\lambda} = \{(x,z) \in [x_1/\lambda, x_2/\lambda] \times \mathbb{R}, \text{ such that} -1+\beta b < z < \varepsilon\zeta\}$, where x_1 and x_2 satisfy the inequality $x_1 < x_2$.

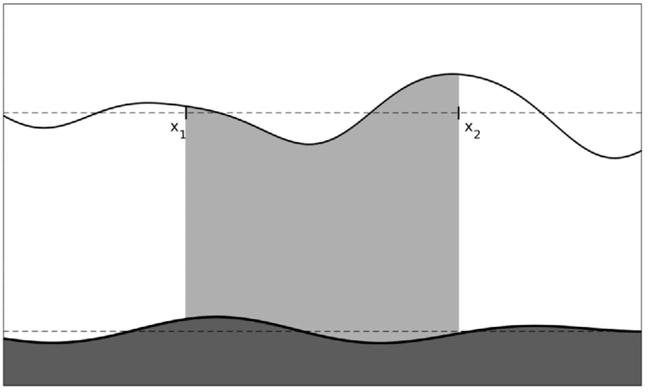

FIGURE 2 Control volume for the analysis of mass, momentum, and energy conservation. The control volume is bounded above by the free surface and below by the fluid bed and reaches from x_1 to x_2 along the horizontal.

3.1 | Mass density and flux

In this subsection, we investigate the mass conservation properties of Equations (26). At any time, the mass inside the control volume shown in Figure 2 is given by

$$M = \int_{\Omega_{t,\lambda}} \rho \, dz \, dx = \lambda h_0 \int_{x_1/\lambda}^{x_2/\lambda} \int_{-1+\beta b}^{\varepsilon\zeta} dz \, dx \, .$$

Since mass is always conserved in classical mechanics, the rate of decrease of mass in $\Omega_{t,\lambda}$ (i.e., $-\partial_t M$) is equal to the rate of outward mass flux of $\Omega_{t,\lambda}$. By definition, due to the incompressibility condition, the horizontal component of the velocity in the fluid domain is $\partial_x \varphi$. Now, since there is no mass flux through the bottom or through the free surface, then by Green's formula, the dimensionless mass conservation can be stated in terms of the flow variables as

$$\partial_t \int_{x_1/\lambda}^{x_2/\lambda} \int_{-1+\beta b}^{\varepsilon\zeta} dz \, dx = -\varepsilon \left[\int_{-1+\beta b}^{\varepsilon\zeta} \partial_x \varphi(t,x,z) \, dz \right]_{x_1/\lambda}^{x_2/\lambda} .$$

Using the change of variables $z' = z + 1 - \beta b$, integrating with respect to z' and replacing the expressions (14), (16), and (21), it can be seen that (we eliminate the primes for the sake of clarity)

$$\partial_t \int_{x_1/\lambda}^{x_2/\lambda} \int_0^h dz \, dx = -\varepsilon \left[\int_0^h \partial_x \varphi_0 + \varepsilon \partial_x \varphi_1 + O(\varepsilon^2) \, dz \right]_{x_1/\lambda}^{x_2/\lambda}$$

$$\partial_t \int_{x_1/\lambda}^{x_2/\lambda} h \, dx = -\left[\int_0^h \varepsilon \psi_x(t,x) + O(\varepsilon^2) \;\, dz \right]_{x_1/\lambda}^{x_2/\lambda}$$

$$\frac{1}{x_2/\lambda - x_1/\lambda} \int_{x_1/\lambda}^{x_2/\lambda} \partial_t h \, dx = -\frac{1}{x_2/\lambda - x_1/\lambda} \left[\varepsilon h_b v + O(\varepsilon^2) \right]_{x_1/\lambda}^{x_2/\lambda} .$$

Taking the limit $x_2/\lambda - x_1/\lambda \to 0$ shows that the nondimensional mass balance equation reads

$$h_t + \varepsilon \partial_x (h_b v) = O(\varepsilon^2) \, . \tag{35}$$

It is important to note that the solutions of the Boussinesq–Peregrine system do not exactly satisfy the mass balance (35), that is, the conservation form (29), primarily because of the assumption $\varepsilon \sim \mu$.

3.2 | Momentum density and flux

In this section, we investigate the momentum conservation properties of the system (26). If momentum is conserved, then the rate of change of the horizontal momentum is equal to the net influx of momentum through the boundaries plus the net work done on the boundary of the control volume. Indeed, applying ∂_x to the second equation of the Bernoulli formulation (8), and then using the incompressibility condition (i.e., the first equation), we obtain

$$\varphi_{tx} + \varepsilon(\varphi_x^2)_x + (\varphi_x\varphi_z)_z + \overline{P}_x = 0\,. \tag{36}$$

Integrating the equation above vertically in the fluid domain, and using the boundary conditions in (8), momentum conservation appears in the local form

$$\begin{aligned}\partial_t \int_{-1+\beta b}^{\varepsilon\zeta} \varphi_x\, dz + \partial_x\left(\varepsilon \int_{-1+\beta b}^{\varepsilon\zeta} \varphi_x^2\, dz + \int_{-1+\beta b}^{\varepsilon\zeta}\left(\overline{P} - \frac{1}{\varepsilon}z\right) dz\right)&\\ = \varepsilon\zeta_x\left(\overline{P}(t,x,z) - \frac{1}{\varepsilon}z\right)_{|z=\varepsilon\zeta} - \beta b_x\left(\overline{P}(t,x,z) - \frac{1}{\varepsilon}z\right)_{|z=-1+\beta b}&\,.\end{aligned} \tag{37}$$

First, we recall that by definition (13), $\int_{-1+\beta b}^{\varepsilon\zeta} \varphi_x\, dz = hv$. Now, replacing expressions (14), (16), (28), (21), and (22) and dropping ε^2-components, the following integrals can be evaluated:

$$\varepsilon \int_{-1+\beta b}^{\varepsilon\zeta} \varphi_x^2\, dz = \varepsilon h_b v^2 + O(\varepsilon^2)$$

$$\int_{-1+\beta b}^{\varepsilon\zeta}\left(\overline{P} - \frac{1}{\varepsilon}z\right) dz = \frac{1}{2\varepsilon}(\varepsilon\zeta + h_b)^2 + \frac{\varepsilon}{3}h_b^3\zeta_{xx} - \frac{\varepsilon}{2}\beta h_b^2 b_x\zeta_x + O(\varepsilon^2)\,.$$

Furthermore, in view of the dynamic pressure expression (28), we see that

$$\varepsilon\zeta_x\left(\overline{P}(t,x,z) - \frac{1}{\varepsilon}z\right)_{|z=\varepsilon\zeta} = O(\varepsilon^2), \qquad \overline{P}(t,x,z)_{|z=-1+\beta b} = \zeta + \frac{\varepsilon}{2}h_b^2\zeta_{xx} - \varepsilon\beta h_b b_x\zeta_x + O(\varepsilon^2)\,.$$

Consequently, the momentum balance law can be expressed in the form

$$\partial_t(hv) + \partial_x\left(\frac{1}{2\varepsilon}(\varepsilon\zeta + h_b)^2 - \frac{1}{2\varepsilon}h_b^2 + \varepsilon h_b v^2 + \frac{1}{3}\varepsilon h_b^3\zeta_{xx}\right) = -\beta b_x\zeta + \frac{1}{2}\varepsilon\beta h_b^2 b_{xx}\zeta_x + O(\varepsilon^2)\,. \tag{38}$$

From the above relation, the corresponding dimensionless expressions for the momentum density and momentum flux can be read off and are given by

$$I = hv \qquad \text{and} \qquad q_I = \frac{1}{2\varepsilon}h^2 - \frac{1}{2\varepsilon}h_b^2 + \varepsilon h_b v^2 + \frac{1}{3}\varepsilon h_b^3\zeta_{xx}\,.$$

3.3 | Energy density and flux

Attention is now turned to the mechanical energy in the fluid. Usually, the energy equation is obtained by taking the scalar product of the vector velocity $\nabla_{x,z}\varphi$ with the vector momentum equation (Bernoulli's equation (5)). Indeed, applying ∂/∂_x to the second equation of (8) and multiplying by φ_x yields

$$\frac{1}{2}\partial_t(\varphi_x^2) + \varphi_x\partial_x\left\{\frac{1}{2}\left(\varepsilon\varphi_x^2 + \varphi_z^2\right) + \overline{P}\right\} = 0\,.$$

Similarly, applying ∂/∂_z to the second equation of (8) and multiplying by φ_z yields

$$\frac{1}{2\varepsilon}\partial_t(\varphi_z^2) + \varphi_z\frac{1}{\varepsilon}\partial_z\left\{\frac{1}{2}\left(\varepsilon\varphi_x^2 + \varphi_z^2\right) + \overline{P}\right\} = 0\,.$$

Summing the two equations above, using the incompressibility condition, and noting that $2\varepsilon^{-2}\partial z/\partial_t = 0$ since z and t are independent variables, yields the energy equation in the form

$$\frac{1}{2}\partial_t\left(\varphi_x^2 + \frac{1}{\varepsilon}\varphi_z^2 + \frac{2}{\varepsilon^2}z\right) + \partial_x\left\{\frac{1}{2}\left(\varepsilon\varphi_x^2 + \varphi_z^2\right)\varphi_x + \overline{P}\varphi_x\right\} + \frac{1}{\varepsilon}\partial_z\left\{\frac{1}{2}\left(\varepsilon\varphi_x^2 + \varphi_z^2\right)\varphi_z + \overline{P}\varphi_z\right\} = 0\,. \tag{39}$$

Integrating vertically in the fluid domain, the energy equation reads

$$\frac{1}{2}\frac{\partial}{\partial_t}\int_{-1+\beta b}^{\varepsilon\zeta}\left(\varphi_x^2 + \frac{1}{\varepsilon}\varphi_z^2 + \frac{2}{\varepsilon^2}z\right)dz + \frac{\partial}{\partial_x}\left(\frac{1}{2}\int_{-1+\beta b}^{\varepsilon\zeta}\left(\varepsilon\varphi_x^2 + \varphi_z^2\right)\varphi_x\,dz + \int_{-1+\beta b}^{\varepsilon\zeta}\overline{P}\varphi_x\,dz\right)$$
$$= \left(\overline{P} - \frac{1}{\varepsilon}z\right)_{|z=\varepsilon\zeta}\left(\varepsilon\zeta_x\varphi_x - \frac{1}{\varepsilon}\varphi_z\right)_{|z=\varepsilon\zeta}. \tag{40}$$

Replacing expressions (14), (16), (28), (21), and (22) and dropping ε^2-components in the above integrals, it holds that

$$\int_{-1+\beta b}^{\varepsilon\zeta}\left(\varphi_x^2 + \frac{1}{\varepsilon}\varphi_z^2 + \frac{2}{\varepsilon^2}z\right)dz = \zeta^2 - \frac{1}{\varepsilon^2}h_b^2 + hv^2 + \varepsilon\left[\frac{1}{3}h_b^3v_x^2 - \beta h_b^2 b_x v v_x + \beta^2 h_b b_x^2 v^2\right] + O(\varepsilon^2)\,, \tag{41}$$

$$\frac{1}{2}\int_{-1+\beta b}^{\varepsilon\zeta}\left(\varepsilon\varphi_x^2 + \varphi_z^2\right)\varphi_x\,dz + \int_{-1+\beta b}^{\varepsilon\zeta}\overline{P}\varphi_x\,dz = \frac{1}{2}\varepsilon h_b v^3 + \zeta hv + \frac{1}{3}\varepsilon h_b^3\zeta_{xx}v - \frac{1}{2}\varepsilon\beta h_b^2 b_x\zeta_x v + O(\varepsilon^2)\,. \tag{42}$$

To compute the right-hand side, we need to look for an asymptotic expansion of the potential velocity up to order ε^3 instead of ε^2 as before. That is, the approximate potential is now of the form

$$\varphi_{app} = \varphi_0 + \varepsilon\tilde{\varphi}_1 + \varepsilon^2\tilde{\varphi}_2\,. \tag{43}$$

As in Section 2, the ε^3-approximate solutions to the boundary-value problem (11) (we refer also to Ref. 21 for straightforward deduction of the expressions) are given by

$$\begin{cases} \tilde{\varphi}_1 = \varphi_1 + \varepsilon\zeta\partial_x(h_b\psi_x) \\ \tilde{\varphi}_2 = z\beta b_x\Big(-h_b\beta b_x\psi_{xx} + \frac{1}{2}h_b^2\psi_{xxx} - \beta h_b\partial_x(b_x\psi_x)\Big) - \Big(\frac{1}{3}z^3 - zh_b^2\Big)\beta b_x\psi_{xxx} \\ \qquad -\frac{1}{2}\Big(\frac{1}{3}z^3 - zh_b^2\Big)\beta b_{xx}\psi_{xx} + \Big(\frac{1}{24}z^4 + \frac{1}{6}h_b^3 z + \frac{1}{2}(\frac{1}{3}z^3 - h_b^2 z)h_b\Big)\psi_{xxxx} \\ \qquad -\frac{1}{2}\Big(\frac{1}{3}z^3 - h_b^2 z\Big)\beta\partial_x^2(b_x\psi_x)\,. \end{cases} \tag{44}$$

Indeed, by (28) and (20), (14), and (16), it holds that

$$\left(\overline{P} - \frac{1}{\varepsilon}z\right)_{|z=\varepsilon\zeta} = O(\varepsilon^2)\,, \qquad \varepsilon(\zeta_x\varphi_x)_{|z=\varepsilon\zeta} = \varepsilon\zeta_x v + O(\varepsilon^2)\,.$$

In addition, combining (43) and (44), it holds that

$$-\frac{1}{\varepsilon}(\varphi_z)_{|z=\varepsilon\zeta} = \partial_x(h_b v) + \varepsilon\zeta v_x + O(\varepsilon^2)\,.$$

Consequently, combining the above expressions it holds that

$$\left(\overline{P} - \frac{1}{\varepsilon}z\right)_{|z=\varepsilon\zeta}\left(\varepsilon\zeta_x\varphi_x - \frac{1}{\varepsilon}\varphi_z\right)_{|z=\varepsilon\zeta} = -\zeta_t\left(\overline{P} - \frac{1}{\varepsilon}z\right)_{|z=\varepsilon\zeta} = O(\varepsilon^2)\,. \tag{45}$$

Finally, gathering (41), (42), and (45) in (40), the energy balance law reads

$$\begin{aligned} &\frac{1}{2}\partial_t\left(\zeta^2 - \frac{1}{\varepsilon^2}h_b^2 + hv^2 + \frac{1}{3}\varepsilon h_b^3 v_x^2 - \varepsilon\beta h_b^2 b_x v v_x + \varepsilon\beta^2 h_b b_x^2 v^2\right) \\ &+ \partial_x\left(\frac{1}{2}\varepsilon h_b v^3 + \zeta h v + \frac{1}{3}\varepsilon h_b^3\zeta_{xx}v - \frac{1}{2}\varepsilon\beta h_b^2 b_x\zeta_x v\right) = O(\varepsilon^2)\,. \end{aligned} \tag{46}$$

4 | ESTIMATES ON THE MECHANICAL BALANCE LAWS

The aim of this section is to show that the L^∞-norms of the mechanical balance laws derived in Section 3 can be bounded by positive constants that depend only on the X^s-norm of the initial data and the nondimensional parameters given in (7). Before stating and proving the main theorems in this section, let us introduce some results on the invertibility of the left-most linear operator of the second equation of (26), denoted by $\mathfrak{I} = h_b - h_b\mathcal{T}[h_b, \beta b]$, on which most of the analysis in this section is based.

Denote by $C_b^\infty(\mathbb{R})$, the space of infinitely differentiable functions that are bounded together with all their derivatives. The following lemma recalls some useful explanations of the operator $\mathfrak{I}$ and its inverse.

Lemma 1. *Suppose that $b \in C_b^\infty(\mathbb{R})$ and under the assumption (32). The operator*

$$\mathfrak{I} : H^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})\,,$$

is well-defined, one-to-one and onto.

Proof. The proof to the same operator is given in Ref. 13 (Lemma 1). □

The following lemma gives good properties to the inverse operator.

Lemma 2. *Suppose that $b \in C_b^\infty(\mathbb{R})$ and let $t_0 > 1/2$ such that $b \in H^{t_0+1}(\mathbb{R})$ under assumption (32). Then, we have*

(i) if $0 \le s \le t_0 + 1$, it holds that

$$|\mathfrak{T}^{-1} f|_{H^s} + \sqrt{\varepsilon}|\partial_x \mathfrak{T}^{-1} f|_{H^s} + \sqrt{\varepsilon}|\mathfrak{T}^{-1}\partial_x f|_{H^s} \le C\left(h_{min}^{-1}, |b|_{H^{t_0+1}}\right)|f|_{H^s}$$

(ii) $\forall\, s \ge t_0 + 1$ and $\zeta \in H^s(\mathbb{R})$, the above inequalities also hold but in this case the constant on the right-hand side depends on $|b|_{H^s}$ instead $|b|_{H^{t_0+1}}$.

Proof. Also, the proof of this lemma can be found in Ref. 13 (Lemma 2). □

In view of the above lemmas, we prove the following estimates on v_t that we shall use intensively in this section.

Lemma 3. *Suppose that $(\zeta, v) \in H^{s+3}(\mathbb{R})^2$ for any $s \in \mathbb{R}$ and assumption (32) holds. We have*

$$|\partial_t v|_{H^s} \le C|\zeta|_{H^{s+1}}|v|_{H^{s+1}} \ , \tag{47}$$

$$|\partial_t v + \zeta_x|_{H^s} \le \varepsilon \underline{C}|\zeta|_{H^{s+3}}|v|_{H^{s+3}} \ , \tag{48}$$

where C is a constant depending on h_{min}^{-1} and $|b|_{H^s}$ and $\underline{C}$ depends in addition to $|b|_{H^{s+2}}$.

Proof. For the first estimate, we apply $\mathfrak{T}^{-1}$ to the second equation on (26) multiplied by h_b, then we have

$$v_t = -\mathfrak{T}^{-1}\partial_x\left(h_b\zeta + \frac{1}{2}\varepsilon h_b v^2\right).$$

Now, using Lemma 2, estimate (47) holds. For the second estimate, remark that from (26) and (25), we have

$$v_t + \zeta_x = \varepsilon\left[-\frac{1}{3}h_b^2\partial_x^2 v_t + \beta h_b b_x \partial_x v_t + \frac{1}{2}\beta h_b b_{xx} v_t - v v_x\right].$$

Now, using (47), estimate (48) holds. □

4.1 | Approximate mass balance

Denote by

$$\mathcal{M}(\zeta, v) = \partial_t h + \varepsilon\partial_x(h_b v)\ . \tag{49}$$

The approximate local mass balance (35) can be formulated as follows.

Theorem 2 (L^∞-momentum approximation). *Let $b \in H^{s+1}(\mathbb{R})$ such that it satisfies (32) and any $s > 3/2$. Suppose $U = (\zeta, v)^\top$ is a solution of (26) with initial data $U_0 = (\zeta_0, v_0)^\top \in X^s$. Then the solution $U \in X^s$ satisfies the following estimates for all $t \in [0, T)$:*

$$\left\| \mathcal{M}(\zeta, v) \right\|_{L^\infty(\mathbb{R})} \le \varepsilon^2 C\ , \tag{50}$$

$$\int_{\mathbb{R}} \mathcal{M}(\zeta, v)\, dx = 0. \tag{51}$$

Here C is a constant depending on $|U_0|_{X^s}$.

Proof. From the expression of the integrands (49) and (29), and after some calculations, it holds that

$$\mathcal{M}(\zeta, v) - \mathcal{C} = -\varepsilon^2\partial_x(\zeta v)\ . \tag{52}$$

Therefore, using the continuous embedding $H^{s-1>\frac{1}{2}}(\mathbb{R}) \subset L^\infty(\mathbb{R})$, with the energy size estimate(34) in hands, the desired estimate (50) holds. Now, integrating (52) in x, (51) holds. □

4.2 | Approximate momentum balance

Denote by

$$\mathcal{I}(\zeta, v) = \partial_t(hv) + \partial_x\left(\frac{1}{2}h\zeta + \frac{1}{2}h_b\zeta + \varepsilon h_b v^2 + \frac{1}{3}\varepsilon h_b^3\zeta_{xx}\right) + \beta b_x\zeta - \frac{1}{2}\varepsilon\beta h_b^2 b_{xx}\zeta_x\ . \tag{53}$$

Here we used $\frac{1}{2\varepsilon}h^2 - \frac{1}{2\varepsilon}h_b^2 = \frac{1}{2}h\zeta + \frac{1}{2}h_b\zeta$. The approximate local momentum balance (38) can be formulated as follows.

Theorem 3 (L^∞-momentum approximation). *Let $b \in H^{s+2}(\mathbb{R})$ such that it satisfies (32) and any $s > 11/2$. Suppose $U = (\zeta, v)^\top$ is a solution of (26) with initial data $U_0 = (\zeta_0, v_0)^\top \in X^s$. Then, the solution $U \in X^s$ satisfies the following estimates for all $t \in [0, T)$:*

$$\left\| \mathcal{I}(\zeta, v) \right\|_{L^\infty(\mathbb{R})} \le \varepsilon^2 C\ . \tag{54}$$

Here C is a constant depending on h_{min}^{-1}, $|b|_{H^{s+2}}$, and $|U_0|_{X^s}$.

Proof. Remark that using (29), we have $\varepsilon\partial_t(\zeta v) = \varepsilon\zeta v_t - \frac{1}{2}\varepsilon\partial_x(hv^2) - \frac{1}{2}\varepsilon^2\zeta_x v^2 + \frac{1}{2}\varepsilon\beta b_x v^2$. Now, keeping the latter identity in hands and from the expression of the integrands (53) and (30), and after some calculations, it holds that

$$
\begin{aligned}
\mathcal{I}(\zeta,v)-\mathcal{C}_* &= \varepsilon\zeta v_t+\frac{1}{3}\varepsilon h_b^3\partial_x^2 v_t-\varepsilon\beta h_b^2 b_x\partial_x v_t-\frac{1}{2}\varepsilon\beta h_b^2 b_{xx}v_t+\partial_x\left(\frac{1}{2}\varepsilon\zeta^2+\frac{1}{3}\varepsilon h_b^3\zeta_{xx}\right)-\frac{1}{2}\varepsilon\beta h_b^2 b_{xx}\zeta_x\\
&\quad -\varepsilon^2(\zeta v v_x+\zeta_x v^2)\\
&= \varepsilon\zeta(v_t+\zeta_x)+\frac{1}{3}\varepsilon h_b^3\partial_x^2(v_t+\zeta_x)-\varepsilon\beta h_b^2 b_x\partial_x(v_t+\zeta_x)-\frac{1}{2}\varepsilon\beta h_b^2 b_{xx}(v_t+\zeta_x)\\
&\quad -\varepsilon^2(\zeta v v_x+\zeta_x v^2).
\end{aligned}
\tag{55}
$$

Therefore, using the continuous embedding $H^{s-5>\frac{1}{2}}(\mathbb{R})\subset L^\infty(\mathbb{R})$, with the energy size estimate (34) in hands and estimate (48), then the desired estimate (54) holds. □

Remark 2. It is worth noting that topographic variations have no significant effect on the parameter estimate. In other words, if $\beta=0$, the classical Boussineq system has a similar ε^2-approximation as (54).

Theorem 4 (L^∞-momentum approximation). *Let $b\in H^{s+2}(\mathbb{R})$ such that it satisfies (32) and any $s>3/2$. Suppose $U=(\zeta,v)^\top$ is a solution of (26) with initial data $U_0=(\zeta_0,v_0)^\top\in X^s$. Then the solution $U\in X^s$ satisfies the following estimates for all $t\in[0,T)$:*

$$
\left\| \int_{\mathbb{R}} \mathcal{I}(\zeta,v)\,dx \right\|_{L^\infty(\mathbb{R})} \le \varepsilon^2 \underline{C}\,. \tag{56}
$$

Here C is a constant depending on h_{min}^{-1}, $|b|_{H^{s+2}}$, and $|U_0|_{X^s}$.

Proof. Note that, one may rewrite (55) as follows:

$$
\begin{aligned}
\mathcal{I}(\zeta,v) &= \partial_x\left(\frac{1}{3}\varepsilon h_b^3\partial_x(v_t+\zeta_x)-\frac{1}{3}\varepsilon(h_b^3)_x(v_t+\zeta_x)-\varepsilon\beta h_b^2 b_x(v_t+\zeta_x)-\varepsilon^2\zeta v^2\right)\\
&\quad+\varepsilon\left(\zeta+\frac{1}{3}\varepsilon(h_b^3)_{xx}+\beta(h_b^2 b_x)_x-\frac{1}{2}\beta h_b^2 b_{xx}\right)(v_t+\zeta_x)+\varepsilon^2\zeta v v_x\,.
\end{aligned}
$$

Now, since (ζ,v) and their derivatives vanish at infinity, it can be seen that

$$
\int_{\mathbb{R}} \mathcal{I}(\zeta,v)\,dx = \int_{\mathbb{R}}\left[\varepsilon\left(\zeta+\frac{1}{3}\varepsilon(h_b^3)_{xx}+\beta(h_b^2 b_x)_x-\frac{1}{2}\beta h_b^2 b_{xx}\right)(v_t+\zeta_x)+\varepsilon^2\zeta v v_x\right]dx\,,
$$

Now, by the Cauchy–Schwarz inequality, Parseval's and Young's identities, it holds that

$$
\begin{aligned}
\varepsilon^2\int_{\mathbb{R}}|\zeta v v_x|\,dx &= \varepsilon^2\int_{\mathbb{R}}(1+\xi^2)^{\frac{s}{2}}|\zeta v|(1+\xi^2)^{-\frac{s}{2}}|v_x|\,dx\Big|\\
&\le \varepsilon^2\left|(1+\xi^2)^{\frac{s}{2}}\widehat{\zeta v}\right|_{L^2(\mathbb{R})}\left|(1+\xi^2)^{-\frac{s}{2}}\widehat{v_x}\right|_{L^2(\mathbb{R})}\\
&\le \varepsilon^2|\zeta v|_{H^s}|v_x|_{H^{-s}}\le\varepsilon^2|\zeta v|_{H^s}|v_x|_{H^{s-1}}\le\varepsilon^2 C(|U|_{X^s})
\end{aligned}
$$

and

$$\frac{1}{2}\varepsilon\beta\int_{\mathbb{R}}\left|h_b^2 b_{xx}(v_t+\zeta_x)\right| dx = \frac{1}{2}\varepsilon\beta\int_{\mathbb{R}}(1+\xi^2)^{\frac{-s+3}{2}}|h_b^2 b_{xx}|(1+\xi^2)^{\frac{s-3}{2}}|v_t+\zeta_x|\, dx$$
$$\leq \frac{1}{2}\varepsilon|h_b^2 b_{xx}|_{H^s}|v_t+\zeta_x|_{H^{s-3}} \leq \varepsilon^2\underline{C}|\zeta|_{H^s}|v|_{H^s} \leq \varepsilon^2 C(|U|_{X^s})\,.$$

Here we used the fact that $-s < s-1$ and $3-s < s$, respectively. In this manner, the other components of the integral are controlled similarly and thus the desired estimate (56) holds. □

Remark 3. Also here the topographic variations have no significant effect on the parameter estimate. In other words, if $\beta = 0$, the classical Boussineq system has a similar ε^2-approximation as (56).

4.3 | Approximate energy balance

Denote by

$$\mathcal{E}(\zeta,v) = \frac{1}{2}\partial_t\left(\zeta^2 - \frac{1}{\varepsilon^2}h_b^2 + hv^2 + \frac{1}{3}\varepsilon h_b^3 v_x^2 - \varepsilon\beta h_b^2 b_x v v_x + \varepsilon\beta^2 h_b b_x^2 v^2\right) \tag{57}$$
$$+\,\partial_x\left(\frac{1}{2}\varepsilon h_b v^3 + \zeta h v + \frac{1}{3}\varepsilon h_b^3\zeta_{xx}v - \frac{1}{2}\varepsilon\beta h_b^2 b_x\zeta_x v\right).$$

The approximate local energy balance (46) can be formulated as follows.

Theorem 5 (L^∞-momentum approximation). *Let $b \in H^{s+2}(\mathbb{R})$ such that it satisfies (32) and any $s > 11/2$. Suppose $U = (\zeta, v)^\top$ is a solution of (26) with initial data $U_0 = (\zeta_0, v_0)^\top \in X^s$. Then the solution $U \in X^s$ satisfies the following estimates for all $t \in [0,T)$:*

$$\left\|\mathcal{E}(\zeta,v)\right\|_{L^\infty(\mathbb{R})} \leq \varepsilon^2 C\,. \tag{58}$$

Here C is a constant depending on h_{min}^{-1}, $|b|_{H^{s+2}}$, and $|U_0|_{X^s}$.

Proof. Remark that using (29), we have $\frac{1}{2}\varepsilon\partial_t(\zeta v^2) = \varepsilon\zeta v v_t - \frac{1}{6}\varepsilon\partial_x(hv^3) - \frac{1}{3}\varepsilon^2\zeta_x v^3 + \frac{1}{3}\varepsilon\beta b_x v^3$. Now, keeping the latter identity in hands and from the expression of the integrands (53) and (30), and after some calculations it holds that

$$\mathcal{E}(\zeta,v) - C_{**} = \varepsilon\zeta v(v_t+\zeta_x) + \varepsilon\partial_x\left(\frac{1}{3}h_b^3 v\partial_x(v_t+\zeta_x) - \frac{1}{2}\beta h_b^2 b_x v(v_t+\zeta_x) - \frac{1}{2}\varepsilon\zeta v^3\right)$$
$$+\,\varepsilon^2\zeta v^2 v_x. \tag{59}$$

Therefore, using the continuous embedding $H^{s-5>\frac{1}{2}}(\mathbb{R}) \subset L^\infty(\mathbb{R})$, with the energy size estimate (34) in hands and estimate (48), then the desired estimate (56) holds. □

Remark 4. It is worth noting that topographic variations have no significant effect on the parameter estimate. In other words, if $\beta = 0$, the classical Boussineq system has a similar ε^2-approximation as (54).

Theorem 6 (L^1-momentum approximation). *Let $b \in H^{s+2}(\mathbb{R})$ such that it satisfies (32) and any $s > 3/2$. Suppose $U = (\zeta, v)^\top$ is a solution of (26) with initial data $U_0 = (\zeta_0, v_0)^\top \in X^s$. Then the solution $U \in X^s$ satisfies the following estimates for all $t \in [0, T)$:*

$$\Big\| \int_{\mathbb{R}} \mathcal{E}(\zeta, v)\, dx \Big\|_{L^\infty(\mathbb{R})} \le \varepsilon^2 \underline{C}\,. \tag{60}$$

Here C is a constant depending on h_{min}^{-1}, $|b|_{H^{s+2}}$, and $|U_0|_{X^s}$.

Proof. Now, since (ζ, v) and their derivatives vanish at infinity, from (59) it can be seen that

$$\int_{\mathbb{R}} \mathcal{E}(\zeta, v)\, dx = \int_{\mathbb{R}} \left[\varepsilon \zeta v(v_t + \zeta_x) + \varepsilon^2 \zeta v^2 v_x\right] dx\,,$$

Now, by Cauchy–Schwarz inequality, Parseval's and Young's identities, it holds that

$$\begin{aligned}
\varepsilon^2 \int_{\mathbb{R}} |\zeta v^2 v_x|\, dx &= \varepsilon^2 \Big| \int_{\mathbb{R}} (1+\xi^2)^s |\zeta v^2| (1+\xi^2)^{-s} |v_x|\, dx \\
&\le \varepsilon^2 \big|(1+\xi^2)^s \widehat{\zeta v^2}\big|_{L^2(\mathbb{R})} \big|(1+\xi^2)^{-s} \widehat{v_x}\big|_{L^2(\mathbb{R})} \\
&\le \varepsilon^2 |\zeta v^2|_{H^s} |v_x|_{H^{-s}} \le \varepsilon^2 |\zeta v^2|_{H^s} |v_x|_{H^{s-1}} \le \varepsilon^2 C(|U|_{X^s})
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon \int_{\mathbb{R}} |\zeta v(v_t + \zeta_x)|\, dx &= \frac{1}{2}\varepsilon \Big| \int_{\mathbb{R}} (1+\xi^2)^{-s+3} |\zeta v| (1+\xi^2)^{s-3} |v_t + \zeta_x|\, dx \\
&\le \frac{1}{2}\varepsilon |\zeta v|_{H^s} |v_t + \zeta_x|_{H^{s-3}} \le \varepsilon^2 \underline{C} |\zeta|_{H^s} |v|_{H^s} \le \varepsilon^2 C(|U|_{X^s})\,.
\end{aligned}$$

Here we used the fact that $-s < s-1$ and $3 - s < s$, respectively. In this manner, the other components of the integral are controlled similarly and thus the desired estimate (56) holds. □

Remark 5. Also here the topographic variations have no significant effect on the parameter estimate. In other words, if $\beta = 0$, the classical Boussineq system has a similar ε^2-approximation as (56).

5 | CONCLUSION

This work has centered on the investigation of the mechanical balance laws which were found to be approximately valid within the framework of the Boussinesq–Peregrine system, specifically in the context of surface waves over complex bathymetry. Our findings not only shed light on the intricate behavior of mass, momentum, and energy mechanical quantities but also establish the precise regime of their approximations, offering valuable insights under well-defined conditions.

Notably, the estimation parameters remain independent of bathymetric variations, underscoring the resilience of our findings to variations in seabed topography. Furthermore, our analysis emphasizes the robustness of the integral estimates of the mechanical quantities, showcasing their capacity to maintain precision without requiring additional regularity with respect to the initial data. However, we do find that a higher level of regularity is necessary to prove the approximate validity of the mechanical quantities.

Future work will include the extension of this analysis to two horizontal dimensions and the direct use of the mechanical balance laws in the numerical discretization of the governing equations.

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DATA AVAILABILITY STATEMENT

No data were used or created in the work on this article.

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